COADJOINT ORBITS OF SYMPLECTOMORPHISM GROUPS OF SURFACES

BY

ILIA KIRILLOV

A thesis submitted in conformity with
the requirements for the degree of
Doctor of Philosophy

Graduate Department of Mathematics
University of Toronto

ABSTRACT

Coadjoint orbits of symplectomorphism groups of surfaces

Ilia Kirillov

Doctor of Philosophy

Graduate Department of Mathematics

University of Toronto

2024

In this thesis, we classify generic coadjoint orbits for the action of symplectic (equivalently, area-preserving) diffeomorphisms of compact symplectic surfaces with or without boundary. This completes V. Arnold's program of studying Casimir invariants of incompressible fluids in 2D. To obtain this classification, we first solve an auxiliary problem, which is of interest by itself: classify generic Morse functions on surfaces with respect to the action of area-preserving diffeomorphisms. As a technical tool, we prove an analog of Morse-Darboux lemma in the case of a singular point on the boundary. We also generalize all the results above to the case of non-orientable surfaces without boundary. The new results in this thesis are based on the following papers [23, 24, 18].

To anyone,
who did something nice.

ACKNOWLEDGEMENTS

I would like to thank...

CONTENTS

1

INTRODUCTION

1.1 OVERVIEW

The classification problem for coadjoint orbits for the action of symplectic (or area-preserving) diffeomorphisms in two dimensions was known to specialists in view of its application in fluid dynamics since the 1960s, and it was explicitly formulated in [2, see Section I.5] in 1998. The same classification problem also arises in Poisson geometry since coadjoint orbits are symplectic leaves of the Lie-Poisson bracket, and also in representation theory in connection with the orbit method of A.Kirillov [22]. The classification of generic coadjoint orbits was obtained in [17, 19] for the case of closed surfaces. In this thesis we generalize the results of [17, 19] to the case of surfaces with boundary, and to the case of closed non-orientable surfaces.

Remark 1.1.1. Note that the classification of coadjoint orbits for diffeomorphisms of the circle was solved by A. Kirillov in [21].

The classification problem for coadjoint orbits for symplectomorphisms of a surface is closely related to the classification of functions up to symplectomorphism. Hence, we are going to address the following two problems:

1. Classify generic smooth functions on symplectic surfaces up to symplectomorphisms.

2. Classify generic coadjoint orbits of symplectomorphism groups of surfaces.

Let us outline the structure of the thesis. In Chapter 2 we give a local classification of functions on symplectic surfaces (see Theorem 2.4.1). Let $F\colon M \to \mathbb{R}$ be a smooth function, and let O be a critical point for the restriction $F|_{\partial M}$. We describe a local normal form for a pair (F, ω) near a point O, where $F\colon M \to \mathbb{R}$ is Morse function and ω is a symplectic form. Next, in Chapter 3 we give a global classification of functions on symplectic surfaces. Roughy speaking, the classification theorem for functions, Theorem 3.4.1, states that there is a one-to-one correspondence between functions up to a symplectomorphism and *measured Reeb graphs* up to an isomorphism. In Chapter 4 we obtain a classification of generic coadjoint orbits for the case of orientable surfaces with boundary. In Chapter 5 we obtain a classification of generic function and orbits for the case of closed non-orientable surfaces. The classification results for coadjoint orbits (Theorems 4.5.1 and 5.3.12) are also given in terms of measured Reeb graphs supplemented with some additional data.

It is worth noting that the classification of functions in [19] is based on the classification of so-called simple Morse fibrations obtained in [13]; while, the proofs in the present thesis use a different method so it gives an alternative proof for Theorem 3.11 from [19] (classification of functions in the case of closed surfaces).

In the recent works [29, 12] the orbit method was applied to the symplectomorphism group of the two-sphere.

Finally, we note that all objects in the present thesis are infinitely smooth.

1.2 MOTIVATION: THE HAMILTONIAN FRAMEWORK OF THE EULER EQUATION

The main motivation for classification of functions and coadjoint orbits is related to description of the first integrals for the Euler equation of ideal hydrodynamics.

Consider a symplectic surface (M, ω) with boundary ∂M. We denote by $\mathrm{SDiff}(M)$ the Lie group of all symplectomorphisms of M, and by $\mathfrak{svect}(M)$ the corresponding Lie algebra of divergence-free vector fields on M. A linear functional I on $\mathfrak{svect}(M)$ is called *regular* if there exists a smooth 1-form ξ_I such that the value of I on a vector field v is the pairing between ξ_I and v :

$$I(v) = \int_M \xi_I(v)\omega.$$

The space $\mathfrak{svect}^*(M)$ of regular functionals on $\mathfrak{svect}(M)$ is a dense subset in the space of all continuous linear functionals on $\mathfrak{svect}(M)$ with respect to the C^2-topology. It turns out that the space of regular functionals $\mathfrak{svect}^*(M)$ can be identified with the space of cosets $\Omega^1(M)/d\Omega^0(M)$, since exact 1-forms give zero functionals on divergence-free vector fields. Moreover, the natural action of the group $\mathrm{SDiff}(M)$ on the space of cosets $\Omega^1(M)/d\Omega^0(M)$ by means of pull-backs coincides with the coadjoint action of the group of symplectomorphisms $\mathrm{SDiff}(M)$. The proof of this fact can be found in [2] (see Section I.8). More information about infinite-dimensional Lie groups can be found in [20].

Now let us fix a Riemannian metric $(\cdot, \cdot)$ on the surface M such that the corresponding area form coincides with the symplectic form ω. The motion of an inviscid incompressible fluid on M is described by the Euler equation

$$\partial_t v + \nabla_v v = -\nabla p \tag{1.1}$$

describing an evolution of a divergence-free velocity field v of a fluid flow in M, where $\mathrm{d}iv\, v = 0$ and the field v is tangent to the boundary ∂M. The pressure function p entering the Euler equation is defined uniquely modulo an additive constant by this equation along with the divergence-free constraint on the velocity v.

The metric $(\cdot, \cdot)$ allows us to identify the Lie algebra and its dual by means of the so-called inertia operator: given a vector field v on M one defines the 1-form $\alpha = v^\flat$ as the pointwise inner product with vectors of the velocity field v: $v^\flat(W) := (v, W)$ for all $W \in T_x M$. The Euler equation (1.1) rewritten on 1-forms is

$$\partial_t \alpha + L_v \alpha = -dP$$

for the 1-form $\alpha = v^\flat$ and an appropriate function P on M. In terms of the cosets of 1-forms $[\alpha] = \{\alpha + df \mid f \in C^\infty(M)\} \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$, the Euler equation looks as follows:

$$\partial_t[\alpha] + L_v[\alpha] = 0 \tag{1.2}$$

on the dual space $\mathfrak{g}^*$, where L_v is the Lie derivative along the field v.

The Euler equation (1.2) shows that the coset of 1-forms $[\alpha]$ evolves by an area-preserving change of coordinates, i.e. it remains in the same coadjoint orbit in $\mathfrak{g}^*$. This is why invariants of coadjoint orbits of cosets $[\alpha]$ describe first integrals, called Casimirs, of the Euler equation, and their complete classification is important in many areas of ideal fluid dynamics.

Notice that the Euler equation is well-defined on any Rimannian manifold, in particular, the manifold can be non-orientable. Hydrodynamics on non-orientable surfaces, while being seemingly hypothetical, can be observed in nature: for instance, soap films on a wire can have the shape of a Möbius band, the simplest non-orientable surface with boundary. One can see that the soapy fluid keeps moving, i.e. retains dynamics, inside this 2D minimal surface. Dynamics of vortices on non-orientable surfaces was considered e.g. in [3, 4, 33]. In chapter 5 we present the Hamiltonian setting for non-orientable hydrodynamics, emphasizing its similarity with and difference from the orientable framework.

1.3 EXPOSITION OF THE MAIN RESULTS

In this section we present the main results of this thesis. Each subsection corresponds to the result from one of the chapters.

1.3.1 *Local classification*

In this subsection we present the results from Chapter 2. Let M be a compact connected surface with boundary ∂M. Consider a pair (F, ω), where $F\colon M \to \mathbb{R}$ is a Morse function, and ω is an area (symplectic) form. Fix a point $O \in M$. If $O \in M \setminus \partial M$ is a regular point for F, then there exist a chart (p, q) such that locally near O we have $F = p$ and $\omega = dp \wedge dq$. On the other hand, if O is non-degenerate point, then the local normal form for a pair (F, ω) is provided by, so called, Morse-Darboux lemma.

Theorem 1.3.1. *(=Theorem 2.2.1) Let (M, ω) be a symplectic surface, and $F\colon M \to \mathbb{R}$ be a Morse function. Let $O \in M \setminus \partial M$ be a critical point for the function F. Then there exists a coordinate chart (p, q) centered at O such that $\omega = dp \wedge dq$ and $F = \lambda \circ S$ where $S = pq$ or $S = p^2 + q^2$. Here λ is a smooth function of one variable defined in some neighborhood of the origin $0 \in \mathbb{R}$ and $\lambda'(0) \neq 0$. Moreover:*

(i) In the case $S = p^2 + q^2$, and the function λ is uniquely determined by the pair (F, ω).

(ii) In the case $S = pq$, only the Taylor series of the function λ is uniquely determined by the pair (F, ω). In other words, if $(\tilde{p}, \tilde{q})$ is another chart as above then $\tilde{p}\tilde{q} = pq + \psi(pq)$ (for sufficiently small $p, q, \tilde{p}, \tilde{q}$), where ψ is a function of one variable flat at the origin. Furthermore, every function of one variable that is flat at the origin can be obtained in this way.

