A Gluing problem for a gauged hyperbolic PDE

by

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Abstract

In this thesis, we study the dynamic Ablien Higgs model in dimension 3 at the critical coupling. This is a system of partial differential equations which enjoys local symmetries known as gauge transformations. The stationary finite energy solutions to these equations in dimension 2 have been classified by Jaffe and Taubes in 1980, the so called vortex configurations. In 1992, Stuart has proved that one can construct solutions near the critical coupling regime in dimension $1+2$ whose dynamics are approximated by a finite dimension Hamiltonian system to the moduli space which reduces to the geodesic flow at the critical coupling.

In this project, we study how one can glue the vortex configurations to find dynamic solutions in dimension 3. More precisely, we prove that one can construct solutions which are approximated by wave maps to the moduli space of vortex configurations. The proof involves an ansatz to construct approximate solutions and then add perturbations. In the ansatz, we go through an iterative mechanism to reduce the error of the approximate solution so that it is prepared to be perturbed to find an honest solution.

In both steps of the project, the ansatz and perturbation, the choice of gauge is crucial. It is noteworthy that the choice of gauge is different in these two steps. We proceed by a choice for gauge, simplify the equations and then we have to decompose the quantities into two components, the zero modes (the tangent vectors in the moduli space) and the orthogonal complement to zero modes. According to the Higgs mechanism, stability is available for the components orthogonal to zero modes. In this regard, in the ansatz, the dynamics of zero modes are designed in such a way that some orthogonality condition is satisfied. In the perturbation part, the dynamics of zero modes is forced by the evolution of orthogonal components. Obtaining desired estimates for the tangential part requires taking advantage of explicit structure of equations, rather than the usual estimates. Also, the number of iterations in the ansatz should be high enough so that the desired estimates hold for the tangential part.

Contents

Chapter 1

Abelian Higgs Model

1.1 Introduction

Abelian Higgs model is a physical model which can be regarded as part of the Higgs sector of the standard model in particle physics. In particle physics, there is a well-known concept of wave-particle duality by which studying the dynamics of particles can be replaced by the study of quantum fields defined over a space-time. In this regard, Abelian Higgs model describes the interaction of the so called Higgs field and the electromagnetic field.
Consider the spacetime $\mathbb{R}^{1+3}$. The Higgs field is a complex function

$$\Phi : \mathbb{R}^{1+3} \to \mathbb{C}$$

and the electromagentic potential is a real one-form

$$A = A_0 dt + A_1 dx^1 + A_2 dx^2 + A_3 dz$$

where t is the time component. The components of the electromganetic field are given by

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1.1}$$

where

$$\partial_0 = \partial_t \quad , \quad \partial_1 = \partial_{x_1} \quad , \quad \partial_2 = \partial_{x_2} \quad , \quad \partial_3 = \partial_z \tag{1.2}$$

The Abelian Higgs model with the coupling parameter λ is described by the following equations:

$$D_0 D_0 \Phi - \sum_{j=1}^{3} D_j D_j \Phi + \frac{\lambda}{2}\big(|\Phi|^2 - 1\big)\Phi = 0 \tag{1.3}$$

$$\Big(\partial_0 \mathcal{F}_{0j} - \sum_{k=1}^{3} \partial_k \mathcal{F}_{kj}\Big) - \big(i\Phi, D_j\Phi\big) = 0 \quad \big(\text{for } j = 0, 1, 2, 3\big) \tag{1.4}$$

where

$$D_\mu = \partial_\mu - iA_\mu \tag{1.5}$$

for $\mu = 0, 1, 2, 3$ and

$$(a, b) = \Re(a\bar{b}) \tag{1.6}$$

for any two complex numbers a, b. These notations will be used throughout this thesis.
The above equations correspond to the finding the critical points of the functional:

$$E = \int_{\mathbb{R}^{1+3}} \Big(-|D_0\Phi|^2 + \sum_{j=1}^{3} |D_j\Phi|^2 \Big) + \frac{1}{2}\sum_{\alpha,\beta} \big(\mathcal{F}_{\alpha\beta}\big)^2 + \frac{\lambda}{4}\big(|\Phi|^2 - 1\big)^2 \tag{1.7}$$

The Abelian Higgs model has a crucial property which is invariance under some local symmetries called gauge transformations. If (ϕ, A) is a solution for the above equations, then for any smooth function χ, $(\tilde{\phi}, \tilde{A})$ obtained by

$$\begin{aligned} &\tilde{\Phi}(t, x) = \Phi(t, x)e^{i\chi(t,x)} \\ &\tilde{A} = A + d\chi \end{aligned} \tag{1.8}$$

will be another solution. The transformation described by (1.8) is called a gauge transformation in the literature of mathematical physics.
One can consider similar equations for the spacetime $\mathbb{R}^{1+2}$. Also, by removing the time variable, one obtains the static two-dimensional Abelian Higgs model described by the following equations:

$$\begin{aligned} &-\sum_{j=1}^{2} D_j D_j \phi + \frac{\lambda}{2}\big(|\phi|^2 - 1\big)\phi = 0 \\ &\sum_{k=1}^{2} \partial_k \mathcal{F}_{kj} + \big(i\phi, D_j\phi\big) = 0 \quad \big(\text{for } j = 1, 2\big) \end{aligned} \tag{1.9}$$

From now on, we use the varibles (ϕ, α) for the static two-dimesnial Abelian Higgs model with

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2$$

The equations (1.9) appear in the theory of superconductivity.
The finite energy stationary solutions to the Abelian Higgs model in $2D$ in the so called critical coupling $\lambda = 1$ have been classified by the work of Jaffe and Taubes in 1990 [1]. They proved that every finite energy solution to these equations can be characterized by the zero set of the Higgs field, up to a gauge field. More precisely, they proved that given any finite set $S = \{z_1, z_2, \cdots, z_N\} \subset$ with possible multiplicities, there exists a unique smooth solution (ϕ, α) up to gauge transformations such that ϕ vanishes exactly on S. The points $\{z_j\}_{j=1}^{N}$ are called vortex centers. These solutions are called vortex configurations. In section 1.2, we give a detailed description of the construction of vortex configurations.
The question that we address in this thesis is a gluing problem; how can one use the vortex solutions to find dynamics solutions in $\mathbb{R}^{1+2}$ or $\mathbb{R}^{1+3}$. This problem has been studied in dimension 2 by Stuart in [2] in 1994 and revisited later on by Palvelev in [4] in 2008. In this thesis, we answer the question in dimension $\mathbb{R}^{1+3}$.
To answer this question, the space of whole N-vortex configurations has been considered as a smooth Riemannain manifold named the moduli space M_N. Then, in [2], it has been proved that if $(q, p) =$

$(q_\mu, p_\mu)_\mu : [0, T) \to (TM_2)^*$ satisfies a Hamiltonian system:

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H_\pm}{\partial q_\mu} \quad , \quad \frac{dq_\mu}{d\tau} = \frac{\partial H_\pm}{\partial p_\mu} \tag{1.10}$$

for some $H_\pm$, then for any $\epsilon > 0$ small enough, corresponding to the coupling constant $\lambda = 1 \pm \epsilon^2$, one can construct solutions to the AHM equations of the form:

$$\begin{pmatrix} \Phi \\ A_1, A_2 \\ A_0 \end{pmatrix} (x,t) = \begin{pmatrix} \phi((x; q(\epsilon t))e^{i\Sigma} \\ \alpha(x; q(\epsilon t)) + d\Sigma \\ 0 \end{pmatrix} + O(\epsilon) \tag{1.11}$$

for some function Σ, valid over an interval of the form $[0, \frac{T'}{\epsilon}^2)$, where the perturbation is measured in some Sobolev norm.

In [4], by similar arguments, Palvelev has proved that if $\lambda = 1$, then one can consider a geodesic $q : [0, T] \to M_N$ and construct solutions to the AHM of the form (1.11). We have given a detailed description of these two results and the main ideas of the proofs in section 1.3.

In this thesis, we prove that if $q : [0, T] \times \mathbb{R} \to M_N$ is a smooth wave map, then for any $\epsilon > 0$ small enough, one can construct solutions to the AHM close to q in the sense that:

$$\begin{pmatrix} \Phi \\ A_1, A_2 \\ A_0, A_3 \end{pmatrix} (t,x,z) = \begin{pmatrix} \phi((x; q(\epsilon t, \epsilon z)) \\ \alpha(x; q(\epsilon t, \epsilon z)) \\ 0 \end{pmatrix} + \begin{pmatrix} O(\epsilon) \\ O(\epsilon) \\ O(\epsilon^{\frac{1}{2}}) \end{pmatrix} \tag{1.12}$$

where the perturbation is measured in some Sobolev norm and the result is valid over a time interval of the form $[0, \frac{T'}{\epsilon})$. This result is proved in chapter 2.

Amongst the many ideas used in this construction, I would like to mention a heuristic on how the gauge symmetry contributes to the the aforementioned gluing problem.

1.1.1 Organization of the Thesis

The rest of chapter 1 is devoted to a detailed description of the construction of vortex configurations based on [1] in section 1.2, explaining the main ideas of the work of Stuart [2] and Palvelev [3] in 1.3 and finally the physical implication of these results in section 1.4.

In chapter 2, we give a proof of the aforementioned main result of the thesis on the approximation of the AHM equations by the wave maps to the moduli space of vortex configurations. Finally, in the appendix, we prove analytical results on the elliptic equations used in the chapter 2.

1.2 Static 2D Abelian Higgs Model

In this section, I review some well-known literature on the time independent Abelian Higgs model in dimension 2, based on [1] The stationary $2D$ equations are of independent interest, however their almost explicit structure is of great usage in the dynamic problem in chapter 2. In subsection 1.2.1, I go over their construction. Following [2], in subsection 1.2.2, I will describe the moduli space of vortex configuration as a smooth Riemannian manifold, and in subsection 1.2.3, I will explain te smooth Riemannian manifold structure on the moduli space.

1.2.1 Construction of vortex configurations

Consider the system of equations (1.9). These equations correspond to the Lagrangian:

$$\mathcal{L} = \int_{\mathbb{R}^2} \left(\sum_{j=1}^{2} |D_j\phi|^2 + \mathcal{F}_{12}^2 + \frac{\lambda}{4}\big(|\phi|^2 - 1\big)^2 \right) \tag{1.13}$$

By some integration by parts,

$$\begin{aligned} \mathcal{L} = & \int_{\mathbb{R}^2} \Big(\big(\partial_1\phi_1 + \alpha_1\phi_2\big) \mp \big(\partial_2\phi_2 - \alpha_2\phi_1\big) \Big)^2 \\ & + \int_{\mathbb{R}^2} \Big(\big(\partial_2\phi_1 + \alpha_2\phi_2\big) \pm \big(\partial_1\phi_2 - \alpha_1\phi_1\big) \Big)^2 \\ & + \int_{\mathbb{R}^2} \Big[\mathcal{F}_{12} \pm \big(|\phi|^2 - 1\big) \Big]^2 \pm \int_{\mathbb{R}^2} \mathcal{F}_{12} \\ & + \frac{(\lambda - 1)}{4} \big(1 - |\phi|^2\big)^2 \end{aligned} \tag{1.14}$$

where ϕ_1, ϕ_2 denote the real and imaginary parts of ϕ.
Suppose that $\lambda = 1$. Then, according to (1.13), finite energy solutions satisfy the conditions:

$$\begin{aligned} &|\phi| \to 1 \\ &|D_j\phi| \to 0 \quad j = 1, 2 \end{aligned} \tag{1.15}$$

at infinity. The behavior of ϕ at infinity can be described by

$$\phi_{|x| \to \infty} : S^1 \to S^1 \tag{1.16}$$

One can assign a winding number to this map which coincides with the quantity:

$$K = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathcal{F}_{12} d^2x \tag{1.17}$$

When $\lambda = 1$, one can fix K and consider the minmization problem for the Lagrnagian $\mathcal{L}$ and obtain the following equations:

$$\begin{aligned} &\big(\partial_1\phi_1 + \alpha_1\phi_2\big) \mp \big(\partial_2\phi_2 - \alpha_2\phi_1\big) = 0 \\ &\big(\partial_2\phi_1 + \alpha_2\phi_2\big) \pm \big(\partial_1\phi_2 - \alpha_1\phi_1\big) = 0 \\ &\mathcal{F}_{12} \pm \frac{1}{2}\big(|\phi|^2 - 1\big) = 0 \end{aligned} \tag{1.18}$$

One can also write these equations in the form

$$\begin{aligned} &\big(D_1 \pm iD_2\big)\phi = 0 \\ &\mathcal{F}_{12} \pm \frac{1}{2}\big(|\phi|^2 - 1\big) = 0 \end{aligned} \tag{1.19}$$

which are called the Bolgomony equations. The sign in the above equations coincides with the sign of K. The first solutions found for these equations were the rotationally symmetric ones [7]. As mentioned in [8], if the winding number is n, these solutions take the form

$$\begin{aligned} &\phi = e^{i\pi n}\rho(r) \\ &A_r = 0 \quad , \quad A_\theta = na(r) \end{aligned} \tag{1.20}$$

in polar coordniates where ρ, a satisfy the equations

$$\begin{aligned} &r\frac{d\rho}{dr} - n(1-a)\rho = 0 \\ &\frac{2n}{r}\frac{da}{dr} + (\rho^2 - 1) = 0 \end{aligned} \tag{1.21}$$

with the boundary conditions

$$\rho(0) = a(0) = 0 \quad , \quad \rho(\infty) = a(\infty) = 1 \tag{1.22}$$

The function ρ behaves as below:

$$\begin{aligned} &\rho(r) \sim Ar^n \qquad r \to 0 \\ &\rho(r) \sim 1 - BK_0(r) \qquad r \to \infty \end{aligned} \tag{1.23}$$

where BK_0 is the zeroth order modified Bessel function.

To study Bogomolny equations, one can linearize it around a given static 2D solution. Due to the gauge invariance property of the equations, onc can accompany the linearized equations with a gauge condition. Corresponding to the gauge transformations (1.8), one can define the infinitesimal gauge transformations

$$\left(\tilde{\phi}, \tilde{a}\right) \to \left(\tilde{\phi} + i\phi\chi, \tilde{a} + d\chi\right) \tag{1.24}$$

Now, Consider a vector $\left(\tilde{\phi}, \tilde{a}\right)$. We say that it satisfies the gauge orthogonality condition with respect to the vortex solution (ϕ, α), if

$$\nabla.\tilde{a} - \left(i\phi, \tilde{\phi}\right) = 0 \tag{1.25}$$

The linearized Bogomolny equations accompanied with the gauge orthogonality condition can be written in the form:

$$\mathcal{D}_{(\phi,\alpha)}\begin{pmatrix}\tilde{\phi} \\ \tilde{a}\end{pmatrix} = \begin{pmatrix}\bar{\partial}\tilde{\phi} - i\bar{a}\tilde{\phi} - i\bar{\tilde{a}}\phi \\ \partial\bar{\tilde{a}} + \frac{i}{4}\bar{\phi}\tilde{\phi}\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \tag{1.26}$$

where

$$\begin{aligned} &a = \frac{1}{2}\left(\alpha_1 - i\alpha_2\right) \\ &\tilde{a} = \frac{1}{2}\left(\tilde{a}_1 - i\tilde{a}_2\right) \\ &\partial = \frac{1}{2}\left(\partial_1 - i\partial_2\right) \end{aligned} \tag{1.27}$$

In [9], it has been shown that around a solution (ϕ, α) with the winding numnber n, there are $2n$ linearly independent solutions of the linearized equations (1.26). These solutions are called zero modes. This suggests that there might be a $2n$-parameter family of solutions with the winding number n. In [1], such solutions have been constructed. They are called "vortices" and "anti-vortices".

Theorem 1. *Given an integer $N \geq 0$ and points $z_1, z_2, \cdots, z_N$ in the complex plane. Suppose that z_j repeats n_j times in the sequence. There exists a smooth finite energy solution (ϕ, α) to equations* (1.19) *with the positive choice for sign, satisfying the following properties:*

1. *The zeros of ϕ are the set of points $z_1, z_2, \cdots, z_N$ and the behavior of ϕ around any z_j is:*

$$\phi(z) \sim c_j \big(z - z_j\big)^{n_j} \tag{1.28}$$

for some $c_j \neq 0$.

2. $$|D_1\phi| + |D_2\phi| \leq C\big(1 - |\phi|\big) \leq C \exp\big(-(1-\delta)|z|\big) \tag{1.29}$$

for any δ where $C = C(\delta)$ is a constant.

3. $$N = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathcal{F}_{12} \tag{1.30}$$

In the above theorem, the points $\big(z_1, z_2, \cdots, z_N\big)$ are called the vortex centers and solutions are called vortices or N-vortex solutions. If (ϕ, α) is an N-vortex solution, then $(\bar{\phi}, -a)$ is an N-antivortex solution for equation (1.26), but with the negative choice for sign.

To prove the above theorem, one can notice that if u solves the equation

$$-\Delta u + e^u - 1 = -4\pi \sum_{k=1}^{N} \delta(z - z_k) \tag{1.31}$$

with

$$\lim_{|z| \to \infty} u = 0 \tag{1.32}$$

then the followings will solve (1.19)

$$\begin{aligned} \phi &= \exp \frac{1}{2}\big[u + i\Theta\big] \\ \alpha_1 &= \frac{1}{2}\big(\partial_2 u + \partial_1 \Theta\big) \\ \alpha_2 &= \frac{-1}{2}\big(\partial_1 u - \partial_2 \Theta\big) \end{aligned} \tag{1.33}$$

where

$$\Theta = 2\sum_{i=1}^{N} \arg(z - z_i) \tag{1.34}$$

To solve (1.31), one can consider the functions

$$\begin{aligned} u_0 &= -\sum_{k=1}^{N} \ln\Big(1+\mu|z-z_k|^{-2}\Big) \\ g_0 &= 4\sum_{k=1}^{N} \frac{\mu}{\big(|z-z_k|^2+\mu\big)^2} \end{aligned} \tag{1.35}$$

for some $\mu > 4N$. Then, regarding u_0 as a distribution, we have:

$$\Delta u_0 = -4\sum_{k=1}^{N} \frac{\mu}{\big(|z-z_k|^2+\mu\big)^2} + 4\pi\sum_{k=1}^{N}\delta\big(z-z_k\big) \tag{1.36}$$

Therefore, by setting

$$v = u - u_0$$

(1.36) is equivalent to

$$\Delta v = e^{v+u_0} + (g_0 - 1) \tag{1.37}$$

with

$$\lim_{|z|\to\infty} |v| = 0 \tag{1.38}$$

Equation (1.37) is equivalent to finding the critical point of the functional

$$\begin{aligned} &T : H^1 \to \mathbb{R} \\ &T(v) = \int_{\mathbb{R}^2} \Big[\frac{1}{2}|\nabla v|^2 + v(g_0-1) + e^{u_0}(e^v - 1)\Big] \end{aligned} \tag{1.39}$$

It has been proved in [1] that this functional has a unique minimizer and therefore the Bolgomony equations have the prescribed solutions.

1.2.2 Moduli Space of vortex configurations

In [2], the moduli space M_N, the space of all N-vortex solutions modulo gauge equivalence, provided by theorem 1, has been discussed. M_N is homemorphic to $\mathbb{R}^{2N}$. This is because corresponding to a vortex solution (ϕ, α) with the vortex centers

$$\Big(z_1, z_2, \cdots, z_N\Big)$$

one can consider the coefficients of the polynomial

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_N)$$

and notice that this defines a bijection between $\mathbb{R}^{2N}$ and M_N.

From now on, we assume that

$$z_j = z_{_{j,1}} + iz_{_{j,2}}$$

for real numbers $z_{j,1}, z_{j,2}$. Also, we write

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_N) = S_0 + S_1 z + S_2 z^2 + \cdots + z^N$$

and assume that

$$S_j = S_{j,1} + i S_{j,2}$$

for real numbers $S_{j,1}, S_{j,2}$. Also, let

$$S = \left(S_0, S_1, \cdots, S_{N-1}\right)$$

Theorem 2. *[3] The vortex solution (ϕ, α) depends smoothly on S; for any $z \in \mathbb{C}$, the function $(\phi, \alpha)(z; S)$ depends smoothly on z.*

To justify this theorem, one can note that the functions u_0 and g_0 in (1.35) depend smoothly on S and therefore, it is expectable that v behaves smoothly with respect to S. To be more precise, one can use (1.33) to write

$$\phi(z) = p(z) f(z)$$

where

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_N) \tag{1.40}$$

and $f(z) > 0$. Let

$$w = 2 \ln f + \ln \left(1 + |p|^2\right)$$

Then,

$$w = v + \ln \left(1 + |p|^2\right) - \sum_{k=1}^{N} \ln \left(\mu + |z - z_k|^2\right) \tag{1.41}$$

and according to equation (1.37),

$$\Delta w = \frac{|p|^2}{1 + |p|^2} e^w - 1 + \Delta \ln \left(1 + |p|^2\right) \tag{1.42}$$

In [1], it has been proved that $v \in H^2$. Therefore, $w \in H^2$. To prove that w behaves smoothly with respect to S, first note that for any $h \in H^2(\mathbb{R}^2)$,

$$(e^h - 1) \in L^2(\mathbb{R}^2)$$

and the map

$$\begin{aligned} &T : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \\ &T(h) = e^h - 1 \end{aligned} \tag{1.43}$$

is smooth. This can be proved by considering the Taylor explansion of e^h and using the Sobolev embedding theorem. This implies that the map

$$F : H^2(\mathbb{R}^2) \times \mathbb{C}^N \to L^2\big(\mathbb{R}^2\big)$$
$$F(v, S) = -\Delta v + \frac{|p(S)|^2}{1 + |p(S)|^2} e^v - 1 + \Delta \ln\big(1 + |p(S)|^2\big) \tag{1.44}$$

is smooth. Now, suppose that $w(S)$ denotes the solution to equation (1.42). In [3], it has been shown that the map

$$I : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$
$$I(h) = F'_v\big(w(S); S\big)h \tag{1.45}$$

is invertible for any $S \in \mathbb{C}^N$, where F'_v denotes differentiation with respect to the first input variable of F. This together with an implicit function theorem for Banach spaces implies that w and henceforth (ϕ, α) behave smoothly with respect to S. Furthermore, one can prove the following:

Proposition 1. *For any multi-index r and every compact subset $K \subset M_N$, there exist numbers $A, B, R > 0$ such that the function $u = \ln|\phi|^2$ corresponding to a point $p \in K$ has the property*

$$|D^r u(x)| \leq Ae^{-B|x|} \tag{1.46}$$

for every $x \in \mathbb{R}^2$ with $|x| \geq R$, where D denotes differentiation with respect to x_1, x_2 and $S_{j,1}, S_{j,2}$ with $j \in \{0, 1, 2, \cdots, N\}$.

1.2.3 A Riemannian metric on the moduli space of vortex configurations

One can define a Riemannian metric on M_N. Consider a smooth curve

$$\gamma : [0, 1] \to M_N$$

In [2], the procedure to define $||\gamma'(0)||$ is as follows:

Suppose that

$$(\phi, \alpha) = \big(\phi(\gamma(0)), \alpha(\gamma(0))\big)$$

First, one finds a smooth function χ such that the vector

$$\big(\tilde{\phi}, \tilde{a}\big) = \Big(\frac{d\phi}{dt}(0), \frac{d\alpha}{dt}(0)\Big) + \big(i\phi\chi, d\chi\big) \tag{1.47}$$

satisfies the gauge-orthogonality condition with respect to (ϕ, α). Following (1.25), this leads to the equation

$$\Delta\chi - |\phi|^2\chi = -\frac{1}{2}\big(\partial_t\Theta\big)|\phi|^2 \tag{1.48}$$

To construct a solution for this equation, one writes

$$\chi = \frac{\rho}{2}\big(\partial_t\Theta\big) + \zeta \tag{1.49}$$

where ρ is a smooth cut-off function which is equal to 1 if $|x|$ is large enough and vanishes inside a disk containing the zeros of ϕ. Then, ζ satisfies:

$$\Delta\zeta - |\phi|^2\zeta = g \tag{1.50}$$

where g is a smooth and compactly supported function. According to lemma 6 in the appendix, this equation has a smooth solution ζ which decays exponentially to zero at infinity. In this way, we find a solution χ for the equation (1.48). According to proposition 1, the function $u = \ln|\phi|^2$ has exponential decay. This implies that the vector $(\tilde{\phi}, \tilde{a})$ defined by (1.47) belongs to L^2 and one defines

$$||\gamma'(0)|| = ||(\tilde{\phi}, \tilde{a})||_{L^2} \tag{1.51}$$

Using the procedure above, one can find the vectors

$$n_{j,\alpha} = \Big(\frac{\partial\phi}{\partial S_{j,\alpha}} + i\phi\chi_{j,\alpha}, \frac{\partial a}{\partial S_{j,\alpha}} + d\chi_{j,\alpha}\Big)$$

which are gauge-orthogonal with respect to $\big(\phi, \alpha\big)$, for $j = 1, 2, \cdots, N$ and $\alpha = 1, 2$. These vectors are called the zero modes. The complex and real parts of n_μ are denoted by $n_{\mu,\phi}$ and $n_{\mu,a}$, respectively. Using proposition 1, one obtains the following:

Proposition 2. *Suppose that K is a compact subset of M_N. For every multi-index r, there exists numbers $A, \delta > 0$ such that for every $p \in M_N$ and every $x \in \mathbb{R}^2$,*

$$|D^r n_{j,\alpha}(p)(x)| \leq Ae^{-\delta|x|} \tag{1.52}$$

where D denotes differentiation with respect to x_1 or x_2 or $S_{j,1}$ or $S_{j,2}$ with $j \in \{0, 1, 2, \cdots, N\}$.

One can also obtain the following estimates by looking at the construction of the vortex solutions and proposition 1

Proposition 3. *Suppose that s is a multi-index and D denotes differentiation with respect to x_1 or x_2 or $S_{j,1}$ or $S_{j,2}$ with $j \in \{0, 1, 2, \cdots, N\}$. Then, for $p \in M_N$, there exists $A, C, \delta, \epsilon > 0$ such that if*

$$|q - p| < \epsilon$$

then

$$\begin{aligned} &|D^s\phi(z;q)| < C(|z|+1)^{-|s|} \\ &|D^s\alpha_j(z;q)| < C(|z|+1)^{-|s|-1} \quad j = 1, 2 \\ &|D^s(|\phi|^2 - 1)|(z;q) < Ae^{-\delta|z|} \\ &|D^s(D_{\alpha_j}\phi)|(z;q) < Ae^{-\delta|z|} \end{aligned} \tag{1.53}$$

1.3 Dynamic 2D Abelian Higgs Model

The vortex configurations are static solutions to the Abelian Higgs Model. In [2] and [4], the authors have constructed solutions for the dynamic Abelian Higgs model in $\mathbb{R}^{1+2}$ by using vortices, when λ is close enough to 1. In this section, we go over these the main ideas of these papers and mention

some of the calcualutions used there.
The general idea is to consider a path

$$q : [0, T) \to M_N$$

and for small enough $\epsilon > 0$, consider the ansatz

$$\begin{aligned} \Phi(t,x) &= \phi(x; q(\epsilon t))e^{i\Sigma} + \epsilon^2 \tilde{\phi} = \varphi\big(x,t\big) + \epsilon^2 \tilde{\phi} \\ A_0 &= \epsilon^3 \tilde{a}_0 \\ A_i &= \alpha_i(x; q(\epsilon t)) + \partial_i \Sigma + \epsilon^2 \tilde{a}_i = a_i\big(x,t\big) + \epsilon^2 \tilde{a}_i \qquad i = 1, 2 \end{aligned} \tag{1.54}$$

to find a solution for the Abelian Higgs Model such that

$$||\big(\tilde{\phi}, \tilde{a}\big)||_{H^3} + ||\big(\tilde{\phi}_t, \tilde{a}_t\big)||_{H^2}$$

remains bounded on a time interval of length $O(\frac{1}{\epsilon})$.

From now one, the input parameter of the path q is denoted by τ and we will assume $\tau = \epsilon t$.

The gauge function Σ is chosen such that the vector $\big(\frac{\partial \varphi}{\partial \tau}, \frac{\partial a}{\partial \tau}\big)$ satisfies the gauge orthogonality condition.

- In [2], the following result has been proved:

  Theorem 3. *Consider the initial value problem for the Abelian Higgs model with*

  $$\lambda = 1 + l\epsilon^2$$

  where $l = \pm 1$. Suppose that the initial data is close to a two vortex $\big(\phi(q(0)), \alpha(q(0))\big)$ in the sense that:

  $$\begin{aligned} A(0,x) &= \alpha\big(x; q(0)\big) + \epsilon^2 \tilde{a}(0) \\ \Phi(o,x) &= \phi\big(x; q(0)\big) + \epsilon^2 \tilde{\phi}(0) \\ A_t(0,x) &= \epsilon \sum_{\mu} q'_{\mu}(0) n_{\mu,a} + \epsilon^2 \tilde{a}_t(0) \\ \Phi_t\big(0,x\big) &= \epsilon \sum_{\mu} q'_{\mu}(0) n_{\mu,\phi} + \epsilon^2 \tilde{\phi}_t(0) \end{aligned} \tag{1.55}$$

  where $\big(\tilde{\phi}(0), \tilde{a}(0)\big)$ satisfy the following conditions:

  $$\big(n_{\mu}, (\tilde{\phi}(0), \tilde{a}(0)\big) = \frac{d}{dt}\Big|_{t=0} \big(n_{\mu}, (\tilde{\phi}, \tilde{a})\big) = 0 \tag{1.56}$$

  $$\nabla . \tilde{a}(0,.) - \big(i\phi(0; q(0)), \tilde{\phi}(0,.)\big) = 0 \tag{1.57}$$

Then, there exists K such that if

$$\left|\left(\tilde{\phi}(0), \tilde{a}(0)\right)\right|_{H^3} + \left|\left(\tilde{\phi}_t(0), \tilde{a}_t(0)\right)\right|_{H^2} \leq K$$

then, for sufficiently small ϵ, there exists a time $T_* = O\left(\frac{1}{\epsilon}\right)$ such that on the interval $[0, T_*]$, there exists a solution of the form

$$\begin{aligned} \Phi &= \phi\big(x; q(t)\big)e^{i\sigma} + \epsilon^2\tilde{\phi} \\ A &= \alpha\big(x; q(t)\big) + d\sigma + \epsilon^2\tilde{a}(t,x) \end{aligned} \tag{1.58}$$

with

f

$$\begin{aligned} q_{\mu}(t) &= q^0_{\mu}(t) + \epsilon\tilde{q}_{\mu}(t) \\ p_{\mu}(t) &= p^0_{\mu}(t) + \epsilon\tilde{p}_{\mu}(t) \end{aligned} \tag{1.59}$$

where $p^0(\tau), q^0(\tau)$ are solutions of the Hamiltonian system

$$\frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i} \quad , \quad \frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i} \tag{1.60}$$

where H is defined by

$$H(p,q) = \frac{1}{2}g^*(p,p) + V(q) \tag{1.61}$$

and g^* is the metric dual to the metric $g : TM_2 \to \mathbb{R}$ and the potential V is defined by

$$V(q) = \frac{l}{8}\int_{\mathbb{R}^2} \left(1 - |\phi|^2\right)^2 \tag{1.62}$$

and the initial conditions are

$$q^0_{\mu}(0) = q_{\mu}(0) \quad , \quad p^0_{\mu}(0) = \sum_{\nu} g_{\mu\nu} q'_{\nu}(0) \tag{1.63}$$

and $\left(\tilde{\phi}, \tilde{a}\right)$ satisfy the condition

$$\begin{aligned} &\nabla.\tilde{a}(t,.) - \left(i\phi\big(.; q(t)\big), \tilde{\phi}(t,.)\right) = 0 \\ &\left(n_{\mu}, (\tilde{\phi}, \tilde{a})\right) = 0 \end{aligned} \tag{1.64}$$

and the maps

$$\begin{aligned} &t \to \tilde{p}(t) \quad , \quad t \to \frac{1}{\epsilon}\frac{dp}{dt} \\ &t \to \tilde{q}(t) \quad , \quad t \to \frac{1}{\epsilon}\frac{dq}{dt} \\ &t \to \left((\tilde{\phi}, \tilde{a}), (\tilde{\phi}_t, \tilde{a}_t)\right) \in H^3 \oplus H^2 \end{aligned} \tag{1.65}$$

are continuous and bounded independent of ϵ. In addition $|A_0(t,.)|_{L^\infty} = O(\epsilon^3)$ and the map

$t \to \sigma(t) \in C^\infty\big(\mathbb{R}^2\big)$ is twice differentiable and the solution has the regularity

$$(\tilde{q}, \tilde{p}) \in C^2\big([0, T_*]\big) \oplus C^1\big([0, T_*]\big) \tag{1.66}$$

and

$$(\tilde{\phi}, \tilde{a}) \in C^1\big([0, T_*], H^1 \oplus L^2\big) \cap C\big([0, T_*], H^3 \oplus L^2\big) \tag{1.67}$$

- In [4], the following result has been proved:

Theorem 4. *Consider a geodesic $Q : [0, \tau_0] \to M_N$. Suppose that*

$$Q(\tau) = \big(\phi(\tau), \alpha_1(\tau), \alpha_2(\tau)\big)$$

Then, there exist positive numbers $\tau_1 \leq \tau_0$ and ϵ_0, M and a smooth family $t \to \Sigma(t; .) \in C^\infty(\mathbb{R}^2; \mathbb{R})$ defined on $[0, \frac{\tau_1}{\epsilon}]$, with the follwoing properties:
For each $\epsilon \in [0, \epsilon_0]$ there exists a solution $\big(\Phi^\epsilon(t), A_0^\epsilon(t), \alpha_1^\epsilon(t), \alpha_2^\epsilon(t)\big)$ of the Abelian Higgs model defined on the interval $[0, \frac{\tau_1}{\epsilon_0}]$ of the form

$$\begin{aligned} A_0^\epsilon(t, x_1, x_2) &= \epsilon^3 a_0^\epsilon(t, x_1, x_2) \\ \alpha_1^\epsilon(t, x_1, x_2) &= \alpha_1(\epsilon t; x_1, x_2) + \partial_{x_1} \Sigma(t; x_1, x_2) + \epsilon^2 \alpha_1^\epsilon(t, x_1, x_2) \\ \alpha_2^\epsilon(t, x_1, x_2) &= \alpha_2(\epsilon t; x_1, x_2) + \partial_{x_2} \Sigma(t; x_1, x_2) + \epsilon^2 \alpha_2^\epsilon(t, x_1, x_2) \\ \phi^\epsilon(t, x_1, x_2) &= \phi(\epsilon t; x_1, x_2) e^{i\Sigma} + \epsilon^2 \phi^\epsilon(t, x,_1, x_2) \end{aligned} \tag{1.68}$$

such that

$$||a_0^\epsilon(t)||_{H^3} + ||a_1^\epsilon(t)||_{H^3} + ||a_2^\epsilon(t)||_{H^3} + ||\phi^\epsilon(t)||_{H^3} \leq M \tag{1.69}$$

for every t with $0 \leq t \leq \frac{\tau_1}{\epsilon}$.

