DISCRETE COMPLEX ANALYSIS AND THE CONVERGENCE OF OBSERVABLES ON ORTHODIAGONAL MAPS

by

David Pechersky

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Graduate Department of Mathematics
University of Toronto

Abstract

David Pechersky[1]

Doctor of Philosophy

Graduate Department of Mathematics

University of Toronto

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[1] david.pechersky@mail.utoronto.ca

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Chapter 0

Introduction

0.1 Introduction and Motivation

Discrete complex analysis is the study of discrete holomorphic functions. These are functions defined on planar graphs embedded in the plane, that satisfy a discrete analogue of the Cauchy-Riemann equations. Given a graph embedded in the plane, there are many ways one could discretize the Cauchy- Riemann equations. As such, the focus is on classes of graphs embedded in the plane and discretizations of the Cauchy- Riemann equations on these graphs so that discrete holomorphic functions have many of the properties that are characteristic of holomorphic functions in the plane: i.e. discrete holomorphic functions integrate to 0 over closed contours, the real and imaginary parts of a discrete holomorphic function are discrete harmonic, etc.

The subject of discrete complex analysis is classical, going back to the work of Isaacs [31] and Lelong-Ferrand [40]. However, in the past 20- 30 years there has been a renewed interest in the subject as a result of the work of Kenyon [32, 33], Mercat [42, 43] and Smirnov [50] who demonstrated the power of discrete complex analysis as a tool for understanding statistical mechanics in two dimensions, at criticality. Landmark results following from the application of discrete complex analysis to 2D statistical physics include the proof of Cardy's formula for critical percolation [50], the proof of the convergence of interfaces in the critical Ising model to SLE_3 [12], and the convergence of critical percolation interfaces to SLE_6 [8]. Here, SLE_κ refers to Schramm- Loewner evolution, a 1- parameter family of random curves parametrized by $\kappa > 0$ that is uniquely characterized by conformal invariance and a certain Markov- type property.

For all of these results, the story is as follows:

1. Many important models in statistical physics are defined on a lattice, rather than in the continuum. Thus, to understand the large- scale picture that comes out of the microscopic, lattice- level interactions, we need to take the limit of our model as the mesh of the lattice

goes to 0.

2. Physicists have this intuition that, at criticality, 2D statistical physics should be conformally invariant in the limit. As such, many of the limiting observables in these models are either harmonic or holomorphic functions, depending on whether they take real or complex values.

3. This is where discrete complex analysis enters the picture. Provided our lattice accommodates a notion of discrete complex analysis, our first step in rigorously establishing the limiting behavior of our statistical physics model is to identify a discrete holomorphic observable in the model. That is, we find some functional of the statistical physics model that satisfies a discrete version of the Cauchy- Riemann equations.

4. A priori regularity estimates for discrete holomorphic functions tell us that discrete holomorphic observables are precompact with respect to the topology of uniform convergence on compacts. This gives us subsequential limits for our observables as the mesh of our lattice tends to 0. Discrete holomorphicity of the observables tells us that any subsequential limit of our observables is a holomorphic function.

5. Since the limiting observable is holomorphic, we can uniquely identify it by its boundary behavior. Since all the subsequential limits agree, we conclude that our discrete holomorphic observables converge to the relevant continuum limit.

6. Convergence of this observable is then leveraged to prove convergence of the discrete random objects of interest to their continuum limits: i.e. percolation interfaces to SLE_6, interfaces in the Ising model to SLE_3, etc.

Isoradial graphs or rhombic lattices, first introduced by Duffin in [19], are a natural setting for discrete complex analysis. At this point, critical statistical physics on isoradial graphs is well-understood. For instance, the conformal invariance of height functions in the dimer model [41], universality of critical exponents for the random cluster model [20], and the convergence of multipoint spin correlations for the Ising model [13] have all been established when the underlying lattice is isoradial. For more on critical statistical physics on isoradial graphs, see the surveys [10] and [26]. All this is in large part due to work of Chelkak and Smirnov who in [15] prove fundamental results for discrete harmonic and holomorphic functions on isoradial graphs. In particular, they prove convergence of discrete harmonic measure, discrete Poisson kernels and discrete Green's functions on isoradial graphs to their continuous counterparts. This builds on prior work of Kenyon [33], which establishes explicit formulas and asymptotics for the whole-plane Green's function on isoradial graphs.

Orthodiagonal maps are another class of graphs embedded in the plane, with edge weights coming from the geometry of this embedding, that accommodate a notion of discrete complex analysis. They are a strict generalization of isoradial graphs: every isoradial graph is orthodiagonal. Until the recent introduction of t- embeddings by Chelkak, Laslier and Russkikh [14] and independently by Kenyon, Lam, Ramassamy and Russkikh [34] orthodiagonal maps were the most general setting for discrete complex analysis. Most of the well-known planar lattices- i.e. the square lattice, the triangular lattice, the hexagonal lattice- are orthodiagonal. More generally, as a consequence of the double circle packing theorem, a wide variety of planar graphs admit an orthodiagonal embedding (see Section 2 of [25]). In contrast to the isoradial setting, critical 2D statistical physics on general orthodiagonal maps is still poorly understood.

In this thesis, the focus will be on proving convergence results for discrete harmonic and discrete holomorphic functions on orthodiagonal maps. In Chapter 2, which is joint work with my advisor Ilia Binder, we show that for finer and finer orthodiagonal approximations of a simply connected domain Ω with four distinguished boundary points, a certain discrete conformal map, defined on the faces of our orthodiagonal map, converges uniformly on compacts to its continuum analogue. In Chapter 3, we show that for Hölder boundary data, solutions to the Dirichlet problem on orthodiagonal maps converge to the solution of the corresponding continuous Dirichlet problem with a polynomial rate of convergence in the mesh of our orthodiagonal map. Finally, in Chapter 4, we use our estimates for Laplacian of the convolution , we prove Along the way, we prove fundamental results for random walks, discrete harmonic functions, discrete holomorphic functions and discrete extremal length on orthodiagonal maps that are of independent interest. These results constitute a toolbox for extending the results of critical 2D statistical physics on isoradial graphs to the more general orthodiagonal setting.

Our motivation for extending the tools of discrete complex analysis to a wider class of discretizations of 2D space is twofold. On the one hand, it confirms our intuition that the underlying physical phenomena are universal. That is, it shouldn't matter whether we define our statistical physics model on a triangular or on a square grid. As the mesh shrinks to 0, the limiting behavior should be the same.

Perhaps more importantly, extending discrete complex analysis techniques to wider classes of discretizations of the 2D space is expected to have applications to the understanding of statistical physics on random surfaces, which is one of the big open problems in modern probability theory. The story here is as follows:

1. sample a large planar graph uniformly at random in any sort of sensible way. For instance, we could sample a p- angulation (planar map where all the faces are polygons with p sides) with n faces uniformly at random.

2. "decorate" this random planar graph with a critical statistical physics model. For instance, we can decorate our planar graph with a critical Ising model. The probability of seeing a particular random graph is then proportional to the partition function of the critical Ising model on that graph.

3. thinking of the resulting random graph as a compact metric space with the graph metric,

under appropriate rescaling, it should converge in Gromov- Hausdorff distance to some limiting object.

4. the intuition from statistical physics is that the resulting random metric space is γ- Liouville quantum gravity for some parameter γ. For details, see [48].

The trouble with this last step is that Liouville quantum gravity is typically defined as a random metric on the sphere. Thus, to relate the object we got in step 3 to γ- Liouville quantum gravity for some appropriate parameter γ, we need to embed our family of random graphs in the Riemann sphere, so that the complex structure of our embedding is somehow compatible with the complex structure of our graph. The first rigorous results in this direction were recently proven for Cardy embeddings of random triangulations by Holden and Sun [29].

Another approach to uniformizing discrete random geometry is circle packings. Namely, if our planar graph is a triangulation, the Koebe-Andreev-Thurston theorem tells us that this graph has a unique representation as the tangency graph of a circle packing, up to Möbius transformations and reflections (see Chapter 3 of [44]). Thus, as long as our random graphs are triangulations, circle packings give us a natural embedding of our random graphs in the Riemann sphere.

If we can extend discrete complex analysis techniques to planar graphs embedded in the plane according to one of the aforementioned schema, we should be able to use the discrete complex analysis techniques that have been so useful for understanding critical statistical physics in 2D Euclidean space to say something about critical statistical physics on random surfaces. In particular, if we embed a triangulation in the Riemann sphere via circle packing, the resulting embedding is orthodiagonal (see Proposition 2.1 of [25]). Thus, extending discrete complex analysis techniques to orthodiagonal maps should already be sufficient to apply these techniques to the understanding of critical statistical physics on random triangulations.

0.2 Orthodiagonal Maps, Tilings of Rectangles, and their Convergence to Conformal Maps

In Chapter 2 we use discrete complex analysis techniques to solve a purely deterministic problem in the general orthodiagonal setting. Namely, a classic paper of Brooks, Smith, Stone and Tutte describes how planar electrical networks give rise to tilings of rectangles by smaller subrectangles [7]. Each subrectangle in the tiling corresponds to an edge of the network and its aspect ratio is precisely the conductance of this corresponding edge. These tilings can be thought of as discrete analogues of the uniformizing conformal map that maps a simply connected domain to a rectangle so that four distinguished points on the boundary of our simply connected domain are mapped to the four corners of the rectangle. We make this idea rigorous by showing that for any simply connected domain, if we have an increasingly fine sequence of orthodiagonal approximations, the associated tilings converge to the corresponding uniformizing conformal map. This significantly improves on a previous result of Georgakopoulos and Panagiotis who prove this convergence in the case where the approximating orthodiagonal map is just a chunk of the square grid [24]. Furthermore, our approach is significantly different from the one in [24] which relies heavily on the fact that reflected random walks on $\delta\mathbb{Z}^2$ converge in law to reflected Brownian motion as $\delta \to 0$. To our knowledge, this result is not known for any other lattices.

In recent work, Albin, Lind and Pietro- Corradini provide an explicit rate of convergence for these tilings to the limiting conformal map in the general orthodiagonal setting (this is effectively Theorem 3 of [3]), subject to certain assumptions on the smoothness of the boundary of the simply connected domain that is being approximated. They then use this to prove convergence of the probabilistic interpretation of modulus as well as convergence of discrete extremal length to continuous extremal length in this setting [3]. By employing a different approach, we manage to avoid making any assumptions about the smoothness of the boundary of our simply connected domain, at the expense of providing an explicit rate of convergence.

Our result can also be interpreted as the rectangle tiling analogue of similar results that are known for circle packings. As we discussed earlier, the Koebe-Andreev-Thurston theorem tells us that any finite triangulation can be realized as the tangency graph of a circle packing in the plane. With this in mind, Bill Thurston made the observation that if you fill a simply connected domain with circles packed together, the Koebe-Andreev-Thurston theorem gives you a natural way to repack these circles in the unit disk in a way that preserves tangency. Since this "repacking map" sends circles to circles, if we fill our simply connected domain with smaller and smaller circles, the corresponding circles in the images should also get smaller and smaller. In the limit, these repacking maps should converge to a function that sends infinitisimal circles to infinitisimal circles. In other words, a conformal map. Thus, Thurston conjectured that circle packings should give us a way to approximate the uniformizing conformal map from a simply connected domain to the unit disk [53]. This was proven by Rodin and Sullivan when the circle packings in the simply connected domain consist of circles, all having the same radii, packed together in a honeycomb pattern [47]. This was later generalized to circle packings with arbitrary combinatorics by He and Schramm [27]. For more on circle packings and their connection to complex analysis, see [52].

Finally, it is worth noting that the closely related tilings of cylinders have been the object of recent study by Benjamini and Schramm [5], Georgakopoulos [23] and Hutchcroft and Peres [30] in connection with the Poisson boundary of infinite planar graphs.

0.3 A Polynomial Rate of Convergence for the Dirichlet Problem on Orthodiagonal Maps

Due to the ubiquity of diffusion phenomenon in the physical world, the Dirichlet problem is one of the most important partial differential equations in mathematical physics. Recently, Gurel-Gurevich, Jerison and Nachmias showed that solutions to the Dirichlet problem on orthodiagonal maps converge to the solution of the corresponding continuous Dirichlet problem [25]. This improves on prior work of Chelkak and Smirnov [15], Skopenkov [49] and Werness [54], where this result is proven for the Dirichlet problem on orthodiagonal maps, subject to various additional regularity assumptions on the underlying lattice. In particular, Theorem 1.1 of [25] provides an explicit rate of convergence for the Dirichlet problem on orthodiagonal maps to the corresponding continuous Dirichlet problem for C^2 boundary data. In Chapter 3, we improve upon the rate of convergence in Theorem 1.1 of [25], by showing that as long as our boundary data is Hölder, we have a polynomial rate of convergence for the Dirichlet problem on orthodiagonal maps to the corresponding continuous Dirichlet problem.

0.4 Lipschitz Regularity on a Mesoscopic Scale for Harmonic Functions on Orthodiagonal Maps

Suppose Ω is a subdomain of $\mathbb{R}^2$ and $h : \Omega \to \mathbb{R}$ is harmonic. Then the classical Harnack estimate says that for any $x, y \in \Omega$ so that $|x - y| \leqslant d = \text{dist}(x, \partial\Omega) \wedge \text{dist}(y, \partial\Omega)$, we have that:

$$|h(y) - h(x)| \leqslant 2 \|h\|_{L^\infty(\Omega)} \left(\frac{|x - y|}{d} \right) \tag{0.4.1}$$

This result follows readily from the mean value property for harmonic functions. Namely, if $x \in \Omega$ and $r < \text{dist}(x, \partial\Omega)$, we have that:

$$h(x) = \frac{1}{\pi r^2} \int_{B(x,r)} h(u) dA(u) \tag{0.4.2}$$

where "$dA(u)$" refers to integration with respect to area on Ω. Insofar as orthodiagonal maps are good approximations of continuous 2D space, we should expect that something like this is true for discrete harmonic functions on orthodiagonal maps. Indeed, in the more restricted setting of isoradial graphs with angles uniformly bounded away from 0 and π, Chelkak and Smirnov show that an anologue of the Harnack estimate holds (see Corollary 2.9 of [15]). Just as in the continuous setting, the result follows from an analogue of the integral mean value property in Equation 0.4.2 for discrete harmonic functions (see Proposition A.2 of [15]). The proof of this mean value property for discrete harmonic functions on isoradial graphs requires asymptotics for the discrete Green's function on isoradial graphs, proven by Kenyon (see Theorem 7.3 of [33]). It is expected that these estimates should also hold for the discrete Green's function on general orthodiagonal maps, however, this has yet to be proven. In light of this, to prove Harnack-type estimates for discrete harmonic functions on orthodiagonal maps, we will use a different approach.

Namely, in Chapter 3, we show that if you convolve a discrete harmonic function with a smooth mollifier, the resulting continuous function is "almost" harmonic in that its Laplacian is small in a precise quantitative sense. It turns out that to have a Harnack-type estimate like the one in Equation 0.4.1, we do not need our function to be harmonic. Being almost harmonic is enough. Thus, we have a Harnack-type estimate for the convolution of a discrete harmonic function with a smooth mollifier.

Regularity estimates for discrete harmonic functions on orthodiagonal maps tell us that that our original discrete harmonic function is close to its convolution with a smooth mollifier, provided the support of this smooth mollifier is small. Since discrete harmonic functions on orthodiagonal maps are close to continuous functions that satisfy a Harnack-type estimate, we conclude that we also have a Harnack-type estimate for discrete harmonic functions on orthodiagonal maps, at least on a mesoscopic scale.

Chapter 1

Preliminaries

1.1 The Theory of Electrical Networks

Following [25], a finite **network** is a finite graph $G = (V, E)$ along with a weight function $c : E \to \mathbb{R}_{>0}$. For any edge $e \in E$ we say that $c(e)$ is the **conductance** of that edge. The reciprocal $r(e) = \frac{1}{c(e)}$ is the **resistance** of that edge.

A function $\theta : \vec{E} \to \mathbb{R}$ is said to be **antisymmetric** if $\theta(-\vec{e}) = -\theta(\vec{e})$ for all $\vec{e} \in \vec{E}$. Intuitively, antisymmetric functions on a network G are the discrete analogues of vector fields. Let $\ell^2_-(\vec{E})$ denote the space of antisymmetric functions on $\vec{E}$ with the inner product:

$$\langle \theta, \psi \rangle_r := \frac{1}{2} \sum_{\vec{e} \in \vec{E}} r(e) \theta(\vec{e}) \psi(\vec{e})$$

The **energy** of $\theta \in \ell^2_-(\vec{E})$ is:

$$\mathcal{E}(\theta) = \|\theta\|_r^2 = \langle \theta, \theta \rangle_r$$

Given $f : V \to \mathbb{R}$ its **gradient** $cdf : \vec{E} \to \mathbb{R}$ is given by:

$$(cdf)(\vec{e}) = c(e)(f(e^+) - f(e^-))$$

For any function $f : V \to \mathbb{R}$, the gradient is antisymmetric. Thus, we can define the energy of a function $f : V \to \mathbb{R}$ as the energy of its gradient:

$$\mathcal{E}(f) = \mathcal{E}(cdf) = \frac{1}{2} \sum_{\vec{e} \in \vec{E}} c(e) \big(f(e^+) - f(e^-)\big)^2$$

Given a function in $\ell^2_-(\vec{E})$, we are often interested in the energy of its restriction to some subgraph of G. To make it clear where it is we are computing the energy, if $\theta \in \ell^2_-(\vec{E})$ and $G' = (V', E')$

is a subgraph of $G = (V, E)$ with the same edge weights, then:

$$\mathcal{E}(\theta; G') = \frac{1}{2} \sum_{\vec{e} \in \vec{E}'} r(e)\theta(\vec{e})^2$$

Similarly, if $f : V \to \mathbb{R}$,

$$\mathcal{E}(f; G') = \frac{1}{2} \sum_{\vec{e} \in \vec{E}'} c(e)\big(f(e^+) - f(e^-)\big)^2$$

A function $\theta \in \ell^2_-(\vec{E})$ satisfies the **cycle law** if for any directed cycle $\gamma = (\vec{e}_1, \vec{e}_2, ..., \vec{e}_m)$ in G,

$$\sum_{i=1}^{m} r(e_i)\theta(\vec{e}_i) = 0$$

It is not hard to see that $\theta \in \ell^2_-(\vec{E})$ satisfies the cycle law if and only if $\theta = cdf$ for some function $f : V \to \mathbb{R}$. Given $\theta \in \ell^2_-(\vec{E})$, its **divergence** $\mathrm{div}(\theta) : V \to \mathbb{R}$ is given by:

$$\big(\mathrm{div}(\theta)\big)(x) = \sum_{e^- = x} \theta(\vec{e})$$

Similar to the continuous setting, the divergence of θ at x measures the net flow out of x by θ. Given distinct vertices $a, z \in V$, a function $\theta \in \ell^2_-(\vec{E})$ is a **flow** from a to z if:

$$\big(\mathrm{div}(\theta)\big)(x) = 0 \text{ for all } x \in V \backslash \{a, z\}.$$

Given a flow θ from a to z its **strength**, denoted by $\|\theta\|$, is defined as follows:

$$\|\theta\| = \sum_{x : x \sim a} \theta(a, x) = \big(\mathrm{div}(\theta)\big)(a)$$

For every flow θ from a to z,

$$\|\theta\| = \sum_{y : y \sim z} \theta(y, z) = -\big(\mathrm{div}(\theta)\big)(z)$$

This is because:

$$\sum_{\vec{e} \in \vec{E}} \theta(\vec{e}) = 0 = \sum_{x \in V} \sum_{y : y \sim x} \theta(x, y) = \sum_{x \in V} \big(\mathrm{div}(\theta)\big)(x) = \big(\mathrm{div}(\theta)\big)(a) + \big(\mathrm{div}(\theta)\big)(z)$$

The first equality follows from the antisymmetry of θ. Given $f : V \to \mathbb{R}$, its **Laplacian** $\Delta f : V \to \mathbb{R}$ is given by:

$$\Delta f(x) = \big(\mathrm{div}(cdf)\big)(x) = \sum_{e^- = x} c(e)\big(f(e^+) - f(e^-)\big) = \sum_{y : y \sim x} c(x, y)\big(f(y) - f(x)\big)$$

If $\Delta f(x) = 0$, we say that f is **harmonic** at x. Equivalently, f is harmonic at x if:

$$f(x) = \frac{1}{\pi_x} \sum_{y:y\sim x} c(x,y) f(y)$$

where:

$$\pi_x = \sum_{y:y\sim x} c(x,y)$$

From this formula, it is immediate that harmonic functions satisfy the maximum principle:

Proposition 1.1.1. Suppose that $G = (V, E, c)$ is a finite network and $h : V \to \mathbb{R}$ is harmonic on $U \subsetneq V$. Define:

$$\partial U = \{w \in V \backslash U : w \sim u \text{ for some } u \in U\}$$

Then:

$$\max_{u\in U} h(u) \leqslant \max_{v\in \partial U} h(v)$$

Following [44], a simple random walk on the network $G = (V, E, c)$ is the discrete time Markov process $(X_n)_{n\geqslant 0}$ with transition probabilities:

$$P(x,y) = \frac{c(x,y)}{\pi_x} 1_{(x\sim y)}$$

Given a function $f : V \to \mathbb{R}$ and vertices $a, z \in V$, it is clear that *cdf* is a flow from a to z if and only if $\Delta f(x) = 0$ for all $x \in V\backslash\{a, z\}$. We call such a function a **voltage**. Since the discrete boundary value problem:

$$\begin{aligned} h(a) &= \alpha \\ h(b) &= \beta \\ \Delta h(x) &= 0 \text{ for all } x \in V\backslash\{a,z\} \end{aligned}$$

has a unique solution for any choice of $\alpha, \beta \in \mathbb{R}$, voltages form a two-parameter family. The flow *cdh* corresponding to any voltage $h : V \to \mathbb{R}$ is known as the corresponding **current flow**. Given distinct vertices $a, z \in V$, the **effective resistance** between a and z in G, denoted by $R_{\text{eff}}(a \leftrightarrow z; G)$, is given by:

$$R_{\text{eff}}(a \leftrightarrow z; G) = \frac{h(z) - h(a)}{\|cdh\|}$$

where h is any nonconstant voltage. To see that this quantity is well- defined, just observed that adding a constant doesn't affect the voltage difference between a and z, $h(z) - h(a)$, or the current flow *cdh*. Similarly, multiplying h by a nonzero constant scales the voltage difference and the strength of the corresponding current flow by the same factor, leaving the effective resistance unchanged.

More generally, given disjoint sets of vertices $A, Z \subseteq V$, we can define a new network by identifying the vertices of A to a single vertex a and identifying the vertices of Z to a single vertex z. Then the effective resistance $R_{\text{eff}}(A \leftrightarrow Z; G)$ between A and Z in G, is given by the electrical resistance between the vertices a and z in this new network.

In this paper, we will frequently need to bound effective resistances from above and below. To do this, we will use the following pair of variational formulas. Dirichlet's Principle allows us to bound effective resistances from below by finding functions with small discrete Dirichlet energy:

Proposition 1.1.2 (Dirichlet's Principle)**.** If $G = (V, E, c)$ is a finite network with distinct vertices $a, z \in V$ then:

$$R_{\text{eff}}(a \leftrightarrow z; G) = \sup \left\{ \frac{1}{\mathcal{E}(h)} : h : V \to \mathbb{R},\ h(a) = 0,\ h(z) = 1 \right\}$$

Thomson's Principle allows us to bound effective resistances from above, by finding low- energy flows:

Proposition 1.1.3 (Thomson's Principle)**.** If $G = (V, E, c)$ is a finite network with distinct vertices $a, z \in V$ then:

$$R_{\text{eff}}(a \leftrightarrow z; G) = \inf \{ \mathcal{E}(\theta) : \|\theta\| = 1,\ \theta \text{ is a flow from } a \text{ to } z \}$$

Given $f : V \to \mathbb{R}$ and $A, B \subseteq V$ nonempty, disjoint sets of vertices, we define the quantity $\text{gap}_{A,B}(f)$ as follows:

$$\text{gap}_{A,B}(f) = \min_{b \in B} f(b) - \max_{a \in A} f(a)$$

Recall Proposition 4.11 of [25] which tells us that:

Proposition 1.1.4. If $G = (V, E, c)$ is a finite network, $A, B \subseteq V$ are disjoint, nonempty sets of vertices then for any flow θ on G and any function $f : V \to \mathbb{R}$ such that $\text{gap}_{A,B}(f) \geqslant 0$,

$$\|\theta\| \cdot \text{gap}_{A,B}(f) \leqslant \mathcal{E}(\theta)^{1/2} \mathcal{E}(f)^{1/2}$$

This inequality follows almost immediately from the Cauchy- Schwarz inequality on $\ell^2_-(\vec{E})$. Furthermore, Dirichlet's Principle and Thomson's Principle can both be recovered cheaply as corollaries to this inequality.

1.2 Extremal Length and Planar Networks

Suppose $G = (V, E, c)$ is a finite network and Γ is a nonempty collection of paths in G. Then the **extremal length** of the collection of paths Γ is given by the following variational formula:

$$\lambda(\Gamma, G) := \sup_{\rho} \frac{\ell^2(\rho, \Gamma)}{A(\rho)} \tag{1.2.1}$$

where our supremum is taken over all nonzero metrics $\rho : E \to \mathbb{R}_{\geqslant 0}$ and:

$$\ell(\rho, \Gamma) := \min\{\sum_{e \in \gamma} \rho(e) : \gamma \in \Gamma\}, \qquad\qquad A(\rho) := \sum_{e \in E} c(e)\rho(e)^2$$

Note that the quantity $\ell^2(\rho, \Gamma)/A(\rho)$ doesn't change if we replace ρ by some scalar multiple $\lambda\rho$ where $\lambda > 0$. Thus:

$$\lambda(\Gamma, G) = \sup_{\rho} \frac{l^2(\rho, \Gamma)}{A(\rho)} = \sup_{A(\rho)=1} l^2(\rho, \Gamma) = \sup_{l(\rho,\Gamma)=1} \frac{1}{A(\rho)} = \big(\inf_{l(\rho,\Gamma)=1} A(\rho)\big)^{-1}$$

The set of metrics ρ on G such that $l(\rho, \Gamma) = 1$ is referred to as the set of **admissible metrics** and is denoted by $\mathcal{A}(\Gamma)$. We say that a metric ρ on G is **extremal** for $\lambda(\Gamma, G)$ if $\lambda(\Gamma, G) = l^2(\Gamma, G)/A(\rho)$. Looking at the second equality above, we see that when we compute the extremal length of the path family Γ, we are optimizing a continuous function, $\rho \mapsto l^2(\rho, \Gamma)$, over the set of $\rho \in \mathbb{R}^E_{\geqslant 0}$ such that $A(\rho) = 1$. This is a compact subset of $\mathbb{R}^E$ with respect to the standard topology on $\mathbb{R}^E$. Thus, in contrast to the continuous setting (see problem IV.9 of [22]), for a finite network we always have an extremal metric.

This extremal metric is unique up to multiplication by a scalar. This follows by the same argument as the in the continuous setting: suppose that ρ_1 and ρ_2 are both extremal for $\lambda(\Gamma, G)$. First we rescale so that $A(\rho_1) = A(\rho_2) = 1$. It follows that $l^2(\Gamma, \rho_1) = l^2(\Gamma, \rho_2) = \lambda(\Gamma, G)$. Consider the metric $\nu := \frac{1}{2}(\rho_1 + \rho_2)$. Trivially,

$$l(\nu, \Gamma) \geqslant \frac{1}{2}\big(l(\rho_1, \Gamma) + l(\rho_2, \Gamma)\big) = \sqrt{\lambda(\Gamma, G)} \implies l^2(\Gamma, \nu) \geqslant \lambda(\Gamma, G) \tag{1.2.2}$$

On the other hand, by Cauchy- Schwartz, $A(\nu) \leqslant \frac{1}{2}\big(A(\rho_1) + A(\rho_2)\big) = 1$ with equality iff ρ_1 is a scalar multiple of ρ_2. If $A(\nu) < 1$ then $l^2(\nu, \Gamma)/A(\nu) > \lambda(\Gamma, G)$. This is not possible since $\lambda(\Gamma, G)$ is the supremum of $l^2(\rho, \Gamma)/A(\rho)$ over all metrics ρ. Thus, ρ_2 is a scalar multiple of ρ_1. Since $A(\rho_1) = A(\rho_2) = 1$ we actually have that $\rho_1 = \rho_2$.

In all of the cases we're interested in, the path family Γ will be the set of paths γ in G that start at a vertex of S and end at a vertex of T for S, T nonempty disjoint subsets of V. We denote the extremal length of this path family by $\lambda(S \leftrightarrow T; G)$. It turns out that the quantity $\lambda(S \leftrightarrow T; G)$ is precisely the effective resistance between S and T from the theory of electrical networks:

Proposition 1.2.1. (Theorem 2 of [18]) Suppose $G = (V, E, c)$ is a finite network and S, T are nonempty, disjoint subsets of V. Let $\Gamma_{S,T}$ denote the set of nearest- neighbor paths in G that start at a vertex of S and end at a vertex of T. Then:

$$\lambda(\Gamma_{S \leftrightarrow T}, G) = R_{\text{eff}}(S \leftrightarrow T; G)$$

One nice property of extremal length is blocking duality. Given, S, T nonempty, disjoint sets of vertices in G, we say that a set $F \subset E$ is an S**-** T **cut** if F separates S from T in G. That is, if we remove the edges of F from G, there is no nearest- neighbor path in G starting at a vertex of S and ending at a vertex of T.

Let $B(S, T; G)$ denote the set of S- T cuts in G. Analogous to how we defined the extremal length of a path family, we can talk about the extremal length of the set of S- T cuts in G. This is denoted by $\lambda(S \nleftrightarrow T; G)$ and defined as follows:

$$\lambda(S \nleftrightarrow T; G) = \sup_{\rho} \frac{\ell^2(\rho, B(S, T; G))}{A(\rho)}$$

where our supremum is taken over all nonzero metrics $\rho : E \to \mathbb{R}_{\geqslant 0}$, and:

$$\ell(\rho, B(S, T; G)) = \min\{\sum_{e \in F} \rho(e) : F \in B(S, T; G)\}, \qquad A(\rho) = \sum_{e \in E} c(e)\rho(e)^2 \tag{1.2.3}$$

More generally, while we initially restricted our attention to path families so as to draw parallels with the continuous theory, it is clear that if we let Γ be any family of multisets of edges in G, definition 1.2.1 still makes sense. Thus, we can actually talk about the extremal length of any family of multisets of edges of G. For instance, the modulus of the set of spanning trees of a network has been the subject of recent study [2].

A classic result of Ford and Fulkerson relates the extremal length of paths from S to T to the extremal length of the set of S- T cuts:

Proposition 1.2.2. (Theorem 1 of [21]) If $G = (V, E, c)$ and $H = (V, E, r)$ are finite networks so that $r : E \to \mathbb{R}_{>0}$ is the resistance function corresponding to the conductance function $c : E \to \mathbb{R}_{>0}$ and S, T are nonempty, disjoint sets of vertices in G, then:

$$\lambda(S \leftrightarrow T; G) \cdot \lambda(S \nleftrightarrow T; H) = 1$$

For a more modern treatment of this result as well as a generalization to the case of p- extremal length, see [1]. This result is particularly useful in the case where our graph G is planar, in which case we can identify the set of S- T cuts with path families in the dual graph.

A **finite planar map** is a finite planar graph (V, E) along with a proper embedding of this graph into the Riemann sphere, viewed up to homeomorphism of the Riemann sphere. Specifying a proper embedding of a graph in the Riemann sphere up to orientation- preserving homeomorphism is equivalent to assigning a coherent system of orientations to the edges about each vertex (for details, see Section 1.1.2 of [17]). Thus, despite the topology present in our initial definition, planar maps can be viewed as purely combinatorial objects. Equivalently, we can think of finite planar maps as gluings of polygons along edges so that the resulting topological manifold is a sphere.

A **quadrangulation with boundary** is a bipartite planar map all of whose faces are quadrilaterals, with the possible exception of some finite number of distinguished faces which we think of as "holes" in our planar map. Notice that requiring our planar map to be bipartite is equivalent to asking that all of these "hole" faces have an even number of sides. Given a quadrangulation with boundary $G = (V, E)$, we refer to these distinguished faces as the **exterior** faces of G. The remaining faces are called the **interior** faces of G. The edges and vertices tangent to the exterior faces of G are known as the boundary vertices and edges of G. We denote these by ∂V and ∂E. A quadrangulation with boundary $G = (V, E)$ is **simply- connected** if G has a unique exterior face whose boundary is a simple, closed curve.

Since our quadrangulations are bipartite, we have a natural bipartition of the vertices $V = V^\bullet \sqcup V^\circ$. The vertices of $V^\bullet$ are known as the **primal** vertices of G and are typically colored black. The vertices of V° are known as the **dual** vertices and are typically colored white. These give rise to the primal and dual graphs $G^\bullet = (V^\bullet, E^\bullet)$ and $G^\circ = (V^\circ, E^\circ)$. $G^\bullet$ is formed by connecting any pair of primal vertices that share an interior face in G. Similarly, G° is formed by connecting any pair of dual vertices that share an interior face in G. Since the interior faces of G are all quadrilaterals, each interior face corresponds to one primal and one dual edge. In this way, there is a natural correspondence between the primal and dual edges.

Based on the paragraph above, it might seem that the setting we are working in is very restrictive. On the contrary, observe that this procedure of recovering a graph $G^\bullet$ (and its dual G°) from a quadrangulation G gives us a one- to- one correspondence between the set of quadrangulations with n faces and the set of planar maps with n edges (see Section 2.2.1 of [17]). In other words, restricting our attention to bipartite quadrangulations with k holes is equivalent to restricting our attention to embeddings of a graph and its dual in the Riemann sphere, up to orientation- preserving homeomorphism, so that the resulting discrete object has the topology of the Riemann sphere with k discs removed.

Given a quadrangulation with boundary $G = (V^\bullet \sqcup V^\circ, E)$, a **conformal metric** on G is a function $c : E^\bullet \sqcup E^\circ \to (0, \infty))$ such that:

$$c(e^\circ) = \frac{1}{c(e^\bullet)}$$

for $e^\circ \in E^\circ$, $e^\bullet \in E^\bullet$ so that e° is the dual edge corresponding to the primal edge $e^\bullet$. Let $c^\bullet : E^\bullet \to (0, \infty)$ and $c^\circ : E^\circ \to (0, \infty)$ denote the conductances on $G^\bullet$ and G° produced by restricting c to $E^\bullet$ and E° respectively. If $\theta \in \ell^2_-(\vec{E}^\bullet)$ and $f : V^\bullet \to \mathbb{R}$, we write:

$$\mathcal{E}^\bullet(\theta) = \mathcal{E}(\theta; G^\bullet), \qquad \mathcal{E}^\bullet(f) = \mathcal{E}(f; G^\bullet)$$

to emphasize that these energies are being computed on the primal graph $G^\bullet$. Similarly, given a subgraph H of $G^\bullet$, we write:

$$\mathcal{E}^\bullet(\theta; H) = \mathcal{E}(\theta; H), \qquad \qquad \mathcal{E}^\bullet(f; H) = \mathcal{E}(f; H)$$

Given $\omega \in l^2_-(\vec{E}^\circ)$, $g : V^\circ \to \mathbb{R}$, and a subgraph H of G°, the quantities $\mathcal{E}^\circ(\omega)$, $\mathcal{E}^\circ(g)$, $\mathcal{E}^\circ(\omega; H)$, $\mathcal{E}^\circ(g; H)$ are defined analogously.

A **discrete conformal rectangle** is a simply- connected, bipartite quadrangulation with boundary endowed with a conformal metric $c : E^\bullet \sqcup E^\circ \to (0, \infty)$ and four distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$, listed in counterclockwise order: since our quadrangulation with boundary is simply- connected, it has a unique exterior face f, so $A^\bullet, B^\bullet, C^\bullet, D^\bullet$ must all lie along f. Furthermore, having embedded our quadrangulation into the Riemann sphere, we can talk about orientation.

The distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ of our discrete conformal rectangle give rise to primal boundary arcs $[A^\bullet, B^\bullet], [C^\bullet, D^\bullet] \subseteq \partial V^\bullet$. $[A^\bullet, B^\bullet]$ refers to the set of primal vertices that lie along the counterclockwise path from $A^\bullet$ to $B^\bullet$ along the boundary of f. Similarly, $[C^\bullet, D^\bullet]$ is the set of primal vertices that lie along the counterclockwise path from $C^\bullet$ to $D^\bullet$ along the boundary of f. These primal boundary arcs have corresponding dual arcs $[B^\circ, C^\circ], [D^\circ, A^\circ] \subseteq \partial V^\circ$ where $[B^\circ, C^\circ]$ consists of the set of dual vertices that lie along the counterclockwise path from $B^\bullet$ to $C^\bullet$ along the boundary of f. Similarly, $[D^\circ, A^\circ]$ is the set of dual vertices that lie along the counterclockwise path from $D^\bullet$ to $A^\bullet$ along the boundary of f.

We say that an S- T cut, $F \subseteq E$, is **minimal** if for any edge $e \in F$, $F \backslash \{e\}$ is no longer an S- T cut.

Lemma 1.2.3. (Lemma VIII.1 of [18]) If (G, c) is a discrete conformal rectangle with distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ giving rise to primal boundary arcs $[A^\bullet, B^\bullet]$, $[C^\bullet, D^\bullet] \subseteq \partial V^\bullet$ with corresponding dual arcs $[B^\circ, C^\circ], [D^\circ, A^\circ] \subseteq \partial V^\circ$, then the set of minimal $[A^\bullet, B^\bullet]$- $[C^\bullet, D^\bullet]$ cuts in $G^\bullet$ is in one- to- one correspondence with the set of simple paths from $[B^\circ, C^\circ]$ to $[D^\circ, A^\circ]$ in G°.

When computing the extremal length of the family of paths between disjoint vertex sets S and T in G, for a fixed metric ρ, we are interested in the quantity:

$$\inf_\gamma \left\{ \sum_{e \in \gamma} \rho(e) \right\}$$

where our infimum is taken over all paths γ in G between S and T. Since any such path that isn't simple has a simple subpath of smaller ρ- weight, when taking this infimum, it actually suffices to restrict our attention to simple paths γ from S to T. Similarly, when we compute the extremal length of the set of S- T cuts in a network, rather than taking an infimum over all S- T cuts, it suffices to restrict our attention only to minimal S- T cuts. Thus, as an immediate corollary of Lemma 1.2.3, we have that:

Corollary 1.2.1. *If (G, c) is a discrete rectangle with distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ giving rise to primal boundary arcs $[A^\bullet, B^\bullet], [C^\bullet, D^\bullet] \subseteq \partial V^\bullet$ with corresponding dual arcs $[B^\circ, C^\circ], [D^\circ, A^\circ] \subseteq \partial V^\circ$, then:*

$$\lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; (G^\bullet, c^\bullet)) = \lambda([B^\circ, C^\circ] \leftrightarrow [D^\circ, A^\circ]; (G^\circ, c^\circ))$$

Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is a bipartite quadrangulation with boundary endowed with a conformal metric $c : E^\bullet \sqcup E^\circ \to \mathbb{R}$. We say that $h : V^\bullet \to \mathbb{R}$ is harmonic on $G^\bullet$ if h is harmonic on $\mathrm{Int}(V^\bullet) = V^\bullet \backslash \partial V^\bullet$. $\widetilde{h} : V^\circ \to \mathbb{R}$ is the **harmonic conjugate** of h on G if for any interior face f of G, we have that:

$$\big(\widetilde{h}(w_2) - \widetilde{h}(w_1)\big) = c^\bullet(v_1, v_2)\big(h(v_2) - h(v_1)\big) \tag{1.2.4}$$

where v_1, w_1, v_2, w_2 are the vertices of f listed in counterclockwise order so that $v_1, v_2 \in V^\bullet$ and $w_1, w_2 \in V^\circ$. Equation 1.2.4 is a discrete analogue of the Cauchy- Riemann equations for a quadrangulation with boundary G, endowed with a conformal metric. It is easy to check that the conjugate of a harmonic function on $G^\bullet$ is harmonic on G°. Additionally, since c is a conformal metric, if $\widetilde{h}$ is the harmonic conjugate of h, then h is the harmonic conjugate of $\widetilde{h}$. The next two propositions are well- known, though they are rarely stated in this generality:

Proposition 1.2.4. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is a bipartite quadrangulation with boundary endowed with a conformal metric $c : E^\bullet \sqcup E^\circ \to \mathbb{R}$. If $h : V^\bullet \to \mathbb{R}$ is harmonic on $G^\bullet$, then the harmonic conjugate of h, if it exists, is unique up to an additive constant.

Proposition 1.2.5. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is a bipartite quadrangulation with boundary endowed with a conformal metric $c : E^\bullet \sqcup E^\circ \to \mathbb{R}$ and $h : V^\bullet \to \mathbb{R}$ is harmonic on $G^\bullet$. Let $H = (V_H^\bullet \sqcup V_H^\circ, E_H)$ be submap of G which is itself a simply- connected, bipartite quadrangulation with boundary. Then h has a harmonic conjugate $\widetilde{h} : V_H^\circ \to \mathbb{R}$ on H.