The first result of this thesis is an analog of Morse-Darboux lemma for the case when $O \in \partial M$ is regular point for F, but critical point for the restriction $F|_{\partial M}$.

Theorem 1.3.2. *(=Theorem 2.3.1) Let (M, ω) be a symplectic surface, and $F\colon M \to \mathbb{R}$ be a Morse function. Let $O \in \partial M$ be a regular point of F and a non-degenerate critical point of its restriction $F|_{\partial M}$. Then there exists a coordinate chart (p, q) centered at O such that $\omega = dp \wedge dq$ and $F = \lambda \circ S$ where $S = q + p^2$ or $S = q - p^2$. Here λ is a smooth function of one variable defined in some neighborhood of the origin $0 \in \mathbb{R}$ and $\lambda'(0) \neq 0$. In this chart M is defined by $q \geq 0$ and the boundary ∂M is given by the equation $\{q = 0\}$, see Figure 2.1. Moreover:*

(i) In the case $S = q + p^2$, the function λ is uniquely determined by the pair (F, ω).

(ii) In the case $S = q - p^2$, only the Taylor series of the function λ is uniquely determined by the pair (F, ω). In other words, if $(\tilde{p}, \tilde{q})$ is another chart as above then $\tilde{q} - \tilde{p}^2 = q - p^2 + \psi(q - p^2)$. Furthermore, every function of one variable that is flat at the origin can be obtained in this way.

1.3.2 *Global classification*

In this subsection we present the results from Chapter 3. Let M be a compact connected surface with boundary ∂M. Consider a pair (F, ω), where $F \colon M \to \mathbb{R}$ is a simple Morse function, and ω is an area (symplectic) form. Our goal is to classify generic (simple Morse) smooth functions on the surface M up to symplectomorphisms. This problem was solved in [19] for the case of a closed surface M. It turns out that invariants of this action on functions are given by the Reeb graphs (see Figure 1.1) of functions equipped with a measure on the graph. The corresponding measures on Reeb graphs are not arbitrary but satisfy certain constraints in terms of asymptotic expansions at all three-valent vertices of the graph.

In this thesis we generalize the notion of a measured Reeb graph to the case of surfaces with boundary (see Figure 1.2). Then we formulate and prove the classification theorem for simple Morse functions. The statement of this theorem (given below) is almost identical to the case of closed surfaces but almost all notions there required some modifications.

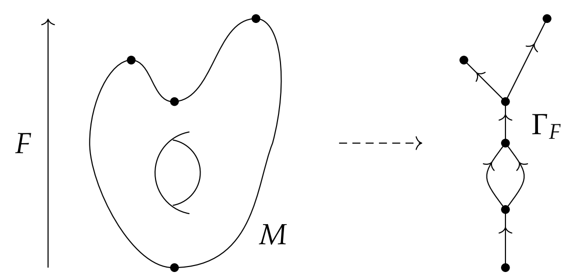


Figure 1.1: Reeb graph for a height function with two maxima on a torus.

Theorem 1.3.3. *(=Theorem 3.4.1) Let M be a compact connected oriented surface with boundary ∂M. Then there is a one-to-one correspondence between simple Morse functions on M, considered up to symplectomorphism, and (isomorphism classes of) measured Reeb graphs compatible with M. In other words, the following statements hold.*

i) Let $F, G \colon M \to \mathbb{R}$ be two simple Morse functions. Then the following conditions are equivalent:

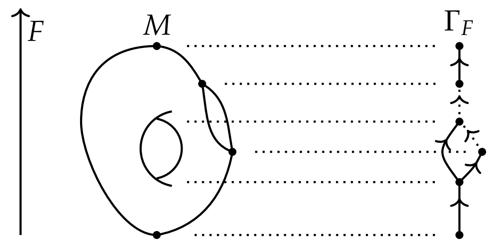


Figure 1.2: A torus with one hole with the height function on it and the corresponding Reeb graph

a) *There exists a symplectomorphism* $\Phi\colon M \to M$ *such that* $\Phi^* F = G$.

b) *Measured Reeb graphs of F and G are isomorphic.*

Moreover, every isomorphism $\phi\colon (\Gamma_F, f, \mu_F) \to (\Gamma_G, g, \mu_G)$ *can be lifted to a symplectomorphism* $\Phi\colon M \to M$ *such that* $\Phi^* F = G$.

ii) *For each measured Reeb graph* (Γ, f, μ) *compatible with* (M, ω) *there exists a simple Morse function* $F : M \to \mathbb{R}$ *such that the corresponding measured Reeb graph* Γ_F *is isomorphic to* (Γ, f, μ).

1.3.3 *Coadjoint orbits*

In this subsection we present the results from Chapter 5. Consider a symplectic surface (M, ω) with boundary ∂M. We denote by SDiff(M) the Lie group of all symplectomorphisms of M, and by $\mathfrak{svect}(M)$ the corresponding Lie algebra of divergence-free vector fields on M. As we mentioned in the Section 1.2, the natural action of the group SDiff(M) on the space of cosets $\Omega^1(M)/d\Omega^0(M)$ by means of pull-backs coincides with the coadjoint action of the group of symplectomorphisms SDiff(M). Define the exterior derivative operator d on the space of cosets $\{\alpha + \mathrm{d}f | f \in C^\infty(M)\}$ by the formula $\mathrm{d}[\alpha] := \mathrm{d}\alpha$. Consider the following mapping:

$$\mathrm{curl}\colon \Omega^1(M)/\mathrm{d}\Omega^0(M) \to C^\infty(M),$$

defined by taking a *vorticity function* $\mathrm{d}\alpha/\omega =: \mathrm{curl}[\alpha]$.

Suppose that cosets $[\alpha]$ and $[\beta]$ belong to the same coadjoint orbit of SDiff(M). Then by definition, there is a symplectomorphism Φ such that $[\Phi^*\beta] = [\alpha]$. With every generic coset $[\alpha] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$ one can associate a measured Reeb graph $\Gamma_{\mathrm{curl}[\alpha]}$. If two simple Morse cosets $[\alpha]$ and $[\beta]$ belong to the same coadjoint orbit, then the corresponding Reeb graphs are isomorphic.

Suppose that cosets $[\alpha]$ and $[\beta]$ have isomorphic Reeb graphs. Then it follows from the classification result for functions (Theorem 3.4.1) that there exists a symplectomorphism Φ such that $\Phi^*\mathrm{curl}[\beta] = \mathrm{curl}[\alpha]$. Therefore, the 1-form $\Phi^*[\beta] - [\alpha]$ is closed. Since this 1-form is not necessarily exact, the cosets $[\alpha]$ and $[\beta]$ do not necessarily belong to the same coadjoint orbit. Nevertheless, we conclude that the space of coadjoint orbits corresponding to the same measured Reeb graph is finite-dimensional and its dimension is at most $\dim \mathrm{H}^1(M)$. The classification result for coadjoint orbits is given in terms of augmented circulation graphs (measured Reeb graphs supplemented with some additional data).

Theorem 1.3.4. *(=Theorem 4.5.1) Let (M, ω) be a connected symplectic surface with or without boundary. Then generic coadjoint orbits of $\mathrm{SDiff}(M)$ are in one-to-one correspondence with (isomorphism classes of) augmented circulation graphs $(\Gamma, f, \mu, \mathcal{C})$ compatible with M. In other words, the following statements hold:*

i) For a symplectic surface M and generic cosets $[\alpha], [\beta] \in \mathfrak{svect}^*(M)$ the following conditions are equivalent:

   a) $[\alpha]$ and $[\beta]$ lie in the same orbit of the $\mathrm{SDiff}(M)$ coadjoint action;

   b) augmented circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to the cosets $[\alpha]$ and $[\beta]$ are isomorphic.

ii) For each augmented circulation graph Γ which is compatible with M, there exists a generic $[\alpha] \in \mathfrak{svect}^*(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, f, \mu, \mathcal{C}, \xi)$.

Corollary 1.3.5. *(=Corollary 4.5.2) The space of coadjoint orbits of the group $\mathrm{SDiff}(M)$ corresponding to the same measured Reeb graph (Γ, f, μ) is a finite-dimensional affine space and its dimension is $\dim \mathrm{H}_1(\Gamma, \Gamma^d) + \dim \mathrm{H}_1(\Gamma^d)$.*

1.3.4 *Non-orientable case*

In this subsection we present the results from Chapter 4. Non-orientable manifolds do not carry non-vanishing volume forms but allow densities (also called pseudo-forms). Densities can be thought of as non-vanishing top-degree forms whose sign changes after returning to the same point along an orientation-reversing loop. Fix a density ρ on a non-orientable manifold N and consider the infinite-dimensional group $\mathrm{Diff}_\rho(N)$ of measure-preserving diffeomorphisms of N. In the present paper we study the group $\mathrm{Diff}_\rho(N)$ in the case when N is a closed non-orientable surface. Our first main result is a classification of generic pseudo-functions on such surfaces with respect to the action of $\mathrm{Diff}_\rho(N)$. The second result is a classification of generic coadjoint orbits of $\mathrm{Diff}_\rho(N)$.