Solving the ansatz (1.54) requires the linearization of the Abelian Higgs model equations. The linearization involves an operator $\mathcal{L}$ defined by

$$\mathcal{L}[\phi, \alpha](\tilde{\varphi}, \tilde{a}) = \begin{pmatrix} -\sum_{i=1}^2 \Big(\big(D_i^{(0)}\big)^2 \tilde{\phi} - 2i\tilde{a}_i D_i^{(0)} \phi\Big) + \frac{1}{2}\big(3|\phi|^2 - 1\big)\tilde{\phi} \\ \\ -\Delta \tilde{a}_i + |\phi|^2 \tilde{a}_i - 2\big(i\tilde{\phi}, D_i^{(0)}\phi\big) \end{pmatrix}_{i=1,2} + \begin{pmatrix} i\phi\big(\nabla.\tilde{a} - \big(i\phi, \tilde{\phi}\big)\big) \\ \\ \partial_i\big(\nabla.\tilde{a} - \big(i\phi, \tilde{\phi}\big)\big) \end{pmatrix}_{i=1,2} \tag{1.70}$$

where

$$D_i^{(0)} = \partial_i - i\alpha_i \tag{1.71}$$

for $i = 1, 2$. If one assumes the gauge orthogonality condition for $(\tilde{\varphi}, \tilde{a})$, the operator $\mathcal{L}$ changes to the following elliptic operator:

$$L[\phi, \alpha](\tilde{\varphi}, \tilde{a}) = \begin{pmatrix} -\sum_{i=1}^2 \Big(\big(D_i^{(0)}\big)^2 \tilde{\phi} - 2i\tilde{a}_i D_i^{(0)} \phi\Big) + \frac{1}{2}\big(3|\phi|^2 - 1\big)\tilde{\phi} \\ \\ -\Delta \tilde{a}_i + |\phi|^2 \tilde{a}_i - 2\big(i\tilde{\phi}, D_i^{(0)}\phi\big) \end{pmatrix}_{i=1,2} \tag{1.72}$$

In the the proof of theorems 3 and 4, a coercivity result about the operator L on the subspace orthogonal to zero modes is crucial. Indeed, the kernel of the elliptic operator L is spanned by the zero modes at $\big(\phi, \alpha\big)$. We have:

$$\int_{\mathbb{R}^2} \big(\psi, L\psi\big) = Hess(E)_{(\phi,\alpha)}\big(\psi, \psi\big) + \int_{\mathbb{R}^2} \Big(\nabla.\tilde{a} - \big(i\phi, \tilde{\phi}\big)\Big)^2 \tag{1.73}$$

where E is the $(0+2)$-dimensional version of the energy functional (1.7). The above quantity is denoted by

$$\overline{Hess}(E)_{(\phi,\alpha)}\big(\psi, \psi\big)$$

Proposition 4. $\big($*Coercivity of the corrected Hessian*$\big)$ *There exists a universal number* $\gamma > 0$ *such that for every*

$$\psi = \big(\tilde{\phi}, \tilde{a}\big) \in H^1$$

with $\langle \psi, n_\mu \rangle_{L^2} = 0$ *for every zero mode* n_μ, *there holds*

$$\overline{Hess}(E)_{(\phi,\alpha)}\big(\psi, \psi\big) \geq \gamma ||(\tilde{\phi}, \tilde{a})||^2_{H^1}$$

This was proved in [2]. It is noteworthy that part of the proof relies on the fact that the corrected Hessian can be written as:

$$\overline{Hess}(E)_{(\phi,\alpha)}\big(\psi, \psi\big) = \int_{\mathbb{R}^2} |\mathcal{D}_{(\phi,\alpha)}(\tilde{\varphi}, \tilde{a})|^2 \tag{1.74}$$

where $\mathcal{D}$ is the first order operator defined by (1.26), regarding the linearization of the Bolgomony equations (1.19) mixed with the gauge orthogonality condition.

1.4 Physical Applications

One can imagine that when the vortices are far away from each other, their dynamics is almost independent, but the question is how vortices interact when they are close to each other? These issues can be studied by inevstigating the metric and using the aformentioned theorems on the geodesic description of dynamics of vortices. Any element in M_2 can be identified with the center of mass and the relative position of its vortices:

$$M_2 = \mathbb{R}^2 \times M_2^0$$

The metric on M_2 is invariant wih respect to translation and rotation of vortices. Therefore, as mentioned in [5], the metric on M_2^0 can be desribed by:

$$ds^2 = F^2(r)dr^2 + G^2(r)d\theta^2$$

Here, (r, θ) are chosen such that $\big(rcos(2\theta), rsin(2\theta)\big)$ represents the relative position of the vortices. When, r is large enough, the metric is almost flat:

$$ds^2 \approx dr^2 + r^2 d\theta^2$$

It has been shown in [6] that near the origin, the metric takes the form

$$ds^2 \approx (cr)^2(dr^2 + r^2 d\theta^2)$$

This is in accordance with the fact that the metric on M_2 behaves smoothly with respect to the local coordinates $(z_1 + z_2, z_1 z_2)$. The above form of the metric implies that the curvature at the origin is positive. The surface M_2^0 can be visualized by a cone which is smoothed at the corner.
In [2], two scenarios for the interaction of vortices when they are close to each other has been discussed:

1. **Repulsion and attraction of vortices:**
   Suppose that $\lambda = 1 + \epsilon^2$ with ϵ small enough. In the settings of theorem **??**, the dynamic of vortices can be approximately modeled by the Hamiltonian dynamics corresponding to the Hamiltionian:
   $$H(p, q) = \frac{1}{2} g^*(p, p) + V(q) \tag{1.75}$$
   where
   $$V(q) = \frac{1}{8} \int_{\mathbb{R}^2} \left(1 - |\phi|^2\right)^2 \tag{1.76}$$
   Suppose that $Z_1(\tau), Z_2(\tau)$ denote the location of the voritices as complex numbers. Consider the coordinate system
   $$\begin{aligned} P(\tau) &= Z_1(\tau) + Z_2(\tau) \\ Q(\tau) &= Z_1(\tau) Z_2(\tau) \end{aligned} \tag{1.77}$$
   Suppose that
   $$\begin{aligned} Z_1(0) &= -Z_2(0) \\ |Z_1|'(0) &= |Z_2'|(0) = 0 \end{aligned} \tag{1.78}$$
   Then, $Z_1(\tau) = -Z_2(\tau)$ for $\tau > 0$. Since the potential V depends only on $|Z_1 - Z_2|$, then we can write
   $$V(q(\tau)) = u(|Z_1 - Z_2|) \tag{1.79}$$
   for some function u. Also, we can write
   $$\frac{1}{2} g(q', q') = f(|Q|)|Q'|^2 \tag{1.80}$$
   where f is nonnegative. Using the conservation of energy, we have:
   $$f(|Q|)|Q'|^2 + u(|Q(\tau)|) = u(|Q(0)|) \tag{1.81}$$
   This implies that $|Q(\tau)|$ is decreasing, if the initial condition is nonzero. It has been conjectured that the above function u is a decreasing function. If this is true, we can deduce that for $\lambda = 1 + \epsilon^2$, the vortices repell under the prescribed initail conditions. Similar arguments imply the attraction of voritces for $\lambda = 1 - \epsilon^2$.

2. **Scattering of vortices:** Consider the initial position where Z_1, Z_2 are located symmterically with respect to origin on the real line. Suppose that $Q = q_1 + iq_2$. Let

$$\begin{aligned} &P(0) = P'(0) = 0 \\ &q_1(0) < 0 \ , \ \ q_1'(0) = M > 0 \ , \ \ q_2(0) = q_2'(0) = 0 \end{aligned} \tag{1.82}$$

The vortcies will remain on the real line and when M is large enough, the dynamics can be described by:

$$q_2(\tau) = 0 \ , \ \ q_1(\tau) = f(\tau) \tag{1.83}$$

where f eventually becomes positive. This implies that

$$Z_1(\tau) = -Z_2(\tau) = \sqrt{-f(\tau)} \tag{1.84}$$

Therefore, the vortices move towards each other along the real line and when f changes sign from negative to positive, they move away from each orther along the y-axis. This is the so-called right angle scattering. This can also be seen by looking at the geodesics on the manifold M_2^0. When a geodesic passes from the origin, the angle θ changes from 0 to $\frac{\pi}{2}$ which is another way to represent the scattering.

Chapter 2

Dynamics of Abelian Higgs vortex lines at the critical coupling

Consider the Minkowski space-time $\mathbb{R}^{1+3}$ and let (t, x_1, x_2, z) denote the coordinates where t is the time. As explained in section 1.1, we aim to glue the vortex configuration solutions along the t and z-direction in the Minkowski space-time $\mathbb{R}^{1+3}$ to construct solutions for the AHM modulated by wave maps. That is given a wave map $q : [0, T) \to \mathbb{R} \to M_N$, for every $\epsilon > 0$ we want to find solutions to AHM close to $q(\epsilon t, \epsilon z)$ in the (Φ, A_1, A_2) variables and also with small (A_0, A_3) variables, over a domain $[0, \frac{T'}{\epsilon}) \times \mathbb{R}^3$ for some $T' > 0$. The precise statement is in section 2.2.
In gluing constructions, the general recipe is to have an ansatz to construct approximate solutions and then add perturbations to find an honest solution to the underlying PDE. The ansatz involves iteration of a procedure to improve the order of error. There is no reason that by repeating the procedure, we converge to an exact solution to the differential equations. This is similar to say that when we solve an ODE by considering a Taylor series for the solution, the series does not necessarily converge to the solution.
In the literature of PDEs, gluing constructions have been applied to various equations in different contexts, including dispersive equations, fluid dynamics and geometric wave equations. In terms of the nonlinear Schrodinger equations, one can look at [11] by Floer and Weinstein. In fluid dynamics, there are several papers including [12]. In hyperbolic PDEs, once can refer to [13].
Gluing construction in a gauged system of partial differential equations is the main matter of concern in this thesis. As we will see, considering a correct gauge condition is the main concept here. The crucial point is that the gauge condition in the ansatz and the perturbation part are different from each other.

2.1 Notation

1. In this chapter, in the context, when there are more than one choice for the gauge fields under discussion, corresponding to the vortex configuration (ϕ, α), we use the following notation for the covariant derivative:

$$D_\mu^{(0)} = \partial_\mu - i\alpha_\mu \quad \mu = 1, 2 \tag{2.1}$$

If there are no other gauge fields available in the context, we use the notation D_μ as in chapter 1.

2. Suppose that (ϕ, α) is a vortex configuration. Then,

$$\mathcal{L}[\phi,\alpha](\tilde{\varphi},\tilde{a}) = \begin{pmatrix} -\sum_{i=1}^{2}\left(D_i^2\tilde{\phi} - 2i\tilde{a}_i D_i\phi\right) + \frac{1}{2}\left(3|\phi|^2-1\right)\tilde{\varphi} \\ -\Delta\tilde{a}_i + |\phi|^2\tilde{a}_i - 2\left(i\tilde{\varphi}, D_i\phi\right) \end{pmatrix}_{i=1,2} + \begin{pmatrix} i\phi\left(\nabla.\tilde{a} - \left(i\phi,\tilde{\varphi}\right)\right) \\ \partial_i\left(\nabla.\tilde{a} - \left(i\phi,\tilde{\varphi}\right)\right) \end{pmatrix}_{i=1,2} \tag{2.2}$$

3. Suppose that (ϕ, α) is a vortex configuration. Then,

$$L[\phi,\alpha](\tilde{\varphi},\tilde{a}) = \begin{pmatrix} -\sum_{i=1}^{2}\left(D_i^2\tilde{\varphi} - 2i\tilde{a}_i D_i\phi\right) + \frac{1}{2}\left(3|\phi|^2-1\right)\tilde{\varphi} \\ -\Delta\tilde{a}_i + |\phi|^2\tilde{a}_i - 2\left(i\tilde{\varphi}, D_i\phi\right) \end{pmatrix}_{i=1,2} \tag{2.3}$$

4. The components of a point in the spacetime $\mathbb{R}^{1+3}$ are denoted as:

$$(t, x) = (t, x_1, x_2, z)$$

and also, we encapsulate x_1 and x_2 by

$$y = (x_1, x_2)$$

Also, we use the coordinate system

$$(y, \tau, \zeta) = ((x_1, x_2), \epsilon t, \epsilon z)$$

where ϵ is a scaling parameter in the context.

5. In the calculations, Δ refers to the Laplacian operator on $\mathbb{R}^3$:

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_z^2$$

and

$$\Delta_y = \partial_{x_1}^2 + \partial_{x_2}^2$$

6. If $(r_\alpha)_\alpha$ denotes the special coordinate system introduced in section 1.2.2 for the moduli space M_N, then the zero modes are denoted by

$$\tilde{n}_\alpha = \frac{\partial}{\partial r_\alpha}$$

7. If we apply the Gram-Schmidt process to $(\tilde{n}_\alpha)_\alpha$, we obtain an orthonormal basis $(n_\alpha)_\alpha$ for each point on the tangent space TM_N which depends smoothly on the base point.

8. Suppose that X, Y, Z are normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$ and U is an open subset of Y. We say that a function $f : X \times U \to Z$ is of class $\mathcal{E}_m(X, U, Z)$ if

for every multi-index r with $|r| \leq m$, one can find $A, \gamma > 0$ such that

$$|D^r f(x,y)| \leq Ae^{-\gamma|x|}$$

for every $(x,y) \in X \times U$.

9. The components of the zero mode n_α or $\tilde{n}_\alpha$ at a vortex configuration is denoted by

$$\big(n_{\alpha,\varphi}, n_{\alpha,1}, n_{\alpha,2}\big)$$

or

$$\big(\tilde{n}_{\alpha,\varphi}, \tilde{n}_{\alpha,1}, \tilde{n}_{\alpha,2}\big)$$

where $n_{\alpha,\phi}, \tilde{n}_{\alpha,\phi}$ correspond to the Higgs field component and $n_{\alpha,1}, \tilde{n}_{\alpha,1}$ and $n_{\alpha,2}, \tilde{n}_{\alpha,2}$ correspond to the gauge field components.

10. For a set S, $\mathcal{R}[S]$ refers to the ring generated by elements of S.

2.2 Main result

The final result of this thesis is the following

Theorem 5. *Suppose that $q : [0,T] \times \mathbb{R} \to M_N$ is a smooth wave map and it is constant at $(+\infty)$ and $(-\infty)$, then there exist numbers T_0, ϵ_0, M such that for every ϵ with $0 < \epsilon \leq \epsilon_0$, there exists a solution*

$$\big(\Phi^\epsilon, \sum_{j=0}^{3} A_j^\epsilon dx_j\big)$$

of the Abelian Higgs model with $\lambda = 1$ on

$$\big[0, \frac{T_0}{\epsilon}\big) \times \mathbb{R}^3$$

with

$$\begin{aligned} \Phi^\epsilon(t,x,z) &= \phi(x; q(\epsilon t, \epsilon z)) + \varphi^\epsilon \\ A_j^\epsilon(t,x,z) &= \alpha_j(x; q(\epsilon t, \epsilon z)) + a_j^\epsilon \qquad j = 1,2 \end{aligned} \tag{2.4}$$

where $(\phi, \alpha_1, \alpha_2)$ are the components of the vortex configuration solutions and:

$$||\varphi^\epsilon(t,.)||_{H^3} + ||\partial_t \varphi^\epsilon(t,.)||_{H^2} + \sum_{j=1}^{2} \Big(||a_j^\epsilon(t,.)||_{H^3} + ||\partial_t a_j^\epsilon(t,.)||_{H^2} \Big) \leq \epsilon^{\frac{3}{2}} M \tag{2.5}$$

and

$$\sum_{j=0,3} ||A_j^\epsilon(t,.)||_{L^\infty} \leq \epsilon M \tag{2.6}$$

and

$$\sum_{j=0,3} ||\nabla A_j^\epsilon(t,.)||_{H^2} + ||\partial_t A_j^\epsilon(t,.)||_{H^2} \leq \epsilon^{\frac{1}{2}} M \tag{2.7}$$

for some M and every t, where ∇ refers to spatial derivatives.

2.3 Sketch of the proof

As described above, the two main steps are

1. An ansatz to construct approximation solutions
2. Perturbation of the constructed approximate solution
   In subsections 2.3.1 and 2.3.2, we are going to explain the big picture of these two steps. The detailed proof will be in sections 2.4 and 2.5.

2.3.1 Ansatz: An overview

In the first step, the ansatz, we look for a series of the form

$$\Phi_m(t,x) = \phi(y; q(\epsilon t, \epsilon z)) + \sum_{i=1}^{m} \epsilon^{2i} \varphi_i(\epsilon t, y, \epsilon z) \tag{2.8}$$

$$A_{m,j}(t,x) = \alpha_j(y; q(\epsilon t, \epsilon z)) + \sum_{i=1}^{m} \epsilon^{2i} a_{j,i}(\epsilon t, y, \epsilon z) \qquad j = 1,2 \tag{2.9}$$

$$A_{m,j}(t,x) = \sum_{i=1}^{m} \epsilon^{2i-1} a_{j,i}(\epsilon t, y, \epsilon z) \qquad j = 0,3 \tag{2.10}$$

such that the following conditions are satisfied:

$$S_\varphi[\Phi_m, A_m] = D_0 D_0 \Phi_m - \sum_{j=1}^{3} D_j D_j \Phi_m + \frac{1}{2}\big(|\Phi_m|^2 - 1\big)\Phi_m = O(\epsilon^{2m+2}) \tag{2.11}$$

$$S_{a_j}[\Phi_m, A_m] = \Big(\partial_0 \mathcal{F}_{0j} - \sum_{k=1}^{3} \partial_k \mathcal{F}_{kj}\Big) - \big(i\Phi_m, D_j \Phi_m\big) = O(\epsilon^{2m+2}) \quad j = 1,2 \tag{2.12}$$

$$S_{a_j}[\Phi_m, A_m] = \Big(\partial_0 \mathcal{F}_{0j} - \sum_{k=1}^{3} \partial_k \mathcal{F}_{kj}\Big) - \big(i\Phi_m, D_j \Phi_m\big) = O(\epsilon^{2m+1}) \quad j = 0,3 \tag{2.13}$$

The process is inductive. The base case goes in this way: By looking at the error condition (2.13), we come up with two equations

$$-\Delta_y a_{j,1} + |\phi|^2 a_{j,1} = f_j \tag{2.14}$$

for $j = 0,3$ where f_j depends on the wave map. These equations are solvable by lemma 6 in the appendix. Then, the error condition (2.11) and (2.12), leads to the following system of equations

$$\mathcal{L}\psi = E \tag{2.15}$$

where $\psi = (\varphi_{1,1}, a_{1,1}, a_{2,1})$ and E depends on the wave map. (It is the error from the step zero in the ansatz where we simply plug-in the wave map q in the equations and let $A_0 = A_3 = 0$) As mentioned in chapter 1, the operator $\mathcal{L}$ has an infinite dimensional kernel spanned by the infinitesimal gauge transformations and the zero modes. We choose the gauge orthogonality condition for ψ:

$$\nabla.\tilde{a} - (i\phi, \tilde{\varphi}) = 0 \tag{2.16}$$

Then, equation (2.15) reads

$$L\psi = E \tag{2.17}$$

Following the coercivity statement 4 from the Stuart's paper, we have proved in lemma 10 in the appendix that the system (2.17) is solvable if E is orthogonal to zero modes and the solution ψ will be again in the orthogonal subspace to zero modes. This orthogonality condition leads to the wave map equation. But, now we need to check that ψ satisfies the gauge orthogonality condition. In lemma 10 in the appendix, we have proved that if E satisfies the gauge orthogonality condition, then ψ also does. We separately proved that E satisfies this condition and this finishes the first step.

The induction step goes in this way: Suppose that we have constructed an approximate solution (Φ_m, A_m) in the form of (2.8). If we want to update the number m and follow the similar procedure as the first step explained above, we encounter the issue that the orthogonality condition may not be satisfied. In this regard, we use some degrees of freedom in the current stage for the approximate solution (Φ_m, A_m) that we have not used so far. In (2.8), we can replace the vector

$$\psi_m = (\varphi_m, a_{1,m}, a_{2,m})$$

by

$$\psi_m + \sum_\mu c_\mu(\epsilon t, \epsilon z) n_\mu(y)$$

for any t and z and this does not disturb the leading order term of the error from the approximate solution (Φ_m, A_m), as the zero modes belong to the kernel of the operators $\mathcal{L}$ and L.

Therefore, we consider the following:

$$\begin{pmatrix} (\Phi_{m+1}, A_{m+1,1}, A_{m+1,2}) \\ \\ (A_{m+1,0}, A_{m+1,3}) \end{pmatrix} = \begin{pmatrix} (\Phi_m, A_{m,1}, A_{m,2}) \\ \\ (A_{m,0}, A_{m,3}) \end{pmatrix} + \epsilon^{2m} \begin{pmatrix} \sum_\mu c_\mu(\epsilon t, \epsilon z) n_\mu(y) \\ \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} \epsilon^{2m+2}(\varphi_{m+1}, a_{1,m+1}, a_{2,m+1})(\epsilon t, y, \epsilon z) \\ \\ \epsilon^{2m+1}\big(a_{0,m+1}, a_{3,m+1}\big)(\epsilon t, y, \epsilon z) \end{pmatrix} \tag{2.18}$$

We let $\{c_\mu\}_\mu$ to be unknown. Now, we repeat the same process and let the new functions in the ansatz depend on $\{c_\mu\}_\mu$. When we want to find $(\varphi_{m+1}, a_{1,m+1}, a_{2,m+1})$, we observe that the orthogonality condition leads to a hyperbolic PDE for the functions $\{c_\mu\}_\mu$ which has local well-posedness. We find $\{c_\mu\}_\mu$ in this way. Afterwards, the other pieces of the ansatz will be automatically determined.

2.3.2 Perturbation of approximate solutions: An overview

Suppose that we have found an approximate solution $v = (\varphi, a)$ as mentioned above. We look for a perturbation $u = (\tilde{\varphi}, \tilde{a})$ such that $v + u$ is a solution for the AHM. The strategy is as follows:

- **Imposing a gauge condition:**

$$-\partial_0 \tilde{a}_0 + \sum_{j=1}^{3} \partial_j \tilde{a}_j - (i\varphi, \tilde{\varphi}) = 0 \tag{2.19}$$

- **Rewriting the equations based on the gauge condition:**
Let

$$\psi = (\tilde{\varphi}, \tilde{a}_1 dx_1 + \tilde{a}_2 dx_2)$$

After imposing the gauge condition 2.19, the equations can be written as:

$$u_{tt} - u_{zz} + Mu = Pu + N + E \tag{2.20}$$

where

$$Mu = \begin{pmatrix} L\psi \\ -\Delta_y \tilde{a}_0 + |\phi|^2 \tilde{a}_0 \\ -\Delta_y \tilde{a}_3 + |\phi|^2 \tilde{a}_3 \end{pmatrix} \tag{2.21}$$

and Pu consists of small linear terms, Nu consists of nonlinear terms and E is the error from the approximate solution v.

- **Local Existence:**
For the system of equations such as (2.21), there are some standard local wellposed-ness results. We apply such an statement to find a solution defined over an interval $[0, T_0)$. These statements come with an apriori estimate.

- **Bootstrap:** Consider the constructed localized in time solution in the previous step. We want to use equations (2.19) and the apriori estimates to find refined estimates for u. The idea is again to look at each slice $\{t = t_0, z = z_0\}$ and look at the orthogonal and tangential components to zero modes.

Let us fix the time as t. Consider the following energy quantities:

$$Q_1(t) = \int_{\mathbb{R}^3} |u_t|^2 + |u_z|^2 + (Mu, u) \tag{2.22}$$

and

$$Q_2(t) = \int_{\mathbb{R}^3} |(Mu)_t|^2 + |(Mu)_z|^2 + (M^2 u, Mu) \tag{2.23}$$

By using equations (2.20), one can verify that

$$Q_1'(t) + Q_2'(t) \leq C\epsilon F(||u||_{H^3}, ||u_t||_{H^2}, ||E||_{H^2}, ||N||_{H^2}) \tag{2.24}$$

where F is an smooth function with $F(0) = DF(0) = 0$

The quantity $Q_1(t) + Q_2(t)$ bounds $\tilde{a}_0, \tilde{a}_3, (\tilde{a}_0)_t, (\tilde{a}_3)_t$ and $\psi^{\perp}$ where for every t, z the function $\psi^{\perp}$ is such that

$$\psi(., t, z) = \psi^{\perp}(., t, z) + \sum_{\mu} c_{\mu}(t, z) n_{\mu}(.) \tag{2.25}$$

where $\{n_{\mu}\}$ are the corresponding zero modes and

$$\langle \psi^{\perp}, n_{\mu} \rangle = 0$$

We write the variables $\tilde{a}_0$ and $\tilde{a}_3$ in the form:

$$\tilde{a}_j = f_j + \partial_0 \chi_j \tag{2.26}$$

where χ_j solves the equations

$$(-\Delta + |\varphi|^2)\chi_j = \partial_0 \tilde{a}_j \tag{2.27}$$

By taking the inner product of equation (2.20) in the Higgs part and the gauge field part for A_1 and A_2, we observe that

$$(c_{\mu})_{tt} - (c_{\mu})_{zz} = \partial_0 h_0 + \partial_3 h_3 + h \tag{2.28}$$

for some function h_0, h_1 and h_3. Then, we use the explicit formula for the wave equations to find c_{μ} and the final expression for c_{μ} can be expressed in terms of h_0, h_3 and h, rather than $\partial_0 h_0$ or $\partial_3 h_3$. The expression of the function h_0 involves χ_j. Based on this, we have good enough estimates for h_0, h_3 and h and hence for c_{μ}.

We also use similar ideas to control $\partial_t \psi$ by decomposing it into zero modes and orthogonal components. In the end, we observe that these estimates are enough to control $||u||_{H^3} + ||u_t||_{H^2}$.

Using the aforementioned estimates, we afford to do a bootstrap argument. That is we prove that if we have an estimate for u on a time interval $[0, T)$:

$$||u||_{H^3} + ||u_t||_{H^2} \leq K \leq \epsilon^3 \tag{2.29}$$

Then,

$$||u||_{H^3} + ||u_t||_{H^2} \leq C(\epsilon t)^{\frac{1}{2}} K + \epsilon^4 \tag{2.30}$$

- **Moving forward in time:**
Estimate 2.30 suffices to go through an iterative process of applying a local existence result and apriori estimate, refining the estimate, again applying a local existence theorem to the end point of the interval and again refining the estimates and so on. In this way, we prove the desired estimates over a time interval of length $\frac{\kappa}{\epsilon}$ for some κ independent of ϵ

2.4 Approximate solutions for the Abelian Higgs Model

For the complex function $\Phi : \mathbb{R}^{1+3} \to \mathbb{C}$ and the real 1-form

$$A = A_0 dt + \sum_{i=1}^{3} A_i dx^i$$

consider the quantities:

$$\begin{aligned} S_\varphi[\Phi, A] &= D_0 D_0 \Phi - \sum_{j=1}^{3} D_j D_j \Phi + \frac{1}{2}\big(|\Phi|^2 - 1\big)\Phi \\ S_{a_j}[\Phi, A] &= \big(\partial_0 \mathcal{F}_{0j} - \sum_{k=1}^{3} \partial_k \mathcal{F}_{kj}\big) - \big(i\Phi, D_j\Phi\big) \qquad j = 0, 1, 2, 3 \end{aligned} \tag{2.31}$$

Let

$$S[\Phi, A] = \big(S_\varphi[\Phi, A], S_{a_0}[\Phi, A], \cdots, S_{a_3}[\Phi, A]\big)$$

For given $\epsilon > 0$, we find a pair (Φ_n, A_n) such that

$$S[\Phi_n, A_n] = O(\epsilon^{2n+1}, \epsilon^{2n-1}, \epsilon^{2n+1}, \epsilon^{2n+1}, \epsilon^{2n-1})$$

and

$$\begin{pmatrix} \Phi_n \\ A_n \end{pmatrix}(t, x) = \begin{pmatrix} (\phi, \alpha)(q(y; \epsilon t, \epsilon z)) \\ 0 \end{pmatrix} + \begin{pmatrix} O(\epsilon^2) \\ O(\epsilon) \end{pmatrix}$$

where $q : [0, T] \times \mathbb{R} \to M_N$ is a wave map. The precise statement is as follows:

Proposition 5. *Suppose that $q : [0, T] \times \mathbb{R} \to M_N$ is a smooth wave map with compactly supported image and bounded derivatives of any order. There exists ϵ_0 such that if $\epsilon < \epsilon_0$, then for every m*

one can find smooth (Φ_m, A_m) *the form*

$$\begin{aligned} \Phi_m(t,x) &= \phi(y;q(\epsilon t,\epsilon z)) + \sum_{i=1}^{m} \epsilon^{2i}\varphi_i(\epsilon t, y, \epsilon z) \\ A_{m,j}(t,x) &= \alpha_j(y;q(\epsilon t,\epsilon z)) + \sum_{i=1}^{m} \epsilon^{2i}a_{j,i}(\epsilon t, y, \epsilon z) \qquad j=1,2 \\ A_{m,j}(t,x) &= \sum_{i=1}^{m} \epsilon^{2i-1}a_{j,i}(\epsilon t, y, \epsilon z) \qquad j=0,3 \end{aligned} \tag{2.32}$$

defined on $[0, \frac{T_m}{\epsilon}] \times \mathbb{R}$ *for some* $T_m > 0$ *such that if*

$$A_m = (A_{m,j})_{j=0}^{3}$$

then

$$\begin{pmatrix} S_\varphi[\Phi_m, A_m] \\ S_{a_1}[\Phi_m, A_m] \\ S_{a_2}[\Phi_m, A_m] \end{pmatrix}(t,x) = \sum_{i=m+1}^{3m} \epsilon^{2i} f_i(\epsilon t, y, \epsilon z) \tag{2.33}$$

and

$$(S_{a_j}[\Phi_m, A_m])(t,x) = \sum_{i=m+1}^{3m} \epsilon^{2i-1} g_{j,i}(\epsilon t, y, \epsilon z) \quad j=0,3 \tag{2.34}$$

where if

$$h \in \{f_i, g_{j,i}, \varphi_i\}_{i,j} \cup \{a_{j,i}\}_{j=1,2} \cup \{a_{j,i}\}_{i>1, j\notin\{1,2\}} \tag{2.35}$$

then for any multi-index r*, one can find* $A, \delta > 0$ *such that*

$$|D^r h(\tau, y, \zeta)| \leq Ae^{-\delta|y|} \tag{2.36}$$

and

$$|D^r a_{j,1}(\tau, y, \zeta)| \leq A(|y|+1)^{-|r|-1} \tag{2.37}$$

for $j \in \{0,3\}$ *and any* y, τ, ζ.

Proof. The proof of this statement goes by induction on m. Following notation 8, $\mathcal{E}$ refers to the class of all functions $f : \mathbb{R}^2 \times [0,S] \times \mathbb{R}^1 \to \mathbb{C}$ for some S such that for every multi-index r, one can find $A, \gamma > 0$ such that

$$D^r f(y, \tau, \zeta) \leq Ae^{-\gamma|y|}$$

for every y, τ, ζ.