Remark 1.2.2. Ford- Fulkerson duality is a general statement that holds for any finite network. However, in the case of discrete rectangles, it has a particularly simple proof stemming from the fact that if h is the function on $V^\bullet$ that is equal to 0 on $[A^\bullet, B^\bullet]$, 1 on $[C^\bullet, D^\bullet]$ and is harmonic elsewhere so that:

$$\lambda^\bullet = \lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; G) = \frac{1}{\mathcal{E}^\bullet(h)}$$

its harmonic conjugate $\widetilde{h}$ is (up to an additive constant) equal to 0 on $[B^\circ, C^\circ]$, $\frac{1}{\lambda^\bullet}$ on $[D^\circ, A^\circ]$ and is harmonic elsewhere. Furthermore, by the discrete Cauchy-Riemann equations, $\mathcal{E}^\circ(\widetilde{h}) = \mathcal{E}^\bullet(h)$. Hence:

$$\lambda^\circ = \lambda([B^\circ, C^\circ] \leftrightarrow [D^\circ, A^\circ]; G^\circ) = \frac{(1/\lambda^\bullet)^2}{\mathcal{E}^\circ(\widetilde{h})} = \frac{(1/\lambda^\bullet)^2}{(1/\lambda^\bullet)} = \frac{1}{\lambda^\bullet}$$

1.3 Tilings of Rectangles

Suppose $\Omega \subseteq \mathbb{C}$ is a Jordan domain with analytic boundary and distinguished boundary points A, B, C, D listed in counterclockwise order. Let $[A, B], [B, C], [C, D], [D, A]$ denote the closed boundary arcs stretching counterclockwise from A to B, B to C, C to D and D to A along $\partial\Omega$. Let $(A, B), (B, C), (C, D), (D, A)$ be the corresponding open boundary arcs. Let $\widetilde{h}$ be the solution to the following boundary value problem:

$$\begin{aligned} \widetilde{h}(x) &= 0 \text{ for } x \in [B, C] \\ \Delta\widetilde{h}(x) &= 0 \text{ for } x \in \Omega \\ \widetilde{h}(x) &= 1 \text{ for } x \in [A, D] \\ \partial_n\widetilde{h}(x) &= 0 \text{ for } x \in (B, C) \cup (D, A) \end{aligned}$$

The existence and uniqueness of the solution to this problem is clear by conformal invariance. Namely, if L is the extremal length from $[A, B]$ to $[C, D]$ in Ω and $\mathcal{R}_L := (0, L) \times (0, 1)$ then there is a unique conformal map $\phi : \Omega \to \mathcal{R}_L$ sending A, B, C, D to the corners of $\mathcal{R}_L$. Since our boundary value problem is conformally invariant, $\widetilde{h}(x) = Im(\phi(x))$. Furthermore, the conjugate harmonic function $h(x) = Re(\phi(x))$ satisfies:

$$\begin{aligned} h(x) &= 0 \text{ for } x \in [A, B] \\ h(x) &= L \text{ for } x \in [C, D] \\ \partial_n h(x) &= 0 \text{ for } x \in (B, C) \cup (D, A) \end{aligned}$$

Thus, if we are presented with a Jordan domain Ω with four distinguished boundary points A, B, C, D listed in counterclockwise order and $\phi : \Omega \to \mathcal{R}_L$ is the conformal map that maps A, B, C, D to the corners of $\mathcal{R}_L$, we can intuitively think of the real and imaginary parts of this conformal map as solving the aforementioned boundary value problems.

Suppose (G, c) is a discrete rectangle with distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ listed in counterclockwise order, giving rise to primal boundary arcs $[A^\bullet, B^\bullet], [C^\bullet, D^\bullet] \subseteq \partial V^\bullet$ and corresponding dual boundary arcs $[B^\circ, C^\circ], [D^\circ, A^\circ] \subseteq \partial V^\circ$. Let $\widetilde{h}$ be the solution to the following boundary value problem on (G°, c°):

$$\begin{aligned} \widetilde{h}(x) &= 0 \;\text{ for } x \in [B^\circ, C^\circ] \\ \widetilde{h}(x) &= 1 \;\text{ for } x \in [D^\circ, A^\circ] \\ \Delta^\circ\widetilde{h}(x) &= 0 \;\text{ for } x \in V^\circ \backslash ([B^\circ, C^\circ] \cup [D^\circ, A^\circ]) \end{aligned}$$

Let h be the solution to the following boundary value problem on $(G^\bullet, c^\bullet)$:

$$\begin{aligned} h(x) &= 0 \text{ for } x \in [A^\bullet, B^\bullet] \\ h(x) &= L \text{ for } x \in [C^\bullet, D^\bullet] \\ \Delta^\bullet h(x) &= 0 \text{ for } x \in V^\bullet \backslash ([A^\bullet, B^\bullet] \cup [C^\bullet, D^\bullet]) \end{aligned}$$

where L is the effective resistance between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $(G^\bullet, c^\bullet)$. Just as in the continuous setting, $\widetilde{h}$ is the harmonic conjugate of h. Since they are defined in terms of analogous boundary value problems, the functions h and $\widetilde{h}$ on G are discrete analogues of the real and imaginary parts of the uniformizing conformal map that takes a simply connected domain with four distinguished prime ends to a rectangle so that the four distinguished prime ends are mapped to the four corners of the rectangle.

Suppose (G, c) is a discrete rectangle with distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ listed in counterclockwise order, giving rise to primal boundary arcs $[A^\bullet, B^\bullet], [C^\bullet, D^\bullet] \subseteq \partial V^\bullet$ and corresponding dual boundary arcs $[B^\circ, C^\circ], [D^\circ, A^\circ] \subseteq \partial V^\circ$. Let $h : V^\bullet \to \mathbb{R}$, $\widetilde{h} : V^\circ \to \mathbb{R}$ be conjugate harmonic functions defined as above. For any interior face f of G with incident vertices x, y, u, v where $x, y \in V^\bullet$, $u, v \in V^\circ$, the image of f under the **tiling map** ϕ is defined as follows:

$$\phi(f) = [h(x), h(y)] \times [\widetilde{h}(u), \widetilde{h}(v)]$$

where the order of x, y and u, v is chosen so that:

$$h(x) \leqslant h(y), \qquad\qquad \widetilde{h}(u) \leqslant \widetilde{h}(v)$$

As the name suggests, ϕ corresponds to a tiling of the rectangle $\mathcal{R}_L$ by smaller subrectangles:

Theorem 1.3.1. (Theorem 4.31 of [7]) Suppose ϕ is the tiling map associated with the discrete rectangle (G, c) with distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \in \partial V^\bullet$ listed in counterclockwise order. Then for any pair of distinct inner faces f, f' of G, the rectangles $\phi(f)$ and $\phi(f')$ have disjoint interiors. Furthermore, if F_{in} is the set of interior faces of G,

$$\bigcup_{f \in F_{\text{in}}} \phi(f) = [0, L] \times [0, 1]$$

Since h and $\widetilde{h}$ are conjugate, the aspect ratio of the rectangle $\phi(f)$ corresponding to the face f of G with incident vertices $x, y \in V^\bullet$ and $u, v \in V^\circ$ is precisely the resistance of the primal edge $\{x, y\} \in E^\bullet$ or equivalently, the conductance of the dual edge $\{u, v\} \in E^\circ$.

1.4 Orthodiagonal Maps

Our reason for working in the level of generality that we did in Sections 1.2 and 1.3 was to showcase the power of the theory of planar electrical networks. That said, if we want the discrete harmonic and discrete holomorphic functions we're looking at to converge to the corresponding continuous harmonic and holomorphic functions in the plane, we need to do two things:

1. Fix an embedding of our graph in the complex plane.

2. Pick a conformal metric that is tied to the geometry of this embedding.

With this in mind, an **orthodiagonal map** is a finite, bipartite quadrangulation with boundary $G = (V^\bullet \sqcup V^\circ, E)$ with a fixed, proper embedding in the plane so that:

- Each edge is a straight line segment.

- Each interior face is a quadrilateral with orthogonal diagonals.

We allow non- convex quadrilaterals, whose diagonals do not intersect. We endow this with a conformal metric $c : E^\bullet \sqcup E^\circ \to (0, \infty)$ defined as follows:

$$c(e^\bullet) = \frac{|e^\circ|}{|e^\bullet|}, \qquad\qquad c(e^\circ) = \frac{|e^\bullet|}{|e^\circ|}$$

for $e^\circ \in E^\circ$, $e^\bullet \in E^\bullet$ so that e° is the dual edge corresponding to the primal edge $e^\bullet$. For any edge $e \in E^\bullet \sqcup E^\circ$, $|e|$ is the length of the edge e in our embedding. Recall that $E^\bullet$ and E° are the edges of $G^\bullet$ and G° respectively, not the edges of G. That is, they correspond to the diagonals of interior faces of G.

To make it clear that the discussion that follows is not totally vacuous, observe that the square lattice, the triangular lattice and the hexagonal lattice all have this property that primal and dual edges are orthogonal. More generally, finite subdomains of isoradial lattices, which have been widely studied in the context of critical statistical physics in 2D (i.e. see [33] and [15]), are precisely orthodiagonal maps whose faces are all rhombii. Furthermore, as a consequence of the double circle packing theorem, a wide variety of planar graphs admit an orthodiagonal embedding (see Section 2 of [25]).

While our choice of conformal metric might seem strange at first, observe that if G is an orthodiagonal map with conformal metric c as above:

- simple random walk on $(G^\bullet, c^\bullet)$ and (G°, c°) is a martingale.

- as a Markov chain, simple random walk on $(G^\bullet, c^\bullet)$ and (G°, c°) is reversible.

The orthogonality of edges and dual edges gives us a natural way to write down the Cauchy-Riemann equations on an orthodiagonal map. Namely, a function $F : V^\bullet \sqcup V^\circ \to \mathbb{C}$ is said to be

discrete holomorphic if for every interior face Q of G with primal diagonal $e^\bullet = \{v_1, v_2\}$ and dual diagonal $e^\circ = \{w_1, w_2\}$ we have:

$$\frac{F(v_2) - F(v_1)}{v_2 - v_1} = \frac{F(w_2) - F(w_1)}{w_2 - w_1} \tag{1.4.1}$$

From this definition it follows that:

- discrete contour integrals vanish if the integrand is discrete holomorphic. That is, if F is discrete holomorphic on G and γ is a simple, closed, directed curve in G so that the faces of G enclosed by γ are all interior faces of G, then:

$$\sum_{\substack{\vec{e} \in \gamma \\ \vec{e} = (e^-, e^+)}} \big(F(e^-) + F(e^+)\big)(e^+ - e^-) = 0$$

- the real and imaginary parts of any discrete holomorphic function are harmonic with respect to the edge weights $c^\bullet$ and c°. That is, $\mathrm{Re}(F)|_{V^\bullet}, \mathrm{Im}(F)|_{V^\bullet}$ are harmonic on $(G^\bullet, c^\bullet)$ and $\mathrm{Re}(F)|_{V^\circ}, \mathrm{Im}(F)|_{V^\circ}$ are harmonic on (G°, c°). Moreover, $\mathrm{Im}(F)|_{V^\circ}$ is the conjugate harmonic function of $\mathrm{Re}(F)|_{V^\bullet}$ and $\mathrm{Re}(F)|_{V^\circ}$ is the conjugate harmonic function of $\mathrm{Im}(F)|_{V^\bullet}$.

In short, orthodiagonal maps provide us with a notion of discrete complex analysis.

Chapter 2

Orthodiagonal Maps, Tilings of Rectangles and their Convergence to Conformal Maps

In this Chapter, we'll deliver on our promise in Section 0.2 and show that the tiling maps associated with finer and finer orthodiagonal approximations of a simply connected domain Ω with four distinguished boundary points, converge to the conformal map that sends Ω to a rectangle, so that the four distinguished boundary points of Ω are mapped to the four corners of the rectangle. Our approach will follow the framework we laid out in Section 0.1 for proving discrete holomorphic observables converge to their continuous counterparts. Before we do this, however, we first need to introduce the terminology to make this precise. In particular, we need to specify what it means for an orthodiagonal map with four distinguished primal boundary vertices to be close to a simply connected domain Ω with four distinguished boundary points.

2.1 Orthodiagonal Approximations of Planar Domains

An **orthodiagonal rectangle** is an orthodiagonal map G with a unique distinguished outer face whose boundary is a simple, closed curve with four distinguished boundary points $A^\bullet, B^\bullet, C^\bullet, D^\bullet \subseteq \partial V^\bullet$ listed in counterclockwise order. As in the case of discrete rectangles, these give rise to primal boundary arcs $[A^\bullet, B^\bullet], [C^\bullet, D^\bullet]$ and corresponding dual arcs $[B^\circ, C^\circ], [D^\circ, A^\circ]$. Given an orthodiagonal map G, let $\widehat{G}$ denote the subdomain of $\mathbb{C}$ formed by taking the interior of the union of the faces of G.

Suppose Ω is a connected, proper subdomain of $\mathbb{C}$. $\gamma \subseteq \overline{\Omega}$ is a **crosscut** of Ω if $\gamma = \eta([0,1])$ for some injective, continuous function $\eta : [0,1] \to \overline{\Omega}$ such that $\eta(0,1) \subseteq \Omega$ and $\eta(0), \eta(1) \in \partial\Omega$ where $\eta(0) \neq \eta(1)$. If γ is a crosscut of Ω, $\Omega \backslash \gamma$ has two connected components. By the Jordan

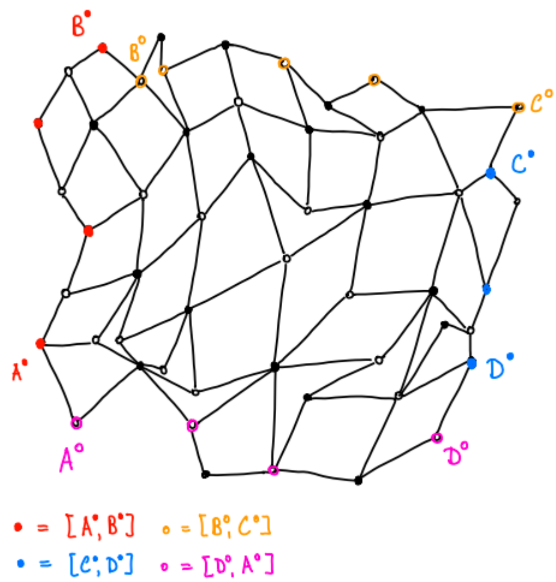

Figure 2.1: An orthodiagonal rectangle with distinguished boundary arcs $[A^\bullet, B^\bullet]$, $[C^\bullet, D^\bullet]$ and corresponding dual arcs $[B^\circ, C^\circ]$, $[D^\circ, A^\circ]$.

curve theorem, the same is true of $\Omega\backslash\gamma$ when γ is a simple, closed curve in Ω. Given disjoint subsets $A, B \subseteq \Omega$ we say that a simple closed curve or crosscut γ **separates** A and B in Ω if $\gamma \cap A = \gamma \cap B = \varnothing$ and A and B lie in distinct connected components of $\Omega\backslash\gamma$.

Fix $z_0 \in \Omega$. For any $z, w \in \Omega\backslash\{z_0\}$, their **Carathéodory distance** with respect to the reference point z_0 is given by:

$$d^{z_0}_{Cara}(z, w) := \inf\{\text{length}(\gamma) : \gamma \text{ is a simple closed curve or crosscut that separates } z \text{ and } w \text{ from } z_0\}$$

$d^{z_0}_{Cara}$ is a metric on $\Omega\backslash\{z_0\}$ that is locally equivalent to the usual Euclidean metric. The **Carathéodory compactification** Ω^* of Ω is the completion of $\Omega\backslash\{z_0\}$ with respect to $d^{z_0}_{Cara}$. As a topological space, the Carathéodory compactification Ω^* is independent of our choice of reference point $z_0 \in \Omega$. $\partial\Omega^*$ is known as the space of **prime ends** of Ω. The prime ends of Ω can be interpretted geometrically as equivalence classes of chains of open sets in Ω converging to a point on the boundary. For details, see Section 3.1 of [6] or Section 2.4 of [45]. Given disjoint subsets $A, B \subseteq \partial\Omega^*$, we say that a crosscut γ of Ω **joins** A and B in Ω if one of the endpoints of γ lies in A and the other lies in B. If $A \subseteq \Omega$, $B \subseteq \partial\Omega^*$, we say that a crosscut γ of Ω **joins** A and B in Ω if $A \cap \gamma = \varnothing$, A is contained in one of the two connected components of $\Omega^*\backslash\gamma$, and B intersects the connected component of $\Omega^*\backslash\gamma$ containing A.

If Ω_1, Ω_2 are proper, connected subdomains of $\mathbb{C}$ and $\phi : \Omega_1 \to \Omega_2$ is conformal, then ϕ extends to a homeomorphism $\phi : \Omega_1^* \to \Omega_2^*$. This tells us that, from the standpoint of complex analysis, the space of prime ends is the right notion of boundary for a proper, connected subdomain of $\mathbb{C}$. In particular, if $\Omega \subsetneq \mathbb{C}$ is simply connected, and $\phi : \Omega \to \mathbb{D}$ is the uniformizing conformal map that maps Ω to the unit disk, we have the following estimates for the modulus of continuity of ϕ and ϕ^{-1} with respect to the Carathéodory metric:

Proposition 2.1.1. Suppose $\Omega \subsetneq \mathbb{C}$ is a bounded simply connected domain, $z_0 \in \Omega$ and $\phi : \Omega \to \mathbb{D}$ is a uniformizing conformal that maps Ω to the unit disk so that $\phi(z_0) = 0$. Then there exists an absolute constant $C_1 > 0$ such that for any $x, y \in \Omega$:

$$|\phi(y) - \phi(x)| \leqslant C_1 \sqrt{\frac{d_{Cara}^{z_0}(x, y)}{|\phi'(z_0)|}} \tag{2.1.1}$$

If additionally we know that Ω is bounded, there exists an absolute constant $C_2 > 0$ such that for any $x, y \in \mathbb{D}$, we have that:

$$d_{Cara}^{z_0}(\phi^{-1}(x), \phi^{-1}(y)) \leqslant C_2 \sqrt{\frac{\text{Area}(\Omega)}{\log\left(\frac{1}{|x-y|}\right)}} \tag{2.1.2}$$

The modulus of continuity for ϕ in Equation 2.1.1 is a consequence of Beurling's estimate (see Proposition 3.85 of [38]). The modulus of continuity for ϕ^{-1} follows from Wolff's lemma (see Proposition 2.2 of [45]). As a consequence, if $\Omega \subsetneq \mathbb{C}$ is simply connected, Ω^* is homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ and the space of prime ends $\partial\Omega^*$ is homeomorphic to S^1. If $x, y \in \partial\Omega^*$ are prime ends, let $[x, y]_{\partial\Omega^*}$ denote the arc along $\partial\Omega^*$ that travels from x to y, counterclockwise.

A **conformal rectangle** is a bounded, simply connected domain Ω along with four distinguished prime ends A, B, C, D, listed in counterclockwise order. Recalling our discussion in Section 0.2, given a conformal rectangle (Ω, A, B, C, D) and a sequence of orthodiagonal rectangles $\left((G_n, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)\right)_{n=1}^\infty$ that are better and better approximations of (Ω, A, B, C, D), we want to show that the associated tiling maps converge to the conformal map from Ω to a rectangle $\mathcal{R}_L$ so that the prime ends A, B, C, D are mapped to the four corners of $\mathcal{R}_L$, where in particular, $\phi(A) = i$. This of course begs the question: what does it mean for an orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ to be a good approximation of (Ω, A, B, C, D)? One natural requirement is that the boundary arcs $[A^\bullet, B^\bullet]$, $[B^\circ, C^\circ]$, $[C^\bullet, D^\bullet]$, $[D^\circ, A^\circ]$ of G should be close to the corresponding continuous boundary arcs $[A, B]_{\partial\Omega^*}$, $[B, C]_{\partial\Omega^*}$, $[C, D]_{\partial\Omega^*}$, $[D, A]_{\partial\Omega^*}$ of Ω. To be precise, since the Carathéodory metric is the right notion of distance for a general simply connected domain, a natural requirement is that the discrete boundary arcs are close to the corresponding continuous boundary arcs in Carathéodory metric. However, to define the Carathéodory metric, we need to introduce a reference point, which a priori isn't part of our setup. To avoid this, we will instead use a closely related quantity, whose definition doesn't require the introduction of a reference point.

For any $z \in \Omega\backslash\gamma$, let $\mathcal{N}_\gamma^z$ denote the component of $\Omega\backslash\gamma$ containing z. Let S, S' be disjoint compact subsets of $\partial\Omega^*$. Then for any $z \in \Omega^*$, we define the **crosscut distance** $d_{cc}^{\Omega}(z, S, S')$ from z to S, away from S' in Ω, by:

$$d_{cc}^{\Omega}(z, S, S') := \inf\{\text{length}(\gamma) : \gamma \text{ is a crosscut of } \Omega \text{ that separates } z \text{ from } S' \text{ such that } \overline{\mathcal{N}_\gamma^z} \cap S \neq \varnothing\}$$

where $\overline{\mathcal{N}_\gamma^z}$ is the closure of $\mathcal{N}_\gamma^z$ with respect to the Carathéodory metric. Similarly, for a compact subset C of Ω^*, its crosscut distance to S, away from S', is given by:

$$d_{cc}^{\Omega}(C, S, S') := \sup_{z \in C} d_{cc}^{\Omega}(z, S, S')$$

The supremum on the right hand side is actually a maximum. To see this, observe that $d_{cc}^{\Omega}(z, S, S')$ is locally Lipschitz as a function of z. Namely,

$$|d_{cc}^{\Omega}(z, S, S') - d_{cc}^{\Omega}(w, S, S')| \leqslant 2d_{\Omega}(z, w)$$

where:

$$d_{\Omega}(z, w) := \inf\{\text{length}(\gamma) : \gamma \text{ is a smooth curve from } z \text{ to } w \text{ in } \Omega\}$$

In particular, if the line segment from z to w is contained in Ω, we have that:

$$|d_{cc}^{\Omega}(z, S, S') - d_{cc}^{\Omega}(w, S, S')| \leqslant 2|z - w|$$

Given a conformal rectangle (Ω, A, B, C, D), we say that the orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a **$(\delta,\ \varepsilon)$- good interior approximation** of (Ω, A, B, C, D) if $\widehat{G} \subseteq \Omega$, $|e| < \varepsilon$ for all edges $e \in E$, and:

$$d_{cc}^{\Omega}([A^\bullet, B^\bullet], [A, B]_{\partial\Omega^*}, [C, D]_{\partial\Omega^*}) < \delta, \qquad d_{cc}^{\Omega}([C^\bullet, D^\bullet], [C, D]_{\partial\Omega^*}, [A, B]_{\partial\Omega^*}) < \delta$$
$$d_{cc}^{\Omega}([B^\circ, C^\circ], [B, C]_{\partial\Omega^*}, [D, A]_{\partial\Omega^*}) < \delta, \qquad d_{cc}^{\Omega}([D^\circ, A^\circ], [D, A]_{\partial\Omega^*}, [B, C]_{\partial\Omega^*}) < \delta$$

In Section 1.3 we defined the tiling map associated with a discrete rectangle. Without a fixed embedding, the faces of a discrete rectangle are purely combinatorial objects. Having fixed an embedding, the faces of an orthodiagonal rectangle are honest- to- goodness subsets of the plane. Hence, we can think of the tiling map ϕ associated with an orthodiagonal rectangle G, as a function $\phi : \widehat{G} \to \mathbb{C}$. We do this by choosing for each interior face f a homeomorphism that maps the quadrilateral f to the corresponding rectangle $\phi(f)$. Unfortunately, this means that the tiling map $\phi : \widehat{G} \to \mathbb{C}$ depends on our choice of homeomorphism. There is also some ambiguity as to the definition of ϕ on the edges of G, since each edge is shared by two distinct faces. That said, by the regularity estimates in Section 2.2.2 we'll see that this isn't a concern, since our choice of homeomorphism doesn't impact the convergence we are looking for.

Having established all the requisite terminology, we can now state precisely the main theorem we intend to prove:

Theorem 2.1.2. Suppose (Ω, A, B, C, D) is a conformal rectangle and $\big((G_n, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)\big)_{n=1}^\infty$ is a sequence of orthodiagonal rectangles so that for each $n \in \mathbb{N}$, $(G_n, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)$ is a $(\delta_n, \varepsilon_n)$-good interior approximation of Ω, where:

$$(\delta_n, \varepsilon_n) \to (0, 0) \text{ as } n \to \infty$$

Let $\phi_n : \widehat{G}_n \to [0, L_n] \times [0, 1]$ be the corresponding tiling maps, where L_n is the discrete extremal length between $[A_n^\bullet, B_n^\bullet]$ and $[C_n^\bullet, D_n^\bullet]$ in $G_n^\bullet$. Then:

$$\phi_n \to \phi \text{ uniformly on compacts as } n \to \infty$$

where ϕ is the conformal map from Ω to the rectangle $(0, L) \times (0, 1)$ so that the prime ends A, B, C, D are mapped to the four corners of the rectangle and in particular, $\phi(A) = i$. Here, L is the extremal length between the arcs $[A, B]_{\partial\Omega*}$ and $[C, D]_{\partial\Omega*}$ in Ω. In particular, it follows that:

$$L_n \to L \text{ as } n \to \infty$$

2.2 Precompactness of the Tiling Maps

To prove Theorem 2.1.2 following the framework outlined in Section 0.1, we need to show that tiling maps corresponding to finer and finer orthodiagonal approximations of a conformal rectangle (Ω, A, B, C, D) are equicontinuous and uniformly bounded on compacts in Ω. In this section, we address this by proving estimates for the norm and modulus of continuity of our tiling maps.

For both the norm and modulus of continuity, when doing this, we begin by proving the corresponding result in the continuous setting. The proof in the continuous setting motivates the proof in the discrete setting. Furthermore, we will need the continuous analogues of our tiling map estimates in the proof of Theorem **??** in Appendix **??**.

2.2.1 Modulus of Continuity for the Limiting Conformal Map

Suppose (Ω, A, B, C, D) is a conformal rectangle. Using the notation of Section 1.4, if $z, w \in \Omega$,

$$\begin{aligned} d_{cc}^\Omega(z, w) = \min\{&d_{cc}^\Omega(\{z, w\}, [A, B]_{\partial\Omega*}; [C, D]_{\partial\Omega*}), d_{cc}^\Omega(\{z, w\}, [B, C]_{\partial\Omega*}; [D, A]_{\partial\Omega*}), \\ &d_{cc}^\Omega(\{z, w\}, [C, D]_{\partial\Omega*}; [A, B]_{\partial\Omega*}), d_{cc}^\Omega(\{z, w\}, [D, A]_{\partial\Omega*}; [B, C]_{\partial\Omega*})\} \end{aligned}$$

In other words, $d_{cc}(z, w)$ is the length of the shortest crosscut of Ω that joins z and w to one boundary arc of (Ω, A, B, C, D) and separates them from the opposite boundary arc. Recall that for $z, w \in \Omega$,

$$d_\Omega(z, w) = \inf\{\text{length}(\gamma) : \gamma \text{ is a smooth curve from } z \text{ to } w \text{ in } \Omega\}$$

That is, d_Ω is the ambient metric on Ω. Let $\phi : \Omega \to \mathcal{R}_L$ be the conformal map from Ω to the rectangle $\mathcal{R}_L$ such that the four prime ends A, B, C, D of Ω, listed in counterclockwise order, are mapped to the four corners of $\mathcal{R}_L$ and in particular, $\phi(A) = i$. The following theorem gives us a modulus of continuity for the real and imaginary parts of ϕ and therefore ϕ itself:

Theorem 2.2.1. Suppose (Ω, A, B, C, D) is a conformal rectangle and $\phi : \Omega \to \mathcal{R}_L$ be the conformal map from Ω to the rectangle $\mathcal{R}_L$ so that the four prime ends A, B, C, D of Ω are mapped to the four corners of $\mathcal{R}_L$ and in particular, $\phi(A) = i$. Here, L is the extremal length between the boundary arcs $[A, B]_{\partial\Omega*}$ and $[C, D]_{\partial\Omega*}$ in Ω. Define:

$$d = \inf\{\text{length}(\gamma) : \gamma \text{ is a crosscut of } \Omega \text{ joining } [A, B]_{\partial\Omega*} \text{ and } [C, D]_{\partial\Omega*} \text{ in } \Omega\}$$
$$d' = \inf\{\text{length}(\gamma) : \gamma \text{ is a crosscut of } \Omega \text{ joining } [B, C]_{\partial\Omega*} \text{ and } [D, A]_{\partial\Omega*} \text{ in } \Omega\}$$

Let h and $\widetilde{h}$ be the real and imaginary parts of ϕ, respectively. Then for any $x, y \in \Omega$ we have that:

$$|h(y) - h(x)| \leqslant \frac{2\pi}{\log\left(\frac{d'}{2(d_\Omega(x,y) \wedge d^\Omega_{cc}(x,y))}\right)}, \qquad |\widetilde{h}(y) - \widetilde{h}(x)| \leqslant \frac{2\pi L}{\log\left(\frac{d}{2(d_\Omega(x,y) \wedge d^\Omega_{cc}(x,y))}\right)}$$

Proof. As per the theorem statement, let h be the real part of ϕ. Fix $x, y \in \Omega$. If $h(x) = h(y)$, the desired result holds. Otherwise, suppose WLOG that $h(x) < h(y)$. We now consider two cases:

Case 1: $d_\Omega(x, y) \leqslant d^\Omega_{cc}(x, y)$.

Consider the region:

$$\Omega_{x,y} = \{z \in \Omega : h(x) < h(z) < h(y)\}$$

This is simply connected. Furthermore, since ϕ maps Ω to the rectangle $(0, L) \times (0, 1)$, ϕ maps $\Omega_{x,y}$ to the rectangle $(h(x), h(y)) \times (0, 1)$. Thus, if we think of $\Omega_{x,y}$ as a conformal rectangle with the distinguished boundary arcs:

$$N = \Omega^*_{x,y} \cap [D, A]_{\partial\Omega*}, \qquad E = \{z \in \Omega : h(z) = h(y)\}$$
$$S = \Omega^*_{x,y} \cap [D, A]_{\partial\Omega*}, \qquad W = \{z \in \Omega : h(z) = h(x)\}$$

Here N, E, S and W stand for "North," "East," "South," and "West." This is to emphasize that our picture is as follows:

In summa:

$$\lambda(W \leftrightarrow E; \Omega) = h(y) - h(x)$$

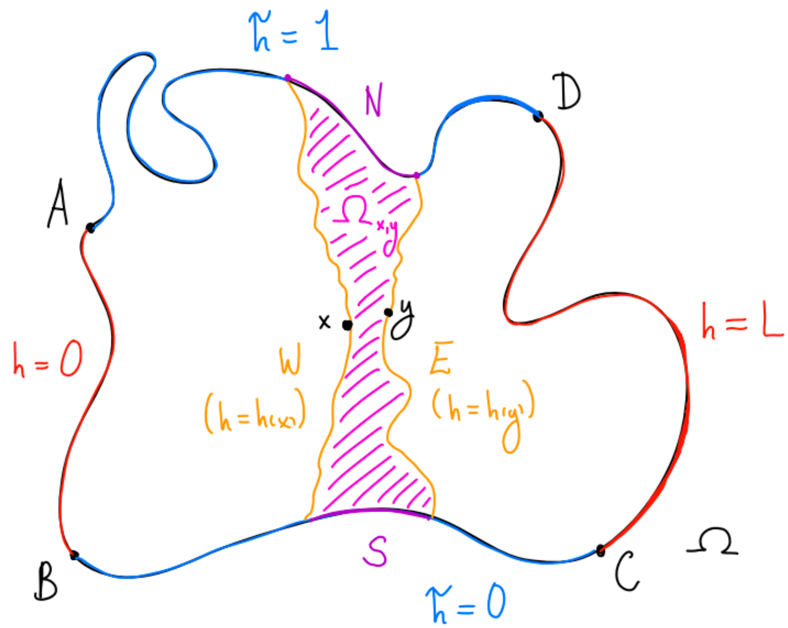

Figure 2.2: The subrectangle $\Omega_{x,y}$ associated with a pair of points x and y in the conformal rectangle (Ω, A, B, C, D).

Having reinterpreted the quantity we're interested in as an extremal length, we can bound it from above by bounding the dual extremal length from below.

Fix ε so that $0 < \varepsilon < \frac{d'}{2} - d_\Omega(x,y)$. By the definition of $d_\Omega(x,y)$, we can find a smooth curve γ in Ω from x to y so that $\text{length}(\gamma) < d_\Omega(x,y) + \varepsilon$. Since x and y lie on opposite boundary arcs of $\Omega_{x,y}$, while γ may not be a crosscut of $\Omega_{x,y}$, there must exist a subarc γ' of γ with endpoints $x' \in W$, $y' \in E$ so that γ' is a crosscut of $\Omega_{x,y}$. Consider the annulus:

$$\mathcal{A} = \{u \in \mathbb{C} : d_\Omega(x,y) + \varepsilon < |u - x'| < \frac{d'}{2}\}$$

where:

$$d' = \inf\{\text{diam}(\gamma) : \gamma \text{ is a curve joining } [B,C]_{\partial\Omega*} \text{ and } [D,A]_{\partial\Omega*} \text{ in } \Omega\}$$

Observe that:

1. Since $\text{length}(\gamma') \leqslant \text{length}(\gamma) < d_\Omega(x,y) + \varepsilon$, the diameter of γ' is at most $d_\Omega(x,y) + \varepsilon$. Hence, $\gamma' \subseteq B(x', d_\Omega(x,y) + \varepsilon)$. Since γ' separates the boundary arcs N and S in $\Omega_{x,y}$, any path from N to S in $\Omega_{x,y}$ must intersect γ' and therefore $B(x', d_\Omega(x,y) + \varepsilon)$.

2. On the other hand, since any curve from N to S in $\Omega_{x,y}$ is a curve from $[B,C]$ to $[D,A]$ in Ω, such a curve must have diameter $\geqslant d'$. Hence, any curve from N to S in $\Omega_{x,y}$ must at some point lie outside the ball $B(x', \frac{d'}{2})$.

Putting all this together, we see that any curve from N to S in $\Omega_{x,y}$ must cross the annulus $\mathcal{A}$.

Consider the metric:

$$\rho(z) = \frac{1}{|z - x'|} 1_{\mathcal{A} \cap \Omega_{x,y}}(z)$$

on $\Omega_{x,y}$. Observe that if η is a C^1 curve that crosses the annulus $\mathcal{A}$ at least once, then:

$$l_\rho(\eta) = \int_\eta \frac{|dz|}{|z - x'|} \geqslant \int_{d_\Omega(x,y)+\varepsilon}^{d'/2} \frac{dr}{r} = \log\left(\frac{d'}{2(d_\Omega(x,y) + \varepsilon)}\right)$$

Furthermore:

$$A(\rho) = \int_{\mathcal{A}\cap\Omega_{x,y}} \frac{1}{|z - x'|} dz_1 dz_2 \leqslant \int_{\mathcal{A}\cap\Omega_{x,y}} \frac{1}{|z - x'|} dz_1 dz_2 = \int_0^{2\pi} \int_{d_\Omega(x,y)+\varepsilon}^{d'/2} \frac{1}{r} dr d\theta = 2\pi \log\left(\frac{d'}{2(d_\Omega(x,y) + \varepsilon)}\right)$$

Hence, plugging ρ into the variational problem for $\lambda(N \leftrightarrow S; \Omega_{x,y})$ we have that:

$$\lambda(N \leftrightarrow S; \Omega_{x,y}) \geqslant \frac{\inf_\eta \left(l_\rho(\eta)\right)^2}{A(\rho)} \geqslant \frac{\left(\log\left(\frac{d'}{2(d_\Omega(x,y)+\varepsilon)}\right)\right)^2}{2\pi \log\left(\frac{d'}{2(d_\Omega(x,y)+\varepsilon)}\right)} = \frac{1}{2\pi} \log\left(\frac{d'}{2(d_\Omega(x,y) + \varepsilon)}\right)$$

By duality for continuous extremal extremal length:

$$\lambda(W \leftrightarrow E; \Omega_{x,y}) \cdot \lambda(N \leftrightarrow S; \Omega_{x,y}) = 1$$

Hence:

$$\lambda(N \leftrightarrow S; \Omega_{x,y}) \leqslant \frac{2\pi}{\log\left(\frac{d'}{2(d_\Omega(x,y)+\varepsilon)}\right)}$$

Since $\varepsilon > 0$ was arbitrary, letting ε tend to 0 in the above inequality, the desired result follows.

Case 2: $d_{cc}^{\Omega}(x,y) \leqslant d_\Omega(x,y)$.

Fix ε so that $0 < \varepsilon < \frac{d'}{2} - d_{cc}^{\Omega}(x,y)$. By the definition of $d_{cc}(x,y)$ we can find a crosscut γ of Ω that joins x and y to one of the four distinguished boundary arcs of (Ω, A, B, C, D) and separates it from the opposite boundary arc, such that length$(\gamma) < d_{cc}^{\Omega}(x,y) + \varepsilon$. We now split our problem into two further cases, depending on whether the relevant boundary arcs of (Ω, A, B, C, D) are Dirichlet arcs where h is constant, or Neumann arcs along which h is monotone.

Case 2.1: γ joins x and y to one of the Dirichlet arcs and separates it from the opposite Dirichlet arc.

WLOG, suppose that γ joins x and y to $[A,B]_{\partial\Omega^*}$ and separates x and y from $[C,D]_{\partial\Omega^*}$. Let $N_{x,y}^{\gamma}$ denote the connected component of $\Omega^* \backslash \gamma$ containing z and w. By the maximum principle for harmonic functions,

$$\sup_{z \in N_{x,y}^{\gamma}} h(z) = \max_{z\in\gamma} h(z)$$

Let v be a point of γ so that:

$$h(v) = \max_{z \in \gamma} h(z)$$

Since $0 < h(x) < h(y) < h(v)$, it follows that:

$$h(y) - h(x) < h(v)$$

Consider the region:

$$\Omega_v = \{z \in \Omega : 0 < h(z) < h(v)\}$$

ϕ maps Ω_v to the rectangle $(0, h(v)) \times (0, 1)$. Thinking of Ω_v as a conformal rectangle with the distinguished boundary arcs:

$$\begin{aligned} N &= \Omega_v^* \cap [D, A]_{\partial\Omega*} & \qquad E &= \Omega_v^* \cap [A, B]_{\partial\Omega*} \\ S &= \Omega_v^* \cap [B, C]_{\partial\Omega*} & \qquad W &= \{z \in \Omega : h(z) = h(v)\} \end{aligned}$$

it follows that:

$$\lambda(W \leftrightarrow E; \Omega_v) = h(v)$$

Similar to case 1, having reinterpreted $h(v)$ as an extremal length, we can bound it from above by bounding the dual extremal length from below.

Since $v \in W$ and v lies along a crosscut γ of Ω that starts and ends along $[A, B]_{\partial\Omega*}$, there must exist a subarc γ' of γ of length at most $\frac{1}{2}(d_{cc}^{\Omega}(x, y) + \varepsilon)$ travelling from E to W and thereby separating N and S in Ω_v. Let v' be the endpoint of γ' that lies in W. Consider the annulus:

$$\mathcal{A} = \{u \in \mathbb{C} : \frac{d_{cc}^{\Omega}(x, y) + \varepsilon}{2} < |u - v'| < \frac{d'}{2}\}$$

Just as in case 1:

- Since $\text{length}(\gamma') \leqslant \frac{d_{cc}^{\Omega}(x,y)+\varepsilon}{2}$, the diameter of γ' is at most $\frac{d_{cc}^{\Omega}(x,y)+\varepsilon}{2}$. Hence, $\gamma \subseteq B(v', \frac{d_{cc}^{\Omega}(x,y)+\varepsilon}{2})$. Since γ' separates the boundary arcs N and S in Ω_v, any path from N to S in Ω_v must intersect γ' and therefore $B(v', \frac{d_{cc}^{\Omega}(x,y)+\varepsilon}{2})$
- Since any curve from N to S in Ω_v is a curve from $[B, C]_{\partial\Omega*}$ to $[D, A]_{\partial\Omega*}$ in Ω_v, such a curve must have diameter $\geqslant d'$. In particular, any such curve must at some point lie outside the ball $B(v', \frac{d'}{2})$.

Putting all this together, we conclude that any curve from N to S in Ω_v must cross the annulus $\mathcal{A}$ at least once. Hence, by the same argument as in case 1, plugging the metric:

$$\rho(z) = \frac{1}{|z - v'|} 1_{\mathcal{A} \cap \Omega_v}(z)$$

into the variational problem for $\lambda(N \leftrightarrow S; \Omega_v)$, we have that:

$$\lambda(N \leftrightarrow S; \Omega_v) \geqslant \frac{1}{2\pi} \log\left(\frac{d'}{d_{cc}(x,y)+\varepsilon}\right)$$

By duality for continuous extremal length:

$$\lambda(W \leftrightarrow E; \Omega_v) \cdot \lambda(N \leftrightarrow S; \Omega_v) = 1$$

Hence:

$$h(y) - h(x) < h(v) \leqslant \frac{2\pi}{\log\left(\frac{d'}{d_{cc}(x,y)+\varepsilon}\right)}$$

Since $\varepsilon > 0$ was arbitrary, letting ε tend to 0 in the above inequality, the desired result follows.

Case 2.2: γ joins z and w to one of the Neumann arcs and separates it from the opposite Neumann arc.

WLOG, suppose that γ joins x and y to $[B, C]_{\partial\Omega*}$ and separates x and y from $[C, D]_{\partial\Omega*}$. Let $N^{\gamma}_{x,y}$ denote the connected component of $\Omega^*\backslash\gamma$ containing x and y. By the maximum principle for harmonic functions,

$$\sup_{z \in N^{\gamma}_{x,y}} h(z) = \max_{z\in\gamma} h(z), \qquad \inf_{z \in N^{\gamma}_{x,y}} h(z) = \min_{z\in\gamma} h(z)$$

Let u, v be points of γ so that:

$$h(u) = \min_{z\in\gamma} h(z), \qquad h(v) = \max_{z\in\gamma} h(z)$$

□

Since $h(u) < h(x) < h(y) < h(v)$, it follows that:

$$h(y) - h(x) < h(v) - h(u)$$

Consider the region:

$$\Omega_{u,v} = \{z \in \mathbb{C} : h(u) < h(z) < h(v)\}$$

ϕ maps $\Omega_{u,v}$ to the rectangle $(h(u), h(v)) \times (0, 1)$. Thinking of $\Omega_{u,v}$ as a conformal rectangle with the distinguished boundary arcs:

$$N = \Omega^*_{u,v} \cap [D, A]_{\partial\Omega*} \qquad E = \{z \in \Omega : h(z) = h(v)\}$$
$$S = \Omega^*_{u,v} \cap [B, C]_{\partial\Omega*} \qquad W = \{z \in \Omega : h(z) = h(u)\}$$

we have that:

$$\lambda(W \leftrightarrow E; \Omega_{u,v}) = h(v) - h(u)$$

Thus, to get an upper bound for $h(v) - h(u)$ and therefore $h(y) - h(x)$, it suffices to bound the dual extremal length $\lambda(N \leftrightarrow S; \Omega_{u,v})$ from below. Observe that:

- Since u and v both lie along a crosscut of Ω of length at most $d_{cc}^{\Omega}(x, y) + \varepsilon$ and $u \in W$, $w \in E$, we can join W and E by a crosscut of $\Omega_{u,v}$ of length at most $d_{cc}^{\Omega}(x, y) + \varepsilon$.