A natural way to describe objects on a non-orientable manifold N is to lift them to the double cover $\widetilde{N}$, which is an oriented manifold. This

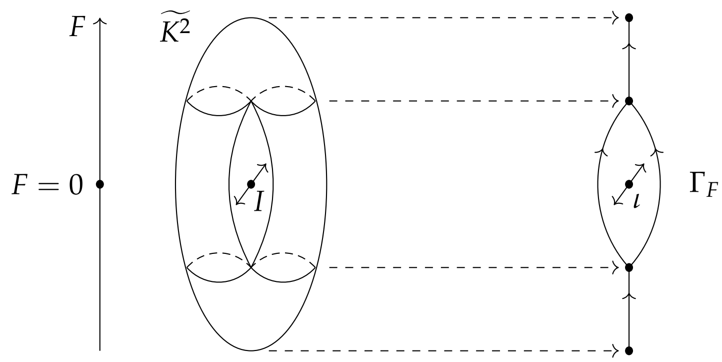


Figure 1.3: Reeb graph for a height function on a torus inducing a pseudo-function on the Klein bottle K^2. The involution I on the torus and ι on the graph are central symmetries.

double cover comes with a fixed-point-free orientation-reversing involution $I : \widetilde{N} \to \widetilde{N}$ interchanging the points in each fiber of the natural projection $\widetilde{N} \to N$. Pseudo-functions F on a non-orientable manifold N are functions on its double cover $\widetilde{N}$ anti-invariant under the involution: $F \circ I = -F$. One can define simple Morse pseudo-functions on N in a natural way: their lifts to $\widetilde{N}$ have to be Morse with distinct critical values. Our first result is the density of such pseudo-functions among all:

Theorem 1.3.6. *(=Theorem 5.2.3) Simple Morse pseudo-functions on a compact non-orientable manifold form an open and dense subset in the space of all smooth pseudo-functions in C^2-topology.*

Our next result is a classification of simple Morse pseudo-functions in 2D. Let N be a closed (i.e. compact and without boundary) non-orientable surface equipped with density ρ. It turns out that invariants of the $\mathrm{Diff}_\rho(N)$-action on pseudo-functions are given by measured Reeb graphs of their lifts to the orientation double cover $\widetilde{N}$, equipped with an involution.

Theorem 1.3.7. *(=Theorem 5.2.11) Let N be a closed connected non-orientable 2D surface equipped with a density ρ. Then there is a one-to-one correspondence between simple Morse pseudo-functions on N, considered up to area-preserving diffeomorphisms, and isomorphism classes of measured Reeb graphs with involution compatible with (N, ρ).*

Example 1.3.8. The height function F on a torus $T^2 = \widetilde{K^2}$ is odd with respect to the central symmetry I and hence induces a pseudo-function on the Klein bottle $K^2 = T^2 \,/\, I$, see Figure 1.3. The measured Reeb graph Γ_F with an involution ι is a complete invariant of the corresponding pseudo-function on the Klein bottle.

The classification of coadjoint orbits of the group $\mathrm{Diff}_\rho(N)$ of area-preserving diffeomorphisms of a non-orientable surface N requires a more subtle set of data than the measured Reeb graph with an involution. Namely, elements of the regular dual space $\mathrm{Vect}^*_\rho(N)$ to the Lie algebra $\mathrm{Vect}_\rho(N)$ are 1-form cosets $[\alpha] \in \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$. One associates to such a coset the vorticity pseudo-function $\mathrm{curl}[\alpha] := \mathrm{d}\alpha/\rho$. This way the classification of such cosets with respect to area-preserving diffeomorphisms can be seen as a refinement of the pseudo-function classification: one needs to augment the measured Reeb graph of $F = \mathrm{curl}[\alpha]$ by additional information, carried by the so-called circulation function described below. This allows one to formulate the full classification of generic coadjoint orbits in terms of circulation graphs.

Theorem 1.3.9. *(=Theorem 5.3.11) Let N be a closed connected non-orientable surface equipped with a density ρ. Then simple Morse coadjoint orbits of* $\mathrm{Diff}_\rho(N)$ *are in one-to-one correspondence with isomorphism classes of circulation graphs compatible with* (N, ρ).

Corollary 1.3.10. *(=Corollary 5.3.13) Let N be a closed connected non-orientable surface equipped with a density ρ. Then the space of coadjoint orbits of the group* $\mathrm{Diff}_\rho(N)$ *corresponding to the same measured Reeb graph* Γ *with involution* ι *is an affine space of dimension*

$$d = \frac{1}{2}(\#\mathrm{Fix}(\iota) + b_1(N) - 1), \tag{1.3}$$

where $\#\mathrm{Fix}(\iota)$ *is the number of fixed points of the involution* ι*, and* $b_1(N) = \dim \mathrm{H}_1(N; \mathbb{R})$ *is the first Betti number of N. In particular,*

$$\frac{1}{2}(b_1(N) - 1) \leq d \leq b_1(N). \tag{1.4}$$

Remark 1.3.11. Here we encounter a completely new phenomenon, not observed for orientable surfaces. Namely, for an orientable surface M the corresponding dimension d is always $\frac{1}{2}b_1(M)$, i.e. the genus of M. On the other hand, for non-orientable surfaces the dimension of the space of coadjoint orbits for a given vorticity is determined not only by the topology of the surface, but also by more subtle information about the involution action on the vorticity Reeb graph.

Example 1.3.12 (for details see Example 5.3.15)**.** Consider the height function on a vertically standing torus shown in Figure 1.3, interpreted as a pseudo-function on the Klein bottle K^2. Since $b_1(K^2) = 1$ and $\#\mathrm{Fix}(\iota) = 0$, the space of coadjoint obits corresponding to the given graph is 0-dimensional, i.e. the graph completely determines the orbit.

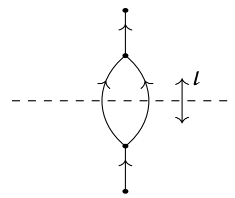


Figure 1.4: A graph involution ι with two fixed points. It is given by symmetry with respect to the dashed line.

On the other hand, consider the height function on the torus lying on a slightly inclined table. It is again odd with respect to the central symmetry and hence defines a pseudo-function on K^2. The corresponding Reeb graph is shown in Figure 1.4. Here one has $\#\mathrm{Fix}(\iota) = 2$, so the space of coadjoint orbits of $\mathrm{Diff}_\rho(K^2)$ corresponding to such a function is 1-dimensional. Note that $b_1(K^2) = 1$, so 0 and 1 are the only possible dimensions d of the orbit space for the Klein bottle.

Note that in the present thesis we didn't consider the case of non-orientable surfaces with boundary e.g. a Möbius strip. This question is left for future research.

2 LOCAL CLASSIFICATION OF FUNCTIONS

The main result of this chapter is a local classification of generic (Morse) functions on two-dimensional symplectic surfaces up to a symplectomorphism. Throughout this section, let M be a surface with boundary ∂M.

We remark on the case of a higher-dimensional manifold M at the end of the chapter.

2.1 SMOOTH CLASSIFICATION

We start by discussing the local classification of Morse functions up to a diffeomorphism.

For a regular point $O \in M \setminus \partial M$ of a function F the implicit function theorem says that there exists a coordinate chart (p, q) centered at O such that $F = F(O) + p$.

For a regular point $O \in \partial M$ of the restriction $F|_{\partial M}$ there exists a coordinate chart (p, q) in M centered at O such that $F = F(O) + p$ and the boundary ∂M is given by the equation $\{q = 0\}$.

Now, let $O \in M \setminus \partial M$ be a non-degenerate critical point for F. The classical Morse lemma implies that there exists a coordinate chart (p, q) centered at O such that $F = F(O) + p^2 + q^2$ or $F = F(O) + pq$.

Finally, let $O \in \partial M$ be a non-degenerate critical point for the restriction $F|_{\partial M}$. The following result is called *boundary Morse lemma.*

Theorem 2.1.1 ([8]). *Let $O \in \partial M$ be a regular point for F and a nondegenerate critical point for $F|_{\partial M}$. Then there exists a chart (p, q) centered at O such that we have $q \geq 0$ wherever q is defined, the boundary ∂M satisfies the equation $q = 0$, and $F = F(O) + q + p^2$ or $F = F(O) + q - p^2$.*

Together, the results above provide a complete list of local normal forms for a Morse function on a two-dimensional surface.

2.2 SYMPLECTIC CLASSIFICATION: THE CASE OF A SINGULAR POINT INSIDE THE SURFACE

Suppose now that M is a symplectic surface with a symplectic 2-form ω. Then for a regular point $O \in M \setminus \partial M$ of the function F there exists a coordinate chart (p, q) centered at O such that $F = F(O) + q$ and $\omega = \mathrm{d}p \wedge \mathrm{d}q$.

Now, let $O \in M \setminus \partial M$ be a non-degenerate critical point for F. The next result is called *Morse-Darboux lemma* and it provides a local normal form for a pair (F, ω) near the point O.

Theorem 2.2.1 ([11, 32])**.** *Let (M, ω) be a symplectic surface, and $F\colon M \to \mathbb{R}$ be a Morse function. Let $O \in M \setminus \partial M$ be a critical point for the function F. Then there exists a coordinate chart (p, q) centered at O such that $\omega = dp \wedge dq$ and $F = \lambda \circ S$ where $S = pq$ or $S = p^2 + q^2$. Here λ is a smooth function of one variable defined in some neighborhood of the origin $0 \in \mathbb{R}$ and $\lambda'(0) \neq 0$. Moreover:*

(i) In the case $S = p^2 + q^2$, and the function λ is uniquely determined by the pair (F, ω).