Suppose that $m = 1$. Let

$$\begin{aligned} \Phi_1(t,x) &= \phi(y;q(\tau,\zeta)) + \epsilon^2\varphi_1(\tau, y, \zeta) \\ A_{1,j}(t,x) &= \alpha_j(y;q(\tau,\zeta)) + \epsilon^2 a_{1,j}(\tau, y, \zeta) \qquad j=1,2 \\ A_{1,j}(t,x) &= \epsilon\tilde{a}_j(\tau, y, \zeta) \qquad j=0,3 \end{aligned} \tag{2.38}$$

For simplicity, we write

$$\begin{aligned} \phi(.) &= \phi(.;q(\tau,\zeta)) \\ \alpha_j(.) &= \alpha_j(.;q(\tau,\zeta)) \qquad j=1,2 \end{aligned} \tag{2.39}$$

Following notation 4, assume that $\tau = \epsilon t$ and $\zeta = \epsilon z$ and we use the coordinate system

$$(y,\tau,\zeta) = ((x_1,x_2),\epsilon t,\epsilon z)$$

By plugging in the ansatz (2.38) into (2.31) and using the fact that

$$\partial_t = \epsilon\partial_\tau \ \ , \ \ \partial_z = \epsilon\partial_\zeta$$

one gets:

$$S_{a_j}[\Phi_1,A_1] = \epsilon g_{j,1} + \epsilon^3 g_{j,2} + \epsilon^5 g_{j,3} \qquad j=0,3 \tag{2.40}$$

where the functions $g_{j,i}(\tau,y,\zeta)$ are independent of ϵ and

$$\begin{aligned} g_{j,1} &= -\big(\Delta_y\tilde{a}_j - |\phi|^2\tilde{a}_j + (i\phi,\frac{\partial\phi}{\partial\tau})\big) \quad j=0 \\ g_{j,1} &= -\big(\Delta_y\tilde{a}_j - |\phi|^2\tilde{a}_j + (i\phi,\frac{\partial\phi}{\partial\zeta})\big) \quad j=3 \end{aligned} \tag{2.41}$$

where Δ_y refers to the laplacian with respect to the variables x_1, x_2. We find the functions $\tilde{a}_j$ such that $g_{j,1} = 0$ for $j = 0,3$ for every τ,ζ. These equations are equivalent to the gauge orthogonality condition for the vectors

$$\Big(\frac{\partial\phi}{\partial\tau} - i\phi\tilde{a}_0, \frac{\partial\alpha}{\partial\tau} - d\tilde{a}_0\Big)$$

and

$$\Big(\frac{\partial\phi}{\partial\zeta} - i\phi\tilde{a}_3, \frac{\partial\alpha}{\partial\zeta} - d\tilde{a}_3\Big)$$

Therefore, by following the strategy in chapter 1 to make the above vectors gauge orthogonal, we can construct the functions $\tilde{a}_j$ for $j = 0,3$. The construction is as follows:
Suppose that

$$\tilde{\partial}_0 = \partial_\tau$$

and

$$\tilde{\partial}_3 = \partial_\zeta$$

Suppose that $R > 0$ is such that the ball $B(0,R)$ contains all of the vortex centers in the image of the wave map q. Suppose that $\rho : \mathbb{R}^2 \to \mathbb{R}$ is a nonnegative smooth cutoff function which is 0 inside the ball $B(0,R)$ and is 1 outside of the ball $B(0,2R)$. Then,

$$\tilde{a}_j = -\frac{1}{2}\rho\tilde{\partial}_j\Theta + b_j \tag{2.42}$$

where Θ is as in (1.34) and b_j solves the equation

$$\Delta_y b_j - |\phi|^2 b_j = c_j \tag{2.43}$$

for each τ, ζ and $c_j = c_j(y, \tau, \zeta)$ is a smooth function of y, q and $\tilde{\partial}_j q$ and is supported inside the ball $B(0, 2R)$ in the y-plane. By the regularity properties of q and using lemma 8 in the appendix, one realizes that $b_j \in \mathcal{E}$ and therefore by (2.42), $\tilde{a}_j$ depends smoothly on y, τ, ζ and

$$(\partial_j \tilde{a}_l - \partial_l \tilde{a}_j) \in \mathcal{E}$$

for any $j, l \in \{0, 3\}$. Furthermore, according to the formula and 2.42 and lemma 5 in the appendix, for any multi-index r, there exists $A > 0$ such that

$$|D^r \tilde{a}_j(y, \tau, \zeta)| \leq A(|y| + 1)^{-|r|+1} \tag{2.44}$$

for every τ, ζ. We define

$$\tilde{D}_j = \tilde{\partial}_j - i\tilde{a}_j$$

for $j = 0, 3$. Also, Suppose that

$$\eta = (\phi, \alpha_1, \alpha_2)$$

and

$$\psi_1 = (\varphi_1, a_{1,1}, a_{1,2})$$

By using the fact that η satisfies the static two dimensional AHM for each τ, ζ, we have:

$$\begin{pmatrix} S_\varphi[\Phi_1, A_1] \\ S_{a_j}[\Phi_1, A_1] \end{pmatrix}_{j=1,2} = \epsilon^2(T + \mathcal{L}\psi_1) + \epsilon^4 f_2 + \epsilon^6 f_3 \tag{2.45}$$

where $f_2(\tau, y, \zeta)$ and $f_3(\tau, y, \zeta)$ do not depend on ϵ and

$$T = \begin{pmatrix} (\tilde{D}_0^2 - \tilde{D}_3^2)\phi \\ \\ (\tilde{\partial}_0^2 - \tilde{\partial}_3^2)\alpha_j - \partial_j(\tilde{\partial}_0 \tilde{a}_0 - \tilde{\partial}_3 \tilde{a}_3) \end{pmatrix}_{j=1,2}$$

Proposition 6. *The vector T is orthogonal to the zero modes at η for any τ, ζ.*

Remark 1. *In fact, we will prove that for $(\phi, \alpha)(., q(\tau, \zeta))$ and $\tilde{a}_j$ for $j = 0, 3$ as above, the vector T is orthogonal to all zero modes, if and only if q is a wave map.*

Proof. In what follows, Greek letters correspond to the indices for the coordinate system on the moduli space M_N. We have:

$$T = \tilde{\partial}_0 \tilde{D}_0 \eta - \tilde{\partial}_3 \tilde{D}_3 \eta - \Big(i\big(\tilde{a}_0 \tilde{D}_0 - \tilde{a}_3 \tilde{D}_3\big)\phi, 0\Big) \tag{2.46}$$

We have:

$$\tilde{D}_j \eta = \sum_\alpha (\tilde{\partial}_j q^\alpha) \tilde{n}_\alpha \tag{2.47}$$

for $j = 0, 3$ where

$$q = (q^\alpha)_\alpha$$

in the coordinate system introduced in the first chapter and $\tilde{n}_\alpha$'s denote the corresponding zero modes introduced in notation 6. Therefore,

$$T = \sum_\alpha \Big((\Box q^\alpha)\tilde{n}_\alpha + (\tilde{\partial}_0 q^\alpha)(\tilde{\partial}_0 \tilde{n}_\alpha) - (\tilde{\partial}_3 q^\alpha)(\tilde{\partial}_3 \tilde{n}_\alpha) \Big) - \sum_\alpha \Big(i(\tilde{a}_0 \tilde{\partial}_0 q^\alpha - \tilde{a}_3 \tilde{\partial}_3 q^\alpha)\tilde{n}_{\alpha,\varphi}, 0) \Big) \tag{2.48}$$

Therefore,

$$\begin{aligned} \langle T, \tilde{n}_\beta \rangle &= \sum_\alpha \Box q^\alpha \langle \tilde{n}_\alpha, \tilde{n}_\beta \rangle + \sum_\alpha \big((\tilde{\partial}_0 q^\alpha)\langle \tilde{\partial}_0 \tilde{n}_\alpha, \tilde{n}_\beta \rangle - (\tilde{\partial}_3 q^\alpha)\langle \tilde{\partial}_3 \tilde{n}_\alpha, \tilde{n}_\beta \rangle \big) \\ &\quad - \sum_\alpha (\tilde{a}_0 \tilde{\partial}_0 q^\alpha - \tilde{a}_3 \tilde{\partial}_3 q^\alpha)\langle i\tilde{n}_{\alpha,\varphi}, \tilde{n}_{\beta,\varphi} \rangle \end{aligned} \tag{2.49}$$

Since q is a wave map, we have:

$$\begin{aligned} \Box q^\alpha \langle \tilde{n}_\alpha, \tilde{n}_\beta \rangle &= \Gamma^\alpha_{\mu\lambda}\big(- \tilde{\partial}_0 q^\mu \tilde{\partial}_0 q^\lambda + \tilde{\partial}_3 q^\mu \tilde{\partial}_3 q^\lambda \big) \langle \tilde{n}_\alpha, \tilde{n}_\beta \rangle \\ &= \big(- \tilde{\partial}_0 q^\mu \tilde{\partial}_0 q^\lambda + \tilde{\partial}_3 q^\mu \tilde{\partial}_3 q^\lambda \big) \langle \nabla_{\tilde{n}_\mu \tilde{n}_\lambda, \tilde{n}_\beta} \rangle \end{aligned} \tag{2.50}$$

where $\{\Gamma^\alpha_{\mu\lambda}\}$ denote the Christoffel symbols of the Levi Civita connection corresponding to the metric on M_N. According to the Koszul formula, we have:

$$\langle \nabla_{\tilde{n}_\mu} \tilde{n}_\lambda, \tilde{n}_\beta \rangle = \frac{1}{2}\big(\partial_\mu \langle \tilde{n}_\lambda, \tilde{n}_\beta \rangle + \partial_\lambda \langle \tilde{n}_\mu, \tilde{n}_\beta \rangle - \partial_\beta \langle \tilde{n}_\lambda, \tilde{n}_\mu \rangle \big) \tag{2.51}$$

By combining (2.50) and (2.51) and then using (2.49), we get:

$$\langle T, \tilde{n}_\beta \rangle = \sum_{k \neq 1,2} s(k) \Big[(\tilde{\partial}_k q^\mu)(\tilde{\partial}_k q^\lambda) \langle \nabla_{\tilde{n}_\mu} \tilde{n}_\lambda - \partial_\lambda \tilde{n}_\mu, \tilde{n}_\beta \rangle + \tilde{a}_k (\tilde{\partial}_k q^\mu) \langle i\tilde{n}_{\mu,\varphi}, \tilde{n}_{\beta,\varphi} \rangle \Big] \tag{2.52}$$

where

$$s(k) = \begin{cases} 1, & \text{if } k = 0. \\ -1, & \text{if } k > 0. \end{cases} \tag{2.53}$$

But,

$$\tilde{a}_k = -(\tilde{\partial}_k q^\lambda)\chi_\lambda \tag{2.54}$$

where $\chi_\lambda : \mathbb{R}^2 \to \mathbb{R}$ is the function which makes the zero mode

$$\tilde{n}_\lambda = (\partial_\lambda \phi + i\chi_\lambda \phi, \partial_\lambda \alpha + d\chi_\lambda)$$

gauge orthogonal. Therefore,

$$\begin{aligned}\langle T,\tilde{n}_\beta\rangle &= \sum_{k\neq 1,2} s(k)(\tilde{\partial}_k q^\mu)(\tilde{\partial}_k q^\lambda)\big(\langle \nabla_{\tilde{n}_\mu}\tilde{n}_\lambda - \partial_\lambda \tilde{n}_\mu, \tilde{n}_\beta\rangle - \chi_\lambda\langle i\tilde{n}_{\mu,\varphi}, \tilde{n}_{\beta,\varphi}\rangle\big)\\ &= \sum_{k\neq 1,2} s(k)\Big[\sum_{\mu<\lambda}(\tilde{\partial}_k q^\mu)(\tilde{\partial}_k q^\lambda)\big(\langle 2\nabla_{\tilde{n}_\mu}\tilde{n}_\lambda - \partial_\lambda\tilde{n}_\mu - \partial_\mu\tilde{n}_\lambda, \tilde{n}_\beta\rangle - \langle i\chi_\lambda\tilde{n}_{\mu,0} + i\chi_\mu\tilde{n}_{\lambda,0}, \tilde{n}_{\beta,0}\rangle\big)\Big]\\ &+ \sum_{k\neq 1,2} s(k)\Big[\sum_{\mu}(\tilde{\partial}_k q^\mu)^2\big(\langle \nabla_{\tilde{n}_\mu}\tilde{n}_\mu - \partial_\mu\tilde{n}_\mu, n_\beta\rangle - \chi_\mu\langle i\tilde{n}_{\mu,\varphi}, \tilde{n}_{\beta,\varphi}\rangle\big)\Big]\end{aligned} \tag{2.55}$$

But according to the Koszul's formula, we have:

$$\begin{aligned}2(\nabla_{\tilde{n}_\mu}\tilde{n}_\lambda, \tilde{n}_\beta) &= \partial_\lambda\langle\tilde{n}_\mu,\tilde{n}_\beta\rangle + \partial_\mu\langle\tilde{n}_\lambda,\tilde{n}_\beta\rangle - \partial_\beta\langle\tilde{n}_\mu,\tilde{n}_\lambda\rangle\\ &= \langle\tilde{n}_\mu,\partial_\lambda\tilde{n}_\beta\rangle + \langle\partial_\mu\tilde{n}_\lambda,\tilde{n}_\beta\rangle + \langle\tilde{n}_\lambda,\partial_\mu\tilde{n}_\beta\rangle + \langle\partial_\lambda\tilde{n}_\mu,\tilde{n}_\beta\rangle - \langle\partial_\beta\tilde{n}_\lambda,\tilde{n}_\mu\rangle - \langle\tilde{n}_\mu,\partial_\beta\tilde{n}_\lambda\rangle\end{aligned} \tag{2.56}$$

Therefore,

$$\langle 2\nabla_{\tilde{n}_\mu}\tilde{n}_\lambda - \partial_\lambda\tilde{n}_\mu - \partial_\mu\tilde{n}_\lambda, \tilde{n}_\beta\rangle = \langle\tilde{n}_\mu, \partial_\lambda\tilde{n}_\beta - \partial_\beta\tilde{n}_\lambda\rangle + \langle\tilde{n}_\lambda, \partial_\mu\tilde{n}_\beta - \partial_\beta\tilde{n}_\mu\rangle \tag{2.57}$$

We have:

$$\tilde{n}_l = \big(\partial_l\phi + i\chi_l\phi, \partial_l\alpha + d\chi_l\big) \tag{2.58}$$

for any l. Therefore,

$$\partial_\lambda\tilde{n}_\beta - \partial_\beta\tilde{n}_\lambda = \Big(i\chi_\beta\partial_\lambda\phi - \chi_\lambda\partial_\beta\phi + i(\partial_\lambda\chi_\beta - i\partial_\beta\chi_\lambda)\phi, d\partial_\lambda\chi_\beta - d\partial_\beta\chi_\lambda\Big) \tag{2.59}$$

for any λ, β. Therefore,

$$\begin{aligned}\langle\tilde{n}_\mu, \partial_\lambda\tilde{n}_\beta - \partial_\beta\tilde{n}_\lambda\rangle + \langle\tilde{n}_\lambda, \partial_\mu\tilde{n}_\beta - \partial_\beta\tilde{n}_\mu\rangle &= \langle\tilde{n}_{\mu,\phi}, i\chi_\beta\partial_\lambda\phi - i\chi_\lambda\partial_\beta\phi\rangle + \langle\tilde{n}_{\lambda,\varphi}, i\chi_\beta\partial_\mu\phi - i\chi_\mu\partial_\beta\phi\rangle\\ &+ \langle\tilde{n}_\mu, (iA_{\lambda,\beta}\phi, dA_{\lambda,\beta})\rangle + \langle\tilde{n}_\lambda, (iA_{\mu,\beta}\phi, dA_{\mu,\beta})\rangle\end{aligned} \tag{2.60}$$

where

$$A_{rs} = \partial_r\chi_s - \partial_s\chi_r$$

Since $\tilde{n}_l$'s satisfy the gauge orthogonality condition, we have:

$$\langle\tilde{n}_\mu, (iA_{\lambda,\beta}\phi, dA_{\lambda,\beta})\rangle = \langle\tilde{n}_\lambda, (iA_{\mu,\beta}\phi, dA_{\mu,\beta})\rangle = 0 \tag{2.61}$$

Therefore,

$$\langle\tilde{n}_\mu, \partial_\lambda\tilde{n}_\beta - \partial_\beta\tilde{n}_\lambda\rangle + \langle\tilde{n}_\lambda, \partial_\mu\tilde{n}_\beta - \partial_\beta\tilde{n}_\mu\rangle = \langle\tilde{n}_{\mu,\varphi}, i\chi_\beta\partial_\lambda\phi - i\chi_\lambda\partial_\beta\phi\rangle + \langle\tilde{n}_{\lambda,\varphi}, i\chi_\beta\partial_\mu\phi - i\chi_\mu\partial_\beta\phi\rangle \tag{2.62}$$

Now, by using (2.57), (2.62) and the identity (2.58) for $l = \mu, \lambda, \beta$, one sees that

$$\langle 2\nabla_{\tilde{n}_\mu}\tilde{n}_\lambda - \partial_\lambda\tilde{n}_\mu - \partial_\mu\tilde{n}_\lambda, \tilde{n}_\beta\rangle - \langle i\chi_\lambda\tilde{n}_{\mu,\varphi} + i\chi_\mu\tilde{n}_{\lambda,\varphi}, \tilde{n}_{\beta,\varphi}\rangle = 0 \tag{2.63}$$

Therefore, by (2.55), one deduces that $\langle T, \tilde{n}_\beta \rangle = 0$. ☐

Lemma 1. *For any choice for Φ, A, we always have*

$$\big(S_\varphi[\Phi, A], i\Phi\big) + \partial_0 S_{a_0}[\Phi, A] - \sum_{j=1}^{3} \partial_j S_{a_j}[\Phi, A] = 0 \tag{2.64}$$

Proof. We compute

$$\big(S_\varphi[\Phi, A], i\Phi\big) = \big(D_0 D_0 \Phi - \sum_{j=1}^{3} D_j D_j \Phi, i\Phi\big) \tag{2.65}$$

Also,

$$\begin{aligned} -\partial_0 S_{a_0} + \sum_{j=1}^{3} \partial_j S_{a_j}[\Phi, A] &= \partial_0\big(\sum_{j=1}^{n} \partial_j \mathcal{F}_{j0} + (i\Phi, D_0\Phi)\big) + \sum_{j=1}^{n} \partial_j \big(\partial_0 \mathcal{F}_{0j} - \sum_{k=1}^{n} \partial_k \mathcal{F}_{kj} - (i\Phi, D_j\Phi)\big) \\ &= \sum_{j=1}^{n} \partial_0 \partial_j (\mathcal{F}_{j0} + \mathcal{F}_{0j}) - \sum_{j=1}^{n}\sum_{k=1}^{n} \partial_j \partial_k \mathcal{F}_{kj} + (i\partial_0\Phi, D_0\Phi) + (i\Phi, \partial_0 D_0 \Phi) \\ &- \sum_{j=1}^{n} \big((i\partial_j \Phi, D_j\Phi) + (i\Phi, \partial_j D_j \Phi)\big) \end{aligned} \tag{2.66}$$

But, we have

$$\mathcal{F}_{\alpha\beta} + \mathcal{F}_{\beta\alpha} = 0$$

for any α, β. Therefore, by (2.66), we have:

$$\begin{aligned} -\partial_0 S_{A_0} + \sum_{j=1}^{3} \partial_j S_{a_j}[\Phi, A] &= (i\partial_0\Phi, D_0\Phi) + (i\Phi, \partial_0 D_0\Phi) - \sum_{j=1}^{3} \big((i\partial_j\Phi, D_j\Phi) - (i\Phi, \partial_j D_j \Phi)\big) \\ &= (i\Phi, D_0 D_0 \Phi) - \sum_{j=1}^{3} (i\Phi, D_j D_j \Phi) \end{aligned} \tag{2.67}$$

(2.64) follows from (2.65) and (2.67). ☐

Proposition 7. *The vector T satisfies the gauge orthogonality condition for any τ, ζ.*

Proof. Let

$$\begin{aligned} \tilde{\Phi}_1(t,x) &= \phi(y; q(\tau,\zeta)) \\ \tilde{A}_{1,j}(t,x) &= \alpha_j(y; q(\tau,\zeta)) \quad j = 1,2 \\ \tilde{A}_{1,j}(t,x) &= \epsilon \tilde{a}_j(\tau, y, \zeta) \quad j = 0,3 \end{aligned} \tag{2.68}$$

where $\tilde{a}_j$ for $j = 0, 3$ is as constructed before. Suppose that $T = (T_0, T_1, T_2)$ where T_0 is the Higgs section component and T_1 and T_2 correspond to the gauge field section components of A_1 and A_2. Let

$$\tilde{A}_1 = \big(\tilde{A}_{1,0}, \tilde{A}_{1,1}, \tilde{A}_{1,2}, \tilde{A}_{1,3}\big)$$

To calculate the error $S[\tilde{\Phi}_1, \tilde{A}_1]$, one can use the results for $S[\Phi_1, A_1]$ by assuming $\psi_1 = 0$. According to (2.45), since T does not depend on ψ_1, we have:

$$S_\varphi[\tilde{\Phi}_1, \tilde{A}_1] = \epsilon^2 T_0 + O(\epsilon^4) \tag{2.69}$$

Also, according to (2.40), we have:

$$S_{a_j}[\tilde{\Phi}_1, \tilde{A}_1] = O(\epsilon^3) \quad j = 0, 3 \tag{2.70}$$

Suppose that the compnents of T are denoted by (T_0, T_1, T_2), where T_0 is the complex part and T_1, T_2 are the real ones. Using (2.69) and (2.70), the coefficient of ϵ^2 in

$$(S_\varphi[\tilde{\Phi}_1, \tilde{A}_1], i\tilde{\Phi}_1) + \partial_0 S_{a_0}[\tilde{\Phi}_1, \tilde{A}_1] - \sum_{j=1}^{3} \partial_j S_{a_j}[\tilde{\Phi}_1, \tilde{A}_1]$$

is

$$\big(T_0, i\phi\big) - \sum_{j=1}^{2} \partial_j T_j$$

But, according to lemma 1,

$$(S_\varphi[\tilde{\Phi}_1, \tilde{A}_1], i\tilde{\Phi}_1) + \partial_0 S_{a_0}[\tilde{\Phi}_1, \tilde{A}_1] - \sum_{j=1}^{3} \partial_j S_{a_j}[\tilde{\Phi}_1, \tilde{A}_1] = 0$$

Therefore T satisfies the gauge orthogonality condition. □

According to (2.46) and (2.47), $T \in \mathcal{E}$. Therefore, by lemmas 10 and 12 in the appendix, we can find $\psi_1 \in \mathcal{E}$ which satisfies the gauge orthogonality condition for every τ, ζ and solves the equation

$$L\psi_1(., \tau, \zeta) = -T(., \tau, \zeta)$$

for each τ and ζ. Therefore, $\mathcal{L}\psi_1 = -T$. According to (2.45), the introduced choices for ψ_1 and $\tilde{a}_j$ for $j = 0, 3$ ensure that

$$S_\varphi[\Phi_1, A_1] = O(\epsilon^4) \quad , \quad S_{a_j}[\Phi_1, A_1] = O(\epsilon^4) \quad j = 1, 2$$

We already established that $S_{a_j}[\Phi_1, A_1] = O(\epsilon^3)$ for $j = 0, 3$. Noting that all of the components of the functions f_2, f_3 in (2.45) and $g_{j,2}$ and $g_{j,3}$ in (2.40) belong to $\langle \mathcal{O}_1 | \mathcal{F}_1 \rangle$ where

$$\mathcal{O}_1 = \mathcal{R}\big[\{\partial_1, \partial_2, \{\tilde{\partial}_j\}, \alpha_1, \alpha_2, \{\tilde{a}_j\}, \phi, \bar{\phi}, 1, i\}\big]$$

and

$$\mathcal{F}_1 = \mathcal{R}\big[\{D_1\phi, \overline{D_1\phi}, D_2\phi, \overline{D_2\phi}, |\phi|^2 - 1, \mathcal{F}_{12}, \{\tilde{\mathcal{F}}_{\alpha\beta}\}, \tilde{D}_j\phi, \overline{\tilde{D}_j\phi}, \varphi_1, \overline{\varphi_1}, a_{1,1}, a_{1,2}\}\big]$$

and

$$\langle \mathcal{O}_1 | \mathcal{F}_1 \rangle = \mathcal{R}\big[\{ab | a \in \mathcal{O}_1, b \in \mathcal{F}_1\}\big]$$

where

$$\tilde{\mathcal{F}}_{\alpha\beta} = \begin{cases} \tilde{\partial}_\alpha \tilde{a}_\beta - \tilde{\partial}_\beta \tilde{a}_\alpha & \alpha, \beta \in \{0, 3, \cdots, n\} \\ \tilde{\partial}_\alpha a_\beta - \partial_\beta \tilde{a}_\alpha & \alpha \notin \{1, 2\} \ , \ \beta \in \{1, 2\} \end{cases} \tag{2.71}$$

Since $\mathcal{F}_1 \subset \mathcal{E}$ and the functions $\alpha_1, \alpha_2, \tilde{a}_j, \phi$ and their multi-derivatives with respect to y, τ, ζ are bounded, one can deduce that

$$\langle \mathcal{O}_1 | \mathcal{F}_1 \rangle \subset \mathcal{E}$$

Therefore

$$f_2, f_3, g_{j,2}, g_{j,3} \in \mathcal{E}$$

Therefore, the statement holds for $m = 1$.

Now, suppose that the statement holds for m and

$$\begin{pmatrix} S_\varphi \\ S_{a_j} \end{pmatrix}_{j=1,2} [\Phi_m, A_m](t, x) = \sum_{i=m+1}^{3m} \epsilon^{2i} f_i(\tau, y, \zeta) \tag{2.72}$$

and

$$S_{a_j}[\Phi_m, A_m](t, x) = \sum_{i=m+1}^{3m} \epsilon^{2i-1} g_{j,i}(\tau, y, \zeta) \tag{2.73}$$

for $j = 0, 3$. Let:

$$\begin{aligned} \Phi_{m+1}(t, x) &= \Phi_m(t, x) + \epsilon^{2m} \sum_{\mu=1}^{2N} c_\mu(\tau, \zeta) n_{\mu,\varphi}(y; \tau, \zeta) + \epsilon^{2m+2} \varphi_{m+1}(\tau, y, \zeta) \\ A_{m+1,j}(t, x) &= A_{m,j}(t, x) + \epsilon^{2m} \sum_{\mu=1}^{2N} c_\mu(\tau, \zeta) n_{\mu,j}(y; \tau, \zeta) + \epsilon^{2m+2} a_{j,m+1}(\tau, y, \zeta) \quad j = 1, 2 \\ A_{m+1,j}(t, x) &= A_{m,j}(t, x) + \epsilon^{2m+1} a_{j,m+1}(\tau, y, \zeta) \quad j = 0, 3 \end{aligned} \tag{2.74}$$

where the new terms are to be found. The procedure to construct the above object is to first let the functions (c_μ) to be undetermined. Then, corresponding to (c_μ), find the functions $a_{j,m+1}$ for $j \in \{0, 3\}$ such that the error term for $S_{a_j}[\Phi_{m+1}, A_{m+1}]$ becomes of order $O(\epsilon^{2m+3})$. Then, to reduce the other error terms, an orthogonality condition to zero modes needs to be satisfied. This leads to a hyperbolic PDE for the functions (c_μ) which is well-posed. After finding suitable (c_μ), one can look back to the above process and find the other terms.

Let

$$E_0 = \sum_\mu c_\mu n_{\mu,\varphi} \tag{2.75}$$

and

$$E_j = \sum_\mu c_\mu n_{\mu,j} \quad j = 1, 2 \tag{2.76}$$

Then, we have:

$$\begin{aligned} S_{a_0}[\Phi_{m+1}, A_{m+1}] = &-\epsilon^{2m+1}\big[\Delta_y a_{0,m+1} - |\phi|^2 a_{0,m+1} + g_{0,m+1}\big] \\ &- \epsilon^{2m+1}\Big[\big(i\phi, \tilde{D}_0 E_0\big) + \big(iE_0, \tilde{D}_0\phi\big) - \tilde{\partial}_0\big(\sum_\mu c_\mu(\partial_1 n_{\mu,1} + \partial_2 n_{\mu,2})\big)\Big] \\ &+ \epsilon^{2m+3}(...) + \cdots \end{aligned} \tag{2.77}$$

$$\tag{2.78}$$

Using the fact that the zero modes n_μ satisfy the gauge orthogonality condition, we have:

$$\big(i\phi, \tilde{D}_0 E_0\big) - \tilde{\partial}_0\big(\sum_\mu c_\mu(\partial_1 n_{\mu,1} + \partial_2 n_{\mu,2})\big) = (iE_0, \tilde{D}_0\phi) \tag{2.79}$$

Therefore,

$$S_{a_0}[\Phi, A] = -\epsilon^{2m+1}\big[\Delta_y a_{0,m+1} - |\phi|^2 a_{0,m+1} + 2\sum_\mu c_\mu(in_{\mu,\varphi}, \tilde{D}_0\phi) + g_{0,m+1}\big] + \epsilon^{2m+3}(...)+ \tag{2.80}$$

Similarly,

$$S_{a_3}[\Phi, A] = -\epsilon^{2m+1}\big[\Delta_y a_{3,m+1} - |\phi|^2 a_{3,m+1} + 2\sum_\mu c_\mu(in_{\mu,\varphi}, \tilde{D}_3\phi) + g_{3,m+1}\big] + \epsilon^{2m+3}(...) + \cdots \tag{2.81}$$

According to lemma 6 in the appendix, one can find

$$h_{j,\mu}, k_j : \mathbb{R}^2 \times [0, T_m) \times \mathbb{R} \to \mathbb{R}$$

of class $\mathcal{E}$ for $j = 0, 3$, regardless of the choice for $(c_\mu)_\mu$, such that

$$a_{j,m+1} = \sum_\mu c_\mu h_{j,\mu} + k_j \quad j = 0, 3 \tag{2.82}$$

make the coefficient of ϵ^{2m+1} in the expressions (2.80) and (2.81) to vanish. (Add details here) Under this assumption for the functions $a_{j,m+1}$ for $j \in \{0, 3\}$, we have:

$$\begin{pmatrix} S_\varphi[\Phi_{m+1}, A_{m+1}] \\ S_{a_j}[\Phi_{m+1}, A_{m+1}] \end{pmatrix}_{j=1,2} = \epsilon^{2m+2}(\mathcal{L}\psi_{m+1} + S_{m+1}) + \epsilon^{2m+4}(...) + \cdots \tag{2.83}$$

and

$$\psi_{m+1} = \big(\varphi_{m+1}, a_{1,m+1}, a_{2,m+1}\big) \tag{2.84}$$

and

$$S_{m+1} = (\tilde{\partial}_0^2 - \tilde{\partial}_3^2)(\sum_\mu c_\mu n_\mu) + p_{m+1}(c_\mu, \tilde{\partial}c_\mu, \tau, \zeta) \tag{2.85}$$

where $\tilde{\partial}$ denotes differentiation with respect to τ, ζ and the function p_{m+1} is a polynomial of degree at most 2 with respect to (c_μ) and $(\tilde{\partial}c_\mu)$ with coefficients of class $\mathcal{E}$ and independent of ψ_{m+1}.

Proposition 8. *One can find smooth functions* (c_μ) *with bounded multi-derivatives such that the vector* S_{m+1} *is orthogonal to the corresponding zero modes for every* τ, ζ *with* $\tau \leq T_{m+1}$ *for some* $T_{m+1} > 0$.

Proof. The orthogonality conditions to the vectors n_λ for S_{m+1} lead to the following equation:

$$\tilde{\Box}(c_\mu)_\mu = F((c_\mu)_\mu, (\tilde{\partial} c_\mu)_\mu, \tau, \zeta) \tag{2.86}$$

where

$$\tilde{\Box} = \tilde{\partial}_0^2 - \sum_{j=3}^{n} \tilde{\partial}_j^2$$

and for each τ, ζ, the function F is linear or quadratic with respect to $(c_\mu), (\tilde{\partial} c_\mu)$ with coefficients of class $\mathcal{E}$. This is a well-posed equation and by considering the zero initial condition for $(c_\mu)_\mu$, one can find smooth solutions with bounded derivatives on the time interval $[0, T_{m+1}]$ for some $T_{m+1} > 0$. □

Now, by choosing $(c_\mu)_\mu$ as in the above proposition and the functions $a_{j,m+1}$ for $j = 0, 3$ according to (2.82), the vector S_{m+1} satisfies the orthogonality condition to the zero modes and one can use lemmas 10 and 14 to find ψ_{n+1} of class $\mathcal{E}$ such that

$$L\psi_{m+1} + S_{m+1} = 0$$

Proposition 9. *The vector* S_{m+1} *satisfies the gauge orthogonality condition.*

Proof. Similar to the proof of proposition 7, we use lemma 1. Let

$$\begin{aligned}
\tilde{\Phi}_m(t,x) &= \Phi_m(t,x) + \epsilon^{2m} \sum_{\mu=1}^{2N} c_\mu(\tau,\zeta) n_{\mu,\varphi}(y;\tau,\zeta) \\
\tilde{A}_{m,j}(t,x) &= A_{m,j}(t,x) + \epsilon^{2m} \sum_{\mu=1}^{2N} c_\mu(\tau,\zeta) n_{\mu,j}(y;\tau,\zeta) \quad j = 1,2 \\
\tilde{A}_{m,j}(t,x) &= A_{m,j}(t,x) + \epsilon^{2m+1} a_{m+1,j}(y;\tau,\zeta) \quad j = 0,3,\cdots,n
\end{aligned} \tag{2.87}$$

By using equation (2.83) and comparing $S[\tilde{\Phi}_m, \tilde{A}_m]$ with $S[\Phi_{m+1}, A_{m+1}]$ and noting that the correspondent of the ψ_{m+1} term of (Φ_{m+1}, A_{m+1}) in $(\tilde{\Phi}_m, \tilde{A}_m)$ is zero and the vector S_{m+1} is independent of ψ_{m+1}, we deduce that

$$\begin{pmatrix} S_\varphi[\tilde{\Phi}_m, \tilde{A}_m] \\ S_{a_j}[\tilde{\Phi}_m, \tilde{A}_m] \end{pmatrix}_{j=1,2} = \epsilon^{2m+2} S_{m+1} + \epsilon^{2m+4}(\cdots) + \cdots \tag{2.88}$$

and according to (2.80) and (2.81), we have:

$$S_{a_j}[\tilde{\Phi}_m, \tilde{A}_m] = \epsilon^{2m+3}(\cdots) + \epsilon^{2m+5}(\cdots) + \cdots \quad j = 0,3 \tag{2.89}$$

Suppose that

$$S_{m+1} = (S_{m+1,0}, S_{m+1,1}, S_{m+1,2})$$

where $S_{m+1,0}$ denotes the complex part. According to (2.88) and (2.89), the coefficient of ϵ^{2m+2} in

$$(i\tilde{\Phi}_m, S_\varphi[\tilde{\Phi}_m, \tilde{A}_m]) + \partial_0 S_{a_0}[\tilde{\Phi}_m, \tilde{A}_m] - \sum_{j=1}^{3} \partial_j S_{a_j}[\tilde{\Phi}_m, \tilde{A}_m]$$

is

$$(i\phi, S_{m+1,0}) - \sum_{j=1}^{2} \partial_j S_{m+1,j}$$

Therefore, according to lemma 1, the vector S_{m+1} satisfies the gauge orthogonality condition. □

Now, according to the above lemma, one can use lemmas 10 and 14 in the appendix to find ψ_{m+1} of class $\mathcal{E}$ which satisfies the gauge orthogonality condition for every τ, ζ and

$$L\psi_{m+1} + S_{m+1} = 0$$

Since ψ_{m+1} is gauge orthogonal, then $\mathcal{L}\psi_{m+1} = L\psi_{m+1}$ and therefore

$$\mathcal{L}\psi_{m+1} + S_{m+1} = 0$$

and by equations (2.80), (2.81), and (2.83), the expected order of error for $S[\Phi_{m+1}, A_{m+1}]$ is obtained. The desired estimates for the error terms hold due to the induction hypothesis and the fact that

$$(\Phi_{m+1}, A_{m+1}) - (\Phi_m, A_m\big)$$

is of class $\mathcal{E}$. This finishes the proof of the theorem. □

2.5 Perturbation of the Approximate solution

Suppose that $v = \big(\varphi, a\big)$ is an approximate solution for the Abelian Higgs model constructed in the first part of the project. Suppose that it is of the form

$$\begin{aligned} \varphi(t,y,z) &= \phi(y; q(\epsilon t, \epsilon z)) + \epsilon^2 \hat{\varphi} \\ a_j(t,y,z) &= \alpha_j(y; q(\epsilon t, \epsilon z)) + \epsilon^2 \hat{a}_j \qquad j = 1,2 \\ a_0(t,y,z) &= \epsilon \hat{a}_0(t,y,z) \\ a_3(t,y,z) &= \epsilon \hat{a}_3(t,y,z) \end{aligned} \tag{2.90}$$

defined on

$$\Big[0, \frac{T_1}{\epsilon}\Big)$$

for some number T_1. We look for an honest solution $(\varphi + \tilde{\varphi}, a + \tilde{a})$ for the Abelian Higgs equations. Let

$$u = (\tilde{\varphi}, \tilde{a}) \tag{2.91}$$

and

$$\tilde{a} = \sum_{j=0}^{3} \tilde{a}_j dx^j \tag{2.92}$$

and

$$\psi = (\tilde{\varphi}, \tilde{a}_1 dx^1 + \tilde{a}_2 dx^2) \tag{2.93}$$

Also, suppose that the error terms of the approximate solutions are denoted by

$$E = \big(E_{\varphi}, E_0, E_1.E_2, E_3\big) \tag{2.94}$$

where E_φ correspond to the Higgs section and E_j correspond to the gauge field A_j. If the number of iterations is large enough, we can ensure that

$$||E(t,.)||_{H^4} + ||E_t(t,.)||_{H^3} \leq C\epsilon^n \tag{2.95}$$

for some C. The number n depends on the number of iterations in the ansatz which will be figured out later.

2.5.1 Gauge Choice

To find u, we need to assume a choice for gauge. Also, we need to justify, why we can assume such a choice:

Proposition 10. *Suppose that*

$$\big(\tilde{\varphi}, \tilde{a}\big) \in C^0\big([0,T'); H^3\big) \cap \big(C^1\big([0,T'); H^2\big) \cap \big(C^2\big([0,T'); H^1\big)$$

is such that

$$w = \big(\varphi, a\big) + \big(\tilde{\varphi}, \tilde{a}\big)$$

is a solution for the system of equations

$$\begin{aligned} S_{\varphi}[w] &= iG(\varphi + \tilde{\varphi}) \\ S_{a_j}[w] &= \partial_j G \qquad j = 0,1,2,3 \end{aligned} \tag{2.96}$$

where

$$G = -\partial_0 \tilde{a}_0 + \partial_1 \tilde{a}_1 + \partial_2 \tilde{a}_2 + \partial_3 \tilde{a}_3 - \big(i\varphi, \tilde{\varphi}\big)$$

Suppose that

$$\tilde{\varphi}(0) = \tilde{a}_1(0) = \tilde{a}_2(0) = \tilde{a}_3(0) = (\partial_t \tilde{\varphi})(0) = (\partial_t \tilde{a})(0) = 0 \tag{2.97}$$

and

$$\Delta \tilde{a}_0(0) - |\varphi|^2(0)\tilde{a}_0(0) + E_0(0) = 0 \tag{2.98}$$

Then, v solves the Abelian Higgs model.