- Since any curve from N to S in $\Omega_{u,v}$ is a curve from $[A, D]_{\partial\Omega^*}$ to $[B, C]_{\partial\Omega^*}$ in Ω, such a curve will have diameter $\geqslant d'$.

Putting all this together, by the same argument as in case 1, verbatim, it follows that:

$$\lambda(N \leftrightarrow S; \Omega_{u,v}) \geqslant \frac{1}{2\pi} \log\left(\frac{d'}{2(d_{cc}^{\Omega}(x, y) + \varepsilon)}\right)$$

Hence:

$$h(y) - h(x) < h(v) - h(u) \leqslant \lambda(W \leftrightarrow E; \Omega_{u,v}) \leqslant \frac{2\pi}{\log\left(\frac{d'}{2(d_{cc}^{\Omega}(x,y)+\varepsilon)}\right)}$$

Letting ε tend to 0 in the above inequality, the desired result follows. The analogous estimate for $\widetilde{h}$ follows by the same argument. $\square$

2.2.2 Modulus of Continuity for Tiling Maps

Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is an orthodiagonal rectangle and let $\widetilde{h}$ be the unique solution to the following boundary value problem on G°:

$$\begin{aligned} \widetilde{h}(x) &= 0 \text{ for all } x \in [D^\circ, A^\circ] \\ \widetilde{h}(x) &= 1 \text{ for all } x \in [B^\circ, C^\circ] \\ \Delta^\circ \widetilde{h}(x) &= 0 \text{ for all } x \in V^\circ \backslash ([D^\circ, A^\circ] \cup [B^\circ, C^\circ]) \end{aligned}$$

Let h be the solution to the following boundary value problem on $G^\bullet$:

$$\begin{aligned} h(x) &= 0 \text{ for all } x \in [A^\bullet, B^\bullet] \\ h(x) &= L \text{ for all } x \in [C^\bullet, D^\bullet] \\ \Delta^\bullet h(x) &= 0 \text{ for all } x \in V^\bullet \backslash ([A^\bullet, B^\bullet] \cup [C^\bullet, D^\bullet]) \end{aligned}$$

where L is the effective resistance between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $G^\bullet$. h and $\widetilde{h}$ are the conjugate discrete harmonic functions that correspond to the real and imaginary parts of the tiling map $\phi : \widehat{G} \to (0, L) \times (0, 1)$. In this section we will prove regularity estimates for h and $\widetilde{h}$ analogous to the regularity estimates we proved for the corresponding conformal map in Section 2.2.1. We do this by adapting our argument in Section 2.2.1 to the discrete, orthodiagonal setting. To do this, we first need to establish the following lemma which gives us an estimate for the gradients of h and $\widetilde{h}$ across an edge:

Lemma 2.2.2. Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is an orthodiagonal rectangle so that the edges of G all have length at most ε. Let $\widetilde{h}$ be the unique solution to the following boundary value problem on G°:

$$\begin{aligned} \widetilde{h}(x) &= 0 \text{ for all } x \in [D^\circ, A^\circ] \\ \widetilde{h}(x) &= 1 \text{ for all } x \in [B^\circ, C^\circ] \\ \Delta^\circ \widetilde{h}(x) &= 0 \text{ for all } x \in V^\circ \backslash ([D^\circ, A^\circ] \cup [B^\circ, C^\circ]) \end{aligned}$$

Let h be the harmonic conjugate of $\widetilde{h}$ which solves the following boundary value problem on $G^\bullet$:

$$\begin{aligned} h(x) &= 0 \text{ for all } x \in [A^\bullet, B^\bullet] \\ h(x) &= L \text{ for all } x \in [C^\bullet, D^\bullet] \\ \Delta^\bullet h(x) &= 0 \text{ for all } x \in V^\bullet \backslash ([A^\bullet, B^\bullet] \cup [C^\bullet, D^\bullet]) \end{aligned}$$

where L is the effective resistance between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $G^\bullet$. Define:

$$\begin{aligned} d &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{G} \text{ from } [A^\bullet, B^\bullet]_{\partial \widehat{G}*} \text{ to } [C^\bullet, D^\bullet]_{\partial \widehat{G}*}\} \\ d' &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{G} \text{ from } [B^\bullet, C^\bullet]_{\partial \widehat{G}*} \text{ to } [D^\bullet, A^\bullet]_{\partial \widehat{G}*}\} \end{aligned}$$

If $\varepsilon \leqslant \frac{d \wedge d'}{16}$, there exists an absolute constant $K > 0$ so that if $x, y \in V^\bullet$ are neighboring edges in $G^\bullet$ and $u, v \in V^\circ$ are neighboring edges in G°, then:

$$|h(y) - h(x)| \leqslant \frac{K}{\log\left(\frac{d'}{\varepsilon}\right)} \qquad\qquad |\widetilde{h}(u) - \widetilde{h}(v)| \leqslant \frac{KL}{\log\left(\frac{d}{\varepsilon}\right)}$$

Proof. Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a orthodiagonal rectangle with edges of length at most ε and $h : V^\bullet \to \mathbb{R}$ be the solution to the Dirichlet- Neumann problem on this orthodiagonal map that is 0 on $[A^\bullet, B^\bullet]$, L on $[C^\bullet, D^\bullet]$ and harmonic elsewhere, where L is the discrete extremal length from $[A^\bullet, B^\bullet]$ to $[C^\bullet, D^\bullet]$ in $G^\bullet$. Define:

$$\chi := \max_{\{x,y\} \in E^\bullet} |h(y) - h(x)|$$

If $\chi = 0$, we're done. Otherwise, select neighboring vertices $x, y \in V^\bullet$ so that $\big(h(y) - h(x)\big) = \chi$. Consider the sets S_x and S_y defined as follows:

$$S_x := \{z \in V^\bullet : h(z) \leqslant h(x)\} \qquad\qquad S_y := \{z \in V^\bullet : h(z) \geqslant h(y)\}$$

By the maximum principle for harmonic functions, S_x, S_y and $V^\bullet \backslash (S_x \sqcup S_y)$ are all connected subsets of $G^\bullet$. Furthermore, $[A^\bullet, B^\bullet] \subseteq S_x$, $[C^\bullet, D^\bullet] \subseteq S_y$. Let $H = (V_H^\bullet \sqcup V_H^\circ, E_H)$ be the suborthodiagonal map of G formed by gluing together all the faces of G that are incident to at least one vertex of $V^\bullet \backslash (S_x \sqcup S_y)$. By the maximum principle, H is simply connected with a unique, distinguished exterior face. Moreover, let:

$$O^\bullet := S_x \cap \partial V_H^\bullet, \qquad W^\bullet := S_y \cap \partial V_H^\bullet$$

Then $O^\bullet$ and $W^\bullet$ are primal boundary arcs of H with corresponding dual arcs:

$$N^\circ := [B^\circ, C^\circ] \cap \partial V_H^\circ, \qquad S^\circ := [D^\circ, A^\circ] \cap \partial V_H^\circ$$

Similar to the proof of Theorem 2.2.1, N, O, S and W stand for "North," "Orient," "South," and "West." We'd have used E for "East," however in the discrete setting, E is already being used to denote the edges of G. Proposition 1.1.4 tells us that for any function $g : V_H^\bullet \to \mathbb{R}$ with $\mathrm{gap}_{O^\bullet, W^\bullet}(g) \geqslant 0$ and any flow θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$:

$$\mathrm{strength}(\theta) \cdot \mathrm{gap}_{O^\bullet, W^\bullet}(g) \leqslant \mathcal{E}^\bullet(\theta; H)^{1/2} \mathcal{E}^\bullet(g; H)^{1/2}$$

Plugging $g = h$ into the inequality above, we have that for any choice of flow θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$:

$$\chi = |h(y) - h(x)| \leqslant \frac{\mathcal{E}^\bullet(\theta; H)^{1/2} \mathcal{E}^\bullet(h; H)^{1/2}}{\mathrm{strength}(\theta)}$$

By Thomson's principle, taking the infimum over all flows θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$ in the expression on the RHS, we have that:

$$\chi \leqslant \mathcal{E}^\bullet(h; H)^{1/2} \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)^{1/2}$$

In Section 1.3 we saw that $\mathcal{E}^\bullet(h)$ is the total area of rectangles in the tiling associated with the orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$. Hence, by the definition of H, the restriction $\mathcal{E}^\bullet(h; H)$ is the total area of rectangles in our tiling that intersect $(h(x), h(y)) \times (0, 1)$. Since $|h(y) - h(x)| = \chi$ and any rectangle in our tiling has width at most χ, it follows that:

$$\mathcal{E}^\bullet(h; H) \leqslant 3\chi$$

Hence:

$$\chi \leqslant 3\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) \tag{2.2.1}$$

By duality:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = 1$$

Thus, to bound $\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)$ and therefore χ from above, it suffices to bound the dual extremal length $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$ from below. We will do this by picking by picking a good metric to plug into the variational problem for $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$.

Consider the metric $\rho : E_H^\circ \to [0,\infty)$ defined as follows:

$$\rho(e^\circ) = \int_{e^\circ} \frac{|dz|}{|z - \frac{x+y}{2}|}$$

for $e^\circ \in E_H^\circ$ contained in the annulus $\widetilde{\mathcal{A}} = \{z : 4\varepsilon \leqslant \left|z - \frac{x+y}{2}\right| \leqslant \frac{d'}{2}\}$, where we think of e° as a line segment in the plane. If e° is not contained in $\widetilde{\mathcal{A}}$, then $\rho(e^\circ) = 0$. Let $\{u, v\} \in E_H^\circ$ be the dual edge corresponding to the primal edge $\{x, y\}$. Observe that:

1. By Lemma 1.2.3, minimal $O^\bullet$- $W^\bullet$ cuts in $H^\bullet$ correspond to simple paths from N° to S° in H° and vice versa. Since $\{x, y\}$ joins $O^\bullet$ and $W^\bullet$, this edge is part of any $O^\bullet$- $W^\bullet$ cut. Hence, any path from N° to S° in H° must use the edge $\{u, v\}$.

2. Any path from N° to S° in H° is a path from $[D^\circ, A^\circ]$ to $[B^\circ, C^\circ]$ in G°. Using the notation in the statement of Lemma 2.2.2, it follows that the diameter of any such path is at least d'.

By 1, since $\{u, v\} \subseteq B(\frac{x+y}{2}, 4\varepsilon)$, any path in H° from N° to S° must at some point lie in the bounded component of $\mathbb{C}\backslash\widetilde{\mathcal{A}}$. By 2, any path in H° from N° to S° must at some point lie in the unbounded component of $\mathbb{C}\backslash\widetilde{\mathcal{A}}$. Putting all this together, we conclude that any path from N° to S° in H° crosses the annulus $\widetilde{\mathcal{A}}$ at least once. If γ is a piecewise C^1 curve that crosses the annulus $\{z : r < |z - z_0| < R\}$ at least once, where $z_0 \in \mathbb{C}$, $R > r > 0$, then:

$$\int_\gamma \frac{|dz|}{|z - z_0|} \geqslant \log\left(\frac{R}{r}\right)$$

If γ is a path from N° to S° in H°, we know that γ must cross the annulus $\widetilde{A}$ at least once. However, when doing this, it is possible that γ uses edges of E_H° that are not entirely contained in $\widetilde{\mathcal{A}}$ and so have zero mass with respect to ρ. Since edges of G all have length at most ε, edges of G° all have length at most 2ε. Thus, we can be sure that all of the edges $e^\circ \in E_H^\circ$ that we use when crossing the slightly smaller annulus $\{z : 6\varepsilon \leqslant \left|z - \frac{x+y}{2}\right| \leqslant \frac{d'}{2} - 2\varepsilon\}$ do indeed have positive mass. Hence:

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{|z - \frac{x+y}{2}|} \geqslant \log\left(\frac{d' - 4\varepsilon}{12\varepsilon}\right)$$

Let $F_{in}(H)$ denote the set of inner faces of H. Given an inner face $Q \in F_{in}(H)$, let $e_Q^\bullet \in E_H^\bullet, e_Q^\circ \in$

E_H° denote the primal and dual diagonals of Q. Next, we estimate the area of the metric ρ:

$$
\begin{aligned}
A(\rho) &= \sum_{\substack{e^\circ \in E_H^\circ \\ \rho(e^\circ)>0}} \frac{|e^\bullet|}{|e^\circ|} \Big(\int_{e^\circ} \frac{|dz|}{|z - \frac{x+y}{2}|} \Big)^2 \leqslant \sum_{\substack{e^\circ \in E_H^\circ \\ \rho(e^\circ)>0}} \frac{|e^\bullet|}{|e^\circ|} \cdot \frac{|e^\circ|^2}{\min_{z\in e^\circ} |z - \frac{x+y}{2}|^2} \overset{(i)}{\leqslant} 2 \sum_{\substack{Q \in F_{in}(H) \\ \rho(e_Q^\circ)>0}} \frac{\text{Area}(Q)}{\min_{z\in Q} |z - \frac{x+y}{2}|^2} \\
&\overset{(ii)}{\leqslant} 2 \sum_{\substack{Q \in F_{in}(H) \\ \rho(e_Q^\circ)>0}} \int_Q \frac{1}{(|z - \frac{x+y}{2}| - 2\varepsilon)^2} dz_1 dz_2 \overset{(iii)}{\leqslant} 2 \int_{3\varepsilon < |z - \frac{x+y}{2}| < \frac{d'}{2} + \varepsilon} \frac{1}{(|z - \frac{x+y}{2}| - 2\varepsilon)^2} dz_1 dz_2 \\
&= 4\pi \int_{3\varepsilon}^{\frac{d'}{2}+\varepsilon} \frac{s}{(s-2\varepsilon)^2} ds = 4\pi \int_{\varepsilon}^{\frac{d'}{2}-\varepsilon} \frac{1}{u} du + 8\pi\varepsilon \int_{\varepsilon}^{\frac{d'}{2}-\varepsilon} \frac{1}{u^2} du \\
&= 4\pi \log\left(\frac{d' - 2\varepsilon}{\varepsilon} \right) + 8\pi\varepsilon \left(\frac{1}{\varepsilon} - \frac{2}{(d'-2\varepsilon)} \right) \\
&\leqslant 4\pi \log\left(\frac{d' - 2\varepsilon}{\varepsilon} \right) + 8\pi
\end{aligned}
$$

In the calculations above:

- (i) follows from the trivial observation that $\min_{z\in Q} |z - \frac{x+y}{2}| \leqslant \min_{z \in e_Q^\circ} |z - \frac{x+y}{2}|$, as well as the fact that for any inner face Q of H, since Q is a quadrilateral with orthogonal diagonals, $\text{Area}(Q) = \frac{1}{2}|e_Q^\bullet||e_Q^\circ|$.
- (ii) follows from the fact that for any inner face Q of H, since the edges of Q all have length at most ε, Q has diameter at most 2ε. Hence, $|z - \frac{x+y}{2}| - 2\varepsilon \leqslant \min_{z\in Q} |z - \frac{x+y}{2}|$ for any $z \in Q$.
- (iii) follows from the fact that if $\rho(e_Q^\circ) > 0$ for some inner face Q of H, this means that the diagonal e_Q° is contained in the annulus $\widetilde{\mathcal{A}}$. Since the corresponding primal diagonal $e_Q^\bullet$ has length at most ε, any point of Q lies in an ε- neighborhood of e_Q°. In particular, any inner face Q of H such that $\rho(e_Q^\circ) > 0$ is contained in the slightly larger annulus $\{z : 3\varepsilon < |z - \frac{x+y}{2}| < \frac{d'}{2} + \varepsilon\}$.

Since $d' \geqslant 16\varepsilon$, it follows that:

$$A(\rho) \leqslant 4\pi \log\left(\frac{d' - 2\varepsilon}{\varepsilon} \right) + 8\pi \asymp \log\left(\frac{d'}{\varepsilon} \right)$$

If γ is a path from N° to S° in H°, using the fact that $d' \geqslant 16\varepsilon$:

$$\ell_\rho(\gamma) \geqslant \log\left(\frac{d' - 4\varepsilon}{16\varepsilon} \right) \asymp \log\left(\frac{d'}{\varepsilon} \right)$$

Putting all this together, we have that:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \geqslant \inf_\gamma \frac{\ell_\rho^2(\gamma)}{A(\rho)} \gtrsim \log\left(\frac{d'}{\varepsilon} \right)$$

where our infimum is over all paths γ in H° from N° to S°. By duality,

$$\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = \frac{1}{\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)} \lesssim \log\left(\frac{d'}{\varepsilon}\right)$$

Plugging this into Equation 2.2.1, the desired result follows. The corresponding estimate for $\widetilde{h}$ follows by the same argument. $\square$

Having established our estimate for the gradient of h and $\widetilde{h}$ across an edge, we are now ready to state and prove the discrete analogue of Theorem 2.2.1:

Theorem 2.2.3. Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is an orthodiagonal rectangle with edges of length at most ε and let h and $\widetilde{h}$ be the real and imaginary parts of the associated tiling map. That is, $\widetilde{h}$ is the unique solution to the following boundary value problem on G°:

$$\begin{aligned} \widetilde{h}(x) &= 0 \ \text{ for all } x \in [D^\circ, A^\circ] \\ \widetilde{h}(x) &= 1 \ \text{ for all } x \in [B^\circ, C^\circ] \\ \Delta^\circ \widetilde{h}(x) &= 0 \ \text{ for all } x \in V^\circ \backslash ([D^\circ, A^\circ] \cup [B^\circ, C^\circ]) \end{aligned}$$

and h be the harmonic conjugate of $\widetilde{h}$ which solves the following boundary value problem on $G^\bullet$:

$$\begin{aligned} h(x) &= 0 \ \text{ for all } x \in [A^\bullet, B^\bullet] \\ h(x) &= L \ \text{ for all } x \in [C^\bullet, D^\bullet] \\ \Delta^\bullet h(x) &= 0 \ \text{ for all } x \in V^\bullet \backslash ([A^\bullet, B^\bullet] \cup [C^\bullet, D^\bullet]) \end{aligned}$$

where L is the effective resistance between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $G^\bullet$. Define:

$$\begin{aligned} d &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{G} \text{ joining } [A^\bullet, B^\bullet]_{\partial \widehat{G}} \text{ and } [C^\bullet, D^\bullet]_{\partial \widehat{G}}\} \\ d' &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{G} \text{ joining } [B^\bullet, C^\bullet]_{\partial \widehat{G}} \text{ and } [D^\bullet, A^\bullet]_{\partial \widehat{G}}\} \end{aligned}$$

Then there exists an absolute constant $K > 0$ so that:

$$|h(y) - h(x)| \leqslant \frac{K}{\log\left(\frac{d'}{(d_{\widehat{G}}(x,y) \wedge d_{cc}^{\widehat{G}}(x,y)) \vee \varepsilon}\right)}, \qquad |\widetilde{h}(u) - \widetilde{h}(v)| \leqslant \frac{KL}{\log\left(\frac{d}{(d_{\widehat{G}}(u,v) \wedge d_{cc}^{\widehat{G}}(u,v)) \vee \varepsilon}\right)}$$

for any $x, y \in V^\bullet$, $u, v \in V^\circ$ such that . In particular, in the bulk (when $|x - y| \leqslant \text{dist}(x, \partial \widehat{G}), \text{dist}(y, \partial \widehat{G})$ and $|u - v| \leqslant \text{dist}(u, \partial \widehat{G}), \text{dist}(v, \partial \widehat{G})$), we have that:

$$|h(y) - h(x)| \leqslant \frac{K}{\log\left(\frac{d'}{|y - x| \vee \varepsilon}\right)}, \qquad |\widetilde{h}(u) - \widetilde{h}(v)| \leqslant \frac{KL}{\log\left(\frac{d}{|u - v| \vee \varepsilon}\right)}$$

A slightly weaker estimate (with $\sqrt{log(\cdot)}$ in place of $\log(\cdot)$) is proven in [3] in the case where our orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a good approximation of a conformal rectangle

(Ω, A, B, C, D) where Ω is a Jordan domain and $[A, B]_{\partial\Omega}, [B, C]_{\partial\Omega}, [C, D]_{\partial\Omega}, [D, A]_{\partial\Omega}$ are analytic arcs that don't meet at cusps. In the context of [3], "good" means that the edges of G have length at most ε and discrete boundary arcs are δ- close to the corresponding continuous boundary arcs in Hausdorff distance for some small $\varepsilon, \delta > 0$. For more details, see Theorem 3 of [3].

We should also point out that while the estimates for the modulus of continuity of h and $\widetilde{h}$ in Theorem 2.2.3 are novel for points near the boundary of G, even when our orthodiagonal map is just a chunk of the square grid, in the bulk we can do significantly better. Namely, in [14], Chelkak, Laslier and Russkikh show that bounded discrete harmonic and discrete holomorphic functions on t- embeddings whose corresponding origami map is κ- Lipschitz on large scales for some $\kappa \in (0, 1)$, are β- Hölder in the bulk for some absolute constant $\beta \in (0, 1)$ (see Proposition 6.13 of [14]). The condition on the origami map here is known as "$Lip(\kappa, \delta)$" where $\delta > 0$ is the scale on which our origami map is κ- Lipschitz. t- embeddings are a more general class of graphs embedded in the plane that accommodate a notion of discrete complex analysis. That is, every orthodiagonal map is a t- embedding (see Section 8.1 of [14]). While it is not explicitly stated in their paper, it is not hard to show that if G is an orthodiagonal map with edges of length at most ε, then for any $\kappa \in (0, 1)$ there exists an absolute constant $c = c(\kappa) > 0$ such that, as a t -embedding, G satisfies the condition $Lip(\kappa, c\varepsilon)$. While the estimates in the bulk provided by Theorem 2.2.3 are sufficient for proving the main result of this Chapter, Theorem 2.1.2, as an immediate consequence of Proposition 6.13 of [14] we have that:

Proposition 2.2.4. Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is an orthodiagonal rectangle with edges of length at most ε and h and $\widetilde{h}$ are the real and imaginary parts of the corresponding tiling map. If $x \in V$, let d_x denote the distance from x to $\partial\widehat{G}$. Then there exists absolute constants $C, C' > 0$, $\beta \in (0, 1)$ such that:

$$|h(y) - h(x)| \leqslant CL\left(\frac{|x - y|}{d_x \wedge d_y}\right)^\beta \qquad\qquad |\widetilde{h}(u) - \widetilde{h}(v)| \leqslant C\left(\frac{|u - v|}{d_u \wedge d_v}\right)^\beta$$

for $x, y \in V^\bullet$, $u, v \in V^\circ$ such that $C'\varepsilon \leqslant |x - y| \leqslant d_x \wedge d_y$, $C'\varepsilon \leqslant |u - v| \leqslant d_u \wedge d_v$.

All that having been said, we now turn to the proof of Theorem 2.2.3:

Proof. Suppose $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a orthodiagonal rectangle with edges of length at most ε and h is the real part of the corresponding tiling map. Fix $x, y \in V^\bullet$. Since we are taking the maximum of $d_{\widehat{G}}(x, y) \wedge d_{cc}^{\widehat{G}}(x, y)$ and ε in our estimate for $|h(y) - h(x)|$ in Theorem 2.2.3, we can assume WLOG that $d_{\widehat{G}}(x, y) \wedge d_{cc}^{\widehat{G}}(x, y) \geqslant \varepsilon$, since the relevant estimate in the case where $d_{\widehat{G}}(x, y) \wedge d_{cc}^{\widehat{G}}(x, y) < \varepsilon$ follows from the case where $d_{\widehat{G}}(x, y) \wedge d_{cc}^{\widehat{G}}(x, y) \geqslant \varepsilon$. If $h(y) = h(x)$, we're done. Otherwise, suppose WLOG that $h(x) < h(y)$. We now consider two cases:

Case 1: $d_{\widehat{G}}(x, y) \leqslant d_{cc}^{\widehat{G}}(x, y)$

Similar to the proofs of Theorem 2.2.1 and Lemma 2.2.2, we begin by reinterpreting the quantity we want to estimate, $\big(h(y) - h(x)\big)$, as an extremal length. Consider the sets S_x and S_y defined as follows:

$$S_x := \{z \in V^\bullet : h(z) \leqslant h(x)\} \qquad\qquad S_y := \{z \in V^\bullet : h(z) \geqslant h(y)\}$$

By the maximum principle for harmonic functions, S_x, S_y and $V^\bullet \backslash (S_x \sqcup S_y)$ are all connected subsets of $G^\bullet$. Furthermore, $[A^\bullet, B^\bullet] \subseteq S_x$, $[C^\bullet, D^\bullet] \subseteq S_y$. Let $H = (V_H^\bullet \sqcup V_H^\circ, E_H)$ be the suborthodiagonal map of G formed by gluing together all the faces of G that are incident to at least one vertex of $V^\bullet \backslash (S_x \sqcup S_y)$. By the maximum principle, H is simply connected with a unique, distinguished exterior face. Furthermore, let:

$$O^\bullet := S_x \cap \partial V_H^\bullet, \qquad\qquad W^\bullet := S_y \cap \partial V_H^\bullet$$

Then $O^\bullet$ and $W^\bullet$ are primal boundary arcs of H with corresponding dual arcs:

$$N^\circ := [D^\circ, A^\circ] \cap \partial V_H^\circ, \qquad\qquad S^\circ := [B^\circ, C^\circ] \cap \partial V_H^\circ$$

Proposition 1.1.4 tells us that for any function $g : V_H^\bullet \to \mathbb{R}$ with $\mathrm{gap}_{O^\bullet, W^\bullet}(g) \geqslant 0$ and any flow θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$:

$$\mathrm{strength}(\theta) \cdot \mathrm{gap}_{O^\bullet, W^\bullet}(g) \leqslant \mathcal{E}^\bullet(\theta; H)^{1/2} \mathcal{E}^\bullet(g; H)^{1/2}$$

Plugging $g = h$ into the inequality above, we have that for any choice of flow θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$:

$$|h(y) - h(x)| \leqslant \frac{\mathcal{E}^\bullet(\theta; H)^{1/2} \mathcal{E}^\bullet(h; H)^{1/2}}{\mathrm{strength}(\theta)}$$

By Thomson's principle, taking the infimum over all flows θ from $O^\bullet$ to $W^\bullet$ in $H^\bullet$ in the expression on the RHS, we have that:

$$|h(y) - h(x)| \leqslant \mathcal{E}^\bullet(h; H)^{1/2} \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)^{1/2} \tag{2.2.2}$$

In Section 1.3 we saw that $\mathcal{E}^\bullet(h)$ is the total area of rectangles in the tiling associated with the orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$. Hence, by the definition of H, the restriction $\mathcal{E}^\bullet(h; H)$ is the total area of rectangles in our tiling that intersect $(h(x), h(y)) \times (0, 1)$. Since the edges of G have length at most ε, Lemma 2.2.2 tells us that the width of any rectangle in our tiling is at most $K\big(\log\big(\frac{d'}{\varepsilon}\big)\big)^{-1}$, where $K > 0$ is an absolute constant. Hence:

$$\mathcal{E}^\bullet(h; H) \leqslant |h(y) - h(x)| + \frac{2K}{\log\big(\frac{d'}{\varepsilon}\big)} \tag{2.2.3}$$

Using the shorthand $\lambda^\bullet = \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)$, plugging our estimate in Equation 2.2.3 into Equation 2.2.2 and rearranging, we have that:

$$|h(y) - h(x)| \leqslant \frac{1}{2}\Big(\lambda^\bullet + \sqrt{(\lambda^\bullet)^2 + \frac{8K\lambda^\bullet}{\log\left(\frac{d'}{\varepsilon}\right)}}\Big) \leqslant \lambda^\bullet + \frac{2K}{\log\left(\frac{d'}{\varepsilon}\right)} = \lambda^\bullet(O^\bullet \leftrightarrow W^\bullet; H^\bullet) + \frac{2K}{\log\left(\frac{d'}{\varepsilon}\right)} \tag{2.2.4}$$

By duality:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = 1$$

Thus, to bound $\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)$ and therefore $|h(y) - h(x)|$ from above, it suffices to bound the dual extremal length $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$ from below. We will do this by picking by picking a good metric to plug into the variational problem for $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$.

By the definition of $d_{\widehat{G}}(x,y)$ we can find a smooth curve γ in $\widehat{G}$ from x to y so that length$(\gamma) < d_\Omega(x,y) + \varepsilon$. By the maximum principle for discrete harmonic functions, $\widehat{G}\backslash\widehat{H}$ consists of two connected components Ω_1 and Ω_2 so that WLOG, $[A^\bullet, B^\bullet] \subseteq \partial\Omega_1$ and $[C^\bullet, D^\bullet] \subseteq \partial\Omega_1$. Thinking of $\widehat{H}$ as a conformal rectangle with the distinguished boundary arcs:

$$\widehat{W} := \partial\Omega_1 \cap \partial\widehat{H} \qquad \widehat{O} := \partial\Omega_2 \cap \partial\widehat{H}$$
$$\widehat{N} := [D^\bullet, A^\bullet]_{\partial\widehat{G}} \cap \partial\widehat{H} \qquad \widehat{S} := [B^\bullet, C^\bullet]_{\partial\widehat{G}} \cap \partial\widehat{H}$$

our curve γ from x to y in $\widehat{G}$ must have a subarc γ' with endpoints $x' \in \widehat{W}, y' \in \widehat{O}$, so that γ' is a crosscut of $\widehat{H}$. By the definition of $\widehat{H}$, every point of $\widehat{W}$ lies within ε of a point of $W^\bullet$ and every point of $\widehat{O}$ lies within ε of a point in $O^\bullet$. Thus, perturbing γ' slightly, we can produce a crosscut γ'' of $\widehat{H}$ with endpoints $x'' \in \widehat{W}, y'' \in \widehat{O}$ so that length$(\gamma'') \leqslant d_{\widehat{G}}(x,y) + 3\varepsilon$. Since γ'' is a crosscut of $\widehat{H}$ joining $O^\bullet$ and $W^\bullet$, γ'' separates N° and S° in $\widehat{H}$. Hence, any path in $\widehat{H}$ and therefore H° from N° to S° must intersect γ''. Consider the annulus:

$$\widetilde{\mathcal{A}} := \{u \in \mathbb{C} : d_{\widehat{G}}(x,y) + 3\varepsilon < |u - x''| < \frac{d'}{2}\}$$

Observe that:

1. Since length$(\gamma'') \leqslant d_{\widehat{G}}(x,y) + 3\varepsilon$, it follows that $\gamma'' \subseteq B(x'', d_{\widehat{G}}(x,y) + 3\varepsilon)$. Since any path from N° to S° in $\widehat{H}$ must intersect γ'', such a path must also intersect the ball $B(x'', d_{\widehat{G}}(x,y) + 3\varepsilon)$.

2. On the other hand, since any path from N° to S° in $\widehat{H}$ is a also path from $[D^\bullet, A^\bullet]_{\partial\widehat{G}}$ to $[B^\bullet, C^\bullet]_{\partial\widehat{G}}$ in $\widehat{G}$, any such curve has diameter $\geqslant d'$. Hence, any curve from N° to S° in $\widehat{H}$ must at some point exit the ball $B(x'', \frac{d'}{2})$.

Putting all this together, we have that any path from N° to S° in H° must cross the annulus $\widetilde{\mathcal{A}}$. Having established this fact, consider the metric $\rho : E_H^\circ \to [0,\infty)$ defined as follows:

$$\rho(e^\circ) = \int_{e^\circ} \frac{|dz|}{|z - x''|}$$

for $e^\circ \in E_H^\circ$ contained in the annulus $\widetilde{\mathcal{A}}$, where we think of e° as a line segment in the plane. If e° is not contained in $\widetilde{\mathcal{A}}$, then $\rho(e^\circ) = 0$. Suppose γ is a path in H° from N° to S°. We saw earlier that γ must cross the annulus $\widetilde{\mathcal{A}}$ at least once. However, when it does this, it is possible that γ uses edges $e^\circ \in E_H^\circ$ that are not entirely contained in $\widetilde{\mathcal{A}}$ and so have zero mass with respect to ρ. Since edges of G have length at most ε, edges of G° have length at most 2ε. Thus, we can be sure that all of the edges we use when crossing the slightly smaller annulus $\{u \in \mathbb{C} : d_{\widehat{G}}(x, y) + 5\varepsilon < |u - x''| < \frac{d'}{2} - 2\varepsilon\}$ do indeed have positive mass. Hence:

$$l_\rho(\gamma) = \int_\gamma \frac{|dz|}{|z - x''|} \geqslant \log\left(\frac{d' - 4\varepsilon}{2(d_{\widehat{G}}(x, y) + 5\varepsilon)}\right)$$

By the same argument as in the proof of Lemma 2.2.2:

$$A(\rho) \leqslant 4\pi \log\left(\frac{d' - 2\varepsilon}{2\, d_{\widehat{G}}(x, y)}\right) + \frac{8\pi\varepsilon}{d_{\widehat{G}}(x, y)}$$

By our assumption that $\varepsilon \leqslant d_{\widehat{G}}(x, y) < \frac{d'}{4}$ and $\varepsilon < \frac{d'}{12}$, we have that:

$$A(\rho) \leqslant 4\pi \log\left(\frac{d' - 2\varepsilon}{2\, d_{\widehat{G}}(x, y)}\right) + \frac{8\pi\varepsilon}{d_{\widehat{G}}(x, y)} \lesssim \log\left(\frac{d'}{d_{\widehat{G}}(x, y)}\right)$$

Similarly, if γ is a path from N° to S° in H°, we have that:

$$l_\rho(\gamma) = \int_\gamma \frac{|dz|}{|z - x''|} \geqslant \log\left(\frac{d' - 4\varepsilon}{2(d_{\widehat{G}}(x, y) + 5\varepsilon)}\right) \gtrsim \log\left(\frac{d'}{d_{\widehat{G}}(x, y)}\right)$$

Putting all this together, we have that:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \geqslant \inf_\gamma \frac{\ell_\rho^2(\gamma)}{A(\rho)} \gtrsim \log\left(\frac{d'}{d_{\widehat{G}}(x, y)}\right)$$

where our infimum is over all paths γ in H° from N° to S°. By duality,

$$\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = \frac{1}{\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)} \lesssim \frac{1}{\log\left(\frac{d'}{d_{\widehat{G}}(x,y)}\right)}$$

Plugging this into Equation 2.2.4, the desired result follows.

Case 2: $d_{cc}^{\widehat{G}}(x, y) \leqslant d_{\widehat{G}}(x, y)$ By the definition of $d_{cc}^{\widehat{G}}(x, y)$, we can find a crosscut γ of $\widehat{G}$ that joins x and y to one of the four distinguished boundary arcs of $(\widehat{G}, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ and separates it from the opposite boundary arc, such that $\text{length}(\gamma) < d_{cc}^{\widehat{G}}(x, y) + \varepsilon$. We now split our problem into two further cases, depending on whether the relevant boundary arcs of $(\widehat{G}, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ are Dirichlet arcs where h is constant, or Neumann arcs along which h is monotone.

Case 2.1: γ joins x and y to one of the Dirichlet arcs and separates them from the opposite Dirichlet arc.

WLOG, suppose γ joins x and y to $[A^\bullet, B^\bullet]_{\partial \widehat{G}}$ and separates them from $[C^\bullet, D^\bullet]_{\partial \widehat{G}}$. Let $\widehat{N}^\gamma_{x,y}$ be the connected component of $\widehat{G} \backslash \gamma$ containing x and y. Define:

$$(N^\gamma_{x,y})^\bullet := V^\bullet \cap \widehat{N}^\gamma_{x,y}, \qquad\qquad \gamma^\bullet := \{z \in V^\bullet : \operatorname{dist}(z, \gamma) \leqslant \varepsilon\}$$

By the maximum principle for discrete harmonic functions,

$$\max_{z \in (N^\gamma_{x,y})^\bullet} h(z) \leqslant \max_{z \in \gamma^\bullet} h(z)$$

Let v be a vertex of $\gamma^\bullet$ so that $h(v) = \max_{z \in \gamma^\bullet} h(z)$ and for some neighboring vertex u of v in $V^\bullet$, $h(u) < h(v)$. Since $x, y \in (N^\gamma_{x,y})^\bullet$, we have that:

$$(h(y) - h(x)) < h(v)$$

Similar to our argument in case 1, let H be the suborthodiagonal map of G formed by gluing together all of the quadrilaterals of G that are tangent to a vertex z of $V^\bullet$ such that $0 \leqslant h(z) < h(v)$. By the maximum principle, H is simply connected with a unique, distinguished exterior face. Furthermore, let:

$$O^\bullet := [A^\bullet, B^\bullet], \qquad\qquad W^\bullet := \{z \in V^\bullet : h(z) \geqslant h(v)\} \cap \partial V^\bullet_H$$

Then $O^\bullet$ and $W^\bullet$ are primal boundary arcs of H with corresponding dual arcs:

$$N^\circ := [B^\circ, C^\circ] \cap \partial V^\circ_H, \qquad\qquad S^\circ := [D^\circ, A^\circ] \cap \partial V^\circ_H$$

With these pairs of distinguished primal and dual arcs, H is an orthodiagonal rectangle. By the same argument as in case 1, almost verbatim, we have that:

$$|h(y) - h(x)| \leqslant h(v) \leqslant \lambda^\bullet(O^\bullet \leftrightarrow W^\bullet; H^\bullet) + \frac{2K}{\log\left(\frac{d'}{\varepsilon}\right)} \tag{2.2.5}$$

By duality:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = 1$$

Thus, to bound $\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)$ and therefore $|h(y) - h(x)|$ from above, it suffices to bound the dual extremal length $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$ from below. We will do this by picking a good metric to plug into the variational formula for $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$.

Observe that γ is a crosscut of $\widehat{G}$ that starts and ends at a point of $[A^\bullet, B^\bullet]_{\partial \widehat{G}}$ and, by the definition of $\gamma^\bullet$, v lies within ε of γ. Since v has a neighboring vertex u with the property that $h(u) < h(v)$, we also have that $v \in W^\bullet$. Since the edges of G have length at most ε, putting all this together, it follows that there exists a crosscut γ' of $\widehat{H}$ of length at most $\frac{1}{2}(d_{cc}^{\widehat{G}}(x,y) + 5\varepsilon)$ joining $W^\bullet$ and $O^\bullet$ in $\widehat{H}$, thereby separating N° and S° in $\widehat{H}$. Pick a point x' in γ' and consider the annulus:

$$\widetilde{A} = \{u \in \mathbb{C} : \frac{1}{2}(d_{cc}^{\widehat{G}}(x,y) + 5\varepsilon) < |u - x'| < \frac{d'}{2}\}$$

Since any path in $\widehat{G}$ from $[B^\circ, C^\circ]_{\partial \widehat{G}}$ to $[A^\circ, D^\circ]_{\partial \widehat{G}}$ has diameter greater than or equal to d', the same is true of any path in $\widehat{H}$ from N° to S°. On the other hand, since γ' separates N° and S° in $\widehat{H}$, any such path must intersect γ' and therefore $B(x', \frac{1}{2}(d_{cc}^{\widehat{G}}(x,y) + 5\varepsilon))$. Putting all this together, we conclude that any path from N° to S° in $\widehat{H}$ must cross the annulus $\widetilde{A}$. Hence, if we define the metric $\rho : E_H^\circ \to (0, \infty)$ by the formula:

$$\rho(e^\circ) = \begin{cases} \int_{e^\circ} \frac{|dz|}{|z-x'|} & \text{for edges } e^\circ \in E_H^\circ \text{ contained in the annulus } \widetilde{A} \\ 0 & \text{otherwise} \end{cases}$$

by the same argument as in case 1, almost verbatim, plugging ρ into the variational formula for $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$, we have that:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \gtrsim \log\left(\frac{d'}{d_{cc}^{\widehat{G}}(x,y)}\right)$$

By duality:

$$\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) \lesssim \frac{1}{\log\left(\frac{d'}{d_{cc}^{\widehat{G}}(x,y)}\right)}$$

Plugging this into Equation 2.2.5, the desired result follows.

<u>**Case 2.2**</u>: γ joins x and y to one of the Neumann arcs and separates them from the opposite Neumann arc.

WLOG, suppose γ joins x and y to $[B^\bullet, C^\bullet]_{\partial \widehat{G}}$ and separates them from $[D^\bullet, A^\bullet]_{\partial \widehat{G}}$. Similar to case 2.1, let $\widehat{N}_{x,y}^\gamma$ be the connected component of $\widehat{G} \backslash \gamma$ containing x and y and define:

$$(N_{x,y}^\gamma)^\bullet := V^\bullet \cap \widehat{N}_{x,y}^\gamma, \qquad \gamma^\bullet := \{z \in V^\bullet : \mathrm{dist}(z, \gamma) \leqslant \varepsilon\}$$

By the maximum principle for harmonic functions, since $h^\bullet$ is harmonic on the part of the boundary of $(N_{x,y}^\gamma)^\bullet$ that intersects $[B^\bullet, C^\bullet]$, we have that:

$$\min_{z \in \gamma^\bullet} h(z) \leqslant \min_{z \in (\widehat{N}_{x,y}^\gamma)^\bullet} h(z), \qquad \max_{z \in (\widehat{N}_{x,y}^\gamma)^\bullet} h(z) \leqslant \max_{z \in \gamma^\bullet} h(z)$$

Let v_1 and v_2 be vertices of $\gamma^\bullet$ so that $h(v_1) = \min_{z \in \gamma^\bullet} h(z)$, $h(v_2) = \max_{z \in \gamma^\bullet} h(z)$ and v_1 and v_2 have neighboring vertices u_1 and u_2 so that $h(v_1) < h(u_1)$ and $h(v_2) > h(u_2)$. Since

$x, y \in (N_\gamma^{x,y})^\bullet$, we have that:

$$\big(h(y) - h(x)\big) \leqslant \big(h(v_2) - h(v_1)\big)$$

Similar to our argument in case 1, let H be the suborthodiagonal map of G formed by gluing together all of the quadrilaterals of G that are tangent to a vertex z of $V^\bullet$ such that $h(v_1) \leqslant h(z) < h(v_2)$. By the maximum principle, H is simply connected with a unique, distinguished exterior face. Furthermore, let:

$$O^\bullet := \{z \in V^\bullet : h(z) \geqslant h(v_1)\} \cap \partial V_H^\bullet, \qquad W^\bullet := \{z \in V^\bullet : h(z) \leqslant h(v_2)\} \cap \partial V_H^\bullet$$

Then $O^\bullet$ and $W^\bullet$ are primal boundary arcs of H with corresponding dual arcs:

$$N^\circ := [D^\circ, A^\circ] \cap \partial V_H^\circ, \qquad S^\circ := [B^\circ, C^\circ] \cap \partial V_H^\circ$$

With these pairs of distinguished primal and dual arcs, H is an orthodiagonal rectangle. By the same argument as in case 1, almost verbatim, we have that:

$$|h(y) - h(x)| \leqslant h(v_2) - h(v_1) \leqslant \lambda^\bullet(O^\bullet \leftrightarrow W^\bullet; H^\bullet) + \frac{2K}{\log\left(\frac{d'}{\varepsilon}\right)} \tag{2.2.6}$$

where $K > 0$ is an absolute constant. By duality:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \cdot \lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) = 1$$

Thus, to bound $\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet)$ and therefore $|h(y) - h(x)|$ from above, it suffices to bound the dual extremal length $\lambda(N^\circ \leftrightarrow S^\circ; H^\circ)$ from below.