(ii) In the case $S = pq$, only the Taylor series of the function λ is uniquely determined by the pair (F, ω). In other words, if $(\tilde{p}, \tilde{q})$ is another chart as above then $\tilde{p}\tilde{q} = pq + \psi(pq)$ (for sufficiently small $p, q, \tilde{p}, \tilde{q}$), where ψ is a function of one variable flat[1] at the origin. Furthermore, every function of one variable that is flat at the origin can be obtained in this way.

The Morse-Darboux lemma is a particular case of Le lemme de Morse isochore, see [11], and also it is a particular case of Eliasson's theorem on the normal form for an integrable Hamiltonian system near a nondegenerate critical point, see [14, 7]. The Morse-Darboux lemma is an important tool in topological hydrodynamics, see [19], and theory of integrable systems, see [13].

Proof. All statements of this theorem but the last one are proved in [11, 32]. The last statement is proved in [32, see Lemma 3.2*a*]. □

2.3 SYMPLECTIC CLASSIFICATION: THE CASE OF A SINGULAR POINT FOR THE RESTRICTION TO THE BOUNDARY

For a regular point $O \in \partial M$ of the restriction $F|_{\partial M}$ there exists a coordinate chart (p, q) in M centered at O such that

1 Here *flat* means that all derivatives of ψ vanish at the origin.

(i) in this chart M is defined by $q \geq 0$ and the boundary ∂M is given by the equation $\{q = 0\}$;

(ii) $F = F(O) + p$;

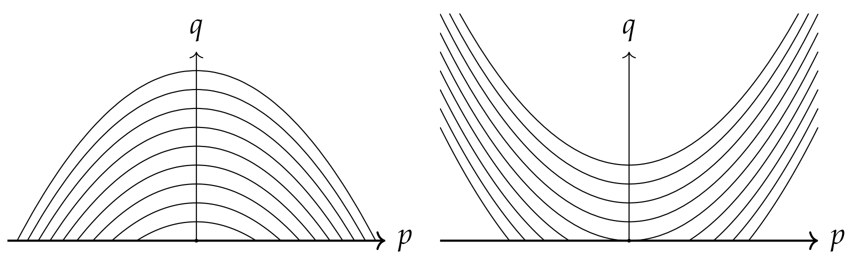
(iii) $\omega = dp \wedge dq$.

Now, let $O \in \partial M$ be a non-critical point for F and a non-degenerate critical point for the restriction $F|_{\partial M}$. The next result is an analog of Morse-Darboux lemma for a point on the boundary.

Theorem 2.3.1 ([23, 25]). *Let (M, ω) be a symplectic surface, and $F\colon M \to \mathbb{R}$ be a Morse function. Let $O \in \partial M$ be a regular point of F and a non-degenerate critical point of its restriction $F|_{\partial M}$. Then there exists a coordinate chart (p, q) centered at O such that $\omega = dp \wedge dq$ and $F = \lambda \circ S$ where $S = q + p^2$ or $S = q - p^2$. Here λ is a smooth function of one variable defined in some neighborhood of the origin $0 \in \mathbb{R}$ and $\lambda'(0) \neq 0$. In this chart M is defined by $q \geq 0$ and the boundary ∂M is given by the equation $\{q = 0\}$, see Figure 2.1. Moreover:*

(i) In the case $S = q + p^2$, the function λ is uniquely determined by the pair (F, ω).

(ii) In the case $S = q - p^2$, only the Taylor series of the function λ is uniquely determined by the pair (F, ω). In other words, if $(\tilde{p}, \tilde{q})$ is another chart as above then $\tilde{q} - \tilde{p}^2 = q - p^2 + \psi(q - p^2)$. Furthermore, every function of one variable that is flat at the origin can be obtained in this way.



(i) Case $S = q + p^2$. (ii) Case $S = q - p^2$.

Figure 2.1: Level sets of the function S. The horizontal axis corresponds to the boundary ∂M.

2.4 EXISTENCE OF MORSE-DARBOUX COORDINATES

In this section we prove the existence of Morse-Darboux coordinates described in Theorem 2.3.1. The remaining statements of Theorem 2.3.1

are shown in the next section. The existence of Morse-Darboux coordinates is equivalent to the following statement:

Theorem 2.4.1. *Let $\omega = \omega(x,y)dx \wedge dy$ be an area form on $\mathbb{R}^2$, and $F = F(x,y)$ be a smooth function such that $F(0,0) = 0$, $F_x(0,0) = 0$, $F_y(0,0) > 0$ and $F_{xx}(0,0) > 0$. Then there exists a chart (p,q) centered at $(0,0)$ such that $\omega = dp \wedge dq$, $F(p,q) = \lambda(p^2+q)$, and $q = 0$ if and only if $y = 0$ (i.e. the boundary is given by the same equation). The function λ of one variable is smooth in the neighborhood of the origin $0 \in \mathbb{R}$ and $\lambda'(0) > 0$.*

In the next subsection we formulate and prove several lemmas necessary for the proof of Theorem 2.4.1.

2.4.1 *Necessary lemmas*

Lemma 2.4.2. *Let*

$$D(F,\varepsilon) := \{(x,y) \in \mathbb{R}^2 \mid F(x,y) \leq \varepsilon \text{ and } y \geq 0\}.$$

Then the function

$$A_F(\varepsilon) := \int\limits_{D(F,\varepsilon)} \omega(x,y)dx \wedge dy$$

is well-defined. It can be expressed as

$$A_F(\varepsilon) = \int\limits_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} dx \int\limits_{0}^{\varepsilon - x^2} \omega(x,y)dy.$$

Remark 2.4.3. The function A_F gives us an invariant of a pair (F,ω). It will play a crucial role in the proof of Theorem 2.4.1

Remark 2.4.4. Consider the upper half-plane H with an area form $\omega = dp \wedge dq$ and a function $F = \lambda(p^2+q)$, where $\lambda'(0) > 0$. Then the function A_F can be expressed as

$$\begin{aligned} A_F(\lambda(\varepsilon)) &= \int\limits_{D(\lambda(p^2+q),\lambda(\varepsilon))} dp \wedge dq \\ &= \int\limits_{D(p^2+q,\varepsilon)} dp \wedge dq = \int\limits_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} dp \int\limits_{0}^{\varepsilon - p^2} dq = \int\limits_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} (\varepsilon - p^2)dx = \frac{4}{3}\varepsilon\sqrt{\varepsilon} = \frac{4}{3}\varepsilon^{3/2}, \end{aligned}$$

so

$$[A_F \circ \lambda](\varepsilon) = 4/3\varepsilon^{3/2}$$

and

$$\lambda(\varepsilon) = A_F^{-1}(4/3\varepsilon^{3/2})$$

or

$$\lambda^{-1}(\varepsilon) = [3/4A_F(\varepsilon)]^{2/3}.$$

So we know how to determine the function λ from Theorem $1'$. Now we want to prove that λ is a smooth function.

Lemma 2.4.5. *The function $\tilde{A}(\varepsilon) := A_F(\varepsilon)^{2/3}$ is smooth in some neighborhood of zero.*

Proof. Let

$$u(x,\varepsilon) := \int_0^{\varepsilon - x^2} \omega(x,y)dy.$$

Note that u is a smooth function of two variables. Further,

$$A_F(\varepsilon) = \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} u(x,\varepsilon)dx.$$

Introducing a new variable $\delta = \sqrt{\varepsilon}$ we obtain

$$A_F(\delta) = \int_{-\delta}^{\delta} u(x,\delta^2)dx.$$

This function is smooth and odd. Let us find the third order Taylor polynomial of $A_F(\delta)$:

$$\begin{aligned}
A_F(\delta) &= \int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} \omega(x,y)dy = \int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} [\omega(0,0) + O(x) + O(y)]dy \\
&= \omega(0,0)\int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} dy + \int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} O(x)dy + \int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} O(y)dy \\
&= \omega(0,0)\frac{4}{3}(\delta^2)^{3/2} + O(\int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} xdy) + O(\int_{-\delta}^{\delta} dx \int_0^{\delta^2-x^2} ydy) \\
&= \omega(0,0)\frac{4}{3}\delta^3 + O(\delta^4) + O(\delta^4) = \omega(0,0)\frac{4}{3}\delta^3 + O(\delta^4).
\end{aligned}$$

It means, that $A_F(\delta) = \delta^3 B(\delta)$, where the function $B(\delta)$ is smooth, even, and $B(0) \neq 0$. So, $A_F(\varepsilon) = \varepsilon^{3/2} B(\sqrt{\varepsilon})$ and $\tilde{A}(\varepsilon) = \varepsilon[B(\sqrt{\varepsilon})]^{2/3}$. □

Remark 2.4.6. The function $\tilde{A}$ is defined only if $\varepsilon \geqslant 0$. But it extends to a smooth function on a neighborhood of zero.

Definition 2.4.7. Recall that one-forms on a surface M with a fixed area form ω may be identified with vector fields, and every smooth function $F : M \to \mathbb{R}$ determines a unique vector field X_F, called the *Hamiltonian vector field* with the *Hamiltonian* F, by requiring that for every vector field Y on M the identity $dF(Y) = \omega(Y, X_F)$ holds. Let P_F be the flow (*Hamiltonian flow*) corresponding to the vector field X_F.