Proof. We have:

$$\Big(S_{\varphi}[w], i(\varphi + \tilde{\varphi})\Big) = -\partial_0 S_{a_0}[w] + \sum_{k=1}^{3} \partial_k S_{a_k}[w] \tag{2.99}$$

Therefore, by (2.96), we have:

$$\big(\partial_{tt} - \Delta\big) G = |\varphi + \tilde{\varphi}|^2 G \tag{2.100}$$

According to the initial conditions, we have:

$$G(0) = 0$$

Note that

$$\begin{aligned} S_{a_0}[w] = E_0 + \Delta\tilde{a}_0 - \sum_{j=1}^{3} \partial_t\partial_j\tilde{a}_j + \Big(i\varphi, \partial_t\tilde{\varphi}\Big) \\ - \tilde{a}_0|\varphi|^2 - 2(\varphi, a_0\tilde{\varphi}) + (i\tilde{\varphi}, \partial_t\varphi) \\ + (i\tilde{\varphi}, \partial_t\tilde{\varphi}) - 2(\varphi, \tilde{a}_0\tilde{\varphi}) - a_0|\tilde{\varphi}|^2 - \tilde{a}_0|\tilde{\varphi}|^2 \end{aligned} \tag{2.101}$$

Therefore, together with equation (2.98) implies that

$$S_{a_0}[w](0) = 0$$

and therefore, by equation (2.96)

$$\partial_t G(0) = 0$$

Therefore, (2.100) implies that $G = 0$ for all times and v solves the Abelian Higgs model. □

2.5.2 Rewriting the equations

According to proposition 10, it suffices to find $u = (\tilde{\varphi}, \tilde{a})$ such that the following is satisfied:

$$\begin{aligned} S_{\varphi}[(\varphi + \tilde{\varphi}, a + \tilde{a})] = iG(\varphi + \tilde{\varphi}) \\ S_{a_j}[\varphi + \tilde{\varphi}, a + \tilde{a}] = \partial_j G \quad\quad j = 0, 1, 2, 3 \end{aligned} \tag{2.102}$$

where

$$G = -\partial_0\tilde{a}_0 + \sum_{j=1}^{3} \partial_j\tilde{a}_j - (i\varphi, \tilde{\varphi}) \tag{2.103}$$

where the initial conditions are:

$$\tilde{\varphi}(0) = \tilde{a}_1(0) = \tilde{a}_2(0) = \tilde{a}_3(0) = (\partial_t\tilde{\varphi})(0) = (\partial_t\tilde{a})(0) = 0 \tag{2.104}$$

and

$$\Delta\tilde{a}_0(0) - |\varphi|^2(0)\tilde{a}_0(0) + E_0(0) = 0 \tag{2.105}$$

The existence of $\tilde{a}_0$ satisfying equation (2.105) is provided by lemma 9 in the appendix.
To write down equation (2.102), first we study the linearization of the equations. To linearize the equations, we use the following calculations:

Lemma 2. *We have:*

$$(D_{A+\tilde{A}})^2(\Phi + \tilde{\Phi}) = D_A^2\Phi + D_A^2\tilde{\Phi} - 2i\tilde{A}D_A\Phi - i(\nabla.\tilde{A})\Phi - D_A(i\tilde{A}\tilde{\Phi}) - i\tilde{A}D_A\tilde{\Phi} - (\tilde{A})^2(\Phi + \tilde{\Phi}) \tag{2.106}$$

and

$$(|\Phi+\tilde{\Phi}|^2-1)(\Phi+\tilde{\Phi}) = (|\Phi|^2-1)\Phi + \Big(2(\Phi,\tilde{\Phi})\Phi + (|\Phi|^2-1)\tilde{\Phi}\Big) + 2(\Phi,\tilde{\Phi})\tilde{\Phi} + |\tilde{\Phi}|^2\tilde{\Phi} \tag{2.107}$$

and

$$\begin{aligned}(i(\Phi+\tilde{\Phi}), D_{A+\tilde{A}}(\Phi+\tilde{\Phi})) = (i\Phi, D_A\Phi) + (i\tilde{\Phi}, D_A\Phi) + (i\Phi, D_A\tilde{\Phi}) - |\Phi|^2\tilde{A} \\ + (i\tilde{\phi}, D_A\tilde{\phi}) - 2(\phi,\tilde{\phi})\tilde{A} - |\tilde{\phi}|^2\tilde{A}\end{aligned} \tag{2.108}$$

Proof. The proof is by straightforward calculations.

$$\begin{aligned}(D_{A+\tilde{A}})^2(\Phi+\tilde{\Phi}) &= (D_A - i\tilde{A})^2(\Phi+\tilde{\Phi}) \\ &= (D_A^2 - i\tilde{A}D_A - iD_A.\tilde{A} - (\tilde{A})^2)(\Phi+\tilde{\Phi}) \\ &= D_A^2\Phi + D_A^2\tilde{\Phi} - i\tilde{A}D_A\Phi - i\tilde{A}D_A\tilde{\Phi} \\ &- iD_A(\tilde{A}\Phi) - iD_A(\tilde{A}\tilde{\Phi}) - (\tilde{A})^2(\Phi+\tilde{\Phi}) \\ &= D_A^2\Phi + D_A^2\tilde{\Phi} - 2i\tilde{A}D_A\Phi - i(\nabla.\tilde{A})\Phi \\ &- iD_A(\tilde{A}\tilde{\Phi}) - i\tilde{A}D_A\tilde{\Phi} - (\tilde{A})^2(\Phi+\tilde{\Phi})\end{aligned} \tag{2.109}$$

Identity (2.107) is simple. To check (2.108), we use the fact that $(iv, iw) = (v, w)$ and proceed as follows:

$$\begin{aligned}(i(\Phi+\tilde{\Phi}), D_{A+\tilde{A}}(\Phi+\tilde{\Phi})) &= (i(\Phi+\tilde{\Phi}), (D_A - i\tilde{A})(\Phi+\tilde{\Phi})) \\ &= (i(\Phi+\tilde{\Phi}), D_A\Phi + D_A\tilde{\Phi} - i\tilde{A}\Phi - i\tilde{A}\tilde{\Phi}) \\ &= (i\Phi, D_A\Phi) + (i\tilde{\Phi}, D_A\Phi) + (i\Phi, D_A\tilde{\Phi}) - |\Phi|^2\tilde{A} \\ &+ (i\tilde{\Phi}, D_A\tilde{\Phi}) - 2(\Phi,\tilde{\Phi})\tilde{A} - |\tilde{\Phi}|^2\tilde{A}\end{aligned} \tag{2.110}$$

□

According to lemma 2, one can observe that

$$S(\Phi+\tilde{\Phi}, A+\tilde{A}) = S(\Phi, A) + \mathcal{H}[\Phi, A](\tilde{\Phi}, \tilde{A}) + N(\Phi, \tilde{\Phi}, A, \tilde{A}) \tag{2.111}$$

where

$$\mathcal{H}[\Phi, A](\tilde{\Phi}, \tilde{A}) = \mathcal{G}[\Phi+\tilde{\Phi}, A][\tilde{\Phi}, \tilde{A}] + \begin{pmatrix} i\Phi\big(\nabla.\tilde{A} - \big(i\Phi, \tilde{\Phi}\big)\big) \\ \partial_i\big(\nabla.\tilde{A} - \big(i\Phi, \tilde{\Phi}\big)\big)\end{pmatrix} \tag{2.112}$$

where

$$\mathcal{G}[\Phi, A][\tilde{\Phi}, \tilde{A}] = \begin{pmatrix} \Box_A\tilde{\Phi} - 2i(\tilde{A}_0 D_0\Phi - \sum_{i=1}^3 \tilde{A}_i D_i\Phi) + \frac{1}{2}\big(3|\Phi|^2 - 1\big)\tilde{\Phi} \\ \Box\tilde{A}_i + |\Phi|^2\tilde{A}_i - 2\big(i\tilde{\Phi}, D_i\Phi\big)\end{pmatrix} \tag{2.113}$$

where

$$\Box_A = D_0 D_0 - \sum_{j=1}^3 D_j D_j$$

and N consists of nonlinear terms with respect to $\tilde{\Phi}$ and $\tilde{A}$.

To write equations (2.102), we use equations (2.111), (2.112) and (2.113), and obtain that

$$\begin{aligned}
S_\varphi(\varphi+\tilde{\varphi},a+\tilde{a})) &= S_\varphi\big((\phi+\epsilon^2\hat{\varphi})+\tilde{\varphi},a+\tilde{a}\big) \\
&= S_\varphi(\varphi,a)+(\partial_t-ia_0)^2\tilde{\varphi}-(\partial_z-ia_3)^2\tilde{\varphi} \\
&-\sum_{j=1}^{2}(\partial_j-i\alpha_j-i\epsilon^2\hat{a}_j)^2\tilde{\varphi} \\
&-2i\tilde{a}_0(\partial_t-ia_0)(\phi+\epsilon^2\hat{\varphi})+2i\tilde{a}_3(\partial_z-ia_3)(\phi+\epsilon^2\hat{\varphi}) \\
&+2i\sum_{j=1}^{2}(\tilde{a}_j(\partial_j-i\alpha_j-i\epsilon^2\hat{a}_j)(\phi+\epsilon^2\hat{\varphi})) \\
&+\frac{1}{2}(3|(\phi+\epsilon^2\hat{\varphi})|^2-1)\tilde{\varphi}+N_0+i\varphi G \\
&= \partial_t^2\tilde{\varphi}-2ia_0\partial_t\tilde{\varphi}-\partial_z^2\tilde{\varphi}+2ia_3\partial_z\tilde{\varphi} \\
&+L_\varphi[\phi,\alpha][\psi]+E_0+N_0 \\
&-2i\tilde{a}_0D_0\varphi+2i\tilde{a}_3D_3\varphi \\
&+R_0u+i\varphi G
\end{aligned} \tag{2.114}$$

where

$$\begin{aligned}
R_0u &= -i(\partial_t a_0)\tilde{\varphi}-a_0^2\tilde{\varphi}+i(\partial_z a_3)\tilde{\varphi}+a_3^2\tilde{\varphi} \\
&+\sum_{j=1}^{2}\Big[2i\epsilon^2\hat{a}_j\partial_j\tilde{\varphi}+i\epsilon^2(\partial_j\hat{a}_j)\tilde{\varphi}-2\epsilon^2\alpha_j\hat{a}_j\tilde{\varphi}+\epsilon^4\hat{a}_j^2\tilde{\varphi}\Big]+3\epsilon^2(\phi,\hat{\varphi})\tilde{\varphi}+\frac{3}{2}\epsilon^4|\hat{\varphi}|^2\tilde{\varphi} \\
&+2i\sum_{j=1}^{2}\epsilon^2\tilde{a}_j\Big(\partial_j\hat{\varphi}-i\hat{a}_j\phi-i\alpha_j\hat{\varphi}-i\epsilon^2\hat{a}_j\hat{\varphi}\Big)
\end{aligned} \tag{2.115}$$

and N_0 consists of nonlinear terms and E_0 corresponds to the error of the approximate solution. Now, for $j=1,2$, we have

$$\begin{aligned}
S_{a_j}[\varphi+\tilde{\varphi},a+\tilde{a}] &= S_{a_j}[\varphi,a]+\partial_j G+\big(\partial_t^2-\partial_z^2-\Delta+|\phi+\epsilon^2\hat{\varphi}|^2\big)\tilde{a}_j \\
&-2(i\tilde{\varphi},(\partial_j-i\alpha_j-i\epsilon^2\hat{a}_j)(\phi+\epsilon^2\hat{\varphi}))+N_j \\
&= \big(\partial_t^2-\partial_z^2-\Delta+|\phi|^2\big)\tilde{a}_j-2(i\tilde{\varphi},D_j\phi) \\
&+S_ju+N_j+E_j+\partial_j G \\
&= (\partial_t^2-\partial_z^2)\tilde{a}_j-L_j[\phi,\alpha](\psi)+S_ju+N_j+E_j+\partial_j G
\end{aligned} \tag{2.116}$$

where

$$S_ju=-2\epsilon^2\Big(i\tilde{\varphi},\partial_j\hat{\varphi}-i\hat{a}_j\phi-i\alpha_j\hat{\varphi}-\epsilon^2\hat{a}_j\hat{\varphi}\Big)+\epsilon^2\tilde{a}_j\Big(2(\phi,\hat{\varphi})+\epsilon^2|\hat{\varphi}|^2\Big) \tag{2.117}$$

and N_j, E_j correspond to the nonlinear terms and error of the approximate solution, respectively.

For $j = 0, 3$, we have:

$$\begin{aligned} S_{a_j}[\varphi + \tilde{\varphi}, a + \tilde{a}] &= S_{a_j}[\varphi, a] + \partial_j G + \left(\partial_t^2 - \partial_z^2 - \Delta + |\phi + \epsilon^2 \hat{\varphi}|^2\right)\tilde{a}_j \\ &\quad - 2(i\tilde{\varphi}, (\partial_j - ia_j)(\phi + \epsilon^2 \hat{\varphi})) + N_j \\ &= \left(\partial_t^2 - \partial_z^2 - \Delta + |\phi|^2\right)\tilde{a}_j - 2(i\tilde{\varphi}, D_j \phi) \\ &\quad + S_j u + N_j + E_j + \partial_j G \end{aligned} \tag{2.118}$$

where

$$S_j u = 2\epsilon^2 (\phi, \hat{\varphi})\tilde{a}_j - 2\epsilon^2 (i\tilde{\varphi}, D_j \hat{\varphi}) \tag{2.119}$$

for $j = 0, 3$ and N_j, E_j are the nonlinearity and the error of the approximate solution. Therefore, one can write down the equations (2.102) for $u = (\tilde{\varphi}, \tilde{a})$ in the following form:

$$u_{tt} + Mu + Pu + N + E = 0 \tag{2.120}$$

where

$$Mu = \begin{pmatrix} -\partial_z^2 \psi + L[\phi, \alpha]\psi \\ \\ -\Delta \tilde{a}_0 + |\phi|^2 \tilde{a}_0 \\ \\ -\Delta \tilde{a}_3 + |\phi|^2 \tilde{a}_3 \end{pmatrix} \tag{2.121}$$

and

$$Pu = \begin{pmatrix} \begin{pmatrix} Ru \\ 0 \end{pmatrix} \\ \\ 2(i\tilde{\varphi}, D_0 \phi) \\ \\ 2(i\tilde{\varphi}, D_3 \phi) \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} R_0 u \\ Su \end{pmatrix} \\ \\ S_0 u \\ \\ S_3 u \end{pmatrix}$$

where

$$Ru = -2ia_0 \partial_t \tilde{\varphi} + 2ia_3 \partial_z \tilde{\varphi} - 2i\tilde{a}_0 D_0 \varphi + 2i\tilde{a}_3 D_3 \varphi \tag{2.122}$$

where

$$Su = (S_1 u, S_2 u) \tag{2.123}$$

where $R_0 u$ and $S_j u$ for $j = 0, 1, 2, 3$ are as before and $N, E.$ consist of nonlinear terms and the error of the approximate solution respectively. The quantity Ru is of the form $\epsilon F(u, Du)$ for some linear function F and the quantities $R_0 u, Su, S_0 u$ and $S_3 u$ are of the form $\epsilon^2 G(u, Du)$ for some linear function G

2.5.3 local existence theorems and a priori estimates

Theorem 6. *Consider the approximate solution $v = (\tilde{\varphi}, \tilde{a})$ described in the beginning of section 2.5 with error E satisfying conditions* (2.95)*. Suppose that*

$$0 \leq s \leq \frac{T_1}{\epsilon}$$

We consider equation (2.120) *for the approximate solution . Consider the starting point to be at $t = s$. For any $b > 0$, there exists $\delta > 0$ such that if $\epsilon < \delta$ and the initial data for equations for u satisfies*

$$(u(s,.), \partial_t u(s,.)) \in H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3)$$

and

$$a = ||u(s,.)||_{H^3} + ||\partial_t u(s,.)||_{H^2} \leq b \tag{2.124}$$

there exists T_2 depending only on the function

$$\tilde{v}(t, x, z) = v(\epsilon t, x, \epsilon z)$$

for the approximate solution v and in particular independent of ϵ, and a solution

$$u \in E_1 = C^0([s, s+T_2]; H^3) \cap C^1([s, s+T_2]; H^2) \cap C^2([s, s+T_2]; H^1)$$

to equation (2.120)*. Furthermore, it satisfies the estimate*

$$||u(t,.)||_{H^3} + ||\partial_t u||_{H^2} \leq Ca + \epsilon^n$$

for some $C > 0$.

Proof. Suppose that $\epsilon < \frac{1}{2}$. Let

$$w = \begin{pmatrix} u \\ \partial_t u \\ \partial_{x_1} u \\ \partial_{x_2} u \\ \partial_{x_3} u \end{pmatrix}$$

Then, the equations for w can be written in the form

$$\partial_t w + \sum_{j=1}^{3} A_j \partial_{x_j} w + F(t, x, w) = f(t, x) \tag{2.125}$$

for some symmetric matrices B_j with constant entries so that the operator

$$K = \frac{\partial}{\partial_t} + \sum_{j=1}^{3} B_j \frac{\partial}{\partial x_j}$$

is symmetric hyperbolic. Also F involves the first order and nonlinear terms of equation (2.120) and f is associated with the error E of the approximate solution and by the property (2.95) of error, we have

$$f(.) \in L^1_{loc}\Big[[0, \frac{T_1}{\epsilon}); H^2(\mathbb{R}^3)\Big] \tag{2.126}$$

for some $C > 0$. Now, we use the so called Friedrichs theorem from page 36, chapter 2 of [14].

Theorem 7 (Friedrichs)**.** *Suppose that L is a symmetric hyperbolic operator and $g \in H^r(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}; H^r(\mathbb{R}^d))$ for some $r > 0$. Then, there is one and only one solution $u \in C(\mathbb{R}; H^r(\mathbb{R}^d))$ to the initial value problem*

$$\begin{aligned} Lu &= f \\ u(0) &= g \end{aligned} \tag{2.127}$$

In addition, there is a constant $C = C(L, r)$ independent of f, g so that for all $f, g > 0$, there holds:

$$||u(t)||_{H^r(\mathbb{R}^d)} \leq Ce^{Ct}||u(0)||_{H^r(\mathbb{R}^d)} + \int_0^t Ce^{C(t-\sigma)}||f(\sigma)||_{H^r(\mathbb{R}^d)}d\sigma \tag{2.128}$$

Let $r = 2$ and $L = K$ in the setting of the above theorem. By using the Sobolev embedding and the fact that the coefficients of F have bounded derivatives of any order and by (2.126), we can use the above theorem to find a sequence

$$w_m \in C\big[[s, \frac{T_1}{\epsilon}); H^2\big]$$

such that

$$\partial_t w_{m+1} + \sum_{j=1}^{3} B_j \partial_{x_j} w_{m+1} + F(t, x, w_m) = f(t, x)$$

and $w_0 = 0$. Suppose that C_1 is the constant provided by the Friedrich's theorem. Suppose that

$$R = 2aC_1 + \epsilon^n$$

There exists a polynomial p of degree 3 such that if

$$||w(t, .)||_{H^2} \leq R$$

then

$$||F(w(t, .))||_{H^2} \leq p(R) \tag{2.129}$$

This is because the nonlinearity in the equation (2.120) is of order 3. By assuming that $w_0 = 0$, we will find T_2 such that for any $t \in [s, s + T_2]$, we have

$$||w_m(t, .)||_{H^2} \leq R \tag{2.130}$$

Note that since $R = 2aC_1 + \epsilon^n$ and $a \leq b$, there exists T_2 such that if $(t - s) \leq T_2$, then

$$C_1 e^{C_1(t-s)}a + \frac{t-s}{C_1}(e^{C_1(t-s)} - 1)(p(R) + C'\epsilon^n) \leq R \tag{2.131}$$

Suppose that for some m, we know that (2.130) holds for any $t \in (s, s + T_2)$. By equation (2.128) and using (2.129), we have:

$$\begin{aligned} ||w_{m+1}(t,.)||_{H^2} \leq & C_1\Big(e^{C_1(t-s)}a + \int_s^t e^{C_1(t-\sigma)}(P(R)+C\epsilon^n)d\sigma\Big) \\ & \leq C_1 e^{C_1(t-s)}a + \frac{t-s}{C_1}(e^{C_1(t-s)}-1)(p(R)+C'\epsilon^n) \end{aligned} \tag{2.132}$$

(2.132) and (2.131) imply that if $(t-s) \leq T_2$, then we have

$$||w_{m+1}(t,.)||_{H^2} \leq R \tag{2.133}$$

and by induction, the above bound holds for any m. Now, we claim that the sequence $\{w_m\}_{m=1}^{\infty}$ converges in $C\big([s, s+T_2); H^2\big)$. Note that

$$K(w_{m+1} - w_m) + F(t,x,w_m) - F(t,x,w_{m-1}) = 0 \tag{2.134}$$

But, since $||w_m(t,.)||_{H^2} \leq R$, there exists a number $\lambda > 0$ such that for any m,

$$||F(t,.,w_m) - F(t,.,w_{m-1})||_{H^2} \leq \lambda ||w_m(t,.) - w_{m-1}(t,.)||_{H^2} \tag{2.135}$$

Therefore, by the Friedrich's theorem, we have:

$$||w_{m+1}(t,.) - w_m(t,.)||_{H^2} \leq C_1 \int_s^t e^{C_1(t-\sigma)}||w_m(t,.) - w_{m-1}(t,.)||_{H^2} d\sigma \tag{2.136}$$

Let

$$\begin{aligned} M_1 &= \sup_{[s,s+T_1]} ||w_1(t,.) - w_0(t,.)||_{H^2} \\ M_2 &= C_1 \lambda e^{C_1 T_2} \end{aligned} \tag{2.137}$$

By induction on m, we have:

$$||w_{m+1}(t,.) - w_m(t,.)||_{H^2} \leq M_1 \frac{(M_2 t)^{m-1}}{(m-1)!} \tag{2.138}$$

Therefore, the sequence $\{w_m\}_{m=0}^{\infty}$ is Cauchy in $C\big([s; s+T_2); H^2\big)$ and the limit satisfies the equation. □

2.5.4 Some quantities and estimates

In this section, we are going to introduce some quantities and estimate their evolution over the course of time. In subsections 2.5.5 and 2.5.6, we will explain how these estimates allow us to do a bootstrap argument.

Some quantities

The new quantities that we want to consider are as follows:

- Let:

$$S = Mu \tag{2.139}$$

and

$$Q_1(t) = \int_{\mathbb{R}^3} |u_t|^2 + (Mu, u) \tag{2.140}$$

and

$$Q_2(t) = \int_{\mathbb{R}^3} |S_t|^2 + (MS, S) \tag{2.141}$$

- For each (t, z), suppose that

$$\psi(t, y, z) = \psi_1(t, y, z) + \sum_{\mu} c_\mu(t, z) n_\mu(y; t, z) \tag{2.142}$$

where

$$\psi_1(t, ., z) \perp T_{(q(\epsilon t, \epsilon z))} M_N \tag{2.143}$$

for any (t, z) where q is the wave map in theorem 5.

- Let:

$$\tilde{a}_j = f_j + \partial_0 \chi_j \tag{2.144}$$

for $j = 0, 3$, where χ_j solves the following equation:

$$\big(\Delta - |\phi|^2\big)\chi_j = \partial_0 \tilde{a}_j \tag{2.145}$$

The existence of such $\chi_j \in H^3(\mathbb{R}^3)$ is guarantied by lemma 9 and remark 3 in the appendix.

Estimates for the above quantities

In this section, we will describe how the quantities Q_1, Q_2, $||c_\mu||_{H^3}$ and $||\partial_t c_\mu||_{H^2}$ change over time, approximately.

-

Proposition 11. *There holds:*

$$(Q_1 + Q_2)'(t) \leq C\big(\epsilon||u||_{H^3} + ||N||_{H^2} + ||E||_{H^2} + \epsilon||u_t||_{H^2}\big)\big(\epsilon||u||_{H^3} + ||u_t||_{H^2}\big) + \epsilon||u||^2_{H^3} \tag{2.146}$$

Proof.

$$Q_1' = 2\int_{\mathbb{R}^3} \big((u_{tt}, u_t) + (Mu, u_t)\big) + (M_t u, u) \tag{2.147}$$

So

$$Q_1'(t) = 2(Pu + N + E, u_t) + (M_t u, u) \tag{2.148}$$

and

$$Q_2' = 2\Big(\int_{\mathbb{R}^3} (S_{tt}, S_t) + (MS, S_t)\Big) + (M_t S, S) \tag{2.149}$$

so

$$Q_2'(t) = 2\big(MPu+MN+ME+2M_tu_t+M_{tt}u,(Mu)_t\big)+(M_tMu,Mu) \quad (2.150)$$

So

$$\begin{aligned}(Q_1+Q_2)'(t) \le & C||Pu||_{L^2}|||u_t||_{L^2} + ||N+E||_{L^2}||u_t||_{L^2} + C\epsilon||u||_{H^1}||u||_{L^2} \\ & +C\big(||Pu+N+E||_{H^2} + \epsilon||u_t||_{H^1} + \epsilon^2||u||_{H^1}\big)\big(\epsilon||u||_{H^1} + ||u_t||_{H^2}\big) + \epsilon||u||^2_{H^3} \\ \le & C\big(\epsilon||u||_{H^3} + ||N||_{H^2} + ||E||_{H^2} + \epsilon||u_t||_{H^2}\big)\big(\epsilon||u||_{H^3} + ||u_t||_{H^2}\big) + \epsilon||u||^2_{H^3}\end{aligned} \quad (2.151)$$

□

•

Proposition 12. *For $j = 0, 3$, we have:*

$$-\Delta f_j + |\phi|^2 f_j = (\partial_0|\phi|^2)\chi_j + 2(i\tilde{\varphi}, D_j\phi) + N_j + E_j \quad (2.152)$$

for $j = 0, 3$, where N_j and E_j are the corresponding components of N and E.

Proof. According to the gauge field section of equation (2.120) and equations (2.144), (2.145), we have:

$$\begin{aligned}\big(-\Delta+|\phi|^2\big)f_j &= \big(-\Delta+|\phi|^2\big)(\tilde{a}_j - \partial_0\chi_j) \\ &= \big(-\Delta+|\phi|^2\big)\tilde{a}_j \\ &- \big(-\Delta+|\phi|^2\big)(\partial_0\chi_j) \\ &= -\partial_t^2\tilde{a}_j + 2(i\tilde{\varphi}, D_j\phi) + N_j + E_j \\ &- \big(-\Delta+|\phi|^2\big)(\partial_0\chi_j) \\ &= -\partial_t^2\tilde{a}_j + 2(i\tilde{\varphi}, D_j\phi) + N_j + E_j \\ &- \partial_0\Big(\big(-\Delta+|\phi|^2\big)(\chi_j)\Big) + (\partial_0|\phi|^2)(\chi_j) \\ &= -\partial_t^2\tilde{a}_j + 2(i\tilde{\varphi}, D_j\phi) + N_j + E_j + \partial_0^2\tilde{a}_j + (\partial_0|\phi|^2)\chi_j \\ &= (\partial_0|\phi|^2)(\chi_j) + 2(i\tilde{\varphi}, D_j\phi) + N_j + E_j\end{aligned} \quad (2.153)$$

□

•

Proposition 13. *There holds:*

$$\partial_{tt}c_\mu - \partial_{zz}c_\mu = \partial_0 h_0 + \partial_3 h_3 + h \quad (2.154)$$

where

$$h_0 = \big(2ia_0\tilde{\varphi} + 2i\chi_0 D_0\varphi - 2i\chi_3 D_3\varphi, n_{\mu,\varphi}\big) + 2\big(\psi, (n_\mu)_t\big) \quad (2.155)$$

and

$$h_3 = -\big(2ia_3\tilde{\varphi}, n_{\mu,\varphi}\big) - 2\big(\psi, (n_\mu)_z\big) \tag{2.156}$$

and

$$\begin{aligned} h = &- (R_0 u, n_{\mu,\varphi}) - \sum_{j=1}^{2} (S_j u, n_{\mu,a_j}) \\ &- (N, n_\mu) - (E, n_\mu) - (\psi, (n_\mu)_{tt}) + (\psi, (n_\mu)_{zz}) \\ &+ \big(2if_0 D_0\varphi - 2if_3 D_3\varphi, n_{\mu,\varphi}\big) \\ &- \big(2i(\partial_0 a_0)\tilde{\varphi}, n_{\mu,\varphi}\big) - \big(2ia_0\tilde{\varphi}, \partial_0 n_{\mu,\varphi}\big) \\ &+ \big(2i(\partial_3 a_3)\tilde{\varphi}, n_{\mu,\varphi}\big) + \big(2ia_3\tilde{\varphi}, \partial_3 n_{\mu,\varphi}\big) \\ &- \big(2i\chi_0(\partial_0 D_0\varphi), n_{\mu,\varphi}\big) - \big(2i\chi_0 D_0\varphi, \partial_0 n_{\mu,\varphi}\big) \\ &+ \big(2i\chi_3 \partial_0 D_3\varphi, n_{\mu,\varphi}\big) + \big(2i\chi_3 D_3\varphi, \partial_0 n_{\mu,\varphi}\big) \end{aligned} \tag{2.157}$$

and therefore for any t, t', we have

$$\begin{aligned} c_\mu(t+t', z) = &\frac{1}{2}\Big[c_\mu(t', z-t) + c_\mu(t', z+t)\Big] + \frac{1}{2}\big[\int_{z-t}^{z+t} \big(\partial_0 c_\mu(t', s) ds\big)\big] \\ &+ \frac{1}{2}\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau, s) ds d\tau \\ &+ \frac{1}{2}\int_0^t \Big[h_3(t'+s, z+t-s) - h_3(t'+s, z-(t-s))\Big] ds \\ &- \frac{1}{2}\int_{z-t}^{z+t} h_0(t', s) ds + \frac{1}{2}\int_0^t h_0(t'+s, z-t+s) ds + \frac{1}{2}\int_0^t h_0(t'+s, z+t-s) ds \end{aligned} \tag{2.158}$$

Proof. We Take the inner product of the equations (2.120) with the zero modes n_μ's:

$$\big(\psi_{tt} - \psi_{zz}, n_\mu\big) + (L\psi, n_\mu) = -(Hu, n_\mu) - (N, n_\mu) - (E, n_\mu) \tag{2.159}$$

where

$$Hu = \begin{pmatrix} Ru \\ 0 \end{pmatrix} + \begin{pmatrix} R_0 u \\ Su \end{pmatrix} \tag{2.160}$$

. We have:

$$(L\psi, n_\mu) = 0 \tag{2.161}$$

Let

$$c_\mu = (\psi, n_\mu)$$

By (2.159) and (2.161), we have:

$$\begin{aligned}(c_\mu)_{tt} - (c_\mu)_{zz} &= 2(\psi_t, (n_\mu)_t) - 2(\psi_z, (n_\mu)_z) - (Hu, n_\mu) - (N, n_\mu) - (E, n_\mu) \\ &+ (\psi, (n_\mu)_{tt} - (n_\mu)_{zz}) \\ &= 2\partial_t(\psi, (n_\mu)_t) - 2\partial_z(\psi, (n_\mu)_z) - (Hu, n_\mu) \\ &- (N, n_\mu) - (E, n_\mu) - (\psi, (n_\mu)_{tt}) + (\psi, (n_\mu)_{zz})\end{aligned} \tag{2.162}$$

Now, we want to write the term (Hu, n_μ) in another form.

$$\begin{aligned}(Hu, n_\mu) &= (Ru, n_{\mu,\varphi}) + \big((R_0 u, Su), n_\mu\big) \\ &= \big(- 2ia_0\partial_t\tilde{\varphi} + 2ia_3\partial_z\tilde{\varphi} - 2i\tilde{a}_0 D_0\varphi + 2i\tilde{a}_3 D_3\varphi, n_{\mu,\varphi}\big) \\ &+ \big((R_0 u, Su), n_\mu\big)\end{aligned} \tag{2.163}$$

Now, by equation (2.163), we have:

$$\begin{aligned}(Hu, n_\mu) &= \big(- 2ia_0\partial_t\tilde{\varphi} + 2ia_3\partial_z\tilde{\varphi} - 2i\tilde{a}_0 D_0\varphi + 2i\tilde{a}_3 D_3\varphi, n_{\mu,\varphi}\big) \\ &+ (R_0 u, n_{\mu,\varphi}) + \sum_{j=1}^{2}(S_j u, n_{\mu,a_j})\end{aligned} \tag{2.164}$$

Therefore, according to (2.144), we have:

$$\begin{aligned}(Hu, n_\mu) = &-\partial_0\big(2ia_0\tilde{\varphi}, n_{\mu,\varphi}\big) + \big(2i(\partial_0 a_0)\tilde{\varphi}, n_{\mu,\varphi}\big) + \big(2ia_0\tilde{\varphi}, \partial_0 n_{\mu,\varphi}\big) \\ &+ \partial_3\big(2ia_3\tilde{\varphi}, n_{\mu,\varphi}\big) - \big(2i(\partial_3 a_3)\tilde{\varphi}, n_{\mu,\varphi}\big) - \big(2ia_3\tilde{\varphi}, \partial_3 n_{\mu,\varphi}\big) \\ &- \big(2if_0 D_0\varphi - 2if_3 D_3\varphi, n_{\mu,\varphi}\big) \\ &- \partial_0\big(2i\chi_0 D_0\varphi, n_{\mu,\varphi}\big) + \big(2i\chi_0(\partial_0 D_0\varphi), n_{\mu,\varphi}\big) + \big(2i\chi_0 D_0\varphi, \partial_0 n_{\mu,\varphi}\big) \\ &+ \partial_0\big(2i\chi_3 D_3\varphi, n_{\mu,\varphi}\big) - \big(2i\chi_3\partial_0 D_3\varphi, n_{\mu,\varphi}\big) - \big(2i\chi_3 D_3\varphi, \partial_0 n_{\mu,\varphi}\big) \\ &+ \big(R_0 u, n_{\mu,\varphi}\big) + \sum_{j=1}^{2}(S_j u, n_{\mu,a_j})\end{aligned} \tag{2.165}$$

Combining equations (2.162) and (2.165), we have:

$$\partial_{tt}c_\mu - \partial_{zz}c_\mu = \partial_0 h_0 + \partial_3 h_3 + h \tag{2.166}$$

To solve this equation, we use the explicit formulas about the linear wave equations. The crucial point is that the term $\partial_0 h_0$ in (2.166) can be handled differently and the time derivative can be dropped in the solution, as in the following lemma.