Since $v_1 \in W^\bullet$, $v_2 \in O^\bullet$ and each of these points lies within ε of γ, there exists a crosscut γ' of $\widehat{H}$ of length at most $d_{cc}^{\widehat{G}}(x, y) + 3\varepsilon$ joining $O^\bullet$ and $W^\bullet$ in $\widehat{H}$ and thereby separating N° and S° in $\widehat{H}$. Hence, by the same argument we used in cases 1 and 2.1, we have that:

$$\lambda(N^\circ \leftrightarrow S^\circ; H^\circ) \gtrsim \log\left(\frac{d'}{d_{cc}^{\widehat{G}}(x, y)}\right)$$

By duality:

$$\lambda(O^\bullet \leftrightarrow W^\bullet; H^\bullet) \lesssim \frac{1}{\log\left(\frac{d'}{d_{cc}^{\widehat{G}}(x,y)}\right)}$$

Plugging this into Equation 2.2.6, the desired result follows.

Having shown that h has the prescribed modulus of continuity, observe that the analogous estimate for $\tilde{h}$ follows by the same argument. This completes our proof. $\square$

2.2.3 Two-Sided Estimates for Extremal Length on Orthodiagonal Maps

In this section, we prove two- sided estimates for the discrete extremal length of an orthodiagonal rectangle that is a (δ, ε)- good approximation of some conformal rectangle (Ω, A, B, C, D). This gives us uniform boundedness of the tiling maps $(\phi_n)_{n=1}^{\infty}$ in Theorem 2.1.2.

Proposition 2.2.5. Suppose (Ω, A, B, C, D) is a conformal rectangle and $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a (δ, ε)- good interior approximation of (Ω, A, B, C, D). Suppose $\delta < \frac{\ell \wedge \ell'}{2}$ where:

$$\ell = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [A, B]_{\partial\Omega*} \text{ and } [C, D]_{\partial\Omega*}\},$$
$$\ell' = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [B, C]_{\partial\Omega*} \text{ and } [D, A]_{\partial\Omega*}\}$$

Then:

$$\frac{(\ell - 2\delta)^2}{2 \cdot \text{Area}(\Omega)} \leqslant \lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; G^\bullet) \leqslant \frac{2 \cdot \text{Area}(\Omega)}{(\ell' - 2\delta)^2}$$

Proof. Let:

$$\lambda^\bullet = \lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; G^\bullet), \qquad \lambda^\circ = \lambda([B^\circ, C^\circ] \leftrightarrow [D^\circ, A^\circ]; G^\circ)$$

Plugging the metric $\rho(e^\bullet) = |e^\bullet|$ into the variational problem for $\lambda^\bullet$, we have that:

$$\begin{aligned}
\lambda^\bullet &\geqslant \frac{\Big(\inf\{\sum_{e^\bullet \in \gamma} |e^\bullet| : \gamma \text{ is a path from } [A^\bullet, B^\bullet] \text{ to } [C^\bullet, D^\bullet] \text{ in } G^\bullet\}\Big)^2}{\sum_{e^\bullet \in E^\bullet} \frac{|e^\circ|}{|e^\bullet|} |e^\bullet|^2} \\
&= \frac{\Big(\inf\{\text{length}(\gamma) : \gamma \text{ is a path from } [A^\bullet, B^\bullet] \text{ to } [C^\bullet, D^\bullet] \text{ in } G^\bullet\}\Big)^2}{2 \cdot \text{Area}(\widehat{G})} \\
&\geqslant \frac{(\ell - 2\delta)^2}{2 \cdot \text{Area}(\Omega)}
\end{aligned}$$

The equality on the second line follows from the fact that if Q is an inner face of G with primal diagonal $e^\bullet$ and dual diagonal e°, then $\text{Area}(Q) = \frac{1}{2}|e^\bullet||e^\circ|$. Thus:

$$\sum_{e^\bullet \in E^\bullet} |e^\bullet||e^\circ| = 2 \sum_{Q \in F_{in}} \text{Area}(Q) = 2 \cdot \text{Area}(\widehat{G})$$

In the inequality on the third line, two things are going on. On the one hand, since G is an interior approximation of Ω, $\text{Area}(\widehat{G}) \leqslant \text{Area}(\Omega)$. On the other hand, suppose γ is a path from $[A^\bullet, B^\bullet]$ to $[C^\bullet, D^\bullet]$ in $G^\bullet$. Since $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a (δ, ε)- good approximation of (Ω, A, B, C, D) (the "δ" is really the relevant part here), we can modify any such path to get a path of length at most $\text{length}(\gamma) + 2\delta$ from $[A, B]_{\partial\Omega*}$ to $[C, D]_{\partial\Omega*}$ in Ω. By the definition of ℓ:

$$\text{length}(\gamma) + 2\delta \geqslant \ell$$

Since this is true of any curve γ from $[A^\bullet, B^\bullet]$ to $[C^\bullet, D^\bullet]$ in $G^\bullet$, the desired result follows.

Similarly, plugging the metric $\rho(e^\circ) = |e^\circ|$ into the variational problem for λ° we have that:

$$\begin{aligned}
\lambda^\circ &\geqslant \frac{\Big(\inf\{\sum\limits_{e^\circ \in \gamma} |e^\circ| : \gamma \text{ is a path from } [B^\circ, C^\circ] \text{ to } [D^\circ, A^\circ] \text{ in } G^\circ\}\Big)^2}{\sum\limits_{e^\bullet \in E^\circ} \frac{|e^\bullet|}{|e^\circ|}|e^\circ|^2} \\
&= \frac{\Big(\inf\{\text{length}(\gamma) : \gamma \text{ is a path from } [B^\circ, C^\circ] \text{ to } [D^\circ, A^\circ] \text{ in } G^\circ\}\Big)^2}{2 \cdot \text{Area}(\widehat{G})} \\
&\geqslant \frac{(\ell' - 2\delta)^2}{2 \cdot \text{Area}(\Omega)}
\end{aligned}$$

By Ford- Fulkerson duality for discrete rectangles (Corollary 1.2.1):

$$\lambda^\bullet = \frac{1}{\lambda^\circ} \leqslant \frac{2 \cdot \text{Area}(\Omega)}{(\ell' - 2\delta)^2}$$

□

The two- sided estimate in Proposition 2.2.5 is very coarse, but it is sufficient for our purposes. It is however worth noting that, as a consequence of our estimates for the modulus of continuity of rectangle tiling maps as well as the corresponding limiting conformal map, we can do significantly better if we are dealing with continuous approximations of a continuous domain or discrete approximations of a discrete domain.

Proposition 2.2.6. If $(\Omega', A', B', C', D')$ is a δ- good approximation of the conformal rectangle (Ω, A, B, C, D), L is the extremal length between $[A, B]_{\partial\Omega*}$ and $[C, D]_{\partial\Omega*}$ in Ω, L' is the extremal length between $[A', B']_{\partial\Omega'}$ and $[C', D']_{\partial\Omega'}$ in Ω', and $\delta > 0$ satisfies $\delta \leqslant \frac{d'}{2}e^{-\frac{4\pi}{L}}$ and $\delta \leqslant \frac{d}{2}e^{-16\pi L}$, where:

$$\begin{aligned}
d &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [A, B]_{\partial\Omega*} \text{ and } [C, D]_{\partial\Omega*}\} \\
d' &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [B, C]_{\partial\Omega*} \text{ and } [D, A]_{\partial\Omega*}\}
\end{aligned}$$

Then:

$$-\frac{8\pi}{\log\left(\frac{d'}{2\delta}\right)} \leqslant L' - L \leqslant \frac{16\pi L^2}{\log\left(\frac{d}{2\delta}\right)}$$

Proof. Let $\phi : \Omega \to \mathcal{R}_L$ be the conformal map from Ω to the rectangle $\mathcal{R}_L$ so that the prime ends A, B, C, D of Ω are mapped to the four corners of $\mathcal{R}_L$ and in particular, $\phi(A) = i$. We write $\phi = h + i\widetilde{h}$ where h is the real part and $\widetilde{h}$ is the imaginary part of ϕ. Then $\|\nabla h\| = \|\nabla \widetilde{h}\|$ is the extremal metric giving us the extremal length between $[A, B]_{\partial\Omega*}$ and $[C, D]_{\partial\Omega*}$ in Ω. By

Theorem 2.2.1,

$$h(z) \leqslant \frac{2\pi}{\log\left(\frac{d'}{2\delta}\right)} \text{ for } z \in [A', B']_{\partial\Omega'} \qquad h(z) \geqslant L - \frac{2\pi}{\log\left(\frac{d'}{2\delta}\right)} \text{ for } z \in [C', D']_{\partial\Omega'}$$
$$\widetilde{h}(z) \leqslant \frac{2\pi L}{\log\left(\frac{d}{2\delta}\right)} \text{ for } z \in [B', C']_{\partial\Omega'} \qquad \widetilde{h}(z) \geqslant 1 - \frac{2\pi L}{\log\left(\frac{d}{2\delta}\right)} \text{ for } z \in [D', A']_{\partial\Omega'}$$

The condition that $\delta \leqslant \frac{d'}{2}e^{-\frac{4\pi}{L}}$ ensures that the values of h on $[C', D']_{\partial\Omega'}$ are larger than the values of h on $[A', B']_{\partial\Omega'}$. Analogously, the condition that $\delta \leqslant \frac{d'}{2}e^{-16\pi L}$ ensures that the values of $\widetilde{h}$ on $[D', A']_{\partial\Omega'}$ are larger than the values of $\widetilde{h}$ on $[B', C']_{\partial\Omega'}$. Plugging the metric $\|\nabla h\|$ into the variational formula for L', we have that:

$$L' \geqslant \frac{\left(L - \frac{4\pi}{\log\left(\frac{d'}{2\delta}\right)}\right)^2}{L} \geqslant L - \frac{8\pi}{\log\left(\frac{d'}{2\delta}\right)} \tag{2.2.7}$$

Similarly, plugging $\|\nabla\widetilde{h}\|$ into the variational formula for the dual problem, we have that:

$$\frac{1}{L'} \geqslant \frac{\left(1 - \frac{4\pi L}{\log\left(\frac{d}{2\delta}\right)}\right)^2}{L} \qquad \Longrightarrow \qquad L' \leqslant \frac{L}{\left(1 - \frac{8\pi L}{\log\left(\frac{d}{2\delta}\right)}\right)} \leqslant L + \frac{16\pi L^2}{\log\left(\frac{d}{2\delta}\right)} \tag{2.2.8}$$

Combining Equations 2.2.7 and 2.2.8, we arrive at the desired result. ☐

Proposition 2.2.7. Suppose $(H, W^\bullet, X^\bullet, Y^\bullet, Z^\bullet)$ is a suborthodiagonal rectangle of $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ so that
$(\widehat{H}, W^\bullet, X^\bullet, Y^\bullet, Z^\bullet)$ is a δ- good interior approximation of $(\widehat{G}, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ and $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε. Let L denote the extremal length between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $G^\bullet$ and let L' denote the extremal length between $[W^\bullet, X^\bullet]$ and $[Y^\bullet, Z^\bullet]$ in $H^\bullet$. Suppose also that δ and ε satisfy $(\delta \vee \varepsilon) \leqslant d'e^{-\frac{2K}{L}}$ and $(\delta \vee \varepsilon) \leqslant de^{-8KL}$, where:

$$d = \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [A^\bullet, B^\bullet]_{\partial\widehat{G}} \text{ and } [C^\bullet, D^\bullet]_{\partial\widehat{G}}\}$$
$$d' = \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } [B^\bullet, C^\bullet]_{\partial\widehat{G}} \text{ and } [D^\bullet, A^\bullet]_{\partial\widehat{G}}\}$$

and $K > 0$ is the absolute constant from Theorem 2.2.3. Then:

$$-\frac{4K}{\log\left(\frac{d'}{\delta\vee\varepsilon}\right)} \leqslant L' - L \leqslant \frac{8KL^2}{\log\left(\frac{d}{\delta\vee\varepsilon}\right)}$$

Proof. Let $h : V^\bullet \to \mathbb{R}$ and $\widetilde{h} : V^\circ \to \mathbb{R}$ be the real and imaginary parts of the tiling map associated to the orthodiagonal rectangle $(G, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$. Then the metric $\rho^\bullet$ on $G^\bullet$ given by the formula $\rho^\bullet(u, v) = |h(u) - h(v)|$ for any edge $\{u, v\} \in E^\bullet$ is the extremal metric giving us the extremal length between $[A^\bullet, B^\bullet]$ and $[C^\bullet, D^\bullet]$ in $G^\bullet$. Similarly, the metric ρ° on G° given by the formula $\rho^\circ(u,v) = |\widetilde{h}(u) - \widetilde{h}(v)|$ for any edge $\{u,v\} \in E^\circ$ is the extremal metric giving us the extremal length between $[B^\circ, C^\circ]$ and $[D^\circ, A^\circ]$ in G°. By Theorem 2.2.3,

$$\begin{aligned}
h(z) &\leqslant \frac{K}{\log\left(\frac{d'}{\delta \vee \varepsilon}\right)} \text{ for } z \in [W^\bullet, X^\bullet] \qquad & h(z) &\geqslant L - \frac{K}{\log\left(\frac{d'}{\delta \vee \varepsilon}\right)} \text{ for } z \in [Y^\bullet, Z^\bullet] \\
\widetilde{h}(z) &\leqslant \frac{KL}{\log\left(\frac{d}{\delta \vee \varepsilon}\right)} \text{ for } z \in [X^\circ, Y^\circ] \qquad & \widetilde{h}(z) &\geqslant 1 - \frac{KL}{\log\left(\frac{d}{\delta \vee \varepsilon}\right)} \text{ for } z \in [Z^\circ, W^\circ]
\end{aligned}$$

The condition that $(\delta \vee \varepsilon) \leqslant d' e^{-\frac{2K}{L}}$ ensures that the values of h on $[Y^\bullet, Z^\bullet]$ are larger than the values of h on $[W^\bullet, X^\bullet]$. Similarly, the condition that $(\delta \vee \varepsilon) \leqslant d e^{-8KL}$ ensures that the values of $\widetilde{h}$ on $[Z^\circ, W^\circ]$ are larger than the values of $\widetilde{h}$ on $[X^\circ, Y^\circ]$. Plugging the metric $\rho^\bullet$ into the variational formula for L', we have that:

$$L' \geqslant \frac{\left(L - \frac{2K}{\log\left(\frac{d'}{\delta \vee \varepsilon}\right)}\right)^2}{L} \geqslant L - \frac{4K}{\log\left(\frac{d'}{\delta \vee \varepsilon}\right)} \tag{2.2.9}$$

Similarly, plugging ρ° into the variational formula for the dual problem, we have that:

$$\frac{1}{L'} \geqslant \frac{\left(1 - \frac{2KL}{\log\left(\frac{d}{\delta \wedge \varepsilon}\right)}\right)^2}{L} \qquad \Longrightarrow \qquad L' \leqslant \frac{L}{\left(1 - \frac{4KL}{\log\left(\frac{d}{\delta \wedge \varepsilon}\right)}\right)} \leqslant L + \frac{8KL^2}{\log\left(\frac{d}{\delta \vee \varepsilon}\right)} \tag{2.2.10}$$

Combining Equations 2.2.9 and 2.2.10, we arrive at the desired result. □

2.3 Limits of Discrete Holomorphic Functions are Holomorphic

Given an orthodiagonal map $G = (V^\bullet \sqcup V^\circ, E)$ recall that a function $F : (V^\bullet \sqcup V^\circ \to \mathbb{C}$ is discrete holomorphic on G if for any interior face Q of G with primal diagonal $e^\bullet = \{u_1, u_2\}$ and dual diagonal $e^\circ = \{v_1, v_2\}$ we have that:

$$\frac{F(u_2) - F(u_1)}{u_2 - u_1} = \frac{F(v_2) - F(v_1)}{v_2 - v_1}$$

In particular, notice that if $F|_{V^\bullet}$ is strictly real then $F|_{V^\circ}$ is strictly imaginary up to an additive constant. This situation is typical of applications of discrete complex analysis. That is, the real part of our discrete holomorphic function typically lives on the primal graph $G^\bullet = (V^\bullet, E^\bullet)$ and the imaginary part lives on the dual graph $G^\circ = (V^\circ, E^\circ)$.

In this section, we prove the following result of independent interest:

Theorem 2.3.1. Let Ω be a subdomain of $\mathbb{C}$ and let $\Omega_n = (V_n^\bullet \sqcup V_n^\circ, E_n)$ be a sequence of orthodiagonal maps so that the edges of Ω_n are of length at most ε_n and $\varepsilon_n \to 0$ as $n \to \infty$. Suppose that for any compact set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $K \subseteq \widehat{\Omega}_n$. For each $n \in \mathbb{N}$, let $F_n : V_n^\bullet \sqcup V_n^\circ \to \mathbb{C}$ be a discrete holomorphic function on Ω_n so that:

$$\operatorname{Re}(F_n(z)) = 0 \text{ for all } z \in V^\circ, \qquad \operatorname{Im}(F_n(z)) = 0 \text{ for all } z \in V^\bullet$$

That is, the real part of F_n lives on the primal graph $\Omega_n^\bullet$ and the imaginary part of F_n lives on the dual graph Ω_n°. Suppose also that the Dirichlet energies of the F_n's are uniformly bounded on compacts. That is, for any compact set $K \subseteq \Omega$ and $N \in \mathbb{N}$ such that $K \subseteq \Omega_n$ for all $n \geqslant N$ we have that:

$$\sup_{n \geqslant N} \mathcal{E}_K^\bullet(\operatorname{Re}(F_n)) = \frac{1}{2} \sup_{n \geqslant N} \Big(\sum_{\substack{Q \in F_{in}(\Omega_n) \\ Q \subseteq K}} \frac{|e_Q^\circ|}{|e_Q^\bullet|} \Big(\operatorname{Re}(F_n((e_Q^\bullet)^+)) - \operatorname{Re}(F_n((e_Q^\bullet)^-)) \Big)^2 \Big) < \infty$$

Or equivalently, since $\operatorname{Re}(F_n)$ and $\operatorname{Im}(F_n)$ are conjugate harmonic functions:

$$\sup_{n \geqslant N} \mathcal{E}_K^\circ(\operatorname{Im}(F_n)) = \frac{1}{2} \sup_{n \geqslant N} \Big(\sum_{\substack{Q \in F_{in}(\Omega_n) \\ Q \subseteq K}} \frac{|e_Q^\bullet|}{|e_Q^\circ|} \Big(\operatorname{Im}(F_n((e_Q^\circ)^+)) - \operatorname{Im}(F_n((e_Q^\circ)^-)) \Big)^2 \Big) < \infty$$

Let $\widehat{F}_n : \widehat{\Omega}_n \to \mathbb{C}$ be a sequence of continuous functions on $\widehat{\Omega}_n$ so that:

$$\operatorname{Re}(\widehat{F}_n(z)) = F_n(z) \text{ for all } z \in V_n^\bullet, \qquad \operatorname{Im}(\widehat{F}_n(z)) = F_n(z) \text{ for all } z \in V_n^\circ$$

That is, for each $n \in \mathbb{N}$, $\widehat{F}_n$ is some sort of sensible extension of F_n to a continuous function on $\widehat{\Omega}_n$. If:

$$\widehat{F}_n \to F \text{ uniformly on compacts in } \Omega$$

Then $F : \Omega \to \mathbb{C}$ is holomorphic.

In Section 2.4 we will use this result to show that any subsequential limit of our tiling maps is holomorphic. This is crucial to showing that our tiling maps converge to the relevant conformal map.

Remark 2.3.1. Note that while Theorem 2.3.1 is sufficient for our purposes, stronger results already exist in the literature. Namely, in [14], Chelkak, Laslier, and Russkikh prove that local uniform limits of discrete holomorphic functions on t- embeddings are holomorphic (see Proposition 6.15 of [14]). Since every orthodiagonal map is a t-embedding (see Section 8.1 of [14]), it follows that the same is true of discrete holomorphic functions on orthodiagonal maps.

Having been initially unfamiliar with their work, we found an independent proof of this result in the more restrictive orthodiagonal setting, with the additional condition that the Dirichlet energies of your discrete holomorphic functions must be uniformly bounded on compacts. While this is a strictly weaker result in a less general setting, we still think it is worthwhile to present the proof. The reasons for this are twofold:

1. The proof is elementary and takes place in a simpler setting.

2. It gives us an excuse to introduce Lemma 2.3.2 which tells us that we can approximate continuous contours by discrete contours in our orthodiagonal map, that are close to the corresponding continuous contour in Hausdorff distance and have comparable length. This will be important in the proof of Proposition 3.4.1 which is the key technical estimate in the proof of Theorem 3.1.1 and Theorem 4.0.1, which are the main results of Chapters 3 and 4.

As per the discussion above, to prove Theorem 2.3.1, we first need the following lemma:

Lemma 2.3.2. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with mesh at most ε, and δ is a positive real number so that $\delta \geqslant 4\varepsilon$. Suppose ℓ is a line segment in $\mathbb{C}$ so that $L = \text{length}(\ell) \geqslant 8\varepsilon$ and $\widehat{G}$ contains a δ- neighborhood of ℓ. Then there exist nearest- neighbor paths $\gamma^\bullet$ in $G^\bullet$ and γ° in G° that are δ- close to ℓ in Hausdorff distance, and:

$$\text{length}(\gamma^\bullet), \text{length}(\gamma^\circ) \leqslant 2L\Big(1 + \frac{4\varepsilon}{\delta}\Big)$$

Furthermore, the endpoints of $\gamma^\bullet$ and γ° both lie within δ of the endpoints of ℓ.

Proof. Let $G = (V^\bullet \sqcup V^\circ, E)$ be an orthodiagonal map with edges of length at most ε. It follows that the edges of $G^\bullet$ and G°, which correspond to diagonals of inner faces of G, have length at most 2ε. Let ℓ be a line segment in $\mathbb{C}$ so that $\widehat{G}$ contains a δ- neighborhood of ℓ. Without loss of generality, suppose $\ell = [0, L]$, the line segment between 0 and L in $\mathbb{C}$. Let $R = [0, L] \times [-\delta, \delta]$. Since $\widehat{G}$ contains a δ- neighborhood of ℓ, $R \subseteq \widehat{G}$.

Let $H = (V_H^\bullet \sqcup V_H^\circ, E_H)$ be the suborthodiagonal map of G formed by taking the union of all the inner faces of G contained in R. Since R is convex, H is simply- connected with a unique, distinguished boundary face. Moreover, observe that any pair of neighboring vertices along the boundary of H, must be part of an inner face of G that intersects the boundary of the rectangle R. Thus, the vertices and edges on the boundary of H all lie within ε of ∂R. Let $A^\bullet, B^\bullet, C^\bullet, D^\bullet$ be the points of $\partial V_H^\bullet$ closest to the four corners $(0, -\delta), (0, \delta), (L, \delta), (L, -\delta)$ of R. Then $(H, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ is a $(2\varepsilon, \varepsilon)$- good interior approximation of R.

By Proposition 2.2.5:

$$\frac{(L - 4\varepsilon)^2}{4\delta L} \leqslant \lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; H^\bullet) \leqslant \frac{4\delta L}{(2\delta - 4\varepsilon)^2}$$

Plugging the metric $\rho(e^\bullet) = |e^\bullet|$ into the variational problem for $\lambda([A^\bullet, B^\bullet] \leftrightarrow [C^\bullet, D^\bullet]; H^\bullet)$ we have that:

$$\frac{4\delta L}{(2\delta-4\varepsilon)^2} \geqslant \lambda([A^\bullet,B^\bullet] \leftrightarrow [C^\bullet,D^\bullet];G^\bullet) \tag{2.3.1}$$

$$\geqslant \frac{\left(\inf\{\text{length}(\gamma^\bullet) : \gamma^\bullet \text{ is a curve in } H^\bullet \text{ joining } [A^\bullet,B^\bullet] \text{ and } [C^\bullet,D^\bullet]\}\right)^2}{A(\rho)} \tag{2.3.2}$$

where:

$$A(\rho) = \sum_{e^\bullet\in E^\bullet} \frac{|e^\circ|}{|e^\bullet|}|e^\bullet|^2 \overset{(*)}{=} 2\sum_{Q\in F_{in}(H)} \text{Area}(Q) = 2\,\text{Area}(\widehat{H}) \leqslant 4\delta L$$

Equality "$(*)$" follows from the fact that if Q is an inner face of G with primal diagonal $e^\bullet$ and dual diagonal e°, then $\text{Area}(Q) = \frac{1}{2}|e^\bullet||e^\circ|$. Plugging our estimate for $A(\rho)$ into Equation 2.3.2, we have that:

$$\inf\{\text{length}(\gamma^\bullet) : \gamma^\bullet \text{ is a curve in } H^\bullet \text{ joining } [A^\bullet,B^\bullet] \text{ and } [C^\bullet,D^\bullet]\} \leqslant \frac{2L}{\left(1-\frac{2\varepsilon}{\delta}\right)} \leqslant 2L\left(1+\frac{4\varepsilon}{\delta}\right)$$

Since any curve in $H^\bullet$ joining $[A^\bullet,B^\bullet]$ and $[C^\bullet,D^\bullet]$ is δ- close to ℓ in Hausdorff distance, this completes our proof. The proof that we can find a curve γ° in G° with the desired properties follows by the same argument, verbatim, with G° in place of $G^\bullet$. ☐

Armed with this lemma, we are now ready to prove Theorem 2.3.1:

Proof. Since F is the local uniform limit of continuous functions, F must be continuous. Hence, to prove that F is holomorphic, by problem 3 of chapter 2 of [51], it suffices to check that:

$$\oint_\gamma F(z)dz = 0$$

for every simple closed curve γ in Ω that traces out the boundary of a rectangle.

With this in mind, let γ be a simple closed curve in Ω, oriented counterclockwise, that traces out the boundary of a rectangle R with length l and width w. Let d be the distance between the rectangle R and the boundary of Ω. Let K be a compact subset of Ω that contains a $d/2$-neighborhood of R and let N be a natural number so that $K \subseteq \widehat{\Omega}_n$ for all $n \geqslant N$.

Since $(\widehat{F}_n)_{n=1}^\infty$ is a sequence of continuous functions that converges to F uniformly on compacts, by Arzela-Ascoli, it follows that the functions $\widehat{F}_n$ are equicontinuous and uniformly bounded on compacts. With this in mind, for any $\delta > 0$, let $\omega(\delta)$ denote the modulus of continuity of the family of functions $(\widehat{F}_n)_{n=N}^\infty$ on K. That is, for any $n \geqslant N$ and any points $x, y \in K$ we have that:

$$|F_n(y) - F_n(x)| \leqslant \omega(|x-y|)$$

where:

$$\lim_{\delta \to 0^+} \omega(\delta) = 0$$

Recall that by discrete Morera's theorem, for each $n \in \mathbb{N}$, the fact that F_n is discrete holomorphic on Ω_n tells us that for any simple closed directed curve η_n in Ω_n:

$$\sum_{\substack{\vec{e} \in \eta_n \\ \vec{e}=(e^-,e^+)}} \big(F_n(e^-) + F_n(e^+)\big)(e^+ - e^-) = 0$$

By Lemma 2.3.2, for $n \geqslant N$ sufficiently large, we can pick simple closed contours γ_n and η_n in Ω_n so that:

1. γ_n and η_n are both oriented counterclockwise.
2. γ_n lies outside of the rectangle R and $d_{Haus}(\gamma_n, \partial R) = O(\varepsilon_n)$.
3. η_n lies inside the rectangle R and $d_{Haus}(\eta_n, \partial R) = O(\varepsilon_n)$.
4. If $\gamma_n = (w_1, x_1, w_2, x_2, ..., w_{k_n}, x_{k_n}, w_1)$ where $w_1, w_2, ..., w_{k_n} \in V_n^\bullet$ and $x_1, x_2, ..., x_{k_n} \in V_n^\circ$, then $w_i \sim w_{i+1}$ for all i, where our indices i are viewed modulo k_n, so that $\gamma_n^\bullet = (w_1, w_2, ..., w_{k_n}, w_1)$ is a simple closed contour in $\Omega_n^\bullet$. Furthermore:

$$\text{length}(\gamma_n^\bullet) = \sum_{i=1}^{k_n} |w_{i+1} - w_i| = O(l + w)$$

5. If $\eta_n = (y_1, z_1, y_2, z_2, .., y_{m_n}, z_{m_n}, y_1)$ where $y_1, y_2, ..., y_{m_n} \in V_n^\bullet$, $z_1, z_2, ..., z_{m_n} \in V_n^\circ$, then $z_i \sim z_{i+1}$ for all i, where our indices are viewed modulo m_n, so that $\eta_n^\circ = (z_1, z_2, ..., z_{m_n}, z_1)$ is a simple closed contour in Ω_n°. Furthermore:

$$\text{length}(\eta_n^\circ) = \sum_{i=1}^{m_n} |z_{i+1} - z_i| = O(l + w)$$

Consider the discrete contour integral of F_n over γ_n:

$$\sum_{\substack{\vec{e} \in \gamma_n \\ \vec{e}=(e^-,e^+)}} \big(F_n(e^-) + F_n(e^+)\big)(e^+ - e^-) = \sum_{i=1}^{k_n} F_n(x_i)(w_{i+1} - w_i) + \sum_{i=1}^{k_n} F_n(w_i)(x_i - x_{i-1}) = 0 \quad (2.3.3)$$

Th first equality follows by rewriting the original sum over directed edges as a sum over vertices. The second equality is just discrete Morera. Since $\text{Im}(\widehat{F}_n)$ agrees with F_n on V°, for each directed edge (w_i, w_{i+1}), we have that:

$$\Big|\int_{w_i}^{w_{i+1}} \text{Im}(\widehat{F}_n(z))dz - F_n(x_i)(w_{i+1} - w_i)\Big| \leqslant \omega(\varepsilon_n) \cdot |w_{i+1} - w_i|$$

provided that n is large enough so that the contour γ_n is contained in K. Summing over directed

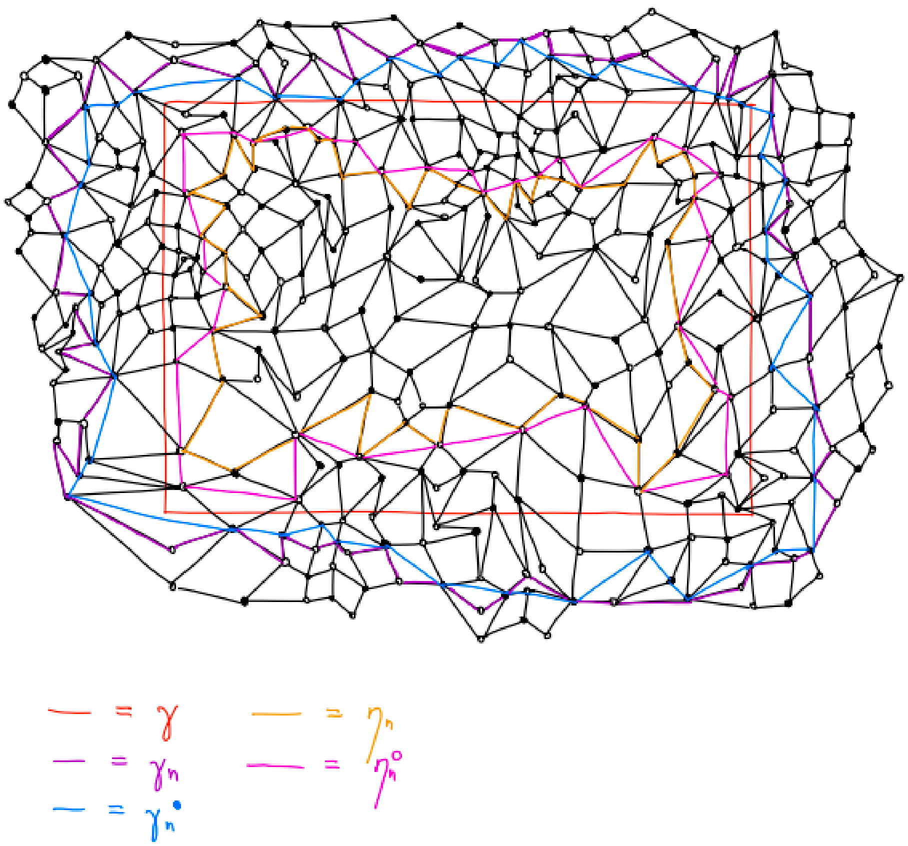

Figure 2.3: The contours γ, γ_n, $\gamma_n^\bullet$, η_n and $\eta_n^\bullet$ for some orthodiagonal map.

edges we have that:

$$\Big| \oint_{\gamma_n^\bullet} \mathrm{Im}(\widehat{F}_n(z))dz - \sum_{i=1}^{k_n} F_n(x_i)(w_{i+1} - w_i) \Big| = O(\omega(\varepsilon_n) \cdot (l + w)) \tag{2.3.4}$$

In other words, we see that the discrete contour integral of $\mathrm{Im}(F_n)$ over γ_n is close to the continuous contour integral of $\mathrm{Im}(\widehat{F}_n)$ over $\gamma_n^\bullet$. In contrast, since the x_i's don't form a simple closed contour in Ω_n° and we don't have any control over the quantity:

$$\sum_{i=1}^{k_n} |x_{i+1} - x_i|$$

it is not clear that we have a similar result comparing the discrete contour integral of $\mathrm{Re}(F_n)$ over γ_n to some continuous contour integral of $\mathrm{Re}(\widehat{F}_n)$. This is where our second contour η_n comes in.

By the same argument we used to handle the behavior of $\mathrm{Im}(F_n)$ on γ_n, since $\mathrm{Re}(\widehat{F}_n)$ agrees with F_n on $V^\bullet$, for any directed edge (z_{i-1}, z_i) we have that:

$$\Big|\int_{z_{i-1}}^{z_i} \mathrm{Re}(\widehat{F}_n(z))dz - F_n(y_i)(z_i - z_{i-1})\Big| \leqslant \omega(\varepsilon_n)\cdot|z_i - z_{i-1}|$$

Summing over directed edges we have that:

$$\Big|\oint_{\eta_n^\circ} \mathrm{Re}(\widehat{F}_n(z))dz - \sum_{i=1}^{m_n} F_n(y_i)(z_i - z_{i-1})\Big| = O(\omega(\varepsilon_n)\cdot(l+w)) \tag{2.3.5}$$

Let H be the suborthodiagonal map of G bounded by the curves η_n and γ_n. By discrete Green's theorem applied to the function $\mathrm{Re}(F_n)$ on H, we have that:

$$\sum_{i=1}^{k_n} F_n(w_i)(x_i - x_{i-1}) - \sum_{i=1}^{m_n} F_n(y_i)(z_i - z_{i-1}) = \sum_{\substack{Q\in F_{in}(H)\\ Q=[u_1,v_1,u_2,v_2]}} \big(F_n(u_2) - F_n(u_1)\big)(v_2 - v_1)$$

Applying the Cauchy-Schwarz inequality, we have that:

$$\begin{aligned}
&\Big|\sum_{\substack{Q\in F_{in}(H)\\ Q=[u_1,v_1,u_2,v_2]}} \big(F_n(u_2) - F_n(u_1)\big)(v_2 - v_1)\Big| \\
&\leqslant \Big(\sum_{\substack{Q\in F_{in}(H)\\ Q=[u_1,v_1,u_2,v_2]}} \frac{|v_2 - v_1|}{|u_2 - u_1|}\big(F_n(u_2) - F_n(u_1)\big)^2\Big)^{1/2}\Big(\sum_{\substack{Q\in F_{in}(H)\\ Q=[u_1,v_1,u_2,v_2]}} \frac{|u_2 - u_1|}{|v_2 - v_1|}|v_2 - v_1|^2\Big)^{1/2} \\
&\leqslant \mathcal{E}_K^\bullet(\mathrm{Re}(F_n))^{1/2}\Big(\sum_{\substack{Q\in F_{in}(H)\\ Q=[u_1,v_1,u_2,v_2]}} |u_2 - u_1||v_2 - v_1|\Big)^{1/2} = \sqrt{2}\,\mathcal{E}_K^\bullet(\mathrm{Re}(F_n))^{1/2}(\mathrm{Area}(\widehat{H}))^{1/2} \\
&= \mathcal{E}_K^\bullet(\mathrm{Re}(F_n))^{1/2}(l+w)^{1/2}\cdot O(\varepsilon_n^{1/2}) \leqslant \big(\sup_{n\geqslant N} \mathcal{E}_K^\bullet(\mathrm{Re}(F_n))\big)^{1/2}(l+w)^{1/2}\cdot O(\varepsilon_n^{1/2})
\end{aligned}$$

Thus:

$$\Big|\sum_{i=1}^{k_n} F_n(w_i)(x_i - x_{i-1}) - \sum_{i=1}^{m_n} F_n(y_i)(z_i - z_{i-1})\Big| \leqslant \big(\sup_{n\geqslant N} \mathcal{E}_K^\bullet(\mathrm{Re}(F_n))\big)^{1/2}(l+w)^{1/2}\cdot O(\varepsilon_n^{1/2}) \tag{2.3.6}$$

By 2.3.4, we have that:

$$\lim_{n\to\infty}\Big(\oint_{\gamma_n^\bullet} \mathrm{Im}(\widehat{F}_n(z))dz - \sum_{i=1}^{k_n} F_n(x_i)(w_{i+1} - w_i)\Big) = 0 \tag{2.3.7}$$

Similarly, since the Dirichlet energies of the F_n's are uniformly bounded on compacts, combining 2.3.5 and 2.3.6, we have that:

$$\lim_{n\to\infty}\Big(\oint_{\eta_n^\circ} \mathrm{Re}(\widehat{F}_n(z))dz - \sum_{i=1}^{k_n} F_n(w_i)(x_i - x_{i-1})\Big) = 0 \tag{2.3.8}$$

Combining 2.3.7 and 2.3.8, we have that:

$$\lim_{n\to\infty}\Big(\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz+\oint\limits_{\gamma_n^\bullet}\mathrm{Im}(\widehat{F}_n(z))dz\Big)=0$$

Thus, to prove that:

$$\oint\limits_{\gamma}F(z)dz=\oint\limits_{\gamma}\mathrm{Re}(F(z))dz+\oint\limits_{\gamma}\mathrm{Im}(F(z))dz=0$$

It suffices to show that:

$$\lim_{n\to\infty}\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz=\oint\limits_{\gamma}\mathrm{Re}(F(z))dz$$

and:

$$\lim_{n\to\infty}\oint\limits_{\gamma_n^\bullet}\mathrm{Im}(\widehat{F}_n(z))dz=\oint\limits_{\gamma}\mathrm{Im}(F(z))dz$$

To this effect, let ψ be any smooth, real- valued function on Ω. By the triangle inequality:

$$\begin{aligned}\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz-\oint\limits_{\gamma}\mathrm{Re}(F(z))dz\Big|\leqslant&\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz-\oint\limits_{\eta_n^\circ}\mathrm{Re}(F(z))dz\Big|+\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(F(z))dz-\oint\limits_{\eta_n^\circ}\psi(z)dz\Big|\\&+\Big|\oint\limits_{\eta_n^\circ}\psi(z)dz-\oint\limits_{\gamma}\psi(z)dz\Big|+\Big|\oint\limits_{\gamma}\psi(z)dz-\oint\limits_{\gamma}\mathrm{Re}(F(z))dz\Big|\end{aligned}$$

We bound the first term as follows:

$$\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz-\oint\limits_{\eta_n^\circ}\mathrm{Re}(F(z))dz\Big|\leqslant\mathrm{length}(\eta_n^\circ)\cdot\|\mathrm{Re}(\widehat{F}_n)-\mathrm{Re}(F)\|=O\big((l+w)\cdot\|\mathrm{Re}(\widehat{F}_n)-\mathrm{Re}(F)\|\big)$$

where $\|\mathrm{Re}(\widehat{F}_n)-\mathrm{Re}(F)\|$ is the sup norm of $\mathrm{Re}(\widehat{F}_n)-\mathrm{Re}(F)$ on K. We can handle the second and fourth terms analogously:

$$\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(F(z))dz-\oint\limits_{\eta_n^\circ}\psi(z)dz\Big|\leqslant\mathrm{length}(\eta_n^\circ)\cdot\|\mathrm{Re}(F)-\psi\|=O\big((l+w)\cdot\|\mathrm{Re}(F)-\psi\|\big)$$

$$\Big|\oint\limits_{\gamma}\psi(z)dz-\oint\limits_{\gamma}\mathrm{Re}(F(z))dz\Big|\leqslant\mathrm{length}(\gamma)\cdot\|\psi-\mathrm{Re}(F)\|=2(l+w)\cdot\|\psi-\mathrm{Re}(F)\|$$

To bound the third term, recall that if σ is a simple closed Lipschitz curve, oriented counterclockwise, and h is a complex- valued function whose real and imaginary part are both C^2, Green's theorem tells us that:

$$\oint\limits_{\sigma}h(z)dz=\int_D\partial_{\overline{z}}h(x+iy)dxdy$$

where D is the region bounded by σ. We know that γ bounds the rectangle R. Let R_n denote the region bounded by η_n°. By Green's theorem,

$$\Big|\oint\limits_{\gamma}\psi(z)dz - \oint\limits_{\eta_n^\circ}\psi(z)dz\Big| = \Big|\int_R \partial_{\overline{z}}\psi(x+iy)dxdy - \int_{R_n}\partial_{\overline{z}}\psi(x+iy)dxdy\Big|$$
$$= O\big((l+w)\cdot\|\partial_{\overline{z}}\psi\|\cdot\varepsilon_n\big)$$