Definition 2.4.8. Recall that in the chart (x, y) we have $F(x, y) = x^2 + y$. Let $t_F(\varepsilon)$ be the time necessary to go from $(-\sqrt{\varepsilon}, 0)$ to the point $(\sqrt{\varepsilon}, 0)$ under the action of P_F, i.e. $t_F(\varepsilon)$ is defined by

$$P_F^{t_F(\varepsilon)}(-\sqrt{\varepsilon}, 0) = (\sqrt{\varepsilon}, 0).$$

Definition 2.4.9. The curve

$$\gamma(\varepsilon) := P_F^{\frac{1}{2} t_F(\varepsilon)}(-\sqrt{\varepsilon}, 0)$$

where $\varepsilon \geqslant 0$ is called a *bisector*.

Lemma 2.4.10. *The bisector is smooth and transversal to the boundary* $\{y = 0\}$.

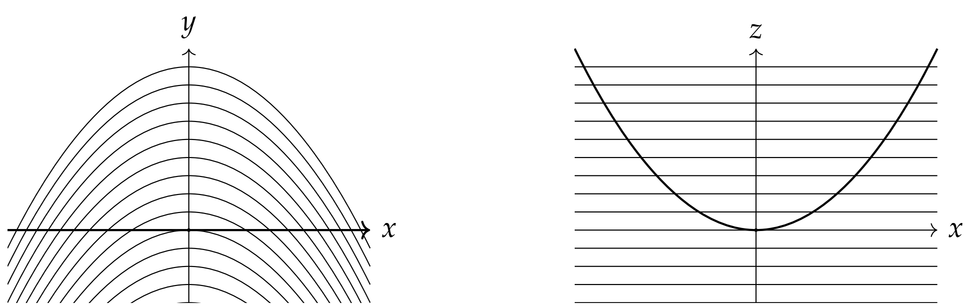


Figure 2.2: Level sets of the function F in charts (x, y) and (x, z). The thick curve is the boundary of M.

Proof. Let us introduce a new coordinate system (x, z), where

$$z(x, y) := F(x, y) = x^2 + y$$

(see Figure 2.2). Then in these new coordinates $F(x,z) = z$, $\omega = \omega(x,z)dx \wedge dz$, while $y = 0$ if and only if $z = x^2$, and $X_F = (-\frac{1}{\omega(x,z)}, 0)$. Let us compute the function t_F. Note that

$$-\omega(x,z)dx = dt \tag{2.1}$$

Integrating (2.1) over the horizontal segment between the points $(-\sqrt{z}, z)$ and $(\sqrt{z}, z)$, we get

$$t_F(z) = -\int_{-\sqrt{z}}^{\sqrt{z}} \omega(\tau, z)d\tau.$$

In the same way we obtain equations for the bisector $(s(z), z)$

$$\int_{-\sqrt{z}}^{s(z)} \omega(\tau, z)d\tau = \frac{1}{2}\int_{-\sqrt{z}}^{\sqrt{z}} \omega(\tau, z)d\tau. \tag{2.2}$$

Introducing a new variable $w = \sqrt{z}$ we obtain an equation for the function $\hat{s}(w) := s(w^2)$:

$$\int_{-w}^{\hat{s}(w)} \omega(\tau, w^2)d\tau = \frac{1}{2}\int_{-w}^{w} \omega(\tau, w^2)d\tau, \tag{2.3}$$

Equation (2.3) allows us to define $\hat{s}(w)$ even if $w < 0$. We claim that $\hat{s}$ is a smooth function and $\hat{s}(-w) = \hat{s}(w)$.

Partial derivative of (2.3) with respect to $\hat{s}$ is $\omega(\hat{s}(w), w^2)$. For any (x,z) we have $\omega(x,z) \neq 0$. It follows from the implicit function theorem that $\hat{s}(w)$ depends smoothly on w. It is easy to see that Equation (2.3) defines $\hat{s}$ as an even function of w. It implies that $s(z) = \hat{s}(\sqrt{z})$ is a smooth function of z. Now it is clear that the bisector is transversal to the boundary $\{z = x^2\}$. □

Remark 2.4.11. It follows from the proof of Lemma 2.4.10 that the bisector curve can be smoothly extended to the lower half plane.

Definition 2.4.12. Let $T_F(x,y)$ be the time necessary to go from the bisector to the point (x,y) under the action of P_F.

Remark 2.4.13. In the chart (x,z), we have:

$$T_F(x,z) = \int_{s(z)}^{x} -\omega(\tau,z)d\tau = \int_{x}^{s(z)} \omega(\tau,z)d\tau,$$

where the function s is defined in Lemma 2.4.10. Now it is clear that T_F is a smooth function.

Also note that since P_F is the flow of the vector field X_F, it follows that $dT_F(X_F) = 1$.

Lemma 2.4.14. $\omega = dF \wedge dT_F$.

Proof. Using that $dT_F(X_F) = 1$, we get

$$i_{X_F} dF \wedge dT_F = dF(X_F)dT_F - dFdT_F(X_F) = -dT_F(X_F)dF = -dF = i_{X_F}\omega,$$

so

$$i_{X_f}(dF \wedge dT_F - \omega) = 0,$$

and, since the ambient surface is 2-dimensional and $X_F \neq 0$, it follows that $\omega = dF \wedge dT_F$. □

Lemma 2.4.15. $\frac{d}{d\varepsilon} A_F(\varepsilon) = |t_F(\varepsilon)|$.

Proof. To proof this, let us use the chart (x, z) from Lemma 2.4.10 Recall that in this chart $F(x, z) = z$. Now it follows from the definition of A_f and from Lemma 4 that

$$A_F(\varepsilon+\delta) - A_F(\varepsilon) = |\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{\varepsilon}^{\varepsilon+\delta} dz \wedge dT_F| + o(\delta) = |\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} dT_F \int_{\varepsilon}^{\varepsilon+\delta} dz| + o(\delta) =$$
$$= \delta|T_F(\sqrt{\varepsilon}, 0) - T_F(-\sqrt{\varepsilon}, 0)| + o(\delta) = \delta|t_F(\varepsilon)| + o(\delta).$$

So

$$\frac{d}{d\varepsilon} A_F(\varepsilon) = |t_F(\varepsilon)|.$$

□

Lemma 2.4.16. *Suppose that after a coordinate transformation $(x, y) \to (p, q)$ the following conditions hold:*

1. $F(p, q) = p^2 + q$.
2. $\omega = dp \wedge dq$.
3. *The equation $p = 0$ describes the bisector.*
4. $A_F(\varepsilon) = \frac{4}{3}\varepsilon\sqrt{\varepsilon}$.

Then $y(p, q) = 0$ if and only if $q = 0$.

Proof. First of all, the Condition 4 states that $A_F(\varepsilon) = \frac{4}{3}\varepsilon\sqrt{\varepsilon}$, hence the function $A'_F(\varepsilon)$ can be written as:

$$\frac{d}{d\varepsilon} 4/3\varepsilon\sqrt{\varepsilon} = 2\sqrt{\varepsilon}. \quad (2.4)$$

Let us check that $y = 0$ if and only if $q = 0$. It is follows from Lemma 2.4.10 that the curve $\{y = 0\}$ is transversal to the bisector $\{p = 0\}$. So, the curve $y = 0$ is a graph of some function $q = r(p)$ (see Figure 2.3). It follows from the definition of bisector that $r(x) = r(-x)$. Let us proof that $r(x) \equiv 0$. Assume that there exists some p_0 such that $q_0 := r(p_0) > 0$ (the case $q_0 < 0$ is analogous).

$$A'_F(q_0 + p_0^2) = [\text{by equation (2.4)}] = 2\sqrt{q_0 + p_0^2} > |2p_0|$$
$$= [\text{by conditions (1),(2),(3) and the definition of } t_F] = |t_F(q_0 + p_0^2)|$$
$$= [\text{by Lemma 2.4.15}] = A'_F(q_0 + p_0^2).$$

This contradiction concludes the proof.

$\square$

2.4.2 *Proof of the Theorem 2.4.1*

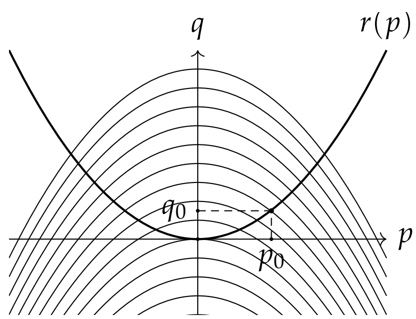


Figure 2.3: An illustration to the proof of Lemma 2.4.16.

Proof. Consider the function

$$\lambda(\varepsilon) := A_F^{-1}(\frac{4}{3}\varepsilon\sqrt{\varepsilon}).$$

It follows from Lemma 2.4.5 that λ is a smooth function. Let also

$$H(x, y) := [\lambda^{-1} \circ F](x, y)$$

$$p(x, y) := -T_H(x, y)$$
$$q(x, y) := H - p^2(x, y).$$

Then

$$dp \wedge dq = -dT_H \wedge d(H - T_H^2) =$$
$$= -dT_H \wedge dH + dT_H \wedge 2T_H dT_H = dH \wedge dT_H = [\text{by Lemma 2.4.14}] = \omega,$$

so dp and dq are linearly independent. Further, in the chart (p, q), we have

1. $H(p, q) = p^2 + q$ and $F(p, q) = \lambda(p^2 + q)$.
2. $\omega = dp \wedge dq$.
3. The equation $p = 0$ describes the bisector, because $p(x, y) = 0$ if and only if $T(x, y) = 0$, while the latter means that the point (x, y) belongs to the bisector.
4. $A_H(\varepsilon) = A_F(\lambda(\varepsilon)) = A_F(A_F^{-1}(\frac{4}{3}\varepsilon\sqrt{\varepsilon})) = \frac{4}{3}\varepsilon\sqrt{\varepsilon}$.