Lemma 3. *The solution to the equation*

$$\partial_t^2 f - \partial_z^2 f = \partial_t w \tag{2.167}$$

with the initial data $f(t,0) = \partial_t f(t,0) = 0$ can be written as

$$f(t,z) = -\frac{1}{2}\int_{z-t}^{z+t} w(s,0)ds + \int_0^t w(s,z-t+s)ds + \int_0^t w(s,z+t-s)ds \quad (2.168)$$

Proof. We have:

$$(\partial_t - \partial_z)(\partial_t + \partial_z)f = \frac{1}{2}(\partial_t - \partial_z)w + \frac{1}{2}(\partial_t + \partial_z)w \quad (2.169)$$

Therefore, by linearity of the equations, the solution to the equations can be written as $f_1 + f_2$ where f_1 solves

$$\begin{aligned} (\partial_t + \partial_z)f_1 &= \frac{1}{2}w \\ f_1(0,z) &= \frac{1}{2}\int_0^z w(0,s)ds \end{aligned} \quad (2.170)$$

and

$$\begin{aligned} (\partial_t - \partial_z)f_2 &= \frac{1}{2}w \\ f_2(0,z) &= -\frac{1}{2}\int_0^z w(0,s)ds \end{aligned} \quad (2.171)$$

Therefore, we have:

$$f(t,z) = -\frac{1}{2}\int_{z-t}^{z+t} w(0,s)ds + \frac{1}{2}\int_0^t w(s,z-t+s)ds + \frac{1}{2}\int_0^t w(s,z+t-s)ds \quad (2.172)$$

□

Using lemma 3 and the d'Alembert's formula, we have:

$$\begin{aligned} c_\mu(t+t',z) &= \frac{1}{2}\Big[c_\mu(t',z-t) + c_\mu(t',z+t)\Big] + \frac{1}{2}\Big[\int_{z-t}^{z+t}\big(\partial_0 c_\mu(t',s)ds\big)\Big] \\ &+ \frac{1}{2}\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} (\partial_3 h_3 + h)(t'+\tau,s)dsd\tau \\ &- \frac{1}{2}\int_{z-t}^{z+t} h_0(t',s)ds + \int_0^t h_0(t'+s,z-t+s)ds + \int_0^t h_0(t'+s,z+t-s)ds \\ &= \frac{1}{2}\Big[c_\mu(t',z-t) + c_\mu(t',z+t)\Big] + \frac{1}{2}\Big[\int_{z-t}^{z+t}\big(\partial_0 c_\mu(t',s)ds\big)\Big] \\ &+ \frac{1}{2}\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)dsd\tau \\ &+ \frac{1}{2}\int_0^t \Big[h_3(t'+s,z+t-s) - h_3(t'+s,z-(t-s))\Big]ds \\ &- \frac{1}{2}\int_{z-t}^{z+t} h_0(t',s)ds + \frac{1}{2}\int_0^t h_0(t'+s,z-t+s)ds + \frac{1}{2}\int_0^t h_0(t'+s,z+t-s)ds \end{aligned} \quad (2.173)$$

□

•

Proposition 14. *For any t, t', we have:*

$$\begin{aligned}||c_\mu(t+t',.)||_{H^3} \leq & \big[||c_\mu(t',.)||_{H^3} + C(t+1)||\partial_0 c_\mu(t')||_{H^2}\big] \\ & + C\epsilon(t+1)\Big(||\psi(t',.)||_{H^2} + ||\partial_0 \tilde{a}_0(t',.)||_{L^2} + ||\partial_0 \tilde{a}_3(t',.)||_{L^2}\Big) \\ & + Ct\epsilon\big[\sup_{(t',t+t')} ||\psi(\tau,.)||_{H^2} + \sup_{(t',t+t')} ||\partial_0 \tilde{a}_0(\tau,.)||_{H^1} + \sup_{(t',t+t')} ||\partial_0 \tilde{a}_3(\tau,.)||_{H^1}\big)\big] \\ & + Ct(t+1)\epsilon^2\big[\sup_{(t',t+t')} ||\psi(\tau,.)||_{H^2} + \sum_{j\in\{0,3\}} \sup_{(t',t+t')} ||\partial_t \tilde{a}_j||_{H^1}\big] \\ & + Ct(t+1)\big[\sum_{j\in\{0,3\}} \sup_{(t',t+t')} ||N_j(\tau,.)||_{H^2} + \sum_{j\in\{0,3\}} \sup_{(t',t+t')} ||E_j(\tau,.)||_{H^2}\big]\end{aligned} \tag{2.174}$$

and

$$\begin{aligned}||\partial_0 c_\mu(t+t',.)||_{H^2} \leq & ||c_\mu(t',.)||_{H^2} + ||\partial_0 c_\mu(t',.)||_{H^2} \\ & + C\epsilon(t+1)\Big[\sup_{(t',t+t')} ||\psi(\tau,.)||_{H^2} + \sup_{(t',t+t')} ||\partial_0 \tilde{a}_0(\tau,.)||_{H^1} + \sup_{(t',t+t')} ||\partial_0 \tilde{a}_3(\tau,.)||_{H^1}\Big] \\ & + C\epsilon(t+1)\Big[\sup_{(t',t+t')} ||N(\tau,.)||_{H^3} + \sup_{(t',t+t')} ||E(\tau,.)||_{H^3}\Big)\Big]\end{aligned} \tag{2.175}$$

Proof. We use proposition 13 and therefore, we estimate the terms on the right hand side of equation (2.158). There holds:

$$||c_\mu(t', z-t) + c_\mu(t', z+t)||_{H^3} \leq 2||c_\mu(t',.)||_{H^3} \tag{2.176}$$

To estimate

$$\int_{z-t}^{z+t} (\partial_0 c_\mu)(t', s)ds$$

we use the following lemma

Lemma 4. *Suppose that*

$$w(z) = \int_{z-t}^{z+t} f(s)ds$$

$$||w||_{H^3} \leq C\big(t||f||_{L^2} + ||f||_{H^2}\big) \tag{2.177}$$

Proof. We have:

$$||w||_{L^2} \leq 2t||f||_{L^2} \tag{2.178}$$

Also,

$$w_z(z) = f(z-t) - f(z+t) \tag{2.179}$$

Therefore,

$$||w_z||_{H^2} \leq C||f||_{H^2} \tag{2.180}$$

Equations (2.178) and (2.180) finish the proof. □

Therefore, by lemma 4, we have:

$$||\int_{z-t}^{z+t} \big(\partial_0 c_\mu(t',s)ds\big)||_{H^3} \leq C\big(t||\partial_0 c_\mu(t',.)||_{L^2} + ||\partial_0 c_\mu(t',.)||_{H^2}\big) \tag{2.181}$$

Similarly, we have:

$$||\int_{z-t}^{z+t} h_0(t',s)ds||_{H^3} \leq C\big(t||\partial_0 c_\mu(t',.)||_{L^2} + ||\partial_0 c_\mu(t',.)||_{H^2}\big) \tag{2.182}$$

Now, we want to estimate

$$\Big|\Big|\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)ds\Big|\Big|_{H^3} \tag{2.183}$$

We have:

Claim 1.

$$\Big|\Big|\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)ds\Big|\Big|_{H^3} \leq Ct \sup_{(t',t'+t)} \Big(t||h(t'+\tau,.)||_{L^2} + ||h(t'+\tau,.)||_{H^2}\Big) \tag{2.184}$$

Now, we want to estimate the above quantities.

Proof.

$$\Big|\Big|\int_0^t \int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)ds\Big|\Big|_{H^3} \leq t \sup_\tau \Big|\Big|\int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)ds\Big|\Big|_{H^3} \tag{2.185}$$

By using lemma 4, we have:

$$\Big|\Big|\int_{z-(t-\tau)}^{z+(t-\tau)} h(t'+\tau,s)ds\Big|\Big|_{H^3} \leq C\Big(t||h(t'+\tau,.)||_{L^2} + ||h(t'+\tau,.)||_{H^2}\Big) \tag{2.186}$$

Equations (2.185) and (2.186) imply (2.184). □

By differentiating with respect to z, we have:

$$\Big|\Big|\frac{1}{2}\int_0^t \big[h_3(t'+s,z+t-s) - h_3(t'+s,z-(t-s))\big]ds\Big|\Big|_{H^3} \leq Ct \sup_{\tau\in(t',t'+t)} ||h_3(\tau,.)||_{H^3} \tag{2.187}$$

Similarly, we have:

$$\Big|\Big|\int_0^t h_0(t'+s,z-t+s)ds+\int_0^t h_0(t'+s,z+t-s)ds\Big|\Big|_{H^3}\leq Ct\sup_{\tau\in(t',t'+t)}||h_0(\tau,.)||_{H^3} \tag{2.188}$$

By using (2.173) and combining estimates (2.176), (2.181), (2.182),(2.184), (2.187) and (2.188), we have:

$$\begin{aligned}||c_\mu(t+t',.)||_{H^3}\leq &||c_\mu(t',.)||_{H^3}+C\big(t||\partial_0 c_\mu(t',.)||_{L^2}+||\partial_0 c_\mu(t',.)||_{H^2}\big)\\ &+C\big(t||h_0(t',.)||_{L^2}+||h_0(t',.)||_{H^2}\big)\\ &+C\big[t\sup_{(t',t+t')}||h_3(\tau,.)||_{H^3}+t\sup_{(t',t+t')}||h_0(\tau,.)||_{H^3}\big]\\ &+Ct\sup_{(t',t'+t)}\Big(t||h(\tau,.)||_{L^2}+||h(\tau,.)||_{H^2}\Big)\end{aligned} \tag{2.189}$$

for some $C>0$. But, by (2.155), we have:

$$||h_0(\tau,.)||_{H^3}\leq C(||a_0\tilde{\varphi}(\tau,.)||_{H^3}+||\chi_0 D_0\varphi(\tau,.)||_{H^3}+||\chi_3 D_3\varphi(\tau,.)||_{H^3}+\epsilon||\psi(\tau,.)||_{H^3}) \tag{2.190}$$

According to equations (2.145) and lemma 9 and remark 3 in the appendix, we have:

$$||\chi_j(\tau,.)||_{H^3}\leq C||\partial_t\tilde{a}_j(\tau,.)||_{H^1} \tag{2.191}$$

for some $C>0$ and any τ and $j=0,3$. According to the estimates that we have for a_0 and $D_0\varphi$ and $D_3\varphi$, by (2.190) and (2.191), we have:

$$||h_0(\tau,.)||_{H^3}\leq C\epsilon(||\psi(\tau,.)||_{H^3}+||\partial_0\tilde{a}_0(\tau,.)||_{H^1}+||\partial_0\tilde{a}_3(\tau,.)||_{H^1}) \tag{2.192}$$

Similarly, by (2.156), we have:

$$||h_3(\tau,.)||_{H^3}\leq C\epsilon||\psi(\tau,.)||_{H^3} \tag{2.193}$$

To estimate $||h(\tau,.)||_{H^2}$ defined in (2.157), we first estimate $||f_j(\tau,.)||_{H^2}$. By equation (2.152) and (2.191), we have:

$$||f_j(\tau,.)||_{H^2}\leq C\big[\epsilon||\psi(\tau,.)||_{L^2}+\epsilon||\partial_t\tilde{a}_j||_{H^1}+||N_j(\tau,.)||_{L^2}+||E_j(\tau,.)||_{L^2}\big] \tag{2.194}$$

Also, by equation (2.115), we have:

$$||\big(R_0u(\tau,.),n_{\mu,\varphi}(\tau,.)\big)||_{H^2}\leq C\epsilon^2||\psi(\tau,.)||_{H^2} \tag{2.195}$$

and by equation (2.117),

$$||(S_ju(\tau,.),n_{\mu,\varphi}(\tau,.)\big)||_{H^2}\leq C\epsilon^2||\psi(\tau,.)||_{H^2} \tag{2.196}$$

Therefore, by equation (2.157), we have:

$$||h(\tau,.)||_{H^2} \leq C\Big[\epsilon^2||\psi(\tau,.)||_{H^2}+\sum_{j\in\{0,3\}}\left[\epsilon^2||\partial_t\tilde{a}_j||_{H^1}+||N_j(\tau,.)||_{H^2}+||E_j(\tau,.)||_{H^2}\right]\Big] \tag{2.197}$$

Therefore, by equations (2.189), (2.192), (2.193) and (2.197), we have:

$$\begin{aligned}
||c_\mu(t+t',.)||_{H^3} \leq & \big[||c_\mu(t',.)||_{H^3} + C(t+1)||\partial_0 c_\mu(t')||_{H^2}\big] \\
& + C\epsilon(t+1)\Big(||\psi(t',.)||_{H^2} + ||\partial_0\tilde{a}_0(t',.)||_{L^2} + ||\partial_0\tilde{a}_3(t',.)||_{L^2}\Big) \\
& + Ct\epsilon\big[\sup_{(t',t+t')}||\psi(\tau,.)||_{H^2} + \sup_{(t',t+t')}||\partial_0\tilde{a}_0(\tau,.)||_{H^1} + \sup_{(t',t+t')}||\partial_0\tilde{a}_3(\tau,.)||_{H^1})\big] \\
& + Ct(t+1)\epsilon^2\big[\sup_{(t',t+t')}||\psi(\tau,.)||_{H^2} + \sum_{j\in\{0,3\}}\sup_{(t',t+t')}||\partial_t\tilde{a}_j||_{H^1}\big] \\
& + Ct(t+1)\big[\sum_{j\in\{0,3\}}\sup_{(t',t+t')}||N_j(\tau,.)||_{H^2} + \sum_{j\in\{0,3\}}\sup_{(t',t+t')}||E_j(\tau,.)||_{H^2}\big]
\end{aligned} \tag{2.198}$$

Now, we want to control $||\partial_0 c_\mu||_{H^2}$.

$$\begin{aligned}
\partial_0 c_{_\mu}(t+t',z) = & \frac{1}{2}\partial_t\Big[c_\mu(t',z-t) + c_\mu(t',z+t)\Big] + \frac{1}{2}\partial_t\big[\int_{z-t}^{z+t}\big(\partial_0 c_\mu(t',s)ds\big)\big] \\
& + \frac{1}{2}\partial_t\big[\int_0^t \big(h_3(t'+\tau,z+t-\tau) - h_3(t'+\tau,z-(t-\tau))\big)d\tau\Big] \\
& + \frac{1}{2}\partial_t\big[\int_0^t\int_{z-(t-\tau)}^{z+t-\tau} h(t'+\tau,s)dsd\tau\big] \\
& - \frac{1}{2}\frac{\partial}{\partial t}\int_{z-t}^{z+t} h_0(t',s)ds + \frac{1}{2}\frac{\partial}{\partial t}\int_0^t h_0(t'+s,z-t+s)ds \\
& + \frac{1}{2}\frac{\partial}{\partial t}\int_0^t h_0(t'+s,z+t-s)ds \\
& = -\frac{1}{2}(\partial_z c_\mu)(t',z-t) + \frac{1}{2}(\partial_z c_\mu)(t',z+t) + \frac{1}{2}(\partial_0 c_\mu)(t',z+t) + \frac{1}{2}(\partial_0 c_\mu)(t',z-t) \\
& + \frac{1}{2}\big[\int_0^t \big((\partial_3 h_3)(t'+\tau,z+t-\tau) - (\partial_3 h_3)(t'+\tau,z-(t-\tau)\big)d\tau\Big] \\
& + \frac{1}{2}\int_0^t\Big[\big(h(t'+\tau,z+t-\tau) + h(t'+\tau,z-(t-\tau)\big)d\tau\Big] \\
& - \frac{1}{2}\big(h_0(t',z+t) + h_0(t',z-t)\big) + h_0(t'+t,z) \\
& - \frac{1}{2}\int_0^t(\partial_3 h_0)(t'+s,z-t+s)ds + \frac{1}{2}\int_0^t(\partial_3 h_0)(t'+s,z+t-s)ds
\end{aligned} \tag{2.199}$$

Therefore, by (2.192), (2.193) and (2.197), we have:

$$\begin{aligned}||\partial_0 c_\mu(t+t',.)||_{H^2} &\leq ||c_\mu(t',.)||_{H^3} + ||\partial_0 c_\mu(t',.)||_{H^2} + t\Big[\sup_{(t',t+t')}||h_3(\tau,.)||_{H^3} + \sup_{(t',t+t')}\Big||h(\tau,.)||_{H^2}\Big] \\ &+ C(t+1)\sup_{(t',t+t')}||h_0(\tau,.)||_{H^3} \\ &\leq ||c_\mu(t',.)||_{H^3} + ||\partial_0 c_\mu(t',.)||_{H^2} \\ &+ C\epsilon(t+1)\left[\sup_{(t',t+t')}||\psi(\tau,.)||_{H^3} + \sup_{(t',t+t')}||\partial_0\tilde{a}_0(\tau,.)||_{H^1} + \sup_{(t',t+t')}||\partial_0\tilde{a}_3(\tau,.)||_{H^1}\right] \\ &+ C\epsilon(t+1)\left[||N(\tau,.)||_{H^3} + ||E(\tau,.)||_{H^3}\Big)\right]\end{aligned} \tag{2.200}$$

□

2.5.5 Returning to estimates for the perturbation

To move from the estimates that we have for the quantities $Q_1, Q_2, \{c_\mu\}_\mu$ to the estimates that we want to have for u, we use propositions 15 and 16.

Proposition 15. *There exists $C > 0$ such that for any t,*

$$\sum_\mu ||c_\mu(.,t)||^2_{H^3} + Q_1(t) + Q_2(t) \geq C||u(.,t)||^2_{H^3} \tag{2.201}$$

Proposition 16. *There exists $C > 0$ such that for any t,*

$$||\partial_t u(.,t)||^2_{H^2} \leq C\Big(||\partial_t c_\mu(.,t)||^2_{H^2} + ||c_\mu(.,t)||^2_{H^2} + Q_1(t) + Q_2(t)\Big) \tag{2.202}$$

The main ingredient in the proof of the above estimates is the coercivity result 4 which implies that

Proposition 17. *Suppose that $\psi \in H^1(\mathbb{R}^2)$ and let $p = (\phi, \alpha) \in M_N$ is an N-vortex configuration. Let*

$$c_\mu = \langle \psi, n_\mu \rangle_{L^2} \tag{2.203}$$

where $\{n_\mu\}_\mu$ is the introduced orthonormal basis for $T_p M_N$. Then,

$$\int_{\mathbb{R}^2} (L\psi, \psi) + \sum_\mu |c_\mu|^2 \geq C||\psi||^2_{H^1} \tag{2.204}$$

We will also use the following estimate:

Proposition 18. *Suppose that $f \in H^1(\mathbb{R}^2)$ and $p = (\phi, \alpha) \in K \subset M_N$ and $K \subset M_N$ is compact. Let*

$$A(f) = -\Delta f + |\phi|^2 f$$

There exist a constant $C = C(K)$ such that

$$Q(f) = \int_{\mathbb{R}^3} \big(Af, f\big) \geq C||f||^2_{H^1}$$

The proof of this estimate can be found in the proof of lemma ??? in the appendix.

Proof of proposition 15

In the proof, we consider the time t fixed. Then, let

$$\gamma(z) = q(t, z)$$

We use the coordinate system $(y, z) = \big((x_1, x_2), z\big)$ for the subspace $\{t\} \times \mathbb{R}^3$. For any

$$w \in H^1\big(\mathbb{R}^3, (\mathbb{C} \oplus \mathbb{R}^2) \oplus \mathbb{R} \oplus \mathbb{R}\big)$$

with

$$w = (\psi, w_0, w_3)$$

Let

$$Kw = -\partial_z^2 w + \begin{pmatrix} L\psi \\ \\ (-\Delta_y + |\phi|^2)w_0 \\ \\ (-\Delta_y + |\phi|^2)w_3 \end{pmatrix} \tag{2.205}$$

where L and ϕ correspond to the base point vortex configuration in $\gamma(z)$, for any $z \in \mathbb{R}$.
Let

$$c_\mu(z) = \langle \psi(., z), n_\mu(., z)\rangle_{L^2} \tag{2.206}$$

and

$$E_1(w) = \int_{\mathbb{R}^3} (Kw, w) + \sum_\mu |c_\mu|^2_{L^2} \tag{2.207}$$

Proposition 19. *There exists a constant $C > 0$ such that for any $w \in H^1(\mathbb{R}^3)$, we have*

$$E_1(w) \geq C||w||^2_{H^1} \tag{2.208}$$

Proof. Note that by proposition 17, for any z, we have:

$$\int_{\mathbb{R}^2} (L\psi, \psi)(y, z)dy + \sum_\mu |c_\mu|^2(z) \geq C||\psi(., z)||^2_{H^1} \tag{2.209}$$

By integrating this over z, we deduce that

$$\begin{aligned} E_1(w) &\geq ||\partial_z \psi||_{L^2}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}^2} (L\psi, \psi)(y,z) dz dy + \sum_\mu \int_{\mathbb{R}} |c_\mu|^2 dz \\ &\geq C\Big(||\partial_z \psi||_{L^2}^2 + \int_{\mathbb{R}} ||\psi(.,z)||_{H^1}^2 dz \Big) \\ &\geq C||\psi||_{H^1}^2 \end{aligned} \tag{2.210}$$

Similarly, by proposition 18, we have:

$$\begin{aligned} E_1(w) &\geq C\Big[\sum_{j\in\{0,3\}} ||\partial_z w_j||_{L^2}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}^2} \big((-\Delta_y w_j + |\phi|^2 w_j) w_j \big) dy dz \Big] \\ &\geq C \sum_{j\in\{0,3\}} \Big[||\partial_z w_j||_{L^2}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}^2} (|\nabla_y w_j|^2 + |\phi|^2 w_j^2) dy dz \Big] \\ &\geq C \sum_{j\in\{0,3\}} \Big[||\partial_z w_j||_{L^2}^2 + \int_{\mathbb{R}} ||w_j(.,z)||_{H^1}^2 dz \Big] \\ &\geq C \sum_{j\in\{0,3\}} ||w_j||_{H^1}^2 \end{aligned} \tag{2.211}$$

Inequalities (2.210) and (2.211) finish the proof of proposition 19. □

Proposition 20. *There exists a constant $C > 0$ such that for any $w \in H^3(\mathbb{R}^3)$, we have*

$$\int_{\mathbb{R}^3} (K^2 w, Kw) + \sum_\mu ||c_\mu||_{H^2}^2 + C\epsilon ||w||_{H^1}^2 \geq C||Kw||_{H^1}^2 \tag{2.212}$$

Proof. We apply proposition 19 to Kw. Let

$$d_\mu(z) = \langle -\partial_z^2 \psi + L\psi, n_\mu(z) \rangle_{L^2} \tag{2.213}$$

Then by proposition 19, we obtain that

$$\int_{\mathbb{R}^3} (K^2 w, Kw) + \sum_\mu ||d_\mu||_{L^2}^2 \geq C||Kw||_{H^1}^2 \tag{2.214}$$

But, note that

$$\begin{aligned} d_\mu(t,z) &= \langle -\partial_z^2 w + Lw, n_\mu \rangle_{L^2} \\ &= -\langle \partial_z^2 w, n_\mu \rangle_{L^2} \\ &= -\partial_z^2 c_\mu + 2\langle w_z, (n_\mu)_z \rangle_{L^2} + \langle w, (n_\mu)_{zz} \rangle_{L^2} \end{aligned} \tag{2.215}$$

Therefore,

$$||d_\mu||_{L^2}^2 \leq ||c_\mu||_{H^2}^2 + C\epsilon ||w||_{H^1}^2 \tag{2.216}$$

Therefore, by (2.214), we have:

$$\int_{\mathbb{R}^3} (K^2 w, Kw) + \sum_\mu ||c_\mu||_{H^2}^2 + C\epsilon ||w||_{H^1}^2 \geq C||Kw||_{H^1}^2 \tag{2.217}$$

□

Proposition 21. *There exists a constant* $C > 0$ *such that*

$$||Mu||_{H^1}^2 + ||u||_{H^1}^2 \geq C||u||_{H^3}^2 \tag{2.218}$$

Proof. It suffices to show that

$$||\Delta u||_{H^1}^2 \leq C\Big(||Mu||_{H^1}^2 + ||u||_{H^1}^2\Big) \tag{2.219}$$

We have:

$$||\Delta u||_{L^2}^2 \leq C\Big(||Mu||_{L^2}^2 + ||u||_{H^1}^2\Big) \tag{2.220}$$

Therefore,

$$||u||_{H^2}^2 \leq C\Big(||Mu||_{L^2}^2 + ||u||_{H^1}^2\Big) \tag{2.221}$$

By replacing u with ∇u in (2.220), we have:

$$\begin{aligned} ||\nabla\Delta u||_{L^2}^2 &\leq C\big(||M\nabla u||_{L^2}^2 + ||\nabla u||_{H^1}^2\big) \\ &\leq C\big(||\nabla Mu||_{L^2}^2 + ||u||_{H^2}^2\big) \\ &\leq C\big(||Mu||_{H^1}^2 + ||u||_{H^1}^2\big) \end{aligned} \tag{2.222}$$

where the last line follows by (2.221). This implies (2.218). □

According to propositions 19 and 20, we have:

$$Q_1 + Q_2 + \sum_\mu ||c_\mu||_{H^2}^2 + C\epsilon||u||_{H^1}^2 \geq C\big[||u||_{H^1}^2 + ||Mu||_{H^1}^2\big] \tag{2.223}$$

This and (2.218) imply (2.201).

Proof of proposition 16

Using proposition 15, we have:

$$\begin{aligned} Q_2(t) + Q_1(t) + \sum_\mu ||\partial_t c_\mu||_{H^2}^2 + \sum_\mu ||c_\mu||_{H^3}^2 &\geq C\Big[||u_t||_{L^2}^2 + ||(Mu)_t||_{L^2}^2 + ||u||_{H^3}^2 \\ &\quad + \sum_\mu ||\partial_t c_\mu||_{H^2}^2 + \sum_\mu ||c_\mu||_{H^3}^2\Big] \\ &\geq C\Big[||u_t||_{L^2}^2 + ||Mu_t||_{L^2}^2 + \sum_\mu ||(\partial_t\psi, n_\mu)_{L^2}||_{H^2}^2\Big] \\ &\geq C\Big[||Mu_t||_{L^2}^2 + \int_{\mathbb{R}^3}(Mu_t, u_t) \\ &\quad + \sum_\mu ||(\partial_t\psi, n_\mu)_{L^2}||_{H^2}^2\Big] \end{aligned} \tag{2.224}$$

But by proposition 19, we have:

$$\int_{\mathbb{R}^3} (Mu_t, u_t) + \sum_{\mu} ||(\partial_t \psi, n_\mu)_{L^2}||^2_{H^2} \geq C||u_t||^2_{H^1} \tag{2.225}$$

According to (2.224) and (2.225), we have:

$$\begin{aligned} Q_1(t) + Q_2(t) + \sum_{\mu} ||\partial_t c_\mu||^2_{H^2} + \sum_{\mu} ||c_\mu||^2_{H^3} &\geq C\big[||Mu_t||^2_{L^2} + ||u_t||^2_{H^1}\big] \\ &\geq C||u_t||^2_{H^2} \end{aligned} \tag{2.226}$$

2.5.6 Bootstrap

Theorem 8. *Suppose that the number n in* (2.95) *satisfies $n \geq 6$. There exists $\epsilon_1 > 0$ such that if $\epsilon < \epsilon_1$ and u solves the equation* (2.120) *on*

$$[0, T') \times \mathbb{R}^3$$

for some $T' < \frac{1}{2\epsilon}$ and

$$||u(t,.)||_{H^3} + ||\partial_t u(t,.)||_{H^2} \leq K \leq \epsilon^3 \tag{2.227}$$

and the initial conditions are as mentioned before,

$$||u(t,.)||_{H^3} + ||\partial_t u(t)||_{H^2} \leq C\big(\epsilon t\big)^{\frac{1}{2}} K + \epsilon^4 \tag{2.228}$$

for some $C > 0$ which depends only on the wave map and the number of iterations in the ansatz.

Proof. We have

$$||N||_{H^2} \leq CK^2 \tag{2.229}$$

and

$$||E||_{H^3} \leq C\epsilon^6 \tag{2.230}$$

Now, we want to control $Q_1 + Q_2$. According to (2.146), (2.227), (2.229) and (2.230), we have:

$$\begin{aligned} Q_1(t) + Q_2(t) &\leq Q_1(0) + Q_2(0) + Ct\Big[\big(\epsilon K + K^2 + \epsilon^6\big)K + \epsilon K^2\Big] \\ &\leq Ct\epsilon K + C\epsilon^8 \end{aligned} \tag{2.231}$$

and by (2.174), we have:

$$\begin{aligned} ||c_\mu(t)||_{H^3} &\leq C\epsilon(t+1)K + Ct\epsilon K + Ct(t+1)\epsilon^2 K + Ct(t+1)(K^2 + \epsilon^6) \\ &\leq Ct\epsilon K + C\epsilon^4 \end{aligned} \tag{2.232}$$

and by (2.175), we have:

$$\begin{aligned} ||\partial_0 c_\mu||_{H^2} &\leq C\epsilon(t+1)K + C\epsilon(t+1)(K^2 + \epsilon^n) \\ &\leq C\epsilon t K + C\epsilon^4 \end{aligned} \tag{2.233}$$

Therefore,

$$Q_1(t) + Q_2(t) + ||c_\mu||^2_{H^3} + ||\partial_t c_\mu||^2_{H^2} \leq Ct\epsilon K^2 + C\epsilon^8 \tag{2.234}$$

According to (2.234) and propositions 15 and 16, the inequality (2.228) holds. □

2.5.7 Last Step: Moving forward in time

To prove theorem 5, we do the iterations in the ansatz 3 times so that by propositions 5, we can ensure that the error E of the approximate solution satisfies

$$||E||_{H^4} + ||E_t||_{H^3} \leq C\epsilon^6 \tag{2.235}$$

and choose the initial data as in (2.104) and (2.105). We use the local existence statement, theorem 6, with $b = 1$ in (2.124) to find a solution u defined over the interval $[0, T_1)$, for some T_1. Then, we use the bootstrap theorem 8 to refine the estimates. Then, we will look at the time $t = T_1$ and if it satisfies condition in the local existence result, we repeat the same process and then look at the time $t = 2T_1$ and continue likewise. We claim that there exists a number κ such that one can use the mentioned procedure and find solutions to (2.120) in a time interval of $[0, \frac{\kappa}{\epsilon})$ such that u satisfies

$$||u||_{H^3} + ||u_t||_{H^2} \leq \epsilon^3 \tag{2.236}$$

Suppose that (2.236) holds up to the step l; interval $[0, lT_1)$, but not for step $(l + 1)$, the interval $[lT_1, (l + 1)T_1)$. Therefore by the bootstrap statement, theorem 8, we have:

$$||u||_{H^3} + ||u_t||_{H^2} \leq C(\epsilon t)^{\frac{1}{2}}\epsilon^3 + \epsilon^4 \tag{2.237}$$

for $0 \leq t \leq lT$. Therefore, by the local existence theorem, we have:

$$||u||_{H^3} + ||u_t||_{H^2} \leq C\Big[C(\epsilon t)^{\frac{1}{2}}\epsilon^3 + \epsilon^4\Big] + \epsilon^5 \tag{2.238}$$

over the interval $[lT, (l + 1)T)$. Since the condition (2.236) fails to be satisfied on this interval, then by (2.238)

$$C(\epsilon(l + 1)T)^{\frac{1}{2}} \geq \frac{1}{2} \tag{2.239}$$

if ϵ is small enough, and therefore,

$$l + 1 \geq \frac{1}{4C^2T^2\epsilon} \tag{2.240}$$

and this finishes the proof of the fact that the aforementioned process can be carried on over a time interval of time interval $[0, \frac{\kappa}{\epsilon}]$ with the desired estimate (2.236). Therefore, we have proved that

Proposition 22. *Provided that the approximate solution $v = (\varphi, a)$ constructed in theorem 5 satisfies the error condition* (2.95) *with $n = 6$, then one can find $u = (\tilde{\varphi}, \tilde{a})$ such that $(v + u)$ solves AHM on a time interval of the form* $[0, \frac{\kappa}{\epsilon})$ *and*

$$||u(., t)||_{H^3} + ||u_t(., t)||_{H^2} \leq \epsilon^3 \tag{2.241}$$

for every t

2.6 Proof of the main theorem

We use theorem 5 with $m = 3$ iterations so that the error term E of the approximate solution is of the form

$$E = (E_\varphi, E_0, E_1, E_2, E_3) = O(\epsilon^8, \epsilon^7, \epsilon^8, \epsilon^8, \epsilon^7) \tag{2.242}$$

in a pointwise sense and since the error term is supported on an interval of length $\frac{C}{\epsilon}$, then we have the estimate

$$||E(.;t)||_{H^4} + ||E_t(.;t)||_{H^3} \leq C\epsilon^6 \tag{2.243}$$

for any t. Assume that $v = (\varphi, a)$ is the obtained approximate solution. Now, by proposition 22, we find u with

$$||u||_{H^3} + ||u_t||_{H^2} \leq \epsilon^3 \tag{2.244}$$

on the interval $[0, \frac{\kappa}{\epsilon})$ such that $v + u$ solves the AHM equation. Property 2.244 and (2.39), (2.36), (2.37) in theorem 2.242 imply that

$$v(t, y, z) = \begin{pmatrix} (\phi, \alpha)(y; q(\epsilon t, \epsilon z)) \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{\psi} \\ \bar{a}_0, \bar{a}_3 \end{pmatrix} \tag{2.245}$$

where

$$||\psi||_{H^3} + ||\psi_t||_{H^2} \leq C\epsilon^{\frac{3}{2}} \tag{2.246}$$

and

$$\sum_{j=0,3} ||\bar{a}_j||_{L^\infty} \leq C\epsilon \tag{2.247}$$

and also

$$\sum_{j=0,3} ||\nabla \bar{a}_j||_{H^2} \leq C\epsilon^{\frac{1}{2}} \tag{2.248}$$

Properties (2.244), (2.245), (2.246), (2.247) and (2.248) imply the desired estimates in theorem 5 for a time interval of the form $[0, \frac{C}{\epsilon})$ where C depends only in the wave map q.

Chapter 3

Appendix

In this appendix, we are going to mention some analytic results and estimates which have been used in the chapters.

Lemma 5. *Corresponding to the vortex centers $z_1, z_2, \cdots, z_N$ contained inside the ball $B(0,R) \subset \mathbb{R}^2$, consider the function $\Theta : \mathbb{C} \to \mathbb{R}$ as*

$$\Theta(z) = 2\sum_{i=1}^{N} \arg(z - z_i) \tag{3.1}$$

Suppose that $z = z^1 + iz^2$ and

$$z_j = z_j^1 + iz_j^2$$

where $z_1, z_2, z_j^1, z_j^2 \in \mathbb{R}$. Then, for every multi-index $r > 0$, there exists $A > 0$ such that

$$|D^r\Theta(z)| \leq A|z|^{-|r|+1}$$

for every $z \in \mathbb{R}^2$ with $|z| > 2R$, where D^r denotes a combination of the differentials $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial z_j^i}$.

Lemma 6. *Consider a compact subset $K \subset M_1$. Let $p = (\phi, \alpha) \in K$. Then for any $\zeta \in L^2(\mathbb{R}^2)$, there exists $u \in H^2(\mathbb{R}^2)$ such that:*

$$\Delta u - |\phi|^2 u = \eta \tag{3.2}$$

Furthermore, it satisfies

$$||u||_{H^2} \leq C||\eta||_{L^2} \tag{3.3}$$

for a constant $C = C(K)$. Moreover, if $\eta \in H^2(\mathbb{R}^2)$ and

$$|\eta|(x) \leq Ae^{-\gamma|x|} \tag{3.4}$$

for some constants $A, \gamma > 0$ with $0 \leq \gamma < 1$, then

$$|u|(x) \leq Be^{-\gamma|x|} \tag{3.5}$$

for some $B = B(K)$.

Proof. First, for any $\eta \in L^2$, consider the following functional:

$$\begin{aligned} &I_\eta : H^1(\mathbb{R}^2) \to \mathbb{R} \\ &I_\eta[v] = \int_{\mathbb{R}^2} \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}|\phi|^2 v^2 - v\eta\Big) dx \end{aligned} \tag{3.6}$$

If $v \in H^1$, then $|I_\eta[v]| < \infty$, since $|\phi|^2 \leq 1$.

Proposition 23. *For $\eta \in L^2(\mathbb{R}^2)$, the functional I_η has a unique minimizer $u \in H^1$. Furthermore:*

$$||u||_{H^1} \leq C||\eta||_{L^2} \tag{3.7}$$

for a constant C.

Proof. In the following proof, all of the constants can be chosen in a way to depend uniformly on p. First, we prove that there exist constants $a \geq 0$ and $b \geq 0$ such that:

$$I_\eta[v] \geq a||v||^2_{H^1} - b||\eta||^2_{L^2} \tag{3.8}$$

To prove this, first consider the following functional:

$$\begin{aligned} &J : H^1(\mathbb{R}^2) \to \mathbb{R} \\ &J[v] = \int_{\mathbb{R}^2} \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}|\phi|^2 v^2\Big) dx \end{aligned} \tag{3.9}$$

Proposition 24. *There exists a constant $C \geq 0$ such that*

$$J[v] \geq C||v||^2_{H^1} \tag{3.10}$$

for any $v \in H^1\big(\mathbb{R}^2\big)$.