Letting $n\to\infty$, the first and third terms vanish, leaving us with:

$$\limsup_{n\to\infty}\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz - \oint\limits_{\gamma}\mathrm{Re}(F(z))dz\Big| = O\big((l+w)\cdot\|\mathrm{Re}(F)-\psi\|\big)$$

Since smooth functions are dense in the space of continuous functions on a compact set with the sup norm, we can choose ψ so that $\|\mathrm{Re}(F)-\psi\|$ is arbitrarily small. Thus:

$$\limsup_{n\to\infty}\Big|\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz - \oint\limits_{\gamma}\mathrm{Re}(F(z))dz\Big| = 0$$

From which we conclude that:

$$\lim_{n\to\infty}\oint\limits_{\eta_n^\circ}\mathrm{Re}(\widehat{F}_n(z))dz = \oint\limits_{\gamma}\mathrm{Re}(F(z))dz$$

Applying the same argument to $\big(\oint_{\gamma_n^\bullet}\mathrm{Im}(\widehat{F}_n(z))dz - \oint_{\gamma}\mathrm{Im}(F(z))dz\big)$, we get that:

$$\lim_{n\to\infty}\oint\limits_{\gamma_n^\bullet}\mathrm{Im}(\widehat{F}_n(z))dz = \oint\limits_{\gamma}\mathrm{Im}(F(z))dz$$

□

2.4 Convergence of Tilings to Conformal Maps

2.4.1 Proof of Theorem 2.1.2

Proof. Suppose $\Omega \subsetneq \mathbb{C}$ is a simply connected domain with distinguished prime ends A, B, C, D listed in counterclockwise order and $\delta_n, \varepsilon_n > 0$ are sequences of positive reals so that:

$$(\delta_n, \varepsilon_n) \to (0,0) \text{ as } n \to \infty$$

For each $n \in \mathbb{N}$, let $\Omega_n = (V_n^\bullet \sqcup V_n^\circ, E)$ be an orthodiagonal map so that $(\Omega_n, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)$ is a $(\delta_n, \varepsilon_n)$- good interior approximation to (Ω, A, B, C, D) for some choice of distinguished boundary points $A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet \in \partial V_n^\bullet$. Let ϕ_n be the tiling map associated to the orthodiagonal rectangle $(\Omega_n, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)$ with real part h_n and imaginary part $\widetilde{h}_n$. That is, $\widetilde{h}_n : V_n^\circ \to \mathbb{R}$ solves the following boundary value problem on Ω_n°:

$$
\begin{aligned}
\widetilde{h}_n(x) &= 1 \text{ for all } x \in [B_n^\circ, C_n^\circ] \\
\widetilde{h}_n(x) &= 0 \text{ for all } x \in [D_n^\circ, A_n^\circ] \\
\Delta^\circ \widetilde{h}_n(x) &= 0 \text{ for all } x \in V_n^\circ \backslash \big([B_n^\circ, C_n^\circ] \cup [D_n^\circ, A_n^\circ]\big)
\end{aligned}
$$

and $h_n : V^\bullet \to \mathbb{R}$ is the harmonic conjugate of $\widetilde{h}_n$ which solves the following boundary value problem on $\Omega_n^\bullet$:

$$
\begin{aligned}
h_n(x) &= 0 \text{ for all } x \in [A_n^\bullet, B_n^\bullet] \\
h_n(x) &= L_n \text{ for all } x \in [C_n^\bullet, D_n^\bullet] \\
\Delta^\bullet h_n(x) &= 0 \text{ for all } x \in V_n^\bullet \backslash \big([A_n^\bullet, B_n^\bullet] \cup [C_n^\bullet, D_n^\bullet]\big)
\end{aligned}
$$

where L_n is the extremal length from $[A_n^\bullet, B_n^\bullet]$ to $[C_n^\bullet, D_n^\bullet]$ in $\Omega_n^\bullet$. Let F_n be the discrete holomorphic function on Ω_n that agrees with h_n on $V_n^\bullet$ and agrees with $i\widetilde{h}_n$ on V_n°. Let $\widehat{F}_n$ be any sensible extension of F_n to a continuous function on $\widehat{\Omega}_n$ so that:

$$
\operatorname{Re}(\widehat{F}_n(z)) = h_n(z) \text{ for all } z \in V_n^\bullet \qquad \operatorname{Im}(\widehat{F}_n(z)) = \widetilde{h}_n(z) \text{ for all } z \in V_n^\circ
$$

One natural way to do this is to triangulate the faces of $\Omega_n^\bullet$ and define the real part of $\widehat{F}_n$ on each triangle by interpolating linearly between the values of h_n at the corner vertices. Analogously, triangulating each face of Ω_n°, we can define the imaginary part of $\widehat{F}_n$ on each triangle by interpolating linearly between the values of $\widetilde{h}_n$ at the corner vertices. If ϕ_n is the tiling map associated with the orthodiagonal rectangle $(\Omega_n^\bullet, A_n^\bullet, B_n^\bullet, C_n^\bullet, D_n^\bullet)$, our estimates for the modulus of continuity of h_n and $\widetilde{h}_n$ in Theorem 2.2.3 tell us that for any $n \in \mathbb{N}$ and any $z \in \widehat{\Omega}_n$, we have that:

$$
|\phi_n(z) - \widehat{F}_n(z)| \leqslant \frac{K(L_n \vee 1)}{\log\left(\frac{d_n \wedge d_n'}{\varepsilon_n}\right)} \tag{2.4.1}
$$

where:

$$
\begin{aligned}
d_n &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{\Omega}_n \text{ joining } [A_n^\bullet, B_n^\bullet]_{\partial\widehat{\Omega_n^\bullet}_{n}{}^{\bullet}_{n}{}_n} \text{ and } [C_n^\bullet, D_n^\bullet]_{\partial\widehat{\Omega}_n}\} \\
d_n' &= \inf\{\text{diameter}(\gamma) : \gamma \text{ is a curve in } \widehat{\Omega}_n \text{ joining } [B_n^\bullet, C_n^\bullet]_{\partial\widehat{\Omega}_n} \text{ and } [D_n^\bullet, A_n^\bullet]_{\partial\widehat{\Omega}_n}\}
\end{aligned}
$$

and $K > 0$ is an absolute constant. Suppose γ is a curve in $\widehat{\Omega}_n$ joining $[A_n^\bullet, B_n^\bullet]_{\partial\widehat{\Omega}_n}$ and $[C_n^\bullet, D_n^\bullet]_{\partial\widehat{\Omega}_n}$. Since the endpoint of γ on $[A_n^\bullet, B_n^\bullet]_{\partial\widehat{\Omega}_n}$ is δ_n- close to $[A, B]_{\partial\Omega}$ in crosscut distance and the endpoint of γ on $[C_n, D_n]_{\partial\widehat{\Omega}_n}$ is δ_n- close to $[C, D]_{\partial\Omega}$ in crosscut distance it follows that:

$$
d \leqslant d_n + 2\delta_n \qquad \Longleftrightarrow \qquad d_n \geqslant d - 2\delta_n \tag{2.4.2}
$$

By the same reasoning:

$$d' \leqslant d'_n + 2\delta_n \qquad \Longleftrightarrow \qquad d'_n \geqslant d' - 2\delta_n \tag{2.4.3}$$

As long as n is sufficiently large so that $\delta_n < \frac{d \wedge d'}{2}$, plugging Equations 2.4.2 and 2.4.3 into Equation 2.4.1, we have that:

$$|\phi_n(z) - \widehat{F}_n(z)| \leqslant \frac{K(L_n \vee 1)}{\log\left(\frac{(d \wedge d') - 2\delta_n}{\varepsilon_n}\right)}$$

By Proposition 2.2.5, the sequence $(L_n)_{n=1}^{\infty}$ is uniformly bounded. Hence, the quantity on the RHS of this inequality tends to 0 uniformly in z as $n \to \infty$. Thus, to show that the tiling maps ϕ_n converge to the relevant conformal map ϕ, it suffices to show this is true of the functions $\widehat{F}_n$.

The uniform boundedness of the L_n's also tells us that the functions $\widehat{F}_n$ are uniformly bounded. By Theorem 2.2.3, these functions are also equicontinuous on compacts. Hence, by Arzela-Ascoli, the functions $\widehat{F}_n$ are precompact with respect to the topology of uniform convergence on compacts in Ω. Since L_n is precisely the discrete Dirichlet energy of $\widehat{F}_n$ and the sequence $(L_n)_{n=1}^{\infty}$ is uniformly bounded, Theorem 2.3.1 tells us that any subsequential limit of our discrete holomorphic functions $\widehat{F}_n$ is holomorphic.

As per our discussion in Section 0.1, in 2D statistical physics, if you have a holomorphic observable that arises as the limit of some discrete holomorphic observables, you can typically recover the identity of the limiting object from the boundary conditions. This is what we will do here. Suppose $(\widehat{F}_{n_k})_{k \geqslant 1}$ is a subsequence of $(\widehat{F}_n)_{n \geqslant 1}$ such that $\widehat{F}_{n_k}$ converges uniformly on compacts to some holomorphic function F and L_{n_k} converges to some $L > 0$ as $k \to \infty$. We can always pick such a subsequence because the sequence $(\widehat{F}_n)_{n \geqslant 1}$ is precompact and the sequence $(L_n)_{n \geqslant 1}$ is uniformly bounded away from 0 and ∞. By Theorem 2.2.3:

$$|\widehat{F}_{n_k}(y) - \widehat{F}_{n_k}(x)| \leqslant \frac{K(L_{n_k} \vee 1)}{\log\left(\frac{d_{n_k} \wedge d'_{n_k}}{(d_{\widehat{\Omega_{n_k}}}(x,y) \wedge d_{cc}^{\widehat{\Omega_{n_k}}}(x,y)) \vee \varepsilon_{n_k}}\right)}$$

for any $x, y \in \Omega$, provided that k is sufficiently large so that $x, y \subseteq \widehat{\Omega}_{n_k}$. Taking a limit as $k \to \infty$ for fixed x and y, we have that:

$$|\widehat{F}(y) - \widehat{F}(x)| \leqslant \frac{K(L \vee 1)}{\log\left(\frac{d \wedge d'}{(d_{\Omega}(x,y) \wedge d_{cc}^{\Omega}(x,y))}\right)}$$

Hence, our limiting function F extends continuously to Ω^*. Furthermore, $\mathrm{Re}(F) = 0$ on $[A, B]_{\partial\Omega^*}$, $\mathrm{Re}(F) = L$ on $[C, D]_{\partial\Omega^*}$, $\mathrm{Im}(F) = 0$ on $[B, C]_{\partial\Omega^*}$, and $\mathrm{Im}(F) = 1$ on $[D, A]_{\partial\Omega^*}$. This follows from the boundary conditions for the $\widehat{F}_{n_k}$'s and the fact that our estimates in Theorem 2.2.3 hold right up to the boundary. By the argument principle, we conclude that our limiting function F

is conformal and so $F = \phi$, where ϕ is the conformal map from Ω to the rectangle $(0, L) \times (0, 1)$ so that the prime ends A, B, C, D are mapped to the four corners of the rectangle and $\phi(A) = i$. In particular, L must be the extremal length between $[A, B]_{\partial\Omega*}$ and $[C, D]_{\partial\Omega*}$ in Ω. Since all convergence subsequences converge uniformly on compacts to ϕ, it follows that the functions $\widehat{F}_n$ converge to ϕ, uniformly on compacts, as desired.

$\square$

Chapter 3

A Polynomial Rate of Convergence for the Dirichlet Problem on Orthodiagonal Maps

3.1 The Result and an Outline of the Proof

As we alluded to in Section 0.3, in this Chapter, we will prove the following:

Theorem 3.1.1. Suppose $\Omega \subseteq \mathbb{R}^2$ is a bounded simply connected domain, $g : \mathbb{R}^2 \to \mathbb{R}$ is a α-Hölder, and $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε so that for each point $z \in \partial\widehat{V}$, $\operatorname{dist}(z, \partial\Omega) \leqslant \varepsilon$. Let h be the solution to the continuous Dirichlet problem on Ω with boundary data given by g:

$$\begin{aligned} \Delta h(x) &= 0 && \text{for all } x \in \Omega \\ h(x) &= g(x) && \text{for all } x \in \partial\Omega \end{aligned}$$

Let $h^\bullet$ be the solution to the discrete Dirichlet problem on $G^\bullet$ with boundary data given by g:

$$\begin{aligned} \Delta^\bullet h^\bullet(v) &= 0 && \text{for all } v \in \operatorname{Int}(V^\bullet) \\ h^\bullet(v) &= g(v) && \text{for all } v \in \partial V^\bullet \end{aligned}$$

Let $\beta \in (0,1)$ be the absolute constant from Lemma 3.1.4. Then for any $v \in V^\bullet$, we have that:

$$|h^\bullet(v) - h(v)| \leqslant \begin{cases} \left(C_1\|g\| + C_2\frac{\beta}{\beta-\alpha}\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\right)\left(\frac{\varepsilon}{\mathrm{diam}(\Omega)}\right)^{\lambda_1(\alpha,\beta)} & \text{if } \alpha \in (0,\beta) \\ \\ \left(C_1\|g\| + C_2\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\right)\left(\log\left(\frac{\mathrm{diam}(\Omega)}{\varepsilon}\right)\right)\left(\frac{\varepsilon}{\mathrm{diam}(\Omega)}\right)^{\lambda_2(\beta)} & \text{if } \alpha = \beta \\ \\ \left(C_1\|g\| + C_2\frac{\alpha}{\beta-\alpha}\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\right)\left(\frac{\varepsilon}{\mathrm{diam}(\Omega)}\right)^{\lambda_2(\beta)} & \text{if } \alpha \in (\beta,1) \end{cases}$$

where $C_1, C_2 > 0$ are absolute constants, $\|g\|_\alpha$ is the α-Hölder norm of g:

$$\|g\|_\alpha = \sup_{\substack{x,y\in\mathbb{R}^2 \\ x\neq y}} \frac{|g(y)-g(x)|}{|x-y|^\alpha}$$

and the functions λ_1, λ_2 are given by:

$$\lambda_1(\alpha,\beta) = \min_{r\in[0,1]}\max_{s\in(r,1)} \Xi_1(\alpha,\beta,r,s), \qquad \lambda_2(\beta) = \min_{r\in[0,1]}\max_{s\in(r,1)} \Xi_2(\beta,r,s)$$

where:

$$\Xi_1(\alpha,\beta,r,s) = \begin{cases} \max\{r\alpha, \min\{\beta(s-r), \frac{1}{2} - \frac{5s}{2} + \frac{r}{2}, s\alpha\}\} & \text{if } s \leqslant \frac{1-\beta}{1+\beta} \\ \\ \max\{r\alpha, \min\{\beta(s-r), \frac{\beta}{1+\beta} - (2+\frac{\beta}{1+\beta})s + \frac{r}{2}, s\alpha\}\} & \text{otherwise} \end{cases}$$

$$\Xi_2(\beta,r,s) = \begin{cases} \max\{r\beta, \min\{\beta(s-r), \frac{1}{2} - \frac{5s}{2} + \frac{r}{2}\}\} & \text{if } s \leqslant \frac{1-\beta}{1+\beta} \\ \\ \max\{r\beta, \min\{\beta(s-r), \frac{\beta}{1+\beta} - (2+\frac{\beta}{1+\beta})s + \frac{r}{2}\}\} & \text{otherwise} \end{cases}$$

The idea behind the proof is as follows. Fix a point $z \in V^\bullet$. Our goal is to show that $h(z)$ and $h^\bullet(z)$ are close. To do this, we will consider two cases.

3.1.1 Proof Outline: z Near the Boundary

If z is close to the boundary of Ω (and therefore the boundary of $\widehat{G}$), the Hölder regularity of g tells us that, near the boundary, the value of $h(z)$ is close to the value of the boundary data g at nearby points of $\partial\Omega$. Similarly, the Hölder regularity of g tells us that the value of $h^\bullet(z)$ is close to the value of the boundary data g at nearby points of $\partial V^\bullet$. Since $\partial V^\bullet$ is close to $\partial\Omega$ and g is Hölder, we conclude that $h(z)$ and $h^\bullet(z)$ are close. Thus, for z near the boundary, the fact that $h(z)$ and $h^\bullet(z)$ are close follows from estimates for the modulus of continuity of solutions to the discrete and continuous Dirichlet problems with Hölder boundary data.

The key ingredient in the proof of these modulus of continuity estimates for solutions to the continuous Dirichlet problem is the following estimate for planar Brownian motion:

Theorem 3.1.2. (Beurling's Estimate) Let $B(0,R) \subseteq \mathbb{R}^2$ be the ball of radius R about the origin in $\mathbb{R}^2$, $z \in B(0,R)$ and $K \subseteq \mathbb{R}^2$ is a connected, compact subset of the plane so that $0 \in K$ and $K \cap \partial B(0,R) \neq \varnothing$. Let $(B_t)_{t\geqslant 0}$ be a planar Brownian motion and let $T_{\partial B(0,R)}$ and T_K be the hitting times of $\partial B(0,R)$ and K by this Brownian motion. Then:

$$\mathbb{P}^z(B[0,T_{\partial B(0,R)}] \cap K = \varnothing) \leqslant C\Big(\frac{|z|}{R}\Big)^{1/2}$$

where $C > 0$ is an absolute constant.

For a proof of Theorem 3.1.2, see Section 3.8 of [38]. In plain language, Theorem 3.1.2 gives us an upper bound for the probability that a planar Brownian motion, started at z, escapes the ball $B(0,R)$ without hitting K. This is known as the strong Beurling estimate. The word "strong" here alludes to the fact that the exponent of 1/2 in this theorem is sharp. This can be seen by evaluating the probability that Brownian motion started at $r \in (0,1)$ exits the unit disk before hitting the line segment $[-1,0]$. In fact, Theorem 3.1.2 is a direct consequence of a stronger result, known as the Beurling projection theorem (see Theorem 9.2 in Chapter III of [22]), which tells us that given a Brownian motion started at a point $z \in B(0,R)$, a line segment stretching from 0 to $Re^{-i\arg(z)}$ is the connected, compact set containing 0 and intersecting $\partial B(0,R)$, that maximizes the probability that a planar Brownian motion, started at z, escapes the ball $B(0,R)$ without hitting K. An equivalent reformulation of the strong Beurling estimate is as follows:

Proposition 3.1.3. Suppose $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, $(B_t)_{t\geqslant 0}$ is a planar Brownian motion and $T_{\partial\Omega}$ is the hitting of $\partial\Omega$ by this planar Brownian motion. Then for any positive real number $r > 0$ and any $z \in \Omega$ we have that:

$$\mathbb{P}^z(|B_{T_{\partial\Omega}} - z| \geqslant r) \leqslant C\Big(\frac{d_z}{r}\Big)^{1/2}$$

where $C > 0$ is an absolute constant and $d_z = \mathrm{dist}(z, \partial\Omega)$.

It is not hard to show that the statement of Proposition 3.1.3 is equivalent to Theorem 3.1.2. Writing the strong Beurling estimate in this way will make it clearer how it is we are applying this result, when we use it to prove Theorem 3.1.1.

Insofar as orthodiagonal maps are good approximations of continuous 2D space, an analogous estimate should be true for simple random walks on orthodiagonal maps. Indeed, in [35], Kesten proves an analogue of the strong Beurling estimate for simple random walks on the square grid. For a nice exposition of this result, see Section 2.5 of [37]. Later, in [39], Lawler and Limic prove an analogue of the strong Beurling estimate for a large class of random walks on periodic lattices. Establishing a strong Beurling estimate for general orthodiagonal maps, or even for the more restricted setting of isoradial graphs, is currently an open problem. However, in this work, we will show that simple random walks on orthodiagonal maps satisfy a weak Beurling estimate:

Lemma 3.1.4. (Weak Beurling Estimate) There exist absolute constants $\beta, C > 0$ such that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $u \in \mathrm{Int}(V^\bullet)$, $r > 0$ is a real number, $(S_n)_{n\geqslant 0}$ is a simple random walk on $G^\bullet$, and $T_{\partial V^\bullet}$ is the hitting time of $\partial V^\bullet$ by this random walk, then:

$$\mathbb{P}^u(|S_{T_{\partial V^\bullet}} - u| \geqslant r) \leqslant C'\Big(\frac{d_u \vee C\varepsilon}{r}\Big)^\beta$$

where $d_u = \mathrm{dist}(u, \partial V^\bullet)$.

Just as in the continuous setting, this weak Beurling estimate gives us estimates for the modulus of continuity of solutions to the Dirichlet problem on orthodiagonal maps. The word "weak" here refers to the fact that the exponent β in this estimate is not sharp. As a consequence, we have the following weak Harnack-type estimate:

Lemma 3.1.5. There exist absolute constants $\beta, C, C' > 0$ so that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h : V^\bullet \to \mathbb{R}$ is a harmonic function on $G^\bullet$ and $d = d_x \wedge d_y = \mathrm{dist}(x, \partial V^\bullet) \wedge \mathrm{dist}(y, \partial V^\bullet)$ for some vertices $x, y \in V^\bullet$, then:

$$|h(y) - h(x)| \leqslant C' \, \|h\|_\infty \Big(\frac{|y-x| \vee C\varepsilon}{d}\Big)^\beta$$

The standard Harnack estimate for continuous harmonic functions can be interpreted as telling us that bounded harmonic functions are Lipschitz in the bulk (away from the boundary of our domain). Hence, our weak Harnack estimate in Lemma 3.1.5 can be interpreted as saying that discrete harmonic functions on orthodiagonal maps are β-Hölder in the bulk.

To our knowledge, Lemma 3.1.4 is not stated explicitly in this level of generality anywhere in the literature. However, it follows readily from Lemma 6.7 of [14]. Namely, as per our discussion in Section 2.2.2, the t- embeddings of [14] are a strict generalization of the setting we are working in. Namely, every orthodiagonal map is a t-embedding (for details, see Section 8.1 of [14]). Furthermore, it is not difficult to show that for any $\kappa \in (0,1)$, there exists an absolute constant $c = c(\kappa) > 1$ so that if G is an orthodiagonal map with edges of length at most ε, then G satisfies the assumption "*Lip*$(\kappa, c\varepsilon)$" of [14] (see Assumption 1.1 of [14]). Hence, all of the results of [14] that are proven for t-embeddings satisfying the assumption *Lip*(κ, δ) for some $\kappa \in (0,1)$ and $\delta > 0$ transfer over immediately to the orthodiagonal setting. In particular, Lemma 3.1.5 follows immediately from Proposition 6.13 of [14].

Having only been made aware that the t-embeddings of [14] are a strict generalization of orthodiagonal maps after having completed this work, we had to prove Lemmas 3.1.4 and 3.1.5 independently. In Section 3.3, using the terminology of [9], we prove "microscale" properties (S) and (T) for simple random walks on orthodiagonal maps. As a consequence, by arguments of

Chelkak (see the Appendix of [9]), a variety of estimates for simple random walks and discrete harmonic functions on orthodiagonal maps immediately follow. In particular, Lemmas 3.1.4 and 3.1.5 follow from Property (S).

Remark 3.1.1. The proof of Lemma 6.13 of [14], which is the generalization of Lemma 3.1.5 for random walks on t-embeddings, follows from an analogue of property (S) (see Lemma 6.7 of [14]), by the exact same argument as the one we use in Section 3.3.3. What is distinct here is our approach to proving property (S), in the more restricted orthodiagonal setting.

3.1.2 Proof Outline: z Away from the Boundary

Away from the boundary, we use the same strategy used by Chelkak to prove Theorem 4.1 of [10]. This theorem establishes gives a polynomial rate of convergence, mesoscopically far away from the boundary, for certain observables on s-embeddings satisfying the regularity conditions "Unif(δ)" and "Flat(δ)" (see Section 1.3 of [10] for details). To see this same argument, written out in the simpler setting of isoradial graphs, see Proposition 4.4.14 of [46]. The idea is that if a function f is almost harmonic in the sense that $\Delta f \approx 0$, then f is close to the harmonic function with the same boundary data. More precisely, suppose $\Omega \subseteq \mathbb{R}^2$ is a bounded, simply connected domain, $g \in C^0(\mathbb{R}^2)$, and h is the solution to the Dirichlet problem on Ω with boundary data given by g. That is:

$$\begin{aligned} \Delta h(x) &= 0 \qquad \text{for all } x \in \Omega \\ h(x) &= g(x) \ \text{ for all } x \in \partial\Omega \end{aligned}$$

If f is any other function in $C_b^2(\Omega) \cap C(\overline{\Omega})$ with the same boundary data,

$$|f(x) - h(x)| = \Big| \int_{\Omega} \Delta f(y) G_{\Omega}(y, x) dA(y) \Big| \tag{3.1.1}$$

for any $x \in \Omega$, where $G_{\Omega}(\cdot, x)$ is the Green function on Ω centered at x, and "$dA(y)$" is integration with respect to area on Ω. As an immediate consequence of this formula, if f has the same boundary behavior as h and Δf is small, f must be close to h.

To apply the formula in Equation 3.1.1 to the problem of estimating the difference between $h(z)$ and $h^{\bullet}(z)$, we need to replace $h^{\bullet}$ with a smooth function. To this effect:

1. We convolve $h^{\bullet}$ with a smooth mollifier ϕ_{δ}, supported on a ball of radius δ about 0, where δ is small.
2. As long as $\delta \ll d_z$, Lemma 3.1.5 tells us that $h^{\bullet}(z)$ is close to $\phi_{\delta} * h^{\bullet}(z)$.
3. Since $h^{\bullet}$ is discrete harmonic, the convolution $\phi_{\delta} * h^{\bullet}$ is almost harmonic in the sense that $\Delta(\phi_{\delta} * h^{\bullet}) \approx 0$. Verifying this is the most subtle part of the whole argument. This is the subject of Section 3.4.

4. From here, we are in a position to apply our formula in Equation 3.1.1 to show that the solution to our continuous Dirichlet problem, h, is close to $\phi_\delta * h^\bullet$ and therefore $h^\bullet$.

There is a minor technicality that $\phi_\delta * h^\bullet$ is not defined on all of Ω, so this argument actually plays out on some smaller subdomain of Ω. Furthermore, the boundary behavior of $\phi * h^\bullet$ doesn't quite agree with the boundary behavior of h, but morally, this is what's going on.

3.2 A Few Comments on Notation

- If $v \in \mathbb{R}^n$,
$$|v| := \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$
- If $f : \Omega \to \mathbb{R}^m$, where Ω is a subdomain of $\mathbb{R}^n$, then:
$$\|f\| := \sup\{|f(x)| : x \in \Omega\}$$
- Given a 2×2 matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, $\|A\|_2$ is the Frobenius norm of A:
$$\|A\|_2 = \sqrt{a_{1,1}^2 + a_{1,2}^2 + a_{2,1}^2 + a_{2,2}^2}$$
- In particular, if $f : \Omega \to \mathbb{R}$ is C^2, where Ω is a subdomain of $\mathbb{R}^2$, then:
$$\|D^2 f(x)\|_2 = \sqrt{(\partial_1^2 f(x))^2 + 2(\partial_1 \partial_2 f(x))^2 + (\partial_2^2 f(x))^2}$$
and:
$$\|D^2 f\| := \sup\{\|D^2 f(x)\|_2 : x \in \Omega\}$$
If f
- Similarly, if $f : \Omega \to \mathbb{R}$ is C^3, where Ω is a subdomain of $\mathbb{R}^2$, then:
$$\|D^3 f(x)\|_2 = \sqrt{(\partial_1^3 f(x))^2 + (\partial_1^2 \partial_2 f(x))^2 + (\partial_1 \partial_2^2 f(x))^2 + (\partial_2^3 f(x))^2}$$
and:
$$\|D^3 f\| := \sup\{\|D^2 f(x)\|_2 : x \in \Omega\}$$
- Recall that for any matrix, its Frobenius norm is an upper bound for its operator norm. In particular, if $v, w \in \mathbb{R}^2$ and A is a 2×2 matrix,
$$|v^T A w| \leqslant |v| \cdot \|A\|_2 \cdot |w|$$

3.3 Random Walks and A Priori Regularity Estimates for Harmonic Functions on Orthodiagonal Maps

In [9], Chelkak considers locally finite embeddings of a weighted graph $\Gamma = (V, E, c)$ in the plane such that:

- neighboring edges have comparable weight.
- neighboring edges have comparable lengths.
- angles are uniformly bounded away from 0 and π.

Keeping our notation consistent with [9], if u is a vertex of Γ, let $B_r^\Gamma(u) = (V_{B_r^\Gamma(u)}, E_{B_r^\Gamma(u)})$ be the subgraph of Γ induced by the vertex set $B(u,r) \cap V$. The boundary of this graph, denoted by $\partial V_{B_r^\Gamma(u)}$, is the set of vertices of $V \backslash V_{B_r^\Gamma(u)}$ that are adjacent to some vertex of $B_r^\Gamma(u)$. In other words,

$$\partial V_{B_r^\Gamma(u)} = \{y \in V \backslash V_{B_r^\Gamma(u)} : y \sim v \text{ for some } v \in V_{B_r^\Gamma(u)}\}$$

In addition to the aforementioned regularity assumptions on the lattice Γ, Chelkak assumes that simple random walks on Γ satisfy properties (S) and (T):

- Property (S): Let $(S_n)_{n \geqslant 0}$ be simple random walk on Γ and let T be the first time at which this random walk exits the ball $B(u,r)$:

  $$T := \inf\{n \in \mathbb{N}_0 : S_n \notin B(u,r)\}$$

  Then there exist constants $\eta_0 \in (0, \pi)$ and $c_0 > 0$ independent of $u \in \Gamma$ and $r > 0$ such that:

  $$\mathbb{P}^u\big(\arg(S_T - u) \in I\big) \geqslant c_0 \tag{3.3.1}$$

  For any interval $I \subseteq S^1$ with length$(I) \geqslant \eta_0$.

- Property (T): There exists an absolute constant $C_0 > 1$ independent of $u \in \Gamma$, $r > 0$ such that:

  $$C_0^{-1} r^2 \leqslant \sum_{v \in B_r^\Gamma(u)} r_v^2 G_{B_r^\Gamma(u)}(v,u) \leqslant C_0 r^2 \tag{3.3.2}$$

  Where $r_v = \min\{|v - u| : v \sim u\}$ and $G_{B_r^\Gamma(u)}(\cdot, u) : V_{B_r^\Gamma(u)} \sqcup \partial V_{B_r^\Gamma(u)} \to \mathbb{R}$ is the Green's function of $B_r^\Gamma(u)$ centered at u, which is the unique solution to the following boundary value problem:

  $$\Delta_v G_{B_r^\Gamma(u)}(v,u) = \sum_{\substack{y \in \Gamma \\ y \sim v}} c(v,y)\big(G_{B_r^\Gamma(u)}(y,u) - G_{B_r^\Gamma(u)}(v,u)\big) = -\delta_u(v) \quad \text{for } v \in B_r^\Gamma(u)$$

  $$G_{B_r^\Gamma}(u,v) = 0 \quad \text{for } v \in \partial B_r^\Gamma(u)$$

(The subscript of v in "Δ_v" is there to emphasize that we are taking the discrete Laplacian with repect to the variable v, not u.)

Informally, property (T) says that it takes roughly r^2 steps for a simple random walk on Γ to travel distance r away from its starting point. Similarly, property (S) tells us that a simple random walk started at the center of the disk $B(u, r)$ has probability at least c_0 of exiting this disk through a discrete arc of length η_0. Assuming properties (S) and (T), in addition to the aforementioned regularity assumptions on our weighted plane graph Γ, Chelkak proves a number of estimates for random walks, harmonic functions and extremal length. In particular, he shows that:

- we have a weak Beurling estimate for the simple random walk on our embedding (Lemma 2.11 of [9]). This tells us that discrete harmonic functions on our embedding are Hölder in the bulk. If we assume our boundary data is also Hölder, this gives us Hölder regularity up to the boundary.
- discrete and continuous extremal lengths of conformal rectangles are comparable (Proposition 6.2 of [9]).
- a discrete analogue of the Ahlfors-Beurling-Carlemann estimate holds, giving us two- sided estimates for harmonic measure in terms of extremal length (Theorem 7.8 of [9]).

In [4] Angel, Barlow, Gurel- Gurevich and Nachmias prove that if $\Gamma = (V, E, c)$ is a weighted plane graph satisfying Chelkak's regularity assumptions, simple random walks on Γ satisfy properties (S) and (T). Hence, in Chelkak's context, one does not actually need to assume properties (S) and (T): these properties of simple random walk follow from the regularity assumptions on our lattice Γ.

Intuitively, orthodiagonal maps are good approximations of continuous 2D space, so it is reasonable to assume that simple random walks on orthodiagonal maps should satisfy properties (S) and (T) along with the rest of the estimates in Chelkak's toolbox paper. However, while orthodiagonal maps provide us with a notion of discrete complex analysis, they can have arbitrarily small angles, arbitrarily small ratios between lengths of neighboring edges, and arbitrarily large vertex degrees. In short, they may fail to satisfy the assumptions of Chelkak's toolbox paper.

In this section, we show that properties (S) and (T) hold for simple random walks on orthodiagonal maps. As a result, we are able to recover at least some of the results of [9] for general orthodiagonal maps, where we have a notion of discrete complex structure, but no additional constraints on the geometry of our embedding.

3.3.1 Microscale Property (S) on Orthodiagonal Maps

Theorem 1.1 of [25] says the following:

Theorem 3.3.1. (Theorem 1.1 of [25]) Suppose Ω is a bounded simply connected domain, $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. Given $\delta, \varepsilon \in (0, \operatorname{diam}(\Omega))$, let $G = (V^\bullet \sqcup V^\circ, E)$ be a simply connected orthodiagonal map, with edges of length at most ε, such that the Hausdorff distance between $\partial V^\bullet$ and $\partial\Omega$ is at most δ. Let h_c be the solution to the continuous Dirichlet problem on Ω with boundary data g, and let $h_d : V^\bullet \to \mathbb{R}$ be the solution to the discrete Dirichlet problem on $G^\bullet$ with boundary data $g|_{\partial V^\bullet}$. Set:

$$C_1 := \sup_{x \in \tilde{\Omega}} |\nabla g(x)|, \qquad C_2 := \sup_{x \in \tilde{\Omega}} \|D^2 g(x)\|_2$$

where $\tilde{\Omega} = \operatorname{conv}(\overline{\Omega} \cup \widehat{G})$. Then there exists an absolute constant $C > 0$ such that for all $x \in V^\bullet \cap \Omega$,

$$|h_d(x) - h_c(x)| \leqslant \frac{C \operatorname{diam}(\Omega)(C_1 + C_2\varepsilon)}{\log^{1/2}(\operatorname{diam}(\Omega)/(\delta \vee \varepsilon))}$$

Property (S) for simple random walks on orthodiagonal maps follows from Theorem 3.3.1 via an elementary argument.

Lemma 3.3.2. (Microscale Property (S)) Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $u \in V^\bullet$ is a primal vertex of G, and $R > 0$ is a positive real number so that $B(u, R) \subseteq \widehat{G}$. Let $(S_n)_{n=0}^\infty$ be a simple random walk on $G^\bullet$ and let $T_{\partial B(u,R)}$ be the first time at which our random walk exits the open ball $B(u, R)$:

$$T_{\partial B(u,R)} := \inf\{n \in \mathbb{N}_0 : S_n \notin B(u, R)\}$$

Then there exist absolute constants $c, C > 0$, $\eta \in (0, \pi)$ so that:

$$\mathbb{P}^u\big(\arg(S_{T_{\partial B(u,R)}} - u) \in I\big) \geqslant c$$

for any interval $I \subset S^1$ with $\operatorname{length}(I) \geqslant \eta$, provided that $R \geqslant C\varepsilon$.

We call this a microscale estimate because it holds for all balls whose radius is at least a constant multiple of the mesh of our orthodiagonal map.

Proof. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $u \in V^\bullet$ is a primal vertex of G, and $R > 0$ is a positive real number so that $B(u, R) \subseteq \widehat{G}$. Let $B_G(u, R) = (V^\bullet_{B_G(u,R)} \sqcup V^\circ_{B_G(u,R)}, E_{B_G(u,R)})$ be the suborthodiagonal map of G formed by taking the union of all the inner faces Q of G whose corresponding primal diagonal $e^\bullet_Q$ intersects the open ball $B(u, R)$. Having defined $B_G(u, R)$ in this way, we see that a simple random walk on $B_G(u, R)^\bullet$ run until it hits $\partial V^\bullet_{B_G(u,R)}$ is the same thing as a simple random walk on $G^\bullet$, run until it leaves the open ball $B(u, R)$. Since the edges of G all have length at most ε, the edges of $G^\bullet$ have length at most 2ε. Hence, the boundary vertices of $B_G(u, R)^\bullet$ all lie within 2ε of $\partial B(u, R)$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with compact support such that:

- $\varphi(x) = 1$ if $\|x\| \leqslant \frac{1}{2}$

- $\varphi(x) \in (0,1)$ if $\frac{1}{2} < \|x\| < 1$.
- $\varphi(x) = 0$ if $\|x\| \geqslant 1$

Suppose $v \in \partial B(u,R)$. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$g(x) = \varphi\Big(\frac{x-v}{R}\Big)$$

Observe that:

- $g(x) = 1$ for $x \in \partial B(u,R)$ such that $|\arg(x-u)| \leqslant \arccos\big(\frac{7}{8}\big)$.
- $g(x) \in (0,1)$ for $x \in \partial B(u,R)$ such that $\arccos\big(\frac{7}{8}\big) < |\arg(x-u)| < \arccos\big(\frac{1}{2}\big)$.
- $g(x) = 0$ for $x \in \partial B(u,R)$ if $|\arg(x-u)| \geqslant \arccos\big(\frac{1}{2}\big)$.

Here $\arccos(x)$ takes values in $[0,\pi]$ for $x \in [-1,1]$. Let $h_d : V^\bullet_{B_G(u,R)} \to \mathbb{R}$ denote the solution to the following boundary value problem on $B_G(u,R)^\bullet$:

$$\begin{aligned} \Delta^\bullet h_d(x) &= 0 \qquad \text{for all } x \in \mathrm{Int}(V^\bullet_{B_G(u,R)}) \\ h_d(x) &= g(x) \;\; \text{for all } x \in \partial V^\bullet_{B_G(u,R)} \end{aligned}$$

If $(S_n)_{n=1}^\infty$ is a simple random walk on $G^\bullet$ and $T_{\partial B(u,R)} = \inf\{n \in \mathbb{N}_0 : S_n \notin B(u,R)\}$ is the time at which our random walk exits $B(u,R)$, then by the maximum principle for discrete harmonic functions:

$$\mathbb{P}^u\big(|\arg(S_{T_{\partial B(u,R)}} - u)| \leqslant \arccos(1/2)\big) \geqslant h_d(u)$$

provided that $R \geqslant 2\varepsilon$. Notice that:

$$\|\nabla g\|_\infty = R^{-1}\|\nabla\phi\|_\infty, \qquad \|D^2 g\|_\infty = R^{-2}\|D^2\phi\|_\infty$$

Let $h_c : \overline{B(u,R)} \to \mathbb{R}$ be the solution to the corresponding continuous boundary value problem on $B(u,R)$:

$$\begin{aligned} \Delta h_c(x) &= 0 \qquad \text{for all } x \in B(u,R) \\ h_c(x) &= g(x) \;\; \text{for all } x \in \partial B(u,R) \end{aligned}$$

Since $g \equiv 1$ on a boundary arc along $\partial B(u,R)$ of length $2R \cdot \arccos(7/8)$,

$$h_c(u) \geqslant \frac{\arccos(7/8)}{\pi}$$

Applying Theorem 3.3.1 with $\Omega = B(u,R)$, $G = B_G(u,R)$ and g as above, we have that:

$$|h_d(u) - h_c(u)| \leqslant \frac{C(2R)(R^{-1}\|\nabla\varphi\|_\infty + R^{-2}\|D^2\varphi\|_\infty\varepsilon)}{\log^{1/2}(2R/\varepsilon)} = \frac{2C(\|\nabla\varphi\|_\infty + \|D^2\varphi\|_\infty\varepsilon R^{-1})}{\log^{1/2}(2R/\varepsilon)}$$

Notice that if we take $R \geqslant C'\varepsilon$ for some absolute constant $C' > 0$ sufficiently large, we can ensure that:

$$\frac{2C(\|\nabla\varphi\|_\infty + \|D^2\varphi\|_\infty \varepsilon R^{-1})}{\log^{1/2}(2R/\varepsilon)} \leqslant \frac{\arccos(7/8)}{2\pi}$$

Putting all this together, we get that:

$$\mathbb{P}^u\big(|\arg(S_{T_{\partial B(u,R)}} - u)| \leqslant \arccos(1/2)\big) \geqslant h_d(u) \geqslant h_c(u) - |h_c(u) - h_d(u)| \geqslant \frac{\arccos(7/8)}{2\pi}$$

provided that $R \geqslant C'\varepsilon$ for some absolute constant $C' > 0$. $\square$

3.3.2 Microscale Property (T) for Orthodiagonal Maps

Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map, $H = (V_H^\bullet \sqcup V_H^\circ, E_H)$ is a suborthodiagonal map of G, and $u \in \mathrm{Int}(V_H^\bullet)$. Then $G_{H^\bullet}(\cdot, u) : V_H^\bullet \to \mathbb{R}$ is the analyst's Green's function on $H^\bullet$ centered at u. This is the unique solution to the boundary value problem:

$$\begin{aligned} \Delta_v^\bullet G_{H^\bullet}(v,u) &= -\delta_u(v) && \text{for all } v \in \mathrm{Int}(V_H^\bullet) \\ G_{H^\bullet}(v,u) &= 0 && \text{for all } v \in \partial V_H^\bullet \end{aligned}$$

The subscript of v in "$\Delta_v^\bullet$" is there to make it clear that we are looking at the discrete Laplacian on $G^\bullet$ with respect to the variable v, not u. The Green's function also has a probabilistic interpretation. Let $(S_n)_{n\geqslant 0}$ be a simple random walk on $H^\bullet$. For any nonempty subset $A \subseteq V_H^\bullet$, let T_A denote the hitting time of A by this random walk:

$$T_A = \inf\{n \in \mathbb{N}_0 : S_n \in A\}$$

Then:

$$G_{H^\bullet}(v,u) = \frac{\mathbb{E}^u|\{0 \leqslant n \leqslant T_{\partial V_H^\bullet} : S_n = v\}|}{\pi_u}$$

where $\pi_u = \sum_{w:w\sim u} c(u,w)$ is the sum of the weights of the edges of $H^\bullet$ incident to u. Using this probabilistic interpretation, since a simple random walk on $G^\bullet$ is a reversible Markov process, the Green's function is symmetric. Namely, if $u, v \in \mathrm{Int}(V_H^\bullet)$, then:

$$G_{H^\bullet}(u,v) = G_{H^\bullet}(v,u)$$

If $x \in \mathrm{Int}(V^\bullet)$, define:

$$A_x := \sum_{\substack{y \in V_H^\bullet \\ y \sim x}} \mathrm{Area}(Q_{\{x,y\}})$$

where $Q_{\{x,y\}}$ is the inner face of G with primal diagonal $\{x, y\}$. Having made these preliminary observations, we are ready to state and prove the analogue of Chelkak's property (T) for orthodiagonal maps:

Lemma 3.3.3. (Microscale Property (T)) Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $u \in V^\bullet$ is a primal vertex of G and $R > 0$ is a positive real number so that $B(u,R) \subseteq \widehat{G}$. Let $B_G(u,R) = (V^\bullet_{B_G(u,R)} \sqcup V^\circ_{B_G(u,R)}, E_{B_G(u,R)})$ be the suborthodiagonal map of G formed by taking the union of all the inner faces Q of G whose corresponding primal diagonal $e^\bullet_Q$ intersects the open ball $B(u,R)$. Having defined $B_G(u,R)$ in this way, notice that $\mathrm{Int}(V^\bullet_{B_G(u,R)}) = V^\bullet \cap B(u,R)$, $\partial V^\bullet_{B_G(u,R)} = V^\bullet \cap \{z : |z-u| \geqslant R\}$. Then:

$$\frac{1}{2}R^2 \leqslant \sum_{v \in V^\bullet \cap B(u,R)} A_v G_{B_G(u,R)^\bullet}(v,u) \leqslant \frac{1}{2}(R+2\varepsilon)^2$$

In particular, if $R \geqslant 2\varepsilon$:

$$\frac{1}{2}R^2 \leqslant \sum_{v \in V^\bullet \cap B(u,R)} A_v G_{B_G(u,R)^\bullet}(v,u) \leqslant 2R^2$$

Proof. To prove property (T) for orthodiagonal maps, we first need to make some preliminary observations. Given an orthodiagonal map $G = (V^\bullet \sqcup V^\circ, E)$, let $[u_1, v_1, u_2, v_2]$ denote the inner face of G with incident vertices u_1, v_1, u_2, v_2 listed in counterclockwise order where $u_1, u_2 \in V^\bullet$, $v_1, v_2 \in V^\circ$. Using this notation, $[u_1, v_1, u_2, v_2] = [u_2, v_2, u_1, v_1]$. Suppose $x \in \mathrm{Int}(V^\bullet)$ and $x_1, x_2, ..., x_m$ are the neighbours of x in $G^\bullet$, listed in counterclockwise order. Let $y_1, y_2, y_3, ..., y_m$ be vertices of G° so that for each i, $[x, y_i, x_i, y_{i+1}]$ is a face of G, where our indices i are considered modulo m.