So, the chart (p, q) fulfils all conditions of Lemma 2.4.16. Now it follows from Lemma 2.4.16 that the chart (p, q) satisfies all conditions of Theorem 2.4.1.

□

2.5 FINISHING THE PROOF OF THEOREM 2.3.1

Before we proceed with the proof let us formulate and prove the following.

Lemma 2.5.1. *Let h_1 and h_2 be two smooth non-negative functions $\mathbb{R}_+ \to \mathbb{R}_+$ such that $h_i(0) = 0$ and $h_i'(0) > 0$ for $i = 1, 2$. Then the following statements are equivalent:*

(i) The difference $h_1 - h_2$ is a function flat at the origin, i.e. the Taylor series $J_0^\infty h_1$ and $J_0^\infty h_2$ are equal to each other.

(ii) The difference $\sqrt{h_1} - \sqrt{h_2}$ is a smooth function $\mathbb{R}_+ \to \mathbb{R}$.

(iii) The difference $\sqrt{h_1} - \sqrt{h_2}$ is a smooth function $\mathbb{R}_+ \to \mathbb{R}$ flat at the origin.

Proof. The implication $(iii) \implies (ii)$ is evident so it enough to show that $(i) \implies (iii)$ and $(ii) \implies (i)$. Let us start with implication $(i) \implies (iii)$. It follows from Hadamard's lemma that there exist smooth functions $\tilde{h}_1$ and $\tilde{h}_2$ such that $h_i = x\tilde{h}_i$ and $\tilde{h}_i(0) > 0$ for $i = 1, 2$. We have the following formula for the difference $\sqrt{h_1} - \sqrt{h_2}$:

$$\sqrt{h_1(x)} - \sqrt{h_2(x)} = \frac{1}{\sqrt{x}} \frac{h_1(x) - h_2(x)}{\sqrt{\tilde{h}_1(x)} + \sqrt{\tilde{h}_2(x)}} \tag{2.5}$$

for small enough $x > 0$. It follows from the formula (2.5) that the difference $\sqrt{h_1} - \sqrt{h_2}$ is smooth and flat at the origin whenever the difference $h_1 - h_2$ is flat the origin.

It remains to show that (ii) implies (i). Denote by g the smooth function $\sqrt{h_1} - \sqrt{h_2}$. Assume that the difference $h_1 - h_2$ is not flat at the origin. Then there exists a number $n \in \mathbb{N}$ and a smooth non-zero function $f : \mathbb{R}_+ \to \mathbb{R} \setminus 0$ such that $h_1(x) - h_2(x) = x^n f(x)$. It is useful to rewrite formula (2.5) in the following form:

$$g(x)\sqrt{x}\left(\sqrt{\tilde{h}_1(x)} + \sqrt{\tilde{h}_2(x)}\right) = x^n f(x). \tag{2.6}$$

Formula (2.6) implies that the function f is flat at the origin whenever the function g is flat at the origin. The function f is a non-zero function so we conclude that the function g is not flat at the origin. Therefore there exists a number $m \in \mathbb{N}$ and a smooth non-zero function $\tilde{g} : \mathbb{R}_+ \to \mathbb{R} \setminus 0$ such that $g(x) = x^m \tilde{g}(x)$. Now we take the square of both sides of (2.6) and obtain the following formula:

$$x^{1+2m}\tilde{g}(x)^2\left(\sqrt{\tilde{h}_1(x)} + \sqrt{\tilde{h}_2(x)}\right)^2 = x^{2n} f(x)^2. \tag{2.7}$$

That gives us a contradiction since the Taylor series of the left hand side starts with an odd power of x and the Taylor series of the right hand side starts with an even power of x. We conclude that the function $h_1 - h_2$ is flat at the origin. □

Now we proceed with proving the remaining statements of Theorem 2.3.1.

Proof of Theorem 2.3.1. Without loss of generality we can assume that $F(O) = 0$. Recall that the main part of this theorem on the existence of a coordinate chart was proved in Theorem 2.4.1.

Let us prove statement (i) of this theorem. We need to prove the equality $\tilde{p} + \tilde{q}^2 = p + q^2$. In this case (see Figure 2.1, (a)) the region $\{F \leq \varepsilon\}$ is diffeomorphic to a closed half ball provided that $\varepsilon > 0$ is sufficiently small. Therefore, the area of this region

$$A_{F,\omega}(\varepsilon) := \int_{F \leq \varepsilon} \omega$$

is well-defined. Let (p,q) be a coordinate chart centered at O such that $F = \lambda(p^2+q)$ and $\omega = \mathrm{d}p \wedge \mathrm{d}q$. Then we can write an explicit formula for the function $A_{F,\omega}(\varepsilon)$:

$$A_{F,\omega}(\varepsilon) = \int_{-\sqrt{\lambda^{-1}(\varepsilon)}}^{\sqrt{\lambda^{-1}(\varepsilon)}} (\lambda^{-1}(\varepsilon) - p^2)\mathrm{d}p = \frac{4}{3}\lambda^{-1}(\varepsilon)^{3/2}.$$

So we conclude that the function λ (and thus the function $q+p^2$) is uniquely determined by the pair (F,ω).

Now let us prove statement (ii) of this theorem. Consider the second case where in some local chart centered at O we have $F = \lambda(q-p^2)$ and $\omega = \mathrm{d}p \wedge \mathrm{d}q$. Consider a smooth curve $\ell \subset M$ such that it is transversal to the right half of the parabola $\{q = p^2\}$ and these two curves intersect each other at exactly one point. In coordinates (p,q) the curve ℓ can be described as a graph of some function g:

$$\ell = \{(p,q) \colon q = g(p)\}.$$

We fix a number $\varepsilon < 0$, and consider the region R_ε (see Figure 2.4) bounded by the boundary curve $\partial M = \{q = 0\}$, the right half of the parabola $\{q = p^2\} \cap \{p \geq 0\}$, the right half of the parabola $\{q = p^2 + \lambda^{-1}(\varepsilon)\} \cap \{p \geq 0\}$, and the curve ℓ. Then the area of this region

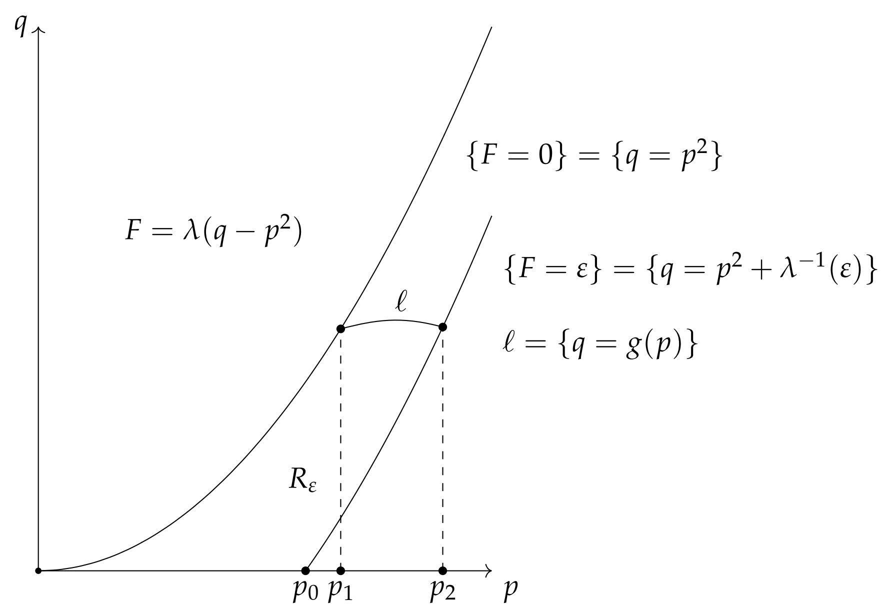


Figure 2.4: The area function $A_{F,\omega,\ell} = \int_{R_\varepsilon} \omega$.

$$A_{F,\omega,\ell}(\varepsilon) := \int_{R_\varepsilon} \omega$$

is well-defined. Denote by p_0 the p-coordinate of the intersection $\{F = \varepsilon\} \cap \partial M$, by p_1 the p-coordinate of the intersection $\{F = 0\} \cap \ell$, and by p_2 the p-coordinate of the intersection $\{F = \varepsilon\} \cap \ell$. The coordinates p_0, p_1, and p_2 depend on ℓ, and also the coordinates p_0 and p_2 depend on ε. It follows from the implicit function theorem that the coordinate p_2 is a smooth function of ε. As for p_0, it is explicitly given by $\sqrt{-\lambda^{-1}(\varepsilon)}$. Note that $p_0(\varepsilon) < p_1 < p_2(\varepsilon)$ provided that $|\varepsilon|$ is sufficiently small. We have the following formula for the function $A_{F,\omega,\ell}(\varepsilon)$:

$$\begin{aligned} A_{F,\omega,\ell}(\varepsilon) &= \int_0^{p_0(\varepsilon)} p^2 \mathrm{d}p - \int_{p_0(\varepsilon)}^{p_1} \lambda^{-1}(\varepsilon)\mathrm{d}p + \int_{p_1}^{p_2(\varepsilon)} (g(p) - p^2 - \lambda^{-1}(\varepsilon))\mathrm{d}p \\ &= p_0^3(\varepsilon)/3 - p_0(\varepsilon)\lambda^{-1}(\varepsilon) + \text{smooth function of } \varepsilon \\ &= \frac{4}{3}(-\lambda^{-1}(\varepsilon))^{3/2} + \text{smooth function of } \varepsilon. \end{aligned} \tag{2.8}$$