Proof. Suppose not. Then we can find a sequence $\{v_n\} \subset H^1(\mathbb{R}^2)$ such that $J[v_n] \to 0$ and $||v_n||_{H^1} = 1$ for any n. We know that $|\phi|(x) \to 1$ as $|x| \to \infty$. So, we can find $r > 0$ such that $|\phi|(x) > \frac{1}{2}$ if $|x| > r$. Now, for any n consider the functions:

$$\begin{aligned} f_n &= v_n\Big|_{B_{2r}(0)} \\ g_n &= v_n\Big|_{\mathbb{R}^2 \setminus B_r(0)} \end{aligned} \tag{3.11}$$

Also, consider the functional

$$\begin{aligned} &J_1 : H^1\big(B_{2r}(0)\big) \to \mathbb{R} \\ &J_1[v] = \int_{B_{2r}(0)} \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}|\phi|^2 v^2\Big) dx \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &J_2 : H^1\big(\mathbb{R}^2 \backslash B_r(0)\big) \to \mathbb{R} \\ &J_2[v] = \int_{\mathbb{R}^2 \backslash B_r(0)} \Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}|\phi|^2 v^2\Big) dx \end{aligned} \tag{3.13}$$

Then, we have:

$$0 \leq J_1[f_n] \leq J[v_n] \tag{3.14}$$

$$0 \leq J_2[g_n] \leq J[v_n] \tag{3.15}$$

Since $J[v_n] \to 0$ as $n \to \infty$, then by the above inequalities , we have:

$$\lim_{n\to\infty} J_1[f_n] = 0 \tag{3.16}$$

$$\lim_{n\to\infty} J_2[g_n] = 0 \tag{3.17}$$

Now, note that since $|\phi| > \frac{1}{2}$ for $|x| > r$, then using the definition of J_2, we can say:

$$J_2[g_n] \geq \frac{1}{8}||g_n||^2_{H^1} \tag{3.18}$$

Using (3.17) and (3.18), we have:

$$\lim_{n\to\infty} ||g_n||_{H^1} = 0 \tag{3.19}$$

Now, we want to prove that:

$$\lim_{n\to\infty} ||f_n||_{H^1} = 0 \tag{3.20}$$

Using the definition of J_1, and (3.16), we can say:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\nabla f_n|^2 dx = 0 \tag{3.21}$$

Therefore, in order to prove (3.20), it suffices to show that:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} f_n^2 = 0 \tag{3.22}$$

Suppose that:

$$c_n = \frac{1}{|B_{2r}(0)|} \int_{B_{2r}(0)} f_n \tag{3.23}$$

and:

$$h_n = f_n - c_n \tag{3.24}$$

Then, according to one of the Poincare inequalities for a ball, we have:

$$||h_n||_{L^2(B_{2r}(0))} \leq C||\nabla f_n||_{L^2(B_{2r}(0))} \tag{3.25}$$

for some constant C. Now, according to (3.21), and (3.25), we have:

$$\lim_{n\to\infty} ||h_n||_{L^2(B_{2r}(0))} = 0 \tag{3.26}$$

According to (3.24), (3.26), in order to prove (3.22), it suffices to show that

$$\lim_{n\to\infty} c_n = 0 \tag{3.27}$$

Suppose not. Then there exists $m > 0$, and a subsequence $\{c_{i_n}\}$ of $\{c_n\}$ such that $|c_{i_n}| > m$ for any n. Without loss of generality, we can assume that $i_n = n$. So for any n, we have:

$$c_n > m > 0 \tag{3.28}$$

Now, according to (3.16), and the definition of J_1, we can say:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 \big(h_n + c_n\big)^2 dx = 0 \tag{3.29}$$

Therefore:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 \big(h_n^2 + c_n^2 + 2h_n c_n\big) dx = 0 \tag{3.30}$$

But, according to (3.26), and the fact that ϕ is continuous, we have:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 h_n^2 dx = 0 \tag{3.31}$$

So, by (3.30), (3.31), we have:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 \big(c_n^2 + 2h_n c_n\big) dx = 0 \tag{3.32}$$

According to (3.28), (3.32), we have:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 \big(c_n + 2h_n\big) dx = 0 \tag{3.33}$$

But according to (3.26), we have:

$$\lim_{n\to\infty} \int_{B_{2r}(0)} |\phi|^2 h_n dx = 0 \tag{3.34}$$

Now, using (3.33), (3.34), we can say:

$$\lim_{n\to\infty} c_n = 0 \tag{3.35}$$

which is contradiction with (3.28). This shows that (3.27) holds, and this implies (3.20), as mentioned before. Now, note that:

$$||v_n||^2_{H^1(\mathbb{R}^2)} \le ||f_n||^2_{H^1} + ||g_n||^2_{H^1} \tag{3.36}$$

Using (3.19), (3.20), and (3.36), we can say that:

$$\lim_{n\to\infty} ||v_n||_{H^1(\mathbb{R}^2)} = 0 \tag{3.37}$$

But this is a contradiction, since in the beginning we assumed that $||v_n||_{H^1} = 1$ for any n. This finishes the proof of proposition 24. □

Now, we want to find $a, b \geq 0$ such that (3.8) holds. We have:

$$I_\eta[v] = J[v] - \int_{\mathbb{R}^2} v\eta dx \tag{3.38}$$

Consider the constant $C > 0$ which satisfies (3.10). Now, note that:

$$\int_{\mathbb{R}^2} v\eta dx \leq \frac{C}{4}||v||^2_{H^1} + \frac{1}{C}||\eta||^2_{L^2} \tag{3.39}$$

Using (3.10), (3.38), and (3.39), we have:

$$I_\eta[v] \geq \frac{3C}{4}||v||^2_{H^1} - \frac{1}{C}||\eta||^2_{L^2} \tag{3.40}$$

Therefore (3.8) holds for some constants $a, b > 0$.

Now, we want to show that the functional I_η has a minimizer in H^1. Note that (3.8) tells us that $\inf(I_\eta) \neq (-\infty)$. Now, consider a minimizing sequence $\{v_n\}$. According to (3.8), we can say that $\{v_n\}$ is bounded in H^1. Therefore, there exists a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of $\{v_n\}$ and $v_0 \in H^1$ such that

$$v_{n_k} \to v_0 \ \textbf{weakly in } H^1 \tag{3.41}$$

Now, note that the functional I_η is lower semicontinuous with respect to the weak topology of H^1. Therefore, v_0 is a minimizer for I_η. To show the uniqueness of the minimizer, suppose that u_1, u_2 are two minimizers of I_η. Then, we have:

$$I_\eta\big[\frac{u_1+u_2}{2}\big] = \frac{1}{2}\big(I_\eta[u_1] + I_\eta[u_2]\big) - \int_{\mathbb{R}^2}\Big[\frac{1}{4}|\nabla(u_1-u_2)|^2 + |\phi|^2\frac{1}{4}(u_1-u_2)^2\Big] \tag{3.42}$$

Therefore, we have $\nabla u_1 = \nabla u_2$ in L^2, and since $u_1, u_2 \in L^2$, then we deduce that $u_1 = u_2$. So the minimizer is unique.

Now, to prove (3.7), note that according to (3.40), we have:

$$\frac{3C}{4}||u||^2_{H^1} - \frac{1}{C}||\eta||^2_{L^2} \leq I[u] \leq I[0] = 0 \tag{3.43}$$

Therefore,

$$||u||^2_{H^1} \leq \frac{4}{3C^2}||\eta||^2_{L^2} \tag{3.44}$$

Note that the constant C above depends only on ϕ, since it was the constant we obtained in proposition (24). This proves (3.7). □

Proposition 25. *Suppose that $\eta \in L^2R^2)$. If u is the minimizer of I_η in $H^1(\mathbb{R}^2)$, then it satisfies the equation*

$$-\Delta u + |\phi|^2 u = \eta \tag{3.45}$$

in the weak sense.

Proof. Suppose that $v \in C_c^\infty(\mathbb{R}^2)$. Then, for any $c \in \mathbb{R}$, we have:

$$I_\eta(u) \leq I_\eta(u + cv)$$

Therefore,

$$\int_{\mathbb{R}^2} \nabla u.\nabla v + |\varphi|^2 uv - v\eta = 0$$

□

Proposition 26. *Suppose that $\eta \in C_c^\infty(\mathbb{R}^2)$, and $u \in H^1(\mathbb{R}^2)$ satisfies the equation*

$$-\Delta u + |\phi|^2 u = \eta \tag{3.46}$$

in the weak sense. Then, $u \in H^2(\mathbb{R}^2)$, and:

$$||u||_{H^2} \leq C||\eta||_{L^2} \tag{3.47}$$

where C is a constant.

Proof. First, we use the following theorem from [10]:

Theorem 9 (Theorem 1 in Page 329 of [10])**.** *Suppose that U is a bounded and open subset of $\mathbb{R}^n$. Consider the elliptic operator*

$$Lv = -\sum_{i,j=1}^{n} \left(a^{ij} v_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x) v_{x_i} + c(x)v \tag{3.48}$$

Assume that

$$a^{ij} \in C^1(U), \;\; b^i, c \in L^\infty(U) \;\; \big(i, j = 1, 2, \cdots, n\big) \tag{3.49}$$

Suppose that $v \in H^1(U)$ is a weak solution of the equation

$$Lv = f \;\; \textbf{\textit{in}} \;\; U \tag{3.50}$$

for some $f \in L^2(U)$. Then

$$v \in H^2_{loc}(U) \tag{3.51}$$

According to theorem (9), we can say that $u \in H^2_{loc}(\mathbb{R}^2)$. Therefore, $\Delta u \in L^2_{loc}(\mathbb{R}^2)$. Now, note that:

$$\Delta u = |\phi|^2 u - \eta \tag{3.52}$$

Therefore $\Delta u \in L^2(\mathbb{R}^2)$. Now, consider a smooth cutoff function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that $\eta = 1$ for

$|x| \leq 1$, and $\eta = 0$ for $|x| \geq 2$. Consider the functions:

$$\eta_r(x) = \eta\Big(\frac{x}{r}\Big) \tag{3.53}$$

Now, define the functions:

$$u_r = u.\eta_r \tag{3.54}$$

Then, note that $u_r \in H^2(\mathbb{R}^2)$, because $u \in H^2_{loc}(\mathbb{R}^2)$. We have:

$$\Delta u_r = \big(\Delta u\big)\eta_r + 2\nabla u.\nabla \eta_r + u\Delta\eta_r \tag{3.55}$$

Therefore,

$$||\Delta u_r||_{L^2} \leq ||\Delta u||_{L^2} + 2||\nabla u.\nabla\eta_r||_{L^2} + ||u\Delta\eta_r||_{L^2} \tag{3.56}$$

But note we can find C such that:

$$\begin{aligned} |\nabla\eta_r| &\leq C \;\; \forall\, r > 1 \\ |\Delta\eta_r| &\leq C \;\; \forall\, r > 1 \end{aligned} \tag{3.57}$$

Therefore, by (3.56), (3.57), we have

$$||\Delta u_r||_{L^2} \leq ||\Delta u||_{L^2} + 3C||u||_{H^1} \;\; \forall\, r > 1 \tag{3.58}$$

But, using (3.52) and the fact that $|\phi| \leq 1$, we have:

$$||\Delta u||_{L^2} \leq ||u||_{L^2} + ||\eta||_{L^2} \tag{3.59}$$

So, by (3.58) and (3.59), we have:

$$||\Delta u_r||_{L^2} \leq C||u||_{H^1} + ||\eta||_{L^2} \;\; \forall\, r > 1 \tag{3.60}$$

for a constant C.

Now, note that according to propositions 23 and 25, the functional I_η has a unique minimizer u_η in H^1 which satisfies the equation:

$$-\Delta u_\eta + |\phi|^2 u_\eta = \eta \tag{3.61}$$

in the weak sense. Using this and (3.46), we have:

$$-\Delta\big(u_\eta - u\big) + |\phi|^2(u_\eta - u) = 0 \tag{3.62}$$

in the weak sense. Since $(u_\eta - u) \in H^1(\mathbb{R}^2)$, then (3.62) implies that:

$$\int_{\mathbb{R}^2} \big|\nabla(u_\eta - u)\big|^2 + |\phi|^2|u_\eta - u|^2 = 0 \tag{3.63}$$

which implies that $u = u_\eta$. Therefore, u is the minimizer of I_η in H^1. Therefore, according to proposition 23, we have:

$$||u||_{H^1} \leq C||\eta||_{L^2} \tag{3.64}$$

where C is a constant. Using (3.60) and (3.64), we have:

$$||\Delta u_r||_{L^2} \leq K||\eta||_{L^2} \ \ \forall \, r > 1 \tag{3.65}$$

for some constant K. Now, we use the following theorem:

Theorem 10. *Suppose that $v \in H^2(\mathbb{R}^n)$. Then, we have:*

$$\int_{\mathbb{R}^n} |\Delta v|^2 dx = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} |v_{x_i x_j}|^2 dx \tag{3.66}$$

Using (3.65), theorem (10), and the fact that $u_r \in H^2(\mathbb{R}^2)$, we have:

$$\int_{\mathbb{R}^2} |(u_r)_{x_i x_j}|^2 \leq K||\eta||_{L^2} \ \ \forall \, r > 1 \tag{3.67}$$

But, note that:

$$\int_{B_{\frac{r}{2}}(0)} |u_{x_i x_j}|^2 \leq \int_{\mathbb{R}^2} |(u_r)_{x_i x_j}|^2 \tag{3.68}$$

Now, using (3.67), and (3.68), we have:

$$\int_{B_{\frac{r}{2}}(0)} |u_{x_i x_j}|^2 \leq K||\eta||_{L^2} \ \ \forall \, r > 1 \tag{3.69}$$

This implies that $u_{x_i x_j} \in L^2(\mathbb{R}^2)$, and therefore $u \in H^2(\mathbb{R}^2)$. Furthermore, (3.64), (3.69) imply (3.47). This finishes the proof of proposition 26. □

Now, suppose that $\{\eta_n\}_{n=1}^{\infty} \subset C_c^{\infty}(\mathbb{R}^2)$ is such that $\eta_n \to \eta$ in $L^2(\mathbb{R}^2)$. According to propositions (23),(25), and (26), there exist $\{u_n\}_{n=1}^{\infty} \subset H^2(\mathbb{R}^2)$ solving the equations:

$$-\Delta u_n + |\phi|^2 u_n = \eta_n \tag{3.70}$$

Furthermore, they satisfy:

$$||u_n||_{H^2} \leq C||\eta_n||_{L^2} \tag{3.71}$$

where C is a constant. According to (3.71), the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in H^2. Therefore, without loss of generality, we can assume that it is Cauchy in the weak topology of $H^2(\mathbb{R}^2)$. So, suppose that $u_n \to u$ in the weak topology of $H^2(\mathbb{R}^2)$, for some $u \in H^2(\mathbb{R}^2)$. Then, since (3.70) holds in the weak sense for every n and $\eta_n \to \eta$ in L^2, then we deduce that u satisfies equation (3.2) in the weak sense. Furthermore, according to (3.71), we have:

$$||u||_{H^2} \leq C||\eta||_{L^2} \tag{3.72}$$

for a constant C. This proves (3.3).

Furthermore, note that if $u_1, u_2 \in H^2$ satisfy equation (3.2), then we have

$$\int_{\mathbb{R}^2} \left|\nabla(u_1 - u_2)\right|^2 + |\phi|^2|u_1 - u_2|^2 = 0 \tag{3.73}$$

which implies that $u_1 = u_2$. Therefore, the solution to equation (3.2) is unique.

Now, suppose that $\zeta \in H^2(\mathbb{R}^2)$ and

$$|\eta|(x) \leq Me^{-\gamma|x|} \tag{3.74}$$

for some $M, \gamma > 0$. First, we prove that:

Proposition 27.

$$\lim_{r\to\infty} \sup_{|x|=r} |u(x)| = 0 \tag{3.75}$$

Proof. According to the Sobolev embedding theorem, we know that $C^{0,\gamma}\big(B_r(x)\big) \subset H^2\big(B_r(x)\big)$ for any ball $B_r(x) \subset \mathbb{R}^2$, and any $0 < \gamma < 1$. Furthermore, there exists a constant $C = C(\gamma, r)$ such that

$$||u||_{C^{0,\gamma}(B_r(x))} \leq C||u||_{H^2(B_r(x))} \tag{3.76}$$

Therefore, since $u \in H^2(\mathbb{R}^2)$, we can say that $u \in C^{0,\gamma}(B_r(x))$ for any ball $B_r(x) \subset \mathbb{R}^2$, and any $0 < \gamma < 1$. Furthermore, there exists a constant $C = C(\gamma, r)$ such that

$$||u||_{C^{0,\gamma}(B_r(x))} \leq C||u||_{H^2(\mathbb{R}^2)} \tag{3.77}$$

for any $x \in \mathbb{R}^2$. Now, take arbitrary $m > 0$. According to (3.77), there exists $\delta > 0$ independent of x such that if $|u(x)| > m$, then $|u(y)| > \frac{m}{2}$ for any y with $|x - y| < \delta$. Therefore, if $|u(x)| > m$, there exists $\delta > 0$ independent of x such that:

$$\int_{B_\delta(x)} |u|^2 \geq \pi\delta\frac{m^2}{4} \tag{3.78}$$

On the other hand, since $u \in L^2(\mathbb{R}^2)$, then there exists $R > 0$ such that:

$$\int_{\mathbb{R}^2\setminus B_R(0)} |u|^2 dx < \pi\delta\frac{m^2}{4} \tag{3.79}$$

According to (3.79), we can say that (3.78) does not hold for $|x| > R + \delta$. This implies that for $|x| > R + \delta$, we have $|u(x)| < m$. (Note that (3.78) holds under the assumption $|u(x)| > m$.) This proves (3.75). □

Now, we prove that:

Proposition 28. *There exists a constant C such that:*

$$||u||_{L^\infty} < CM \tag{3.80}$$

Proof. Similar to the proof of proposition (27), note that for any ball $B_r(x) \subset \mathbb{R}^2$, and any $0 < \gamma < 1$, according to the Sobolev embedding theorem, we have:

$$||u||_{C^{0,\gamma}(B_r(x))} \leq C||u||_{H^2(\mathbb{R}^2)} \tag{3.81}$$

where $C = C\big(r, \gamma\big)$ is a constant. This implies that:

$$||u||_{L^\infty(B_r(x))} \leq C||u||_{H^2(\mathbb{R}^2)} \tag{3.82}$$

for some universal constant C. Therefore:

$$||u||_{L^\infty(\mathbb{R}^2)} \leq C||u||_{H^2(\mathbb{R}^2)} \tag{3.83}$$

for the same universal constant $C > 0$. Now, according to (3.72), and (3.83), we get:

$$||u||_{L^\infty(\mathbb{R}^2)} \leq C||\zeta||_{L^2} \tag{3.84}$$

where C is a constant. But note that

$$||\eta||_{L^2} \leq M||e^{-\gamma|x|}||_{L^2(\mathbb{R}^2)} \tag{3.85}$$

Using (3.84) and (3.85), we get the statement of the proposition. □

Now, to prove the exponential decay of u, consider the function:

$$s(x) = Ne^{-\gamma|x|} \tag{3.86}$$

where $N = N_\gamma(M, \phi)$ is a constant which we will find it later. Then, we have:

$$\Delta s(x) = \big(-\gamma|x|^{-1} + \gamma^2\big)s(x) \tag{3.87}$$

Therefore:

$$\Delta s(x) - |\phi|^2 s(x) = \big(-\gamma|x|^{-1} + \gamma^2 - |\phi|^2\big)s(x) \tag{3.88}$$

Using (3.88), and the fact that u satisfies equation (3.2), we have:

$$\begin{aligned} \Delta(s \pm u)(x) - |\phi|^2(s \pm u)(x) &= \big(-\gamma|x|^{-1} + \gamma^2 - |\phi|^2\big)s(x) \mp \zeta(x) \\ &\leq N\big(-\gamma|x|^{-1} + \gamma^2 - |\phi|^2\big)e^{-\gamma|x|} + Me^{-\gamma|x|} \end{aligned} \tag{3.89}$$

Therefore:

$$\Delta(s \pm u)(x) - |\phi|^2(s \pm u)(x) \leq N\big(-\gamma|x|^{-1} + \gamma^2 - |\phi|^2\big)e^{-\gamma|x|} + Me^{-\gamma|x|} \tag{3.90}$$

Proposition 29. *There exist constants $R = R(\gamma)$, and $N = N(M, \gamma)$ such that*

$$N\big(-\gamma|x|^{-1} + \gamma^2 - |\phi|^2\big)e^{-\gamma|x|} + Me^{-\gamma|x|} < 0 \quad \textit{if} \ |x| > R \tag{3.91}$$

and

$$\left(s \pm u\right)\Big|_{|x|=R} > 0 \tag{3.92}$$

Proof. According to theorem 8.1 of [1], we have:

Theorem 11. *For every coupling constant $\lambda > 0$, given $\epsilon > 0$, we can find $K = K(\lambda, \phi, \epsilon) > 0$ such that:*

$$1 - |\phi|^2 \leq K e^{-(1-\epsilon) m_L |X|} \tag{3.93}$$

where $m_L = \min\{\lambda^{\frac{1}{2}}, 2\}$.

Now, using (3.93), and the fact that $\gamma < 1$, we can find $R = R(p, \gamma)$ such that:

$$-\gamma |x|^{-1} + \gamma^2 - |\phi|^2 < \frac{\gamma^2 - 1}{2} \quad \textbf{if} \ |x| > R \tag{3.94}$$

Therefore, if we choose $N = N(M, \gamma)$ such that:

$$N > \frac{2M}{1 - \gamma^2} \tag{3.95}$$

then, condition (3.91) is satisfied.

Now, note that according to proposition (28), we have:

$$\sup_{|x|=R} u < L \tag{3.96}$$

where $L = L\big(M, \gamma\big)$ is a constant. Therefore, if we choose $N = N(M, \gamma)$ such that:

$$N > e^{\gamma R} L \tag{3.97}$$

then condition (3.92) is satisfied. Therefore, if we choose $R = R(\gamma)$ as described, and $N = N(M, \gamma)$ such that both conditions (3.95), (3.97) are satisfied, then both conditions (3.91), (3.92) will be satisfied. Now, using (3.90), and proposition (29), we see that for the constants R, and $N = N(M, \gamma)$, we have

$$\Delta\big(s \pm u\big)(x) - |\phi|^2 \big(s \pm u\big)(x) < 0 \quad \textbf{if} \ |x| > R \tag{3.98}$$

$$\big(s \pm u\big)\Big|_{|x|=R} > 0 \tag{3.99}$$

But according to proposition (27), we deduce that:

$$\lim_{r \to \infty} \sup_{|x|=r} \big(s \pm u\big)\Big|_{|x|=r} = 0 \tag{3.100}$$

Using equations (3.98), (3.99) and (3.100), and the maximum principle, we have:

$$(s \pm u)(x) > 0 \quad \textbf{if} \ |x| > R \tag{3.101}$$

□

Equation (3.3) and proposition 29 implies (3.5) and this finishes the proof of lemma 6. □

Proposition 30. *Suppose that $U \subset \mathbb{R}^N$ and $f : U \to L^2\big(\mathbb{R}^2, \mathbb{R}\big)$ and $h : U \to M_N$ are differentiable. Suppose that $u : U \to H^2\big(\mathbb{R}^2, \mathbb{R}\big)$ solves the equation*

$$\Delta u(p) - |\phi|^2(h(p))u(p) = f(p) \tag{3.102}$$

for every $p \in U$. Then, u is differentiable and for every $p \in U$, we have:

$$\Delta Du(p) - |\phi|^2\big(h(p)\big)Du(p) = Df(p) + \big(D|\phi|^2(h(p))\big)u(p) \tag{3.103}$$

where D represents differentiation with respect to any of the spatial variables in U.

Proof.

Claim 2. *For every $p \in U$, there exists $C > 0$ and $\delta > 0$ such that for every $q \in U$ with $|p - q| < \delta$, we have*

$$\big|\big|u(p) - u(q)\big|\big|_{H^2} < C|p - q| \tag{3.104}$$

Proof. Let

$$w_{p,q} = u(q) - u(p)$$

We have:

$$\begin{aligned} \Delta w_{p,q} - |\phi|^2\big(h(p)\big)w_{p,q} = &\Big(f(q) - f(p)\Big) \\ &+ \Big(|\phi|^2\big(h(q)\big) - |\varphi|^2\big(h(p)\big)\Big)u(q) \end{aligned} \tag{3.105}$$

There exists $\delta_1 = \delta_1(p) > 0$ and $C_1 > 0$ such that if $|p - q| < \delta_1$, then

$$\big|\big||\phi|^2\big(h(q)\big) - |\phi|^2\big(h(p)\big|\big|_{L^\infty} < C_1|p - q| \tag{3.106}$$

Therefore, since f is differentiable, for every $p \in U$, there exists $\delta_2, C_2 > 0$ such that for every $q \in U$ with $|p - q| < \delta_2$,

$$\Big|\Big|\big(f(q) - f(p)\big) + \Big(|\phi|^2\big(h(q)\big) - |\phi|^2\big(h(p)\big)\Big)u(q)\Big|\Big|_{L^2(\mathbb{R}^2)} \leq C_2|p - q| \tag{3.107}$$

According to lemma 6, (3.105), and (3.186), we deduce the statement of claim. □

Using the fact that f is differentiable, for every $p \in U$ and $\tau \in T_pU$, there exists a function $v_{_{p,\tau}} \in H^2$ which satisfies:

$$\Delta v_{_{p,\tau}} - |\phi|^2\big(h(p)\big)v_{_{p,\tau}} = Df(p)(\tau) + D|\phi|^2(p)(\tau)u(p) \tag{3.108}$$

Suppose that $p + \tau \in U$. Let:

$$w_{p,\tau} = u(p + \tau) - u(p) - v_{_{p,\tau}} \tag{3.109}$$

We have:

$$\Delta w_{p,\tau} - |\phi|^2 w_{p,\tau} = \Big(f(p+\tau) - f(p) - Df(p)(\tau)\Big) + \Big(|\phi|^2(p+\tau) - |\varphi|^2(p) - D|\phi|^2(p)(\tau)\Big)u(p) + \Big(|\phi|^2(p+\tau) - |\phi|^2(p)\Big)\Big(u(p+\tau) - u(p)\Big) \tag{3.110}$$

According to the facts that f and h are differentiable, and claim 5, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then

$$\begin{aligned} &||f(p+\tau) - f(p) - Df(p)(\tau)||_{L^2(\mathbb{R}^2)} \\ +&||\Big(|\phi|^2(p+\tau) - |\phi|^2(p) - D|\phi|^2(p)(\tau)\Big)u(p)||_{L^2(\mathbb{R}^2)} \\ +&||\Big(|\phi|^2(p+\tau) - |\phi|^2(p)\Big)\Big(u(p+\tau) - u(p)\Big)||_{L^2(\mathbb{R}^2)} \\ \leq&\epsilon|\tau| \end{aligned} \tag{3.111}$$

According to (3.110) and (3.111) and lemma 6, we deduce that for every $\epsilon > 0$, there exists $\delta_1 > 0$ such that if $|\tau| < \delta_1$, then

$$||w_{p,\tau}||_{H^2(\mathbb{R}^2)} \leq \epsilon|\tau| \tag{3.112}$$

Therefore, u is differentiable. □

Lemma 7. *Suppose that $U \subset \mathbb{R}^N$ and $f : U \to L^2\big(\mathbb{R}^2, \mathbb{R}\big)$ is m-times differentiable for $m \in \mathbb{N}$ and $h : U \to M_N$ is $(m+2)$-times differentiable. Suppose that $u : U \to H^2\big(\mathbb{R}^2, \mathbb{R}\big)$ solves the equation*

$$\Delta u(p) - |\phi|^2(h(p))u(p) = f(p) \tag{3.113}$$

for every $p \in U$. Then, u is m-times differentiable

Proof. The base case holds by the previous statement. Suppose that the statement holds for m. According to the base case and proposition 30, we have:

$$\Delta Du(p) - |\phi|^2\big(h(p)\big)Du(p) = Df(p) + \big(D|\phi|^2\big(h(p)\big)u(p) \tag{3.114}$$

where D represents differentiation with respect to any spatial direction in U. According to the base case, we have:

$$\Delta Du(p) - |\phi|^2\big(h(p)\big)Du(p) = Df(p) + \big(D|\phi|^2\big(h(p)\big)u(p) \tag{3.115}$$

where D represents differentiation with respect to any spatial direction in U. The function $g : U \to H^2\big(\mathbb{R}^2, \mathbb{R}\big)$ is defined by:

$$g(p) = \Big(D|\phi|^2(h(p))\Big)u(p)$$

Claim 3. *The function g is m-times differentiable.*

Proof. According to the induction hypothesis, u is m-times differentiable.The function

$$e : U \to H^2(\mathbb{R}^2, \mathbb{R})$$

defined by

$$e(p) = \big(D|\phi|^2(h(p))\big)$$

is m-times differentiable. Therefore, by the Sobolev embedding theorem, g is m-times differentiable. □

Claim 3, (3.115) and the induction hypothesis imply that Du is m-times differentiable. Therefore, u is $(m+1)$-times differentiable. □

Lemma 8. *Suppose that U is an open subset of $\mathbb{R}^2$, $f \in \mathcal{E}_m(\mathbb{R}^2, U, \mathbb{R}^2)$ and $h : U \to M_N$ is m-times differentiable and $h(U)$ is a precompact subset of M_N and $D^s(h)$ is bounded for every s with $|s| \le m$. Suppose that $u : \mathbb{R}^2 \times U \to \mathbb{R}$ satisfies the equation*

$$\Delta u(,p) - |\phi|^2\big(h(p)\big)u(,p) = f(,p) \tag{3.116}$$

for every $p \in U$ and $u(,p) \in H^2$ for every $p \in U$. Then, $u \in \mathcal{E}_{m-3}(\mathbb{R}^2, U, \mathbb{R}^2)$.

Proof. Suppose that $V = \mathbb{R}^2 \times U$. Consider the functions

$$\begin{aligned} &\hat{f} : V \to L^2\big(\mathbb{R}^2, \mathbb{R}\big) \\ &\hat{f}(x,p)(y) = f\big(y - x, p\big) \quad \forall x, y \in \mathbb{R}^2, p \in U \end{aligned} \tag{3.117}$$

$$\begin{aligned} &\hat{u} : V \to H^2\big(\mathbb{R}^2, \mathbb{R}\big) \\ &\hat{u}(x,p)(y) = u\big(y - x, p\big) \quad \forall x, y \in \mathbb{R}^2, p \in U \end{aligned} \tag{3.118}$$

$$\begin{aligned} &\hat{h} : V \to M_2 \\ &\hat{h}(x,p)(y) = h\big(y - x, p\big) \quad \forall x, y \in \mathbb{R}^2, p \in U \end{aligned} \tag{3.119}$$

The function $\hat{h}$ is m-times differentiable. For every $q \in V$, we have

$$\Delta \hat{u}(q) - |\varphi|^2\big(\hat{h}(q)\big)\hat{u}(q) = \hat{f}(q) \tag{3.120}$$

Suppose that $q \in V$. For any multi-index r with $|r| \le m$, the function $x \to \big(D^r \hat{f}(q)\big)(x)$ has exponential decay as $|x| \to \infty$. Therefore, $\hat{f}$ is m-times differentiable. Therefore, according to lemma 7, the function $\hat{u}$ is m-times differentiable. Therefore, the function u is m-times differentiable and $D^r u(.,p) \in H^2$ for any multi-index r with $|r| \le m$, and for any multi-index s with $|s| \le (m-3)$, we have:

$$\Delta\Big(D^s u(q)\Big)(x) - \Big(D^s\big(|\varphi|^2\big(h(q)\big)u(q)\big)\Big)(x) = \big(D^s f(q)\big)(x) \tag{3.121}$$

Therefore, by induction on $|s|$, the facts that $h(U)$ is a precompact subset and $f \in \mathcal{E}_m(\mathbb{R}^2, U, \mathbb{R}^2)$, we deduce that $f \in \mathcal{E}_{m-3}(\mathbb{R}^2, U, \mathbb{R}^2)$. □

Lemma 9. *Consider a smooth curve $\gamma : \mathbb{R} \to K$ where K is a compact subset of M_N. Suppose that*

$$\phi(y,z) = \phi(y;\gamma(z)) \tag{3.122}$$

Also, suppose that $|D\gamma| \leq m$ *for some* $m > 0$. *Then, there exists a number* $\delta > 0$ *such that if* $f \in H^2(\mathbb{R}^3)$ *with* $||f||_{L^\infty} \leq \delta$, *then the equation*

$$(-\Delta + |\phi + f|^2)u = g \tag{3.123}$$

for $g \in L^2(\mathbb{R}^3)$, *has a solution* $u \in H^2(\mathbb{R}^3)$ *with*

$$||u||_{H^2} \leq C||g||_{L^2} \tag{3.124}$$

where $C = C(K, m, \delta)$.

Proof. The proof goes by the same arguments as in lemma 6. But, here we consider the energy quantity:

$$Tu = \int_{\mathbb{R}^3} |\nabla u|^2 + |\phi + f|^2 u^2 \tag{3.125}$$

and we prove a coercivity for that:

Claim 4. *If* δ *is small enough, then there exists* $C = C(K, m, \delta) > 0$ *such that*

$$Tu \geq C||u||^2_{H^1} \tag{3.126}$$

Proof. We have:

$$\begin{aligned} Tu &\geq \int_{\mathbb{R}} \int_{\mathbb{R}^2} (|\nabla_y u|^2 + |\phi + f|^2 u) \\ &\geq \int_{\mathbb{R}} \int_{\mathbb{R}^2} \big(|\nabla_y u|^2 + |\phi|^2 u^2 - \delta^2 u^2\big) dydz \\ &\geq \int_{\mathbb{R}} \int_{\mathbb{R}^2} (C - \delta^2) u^2 dydz \\ &\geq (C - \delta^2)||u||^2_{L^2} \end{aligned} \tag{3.127}$$

where C is the constant provided by claim 24 in the proof of lemma 6.
On the other hand, we have

$$Tu \geq ||\nabla u||^2_{L^2} \tag{3.128}$$

Therefore, by (3.127) and (3.128), we have $Tu \geq C||u||^2_{H^1}$. □

Now, the rest of the proof goes in the same way as in lemma 6. We look for the minimizers of the functional

$$\begin{aligned} &T : H^1(\mathbb{R}^3) \to \mathbb{R} \\ &T_g u = Tu - \int_{\mathbb{R}^3} gu \end{aligned} \tag{3.129}$$

Following claim 4, one can prove that if one considers a minimizing sequence for T_g, then it would be bounded in H^1 and one can pass to a weakly convergent subsequence whose limit is a minimizer of T_g. But minimizers of T_g satisfy equations (3.123), and then by elliptic regularity one obtains that the minimizer $u \in H^2(\mathbb{R}^3)$. □

Remark 2. *: The weakly lower semicontinuity of the functional T follows from its convexity and differentiability of T.*

Remark 3. *By using standard elliptic regularity results, if in the statement of lemma 9, $f \in H^m(\mathbb{R}^3)$, then $u \in H^(m+2)(\mathbb{R}^3)$ and*

$$||u||_{H^{m+2}} \leq C||f||_{H^m} \tag{3.130}$$

Lemma 10. *Consider a compact set $K \subset M_N$. Let $p = (\phi, \alpha) \in K$. Then, for any*

$$\zeta = (\zeta_0, \zeta_1, \zeta_2) \in L^2(\mathbb{R}^2, \mathbb{C}) \oplus L^2(\mathbb{R}^2, \mathbb{R}) \oplus L^2(\mathbb{R}^2, \mathbb{R})$$

which is orthogonal to the zero modes at p, there exists a unique

$$\psi = (\tilde{\varphi}, \tilde{a}_1, \tilde{a}_2) \in H^2(\mathbb{R}^2, \mathbb{C}) \oplus H^2(\mathbb{R}^2, \mathbb{R}) \oplus H^2(\mathbb{R}^2, \mathbb{R})$$

which is orthogonal to zero modes at p and solves the equation

$$L[\phi, \alpha]\psi = \zeta \tag{3.131}$$

and

$$||\psi||_{H^2} \leq C||\zeta||_{L^2} \tag{3.132}$$

Furthermore, if the vector ζ satisfies the gauge orthogonality condition

$$\partial_{x_1}\zeta_1 + \partial_{x_2}\zeta_2 = (i\phi, \zeta_0) \tag{3.133}$$

then ψ satisfies the gauge orthogonality condition:

$$\partial_{x_1}\tilde{a}_1 + \partial_{x_2}\tilde{a}_2 = (i\phi, \tilde{\varphi}) \tag{3.134}$$

Furthermore, if the vector ζ satisfies the exponential decay

$$|\zeta(x)| \leq Ae^{-\beta|x|} \tag{3.135}$$

for some γ with $0 < \gamma < 1$, then there exists numbers B and $R = R(K)$ such that

$$\begin{aligned} |\tilde{\varphi}(x)| &\leq Be^{-\frac{\beta}{2}|x|} \\ |\tilde{a_j}(x)| &\leq Be^{-\beta|x|} \end{aligned} \tag{3.136}$$

and if $|x| > R$, then

$$\begin{aligned} |\partial_j\tilde{\varphi}(x)| &\leq Be^{-\frac{\beta}{2}|x|} \\ |\partial_j\tilde{a_j}(x)| &\leq Be^{-\beta|x|} \end{aligned} \tag{3.137}$$

for $j = 1, 2$.