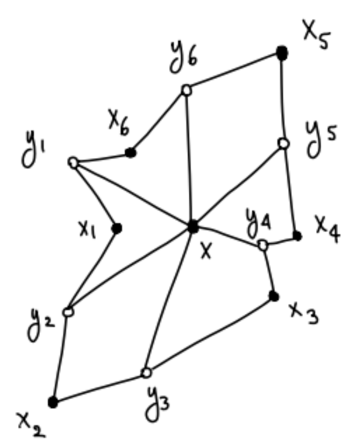

Figure 3.1: A vertex x and its neighbors in an orthodiagonal map.

Then for any $z_0 \in \mathbb{C}$:

$$\Delta^\bullet\big(z \mapsto |z-z_0|^2\big)\big|_{z=x} = \sum_{y \in V^\bullet : y \sim x} c(x,y)\Big(|y-z_0|^2 - |x-z_0|^2\Big) = \sum_{i=1}^m \frac{|y_{i+1}-y_i|}{|x_i - x|}\Big(|x_i - z_0|^2 - |x-z_0|^2\Big)$$
$$= \sum_{i=1}^m \frac{|y_{i+1}-y_i|}{|x_i - x|}\Big(|x_i - x|^2 + 2\,\mathrm{Re}\big((x_i - x)\overline{(x-z_0)}\big)\Big)$$
$$= \sum_{i=1}^m |y_{i+1}-y_i||x_i - x| + 2\,\mathrm{Re}\Big(\overline{(x-z_0)}\sum_{i=1}^m \frac{|y_{i+1}-y_i|}{|x-x_i|}(x_i - x)\Big)$$

Since $[x, y_i, x_i, y_{i+1}]$ is a quadrilateral with orthogonal diagonals for each index i, we have that:

$$\mathrm{Area}([x, y_i, x_i, y_{i+1}]) = \frac{1}{2}|y_{i+1}-y_i||x_i - x| \quad \text{and} \quad (x_i - x) = i\frac{|y_{i+1}-y_i|}{|x_i - x|}(y_{i+1}-y_i)$$

Plugging these into our formula for $\Delta^\bullet\big(z \mapsto |z-z_0|^2\big)\big|_{z=x}$ we have that:

$$\Delta^\bullet\big(z \mapsto |z-z_0|^2\big)\big|_{z=x} = 2\sum_{i=1}^m \mathrm{Area}([x, y_i, x_i, y_{i+1}]) + 2\,\mathrm{Re}\Big(i\overline{(x-z_0)}\sum_{i=1}^m (y_{i+1}-y_i)\Big) = 2A_x$$

Suppose $u \in \mathrm{Int}(V^\bullet)$ is an interior vertex of G and $R > 0$ is a real number so that $B(u,R) \subseteq \widehat{G}$. Let $B_G(u,R) = (V^\bullet_{B_G(u,R)} \sqcup V^\circ_{B_G(u,R)}, E_{B_G(u,R)})$ be the suborthodiagonal map of G described in the statement of Lemma 3.3.3. Notice that a simple random walk on $B_G(u,R)^\bullet$ run until it hits $\partial V^\bullet_{B_G(u,R)}$ is the same thing as a simple random walk on $G^\bullet$, run until it leaves the open ball $B(u,R)$. Furthermore, $\mathrm{Int}(V^\bullet_{B_G(u,R)}) = V^\bullet \cap B(u,R)$.

If $(S_n)_{n \geqslant 0}$ is a simple random walk on $G^\bullet$ with canonical filtration $(\mathcal{F}_n)_{n\geqslant 0}$, for any $z_0 \in \mathbb{C}$:

$$\mathbb{E}(|S_{n \wedge T_{\partial V^\bullet}} - z_0|^2 \,|\mathcal{F}_{n-1}) = |S_{(n-1)\wedge T_{\partial V^\bullet}} - z_0|^2 + \frac{\Delta^\bullet\big(z \mapsto |z-z_0|^2\big)\big|_{z=S_{(n-1)\wedge T_{\partial V^\bullet}}}}{\pi_{S_{(n-1)\wedge T_{\partial V^\bullet}}}}$$
$$= |S_{(n-1)\wedge T_{\partial V^\bullet}} - z_0|^2 + \frac{2A_{S_{(n-1)\wedge T_{\partial V^\bullet}}}}{\pi_{S_{(n-1)\wedge T_{\partial V^\bullet}}}}$$

It follows that the process:

$$|S_{n\wedge T_{\partial B(u,R)}} - u|^2 - 2\sum_{k=0}^{n\wedge(T_{\partial B(u,R)}-1)} \frac{A_{S_{k\wedge T_{\partial B(u,R)}}}}{\pi_{S_{k\wedge T_{\partial B(u,R)}}}}$$

is a martingale. By the optional stopping theorem:

$$\mathbb{E}^u|S_{T_{\partial B(u,R)}} - u|^2 = 2\mathbb{E}^u\Big(\sum_{k=0}^{n\wedge(T_{\partial B(u,R)}-1)} \frac{A_{S^u_{k\wedge T_{\partial B(u,R)}}}}{\pi_{S^u_{k\wedge T_{\partial B(u,R)}}}}\Big) = 2\sum_{v \in V^\bullet \cap B(u,R)} \frac{A_v}{\pi_v}\mathbb{E}|\{0 \leqslant k \leqslant T_{\partial B(u,R)} : S^u_k = v\}|$$
$$= 2\sum_{v \in V^\bullet \cap B(u,R)} A_v G_{B_G(u,R)^\bullet}(u,v)$$

On the other hand, since the edges of G have length at most ε, edges of $G^\bullet$ have length at most 2ε from which we infer that:

$$R^2 \leqslant \mathbb{E}^u|S_{T_{\partial B(u,R)}} - u|^2 \leqslant (R+2\varepsilon)^2$$

Putting all this together, we have that:

$$\frac{1}{2}R^2 \leqslant \sum_{v \in V^\bullet \cap B(u,R)} A_v G_{B_G(u,R)^\bullet}(u,v) \leqslant \frac{1}{2}(R+2\varepsilon)^2$$

If $R \geqslant 2\varepsilon$ it follows that:

$$\frac{1}{2}R^2 \leqslant \sum_{v \in V^\bullet \cap B(u,R)} A_v G_{B_G(u,R)^\bullet}(v,u) \leqslant 2R^2$$

□

3.3.3 The Toolbox

Our goal in this section is to establish analogues of the estimates for random walks and harmonic functions found in the Appendix of [9], including Lemmas 3.1.4 and 3.1.5. The proofs of these results only require properties (S) and (T) as input and so they follow by the same argument as in [9], verbatim. Nevertheless, for the convenience of the reader and for the sake of completeness, we will include proofs or at the very least sketches of proofs herein.

The following result is an analogue of Lemma A.1 of [9]:

Lemma 3.3.4. There exist absolute constants $c, C > 0$ such that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with maximal edge length at most ε, $r > C\varepsilon$ and $u \in V^\bullet$ is a primal vertex of G so that $B(u,r) \subseteq \widehat{G}$, then the probability that a simple random walk on G started at $w \in V^\bullet$, where $\frac{2}{3}r \leqslant |w-u| \leqslant \frac{5}{6}r$, makes a whole turn inside the annulus $B(u,r)\backslash B(u,\frac{1}{2}r)$ and then crosses its own trajectory afterwards, is uniformly bounded below by $c > 0$.

The idea behind the proof is to set up a network of $O(1)$ balls inside our annulus so that:

- by property (S), the probability of travelling from one ball to the next is bounded below by some absolute constant.
- having exhausted all of the $O(1)$ balls, our random walk must have made a loop and thereby crossed its own trajectory.

The only technicality is that we need the radius of our annulus to be sufficiently large so that all of the balls in our network are large enough to ensure that property (S) holds.

Lemma 3.3.5. (Harnack Inequality) There exist absolute constants $c \in (0,1)$, $C > 0$ so that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h : V^\bullet \to [0,\infty)$

is a nonnegative discrete harmonic function on $G^\bullet$ and $u \in V^\bullet$ is a vertex such that $d_u = \operatorname{dist}(u, \partial V^\bullet) \geqslant C\varepsilon$, then:

$$ch(v) \leqslant h(v') \leqslant c^{-1}h(v) \text{ for all } v, v' \in B(u, d_u/2)$$

Proof. By the maximum principle for harmonic functions, there exists a nearest- neighbour path $\gamma = (w_0, w_1, w_2, ..., w_m)$ of vertices in $G^\bullet$ such that $w_0 = v$, $w_m \in \partial V^\bullet$, and $h(w_i) \geqslant h(v)$ for all i. By Lemma 3.3.4, we have that $h(w) \geqslant ch(v)$ for all vertices w such that $\frac{2}{3}d_u \leqslant |w - u| \leqslant \frac{5}{6}d_u$ (any simple random walk path that forms a loop in the annulus $\{z : \frac{1}{2}d_u < |z - u| < d_u\}$ must intersect γ). By the maximum principle, we conclude that $H(v') \geqslant cH(v)$. The proof of the reverse inequality is analogous. □

By an analogous argument, we have the boundary Harnack principle:

Lemma 3.3.6. (Boundary Harnack Principle) There exists an absolute constant $C > 0$ so that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $x \in \partial V^\bullet$, $r > 0$ is a number so that $r \geqslant C\varepsilon$ and $h : V^\bullet \to [0, \infty)$ is a nonnegative discrete harmonic function on G that vanishes on $B(x, r) \cap \partial V^\bullet$, it follows that:

$$h(v) \asymp h(v') \text{ uniformly for all } v, v' \in B(x, r) \cap V^\bullet \text{ such that } \frac{2}{3}r \leqslant |v - v'| \leqslant \frac{5}{6}r$$

Now we are ready to prove the weak Beurling estimate for simple random walks on orthodiagonal maps, Lemma 3.1.4:

Proof. (of Lemma 3.1.4) Let $c, C > 0$ be the absolute constants in the statement of Lemma 3.3.4. Suppose $d_u = \operatorname{dist}(u, \partial V^\bullet) \geqslant C\varepsilon$. Then we can build a network of $\lfloor \log_2\left(\frac{d_u}{r}\right) \rfloor$ annuli with disjoint interiors, so that each annulus separates u from $\partial B(u, r)$ and the ratio between the outer radius and inner radius of each annulus is 2. By Lemma 3.3.4, the probability of a simple random walk on $G^\bullet$ crossing each annulus without first forming a loop separating the inside and outside of the annulus, is bounded above by $(1 - c)$. Of course, if our random walk does form such a loop, it must intersect $\partial V^\bullet$ in the process. Thus:

$$\mathbb{P}^u(|S_{T_{\partial V^\bullet}} - u| \geqslant r) \leqslant (1 - c)^{\lfloor \log_2(\frac{d_u}{r}) \rfloor} \leqslant (1 - c)^{-1}\left(\frac{d_u}{r}\right)^{\log_2(\frac{1}{1-c})}$$

To apply Lemma 3.3.4, we needed the inner radius of each annulus to be at least $C\varepsilon$. This is where the requirement that $d_u > C\varepsilon$ came in. Otherwise, the best we can do is to use the maximum principle to conclude that:

$$\mathbb{P}^u(|S_{T_{\partial V^\bullet}} - u| \geqslant r) \leqslant (1 - c)^{-1}\left(\frac{C\varepsilon}{r}\right)^{\log_2(\frac{1}{1-c})}$$

□

Having established Lemma 3.1.4, Lemma 3.1.5, which tells us that discrete harmonic functions on orthodiagonal maps are β- Hölder in the bulk for some absolute constant $\beta \in (0,1)$, follows as an immediate corollary:

Proof. (of Lemma 3.1.5) Let $C > 0$ be the absolute constant with the same name in Lemma 3.1.4. Observe that if $|x-y| \vee C\varepsilon > \frac{1}{2}d$, the desired result holds trivially, since $|h(y)-h(x)| \leqslant 2\|h\|_\infty$. With this in mind, suppose that $|y-x| \vee C\varepsilon \leqslant \frac{1}{2}d$ and WLOG, $h(y) \geqslant h(x)$. Additionally, suppose that $|x-y| > C\varepsilon$. By the maximum principle, there exists a path $\gamma = (w_0, w_1, ..., w_m)$ of vertices in $G^\bullet$ so that $w_0 = y$, $w_m \in \partial V^\bullet$ and $h(w_i) \geqslant h(y)$. By Lemma 3.1.4, with high probability, a simple random walk started at x will hit γ before exiting the ball of radius d centred at y. Thus:

$$\begin{aligned} h(x) &\geqslant \left(1 - C'\Big(\frac{|y-x|}{d-|y-x|}\Big)^\beta\right) h(y) - C'\Big(\frac{|y-x|}{d-|y-x|}\Big)^\beta \|h\|_\infty \\ &\geqslant \Big(1 - C'\Big(\frac{2|y-x|}{d}\Big)^\beta\Big) h(y) - C'\Big(\frac{2|y-x|}{d}\Big)^\beta \|h\|_\infty \end{aligned}$$

Since $h(y) \geqslant h(x)$, it follows that:

$$|h(y)-h(x)| \leqslant K\|h\|_\infty \Big(\frac{|y-x|}{d}\Big)^\beta$$

where $K > 0$ is an absolute constant. If on the other hand $|x-y| \leqslant C\varepsilon$, where $C\varepsilon \leqslant \frac{1}{2}d$, the maximum principle tells us that:

$$|h(y)-h(x)| \leqslant K\|h\|_\infty \Big(\frac{C\varepsilon}{d}\Big)^\beta$$

This completes our proof. □

While we will not need them for the proof of Theorem 3.1.1, the main result of this chapter, the following two lemmas establish estimates for the Green's function on orthodiagonal maps, analogous to estimates that exist in the continuum. Lemma 3.3.7 is the analogue of Lemma 2.13 of [9] for orthodiagonal maps, and follows by the same argument, verbatim. While an analogue of Lemma 3.3.8 is not explicitly stated in [9], it follows readily from an analogue of Lemma 2.9 of [9] and the weak Beurling estimate. For the corresponding Green's function estimates in the continuum, see Sections 2.4 and 2.5 of [36].

Lemma 3.3.7. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with maximal edge length at most ε, $\Omega = (V_\Omega^\bullet \sqcup V_\Omega^\circ, E_\Omega)$ is a suborthodiagonal map of G, $u \in V_\Omega^\bullet$, $r = \mathrm{dist}(u, \partial V_\Omega^\bullet)$, and $R > 0$ is a real number so that $B(x,R) \subseteq \widehat{G}$. Then there exist absolute constants $A, C > 0$, $c \in (0,1)$ so that if $r \geqslant C\varepsilon$ and $2^k r \leqslant R$, where $k \in \mathbb{N}$, we have that:

$$G_{B_G(u,r)^\bullet}(v,u) \leqslant G_{\Omega^\bullet}(v,u) \leqslant \Big(1 + \frac{A}{(1-c)^{-k}-1}\Big) G_{B_G(u,2^k r)^\bullet}(x,u)$$

Proof. Suppose $H = (U^\bullet \sqcup U^\circ, F)$ is a suborthodiagonal map of $G = (V^\bullet \sqcup V^\circ, E)$. Recall that we have the following probabilistic interpretation for the Green's function:

$$G_{H^\bullet}(x,y) = \frac{\mathbb{E}^x|\{n < \tau_{\partial U^\bullet} : S_n = y\}|}{\pi_x}$$

where $x \in U^\bullet$, $(S_n)_{n\geqslant 1}$ is a simple random walk on $G^\bullet$, $\tau_{\partial U^\bullet}$ is the hitting time of $\partial U^\bullet$ by this random walk, and:

$$\pi_x = \sum_{y:y\sim x} c(x,y)$$

With this in mind, the left-hand inequality in Lemma 3.3.7 is trivial: it just follows from the fact that $\tau_{\partial B_G(u,r)^\bullet} \leqslant \tau_{\partial\Omega^\bullet}$.Regarding the right-hand inequality, by the strong Markov property for simple random walk on $G^\bullet$, we have that:

$$G_{\Omega^\bullet}(x,u) \leqslant G_{B_G(u,2^k r)^\bullet}(x,u) + \max_{y\in\partial V^\bullet_{B_G(u,2^k r)}} \mathbb{P}(\text{SRW started at } y \text{ hits } \partial V^\bullet_{B_G(u,r)} \text{ before } \partial V^\bullet_\Omega) \cdot \max_{z\in\partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(z,u)$$

As far as handling the second term, observe that if our random walk starts at a point of $\partial V^\bullet_{B_G(u,2^k r)}$ and hits $\partial V^\bullet_{B_G(u,r)}$ before $\partial V^\bullet_\Omega$, it must have crossed a series of annuli $A(u, 2^j r, 2^{j+1} r)$ for $j = 0, 1, 2, ..., k-1$ without making a whole turn and then crossing its trajectory, thereby hitting $\partial V^\bullet_\Omega$. By Lemma 3.3.4, as long as $r \geqslant C\varepsilon$ for some absolute constant $C > 0$,

$$\max_{y\in\partial V^\bullet_{B_G(u,2^k r)}} \mathbb{P}(\text{SRW started at } y \text{ hits } \partial V^\bullet_{B_G(u,r)} \text{ before } \partial V^\bullet_\Omega) \leqslant (1-c)^k$$

where $c \in (0,1)$ is a constant, independent of the geometry of our orthodiagonal map. Thus, we have that:

$$G_{\Omega^\bullet}(x,u) \leqslant G_{B_G(u,2^k r)^\bullet}(x,u) + (1-c)^k \max_{z\in\partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(z,u)$$

Applying this estimate to $G_{\Omega^\bullet}(z,u)$, where $z \in \partial V^\bullet_{B_G(u,r)}$, in the expression above and then iterating this process, we have that:

$$\begin{aligned}
G_{\Omega^\bullet}(x,u) &\leqslant G_{B_G(u,2^k r)^\bullet}(x,u) + (1-c)^k \max_{z\in\partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(z,u) \\
&\leqslant G_{B_G(u,2^k r)^\bullet}(x,u) + (1-c)^k \max_{z\in\partial V^\bullet_{B_G(u,r)}} \left(G_{B_G(u,2^k r)^\bullet}(z,u) + (1-c)^k \max_{w\in\partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(w,u)\right) \\
&= G_{B_G(u,2^k r)^\bullet}(x,u) + (1-c)^k \max_{z\in\partial V^\bullet_{B_G(u,r)}} G_{B_G(u,2^k r)^\bullet}(z,u) + (1-c)^{2k} \max_{w\in\partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(w,u) \\
&\leqslant ... \\
&\leqslant G_{B_G(u,2^k r)^\bullet}(x,u) + \frac{(1-c)^k}{1-(1-c)^k}\left(\max_{z\in\partial V^\bullet_{B_G(u,r)}} G_{B_G(u,2^k r)^\bullet}(z,u)\right)
\end{aligned}$$

Observe that $G_{B_G(u,2^k r)^\bullet}(\cdot,u)$ is harmonic in $B_G(u,2^k r)^\bullet$ away from u, and we can cover $\partial V^\bullet_{B_G(u,r)}$ with $O(1)$ balls of radius $r/2$, centered at points of $\partial V^\bullet_{B_G(u,r)}$. Applying the elliptic Harnack inequality (Lemma 3.3.5) on each ball and chaining these estimates together, we have that:

$$\min_{z \in \partial V^\bullet_{B_G(u,r)}} G_{B_G(u,2^k r)^\bullet}(z,u) \asymp \max_{z \in \partial V^\bullet_{B_G(u,r)}} G_{B_G(u,2^k r)^\bullet}(z,u)$$

Since $G_{B_G(u,2^k r)^\bullet}(\cdot,u)$ is superharmonic on $B_G(u,r)^\bullet$,

$$\min_{z \in \partial V^\bullet_{B_G(u,r)}} G_{B_G(u,2^k r)^\bullet}(z,u) \leqslant G_{B_G(u,2^k r)^\bullet}(x,u)$$

for any $x \in V^\bullet_{B_G(u,r)}$. Thus, we have absolute constants $c \in (0,1)$, $A > 0$ so that:

$$G_{\Omega^\bullet}(x,u) \leqslant \Big(1 + \frac{A}{(1-c)^{-k} - 1}\Big) G_{B_G(u,2^k r)^\bullet}(x,u)$$

for any $x \in V^\bullet_{B_G(u,r)}$, provided that $r = \mathrm{dist}(u, \partial V^\bullet_\Omega) \geqslant C\varepsilon$ for some absolute constant $C > 0$ and $B(u, 2^k r) \subseteq \widehat{G}$. $\square$

Lemma 3.3.8. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with maximal edge length at most ε, $\Omega = (V^\bullet_\Omega \sqcup V^\circ_\Omega, E_\Omega)$ is a suborthodiagonal map of G, $u \in V^\bullet_\Omega$, $r = \mathrm{dist}(u, \partial V^\bullet_\Omega)$ and $x \in V^\bullet_\Omega$ satisfies $|x - u| = R > ((K\varepsilon) \vee (2r))$ where $B(u, R+\varepsilon) \subseteq \widehat{G}$ and $K > 0$ is an absolute constant. Then there exist absolute constants $\beta, B > 0$ such that:

$$G_{\Omega^\bullet}(x,u) \leqslant B\Big(\frac{r \vee \varepsilon}{R}\Big)^\beta$$

Proof. Without loss of generality, assume that $r \geqslant C\varepsilon$ where $C > 0$ is the absolute constant from lemma 4.1. We can do this because if $r < C\varepsilon$, adding the vertices and edges of $G \cap B(u, C\varepsilon)$ to Ω gives us an orthodiagonal map that satisfies $r \geqslant C\varepsilon$ whose Green function at u is strictly larger than that of Ω.

By the optional stopping theorem applied to our random walk, stopped upon hitting $\partial V^\bullet_\Omega$ or $\partial V^\bullet_{B_G(u,r)}$:

$$G_{\Omega^\bullet}(x,u) \leqslant \mathbb{P}^x(\text{SRW started at } x \text{ hits } \partial V^\bullet_{B_G(u,r)} \text{ before it hits } \partial V^\bullet_\Omega) \max_{z \in \partial V^\bullet_{B_G(u,r)}} G^\bullet_\Omega(z,u)$$

Just as in the proof of Lemma 3.3.7 above, for a simple random walk starting at x to hit $\partial V^\bullet_{B_G(u,r)}$ before $\partial V^\bullet_\Omega$, our random walk must cross $k = \lfloor \log_2(R/r) \rfloor$ annuli $A(u, 2^j r, 2^{j+1} r)$ for $j = 0, 1, 2, ..., k-1$ without making a whole turn and then crossing its trajectory in $G^\bullet$. By Lemma 3.3.4, it follows that:

$$\mathbb{P}^x(\text{SRW started at } x \text{ hits } \partial V^\bullet_{B_G(u,r)} \text{ before it hits } \partial V^\bullet_\Omega) \leqslant (1-c)^k$$

where $c \in (0,1)$ is an absolute constant. By Lemma 3.3.7, there exists an absolute constant $C' > 0$

such that:

$$G_{\Omega^\bullet}(x,u) \leqslant C' G_{B_G(u,2r)^\bullet}(x,u)$$

for any $x \in V^\bullet_{B_G(u,r)}$. By property (T):

$$\sum_{x \in B_G(u,r/2)^\bullet} \mathrm{Area}(N_v) G_{B_G(u,2r)^\bullet}(x,u) \leqslant \sum_{x \in B_G(u,2r)^\bullet} \mathrm{Area}(N_v) G_{B_G(u,2r)^\bullet}(x,u) \leqslant (2r+\varepsilon)^2$$

Additionally, we have that:

$$\sum_{x \in B_G(u,r/2)^\bullet} \mathrm{Area}(N_v) \geqslant (r/2-\varepsilon)^2$$

As long as $C > 2$, it follows that there exists an absolute constant $c_0 > 0$ so that:

$$G_{\Omega^\bullet}(x_0,u) \leqslant c_0$$

for some $x_0 \in B(u,r/2)$. However, as we discussed in the proof of Lemma 3.3.7, since $G_{\Omega^\bullet}(\cdot,u)$ is superharmonic on $\Omega^\bullet$ and harmonic away from u, we have that:

$$G_{\Omega^\bullet}(x_0,u) \geqslant \min_{x \in \partial V^\bullet_{B_G(u,r/2)}} G_{\Omega^\bullet}(x,u) \asymp \max_{x \in \partial V^\bullet_{B_G(u,r/2)}} G_{\Omega^\bullet}(x,u) \geqslant \max_{x \in \partial V^\bullet_{B_G(u,r)}} G_{\Omega^\bullet}(x,u)$$

Putting all this together, we have that:

$$G_{\Omega^\bullet}(x,u) \lesssim (1-c)^{\lfloor \log_2(R/r) \rfloor} \lesssim (r/R)^{\log_2(\frac{1}{1-c})}$$

Observe that the exponent $\beta \in (0,1)$ we get from this argument is the same exponent $\beta \in (0,1)$ in Lemmas 3.1.4 and 3.1.5. □

3.4 The Convolution of a Discrete Harmonic Function with a Smooth Mollifier is Almost Harmonic

In the section, we will show that the convolution of a discrete harmonic function with a smooth mollifier has small Laplacian. This is the key estimate that will allow us to compare our mollified discrete harmonic function to the corresponding continuous harmonic function. We will also use this result in Chapter 4 to show that discrete harmonic functions are Lipschitz in the bulk on a mesoscopic scale. To prove this, we'll need the following analogue of Proposition 3.12 of [11]:

Proposition 3.4.1. Suppose $S \subseteq \mathbb{R}^2$ is a square with side length l, $f \in C^3_b(S)$ and $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε such that $S \subseteq \widehat{G}$ and $l \gtrsim \varepsilon$. Then:

$$\sum_{v \in V^\bullet \cap S} \Delta^\bullet f(v) = \int_S \Delta f(x) dA(x) + O(\varepsilon \cdot l \cdot \|D^2 f\|) + O(l^3 \cdot \|D^3 f\|)$$

For context, notice that if our orthodiagonal map $G = (V^\bullet \sqcup V^\circ, E)$ is isoradial, with edges of length at most ε and $f \in C_b^3(\widehat{G})$, a straightforward computation (see Lemma 2.2 of [15]) tells us that for any vertex $x \in \mathrm{Int}(V^\bullet)$ we have that:

$$\Delta^\bullet f(x) = \Delta f(x) \cdot \mathrm{Area}(N_x) + O(\varepsilon \cdot \mathrm{Area}(N_x) \cdot \|D^3 f\|) \tag{3.4.1}$$

$$= \Delta f(x) \cdot \mathrm{Area}(N_x) + O(\varepsilon^3 \cdot \|D^3 f\|) \tag{3.4.2}$$

where for any vertex $x \in \mathrm{Int}(V^\bullet)$,

$$\mathrm{Area}(N_x) = \frac{1}{2} \sum_{\substack{y \in V^\bullet \\ y \sim x}} \mathrm{Area}(Q_{\{x,y\}})$$

where $Q_{\{x,y\}}$ is the face of G with primal diagonal $\{x, y\}$. The factor of $\frac{1}{2}$ here, comes from the fact that every inner face of G has two primal vertices, so it is natural that the area of this face should be split evenly between them. Equation 3.4.2 tells us that on isoradial graphs, at any vertex, the discrete Laplacian of a smooth function looks like the continuous Laplacian. In particular, any continuous harmonic function defined in a neighbourhood of $\widehat{G}$ is "almost" discrete harmonic in that its discrete Laplacian is small.

Repeating this computation for a general orthodiagonal map, one finds that this result is no longer true: at any fixed vertex, the discrete Laplacian does not look like the continuous Laplacian. However, Proposition 3.4.1 tells us that, at least on average, the discrete Laplacian does indeed look like the continuous Laplacian.

Proof. By Lemma 2.3.2 we can pick a suborthodiagonal map S' of G so that:

- $d_{\mathrm{Haus}}(\partial\widehat{S'}, \partial S) = O(\varepsilon)$.
- if $x_1, x_2, ..., x_m$ are the vertices of $\partial V^\circ_{S'}$ listed in counterclockwise order, then these vertices form a contour. That is, $\{x_i, x_{i+1}\}$ is an edge of $(S')^\circ$ for all i, where the indices i are being considered modulo m. It follows that if $x \in \partial V^\bullet_{S'}$, then x has exactly one neighbouring vertex y in $(S')^\bullet$, where $y \in \mathrm{Int}(V^\bullet_{S'})$.
- For $x_1, x_2, ..., x_m$ as above, $\sum_{i=1}^{m} |x_{i+1} - x_i| \asymp l$.

In other words, S' is an orthodiagonal approximation to S that is close in Hausdorff distance and whose perimeter (at least in the dual lattice) is comparable to that of S. We will see why this is important later. Then:

$$\sum_{v \in V^\bullet \cap S} \Delta^\bullet f(v) = \sum_{v \in V^\bullet \cap S} \sum_{u : u \sim v} c(u,v)\big(f(u) - f(v)\big)$$

$$= \sum_{v \in V^\bullet \cap S} \sum_{u : u \sim v} c(u,v)\big(\nabla f(v)(u - v) + \frac{1}{2}(u - v)^T D^2 f(v)(u - v) + O(\|D^3 f\| \cdot |u - v|^3)\big)$$

Since $\nabla f(v) \cdot u$ is a linear function of u for fixed v and linear functions are discrete harmonic,

$$\sum_{u:u\sim v} c(u,v)\nabla f(v)\cdot(u-v) = 0$$

for all $v \in \mathrm{Int}(V^\bullet)$. If $\{u,v\}$ is a primal edge of our orthodiagonal map, let $Q = Q_{u,v}$ denote the quadrilateral in our orthodiagonal map that has $\{u,v\}$ as its primal diagonal. Equivalently, we write $Q = [u,r,v,s]$, where u,r,v,s are the vertices of Q listed in counterclockwise order so that $u \in V^\bullet$. Notice that using this notation, $[u,r,v,s] = [v,s,u,r]$. Since our quadrilaterals all have orthogonal diagonals,

$$\mathrm{Area}([u,r,v,s]) = \frac{1}{2}|u-v||r-s| = \frac{1}{2}c(u,v)|u-v|^2$$

With this in mind,

$$\begin{aligned}\sum_{v\in V^\bullet\cap S}\sum_{u:u\sim v} c(u,v)O(\|D^3 f\|\cdot|u-v|^3) &= \sum_{v\in V^\bullet\cap S}\sum_{u:u\sim v} O(\|D^3 f\|\cdot\mathrm{Area}(Q_{u,v})\cdot|u-v|)\\ =&O(\|D^3 f\|\cdot\varepsilon\sum_{v\in V^\bullet\cap S}\sum_{u:u\sim v}\mathrm{Area}(Q_{u,v})) \overset{(*)}{=} O(\|D^3 f\|\cdot\varepsilon\cdot\mathrm{Area}(S)) = O(\|D^3 f\|\cdot\varepsilon\cdot l^2)\end{aligned}$$

The equality, $(*)$, follows from the fact that any quadrilateral of G that lies within ε of S has at most two corresponding primal vertices in $V^\bullet \cap S$ and so is counted at most twice in our sum. Thus:

$$\sum_{v\in V^\bullet\cap S}\Delta^\bullet f(v) = \frac{1}{2}\sum_{v\in V^\bullet\cap S}\sum_{u:u\sim v} c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big) + O(\|D^3 f\|\cdot\varepsilon\cdot l^2)$$

By the same reasoning,

$$\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\Delta^\bullet f(v) = \frac{1}{2}\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big) + O(\|D^3 f\|\cdot\varepsilon\cdot l^2)$$

Hence:

$$\begin{aligned}\Big|\sum_{v\in V^\bullet\cap S}\Delta^\bullet f(v) - \sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\Delta^\bullet f(v)\Big| &= \frac{1}{2}\sum_{\substack{v\in V^\bullet\cap S\\ v\notin \mathrm{Int}(V^\bullet_{S'})}}\sum_{u:u\sim v} c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big) + O(\|D^3 f\|\cdot\varepsilon\cdot l^2)\\ &= O\big(\|D^2 f\|\sum_{\substack{v\in V^\bullet\cap S\\ v\notin \mathrm{Int}(V^\bullet_{S'})}}\sum_{u:u\sim v} c(u,v)|u-v|^2\big) + O(\|D^3 f\|\cdot\varepsilon\cdot\mathrm{Area}(S))\\ &= O\big(\varepsilon\cdot l\cdot\|D^2 f\|\big) + O(\|D^3 f\|\cdot\varepsilon\cdot l^2)\end{aligned}$$

In other words, we see that we can approximate the sum of $\Delta^\bullet f$ over $V^\bullet \cap S$ by the sum of $\Delta^\bullet f$

over $\mathrm{Int}(V^\bullet_{S'})$ and the error we incur when we do this is small. In summa, we have that:

$$\sum_{v\in V^\bullet\cap S}\Delta^\bullet f(v)=\frac{1}{2}\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big)+O\big(\varepsilon\cdot l\cdot\|D^2 f\|\big)+O(\varepsilon\cdot l^2\cdot\|D^3 f\|)$$

For any $v\in V^\bullet\cap S$, Taylor- expanding the second derivatives of f about z_S, the center of the square S, we have that:

$$\begin{aligned}\partial_1^2 f(v)&=\partial_1^2 f(z_S)+O(l\cdot\|D^2 f\|)\\ \partial_1\partial_2 f(v)&=\partial_1\partial_2 f(z_S)+O(l\cdot\|D^2 f\|)\\ \partial_2^2 f(v)&=\partial_2^2 f(z_S)+O(l\cdot\|D^2 f\|)\end{aligned}$$

Applying these estimates to the sum above (effectively, we are treating the second derivatives of f as roughly constant on each square) we have that:

$$\begin{aligned}&\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big)=\\
&=\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)\big(\partial_1^2 f(v)(u_1-v_1)^2+2\partial_1\partial_2 f(v)(u_1-v_1)(u_2-v_2)+\partial_2^2 f(v)(u_2-v_2)^2\big)=\\
&=\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\partial_1^2 f(v)\sum_{u:u\sim v}c(u,v)(u_1-v_1)^2+2\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\partial_1\partial_2 f(v)\sum_{u:u\sim v}c(u,v)(u_1-v_1)(u_2-v_2)\\
&\quad+\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\partial_2^2 f(v)\sum_{u:u\sim v}c(u,v)(u_2-v_2)^2=\\
&=\partial_1^2 f(z_S)\Big(\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)(u_1-v_1)^2\Big)+O\big(l\cdot\|D^3 f\|\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)(u_1-v_1)^2\big)\\
&\quad+2\partial_1\partial_2 f(z_S)\Big(\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)(u_1-v_1)(u_2-v_2)\Big)\\
&\quad+O\Big(l\cdot\|D^3 f\|\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)|u_1-v_1||u_2-v_2|\Big)\\
&\quad+\partial_2^2 f(z_S)\Big(\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)(u_2-v_2)^2\Big)+O\big(l\cdot\|D^3 f\|\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v}c(u,v)(u_2-v_2)^2\big)\end{aligned}$$

Observe that $(u_1-v_1)^2,(u_2-v_2)^2,|u_1-v_1||u_2-v_2|\leqslant|u-v|^2$. Furthermore, using the fact that $c(u,v)|u-v|^2=2\mathrm{Area}(Q_{u,v})$ and observing that every quadrilateral Q within $O(\varepsilon)$ of S is counted at most twice in our sum, we have that each of our error terms is of size at most

$O(l^3 \cdot \|D^3 f\|)$. Thus:

$$\sum_{v\in\mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)\big((u-v)^T D^2 f(v)(u-v)\big) =$$

$$= \partial_1^2 f(z_S)\Big(\underbrace{\sum_{v\in\mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)(u_1-v_1)^2}_{(1)}\Big) + 2\partial_1\partial_2 f(z_S)\Big(\underbrace{\sum_{v\in\mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)(u_1-v_1)(u_2-v_2)}_{(2)}\Big)$$

$$+ \partial_2^2 f(z_S)\Big(\underbrace{\sum_{v\in\mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)(u_2-v_2)^2}_{(3)}\Big) + O(l^3 \cdot \|D^3 f\|)$$

To complete the proof of Proposition 3.4.1, we need to understand the behavior of terms (1), (2) and (3). The idea is to use discrete integration by parts to show that each of (1), (2), (3) is equal to the discretization of a certain contour integral. The fact that the perimeter of S' in the dual lattice is comparable to that of S will allow us to show that this discrete contour integral is close to the corresponding continuous contour integral. Consider term (1). Rewriting this as a sum over directed edges we have that:

$$\sum_{v\in\mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)(u_1-v_1)^2 = \sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in\mathrm{Int}(V^\bullet_{S'})}} c(e)(e_1^+-e_1^-)^2 =$$

$$= \underbrace{\sum_{\vec{e}\in\vec{E}^\bullet_{S'}} c(e)(e_1^+-e_1^-)^2}_{(1a)} + \underbrace{\sum_{\substack{\vec{e}\in\vec{E}\\ e^-\in\partial V^\bullet_{S'}}} c(e)(e_1^+-e_1^-)^2}_{(1b)}$$

Term (1a) is just the discrete Dirichlet energy of the function $z \mapsto \mathrm{Re}(z)$ on S'. Since $\mathrm{Re}(z)$ is discrete harmonic, applying discrete integration by parts we have that:

$$\begin{aligned}
\sum_{\vec{e}\in \vec{E}^\bullet_{S'}} c(e)(e_1^+ - e_1^-)^2 &= \sum_{\vec{e}\in \vec{E}^\bullet_{S'}} c(e)e_1^+(e_1^+ - e_1^-) - \sum_{\vec{e}\in \vec{E}^\bullet_{S'}} e_1^-(e_1^+ - e_1^-) \\
&= \sum_{\vec{e}\in \vec{E}^\bullet_{S'}} c(e)e_1^+(e_1^+ - e_1^-) - \sum_{\vec{e}\in \vec{E}^\bullet_{S'}} e_1^+(e_1^- - e_1^+) \\
&= 2\sum_{\vec{e}\in \vec{E}^\bullet_{S'}} c(e)e_1^+(e_1^+ - e_1^-) = -2\sum_{\vec{e}\in \vec{E}^\bullet_{S'}} c(e)e_1^-(e_1^+ - e_1^-) \\
&= -2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \mathrm{Int}(V^\bullet_{S'})}} c(e)e_1^-(e_1^+ - e_1^-) - 2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}}} c(e)e_1^-(e_1^+ - e_1^-) \\
&= -2\sum_{v\in \mathrm{Int}(V^\bullet_{S'})}\sum_{u:u\sim v} c(u,v)v_1(u_1 - v_1) - 2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}}} c(e)e_1^-(e_1^+ - e_1^-) \\
&= -2\sum_{v\in \mathrm{Int}(V^\bullet_{S'})} \mathrm{Re}(v)\Delta^\bullet(\mathrm{Re}(\cdot))(v) - 2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}}} c(e)\mathrm{Re}(e^-)(\mathrm{Re}(e^+) - \mathrm{Re}(e^-)) \\
&= -2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}}} c(e)e_1^-(e_1^+ - e_1^-)
\end{aligned}$$

Suppose $Q = [e^-, f^-, e^+, f^+]$ is a quadrilateral face of G, where the vertices e^-, f^-, e^+, f^- are listed in counterclockwise order. Since $\mathrm{Im}(z)$ is the discrete harmonic conjugate of $\mathrm{Re}(z)$ (up to an additive constant), we have that:

$$c(e)(e_1^+ - e_1^-) = (f_2^+ - f_2^-)$$

More simply, since the diagonals of Q are orthogonal,

$$\frac{e^+ - e^-}{|e^+ - e^-|} = -i\frac{f^+ - f^-}{|f^+ - f^-|} \iff \frac{e_1^+ - e_1^-}{|e^+ - e^-|} = \frac{f_2^+ - f_2^-}{|f^+ - f^-|},\ \frac{e_2^+ - e_2^-}{|e^+ - e^-|} = -\frac{f_1^+ - f_1^-}{|f^+ - f^-|}$$

Since $c(e) = \frac{|f^+ - f^-|}{|e^+ - e^-|}$, $f_1^+ - f_1^- = -c(e)(e_2^+ - e_2^-)$, $f_2^+ - f_2^- = c(e)(e_1^+ - e_1^-)$. Applying this to the problem at hand, we have that:

$$-2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}}} c(e)e_1^-(e_1^+ - e_1^-) = -2\sum_{\substack{\vec{e}\in \vec{E}^\bullet_{S'}\\ e^-\in \partial V^\bullet_{S'}\\ Q_{e^-,e^+} = [e^-,f^-,e^+,f^+]}} e_1^-(f_2^+ - f_2^-)$$

Let $\partial\vec{E}^\bullet_{S'}$ denote the set of directed edges of S' that go from a vertex of $\partial V^\bullet_{S'}$ to a vertex of $\mathrm{Int}(V^\bullet_{S'})$. Because of how we defined S', for all directed edges $\vec{e} = (e^-, e^+)$ under consideration in the sum above, $e^- \in \partial V^\bullet_{S'}$, $e^+ \in \mathrm{Int}(V^\bullet_{S'})$ and $f^-, f^+ \in \partial V^\circ_{S'}$ where $Q_{e^-,e^+} = [e^-, f^-, e^+, f^+]$. Furthermore, the directed edges $\vec{f} = (f^-, f^+)$ dual to directed edges $\vec{e} \in \partial E^\bullet_{S'}$, form a closed contour in $(S')^\circ$, oriented clockwise, with length is comparable to l. We use $\partial^{\circlearrowright} E^\circ_{S'}$ to denote the set of directed edges in this contour. Intuitively, the sum above is a discretization of the integral of xdy over the contour $\partial^{\circlearrowright} E^\circ_{S'}$. We will now make this intuition precise.