Now consider some other chart $(\tilde{p}, \tilde{q})$ such that $F = \tilde{\lambda}(q - p^2)$, $\omega = \mathrm{d}\tilde{p} \wedge \mathrm{d}\tilde{q}$, and the boundary ∂M is given by $\{\tilde{q} = 0\}$. Then it is follows from above that

$$\frac{4}{3}(-\lambda^{-1}(\varepsilon))^{3/2} - \frac{4}{3}(-\tilde{\lambda}^{-1}(\varepsilon))^{3/2} = f(\varepsilon) \tag{2.9}$$

where f is a smooth function of one variable. We want to prove that the Taylor series $J_0^\infty \lambda$ is equal to the Taylor series of $J_0^\infty \tilde{\lambda}$. It follows from Lemma 2.5.1 that the Taylor series $J_0^\infty \lambda^{-1}(\varepsilon)^3$ is equal to the Taylor series $J_0^\infty \tilde{\lambda}^{-1}(\varepsilon)^3$. From here we conclude that $J_0^\infty \lambda = J_0^\infty \tilde{\lambda}$.

It remains to prove the last part of statement (ii). Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be a function of one variable flat at the origin. The goal is to find a symplectomorphism Φ defined in some neighbourhood of the origin such that O is a fixed point for Φ, the symplectomorphism Φ preserves the boundary ∂M, and

$$\Phi^*[q + \psi(q)] = q.$$

For this part of the proof we are going to use a different coordinate chart (see Figure 2.5):

$$(P := p, Q := q - p^2).$$

First of all, notice that after this "parabolic" change of coordinates the symplectic form still has the standard form:

$$\mathrm{d}P \wedge \mathrm{d}Q = \mathrm{d}p \wedge \mathrm{d}q.$$

Secondly, in the chart $(P, Q)$the function F "straightens" i.e. $F(P, Q) = Q + \psi(Q)$, and the boundary becomes a parabola $\partial M = \{(P + Q^2 = 0\}$.

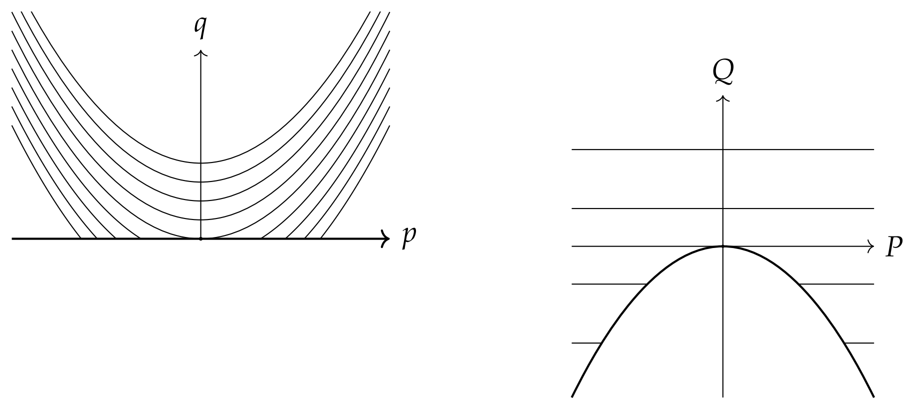


Figure 2.5: The parabolic change of coordinates

Now let us proceed with the proof. Apply Moser's path method and consider the family of functions

$$f^t := Q + t\psi(Q)$$

for each $t \in [0, 1]$. Instead of looking for one symplectomorphism Φ, we will be looking for a family of Hamiltonian symplectomorphisms Φ^t such that

$$\Phi^{t*} f^t = Q, \tag{2.10}$$

$\Phi^t(\partial M) \subset \partial M$ for each $t \in [0, 1]$, and $\Phi^t(O) = O$ for each $t \in [0, 1]$. Let v^t be the vector field corresponding to the flow Φ^t :

$$\frac{d}{dt}\Phi^t = v^t \circ \Phi^t.$$

Differentiating (2.10) with respect to t, we obtain the following differential equation

$$\Phi^{t*} L_{v^t} f^t + \Phi^{t*} \frac{df^t}{dt} = 0,$$

which we rewrite as

$$\Phi^{t*} \left(L_{v^t} f^t + \frac{df^t}{dt} \right) = 0.$$

Since Φ^t is a diffeomorphism, it is equivalent to

$$L_{v^t} f^t + \psi(Q) = 0. \tag{2.11}$$

Since the flow of the field v^t has to preserve the symplectic structure ω, we will be looking for the field v^t in the Hamiltonian form

$$v^t = H^t_Q \frac{\partial}{\partial P} - H^t_P \frac{\partial}{\partial Q} \tag{2.12}$$

where $H^t_Q := \frac{\partial H^t}{\partial Q}$ and $H^t_P := \frac{\partial H^t}{\partial P}$. Substitute the right-hand side of (2.12) into (2.11) to obtain the following partial differential equation

$$\psi(Q) - H^t_P(1 + t\dot{\psi}(Q)) = 0.$$

Rewrite it as

$$H^t_P = -\frac{\psi(Q)}{1 + t\dot{\psi}(Q)}.$$

Consider the family of functions

$$\psi^t(x) := \frac{\psi(x)}{1 + t\dot{\psi}(x)}$$

for each $t \in [0, 1]$. We have $\dot{\psi}(0) = 0$ so the denominator $1 + t\dot{\psi}(x)$ is non-zero in sufficiently small neighbourhood of the origin. Then our equation assumes the form

$$H^t_P = \psi^t(Q). \tag{2.13}$$

It is clear that the general solution to this equation has the form

$$H^t(P, Q) = P\psi^t(Q) + g^t(Q)$$

where g^t is a smooth function of one variable. Our goal is to find a particular solution to (2.13) that is constant along the boundary $\partial M = \{Q + P^2 = 0\}$. That implies the following condition on the function g^t:

$$P\psi^t(-P^2) + g^t(-P^2) = 0.$$

Hence, for any non-positive $x \in \mathbb{R}$ we have

$$g^t(x) = -\sqrt{-x}\psi^t(x).$$

Now define a function g^t in the following way:

$$\begin{cases} g^t(x) = -\sqrt{-x}\psi^t(x) & x \leq 0 \\ g^t(x) = 0 & x > 0 \end{cases}$$

It follows from the flatness of $\psi^t(\cdot)$ that the function g^t defined as above is a smooth function flat at the origin. Now define the function H to be the corresponding solution:

$$H^t(P,Q) = P\psi^t(Q) + g^t(Q)$$

The family of symplectomorphisms Φ^t can be recovered as the flow of the corresponding field $v^t = H^t_Q \frac{\partial}{\partial P} - H^t_P \frac{\partial}{\partial Q}$. The condition $H^t|_{\partial M} = 0$ implies that the field v^t has a zero restriction on the boundary $\partial M = \{Q + P^2 = 0\}$, and we conclude that the corresponding family Φ^t preserves the boundary, and $\Phi^t(O) = O$ for each $t \in [0,1]$. Now applying the theorem on the smooth dependence of the flow on initial data one can conclude that the flow Φ^t is well-defined for $t \in [0,1]$. Hence, the diffeomorphism Φ^1 has the desired properties. $\square$

2.6 THE CASE OF A HIGHER-DIMENSIONAL MANIFOLD

We mentioned in Section 2.2 that Morse-Darboux lemma 2.2.1 is partial case of Le lemme de Morse isochore. Here we present the precise statement of that lemma.

Theorem 2.6.1 ([11]). *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that 0 is a non-degenerate critical point of F i.e. the quadratic form associated with the Hessian*

$$Q(x_1, \dots x_n) = \frac{1}{2} \sum_{1 \leq i,j \leq n} \frac{\partial^2 F}{\partial x_i \partial x_j}(0, \dots, 0) x_i x_j$$

is a non-degenerate quadratic form. Let also $\omega \in \Omega^n(\mathbb{R}^n)$ be a volume form.

Then there exits a diffeomorphism $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\Phi^*\omega = \alpha(Q) d^n x$ and $\Phi^* F = Q$. Here α is a smooth single variable function. Moreover, if Q is positive-definite or negative-definite form, then function α is determined by the pair (F, ω). Otherwise, only Taylor series of α is determined by the pair (F, ω).

It is a natural problem to generalize Le lemme de Morse isochore to the case of a higher-dimensional manifold. We expect the following conjecture to be true.

Conjecture 2.6.2. *Let $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function, and let Γ be the hyperplane $\{x_0 = 0\}$. Assume that $\nabla F(0) \neq 0$ and 0 is non-degenerate critical point for the restriction $F|_\Gamma$ i.e. the quadratic form associated with the Hessian for the restriction $F|_\Gamma$*

$$Q(x_1, \dots x_n) = \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2 F}{\partial x_i \partial x_j}(0, \dots, 0) x_i x_j$$

is a non-degenerate quadratic form. Let also $\omega \in \Omega^{n+1}(\mathbb{R}^{n+1})$ be a volume form.