Notation 1. *For any $r > 0$, let:*

$$(H^r)^\perp = H^r \cap \big(T_{(\phi,\alpha)} M_N\big)^\perp \tag{3.138}$$

Proof. Consider the functional $I_\zeta : (H^1)^\perp \to \mathbb{R}$ defined by:

$$\begin{aligned} I_\zeta[\eta, b] = \int_{\mathbb{R}^2} \Big[\frac{1}{2}|\nabla b|^2 + \frac{1}{2}|D_\alpha \eta|^2 + \frac{1}{4}(3|\phi|^2 - 1)|\eta|^2 + \frac{1}{2}|b|^2|\phi|^2 \\ - 2\sum_{j=1}^{2} \big(i\eta, D_{\alpha_j}\phi\big) b_j - \sum_{j=1}^{2} \zeta_j b_j - (\zeta_0, \eta) \Big] \end{aligned} \tag{3.139}$$

Note that if $(\eta, b) \in (H^1)^\perp$, then using the fact that $|\phi|$, $D_{\alpha_j}\phi$ are bounded, and $\zeta_j \in L^2$, and $\zeta_0 \in L^2$, we see that $\big|I_\zeta[\eta, b]\big| < \infty$.

Claim 5. *The functional I_ζ has a unique minimizer $(\tilde{\phi}, \tilde{a})$ in $(H^1)^\perp$. Furthermore,*

$$||(\tilde{\phi}, \tilde{a})||_{H^1} \leq C||\zeta||_{L^2} \tag{3.140}$$

where C is a constant.

Proof. Consider the functional $J : H^1 \to \mathbb{R}$ defined by:

$$\begin{aligned} J[\eta, b] = \int_{\mathbb{R}^2} \Big[\frac{1}{2}|\nabla b|^2 + \frac{1}{2}|D_\alpha \eta|^2 + \frac{1}{4}(3|\phi|^2 - 1)|\eta|^2 + \frac{1}{2}|b|^2|\phi|^2 \\ - 2\sum_{j=1}^{2} \big(i\eta, D_{\alpha_j}\phi\big) b_j \Big] \end{aligned} \tag{3.141}$$

Then, according to (theorem 3.1 of the Stuart's paper)(put the statement somewhere), we have:

$$J[\eta, b] \geq \gamma \big|(\eta, b)\big|^2_{H^1} \qquad \forall \ (\eta, b) \in (H^1)^\perp \tag{3.142}$$

for some constant γ. Using (3.142), we have:

$$\begin{aligned} I_\zeta[\eta, b] &\geq \gamma|(\eta, b)|^2_{H^1} - \int_{\mathbb{R}^2} \big[\sum_{j=1}^{2} \zeta_j b_j + (\zeta_0, \eta)\big] \\ &\geq \frac{\gamma}{2}|(\eta, b)|^2_{H^1} - \frac{1}{2\gamma}||\zeta_j||^2_{L^2} - \frac{1}{2\gamma}||\zeta_0||^2_{L^2} \end{aligned} \tag{3.143}$$

for the constant $\gamma > 0$ described above.

Now, take a minimizing sequence $\{(\eta_i, b_i)\}_{i=1}^\infty \subset H^1$ for the functional I_η. According to (3.143), this sequence is bounded in H^1. Therefore, there exists $(\eta_0, b_0) \in H^1$ such that $(\eta_i, b_i) \rightharpoonup (\eta_0, b_0)$ in H^1. Furthermore, since $\{(\eta_i, b_i)\} \subset \big(T_p M_N\big)^\perp$, then for any $n \in T_p M_N$, we have:

$$\big((\eta_i, b_i), n\big)_{L^2} = 0 \tag{3.144}$$

Now, since $(\eta_i, b_i) \rightharpoonup (\eta_0, b_0)$ in H^1, by (3.144) we have:

$$\big((\eta_0, b_0), n\big)_{L^2} = 0 \tag{3.145}$$

Therefore, $(\eta_0, b_0) \in (H^1)^{\perp}$. Now, using the fact that I_ζ is lower semicontinuous with respect to the weak topology of H^1, we deduce that (η_0, b_0) is a minimizer for I_ς on $(H^1)^{\perp}$.

Now, suppose that (η_1, b^1), (η_2, b^2) are both minimizers of the functional I_ζ in $(H^1)^{\perp}$. Then, we have:

$$I_\zeta\Big[\frac{\eta^1+\eta^2}{2}, \frac{b^1+b^2}{2}\Big] - \frac{1}{2}\big(I_\varsigma[\eta^1, b^1] + I_\varsigma[\eta^2, b^2]\big) = -J\Big[\frac{\eta^1-\eta^2}{2}, \frac{b^1-b^2}{2}\Big] \tag{3.146}$$

Now, according to (3.142), (3.146), and the fact that (η^1, b^1), (η^2, b^2) are both minimizers of the functional I_ζ in $(H^1)^{\perp}$, we deduce that $b^1 = b^2$ and $\eta^1 = \eta^2$.

Now, suppose that $(\tilde{\phi}, \tilde{a},)$ is the minimizer of I_ς in $(H^1)^{\perp}$. Then, by (3.143), we have:

$$\begin{aligned}
\big|(\tilde{\varphi}, \tilde{a})\big|^2_{H^1} &\le \frac{2}{\gamma}\Big[I_\varsigma[\tilde{\phi}, \tilde{a}] + \frac{1}{2\gamma}\sum_{j=1}^{2}||\zeta_{_j}||^2_{L^2} + \frac{1}{2\gamma}||\eta_{_0}||^2_{L^2}\Big] \\
&\le \frac{2}{\gamma}\Big[I_\varsigma[0,0] + \frac{1}{2\gamma}\sum_{j=1}^{2}||\zeta_{_j}||^2_{L^2} + \frac{1}{2\gamma}||\zeta_{_0}||^2_{L^2}\Big] \\
&= \frac{1}{\gamma^2}\Big(\sum_{j=1}^{2}||\zeta_{_j}||^2_{L^2} + ||\zeta_{_0}||^2_{L^2}\Big)
\end{aligned} \tag{3.147}$$

This implies (3.140). □

Definition 1. *Suppose that for for $i, j, k \in \{1, 2\}$*

$$m^k_{ij}, n^k_i, p^k, \hat{q}_i \in L^\infty(\mathbb{R}^2, \mathbb{R})$$

and

$$\hat{m}_{ij}, \hat{n}^k_i, \hat{p}^k, q^k \in L^\infty(\mathbb{R}^2, \mathbb{C})$$

and

$$\zeta^k_1 \in L^2(\mathbb{R}^2)$$

and $\zeta_2 \in L^2(\mathbb{R}^2, \mathbb{C})$. Then, we say that $(u, v) \in H^1$ satisfies the equations:

$$-\sum_{i,j} m^k_{ij}(u_k)_{x_i x_j} + \sum_i n^k_i (u_k)_{x_i} + p^k u_k + (q^k, v) = \zeta^k_1 \quad k = 1, 2 \tag{3.148}$$

$$-\sum_{i,j} \hat{m}_{ij} D_{\alpha_i} D_{\alpha_j} v + \sum_i \hat{n}_i D_{\alpha_i} v + \hat{p} v + \sum_i \hat{q}_i u^i = \zeta_2 \tag{3.149}$$

in the weak sense if for any $(f,g) \in H^1$, we have:

$$\int_{\mathbb{R}^2} \sum_{i,j} (u_k)_{x_i} (m_{ij}^k f_k)_{x_j} + \Big(\sum_i n_i^k (u_k)_{x_i} + p^k u_k + (q^k, v) \Big) f_k = \int_{\mathbb{R}^2} \zeta_1^k f_k \quad k = 1,2 \tag{3.150}$$

$$\int_{\mathbb{R}^2} \sum_{i,j} \big(D_{\alpha_j} v, D_{\alpha_i} (\hat{m}_{ij} g) \big) + \Big(\sum_i \hat{n}_i D_{\alpha_i} v + \hat{p} v + \sum_i \hat{q}_i u^i, g \Big) = \int_{\mathbb{R}^2} (\zeta_2, g) \tag{3.151}$$

Claim 6. *The equation*

$$L[\phi, \alpha](\eta, b) = \zeta \tag{3.152}$$

has at most one weak solution in the space $(H^1)^{\perp}$.

Proof. Suppose that $(\eta_1, b_1), (\eta_2, b_2) \in (H^1)^{\perp}$ are two solutions for the equation (3.152) in the weak sense. Let

$$(\eta, b) = (\eta_1, b_1) - (\eta_2, b_2)$$

Then, (η, b) satisfies the equations

$$L[\phi, \alpha](\eta, b) = 0 \tag{3.153}$$

in the weak sense. Therefore, we have:

$$-\Delta b_j + |\phi|^2 b_j - 2\big(i\eta, D_{\alpha_j}\phi\big) = 0 \tag{3.154}$$

$$-D_{\alpha_j} D_{\alpha_j} \eta + \frac{1}{2}\big(3|\phi|^2 - 1\big)|\eta|^2 + 2i \sum_{j=1}^{2} (D_{\alpha_j}\phi) b_j = 0 \tag{3.155}$$

in the weak sense. Now, if we integrate the above equations against the test function $(\eta, b) \in H^1$, we deduce that:

$$\int_{\mathbb{R}^2} \big|\nabla b_j\big|^2 + |\phi|^2 b_j^2 - 2\big(i\eta, D_{\alpha_j}\phi\big) b_j = 0 \tag{3.156}$$

$$\int_{\mathbb{R}^2} \big|D_\alpha \eta\big|^2 + \frac{1}{2}\big(3|\phi|^2 - 1\big)|\eta|^2 - 2\sum_{j=1}^{2} \big(i\eta, D_{\alpha_j}\phi\big) b_j = 0 \tag{3.157}$$

Now, adding equation (3.156) to the equation (3.157), we deduce that $J[\eta, b] = 0$. (The functional J was defined in (3.141).) Now, according to (3.142), we deduce that $(\eta, b) = 0$. Therefore, $(\eta_1, b_1) = \eta_2, b_2)$. □

Claim 7. *Suppose that $(\eta, b) \in (H^1)^{\perp}$ is the minimizer of the functional I_ζ in $(H^1)^{\perp}$. Then (η, b) satisfies the equations*

$$L[\phi, \alpha](\eta, b) = \zeta \tag{3.158}$$

in the weak sense.

Proof. Consider the functionals $T_j : H^1 \to \mathbb{R}$ for $j = 1, 2$ and $T_0 : H^1 \to \mathbb{R}$ defined by:

$$T_{_j}[n_0, n_1, n_2] = \int_{\mathbb{R}^2} \big(\nabla n_{_j} . \nabla b_j + \Big(|\phi|^2 b_j + \big(2iD_{\alpha_j}\phi, \eta\big) - \zeta_{_j}\Big) n_{_j} \tag{3.159}$$

$$T_{_0}[n_0, n_1, n_2] = \int_{\mathbb{R}^2} \sum_{j=1}^{2} \big(2iD_{\alpha_j}\phi, n_0\big) b_j + \sum_{j=1}^{2} \big(D_{\alpha_j}\eta, D_{\alpha_j} n_0\big) + \Big(\frac{1}{2}\big(3|\phi|^2 - 1\big)\eta - \zeta_0, n_0\Big) \tag{3.160}$$

Suppose that $u = \big(u_0, u_1, u_2\big) \in H^1$. We can write:

$$u = v + w \tag{3.161}$$

where

$$v = \big(v_0, v_1, v_2\big) \in (H^1)^{\perp}$$

and

$$w = \big(w_0, w_1, w_2\big) \in H^1 \cap T_p M_N$$

Since (η, b) is the minimizer of $I_{_\zeta}$ in $(H^1)^{\perp}$, we have:

$$\frac{d}{dt}\Big|_{t=0} I_{\zeta}\big((\eta, b) + tv\big) = 0 \tag{3.162}$$

On the other hand, we have:

$$\frac{d}{dt}\Big|_{t=0} I_{\zeta}\big((\eta, b) + tv\big) = T_{_{A_1}}(v) + T_{_{A_2}}(v) + T_{_\phi}(v) \tag{3.163}$$

According to (3.162) and (3.163), we have:

$$T_{_{A_1}}(v) + T_{_{A_2}}(v) + T_{_\phi}(v) = 0 \tag{3.164}$$

We know $w \in H^2$. Using this and the fact that $(\eta, b) \in H^1$, we can do some integration by parts to obtain:

$$T_{_1}(w) + T_{_2}(w) + T_{_0}(w) = \int_{\mathbb{R}^2} L_{_1}(w) b_1 + L_{_2}(w) b_2 + \big(L_{_0}(w), \eta\big) \tag{3.165}$$

But since $w \in T_{_p} M_N$, we have

$$L_{_1}(w) = L_{_2}(w) = L_{_0}(w) = 0 \tag{3.166}$$

Using (3.165) and (3.166), we have:

$$T_{_0}(w) + T_{_1}(w) + T_{_2}(w) = 0 \tag{3.167}$$

Now, according to (3.161), (3.164) and (3.167), we have:

$$T_{_0}(u) + T_{_1}(u) + T_{_2}(u) = 0 \tag{3.168}$$

Since $u \in H^1$ is arbitrary, (3.167) and (3.168) imply that:

$$T_0(u) = T_1(u) = T_2(u) = 0 \tag{3.169}$$

Therefore, equation (3.158) holds in the H^1-weak sense. □

Claim 8. *Suppose that* $(\tilde{\varphi}, \tilde{a}) \in (H^1)^\perp$ *satisfies the equations*

$$L[\phi, \alpha](\tilde{\varphi}, \tilde{a}) = \zeta \tag{3.170}$$

in the weak sense. Then, $(\tilde{\varphi}, \tilde{a}) \in H^2$, *and:*

$$||(\tilde{\varphi}, \tilde{a})||_{H^2} \leq C||\zeta||_{L^2} \tag{3.171}$$

where C *is a constant.*

Proof. First, note that according to the claims (5), (6) and (7), we can say that $(\tilde{\varphi}, \tilde{a})$ is the unique minimizer of I_ζ in $(H^1)^\perp$. Therefore, by equation (3.140) in claim (5), we have:

$$||\tilde{\varphi}||_{L^2} \leq C||\zeta||_{L^2} \tag{3.172}$$

for some constant C. Set

$$h_j = 2\big(i\tilde{\varphi}, D_{\alpha_j}\phi\big) + \zeta_j \tag{3.173}$$

Now, note that $D_{\alpha_j}\phi \in L^2$, since it has exponential decay at infinity. Therefore, according to (3.172), we can say

$$||h||_{L^2} \leq C||\zeta||_{L^2} \tag{3.174}$$

for some constant C. Now, note that according to claim (7), we have:

$$-\Delta\tilde{a}_j + |\phi|^2\tilde{a}_j = h \quad \textbf{in the weak sense} \tag{3.175}$$

On the other hand, according to lemma 6, we can say that there exists unique $u_j \in H^2$ such that:

$$-\Delta u_j + |\phi|^2 u_j = h_j \quad \textbf{in the weak sense} \tag{3.176}$$

Now, by (3.175) and (3.176), we can say:

$$-\Delta\big(u_j - \tilde{a}_j\big) + |\phi|^2\big(u_j - a_j\big) = 0 \quad \textbf{in the weak sense} \tag{3.177}$$

in the weak sense. This implies that:

$$\int_{\mathbb{R}^2} \big|\nabla(u_j - \tilde{a}_j)\big|^2 + |\phi|^2\big(u_j - \tilde{a}_j\big)^2 = 0 \tag{3.178}$$

Therefore, $\tilde{a}_j = u_j$. Therefore, $\tilde{a_j} \in H^2$, and according to lemma 6 and (3.174), we have:

$$||\tilde{a}_j||_{H^2} \leq C||\zeta||_{L^2} \tag{3.179}$$

for a constant C.

Now, note that according to equation (3.170), and using the standard elliptic regularity, we can say that:

$$\tilde{\varphi} \in H^2_{loc}(\mathbb{R}^2) \tag{3.180}$$

According to equation (3.170) and claim (7), we have:

$$D_{\alpha_j} D_{\alpha_j} \tilde{\varphi} = \frac{1}{2}\big(3|\phi|^2 - 1\big)\tilde{\varphi} + 2i(D_{\alpha_j}\phi)\tilde{a}_j - \zeta_0 \tag{3.181}$$

in the weak sense. Therefore,

$$\Big(\sum_{j=1}^{2} D_{\alpha_j} D_{\alpha_j} \tilde{\varphi}\Big) \in L^2(\mathbb{R}^2) \tag{3.182}$$

We already know that $\tilde{\varphi} \in H^1$ and

$$||\tilde{\varphi}|| \leq C||\zeta||_{L^2} \tag{3.183}$$

Therefore, (3.182) implies that $\varphi \in H^2$ and provides a constant C such that

$$||\varphi||_{H^2} \leq C||\zeta||_{L^2} \tag{3.184}$$

□

Now, according to claims (5), (6), (7) and (8), we deduce that there exists unique $\psi = (\tilde{\varphi}, \tilde{a}) \in (H^2)^{\perp}$ which satisfies (3.131). Furthermore, (3.132) holds for a constant C.

Now, suppose that $\zeta \in \big(T_{(p)} M_N\big)^{\perp} \cap H^2_{loc}$, and the inequalities in (3.135) hold for the two constants A, γ. Note that using standard elliptic regularity and the fact that $(\tilde{\varphi}, \tilde{a}) \in H^2_{loc}$, we deduce that $(\tilde{\varphi}, \tilde{a}) \in H^4_{loc}$. According to equations (3.131), and the fact that $\nabla.\alpha = 0$, we have:

$$-\Delta\tilde{\varphi} + 2i\alpha_j\partial_j\tilde{\varphi} + \Big(\frac{1}{2}\big(3|\phi|^2 - 1\big) + |\alpha|^2\Big)\tilde{\varphi} + 2i\sum_{j=1}^{2}\big(D_{\alpha_j}\phi\big)\tilde{a}_j = \zeta_0 \tag{3.185}$$

$$-\Delta\tilde{a}_j - 2(i\tilde{\varphi}, D_{\alpha_j}\phi) + \tilde{a}_j|\phi|^2 = \zeta_j \tag{3.186}$$

Claim 9.

$$||\tilde{\varphi}||_{L^\infty} \leq CA \tag{3.187}$$

$$||\tilde{a}_j||_{L^\infty} \leq CA \tag{3.188}$$

for some constant C.

Proof. By the Sobolev embedding theorem, we know that

$$||\big(\tilde{\varphi}, \tilde{a}\big)||_{L^\infty} \leq C||\big(\tilde{\varphi}, \tilde{a}\big)||_{H^2} \tag{3.189}$$

where C is a constant. Using (3.132), (3.135), and (3.189) we deduce (3.187) and (3.188). □

Using the fact that $\psi = (\tilde{\varphi}, \tilde{a}) \in H^2$ and the Sobolev embedding theorem, we deduce:

Claim 10.

$$\lim_{r\to\infty} \max_{|x|=r} \left|\tilde{a}_j(x)\right| = 0 \tag{3.190}$$

$$\lim_{r\to\infty} \max_{|x|=r} \left|\tilde{\varphi}(x)\right| = 0 \tag{3.191}$$

Now, if we rewrite 3.185,3.186 as below:

$$-\Delta\tilde{\phi} + 2i\alpha_j\partial_j\tilde{\phi} + \Big(\frac{1}{2}\big(3|\phi|^2 - 1\big) + |\alpha|^2\Big)\tilde{\phi} = \eta_\phi - 2i\big(D_{\alpha_j}\phi\big)\tilde{a}_j \tag{3.192}$$

$$-\Delta\tilde{a_j} + \tilde{a}_j|\phi|^2 = \eta_{A_j} + 2(i\tilde{\phi}, D_{\alpha_j}\phi) \tag{3.193}$$

then, using the estimates in the above propositions, and the exponential decay for ζ, we can say that there exists N such that

$$\left| - \Delta\tilde{\varphi} + 2i\alpha_j\partial_j\tilde{\varphi} + \Big(\frac{1}{2}\big(3|\phi|^2 - 1\big) + |\alpha|^2\Big)\tilde{\varphi}\right| \leq Ne^{-\beta|X|} \tag{3.194}$$

$$\left| - \Delta\tilde{a_j} + \tilde{a}_j|\phi|^2\right| \leq Ne^{-\beta|X|} \tag{3.195}$$

Now, note that 3.195 is in the form of lemma 6. Based on lemma ???, we get the expected exponential decay estimate for $\tilde{a}_j$.

Now, we want to prove the exponential decay of $\tilde{\varphi}$. Note that

$$\Delta|\tilde{\varphi}|^2 = 2(\Delta\tilde{\varphi}, \tilde{\varphi}) + 2|\nabla\tilde{\varphi}|^2 \tag{3.196}$$

According to equation 3.194, by taking the inner product with $\tilde{\varphi}$, we have:

$$\Big(- \Delta\tilde{\varphi} + 2i\alpha_j\partial_j\tilde{\varphi} + \big(\frac{1}{2}(3|\phi|^2 - 1) + |\alpha|^2\big)\tilde{\varphi}, \tilde{\varphi}\Big) \leq Ne^{-\beta|X|}|\tilde{\varphi}| \tag{3.197}$$

This implies that:

$$(\Delta\tilde{\phi}, \tilde{\phi}) \geq \big(\frac{1}{2}(3|\phi|^2 - 1) + |\alpha|^2\big)|\tilde{\varphi}|^2 - \left|\Big(2i\alpha_j\partial_j\tilde{\varphi}, \tilde{\varphi}\Big)\right| - Ne^{-\beta|X|}|\tilde{\varphi}| \tag{3.198}$$

Now, note that

$$\begin{aligned} \left|\Big(2i\alpha_j\partial_j\tilde{\varphi}, \tilde{\varphi}\Big)\right| &= \left|\Big(2i\partial_j\tilde{\varphi}, \alpha_j\tilde{\varphi}\Big)\right| \\ &\leq |\partial_j\tilde{\varphi}|^2 + |\alpha_j|^2|\tilde{\varphi}|^2 \\ &= |\nabla\tilde{\varphi}|^2 + |\alpha|^2|\tilde{\varphi}|^2 \end{aligned} \tag{3.199}$$

Now, suppose that R_2 is chosen such that and if $|x| > R_2$, then $|\phi(x)| > \frac{1}{3}$. Then combining 3.196, 3.198, 3.199, we can say that if $|x| > R_2$, we have:

$$\begin{aligned} \Delta|\tilde{\varphi}|^2 &\geq \big(\frac{1}{2}(3|\phi|^2 - 1)\big)|\tilde{\varphi}|^2 - Ne^{-\beta|x|}|\tilde{\varphi}| \\ &= |\tilde{\varphi}|^2 + \frac{3}{2}\big(|\phi|^2 - 1\big)|\tilde{\varphi}|^2 - Ne^{-\beta|x|}|\tilde{\varphi}| \end{aligned} \tag{3.200}$$

But we know that $\big(|\phi|^2 - 1\big)$ decays to 0 faster then the exponential $e^{-r|x|}$ for any $0 < r < 1$. Using this and claim 9 in (3.200), we can say that if $|x| > R_2$, then:

$$\Delta|\tilde{\varphi}|^2 \geq |\tilde{\varphi}|^2 - N'e^{-\beta|x|} \tag{3.201}$$

for some $N' > 0$. Consider the function

$$v = |\tilde{\varphi}|^2 - Ke^{-\beta|X|} \tag{3.202}$$

where the constant K is such that

1. $$v\Big|_{|x|=R_2} \leq 0$$

2. $$K \geq \frac{N'}{1-\beta^2}$$

Using 3.201, if $|x| > R_2$, we have:

$$\begin{aligned} \Delta v &\geq |\tilde{\varphi}|^2 - N'e^{-\beta|x|} - K\Delta(e^{-\beta|x|}) \\ &= |\tilde{\varphi}|^2 - N'e^{-\beta|x|} - K\Big(-\frac{\beta}{|x|} + \beta^2\Big)e^{-\beta|x|} \\ &\geq |\tilde{\varphi}|^2 - N'e^{-\beta|x|} - K\beta^2 e^{-\beta|x|} \\ &\geq |\tilde{\varphi}|^2 - Ke^{-\beta|x|} \\ &= v \end{aligned} \tag{3.203}$$

Using (3.203) and the fact that $|v| \to 0$ as $|x| \to \infty$, and the maximum principle, we deduce that $v \leq 0$ on $|x| \geq R_2$. Therefore, if $|x| > R_2$, then

$$|\tilde{\varphi}|^2 \leq Ke^{-\beta|x|}$$

This and equation (3.132) proves the exponential decay estimates (3.136). We will use the following standard lemma in elliptic PDEs:

Lemma 11. *For any $n \in \mathbb{N}$, there exists $C_n > 0$ such that for any $f \in C^2(\mathbb{R}^n)$,*

$$||\nabla f||_{L^\infty(B_1)} < C_n\big(||f||_{L^\infty(B_2)} + ||\Delta f||_{L^\infty(B_2)}\big) \tag{3.204}$$

According to equation 3.193, there exists N and R such that if $|x| > R$, then

$$|\Delta\tilde{a}_j(x)| < Ne^{-\gamma|x|} \quad j = 1, 2 \tag{3.205}$$

Therefore, according to lemma 11, there exists R and N such that if $|x| > R$, then

$$|\partial_k \tilde{a}_j(x)| < Ne^{-\gamma|x|} \quad j = 1,2 \quad , \quad k = 1,2 \tag{3.206}$$

According to equation 3.192, the fact that $|\alpha_j|(x) \to 0$ as $|x| \to \infty$, and lemma 11, we deduce that there exists R and N such that if $|x| > R$, then

$$|\partial_k \tilde{\varphi}(x)| < Ne^{-\gamma|x|} \quad k = 1,2 \tag{3.207}$$

This proves (3.137).
Now, suppose that $\zeta \in \big(T_p M_N\big)^{\perp} \cap H^2_{loc}$ satisfies (3.133). We take the jth partial derivative of equation (3.186):

$$-\Delta\partial_j\tilde{a}_j - 2(i\partial_j\tilde{\varphi}, D_{\alpha_j}\phi) - 2(i\tilde{\varphi}, \partial_j D_{\alpha_j}\phi) + (\partial_j\tilde{a}_j)|\phi|^2 + \tilde{a}_j\partial_j|\phi|^2 = \partial_j(\zeta)_j \tag{3.208}$$

But we have:

$$\begin{aligned} &\partial_j D_{\alpha_j}\phi = \partial_j^2\phi - i(\partial_j a_j)\phi - ia_j\partial_j\phi \\ &\partial_j|\phi|^2 = 2(\phi, \partial_j\phi) \end{aligned} \tag{3.209}$$

If we use equations in (3.209) in (3.208), and add them up for all j, we get:

$$\begin{aligned} \nabla.(\zeta_1,\zeta_2) =& -\Delta\nabla.\tilde{a} - 2(i\partial_j\tilde{\varphi}, \partial_j\phi) + 2\sum_{j=1}^{2} a_j(\partial_j\tilde{\varphi}, \phi) - 2\big(i\tilde{\varphi}, \Delta\phi\big) \\ &+ 2\big(\tilde{\varphi}, (\nabla.a)\phi + \sum_{j=1}^{2} a_j\partial_j\phi\big) + (\nabla.\tilde{a})|\phi|^2 + 2\sum_{j=1}^{2}\tilde{a}_j(\phi, \partial_j\phi) \end{aligned} \tag{3.210}$$

Now, using the fact that

$$L_\phi[\Phi, A](\tilde{\phi}, \tilde{a}) = \zeta_0 \tag{3.211}$$

there holds:

$$\Big(i\phi, -\sum_{j=1}^{2} D_{\alpha_j}D_{\alpha_j}\tilde{\varphi} + \frac{1}{2}\big(3|\phi|^2 - 1\big)\tilde{\phi} + 2i\sum_{j=1}^{2}\tilde{a}_j D_{\alpha_j}\phi\Big) = (i\phi, \zeta_0) \tag{3.212}$$

But we have:

$$D_{\alpha_j}D_{\alpha_j}\tilde{\varphi} = \partial_j^2\tilde{\varphi} - a_j^2\tilde{\varphi} - i(\partial_j a_j)\tilde{\varphi} - 2ia_j\partial_j\tilde{\varphi} \tag{3.213}$$

If we use equation (3.213) in (3.212), we get:

$$\begin{aligned} (i\phi, \zeta_0) =& (-i\phi, \Delta\tilde{\varphi}) + |a|^2(i\phi, \tilde{\varphi}) + \nabla.a(\phi, \tilde{\varphi}) + 2\sum_{j=1}^{2} a_j(\phi, \partial_j\tilde{\varphi}) \\ &+ \frac{1}{2}\big(3|\phi|^2 - 1\big)(i\phi, \tilde{\varphi}) + 2\sum_{j=1}^{2}\tilde{a}_j(\phi, \partial_j\phi) \end{aligned} \tag{3.214}$$

Now if we subtract (3.214) from (3.210), we have:

$$\begin{aligned} &-\Delta\nabla.\tilde{a} + 2\sum_{j=1}^{2}(i\partial_j\phi, \partial_j\tilde{\varphi}) + 2(i\Delta\phi, \tilde{\phi}) + 2\sum_{j=1}^{2}\big(\tilde{\varphi}, (\nabla.a)\phi + a_j\partial_j\phi\big) + (\nabla.\tilde{a})|\phi|^2 \\ &+ (i\phi, \Delta\tilde{\phi}) - |a|^2(i\phi, \tilde{\varphi}) - \nabla.a(\phi, \tilde{\varphi}) - \frac{1}{2}\big(3|\phi|^2 - 1\big)(i\phi, \tilde{\varphi}) = 0 \end{aligned} \tag{3.215}$$

But we have:

$$\Delta(i\phi, \tilde{\varphi}) = (i\Delta\phi, \tilde{\varphi}) + 2\sum_{j=1}^{2}(i\partial_j\phi, \partial_j\tilde{\varphi}) + (i\phi, \Delta\tilde{\varphi}) \tag{3.216}$$

Using (3.216) in (3.215), we have:

$$-\Delta\nabla.\tilde{a} + \Delta(i\phi, \tilde{\varphi}) + (\nabla.\tilde{a})|\phi|^2 + \Big(i\Delta\phi + (\nabla.a)\phi + 2\sum_{j=1}^{2} a_j\partial_j\phi - i|a|^2\phi - \frac{i}{2}\big(3|\phi|^2 - 1\big)\phi, \tilde{\varphi}\Big) = 0 \tag{3.217}$$

But we know (ϕ, a) is a steady state solution for the Abelian Higgs model. Therefore, it satisfies the equation:

$$-\sum_{j=1}^{2} D_j D_j\phi + \frac{1}{2}\big(|\phi|^2 - 1\big)\phi = 0 \tag{3.218}$$

Which can be simplified as:

$$-\Delta\phi + |a|^2\phi + i(\nabla.a)\phi + 2i\sum_{j=1}^{2} a_j\partial_j\phi + \frac{1}{2}\big(|\phi|^2 - 1\big)\phi = 0 \tag{3.219}$$

Now, using equation (3.219) in (3.217), we get:

$$-\Delta\nabla.\tilde{a} + \Delta(i\phi, \tilde{\varphi}) + |\phi|^2\big(\nabla.\tilde{a} - (i\phi, \tilde{\varphi})\big) = 0 \tag{3.220}$$

Or:

$$\big(-\Delta + |\phi|^2\big)\big(\nabla.\tilde{a} - (i\phi, \tilde{\varphi})\big) = 0 \tag{3.221}$$

Since $\tilde{a} \in H^2$, $(\nabla.\tilde{a}) \in H^1$. Also, since $|\phi|$ is bounded and $\tilde{\varphi} \in L^2$, $(i\phi, \tilde{\varphi}) \in L^2$. We have:

$$\partial_j(i\phi, \tilde{\varphi}) = \big(iD_{\alpha_j}\phi, \tilde{\varphi}\big) + \big(i\phi, D_{\alpha_j}\tilde{\varphi}\big) \tag{3.222}$$

Since $\tilde{\varphi} \in H^1$ and $D_{\alpha_j}\phi$ has exponential decay and $|\phi|$ is bounded, then by (3.222), $\big(i\phi, \tilde{\varphi}\big) \in H^1$. Therefore, $\big(\nabla.\tilde{a} - (i\phi, \tilde{\varphi})\big) \in H^1$. Now, according to (3.221) and lemma 6,

$$\nabla.\tilde{a} - (i\phi, \tilde{\varphi}) = 0$$

□

Proposition 31. *Suppose that $U \subset \mathbb{R}^k$ for some $k > 0$ is an open set and $s : U \to M_N$ is differentiable. Suppose that*

$$f = \big(f_0, f_1, f_2\big) : U \to \Big(L^2\big(\mathbb{R}^2, \mathbb{C}\big), L^2\big(\mathbb{R}^2, \mathbb{R}\big), L^2\big(\mathbb{R}^2, \mathbb{R}\big)\Big) \tag{3.223}$$

is differentiable. and

$$f(p) \perp T_{s(p)} M_N \tag{3.224}$$

for each $p \in U$. Suppose that

$$v : U \to \Big(H^2\big(\mathbb{R}^2, \mathbb{C}\big), H^2\big(\mathbb{R}^2, \mathbb{R}\big), H^2\big(\mathbb{R}^2, \mathbb{R}\big) \Big)$$

has the property

$$v(p) \in \big(T_{s(p)} M_N \big)^{\perp} \tag{3.225}$$

for any $p \in U$ and

$$\begin{aligned} L_{\varphi}[\phi(s(p)), \alpha(s(p))](v(p)) &= f_0(p) \\ L_{A_j}[\phi(s(p)), \alpha(s(p))](v(p)) &= f_j(p) \end{aligned} \tag{3.226}$$

for $j = 1, 2$ and any $p \in U$. Then v is differentiable and for each $p \in U$ and $\tau \in T_p U$, we have:

$$Dv(p)(\tau) = m_p(\tau) + E_p(\tau) \tag{3.227}$$

where $m_p(\tau)$ is an element of $T_{s(p)}(M_N)$ which satisfies

$$\Big(m_p(\tau), n_{\mu}\big(s(p)\big) \Big) = -\Big(v(p), Dn_{\mu}\big(s(p)\big)\big(Ds(p)(\tau)\big) \Big) \tag{3.228}$$

for each $\mu \in \{1, 2, 3, 4\}$ and $E_p(\tau)$ is an element of

$$H^2\big(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2\big) \cap \big(T_{s(p)} M_N \big)^{\perp}$$

which satisfies

$$\begin{aligned} L_{\varphi}[\phi(s(p)), \alpha(s(p))](Dv_p(\tau)) =& Df_0(p)(\tau) \\ & -2i \sum_{j=1}^{2} \Big(D\alpha_j\big(s(p)\big)\big(Ds(p)(\tau)\big) \Big) \Big(\big(D_{\alpha_j} v_0\big)(p)(\tau) \Big) \\ & -2 \sum_{j=1}^{2} \alpha_j\big(s(p)\big) \Big(D\alpha_j\big(s(p)\big)\big(Ds(p)(\tau)\big) \Big) v_0(p) \\ & -3 \Big(\phi(p), D\phi\big(s(p)\big)\big(Ds(p)(\tau)\big) \Big) v_0(p) \\ & -2i \sum_{j=1}^{2} v_j(p) D(D_{\alpha_j}\phi)\big(s(p)\big)\big(Ds(p)(\tau)\big) \end{aligned} \tag{3.229}$$

and:

$$\begin{aligned} L_{A_j}[\phi(s(p)), \alpha(s(p))](Dv_p(\tau)) =& Df_j(p)(\tau) \\ & +2\Big(iv_0(p), D(D_{\alpha_j}\phi)\big(s(p)\big)\big(Ds(p)(\tau)\big) \Big) \\ & -v_j(p) D|\phi|^2\big(s(p)\big)\big(Ds(p)(\tau)\big) \end{aligned} \tag{3.230}$$

for $j = 1, 2$.