Suppose $\vec{f} \in \partial^{\circlearrowright} E^\circ_{S'}$ and $Q_{f^-,f^+} = [e^-, f^-, e^+, f^+]$. Then:

$$|e_1^-(f_2^+ - f_2^-) - \oint_{f^-}^{f^+} xdy| \leqslant \varepsilon |f^+ - f^-|$$

Summing over directed edges $\vec{f} \in \partial^{\circlearrowright} E^\circ_{S'}$,

$$\Big| \sum_{\substack{\vec{e} \in \vec{E^\bullet_{S'}} \\ e^- \in \partial V^\bullet_{S'} \\ Q_{e^-,e^+} = [e^-, f^-, e^+, f^+]}} e_1^+(f_2^+ - f_2^-) - \oint_{\partial^{\circlearrowright} E^\circ_{S'}} xdy \Big| \leqslant \varepsilon \cdot \mathrm{length}(\partial^{\circlearrowright} E^\circ_{S'}) \lesssim \varepsilon \cdot l$$

Thus, we see that term (1a) is close to the contour integral $\oint_{\partial^{\circlearrowright} E^\circ_{S'}} xdy$. By Green's theorem,

$$\oint_{\partial^{\circlearrowright} E^\circ_{S'}} xdy = -l^2 + O(\varepsilon \cdot l)$$

Thus:

$$(1a) = 2l^2 + O(\varepsilon \cdot l)$$

Term (1b) can be dealt with the same way as all the error terms we saw previously:

$$\sum_{\substack{\vec{e} \in \vec{E} \\ e^- \in \partial V^\bullet_{S'}}} c(e)(e_1^+ - e_1^-)^2 \leqslant \sum_{\substack{\vec{e} \in \vec{E} \\ e^- \in \partial V^\bullet_{S'}}} c(e)|e^+ - e^-|^2 = 2 \sum_{\substack{\vec{e} \in \vec{E} \\ e^- \in \partial V^\bullet_{S'}}} \mathrm{Area}(Q_e) = O(\varepsilon \cdot l)$$

Putting all this together, we get that:

$$(1) = 2l^2 + O(\varepsilon \cdot l)$$

A similar story plays out in the case of terms (2) and (3). Rewriting terms (2) and (3) as sums over directed edges we have that:

$$(2) = \sum_{v \in \mathrm{Int}(V^\bullet_{S'})} \sum_{u : u \sim v} c(u,v)(u_1 - v_1)(u_2 - v_2) = \sum_{\substack{\vec{e} \in \vec{E^\bullet_{S'}} \\ e^- \in \mathrm{Int}(V^\bullet_{S'})}} c(e)(e_1^+ - e_1^-)(e_2^+ - e_2^-) =$$

$$= \underbrace{\sum_{\vec{e} \in \vec{E^\bullet_{S'}}} c(e)(e_1^+ - e_1^-)(e_2^+ - e_2^-)}_{(2a)} + \underbrace{\sum_{\substack{\vec{e} \in \vec{E} \\ e^- \in \partial V^\bullet_{S'}}} c(e)(e_1^+ - e_1^-)(e_2^+ - e_2^-)}_{(2b)}$$

$$(3) = \sum_{v \in \mathrm{Int}(V^\bullet_{S'})} \sum_{u : u \sim v} c(u,v)(u_2 - v_2)^2 = \sum_{\substack{\vec{e} \in \vec{E}^\bullet_{S'} \\ e^- \in \mathrm{Int}(V^\bullet_{S'})}} c(e)(e_2^+ - e_2^-)^2 =$$

$$= \underbrace{\sum_{\vec{e} \in \vec{E}^\bullet_{S'}} c(e)(e_2^+ - e_2^-)^2}_{(3a)} + \underbrace{\sum_{\substack{\vec{e} \in \vec{E} \\ e^- \in \partial V^\bullet_{S'}}} c(e)(e_2^+ - e_2^-)^2}_{(3b)}$$

By the same argument we used to handle term (1b), terms (2b) and (3b) are of size $O(\varepsilon \cdot l)$. Applying discrete integration by parts to terms (2a) and (3a) and using the fact that $f_1^+ - f_1^- = -c(e)(e_2^+ - e_2^-)$ and $f_2^+ - f_2^- = c(e)(e_1^+ - e_1^-)$ for any quadrilateral $Q = [e^-, f^-, e^+, f^+]$ in G, we have that:

$$(2a) = \sum_{\vec{e} \in \vec{E}^\bullet_{S'}} c(e)(e_1^+ - e_1^-)(e_2^+ - e_2^-) = -2 \sum_{v \in \mathrm{Int}(V^\bullet_{S'})} \mathrm{Re}(v) \Delta^\bullet(\mathrm{Im}(\cdot))(v) - 2 \sum_{\substack{\vec{e} \in \vec{E}^\bullet_{S'} \\ e^- \in \partial V^\bullet_{S'}}} c(e) e_1^- (e_2^+ - e_2^-)$$

$$= -2 \sum_{\substack{\vec{e} \in \vec{E}^\bullet_{S'} \\ e^- \in \partial V^\bullet_{S'}}} c(e) e_1^- (e_2^+ - e_2^-) = 2 \sum_{\substack{\vec{f} \in \partial^{\circlearrowleft} E^\circ_{S'} \\ Q_{f^-, f^+} = [e^-, f^-, e^+, f^+]}} e_1^- (f_1^+ - f_1^-)$$

$$(3a) = \sum_{\vec{e} \in \vec{E}^\bullet_{S'}} c(e)(e_2^+ - e_2^-)^2 = -2 \sum_{v \in \mathrm{Int}(V^\bullet_{S'})} \mathrm{Im}(v) \Delta^\bullet(\mathrm{Im}(\cdot))(v) - 2 \sum_{\vec{e} \in \vec{E}^\bullet_{S'}} c(e) e_2^- (e_2^+ - e_2^-)$$

$$= -2 \sum_{\vec{e} \in \vec{E}^\bullet_{S'}} c(e) e_2^- (e_2^+ - e_2^-) = 2 \sum_{\substack{\vec{f} \in \partial^{\circlearrowleft} E^\circ_{S'} \\ Q_{f^-, f^+} = [e^-, f^-, e^+, f^+]}} e_2^- (f_1^+ - f_1^-)$$

where:

$$\Big| \sum_{\substack{\vec{f} \in \partial^{\circlearrowleft} E^\circ_{S'} \\ Q_{f^-, f^+} = [e^-, f^-, e^+, f^+]}} e_1^- (f_1^+ - f_1^-) - \oint_{\partial^{\circlearrowleft} E^\circ_{S'}} x dx \Big| \lesssim \varepsilon \cdot l$$

$$\Big| \sum_{\substack{\vec{f} \in \partial^{\circlearrowleft} E^\circ_{S'} \\ Q_{f^-, f^+} = [e^-, f^-, e^+, f^+]}} e_2^- (f_1^+ - f_1^-) - \oint_{\partial^{\circlearrowleft} E^\circ_{S'}} y dx \Big| \lesssim \varepsilon \cdot l$$

By Green's theorem:

$$\oint_{\partial^{\circlearrowleft} E^\circ_{S'}} x dx = 0$$

$$\oint_{\partial^{\circlearrowleft} E^\circ_{S'}} y dx = l^2 + O(\varepsilon \cdot l)$$

And so:

$$(2) = O(\varepsilon \cdot l), \qquad (3) = 2l^2 + O(\varepsilon \cdot l)$$

Armed with these estimates for terms (1), (2) and (3) in our sum from earlier, we have that:

$$\begin{aligned}\sum_{V^\bullet \cap S} \Delta^\bullet f(v) &= \partial_1^2 f(z_S)\big(l^2 + O(\varepsilon \cdot l)\big) + \partial_1\partial_2 f(z_S)\cdot O(\varepsilon\cdot l) + \partial_2^2 f(z_S)(l^2 + O(\varepsilon\cdot l)) + O(l^3\|D^3 f\|)\\ &= \Delta f(z_S)\cdot l^2 + O(\varepsilon\cdot l\cdot \|D^2 f\|) + O(l^3\cdot\|D^3 f\|)\\ &= \int_S \Delta f(x)dA(x) + O(\varepsilon\cdot l\cdot\|D^2 f\|) + O(l^3\cdot\|D^3 f\|)\end{aligned}$$

□

Having shown that for a smooth function, the continuous Laplacian agrees with the discrete Laplacian, on average, we are ready to prove the main result of this section:

Proposition 3.4.2. Suppose $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h^\bullet : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and φ is a smooth mollifier supported on the unit ball $B(0,1) \subseteq \mathbb{R}^2$. We can think of $h^\bullet$ as a function on $\widehat{G}$ by extending $h^\bullet$ to $\widehat{G}$ in any sort of sensible way. For instance, we could triangulate the faces of $G^\bullet$ and then define $h^\bullet$ on each face of $G^\bullet$ by linear interpolation. By our a priori regularity estimates for discrete harmonic functions on orthodiagonal maps (see Lemma 3.1.5), the exact details of how we choose to extend $h^\bullet$ to $\widehat{G}$ don't matter.

For any $\delta > 0$, define $\varphi_\delta(z) := \delta^{-2}\varphi(\delta^{-1}z)$. Then φ_δ is a smooth mollifier supported on the ball $B(0,\delta) \subseteq \mathbb{R}^2$. Fix $\delta > 0$, $z \in \widehat{G}$ so that $\varepsilon \lesssim \delta \leqslant \frac{d}{2}$ where $d = d_z = \mathrm{dist}(z, \partial\widehat{G})$. Then:

$$\Delta(\varphi_\delta * h^\bullet)(z) = O(\varepsilon^{\frac{1}{2}}\cdot\delta^{-\frac{5}{2}}\cdot(\|D^2\varphi\|+\|D^3\varphi\|)\cdot\|h^\bullet\|) + O(\varepsilon^{\frac{\beta}{1+\beta}}\cdot\delta^{-2}\cdot d^{-\frac{\beta}{1+\beta}}\cdot(\|D^2\varphi\|+\|D^3\varphi\|)\cdot\|h^\bullet\|)$$

In particular, if $\delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} d^{\frac{2\beta}{1+\beta}}$, we have that:

$$\Delta(\varphi_\delta * h^\bullet)(z) = O(\varepsilon^{\frac{1}{2}}\cdot\delta^{-\frac{5}{2}}\cdot(\|D^2\varphi\|+\|D^3\varphi\|)\cdot\|h^\bullet\|)$$

Otherwise:

$$\Delta(\varphi_\delta * h^\bullet)(z) = O(\varepsilon^{\frac{\beta}{1+\beta}}\cdot\delta^{-2}\cdot d^{-\frac{\beta}{1+\beta}}\cdot(\|D^2\varphi\|+\|D^3\varphi\|)\cdot\|h^\bullet\|)$$

Proof. We compute:

$$\begin{aligned}\Delta(\varphi_\delta * h^\bullet)(z) &= \int_{\mathbb{R}^2}\Delta_z\varphi_\delta(z-w)h^\bullet(w)dA(w) = \int_{\mathbb{R}^2}\Delta_w\varphi_\delta(z-w)h^\bullet(w)dA(w)\\ &= \int_{B(z,\delta)}\Delta_w\varphi_\delta(z-w)h^\bullet(w)dA(w)\end{aligned}$$

Let $\mathcal{S}$ be a family of pairwise disjoint squares of side length ℓ that cover $B(z,\delta)$ up to a region of area $O(\delta\ell)$. Here ℓ is a real parameter such that $\varepsilon \ll \ell \ll \delta$, whose exact value will be specified later. Since each square has area ℓ^2, $\mathcal{S}$ consists of $O((\delta/\ell)^2)$ squares of side length ℓ. Then:

$$\begin{aligned}\Delta(\varphi_\delta * h^\bullet)(z) &= \int_{B(z,\delta)} \Delta_w \varphi_\delta(z-w)h^\bullet(w)dA(w) \\ &= \underbrace{\sum_{S\in\mathcal{S}} \Big(\int_S \Delta_w \varphi_\delta(z-w)h^\bullet(w)dA(w)\Big)}_{(1)} - \underbrace{\int_{(\cup_{S\in\mathcal{S}} S)\setminus B(z,\delta)} \Delta_w \varphi_\delta(z-w)h^\bullet(w)dA(w)}_{(2)}\end{aligned}$$

where:

$$|(2)| = O(\delta \cdot \ell \cdot \|D^2\varphi_\delta\| \cdot \|h^\bullet\|) = O(\ell \cdot \delta^{-3} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|)$$

To handle term (1), fix $S \in \mathcal{S}$ and select a point $w_S \in S \cap V^\bullet$. Then we write:

$$\int_S \Delta_w \varphi_\delta(z-w)h^\bullet(w)dA(w) = \underbrace{h^\bullet(w_S)\int_S \Delta_w \varphi_\delta(z-w)dA(w)}_{(a)} + \underbrace{\int_S \Delta_w \varphi_\delta(z-w)\big(h^\bullet(w) - h^\bullet(w_S)\big)dA(w)}_{(b)} \tag{3.4.3}$$

Since the square S has area ℓ^2 and Lemma 3.1.5 tells us that $|h^\bullet(w) - h^\bullet(w_S)| = O(\|h^\bullet\|\ell^\beta d^{-\beta})$ for any $w \in S$, it follows that:

$$(b) = \int_S \Delta_w \varphi_\delta(z-w)\big(h^\bullet(w)-h^\bullet(w_S)\big)dA(w) = O(\ell^2 \cdot \|D^2\varphi_\delta\| \cdot \|h^\bullet\|(\ell/\delta)^\beta) = O(\ell^{2+\beta} \cdot \delta^{-4} \cdot d^{-\beta}\|D^2\varphi\| \cdot \|h^\bullet\|)$$

As far as handling term (a) in Equation 3.4.3:

$$\begin{aligned}(a) =& h^\bullet(w_S)\int_S \Delta_w \varphi_\delta(z-w)dA(w) \\ =& h^\bullet(w_S)\Big(\Big(\sum_{w\in V^\bullet\cap S} \Delta^\bullet_w \varphi_\delta(z-w)\Big) + O(\varepsilon \cdot \ell \cdot \|D^2\varphi_\delta\|) + O(\ell^3 \cdot \|D^3\varphi_\delta\|)\Big) \\ =& \Big(\sum_{w\in V^\bullet\cap S} \Delta^\bullet_w \varphi_\delta(z-w)h^\bullet(w)\Big) + \Big(\sum_{w\in V^\bullet\cap S} \Delta^\bullet_w \varphi_\delta(z-w)\big(h^\bullet(w_S - h^\bullet(w))\big)\Big) \\ &+ O(\varepsilon \cdot \ell \cdot \delta^{-4} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|) + O(\ell^3 \cdot \delta^{-5} \cdot \|D^3\varphi\| \cdot \|h^\bullet\|) \\ =& \Big(\sum_{w\in V^\bullet\cap S} \Delta^\bullet_w \varphi_\delta(z-w)h^\bullet(w)\Big) + O(\ell^{2+\beta} \cdot \delta^{-4}d^{-\beta} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|) \\ &+ O(\varepsilon \cdot \ell \cdot \delta^{-4} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|) + O(\ell^3 \cdot \delta^{-5} \cdot \|D^3\varphi\| \cdot \|h^\bullet\|)\end{aligned}$$

Briefly,

- the second equality follows by Proposition 3.4.1.
- the fourth equality follows by the following crude estimate:

$$\Big|\sum_{w\in V^\bullet\cap S} \Delta^\bullet_w \varphi_\delta(z-w)\big(h^\bullet(w_S - h^\bullet(w))\big)\Big| \leqslant \sum_{w\in V^\bullet\cap S} |\Delta^\bullet_w \varphi_\delta(z-w)| \cdot \|h^\bullet\| \cdot \ell^\beta \cdot d^{-\beta}$$

where:

$$\begin{aligned}
|\Delta^\bullet_w \varphi_\delta(z-w)| &= \Big|\sum_{\substack{u\in V^\bullet\\ u\sim w}} c(u,w)(\varphi_\delta(z-u)-\varphi_\delta(z-w))\Big| \\
&= \Big|\sum_{\substack{u\in V^\bullet\\ u\sim w}} c(u,w)\big(-\nabla\varphi_\delta(z-w)(w-u)+O(\|D^2\varphi_\delta\|\cdot|u-w|^2)\big)\Big| \\
&= \Big|\sum_{\substack{u\in V^\bullet\\ u\sim w}} c(u,w)\nabla\varphi_\delta(z-w)(w-u)\Big| + O\Big(\|D^2\varphi_\delta\|\sum_{\substack{u\in V^\bullet\\ u\sim w}}\mathrm{Area}(Q_{u,v})\Big) \\
&= O\Big(\delta^{-4}\cdot\|D^2\varphi\|\sum_{\substack{u\in V^\bullet\\ u\sim w}}\mathrm{Area}(Q_{u,v})\Big)
\end{aligned}$$

The first term on the third line above vanishes, since any linear function is discrete harmonic. Summing over $w\in V^\bullet\cap S$, every quadrilateral face Q of G that lies within ε of S is counted at most twice, where $\varepsilon \ll \ell$. Hence:

$$\sum_{w\in V^\bullet\cap S}|\Delta^\bullet_w\varphi_\delta(z-w)| = O(\ell^2\cdot\delta^{-4}\cdot\|D^2\varphi\|)$$

Summing over $S\in\mathcal{S}$, since $\mathcal{S}$ consists of $O(\delta^2\cdot\ell^{-2})$ squares, we have that:

$$\begin{aligned}
(1) = \sum_{S\in\mathcal{S}}\Big(\sum_{w\in V^\bullet\cap S}\Delta^\bullet_w\varphi_\delta(z-w)h^\bullet(w)\Big) &+ O(\varepsilon\cdot\ell^{-1}\cdot\delta^{-2}\cdot\|D^2\varphi\|\cdot\|h^\bullet\|) \\
+ O(\ell^\beta\cdot\delta^{-2}d^{-\beta}\cdot\|D^2\varphi\|\cdot\|h^\bullet\|) &+ O(\ell\cdot\delta^{-3}\cdot\|D^3\varphi\|\cdot\|h^\bullet\|)
\end{aligned}$$

Since the squares in $\mathcal{S}$ cover $B(z,\delta)$ and $\phi_\delta(z-w)$ is supported on $B(z,\delta)$ (as a function of w), we have that:

$$\sum_{S\in\mathcal{S}}\Big(\sum_{w\in V^\bullet\cap S}\Delta^\bullet_w\varphi_\delta(z-w)h^\bullet(w)\Big) = \sum_{w\in\mathrm{Int}(V^\bullet)}\Delta^\bullet_w\varphi_\delta(z-w)h^\bullet(w) = \sum_{w\in\mathrm{Int}(V^\bullet)}\varphi_\delta(z-w)\underbrace{\Delta^\bullet h^\bullet(w)}_{=0} = 0$$

Putting all this together, we get that:

$$\Delta(\varphi_\delta * h^\bullet)(z) = O(\varepsilon\cdot\ell^{-1}\cdot\delta^{-2}\cdot\|D^2\varphi\|\cdot\|h^\bullet\|)+O(\ell^\beta\cdot\delta^{-2}d^{-\beta}\cdot\|D^2\varphi\|\cdot\|h^\bullet\|)+O(\ell\cdot\delta^{-3}(\|D^3\varphi\|+\|D^2\varphi\|)\|h^\bullet\|) \tag{3.4.4}$$

In the estimate above, d,δ and ε are given to us, whereas ℓ is just some parameter satisfying $\varepsilon\ll\ell\ll\delta$. To complete our proof, the last thing we need to do is optimize in ℓ.

First observe that taking $\ell\asymp\delta$ or $\ell\asymp\varepsilon$ doesn't give us an effective estimate. Hence, the optimal choice of ℓ must lie on some intermediate scale. In particular, the asymptotically optimal choice of ℓ corresponds to a critical point of the function $f(\ell)=\ell^{-1}\varepsilon+\ell^\beta d^{-\beta}+\ell\delta^{-1}$.

$$f'(\ell) = \delta^{-1}+\beta\ell^{\beta-1}d^{-\beta}-\ell^{-2}\varepsilon = 0 \iff \delta\varepsilon = \ell^2+\beta\ell^{1+\beta}\delta d^{-\beta}$$

When this equality holds, we either have:

$$\ell^2 \asymp \delta\varepsilon \iff \ell \asymp \sqrt{\delta\varepsilon}$$

or:

$$\ell^{1+\beta}\delta d^{-\beta} \asymp \delta\varepsilon \iff \ell \asymp d^{\frac{\beta}{1+\beta}}\varepsilon^{\frac{1}{1+\beta}}$$

which of these asymptotics holds at the critical point depends on which of the relevant quantities- $\sqrt{\delta\varepsilon}$ or $d^{\frac{\beta}{1+\beta}}\varepsilon^{\frac{1}{1+\beta}}$- is smaller. When $\sqrt{\delta\varepsilon} \leqslant d^{\beta/(1+\beta)}\varepsilon^{1/(1+\beta)} \iff \delta \leqslant d^{\frac{2\beta}{1+\beta}}\varepsilon^{\frac{1-\beta}{1+\beta}}$, we have that $\ell \asymp \sqrt{\delta\varepsilon}$ near the critical point, giving us the estimate:

$$\begin{aligned}\Delta(\varphi_\delta * h^\bullet)(z) &= O(\varepsilon^{\frac{1}{2}} \cdot \delta^{-\frac{5}{2}} \cdot (\|D^2\varphi\| + \|D^2\varphi\|) \cdot \|h^\bullet\|) + O(\varepsilon^{\frac{\beta}{2}} \cdot \delta^{(\frac{\beta}{2}-2)} \cdot d^{-\beta} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|) \\ &= O(\varepsilon^{\frac{1}{2}} \cdot \delta^{-\frac{5}{2}} \cdot (\|D^2\varphi\| + \|D^3\varphi\|) \cdot \|h^\bullet\|)\end{aligned}$$

The second equality follows from the fact that when $\delta \leqslant d^{\frac{2\beta}{1+\beta}}\varepsilon^{\frac{1-\beta}{1+\beta}}$, we have that $\varepsilon^{\frac{1}{2}}\delta^{-\frac{5}{2}} \geqslant \varepsilon^{\frac{\beta}{2}}\delta^{(\frac{\beta}{2}-2)}d^{-\beta}$. When $\delta \geqslant d^{\frac{2\beta}{1+\beta}}\varepsilon^{\frac{1-\beta}{1+\beta}}$, we have that $\ell \asymp d^{\frac{\beta}{1+\beta}}\varepsilon^{\frac{1}{1+\beta}}$ at the critical point, giving us the estimate:

$$\begin{aligned}\Delta(\varphi_\delta * h^\bullet)(z) &= O(\varepsilon^{\frac{\beta}{1+\beta}} \cdot \delta^{-2} \cdot d^{-\frac{\beta}{1+\beta}} \cdot \|D^2\varphi\| \cdot \|h^\bullet\|) + O(\varepsilon^{\frac{1}{1+\beta}} \cdot \delta^{-3} \cdot d^{\frac{\beta}{1+\beta}}(\|D^2\varphi\| + \|D^3\varphi\|) \cdot \|h^\bullet\|) \\ &= O(\varepsilon^{\frac{\beta}{1+\beta}} \cdot \delta^{-2} \cdot d^{-\frac{\beta}{1+\beta}} \cdot (\|D^2\varphi\| + \|D^3\varphi\|) \cdot \|h^\bullet\|)\end{aligned}$$

Again, the second equality follows from the fact that when $\delta \geqslant d^{\frac{2\beta}{1+\beta}}\varepsilon^{\frac{1-\beta}{1+\beta}}$, we have that $\varepsilon^{\frac{\beta}{1+\beta}}\delta^{-2}d^{-\frac{\beta}{1+\beta}} \geqslant \varepsilon^{\frac{1}{1+\beta}}\delta^{-3}d^{\frac{\beta}{1+\beta}}$. This completes our proof. □

3.5 Proof of Theorem 3.1.1

Suppose $\Omega \subset \mathbb{R}^2$ is a simply connected domain and $g : \mathbb{R}^2 \to \mathbb{R}$ is α- Hölder for some $\alpha \in (0,1)$. Let h be the solution to the continuous Dirichlet problem on Ω with boundary data given by g. That is:

$$\begin{aligned}\Delta h(x) &= 0 \qquad \text{for all } x \in \Omega \\ h(x) &= g(x) \quad \text{for all } x \in \partial\Omega\end{aligned}$$

Fix $z_0 \in \Omega$. This will be the point at which we compare the solutions to the continuous and discrete Dirichlet problems. Let $G = (V^\bullet \sqcup V^\circ, E)$ be an orthodiagonal map with edges of length at most ε so that $z_0 \in \widehat{G} \subseteq \Omega$ and $\mathrm{dist}(x, \partial\Omega) \leqslant \varepsilon$ for all $x \in \partial V^\bullet$. Let $h^\bullet : V^\bullet \to \mathbb{R}$ be the solution to the discrete Dirichlet problem on $G^\bullet$ with boundary data given by g. That is,

$$\begin{aligned}\Delta^\bullet h^\bullet(x) &= 0 \qquad \text{for all } x \in \mathrm{Int}(V^\bullet) \\ h^\bullet(x) &= g(x) \quad \text{for all } x \in \partial V^\bullet\end{aligned}$$

Extend $h^\bullet$ affinely to a continuous function on $\widehat{G}$. For instance, we can triangulate each interior face of $G^\bullet$ and extend $h^\bullet$ to each triangle by linear interpolation. Obviously this extension is not unique, however, since discrete harmonic functions on orthodiagonal maps are Hölder in the bulk (Lemma 3.1.5) and near the boundary (see our argument in Case 1 below), the choice of affine extension won't affect our rate of convergence. For any $z \in \Omega$, let $d_z = \operatorname{dist}(z, \partial\Omega)$. As we discussed in Section 0.3, we will now estimate the difference, $|h(z_0) - h^\bullet(z_0)|$, in two different ways. One approach will give us a superior estimate when z_0 is close to $\partial\Omega$. The other will give a superior estimate when z_0 is far away from $\partial\Omega$.

3.5.1 Case 1: z_0 is close to $\partial\Omega$

Let w be a point of $\partial\Omega$ so that $d = d_{z_0} = |z_0 - w|$. By considering the point of intersection between $\partial\widehat{G}$ and the line segment from z_0 to w, it follows that we can find a point $w' \in \partial V^\bullet$ so that $|z_0 - w'| \leqslant d + \varepsilon$ and $|w - w'| \leqslant 2\varepsilon$. By the triangle inequality:

$$|h(z_0) - h^\bullet(z_0)| \leqslant |h(z_0) - g(w)| + |g(w) - g(w')| + |g(w') - h^\bullet(z_0)|$$

Since g is α-Hölder:

$$|g(w) - g(w')| \leqslant 2^\alpha \|g\|_\alpha \varepsilon^\alpha$$

To estimate $|h(z_0) - g(w)|$, we write this quantity as an expectation. Using the layer-cake representation of this expectation along with the strong Beurling estimate (Proposition 3.1.3), we have that:

$$\begin{aligned}
|h(z_0) - g(w)| = |\mathbb{E}^{z_0} g(B_{T_{\partial\Omega}}) - g(w)| \leqslant \mathbb{E}^{z_0}|g(B_{T_{\partial\Omega}}) - g(w)| &= \int_0^\infty \mathbb{P}^{z_0}(|g(B_{T_{\partial\Omega}}) - g(w)| \geqslant \lambda) d\lambda \\
&\leqslant \int_0^\infty \mathbb{P}^{z_0}(\|g\|_\alpha |B_{T_{\partial\Omega}} - w|^\alpha \geqslant \lambda) d\lambda = \alpha\|g\|_\alpha \int_0^{\operatorname{diam}(\Omega)} u^{\alpha-1}\mathbb{P}^{z_0}(|B_{T_{\partial\Omega}} - w| \geqslant u) du \\
&\leqslant \alpha\|g\|_\alpha \int_0^d u^\alpha du + \alpha\|g\|_\alpha \int_d^{\operatorname{diam}(\Omega)} u^{\alpha-1} C_1 \left(\frac{d}{u}\right)^{1/2} du \\
&= \|g\|_\alpha d^\alpha + C_1 \alpha \|g\|_\alpha d^{1/2} \int_d^{\operatorname{diam}(\Omega)} u^{\alpha-3/2} du
\end{aligned}$$

where $C_1 > 0$ is an absolute constant, $(B_t)_{t \geqslant 0}$ is a standard 2D Brownian motion, and $T_{\partial\Omega}$ is the hitting time of $\partial\Omega$ by this Brownian motion. Observe that:

$$\int_d^{\operatorname{diam}(\Omega)} u^{\alpha-3/2} du \leqslant \begin{cases} \frac{2d^{\alpha-1/2}}{(1-2\alpha)} & \text{if } \alpha \in (0, 1/2) \\ \log\left(\frac{\operatorname{diam}(\Omega)}{d}\right) & \text{if } \alpha = 1/2 \\ \frac{2\operatorname{diam}(\Omega)^{\alpha-1/2}}{(2\alpha-1)} & \text{if } \alpha \in (1/2, 1] \end{cases}$$

Hence:

$$|h(z_0) - g(w)| \lesssim \begin{cases} \frac{\alpha}{1-2\alpha}\|g\|_\alpha d^\alpha & \text{if } \alpha \in (0, 1/2) \\ \\ \alpha\|g\|_\alpha d^\alpha \log\left(\frac{\mathrm{diam}(\Omega)}{d}\right) & \text{if } \alpha = 1/2 \\ \\ \frac{\alpha}{2\alpha-1}\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\left(\frac{d}{\mathrm{diam}(\Omega)}\right)^{1/2} & \text{if } \alpha \in (1/2, 1] \end{cases}$$

Using the weak Beurling estimate for simple random walks on orthodiagonal maps (Lemma 3.1.4) in place of the strong Beurling estimate for planar Brownian motion, we can estimate $|g(w') - h^\bullet(z_0)|$ by the same argument:

$$\begin{aligned} |h^\bullet(z_0) - g(w')| &= |\mathbb{E}^{z_0} g(S_{T_{\partial V^\bullet}}) - g(w')| \leqslant \mathbb{E}^{z_0}|g(S_{T_{\partial V^\bullet}}) - g(w')| = \int_0^\infty \mathbb{P}^{z_0}(|g(S_{T_{\partial V^\bullet}}) - g(w')| \geqslant \lambda)d\lambda \\ &\leqslant \int_0^\infty \mathbb{P}^{z_0}(\|g\|_\alpha |S_{T_{\partial V^\bullet}} - w'|^\alpha \geqslant \lambda)d\lambda = \alpha\|g\|_\alpha \int_0^{\mathrm{diam}(\Omega)} u^{\alpha-1}\mathbb{P}^{z_0}(|S_{T_{\partial V^\bullet}} - w'| \geqslant u)du \\ &\leqslant \alpha\|g\|_\alpha \int_0^{2(d\vee\varepsilon)} u^{\alpha-1}du + \alpha\|g\|_\alpha \int_{2(d\vee\varepsilon)}^{\mathrm{diam}(\Omega)} u^{\alpha-1}C_2\left(\frac{d\vee\varepsilon}{u}\right)^\beta du \\ &\lesssim \|g\|_\alpha (d\vee\varepsilon)^\alpha + \alpha\|g\|_\alpha (d\vee\varepsilon)^\beta \int_{2(d\vee\varepsilon)}^{\mathrm{diam}(\Omega)} u^{\alpha-\beta-1}du \end{aligned}$$

where $C_2 > 0$ is an absolute constant, $(S_n)_{n\geqslant 0}$ is a simple random walk on $G^\bullet$, and $T_{\partial V^\bullet}$ is the hitting time of $\partial V^\bullet$ by this random walk. The appearance of "$(d \vee \varepsilon)$" in our estimates comes from the fact that $|z_0 - w'| \leqslant d + \varepsilon \leqslant 2(d\vee\varepsilon)$. Observe that:

$$\int_{2(d\vee\varepsilon)}^{\mathrm{diam}(\Omega)} u^{\alpha-\beta-1}du \lesssim \begin{cases} \frac{(d\vee\varepsilon)^{\alpha-\beta}}{\beta-\alpha} & \text{if } \alpha \in (0,\beta) \\ \\ \log\left(\frac{\mathrm{diam}(\Omega)}{d\vee\varepsilon}\right) & \text{if } \alpha = \beta \\ \\ \frac{\mathrm{diam}(\Omega)^{\alpha-\beta}}{\alpha-\beta} & \text{if } \alpha \in (\beta, 1] \end{cases}$$

Hence:

$$|h^\bullet(z_0) - g(w')| \lesssim \begin{cases} \frac{\alpha}{\beta-\alpha}\|g\|_\alpha (d\vee\varepsilon)^\alpha & \text{if } \alpha \in (0,\beta) \\ \\ \|g\|_\alpha (d\vee\varepsilon)^\alpha \log\left(\frac{\mathrm{diam}(\Omega)}{d\vee\varepsilon}\right) & \text{if } \alpha = \beta \\ \\ \frac{\alpha}{\alpha-\beta}\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\left(\frac{d\vee\varepsilon}{\mathrm{diam}(\Omega)}\right)^\beta & \text{if } \alpha \in (\beta, 1] \end{cases}$$

Since we used a strictly weaker version of the Beurling estimate, our estimate for $|h^\bullet(z_0) - h(w')|$ is necessarily worse than our estimate for $|h(z_0) - g(w)|$. Hence, putting all this together, we have that:

$$|h(z_0) - h^\bullet(z_0)| \lesssim \begin{cases} \frac{\beta}{\beta-\alpha}\|g\|_\alpha (d \vee \varepsilon)^\alpha & \text{if } \alpha \in (0,\beta) \\ \\ \|g\|_\alpha (d \vee \varepsilon)^\alpha \log\left(\frac{\text{diam}(\Omega)}{d\vee\varepsilon}\right) & \text{if } \alpha = \beta \\ \\ \frac{\alpha}{\alpha-\beta}\|g\|_\alpha \text{diam}(\Omega)^\alpha \left(\frac{d\vee\varepsilon}{\text{diam}(\Omega)}\right)^\beta & \text{if } \alpha \in (\beta, 1] \end{cases}$$

3.5.2 Case 2: z_0 is far away from $\partial\Omega$

Let $\delta > 0$ be some mesoscopic scale, whose exact value we will fix later. If ϕ is a radially symmetric smooth mollifier supported on the unit ball $B(0,1) \subseteq \mathbb{R}^2$, then $\phi_\delta(x) = \delta^{-2}\phi(\delta^{-1}x)$ is a radially symmetric smooth mollifer, supported on $B(0,\delta)$. Extend $h^\bullet : V^\bullet \to \mathbb{R}$ to a function on $\widehat{G}$ in any sort of sensible way. For instance, we could triangulate the faces of $G^\bullet$ and define $h^\bullet$ on each triangle by linear interpolation. In this way, we can think of $h^\bullet$ as a function on $\widehat{G}$. This allows us to consider the convolution $\phi_\delta * h^\bullet$. Notice that this is only well-defined for points of $\widehat{G}$ that are at least δ far away from $\partial\widehat{G}$. With this in mind, let Ω^δ be a simply connected domain so that:

- $z_0 \in \Omega^\delta$
- $\Omega^\delta \subseteq \widehat{G} \subseteq \Omega$
- every point of $\partial\Omega^\delta$ is at least 2δ far away from $\partial\widehat{G}$.
- every point of $\partial\Omega^\delta$ lies within $O(\delta)$ of $\partial\Omega$ and therefore $\partial\widehat{G}$.