Proof.

Claim 11. *For every $p \in U$, there exists $C, \delta > 0$ such that for every $q \in U$ with $|p-q| \leq \delta$, we have*

$$||v(.;q) - v(.;p)||_{H^2(\mathbb{R}^2)} < C|p-q| \tag{3.231}$$

Proof. Suppose that $p_1, p_2 \in U$. Let

$$w = v(p_1) - v(p_2)$$

Suppose that

$$w = w_1 + w_2$$

where $w_1 \in T_{s(p_1)}M_N$ and $w_2 \in \big(T_{s(p_1)}M_N\big)^{\perp}$.

Claim 12. *There exists $C_1, \delta_1 > 0$ such that if $|p_1 - p_2| < \delta_1$, then $||w_1||_{H^2} < C_1|p_1 - p_2|$.*

Proof. Consider $C_1, \delta_1 > 0$ such that if $|p_1 - p_2| < \delta_1$, then for every $\mu \in \{1, 2, 3, 4\}$, we have:

$$||n_{\mu}(p_1) - n_{\mu}(p_2)||_{L^2} < C_1|p_1 - p_2| \tag{3.232}$$

We have:

$$\langle v(p_1) - v(p_2), n_{\mu}(p_1)\rangle = -\langle v(p_2), n_{\mu}(p_1) - n_{\mu}(p_2)\rangle \tag{3.233}$$

Therefore, there exists $C_2 > 0$ such that if $|p_1 - p_2| < \delta$, then

$$\langle v(p_1) - v(p_2), n_{\mu}(p_1)\rangle < C_2|p_1 - p_2| \tag{3.234}$$

Therefore, there exists $C_3 > 0$ such that if $|p_1 - p_2| < \delta$, then

$$||w_1||_{H^2} < C_3|p_1 - p_2| \tag{3.235}$$

□

There holds

$$\begin{aligned} L_{\phi}[\phi(s(p_1)), \alpha(s(p_1))](w_2) =& f_0(p_1) - f_0(p_2) \\ & -2i\sum_{j=1}^{2}\big(\alpha_j(s(p_1)) - \alpha_j(s(p_2))\big)\partial_j v_0(p_2) \\ & -\big(|\alpha|^2(s(p_1)) - |\alpha|^2(s(p_2))\big)v_0(p_2) \\ & -\frac{3}{2}\big(|\varphi|^2(s(p_1)) - |\varphi|^2(s(p_2))\big)v_{\varphi}(p_2) \\ & -2i\sum_{j=1}^{2}\Big(\big(D_{\alpha_j}\phi\big)(s(p_1)) - \big(D_{\alpha_j}\phi\big)(s(p_2))\Big)v_j(p_2) \end{aligned} \tag{3.236}$$

and:

$$\begin{aligned} L_{A_j}[\phi(s(p_1)), \alpha(s(p_1))](w_2) =& f_j(p_1) - f_j(p_2) \\ & +2\big(iv_0(s(p_2)), D_{\alpha_j}\phi(s(p_1)) - D_{\alpha_j}\phi(s(p_2))\big) \\ & -v_j(s(p_2))\big(|\phi|^2(s(p_1)) - |\phi|^2(s(p_2))\big) \end{aligned} \tag{3.237}$$

Claim 13. *For every $p_1 \in U$, there exists $C, \delta > 0$ such that for every $p_2 \in U$ with $|p_1 - p_2| \leq \delta$*

$$||\alpha_j(s(p_1)) - \alpha_j(s(p_2))||_{L^\infty} \leq C|p_1 - p_2| \quad j = 1, 2 \tag{3.238}$$

$$|||\alpha|^2(s(p_1)) - |\alpha|^2(s(p_2))||_{L^\infty} \leq C|p_1 - p_2| \tag{3.239}$$

$$|||\phi|^2(s(p_1)) - |\phi|^2(s(p_2))||_{L^\infty} \leq C|p_1 - p_2| \tag{3.240}$$

$$||\big(D_{\alpha_j}\phi\big)(s(p_1)) - \big(D_{\alpha_j}\phi\big)(s(p_2))||_{L^\infty} \leq C|p_1 - p_2| \tag{3.241}$$

Proof. This follows from the estimates in proposition 3. □

Claim 14. *For every $p_1 \in U$, there there exists $C, \delta > 0$ such that for every $p_2 \in U$ with $|p_1 - p_2| < \delta$,*

$$||v(.;p_2)||_{H^2(\mathbb{R}^2)} < C \tag{3.242}$$

Proof. This simply follows from lemma 10. □

According to claims 13, 14, equations 3.236, 3.237, and lemma 10, there exists $\delta_1, C_1 > 0$ such that if $|p_1 - p_2| < \delta_1$, then

$$||w_2||_{H^2} < C_1|p_1 - p_2|$$

Therefore, according to claim 12, claim 11 follows. □

Claim 15. *Suppose that $p \in U$, $\tau \in T_pU$, and $\zeta = \big(Ds(p)\big)(\tau)$. There exists a function*

$$E_p(\tau) \in H^2 \cap \big(T_{s(p)}M_N\big)^\perp \tag{3.243}$$

such that

$$\begin{aligned} L_\varphi[\phi(s(p)), \alpha(s(p))](E_p(\tau)) =& \big(Df_0(p)\big)(\tau) - 2i\sum_{j=1}^{2}\big((D\alpha_j)(s(p))(\zeta)\big)\big(D_{\alpha_j}v_\varphi\big) \\ & -3\big(\phi, D\phi(s(p))(\zeta)\big)v_0 - 2\sum_{j=1}^{2} v_j\big(D(D_{\alpha j}\phi)\big)(s(p))(\zeta) \end{aligned} \tag{3.244}$$

and

$$\begin{aligned} L_{A_j}[\phi(s(p)), \alpha(s(p))](E_p(\tau)) =& \big(Df_j(p)\big)(\tau) + 2\Big(iv_0, \big(D(D_{\alpha j}\phi)\big)(s(p))(\zeta)\Big) \\ & -v_jD\big(|\phi|^2\big)(s(p))(\zeta) \end{aligned} \tag{3.245}$$

Proof. Let

$$\begin{aligned} R_0 =& \big(Df_0(p)\big)(\tau) - 2i\sum_{j=1}^{2}\big((D\alpha_j)(s(p))(\zeta)\big)\big(D_{\alpha_j}v_0\big) \\ & -3\big(\phi, (D\phi(s(p))(\tau)\big)v_0 - 2i\sum_{j=1}^{2} v_j\big(D(D_{\alpha j}\phi)\big)(s(p))(\zeta) \end{aligned} \tag{3.246}$$

and

$$\begin{aligned} R_j =& \big(Df_j(p)\big)(\tau) + 2\Big(iv_0, \big(D(D_{\alpha j}\phi)\big)(s(p))(\zeta)\Big) \\ & - v_j D\big(|\phi|^2\big)(s(p))(\zeta) \end{aligned} \tag{3.247}$$

for $j = 1, 2$.

Claim 16. $R = (R_0, R_1, R_2) \in L^2(\mathbb{R}^2)$ *and*

$$R \perp T_{_{s(p)}} M_N \tag{3.248}$$

Proof. According to estimates 3, the facts that $v \in H^2$ and $\alpha(s(p)) \in L^\infty$, and the fact that f is differentiable, we deduce that $R \in L^2$.

We have:

$$f \perp T_{_{s(p)}} M_N \tag{3.249}$$

Therefore,

$$\big\langle Df(p)(\tau), n_\mu(s(p))\big\rangle_{L^2} + \big\langle (f(p), \big(Dn_\mu\big)(s(p))(\zeta)\big\rangle_{L^2} = 0 \tag{3.250}$$

We have:

$$\begin{aligned} f_0(p) = -\Delta v_0(p) + 2i\sum_{j=1}^{2} \alpha_j \partial_j v_0 + |\alpha|^2 v_0 \\ + \frac{1}{2}\big(3|\phi|^2 - 1\big)v_0 + 2i\sum_{j=1}^{2} v_j D_{\alpha_j}\phi \end{aligned} \tag{3.251}$$

and

$$f_j(p) = -\Delta v_j(p) - 2\big(iv_0, D_{\alpha j}\phi(s(p))\big) + v_j(p)|\phi|^2(s(p)) \tag{3.252}$$

According to (3.250), (3.251), and (3.252), we have

$$
\begin{aligned}
\Big(Df(p)(\tau), n_\mu(s(p))\Big)_{L^2} &= +\bigg(v_0(p), \Big(D(\Delta n_{\mu,\varphi})(s(p))\Big)(\zeta)\bigg)_{L^2} \\
&- 2\sum_{j=1}^{2}\Big(v_0, i\alpha_j D\big(\partial_j n_{\mu,\varphi}\big)(s(p))(\zeta)\Big)_{L^2} \\
&- \Big(v_0, |\alpha|^2(s(p))\big(Dn_{\mu,\varphi}\big)(s(p))(\zeta)\Big)_{L^2} \\
&- \Big(v_0, \frac{1}{2}\big(3|\phi|^2-1\big)\big(Dn_{\mu,0}\big)(s(p))(\zeta)\Big)_{L^2} \\
&- 2\sum_{j=1}^{2}\Big(v_j, \Big(i\big(D_{\alpha_j}\phi(s(p)), \big(Dn_{\mu,\varphi}\big)(s(p))(\zeta)\big)\Big)\Big)_{L^2} \\
&+ 2\sum_{j=1}^{2}\Big(v_j(p), \big(D(\Delta n_{\mu,j})\big)(s(p))(\zeta)\Big)_{L^2} \\
&- 2\sum_{j=1}^{2}\Big(v_0, iD_{\alpha_j}\phi(s(p))\big(Dn_{\mu,A_j}\big)(s(p))(\zeta)\Big)_{L^2} \\
&- \Big(v_j(p), |\phi|^2(s(p))\big(Dn_{\mu,j}\big)(s(p))(\zeta)\Big)_{L^2}
\end{aligned}
\tag{3.253}
$$

$$
\begin{aligned}
\Big(2i\big(D\alpha_j\big)(s(p))(\zeta))(D_{\alpha_j}v_\varphi), n_{\mu,\varphi}\Big)_{L^2} &= \Big(2i\big(D\alpha_j\big)(s(p))(\zeta))\partial_j v_0, n_{\mu,0}\Big)_{L^2} \\
&+ \Big(\big(D|\alpha|^2\big)(s(p))(\zeta))v_0, n_{\mu,\varphi}\Big)_{L^2}
\end{aligned}
\tag{3.254}
$$

According to the fact that $v_0 \in H^2(\mathbb{R}^2)$

$$
\big(D\alpha_j\big)(s(p))(\zeta))v_0 \in H^2(\mathbb{R}^2)
\tag{3.255}
$$

Therefore,

$$
\Big(2i\big(D\alpha_j\big)(s(p))(\zeta))\partial_j v_0, n_{\mu,0}\Big)_{L^2} = -\Big(2i\big(D\alpha_j\big)(s(p))(\zeta))v_0, \partial_j n_{\mu,0}\Big)_{L^2}
\tag{3.256}
$$

According to (3.250),(3.251), (3.253), (3.254), and (3.256), we have:

$$
\begin{aligned}
(R, n_\mu)_{L^2} &= -\bigg(v_0, \big(DL_\varphi[\phi(s(p)), \alpha(s(p))](n_\mu)\big)(\zeta)\bigg)_{L^2} \\
&- \sum_{j=1}^{2}\bigg(v_j, \big(DL_{A_j}[\phi(s(p)), \alpha(s(p))](n_\mu)\big)(\zeta)\bigg)_{L^2} \\
&= 0
\end{aligned}
\tag{3.257}
$$

□

Therefore, according to Lemma 10, the statement holds. □

Suppose that $p \in U$ and $\tau \in T_pU$ and $\zeta = Ds(p)(\tau)$. Suppose that the function $E_p(\tau) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$ is the function which satisfies (3.243), (3.244), and (3.245). Let $m_p(\tau) \in T_{s(p)}M_N$ be such that

$$\big(m_p(\tau), n_\mu(s(p))\big)_{L^2} = -\Big(v(p), \big(Dn_\mu(s(p))(\zeta)\big)\Big)_{L^2} \tag{3.258}$$

for any $\mu \in \{1,2,3,4\}$. Let:

$$S_p(\tau) = E_p(\tau) + m_p(\tau) \tag{3.259}$$

Let

$$w_p(\tau) = v(p+\tau) - v(p) - S_p(\tau) \tag{3.260}$$

Claim 17. *For any $p \in U, \epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then $|w_p(\tau)|_{H^2} < \epsilon|\tau|$.*

Proof. Suppose that $A_\mu(\tau) = \langle w_p(\tau), n_\mu(s(p))\rangle_{L^2}$.

Claim 18. *For any $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then $|A_\mu(\tau)| < \epsilon|\tau|$.*

Proof.

$$\begin{aligned} A_\mu(\tau) = &- \Big(v(p), n_\mu\big(s(p+\tau)\big) - n_\mu\big(s(p)\big) - Dn_\mu\big(s(p)\big)(\zeta)\Big)_{L^2} \\ &+ \Big(v(p) - v(p+\tau), n_\mu(s(p+\tau)) - n_\mu(s(p))\Big)_{L^2} \end{aligned} \tag{3.261}$$

Therefore,

$$\begin{aligned} |A_\mu(\tau)| \leq &||v(p)||_{L^2} \Big|\Big|n_\mu\big(s(p+\tau)\big) - n_\mu\big(s(p)\big) - Dn_\mu\big(s(p)\big)(\zeta)\Big|\Big|_{L^2} \\ &+ \big|\big|v(p) - v(p+\tau)\big|\big|_{L^2} \big|\big|n_\mu(s(p+\tau)) - n_\mu(s(p))\big|\big|_{L^2} \end{aligned} \tag{3.262}$$

According to claim 11, the fact that s is differentiable, and (3.262), the statement follows. □

Suppose that $\hat{w}_p(\tau)$ is the projection of $w_p(\tau)$ on $\big(T_{s(p)}M_N\big)^\perp$, with respect the L^2 inner product. There holds:

$$\begin{aligned} L_\phi[\phi(s(p)), \alpha(s(p))](\hat{w}_p(\tau)) &= F_{0,p}(\tau) \\ L_{A_j}[\phi(s(p)), \alpha(s(p))](\hat{w}_p(\tau)) &= F_{j,p}(\tau) \end{aligned} \tag{3.263}$$

for $j = 1, 2$ where:

$$\begin{aligned} F_{0,p}(\tau) =& f_0(p+\tau) - f_0(p) - Df_0(p)(\tau) \\ & -2i\sum_{j=1}^{2}\big(\alpha_j(s(p+\tau)) - \alpha_j(s(p))\big)\big(\partial_j v_0(p+\tau) - \partial_j v_0(p)\big) \\ & -2i\sum_{j=1}^{2}\Big(\alpha_j\big(s(p+\tau)\big) - \alpha_j\big(s(p)\big) - \big(D\alpha_j\big)\big(s(p)\big)(\zeta)\Big)\big(\partial_j v_0(p)\big) \\ & -\big(|\alpha|^2(s(p+\tau)) - |\alpha|^2(s(p))\big)\big(v_0(p+\tau) - v_0(p)\big) \\ & -\Big(|\alpha|^2\big(s(p+\tau)\big) - |\alpha|^2\big(s(p)\big) - \big(D|\alpha|^2\big)\big(s(p)\big)(\zeta)\Big)v_0(p) \\ & -\frac{3}{2}\big(|\phi|^2(s(p+\tau)) - |\phi|^2(s(p))\big)\big(v_0(p+\tau) - v_0(p)\big) \\ & -\frac{3}{2}\Big(|\phi|^2(s(p+\tau)) - |\phi|^2(s(p)) - \big(D|\phi|^2\big)\big(s(p)\big)(\zeta)\Big)v_\phi(p) \\ & -2i\sum_{j=1}^{2}\Big(\big(D_{\alpha_j}\phi\big)(s(p+\tau)) - \big(D_{\alpha_j}\phi\big)(s(p))\Big)\big(v_j(p+\tau) - v_j(p)\big) \\ & -2i\sum_{j=1}^{2}\Big(\big(D_{\alpha_j}\phi\big)(s(p+\tau)) - \big(D_{\alpha_j}\phi\big)(s(p)) - D\big(D_{\alpha_j}\phi\big)\big(s(p)\big)(\zeta)\Big)v_{a_j}(p) \end{aligned} \tag{3.264}$$

and:

$$\begin{aligned} F_{j,p}(\tau) =& f_j(p+\tau) - f_j(p) - \big(Df_j(p)\big)(\tau) \\ & +2\big(iv_0(p+\tau) - iv_0(p), D_{\alpha_j}\phi(s(p+\tau)) - D_{\alpha_j}\phi(s(p))\big) \\ & +2\Big(iv_0(p), \big(D_{\alpha_j}\phi\big)\big(s(p+\tau)\big) - \big(D_{\alpha_j}\phi\big)\big(s(p)\big) - D\big(D_{\alpha_j}\phi\big)\big(s(p)\big)(\zeta)\Big) \\ & -\Big(v_j(p+\tau) - v_j(p)\Big)\big(|\phi|^2(s(p+\tau)) - |\phi|^2(s(p))\big) \\ & -v_j(p)\Big(|\phi|^2\big(s(p+\tau)\big) - |\phi|^2\big(s(p)\big) - \big(D|\varphi|^2\big)\big(s(p)\big)(\zeta)\Big) \end{aligned} \tag{3.265}$$

According to estimates 3, the fact that $\alpha \in L^\infty$ and differentiability of s, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then

$$\begin{aligned} & \Big|\Big|\alpha_j\big(s(p+\tau)\big) - \alpha_j\big(s(p)\big) - \big(D\alpha_j\big)\big(s(p)\big)(\zeta)\Big)\Big|\Big|_{L^\infty} < \epsilon|\tau| \\ & \Big|\Big||\alpha|^2\big(s(p+\tau)\big) - |\alpha|^2\big(s(p)\big) - \big(D|\alpha|^2\big)\big(s(p)\big)(\zeta)\Big|\Big|_{L^\infty} < \epsilon|\tau| \\ & \Big|\Big||\phi|^2(s(p+\tau)) - |\phi|^2(s(p)) - \big(D|\phi|^2\big)\big(s(p)\big)(\zeta)\Big|\Big|_{L^\infty} < \epsilon|\tau| \\ & \Big|\Big|\big(D_{\alpha_j}\phi\big)(s(p+\tau)) - \big(D_{\alpha_j}\phi\big)(s(p)) - D\big(D_{\alpha_j}\phi\big)\big(s(p)\big)(\zeta)\Big|\Big|_{L^\infty} < \epsilon|\tau| \end{aligned} \tag{3.266}$$

Therefore, according to claims 11 and 13, equations (3.264), (3.265), and differentiability of f for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then

$$\big|\big|F_{0,p}(\tau)\big|\big|_{L^2}, \big|\big|F_{1,p}(\tau)\big|\big|_{L^2}, \big|\big|F_{2,p}(\tau)\big|\big|_{L^2} < \epsilon|\tau| \tag{3.267}$$

Since $\hat{w}_p(\tau) \in H^2$, according to (3.263),

$$\begin{pmatrix} F_{0,p}(\tau) \\ F_{1,p}(\tau) \\ F_{2,p}(\tau) \end{pmatrix} \perp T_{s(p)} M_N \tag{3.268}$$

Therefore, according to lemma 10, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then $||\hat{w}_p(\tau)||_{H^2} < \epsilon|\tau|$. Therefore, according to claim 18, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\tau| < \delta$, then $|w_p(\tau)|_{H^2} < \epsilon|\tau|$. □

Claim 17 implies proposition 31. □

Lemma 12. *Suppose that*

$$\eta = \Big(\eta_0, \eta_1, \eta_2\Big) \in \Big(H^4(\mathbb{R}^2, \mathbb{C}), H^4(\mathbb{R}^2, \mathbb{R}), H^4(\mathbb{R}^2, \mathbb{R})\Big)$$

Suppose that

$$p = (\phi, \alpha) \in K \subset M_N$$

where K is compact in M_N. Suppose that

$$\eta \perp T_p M_N$$

and

$$\begin{aligned} L_\varphi[\tilde{\varphi}, \tilde{\alpha}] &= \eta_0 \\ L_{A_j}[\tilde{\varphi}, \tilde{\alpha}] &= \eta_j \end{aligned} \tag{3.269}$$

for $j = 1, 2$. Suppose that $|\eta|_{H^1} < M$ and

$$\eta(x) < Ae^{-\gamma|x|}$$

Then, there exist a constant

$$B = B(K, A, M)$$

such that

$$|u(x)| < Be^{-\frac{\gamma}{2}|x|}$$

and

$$|Du(x)| < Be^{-\frac{\gamma}{2}|x|}$$

Proof. According to lemma 10, there exists $R = R(K)$ and $A' = A'(K, A)$ such that if $|x| > R$, then

$$|u(x)| < A'e^{-\frac{\gamma}{2}|x|} \tag{3.270}$$

Suppose that

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \tag{3.271}$$

We define the functions

$$\begin{aligned}
&s : \mathbb{R}^2 \to M_N \\
&\eta_0 : \mathbb{R}^2 \to L^2\big(\mathbb{R}^2, \mathbb{C}\big) \\
&\eta_{_j} : \mathbb{R}^2 \to L^2\big(\mathbb{R}^2, \mathbb{R}\big) \quad\ j = 1,2 \\
&v_0 : \mathbb{R}^2 \to H^2\big(\mathbb{R}^2, \mathbb{C}\big) \\
&v_{_j} : \mathbb{R}^2 \to H^2\big(\mathbb{R}^2, \mathbb{R}\big) \quad\ j = 1,2
\end{aligned} \tag{3.272}$$

by

$$\begin{aligned}
&s(y) = (p - y) \\
&\eta_{_k}(y)(x) = \eta_{A_j}(x - y) \ \ k = 0,1,2 \\
&v_{_y}(y)(x) = u_k(x - y) \ \ k = 0,1,2
\end{aligned} \tag{3.273}$$

where $(p - y)$ refers to translation of the vortex centers by the vector y in the standard coordinate system in $\mathbb{R}^2$. Then, for any $z \in \mathbb{R}^2$, we have:

$$\begin{aligned}
&L_\varphi[\phi(s(z), \alpha(s(z)](v(z)) = \eta_{_0}(z) \\
&L_{A_j}[\phi(s(z)), \alpha(s(z)](v(z)) = \eta_{_j}(z) \ \ j = 1,2
\end{aligned} \tag{3.274}$$

Since $\eta \in H^4$, the function $\eta : \mathbb{R}^2 \to L^2\big(\mathbb{R}^2, \mathbb{R}\big)$ is differetiable. Therefore, by proposition 31, the function $v : \mathbb{R}^2 \to H^2\big(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2\big)$ is differentiable. Therefore, by the Sobolev embedding, the function $u : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$ is differentiable. According to proposition 31 and the Sobolev embedding, there is a constant $C_1 = C_1(K, A)$ such that

$$|\partial_j u(x)| \le C||Dv||_{H^2} \le C_1$$

for any x. Lemma 10 and this imply the statement. □

Lemma 13. *Suppose that $U \subset \mathbb{R}^k$ for some $k > 0$ is an open set and $s : U \to M_N$ is n-times differentiable. Suppose that*

$$f = \big(f_{_0}, f_{_1}, f_{_2}\big) : U \to \Big(L^2\big(\mathbb{R}^2, \mathbb{C}\big), L^2\big(\mathbb{R}^2, \mathbb{R}\big), L^2\big(\mathbb{R}^2, \mathbb{R}\big)\Big) \tag{3.275}$$

is m-times differentiable. Assume

$$f(p) \perp T_{_{s(p)}} M_N \tag{3.276}$$

for each $p \in U$. Suppose that

$$v : U \to \Big(H^2\big(\mathbb{R}^2, \mathbb{C}\big), H^2\big(\mathbb{R}^2, \mathbb{R}\big), H^2\big(\mathbb{R}^2, \mathbb{R}\big)\Big)$$

has the property

$$v(p) \in \big(T_{s(p)} M_N\big)^\perp \tag{3.277}$$

for any $p \in U$ and

$$L[\phi(s(p)), \alpha(s(p))]v(p) = f(p) \tag{3.278}$$

for $j = 1, 2$ and any $p \in U$. Then, v is m-times differentiable, provided $n \geq m$.

Proof. We use induction on m. The base case holds by lemma 13. Suppose that the lemma holds when $m = l$. Let $m = l + 1$. Suppose that $p \in U$. According to proposition 31,

$$\partial_k v = m + E \tag{3.279}$$

where m is an element of $T_{s(p)} M_N$ which satisfies

$$\Big(m, n_\mu\big(s(p)\big)\Big) = -\Big(v(p), \Big(\partial_k\big(n_\mu\big(s(p)\big)\big)\Big) \tag{3.280}$$

for each $\mu \in \{1, 2, 3, 4\}$ and $E = \big(E_0, E_1, E_2\big)$ is an element of

$$H^2\big(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2\big) \cap \big(T_{s(p)} M_N\big)^\perp$$

which satisfies

$$\begin{aligned} L_\varphi[\phi(s(p)), \alpha(s(p))](E_0) =& \partial_k f_0(p) \\ & - \sum_{j=1}^{2} 2i\Big(\partial_k\big(\alpha_j(s(p))\big)\Big)\Big(\big(D_{\alpha_j} v_0\big)(p)\Big) \\ & -2\sum_{j=1}^{2} \alpha_j\big(s(p)\big)\Big(\partial_k\big(\alpha_j(s(p))\big)\Big) v_0(p) \\ & -3\Big(\varphi(p), \partial_k\big(\varphi\big(s(p)\big)\big)\Big) v_0(p) \\ & -2i\sum_{j=1}^{2} v_j(p)\partial_k\big((D_{\alpha_j}\phi)\big(s(p)\big)\big) \end{aligned} \tag{3.281}$$

and:

$$\begin{aligned} L_{A_j}[\phi(s(p)), \alpha(s(p))](E_j) =& \partial_k f_j(p) \\ & +2\Big(iv_\phi(p), \partial_k\big((D_{\alpha_j}\phi)\big(s(p)\big)\big)\Big) \\ & -v_j(p)\partial_k\big(|\phi|^2\big(s(p)\big)\big) \end{aligned} \tag{3.282}$$

for $j = 1, 2$. According to the estimates 3, the functions

$$\partial_k\big(\alpha_1 \circ s\big), \partial_k\big(\alpha_2 \circ s\big), \partial_k\big(\phi \circ s\big), \partial_k\big(|\phi|^2 \circ s\big), \partial_k\big(\big(D_{\alpha_j}\phi\big) \circ s\big) : U \to L^\infty\big(\mathbb{R}^2, \mathbb{R}\big)$$

are $(n-1)$-times differentiable. On the other hand, by the induction hypothesis, the functions

$$v_0, v_1, v_2, \partial_1 v_0, \partial_2 v_0, \partial_k f_0, \partial_k f_1, \partial_k f_2 : U \to L^2\big(\mathbb{R}^2, \mathbb{R}\big)$$

are l-times differentiable. Therefore, according to equations (3.281) and (3.282), we deduce that the functions

$$L_{\varphi}[\phi(s(p)),\alpha(s(p))](E):U\to L^2\big(\mathbb{R}^2,\mathbb{R}\big) \tag{3.283}$$
$$L_{A_j}[\phi(s(p)),\alpha(s(p))](E):U\to L^2\big(\mathbb{R}^2,\mathbb{R}\big) \tag{3.284}$$

for $j=1,2$, are l-times differentiable. Therefore, according to lemma 13, the function $E:U\to H^2\big(\mathbb{R}^2,\mathbb{R}\big)$ is l-times differentiable. Since (3.280) holds for any $\mu\in\{1,2,3,4\}$, by the induction hypothesis, the function $m:U\to H^2\big(\mathbb{R}^2,\mathbb{R}\big)$ is l-times differentiable. Therefore, the function $\partial_k v:U\to H^2\big(\mathbb{R}^2,\mathbb{R}\big)$ is l-times differentiable, for any k. Therefore, the function v is $(l+1)$-times differentiable.

□

Lemma 14. *Suppose that $m\geq 4$, $U\subset\mathbb{R}^k$ is an open set and*

$$f=\begin{pmatrix}f_0\\f_1\\f_2\end{pmatrix}\in\mathcal{E}_m(\mathbb{R}^2,U,\mathbb{C}\times\mathbb{R}^2) \tag{3.285}$$

for each $p\in U$. Suppose that $q:U\to M_N$ is n-times differentiable and $q(U)$ is a precompact subset of M_N and $D^s(q)$ is bounded for every multi-index s with $|s|\leq n$ and

$$f(.;p)\perp T_{q(p)}M_N \tag{3.286}$$

and:

$$\begin{aligned}L_{\varphi}[\phi(q(p))),\alpha(q(p))](u(.;p))&=f_0(.;p)\\L_{A_j}[\phi(q(p)),\alpha(q(p))](u(.;p))&=f_j(.;p)\end{aligned} \tag{3.287}$$

for $j=1,2$, where

$$u(.;p)\in H^2\cap\big(T_{q(p)}M_N\big)^{\perp} \tag{3.288}$$

for each $p\in U$. Then,

$$u\in\mathcal{E}_{m-4}(\mathbb{R}^2,U,\mathbb{C}\times\mathbb{R}^2) \tag{3.289}$$

if n is large enough.

Proof. Suppose that $V=\mathbb{R}^2\times U$ and

$$u=\begin{pmatrix}u_0\\u_1\\u_2\end{pmatrix} \tag{3.290}$$

We define the functions

$$\begin{aligned} &\hat{q} : V \to M_N \\ &g_0 : V \to L^2\big(\mathbb{R}^2, \mathbb{C}\big) \\ &g_j : V \to L^2\big(\mathbb{R}^2, \mathbb{R}\big) \quad j = 1, 2 \\ &v_0 : V \to H^2\big(\mathbb{R}^2, \mathbb{C}\big) \\ &v_j : V \to H^2\big(\mathbb{R}^2, \mathbb{R}\big) \quad j = 1, 2 \end{aligned} \tag{3.291}$$

by

$$\begin{aligned} &\hat{q}(y, p)(x) = q(p)(x - y) \\ &g_k(y, p)(x) = f_j(x - y, p) \ \ k = 0.1, 2 \\ &v_k(y, p)(x) = u_k(x - y, p) \ \ k = 0, 1, 2 \end{aligned} \tag{3.292}$$

for any $x, y \in \mathbb{R}^2$ and any $p \in U$. Then, for any $M \in V$, we have:

$$\begin{aligned} &L_\varphi[\varphi(\hat{q}(M), \alpha(\hat{q}(M)](v(M)) = g_\varphi(M) \\ &L_{A_j}[\varphi(\hat{q}(M)), \alpha(\hat{q}(M)](v(M)) = g_{A_j}(M) \ \ j = 1, 2 \end{aligned} \tag{3.293}$$

Since $f \in \mathcal{E}_m(\mathbb{R}^2, U, \mathbb{C} \times \mathbb{R}^2)$, then the function $g : V \to L^2\big(\mathbb{R}^2, \mathbb{R}\big)$ is m-times differentiable. Therefore, by lemma 13, the function $v : V \to H^2\big(\mathbb{R}^2, \mathbb{R}\big)$ is m-times differentiable. Therefore, the function $u : U \to H^2(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2)$ is m-time differentiable.

Claim 19. *There exists a number $a > 0$ such that for any multi-index r with $|r| < (m - 2)$*

$$||D^r u(x, p)||_{H^2} < a$$

Proof. We proceed by induction on $|r|$. If $|r| = 0$, the claim follows by lemma 10, the fact that $s(U)$ is a precompact subset of M_N and $f \in \mathcal{E}_m(\mathbb{R}^2, U, \mathbb{C} \times \mathbb{R}^2)$. Suppose that the statement holds for $|r| \leq l$ where $l < (m - 3)$. Suppose that r is a multi-index with $|r| = l$. Then, the statement follows by the facts that $q(U)$ is a precompact subset of M_N and $D^s(q)$ is bounded for every multi-index s with $|s| \leq (m - 2)$, the induction hypothesis and lemma 12. □

Claim 20. *For any multi-index r with $|r| < (m - 3)$, there exists $A, \gamma > 0$ such that*

$$D^r u(x, p) < Ae^{-\gamma|x|}$$

and

$$\partial_j D^r u(x, p) < Ae^{-\gamma|x|}$$

for $j = 1, 2$.

Proof. We use induction on $|r|$. Suppose that $|r| = 0$. Then, since $f \in \mathcal{E}_m(\mathbb{R}^2, U, \mathbb{C} \times \mathbb{R}^2)$, according to lemma 10, lemma 12 and claim 19, there exists $A_1, \gamma_1 > 0$ such that

$$u(x, p) < A_1 e^{-\gamma_1|x|}$$

and

$$\partial_j u(x,p) < A_1 e^{-\gamma_1|x|}$$

for $j = 1,2$, for any $p \in \mathbb{R}^2$. Suppose that the statement holds for $|r| \leq k$ where $k < (m-4)$. Let $|r| = k+1$. According to equations (3.287) and the induction hypothesis, we have:

$$\begin{aligned} L_\phi\big[\phi(q(p)),\alpha(q(p))\big](D^r u(.,p)) &= g_0^r(p) \\ L_j\big[\phi(q(p)),\alpha(q(p))\big](D^r u(.,p)) &= g_j^r(p) \quad j=1,2 \end{aligned} \tag{3.294}$$

where

$$g_0^r(p), g_j^r(p) < A_2 e^{-\gamma_2} x$$

for every $p \in U$ and by claim 19, there exists a number $a > 0$ such that

$$||g_0^r(p)||_{H^2}, ||g_j^r(p)||_{H^2} < a$$

for any $p \in U$. Suppose that

$$D^r u(.,p) = m_r(p) + E_r(p) \tag{3.295}$$

where $m_r \in \big(T_{q(p)}M_N\big)^\perp$ and $E_r \in T_{q(p)}M_N$. By claim 19, there exist $A_3, \gamma_3 > 0$ such that

$$E_r(p)(x), \partial_1 E_r(p)(x), \partial_2 E_r(p)(x) < A_3 e^{-\gamma_3|x|}$$

for every $p \in U$. By lemmas 10 and 12, there exist $A_4, \gamma_4 > 0$ such that for any $p \in U$

$$m_r(p)(x), \partial_1 m_r(p)(x), \partial_2 m_r(p)(x) < A_4 e^{-\gamma_4|x|}$$

This finishes the proof. □

Claim 20 imply lemma 14. □

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