In this way, $\phi_\delta * h^\bullet$ is well-defined as a function on Ω^δ. Let $\widetilde{h}$ be the solution to the continuous Dirichlet problem on Ω^δ with boundary data given by $\phi_\delta * h^\bullet$. That is:

$$\begin{aligned} \Delta\widetilde{h}(z) &= 0 && \text{for all } z \in \Omega^\delta \\ \widetilde{h}(z) &= (\phi_\delta * h^\bullet)(z) && \text{for all } z \in \partial\Omega^\delta \end{aligned}$$

By the triangle inequality:

$$|h^\bullet(z_0) - h(z_0)| \leqslant |h^\bullet(z_0) - (\phi_\delta * h^\bullet)(z_0)| + |(\phi_\delta * h^\bullet)(z_0) - \widetilde{h}(z_0)| + |\widetilde{h}(z) - h(z)| \tag{3.5.1}$$

By Lemma 3.1.5:

$$|h^\bullet(z_0) - (\phi_\delta * h^\bullet)(z_0)| \lesssim \|g\| \left(\frac{\delta}{d}\right)^\beta$$

To handle the second term in Equation 3.5.1, observe that $(\phi_\delta * h^\bullet)$ is a smooth function on Ω^δ that extends continuously to $\partial\Omega^\delta$ and $\widetilde{h}$ is the harmonic function on Ω^δ that agrees with $\phi_\delta * h^\bullet$ on $\partial\Omega^\delta$. Hence:

$$|(\phi_\delta * h^\bullet)(z_0) - \widetilde{h}(z_0)| = \Big|\int_{\Omega_\delta} (\Delta(\phi_\delta * h^\bullet))(w) G_{\Omega^\delta}(w, z_0) dA(w)\Big| \tag{3.5.2}$$

Proposition 3.4.2 tells us that the convolution of a discrete harmonic function with a smooth mollifier is almost harmonic. Namely, we have that:

$$|(\Delta(\phi_\delta * h^\bullet))(w)| \lesssim \|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} \tag{3.5.3}$$

if $\delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} d_w^{\frac{2\beta}{1+\beta}}$. Otherwise:

$$|(\Delta(\phi_\delta * h^\bullet))(w)| \lesssim \|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2} d_w^{-\frac{\beta}{1+\beta}} \tag{3.5.4}$$

where $d_w = \mathrm{dist}(w, \partial\widehat{G}) \asymp \mathrm{dist}(w, \partial\Omega)$. The fact that $\mathrm{dist}(w, \partial\widehat{G})$ and $\mathrm{dist}(w, \partial\Omega)$ are comparable and therefore interchangeable here, follows from the fact that δ is mesoscopic and we are only considering points $w \in \Omega^\delta$. Plugging our Laplacian estimate in Equation 3.5.3 into Equation 3.5.4, we have that:

$$|(\phi_\delta * h^\bullet)(z_0) - \widetilde{h}(z_0)| \lesssim \|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} \int_{\Omega^\delta} G_{\Omega^\delta}(w, z_0) dA(w) \tag{3.5.5}$$

if $\delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} \mathrm{diam}(\Omega)^{\frac{2\beta}{1+\beta}}$. Otherwise:

$$\begin{aligned} |(\phi_\delta * h^\bullet)(z_0) - \widetilde{h}(z_0)| &\lesssim \|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2} \int_{\Omega^\delta} d_w^{-\frac{\beta}{1+\beta}} G_{\Omega^\delta}(w, z_0) dA(w) && (3.5.6)\\ &\lesssim \|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2-\frac{\beta}{1+\beta}} \int_{\Omega^\delta} G_{\Omega^\delta}(w, z_0) dA(w) && (3.5.7) \end{aligned}$$

Observe that the integral appearing on the RHS of both of the inequalities above can be interpreted probabilistically as the expected amount of time spent a planar Brownian motion started at z_0 spends in Ω^δ, before hitting $\partial\Omega^\delta$. That is:

$$\int_{\Omega^\delta} G_{\Omega^\delta}(w, z_0) dA(w) = \mathbb{E}^{z_0} T_{\partial\Omega^\delta} \leqslant \mathbb{E}^{z_0} T_{\partial\Omega}$$

where $T_{\partial\Omega}$ and $T_{\partial\Omega^\delta}$ are the hitting times of $\partial\Omega$ and $\partial\Omega^\delta$ by our planar Brownian motion. Let $(B_t)_{t\geqslant 0}$ be a planar Brownian motion. Then the process $\left(|B_{t\wedge T_{\partial\Omega}} - z_0|^2 - 2t \wedge T_{\partial\Omega}\right)_{t\geqslant 0}$ is a martingale. By the optional stopping theorem:

$$\mathbb{E}^{z_0}|B_{T_{\partial\Omega}} - z_0|^2 = 2\mathbb{E}^{z_0} T_{\partial\Omega}$$

Using the layer-cake representation of the expectation on the LHS along with the strong Beurling estimate (Proposition 3.1.3), we have that:

$$\begin{aligned} \mathbb{E}^{z_0}|B_{T_{\partial\Omega}} - z_0|^2 &= \int_0^\infty \mathbb{P}^{z_0}(|B_{T_{\partial\Omega}} - z_0|^2 \geqslant \lambda) d\lambda = 2\int_0^\infty u\,\mathbb{P}^{z_0}(|B_{T_{\partial\Omega}} - z_0| \geqslant u) du \\ &= 2\int_0^d u\,du + 2\int_d^{\mathrm{diam}(\Omega)} C\left(\frac{d}{u}\right)^{1/2} u\,du \lesssim d^{1/2}\mathrm{diam}(\Omega)^{3/2} \end{aligned}$$

where $C > 0$ is an absolute constant. Plugging this estimate for $\mathbb{E}^{z_0}|B_{T_{\partial\Omega}} - z_0|^2 = \mathbb{E}^{z_0} T_{\partial\Omega} \geqslant \mathbb{E}^{z_0} T_{\partial\Omega^\delta}$ into Equations 3.5.5 and 3.5.7, we have that:

$$|(\phi_\delta * h^\bullet)(z_0) - \widetilde{h}(z_0)| \lesssim \begin{cases} \|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} d^{1/2} \mathrm{diam}(\Omega)^{3/2} & \text{if } \delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} \mathrm{diam}(\Omega)^{\frac{2\beta}{1+\beta}} \\ \\ \|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2-\frac{\beta}{1+\beta}} d^{1/2} \mathrm{diam}(\Omega)^{3/2} & \text{otherwise} \end{cases}$$

To estimate the third term in Equation 3.5.1, $|h(z_0) - \widetilde{h}(z_0)|$, we use the maximum principle. Suppose $w \in \partial\Omega^\delta$. Then $\widetilde{h}(w) = (\phi_\delta * h^\bullet)(w)$, since the boundary data of $\widetilde{h}$ on $\partial\Omega^\delta$ is given by $\phi_\delta * h^\bullet$. On the other hand, since h is harmonic on Ω, and the smooth mollifier ϕ_δ is radially symmetric, $(\phi_\delta * h)(w) = h(w)$. Hence:

$$|h(w) - \widetilde{h}(w)| = |(\phi_\delta * h)(w) - (\phi_\delta * h^\bullet)(w)| \leqslant \max_{w' \in B(w,\delta)} |h(w') - h^\bullet(w')|$$

Since δ is small and the points $w' \in B(w,\delta)$ are δ-close to the boundary of Ω, by the same argument as in Case 1, we have that:

$$|h(w) - \widetilde{h}(w)| \lesssim \begin{cases} \frac{\beta}{\beta-\alpha} \|g\|_\alpha \delta^\alpha & \text{if } \alpha \in (0,\beta) \\ \\ \|g\|_\alpha \delta^\alpha \log\left(\frac{\mathrm{diam}(\Omega)}{\delta}\right) & \text{if } \alpha = \beta \\ \\ \frac{\alpha}{\alpha-\beta} \|g\|_\alpha \mathrm{diam}(\Omega)^\alpha \left(\frac{\delta}{\mathrm{diam}(\Omega)}\right)^\beta & \text{if } \alpha \in (\beta, 1] \end{cases}$$

Putting all this together, if $\alpha \in (0,\beta)$, we have that:

$$|h(z_0) - h^\bullet(z_0)| = \begin{cases} O(\|g\|_\infty \delta^\beta d^{-\beta}) + O(\|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} d^{1/2} \mathrm{diam}(\Omega)^{3/2}) & \\ \qquad + O(\frac{\beta}{\beta-\alpha} \|g\|_\alpha \delta^\alpha) & \text{if } \delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} \mathrm{diam}(\Omega)^{\frac{2\beta}{1+\beta}} \\ \\ O(\|g\|_\infty \delta^\beta d^{-\beta}) + O(\|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2-\frac{\beta}{1+\beta}} d^{1/2} \mathrm{diam}(\Omega)^{3/2}) & \\ \qquad + O(\frac{\beta}{\beta-\alpha} \|g\|_\alpha \delta^\alpha) & \text{otherwise} \end{cases}$$

If $\alpha = \beta$:

$$|h(z_0) - h^\bullet(z_0)| = \begin{cases} O(\|g\|_\infty \delta^\beta d^{-\beta}) + O(\|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} d^{1/2} \mathrm{diam}(\Omega)^{3/2}) & \\ \qquad + O(\|g\|_\alpha \delta^\alpha \log\left(\frac{\mathrm{diam}(\Omega)}{\delta}\right)) & \text{if } \delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} \mathrm{diam}(\Omega)^{\frac{2\beta}{1+\beta}} \\ \\ O(\|g\|_\infty \delta^\beta d^{-\beta}) + O(\|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2-\frac{\beta}{1+\beta}} d^{1/2} \mathrm{diam}(\Omega)^{3/2}) & \\ \qquad + O(\|g\|_\alpha \delta^\alpha \log\left(\frac{\mathrm{diam}(\Omega)}{\delta}\right)) & \text{otherwise} \end{cases}$$

If $\alpha \in (\beta, 1)$:

$$|h(z_0) - h^\bullet(z_0)| = \begin{cases} O(\|g\|_\infty \delta^\beta d^{-\beta}) + \|g\| \varepsilon^{\frac{1}{2}} \delta^{-\frac{5}{2}} d^{1/2} \mathrm{diam}(\Omega)^{3/2} & \\ \qquad + O(\frac{\alpha}{\alpha-\beta} \|g\|_\alpha \mathrm{diam}(\Omega)^\alpha \big(\frac{\delta}{\mathrm{diam}(\Omega)}\big)^\beta) & \text{if } \delta \geqslant \varepsilon^{\frac{1-\beta}{1+\beta}} \mathrm{diam}(\Omega)^{\frac{2\beta}{1+\beta}} \\ & \\ O(\|g\|_\infty \delta^\beta d^{-\beta}) + O(\|g\| \varepsilon^{\frac{\beta}{1+\beta}} \delta^{-2-\frac{\beta}{1+\beta}} d^{1/2} \mathrm{diam}(\Omega)^{3/2}) & \\ \qquad + O(\frac{\alpha}{\alpha-\beta} \|g\|_\alpha \mathrm{diam}(\Omega)^\alpha \big(\frac{\delta}{\mathrm{diam}(\Omega)}\big)^\beta) & \text{otherwise} \end{cases}$$

3.5.3 Choosing an Optimal δ

In Section 3.5.1, we derived an estimate for $|h(z_0) - h^\bullet(z_0)|$ for z_0 close to the boundary. In Section 3.5.2, we derived an estimate for $|h(z_0) - h^\bullet(z_0)|$ for z_0 far away from the boundary. To complete our proof, we need to:

1. Find the optimal choice of δ for our estimate from Section 3.5.2.
2. Combine these estimates to get an estimate that works for all $z_0 \in V^\bullet$.

We will do this in detail for $\alpha \in (0, \beta)$. The corresponding estimates when $\alpha \in [\beta, 1]$ follow by the same argument. Armed with the intuition that our rate of convergence should be polynomial in ε, we take $\delta = \varepsilon^s \mathrm{diam}(\Omega)^{1-s}$, $d = \varepsilon^r \mathrm{diam}(\Omega)^{1-r}$, where $0 < r < s < 1$. By our estimates from Sections 3.5.1 and 3.5.2 we have that:

$$|h(z_0) - h^\bullet(z_0)| = \begin{cases} \min\{O(\frac{\beta}{\beta-\alpha} \|g\|_\alpha \varepsilon^{r\alpha} \mathrm{diam}(\Omega)^{\alpha - r\alpha}), O(\|g\| \varepsilon^{\beta(s-r)} \mathrm{diam}(\Omega)^{-\beta(s-r)}) & \\ + O(\|g\| \varepsilon^{\frac{1}{2} - \frac{5s}{2} + \frac{r}{2}}) \mathrm{diam}(\Omega)^{-\frac{1}{2} + \frac{5s}{2} - \frac{r}{2}} + O(\frac{\beta}{\beta-\alpha}) \|g\|_\alpha \varepsilon^{\alpha s} \mathrm{diam}(\Omega)^{\alpha - \alpha s}\} & \text{if } s \leqslant \frac{1-\beta}{1+\beta} \\ & \\ \min\{O(\frac{\beta}{\beta-\alpha} \|g\|_\alpha \varepsilon^{r\alpha} \mathrm{diam}(\Omega)^{\alpha - r\alpha}), O(\|g\| \varepsilon^{\beta(s-r)} \mathrm{diam}(\Omega)^{-\beta(s-r)}) & \\ + O(\|g\| \varepsilon^{\frac{\beta}{1+\beta}(1-s) - 2s + \frac{r}{2}}) \mathrm{diam}(\Omega)^{-\frac{\beta}{1+\beta}(1-s) + 2s - \frac{r}{2}} + O(\frac{\beta}{\beta-\alpha}) \|g\|_\alpha \varepsilon^{\alpha s} \mathrm{diam}(\Omega)^{\alpha - \alpha s}\} & \text{otherwise} \end{cases}$$

for $\alpha \in (0, \beta)$. We want to find the fixed choice of δ that minimizes our error for a point which is distance d from the boundary of Ω. This amounts to finding a value of s that minimizes our error for a fixed choice of r, in each of the cases above. In other words, we are interested in the maximum of the function:

$$\Xi_1(\alpha, \beta, r, s) = \begin{cases} \max\{r\alpha, \min\{\beta(s-r), \frac{1}{2} - \frac{5s}{2} + \frac{r}{2}, s\alpha\}\} & \text{if } s \leqslant \frac{1-\beta}{1+\beta} \\ & \\ \max\{r\alpha, \min\{\beta(s-r), \frac{\beta}{1+\beta} - (2 + \frac{\beta}{1+\beta})s + \frac{r}{2}, s\alpha\}\} & \text{otherwise} \end{cases}$$

in s, treating α, β and r as constants. From here, we take the minimum of the resulting function, $\max_{s \in (r,1)} \Xi_1(\alpha, \beta, r, s)$, in r, treating α and β as constants. This corresponds to finding an estimate that works for all d. In this way, we conclude that:

$$|h(z_0) - h^{\bullet}(z_0)| \leqslant \left(C_1\|g\| + C_2\frac{\beta}{\beta-\alpha}\|g\|_\alpha \mathrm{diam}(\Omega)^\alpha\right)\left(\frac{\varepsilon}{\mathrm{diam}(\Omega)}\right)^{\lambda_1(\alpha,\beta)}$$

where:

$$\lambda_1(\alpha,\beta) = \min_{r\in[0,1]}\max_{s\in(r,1)} \Xi_1(\alpha,\beta,r,s)$$

Note that $\lambda_1(\alpha,\beta) > 0$ for any $\alpha \in (0,\beta)$ and $\beta \in (0,1/2]$. To see this, observe that if we take $s = \frac{\beta}{4+6\beta}$ and $r \in [\frac{\beta}{8+12\beta}, 1]$, clearly:

$$\Xi_1(\alpha,\beta,r,\frac{\beta}{4+6\beta}) \geqslant \frac{\beta\alpha}{8+12\beta}$$

On the other hand, if $r \in [0, \frac{\beta}{8+12\beta}]$:

$$\Xi_1(\alpha,\beta,r,\frac{\beta}{4+6\beta}) \geqslant \min\{\beta\big(\frac{\beta}{4+6\beta} - r\big), \frac{\beta}{2+2\beta} + \frac{r}{2}, \frac{\beta\alpha}{4+6\beta}\} \geqslant \min\{\frac{\beta^2}{8+12\beta}, \frac{\beta}{2+\beta}, \frac{\alpha\beta}{4+6\beta}\}$$

Hence:

$$\lambda_1(\alpha,\beta) \geqslant \frac{\alpha\beta}{8+12\beta}$$

□

Chapter 4

Lipschitz Regularity on a Mesoscopic Scale for Harmonic Functions on Orthodiagonal Maps

As we alluded to in Section 0.4, in this chapter, we will prove the following Harnack-type estimate for discrete harmonic functions on orthodiagonal maps:

Theorem 4.0.1. If $\beta \in (0, \frac{1}{2})$ is the absolute constant from Lemma 3.1.4, for any $\alpha \in (0, \frac{\beta}{1+3\beta})$, we have an absolute constant $C_\alpha > 0$ so that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and $z, w \in V^\bullet$ satisfy $|z - w| \geqslant d\big(\frac{\varepsilon}{d}\big)^\alpha$, where $d = \mathrm{dist}(z, \partial\widehat{G}) \wedge \mathrm{dist}(w, \partial\widehat{G})$, then:

$$|h(z) - h(w)| \leqslant C_\alpha \|h\|_\infty \Big(\frac{|z - w|}{d}\Big)$$

We say that this estimate holds on a mesoscopic scale because it requires that the points $z, w \in V^\bullet$ we're looking at are at least $d\big(\frac{\varepsilon}{d}\big)^\alpha$ apart, where $\alpha \in (0, 1)$, implying that $\varepsilon \ll d\big(\frac{\varepsilon}{d}\big)^\alpha \ll 1$. One interpretation of this result is that it tells us that discrete harmonic functions on orthodiagonal maps are Lipschitz in the bulk on a mesoscopic scale. We do not believe this result is sharp. Namely, since orthodiagonal maps are good approximations of continuous 2D space, the Harnack estimate should hold even on a microscopic scale. That is, for any pair of points $z, w \in V^\bullet$ that are at least $C\varepsilon$ apart, for some absolute constant $C > 0$. As we remarked in Section 0.4, this is known to be true for any isoradial graph. This includes subsets of the triangular, the hexagonal, and the square grid. Furthermore, in [14], Chelkak, Laslier and Russkikh show that we have a Harnack estimate on microscopic scales for discrete harmonic functions on t-embeddings satisfying the assumptions "*Lip*(κ, δ)" and "*Exp-Fat*(δ)" for some $\kappa \in (0, 1)$, $\delta > 0$ (see Corollary 6.18 of [14]). For a precise definition of the assumptions "*Lip*(κ, δ)" and "*Exp-Fat*(δ)," see Section 1.2

of [14]. As we discussed in Section 3.1.1, for any $\kappa \in (0,1)$, there exists $c = c(\kappa) > 0$ so that any orthodiagonal map of edge length at most ε, satisfies the assumption "$Lip(c\varepsilon, \varepsilon)$." In contrast, it is not known whether an arbitrary orthodiagonal map satisfies the condition "$Exp\text{-}Fat(\delta)$" for some $\delta > 0$ that only depends on the mesh of our orthodiagonal map. Thus, we do not have a Harnack estimate on microscopic scales for discrete harmonic functions on orthodiagonal maps as an immediate consequence of Corollary 6.18 of [14].

4.1 Lipschitz Regularity on a Mesoscopic Scale (Base Case)

The key idea behind the proof of Theorem 4.0.1 is the following regularity estimate for C^2 functions, in terms of their norm and Laplacian:

Proposition 4.1.1. Suppose Ω is a simply connected domain and $h \in C_b^2(\Omega) \cap C(\overline{\Omega})$. Then:

$$|h(x_2) - h(x_1)| \lesssim \|h\|_\infty \Big(\frac{|x_2 - x_1|}{d}\Big) + \|\Delta h\|_\infty d |x_2 - x_1| \tag{4.1.1}$$

for any $x_1, x_2 \in \Omega$, where $d = d_{x_1} \wedge d_{x_2} = \mathrm{dist}(x_1, \partial\Omega) \wedge \mathrm{dist}(x_2, \partial\Omega)$.

Proof. Suppose $B(x,R) \subseteq \Omega$ is a ball contained in Ω. By Green's identity applied to h and $G_{B(x,R)}(y,x) = -\frac{1}{2\pi}\log\left(\frac{|y-x|}{R}\right)$, we have that:

$$h(x) = \frac{1}{2\pi R}\int_{\partial B(x,R)} h(y)d\sigma(y) - \frac{1}{2\pi}\int_{B(x,R)} \Delta h(y)\log\Big(\frac{|y-x|}{R}\Big)dA(y)$$

where "dA(y)" denotes integration with respect to area in Ω. Similarly, applying Green's identity with h and $R^2 - |y-x|^2$, we have that:

$$\frac{1}{2\pi R}\int_{\partial B(x,R)} h(y)d\sigma(y) = \frac{1}{\pi R^2}\int_{B(x,R)} h(y)dA(y) + \frac{1}{4\pi R^2}\int_{B(x,R)} \Delta h(y)\big(R^2 - |y-x|^2\big)dA(y)$$

Putting all this together, we have that:

$$h(x) = \frac{1}{\pi R^2}\int_{B(x,R)} h(y)dA(y) + \frac{1}{4\pi R^2}\int_{B(x,R)} \Delta h(y)\big(R^2 - |y-x|^2\big)dA(y) - \frac{1}{2\pi}\int_{B(x,R)} \Delta h(y)\log\Big(\frac{|y-x|}{R}\Big)dA(y)$$

Suppose x_1, x_2 are points of Ω. Observe that if $|x_2 - x_1| > \frac{d}{2}$, then trivially, $|h(x_2) - h(x_1)| \leqslant 4\|h\|_\infty\big(\frac{|x_2-x_1|}{d}\big)$ and so the desired result follows. With this in mind, suppose that $|x_2 - x_1| \leqslant \frac{d}{2}$. Then:

$$h(x_2) - h(x_1) = (1) + (2) + (3)$$

where:

$$
\begin{aligned}
|(1)| =& \Big|\frac{1}{\pi d^2}\int_{B(x_1,d)\Delta B(x_2,d)} h(y)dA(y)\Big| \leqslant \frac{5\pi|x_2-x_1|d}{\pi d^2}\|h\|_\infty = 5\|h\|_\infty\Big(\frac{|x_2-x_1|}{d}\Big)\\
|(2)| =& \frac{1}{4\pi}\Big|\int_{B(x_1,d)\cap B(x_2,d)} \Delta h(y)\Big(\frac{|x_2-y|^2-|x_1-y|^2}{d^2}\Big)dA(y) + \int_{B(x_2,d)\setminus B(x_1,d)} \Delta h(y)\Big(1-\frac{|x_2-y|^2}{d^2}\Big)dA(y)\\
&- \int_{B(x_1,d)\setminus B(x_2,d)} \Delta h(y)\Big(1-\frac{|x_1-y|^2}{d^2}\Big)dA(y)\Big|\\
\leqslant& \frac{15\|\Delta h\|_\infty d|x_2-x_1|}{8}\\
|(3)| =& \frac{1}{2\pi}\Big|\int_{B(x_1,d)\cap B(x_2,d)} \Delta h(y)\log\Big(\frac{|y-x_2|}{|y-x_1|}\Big)dA(y) + \int_{B(x_2,d)\setminus B(x_1,d)} \Delta h(y)\log\Big(\frac{|x_2-y|}{d}\Big)dA(y)\\
&- \int_{B(x_1,d)\setminus B(x_2,d)} \Delta h(y)\log\Big(\frac{|x_1-y|}{d}\Big)dA(y)\Big|\\
\leqslant& \frac{\|\Delta h\|_\infty}{2\pi}\Big(\int_{B(x_2,d)} \log\Big(1+\frac{|x_2-x_1|}{|y-x_2|}\Big)dA(y) + \int_{B(x_2,d)\setminus B(x_2,d-|x_2-x_1|)} \log\Big(\frac{d}{|x_2-y|}\Big)dA(y)\\
&+ \int_{B(x_1,d)\setminus B(x_1,d-|x_2-x_1|)} \log\Big(\frac{d}{|x_1-y|}\Big)dA(y)\Big)\\
=& \|\Delta h\|_\infty\Big(\int_0^d r\log\Big(1+\frac{|x_1-x_2|}{r}\Big)dr + 2\int_{d-|x_2-x_1|}^d r\log\Big(\frac{d}{r}\Big)dr\Big) \lesssim \|\Delta h\|_\infty d|x_2-x_1|
\end{aligned}
$$

Putting all this together, we have that:

$$|h(x_2)-h(x_1)| \lesssim \|h\|_\infty\Big(\frac{|x_2-x_1|}{d}\Big) + \|\Delta h\|_\infty d|x_2-x_1|$$

□

Recall that Proposition 3.4.2 tells us that if $h^\bullet$ is a discrete harmonic function, its convolution with a smooth mollifier, $\phi * h^\bullet$, is almost harmonic in that $\Delta(\phi * h^\bullet) \approx 0$. Hence, taking $h = (\phi * h^\bullet)$ in our estimate from Proposition 4.1.1, we have that the convolution of a discrete harmonic function with a smooth mollifier, satisfies a Harnack-type estimate. As a consequence, we can recover a Harnack-type estimate for discrete harmonic functions on orthodiagonal maps, on a mesoscopic scale:

Proposition 4.1.2. There exists an absolute constant $C > 0$ so that if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h^\bullet : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and $z, w \in V^\bullet$ are vertices of $V^\bullet$ so that $|z-w| \geqslant d\big(\frac{\varepsilon}{d}\big)^{\frac{\beta^2}{2(1+\beta)}}$ where $d = \mathrm{dist}(z, \partial\widehat{G}) \wedge \mathrm{dist}(w, \partial\widehat{G})$, then:

$$|h^\bullet(z) - h^\bullet(w)| \leqslant C\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)$$

Proof. Let ϕ be a smooth mollifier supported on the unit ball $B(0,1) \subseteq \mathbb{R}^2$. Then for any $\delta > 0$, $\phi_\delta(x) := \delta^{-2}\phi(\delta^{-1}x)$ is a smooth mollifier, supported on $B(0,\delta)$. Suppose z and w are vertices of $V^\bullet$. By the triangle inequality:

$$|h^\bullet(w) - h^\bullet(z)| \leqslant |h^\bullet(w) - (\phi_\delta * h^\bullet)(w)| + |(\phi_\delta * h^\bullet)(w) - (\phi_\delta * h^\bullet)(z)| + |(\phi_\delta * h^\bullet)(z) - h^\bullet(z)|$$

By Lemma 3.1.5:

$$|(\phi_\delta * h^\bullet)(z) - h^\bullet(z)|, |(\phi_\delta * h^\bullet)(w) - h^\bullet(w)| = O(\delta^\beta \cdot d^{-\beta} \cdot \|h^\bullet\|) \tag{4.1.2}$$

On the other hand, taking $h = (\phi_\delta * h^\bullet)$ in our estimate from Proposition 4.1.1 and using our estimate for the Laplacian of $(\phi_\delta * h^\bullet)$ from Proposition 3.4.2, we have that:

$$\begin{aligned} |(\phi_\delta * h^\bullet)(w) - (\phi_\delta * h^\bullet)(z)| =& O(\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)) + O(\|\Delta(\varphi_\delta * h^\bullet)\| \cdot d \cdot |z-w|) \\ =& O\big(\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) + O\big(\varepsilon^{\frac{\beta}{1+\beta}}\delta^{-2}d^{2-\frac{\beta}{1+\beta}}\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) \\ &+ O\big(\varepsilon^{\frac{1}{2}}\delta^{-\frac{5}{2}}d^2\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) \end{aligned} \tag{4.1.3}$$

Note that having chosen a particular smooth mollifer ϕ, we can disregard the $\|D^2\phi\|$ and $\|D^3\phi\|$ terms in Proposition 3.4.2, since they are just some constants. Looking at the error term in Equation 4.1.2, to get the kind of estimate for $h^\bullet$ we are looking for, we need it to be the case that $\delta^\beta d^{-\beta} \leqslant \big(\frac{|z-w|}{d}\big) \iff \delta \leqslant d\big(\frac{|z-w|}{d}\big)^{1/\beta}$. On the other hand, looking at the estimate for the modulus of continuity of $(\varphi_\delta * h^\bullet)$ in Equation 4.1.3, all of the powers of δ are negative. Hence, to get the best estimate possible, we should take δ to be as large as possible. Namely, $\delta = d\big(\frac{|z-w|}{d}\big)^{1/\beta}$. Plugging this choice of δ into Equations 4.1.2 and 4.1.3 and putting all this together, we get that:

$$\begin{aligned} |h^\bullet(w) - h^\bullet(z)| = O\big(\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) + O\big(\varepsilon^{\frac{\beta}{1+\beta}} \cdot |z-w|^{-\frac{2}{\beta}} \cdot d^{\frac{2}{\beta}-\frac{\beta}{1+\beta}} \cdot \|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) \\ + O\big(\varepsilon^{\frac{1}{2}} \cdot |z-w|^{-\frac{5}{2\beta}} \cdot d^{\frac{5}{2\beta}-\frac{1}{2}} \cdot \|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)\big) \end{aligned}$$

To get an effective estimate, we need it to be the case that:

$$\begin{aligned} \varepsilon^{\frac{\beta}{1+\beta}} \cdot |z-w|^{-\frac{2}{\beta}} \cdot d^{\frac{2}{\beta}-\frac{\beta}{1+\beta}} \leqslant 1 \qquad &\iff \qquad |z-w| \geqslant d\Big(\frac{\varepsilon}{d}\Big)^{\frac{\beta^2}{2(1+\beta)}} \\ \varepsilon^{\frac{1}{2}} \cdot |z-w|^{-\frac{5}{2\beta}} \cdot d^{\frac{5}{2\beta}-\frac{1}{2}} \leqslant 1 \qquad &\iff \qquad |z-w| \geqslant d\Big(\frac{\varepsilon}{d}\Big)^{\frac{\beta}{5}} \end{aligned}$$

Thus, as long as $|z-w| \geqslant d\big(\frac{\varepsilon}{d}\big)^{\frac{\beta}{5} \wedge \frac{\beta^2}{2(1+\beta)}}$, we have that:

$$|h^\bullet(z) - h^\bullet(w)| \leqslant C\|h^\bullet\|\Big(\frac{|z-w|}{d}\Big)$$

where $C > 0$ is some absolute constant. Since the absolute constant $\beta > 0$ from Lemma 3.1.5 is small, $\frac{\beta^2}{2(1+\beta)} < \frac{\beta}{5}$. $\square$

Notice that Proposition 4.1.2 is a weaker version of Theorem 4.0.1 that requires a larger mesoscopic scale for our Harnack-type estimate to kick in. In Section 4.2, we will see how, using Proposition 4.1.2 as a starting point, we can use a bootstrap argument to successively improve the scale on which our Harnack estimate holds, giving us Theorem 4.0.1.

4.2 Refining our Mesoscopic Scale

In this section, we will refine our estimate in Proposition 4.1.2 by improving the mesoscopic scale on which our Harnack-type estimate holds. To do this, we first observe that discrete harmonic functions on orthodiagonal maps are β- Hölder in the bulk on small scales and Lipschitz in the bulk on large scales. This gives us improved Hölder regularity in the bulk, on intermediate scales:

Proposition 4.2.1. (refined Hölder regularity on intermediate scales) Suppose that for any orthodiagonal map G with edges of length at most ε and any function $h : V^\bullet \to \mathbb{R}$ that is harmonic on $\mathrm{Int}(V^\bullet)$ we have that:

- (Lipschitz regularity on large scales) if $z, w \in V^\bullet$ satisfy $|z - w| \geqslant d\left(\frac{\varepsilon}{d}\right)^\alpha$ for some fixed $\alpha \in (0, 1)$, then:
$$|h(z) - h(w)| \leqslant C\|h\|\Big(\frac{|z-w|}{d}\Big)$$
where $C > 0$ is an absolute constant.

Then:

- (Improved Hölder regularity on intermediate scales) for any $\gamma \in (\alpha, 1)$, if G is an orthodiagonal map with edges of length at most ε, $h : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and $z, w \in V^\bullet$ satisfy $|z - w| = d\left(\frac{\varepsilon}{d}\right)^\gamma$ where $d = \mathrm{dist}(z, \partial\widehat{G}) \wedge \mathrm{dist}(w, \partial\widehat{G})$, then:
$$|h(z) - h(w)| \leqslant C\|h\|\Big(\frac{|z-w|}{d}\Big)^{\beta + \frac{\alpha}{\gamma}(1-\beta)}$$

Proof. Suppose G is an orthodiagonal map with edges of length at most ε and $z, w \in V^\bullet$ satisfy $|z-w| = d\left(\frac{|z-w|}{d}\right)^\gamma$ for some $\gamma \in (\alpha, 1)$. Suppose $h : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and WLOG, $h(w) \leqslant h(z)$. Let $B = B_G(w, d\left(\frac{|z-w|}{d}\right)^\alpha)$ denote the discrete ball of radius $d\left(\frac{|z-w|}{d}\right)^\alpha$ centered at w in G. By the maximum principle, we can find a nearest-neighbor path $\gamma = (w_0, w_1, ..., w_m)$ of vertices in $G^\bullet$ so that $w_0 = w$, $w_m \in \partial V_B^\bullet$ and $h(w_{i+1}) \leqslant h(w_i)$ for all i. In particular, it follows that $h(w_i) \leqslant h(w)$ for all i. If $(S_n)_{n\geqslant 0}$ is a simple random walk on $G^\bullet$, and $\tau_\gamma and \tau_{\partial V_B^\bullet}$ are the hitting times of γ and $\partial V_B^\bullet$ by our random walk, by the optional stopping theorem, we have that:

$$h(z) - h(w) = \mathbb{E}^z\big(h(S_{\tau_\gamma \wedge \tau_{\partial V_B^\bullet}}) - h(w)\big) = \mathbb{E}^z\Big(\underbrace{(h(S_{\tau_\gamma}) - h(w))}_{\leqslant 0} 1_{\tau_\gamma \leqslant \tau_{\partial V_B^\bullet}}\Big) + \mathbb{E}^z\big((h(S_{\tau_{\partial B}}) - h(w))1_{\tau_{\partial V_B^\bullet} < \tau_\gamma}\big)$$

$$\leqslant \max_{u \in \partial B} |h(u) - h(w)| \cdot \mathbb{P}(\tau_{\partial V_B^\bullet} < \tau_\gamma) \lesssim \|h\| \Big(\frac{d\big(\frac{\varepsilon}{d}\big)^\alpha}{d}\Big)\Big(\frac{d\big(\frac{\varepsilon}{d}\big)^\gamma}{d\big(\frac{\varepsilon}{d}\big)^\alpha}\Big)^\beta$$

$$\lesssim \|h\|\Big(\frac{\varepsilon}{d}\Big)^{\alpha+\beta(\gamma-\alpha)} = \|h\|\Big(\frac{d\big(\frac{\varepsilon}{d}\big)^\gamma}{d}\Big)^{\beta + \frac{\alpha}{\gamma}(1-\beta)} = \|h\|\Big(\frac{|z-w|}{d}\Big)^{\beta+\frac{\alpha}{\gamma}(1-\beta)}$$

□

From here. the story is as follows:

1. Observe that in our estimate for $\Delta(\phi_\delta * h^\bullet)(z)$ in Proposition 3.4.2, the $\frac{\beta}{1+\beta}$ exponents in the second term come from the fact that harmonic functions are β- Hölder in the bulk on scales $\asymp \ell$ where $\varepsilon \ll \ell \ll \delta \leqslant \frac{d}{2}$.

2. Hence, if we use Proposition 4.2.1 in place of Lemma 3.1.5, we can improve our estimate for $\Delta(\phi_\delta * h^\bullet)(z)$.

3. However, the scale on which we have that discrete harmonic functions on orthodiagonal maps are Lipschitz in the bulk in Proposition 4.1.2, comes from:

   (a) the fact that discrete harmonic functions on orthodiagonal maps are β-Hölder in the bulk, which is used on the intermediate scale δ.

   (b) our estimate for $\Delta(\phi_\delta * h^\bullet)(z)$ in Proposition 3.4.2.

4. Thus, our improved Hölder regularity on intermediate scales in Proposition 4.2.1 can be used to improve the scale at which we can ensure discrete harmonic functions are Lipschitz in the bulk in Proposition 4.1.2.

5. But then we could use the fact that harmonic functions are Lipschitz on smaller scales to improve our estimate for the Hölder regularity of discrete harmonic functions on intermediate scales!

In short, we have a bootstrap argument for refining the scale at which we know that discrete harmonic functions are Lipschitz in the bulk. This is encapsulated in the following result:

Proposition 4.2.2. (the bootstrap) Suppose we know that for some $\alpha \in (0,1)$, there exists an absolute constant $C > 0$ so that for any orthodiagonal map $G = (V^\bullet \sqcup V^\circ, E)$ with edges of length at most ε, any function $h : V^\bullet \to \mathbb{R}$ that is harmonic on $\mathrm{Int}(V^\bullet)$, and any $z, w \in V^\bullet$ satisfying $|z-w| \geqslant d\big(\frac{\varepsilon}{d}\big)^\alpha$, where $d = \mathrm{dist}(z, \partial\widehat{G}) \wedge \mathrm{dist}(w, \partial\widehat{G})$, we have that:

$$|h(z) - h(w)| \leqslant C\|h\|\Big(\frac{|z-w|}{d}\Big)$$

Then taking $\alpha' = (1-\beta)\alpha + \beta\min\{\frac{1}{5}, \frac{\beta+\alpha-\beta\alpha}{2(1+\beta)}\}$, there exists an absolute constant $C' > 0$ so that for any orthodiagonal map $G = (V^\bullet \sqcup V^\circ, E)$ with edges of length at most ε, any function $h : V^\bullet \to \mathbb{R}$ that is harmonic on $\mathrm{Int}(V^\bullet)$ and any $z, w \in V^\bullet$ satisfying $|z-w| \geqslant d\left(\frac{\varepsilon}{d}\right)^{\alpha'}$, where $d = \mathrm{dist}(z, \partial\widehat{G}) \wedge \mathrm{dist}(w, \partial\widehat{G})$, we have that:

$$|h(z) - h(w)| \leqslant C'\|h\|\left(\frac{|z-w|}{d}\right)$$

Proof. Let ϕ be a smooth mollifier supported on the unit ball $B(0,1) \subseteq \mathbb{R}^2$. Then for any $\delta > 0$, $\phi_\delta(x) = \delta^{-2}\phi(\delta^{-1}x)$ is a smooth mollifier, supported on $B(0,\delta)$. Just as in the proof of Proposition 4.1.1, we will convolve our discrete harmonic function h with a smooth mollifier ϕ_δ, where δ is mesoscopic. With this in mind, we write $\delta = d\left(\frac{\varepsilon}{d}\right)^c$ and $\ell = d\left(\frac{\varepsilon}{d}\right)^\gamma$, where $0 < c < \gamma < 1$. Here, ℓ is some intermediate scale. Using Proposition 4.2.1 in place of Lemma 3.1.5, we have that:

$$|(\phi_\delta * h)(z) - h(z)|, |(\phi_\delta * h)(w) - h(w)| = O(\|h\|\left(\frac{\varepsilon}{d}\right)^{\beta c + \alpha(1-\beta)})$$

if $c > \alpha$. If $c \leqslant \alpha$, we have that:

$$|(\phi_\delta * h)(z) - h(z)|, |(\phi_\delta * h)(w) - h(w)| = O(\|h\|\left(\frac{\varepsilon}{d}\right)^{c})$$

In particular, notice that if we repeat our argument in Proposition 4.1.2, picking a value of c that is less than or equal to α will gives us, at best, Lipschitz regularity on scale $d\left(\frac{\varepsilon}{d}\right)^c \geqslant d\left(\frac{\varepsilon}{d}\right)^\alpha$. In short, we end up with an estimate that is inferior to the one we started with. Thus, if we want to improve on our initial estimate, we need to choose c and γ so that $0 < \alpha < c < \gamma < 1$.

Repeating our argument in Proposition 3.4.2 using Proposition 4.2.1 in place of Lemma 3.1.5, we have the following analogue of Equation 3.4.4:

$$\Delta(\phi_\delta * h)(z) = O(\|h\|\left(\frac{\varepsilon}{d}\right)^{1-\gamma-2c} d^{-2}) + O(\|h\|\left(\frac{\varepsilon}{d}\right)^{\beta\gamma + \alpha(1-\beta) - 2c} d^{-2}) + O(\|h\|\left(\frac{\varepsilon}{d}\right)^{\gamma - 3c} d^{-2})$$

Optimizing in $\ell = d\left(\frac{\varepsilon}{d}\right)^\gamma$ for fixed $\delta = d\left(\frac{\varepsilon}{d}\right)^c$, we get that:

$$\Delta(\phi_\delta * h)(z) = O(\|h\|\left(\frac{\varepsilon}{d}\right)^{\frac{1-5c}{2}} d^{-2}) + O(\|h\|\left(\frac{\varepsilon}{d}\right)^{\frac{\beta+\alpha-\alpha\beta}{(1+\beta)} - 2c} d^{-2}) \tag{4.2.1}$$

Applying the estimate in Proposition 4.1.1 to $\phi_\delta * h$ with this Laplacian estimate, we have that:

$$|(\phi_\delta * h)(z) - (\phi_\delta * h)(w)| = O\big(\|h\|\left(\frac{|z-w|}{d}\right)\big) + O\big(\|h\|\left(\frac{|z-w|}{d}\right)\left(\frac{\varepsilon}{d}\right)^{(\frac{1-5c}{2}) \wedge (\frac{\beta+\alpha-\alpha\beta}{(1+\beta)} - 2c)}\big)$$

Combining this with our estimate for the difference between h and $\phi_\delta * h$ from earlier, we have that:

$$|h(z)-h(w)| = O\Big(\|h\|\Big(\frac{|z-w|}{d}\Big)\Big)+O\Big(\|h\|\Big(\frac{|z-w|}{d}\Big)\Big(\frac{\varepsilon}{d}\Big)^{(\frac{1-5c}{2})\wedge(\frac{\beta+\alpha-\alpha\beta}{(1+\beta)}-2c)}\Big)+O\Big(\|h\|\Big(\frac{\varepsilon}{d}\Big)^{\beta c+\alpha(1-\beta)}\Big)$$

To find the scale on which we can ensure Lipschitz regularity of h, we want to pick c as large as possible subject to the constraits:

$$1-5c \geqslant 0, \qquad\qquad \frac{\beta+\alpha(1-\beta)}{1+\beta}-2c \geqslant 0$$

Thus, taking $c = \min\{\frac{1}{5}, \frac{\beta+\alpha-\alpha\beta}{2(1+\beta)}\}$, our estimate for $|h(z)-h(w)|$ above tells us that there exists an absolute constant $C' > 0$ such that:

$$|h(z)-h(w)| \leqslant C'\Big(\frac{|z-w|}{d}\Big)$$

for any $z, w \in V^\bullet$ such that $|z-w| \geqslant d\big(\frac{\varepsilon}{d}\big)^{\beta\alpha'+(1-\beta)\alpha}$, where $\alpha' = \min\{\frac{1}{5}, \frac{\beta+\alpha-\alpha\beta}{2(1+\beta)}\}$. □

Theorem 4.0.1 now follows as a straightforward corollary of Proposition 4.1.2, which serves as our base case, and Proposition 4.2.2, which tells us how we can successively refine our mesoscopic scale:

Proof. (of Theorem 4.0.1) Consider the sequence $(\alpha_n)_{n\geqslant 0}$ such that $\alpha_0 = \frac{\beta^2}{2(1+\beta)}$, and $\alpha_{n+1} = (1-\beta)\alpha_n + \beta\cdot\min\{\frac{1}{5}, \frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)}\}$ for all $n \geqslant 0$. Combining our results in Proposition 4.1.2 and Proposition 4.2.2, we have that for all $n \in \mathbb{N}_0$, if $G = (V^\bullet \sqcup V^\circ, E)$ is an orthodiagonal map with edges of length at most ε, $h : V^\bullet \to \mathbb{R}$ is harmonic on $\mathrm{Int}(V^\bullet)$ and $z, w \in V^\bullet$ satisfy $|z-w| \geqslant d\big(\frac{\varepsilon}{d}\big)^{\alpha_n}$, where $d = \mathrm{dist}(z, \partial G) \wedge \mathrm{dist}(w, \partial G)$, then:

$$|h(z)-h(w)| \leqslant C_n\|h\|\Big(\frac{|z-w|}{d}\Big)$$

where $C_n > 0$ is some absolute constant. Hence, to prove the desired result, it suffices to show that $\lim_{n\to\infty} \alpha_n = \frac{\beta}{1+3\beta}$.

Observe from our recursion that α_{n+1} is the weighted average of α_n and the minimum of $\frac{1}{5}$ and $\frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)}$. Hence, if $\alpha_0 < \frac{1}{5}$, it follows that $\alpha_n < \frac{1}{5}$ for all $n \in \mathbb{N}_0$. Additionally, observe that:

$$\frac{1}{5} < \frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)} \iff \alpha_n > \frac{2-3\beta}{5(1-\beta)}$$

Since the function $f(\beta) = \frac{2-3\beta}{5(1-\beta)}$ is strictly decreasing on $(0,1)$, $\beta \in (0, 1/2]$ and $f(1/2) = 1/5$, we see that the minimum of $\frac{1}{5}$ and $\frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)}$ being equal to $\frac{1}{5}$ requires that $\alpha_n \geqslant \frac{1}{5}$. Thus, for $\alpha_0 = \frac{\beta^2}{2(1+\beta)} < \frac{1}{5}$, our recurrence simplifies to:

$$\alpha_{n+1} = (1-\beta)\alpha_n + \beta\Big(\frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)}\Big)$$

Again, since α_{n+1} is a weighted average of α_n and $\frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)}$,

$$\alpha_{n+1} \geqslant \alpha_n \iff \frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)} \geqslant \alpha_n \iff \alpha_n \leqslant \frac{\beta}{1+3\beta}$$

On the other hand, notice that if $\alpha_n < \frac{\beta}{1+3\beta}$ then $\frac{\beta+\alpha_n-\beta\alpha_n}{2(1+\beta)} = \frac{\beta+(1-\beta)\alpha_n}{2(1+\beta)} < \frac{\beta+(1-\beta)\frac{\beta}{1+3\beta}}{2(1+\beta)} = \frac{\beta}{1+3\beta}$ which tells us that $\alpha_{n+1} < \frac{\beta}{1+3\beta}$, since α_{n+1} is the weighted average of this quantity and α_n. Since $\beta \in (0, 1/2)$, $\alpha_0 = \frac{\beta^2}{2(1+\beta)} < \frac{\beta}{1+3\beta}$. Thus, in our case, the sequence $(\alpha_n)_{n\geqslant 0}$ is strictly increasing and bounded above by $\frac{\beta}{1+3\beta}$, and so converges to a limit λ as n tends to ∞. This limit satisfies:

$$\lambda = (1-\beta)\lambda + \beta\Big(\frac{\beta+\lambda-\beta\lambda}{2(1+\beta)}\Big) \iff \lambda = \frac{\beta}{1+3\beta}$$

This completes our proof. $\square$

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