Then there exits a diffeomorphism $\Phi\colon (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ such that $\Phi^*\omega = \alpha(\frac{\partial F}{\partial x_0}(0, \dots, 0)x_0 + Q) dy \wedge d^n x$, $\Phi^* F = \frac{\partial F}{\partial x_0}(0, \dots, 0)x_0 + Q$, and here α is a smooth single variable function. Moreover, the diffeomorphism Φ maps the hyperplane Γ to itself.

3

GLOBAL CLASSIFICATION OF FUNCTIONS

Smooth functions with non-degenerate singularities are a well-known field of research. There is a number of papers devoted to the functions with non-degenerate critical points on closed 2-dimensional manifolds [30, 1, 6, 19].

In this chapter, we obtain a global classification of generic (simple Morse) smooth functions on compact two-dimensional surfaces with boundary with respect to the action of the group $\mathrm{SDiff}(M)$ of symplectomorphisms of M.

3.1 THE REEB GRAPH OF A FUNCTION

Let M be a compact connected orientable surface (with or without boundary), and let $F\colon M \to \mathbb{R}$ be a Morse function on M. In what follows, by a level we mean a connected component of a level set of F. The graph Γ_F, called the Reeb graph[1] is the set of levels of a function F on a surface M, see Figure 1.1.

3.1.1 *The case of a closed surface*

Originally the notion of a Reeb graph was introduced for Morse functions on closed orientable two-dimensional surfaces. In this case all non-critical levels of a Morse function $F\colon M \to \mathbb{R}$ are circles and the Reeb graph is finite graph whose vertices correspond to critical points of F. In particular, saddle points correspond to 3-valent vertices, and min/max points correspond to 1-valent vertices, see Figure 1.1. The Reeb graph determines the topological type of surface.

Theorem 3.1.1 ([6], Chapter 2, Theorem 2.1). *Let F be a simple Morse function on a closed two-dimensional orientable surface M. Then its Reeb graph determines this surface uniquely up to a diffeomorphism.*

1 This graph is also called the Kronrod graph of a function, see [30, 1].

3.1.2 *The case of a surface with boundary*

The notion of a Reeb graph was generalized in [15] to the case of surfaces with boundary. Let M be a compact connected orientable surface with boundary ∂M, and let $F\colon M \to \mathbb{R}$ be a Morse function on M. In this case all non-critical levels are diffeomorphic to a circle or a line segment. The surface M can be considered as a union of levels, and we get a foliation with singularities. The base space of this foliation with the quotient topology is homeomorphic to a finite connected graph Γ_F (see Figure 1.2) whose vertices correspond to critical values of F or $F|_{\partial M}$. We view this graph as a topological object (rather than combinatorial). By π we denote the projection $M \to \Gamma_F$. Let e be an (open) edge of Γ_F. We denote e by a *solid* line if $\pi^{-1}(e)$ is a cylinder and by a *dashed* line if $\pi^{-1}(e)$ is a strip. We denote the closure of the union of solid (respectively, dashed) edges in Γ_F by Γ_F^s and Γ_F^d, respectively. We denote the preimages $\pi^{-1}(\Gamma_F^s)$ and $\pi^{-1}(\Gamma_F^d)$ by M_F^s and M_F^d. Thus $\Gamma_F = \Gamma_F^s \cup \Gamma_F^d$, and $M = M_F^s \cup M_F^d$. There are 7 possible types of vertices in the graph Γ_F (see Table 3.1). The function F on M descends to a function f on the Reeb graph Γ_F. It is also convenient to assume that Γ_F is oriented: edges are oriented in the direction of increasing f.

Let v be a vertex of the Reeb graph Γ_F. Let us fix a number $\varepsilon > 0$ such that

$$f^{-1}([f(v) - \varepsilon, f(v) + \varepsilon]) \cap e$$

is a proper subset of e for each edge e incident to v. The preimage $P_v^\varepsilon := \pi^{-1}(f^{-1}([f(v) - \varepsilon, f(v) + \varepsilon])) \subset M.$ is connected oriented surface and its boundary ∂P_v^ε is a piecewise smooth closed oriented curve that in general consists of some level curves and some boundary curves (see pictures in the left column of Table 3.1). The curve ∂P_v^ε is connected in the case where the vertex v is incident only to dashed edges, and its image $\pi[\partial P_v^\varepsilon]$ is a closed oriented curve that passes edges incident to the vertex v in a certain cyclic order. This construction is nontrivial only in the case when there are at least three dashed edges incident to the vertex v (otherwise, there is only one cyclic order at the set of edges incident to v). Thus for an arbitrary II-vertex or IV-vertex (see Table 3.1) of the graph Γ_F we have a natural cyclic order for the edges incident to this vertex. The above properties of the graph Γ_F make it natural to introduce the following definition of an abstract Reeb graph.

Definition 3.1.2. An (abstract) Reeb graph (Γ, f) is an oriented connected graph Γ with solid or dashed edges, and a continuous function $f : \Gamma \to \mathbb{R}$, with the following properties and additional data:

(i) Each vertex of Γ is of one of the 7 types from Table 3.1.

(ii) There is a cyclic order on the set of edges incident to II- or IV-vertices (see Table 3.1)

(iii) The function f is strictly monotonic on each edge of Γ, and the edges of Γ are oriented towards the direction of increasing f.

Definition 3.1.3. Abstract Reeb graphs (Γ_1, f) and (Γ_2, g) are said to be *equivalent* by means of the isomorphism $\phi\colon \Gamma_1 \to \Gamma_2$ if the map ϕ :

(i) maps solid (respectively, dashed) edges to solid (respectively, dashed) edges;

(ii) preserves the cyclic order on the set of edges incident to each II- or IV-vertex, i.e. if e_2 follows e_1 in the cyclic order, then $\phi(e_2)$ follows $\phi(e_1)$;

(iii) takes the function g to the function f (i.e. $f = g \circ \phi$).

3.2 RECOVERING THE TOPOLOGY OF A SURFACE FROM THE REEB GRAPH

In this subsection we follow [15, Section 5]. Let M be a compact connected oriented surface with boundary ∂M, and let $F\colon M \to \mathbb{R}$ be a simple Morse function. The restriction of the projection π to each boundary component of M is a closed curve (a map from a circle to the graph) in the graph Γ_F. Informally speaking, the following definition describes those closed curves for an abstract Reeb graph.

Definition 3.2.1. Let (Γ, f) be an abstract Reeb graph. A non-empty sequence of edges $(e_1, e_2, \ldots, e_n)$ together with a sequence $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ of vertices is called a *boundary cycle* if the following three conditions hold:

(i) All edges in the sequence are dashed.

(ii) Each edge e_i is incident to the vertices v_i and v_{i+1} for every $i \in \{1, \ldots, n\}$.

(iii) If the vertex v_i has three or more adjacent dashed edges, then the pair (e_{i-1}, e_i) of consecutive edges is also a consecutive pair of edges with respect to the cyclic order on the set of edges incident to the vertex v_i for every $i \in \{1, \ldots, n\}$.

We call two boundary cycles equivalent if they differ by the action of a cyclic group, i.e. the sets of vertices $v_1 \ldots v_n v_1$ and $v_i \ldots v_n v_1 \ldots v_{i-1} v_i$ define the same topological cycle for each $i \in \{1, \ldots, n\}$. In addition, in the case when a boundary cycle consists only of 1 or 2-valent vertices (i.e. of vertices of type III and IV) we also call two boundary cycles $v_1v_2\dots v_{n-1}v_nv_1$ and $v_nv_{n-1}\dots v_2v_1v_n$ equivalent. We denote by $\sigma(\Gamma)$ the number of (equivalence classes of) boundary cycles in Γ.

Example 3.2.2. Consider a disk with two holes and a torus with one hole, and consider the height function on them (as shown in Figure 3.1). The corresponding Reeb graphs are identical except for the cyclic orders at the vertices C_1 and C_2. In case (a) of a disk with two holes there are three boundary cycles: $B_1C_1D_1B_1$, $C_1E_1D_1C_1$, and $A_1B_1D_1E_1F_1E_1C_1B_1A_1$. In case (b) of a torus with one hole there is only one boundary cycle: $A_2B_2D_2E_2F_2E_2C_2D_2B_2C_2E_2D_2C_2D_2A_2$.

Proposition 3.2.3 ([15, Section 5])**.** *Let M be a compact connected oriented surface with boundary ∂M, and let $F\colon M \to \mathbb{R}$ be a simple Morse function. Then the number of boundary cycles $\sigma(\Gamma_F)$ is equal to the number of boundary components* $\dim H_0(\partial M)$ *of the surface M.*

Theorem 3.2.4 ([15, Theorem 5.3])**.** *The genus $g(M)$ of a surface M is given by the following formula:*

$$\begin{aligned} g(M) = -\chi(\Gamma_F^s) + \frac{-\chi(\Gamma_F^d) + 5\dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d) - \sigma(\Gamma_F)}{2} \\ - \dim \mathrm{H}_0(\Gamma_F^s) - \dim \mathrm{H}_0(\Gamma_F^d) + 3, \end{aligned} \tag{3.1}$$

where $\chi(\Gamma_F) = \dim \mathrm{H}_0(\Gamma_F) - \dim \mathrm{H}_1(\Gamma_F)$ is the Euler characteristic and $\sigma(\Gamma_F)$ is the number of boundary cycles.

Theorem 3.2.4 motivates us to give the following definition.

Definition 3.2.5. Let (Γ, f) be an abstract Reeb graph. Define the *genus* $g(\Gamma)$ as the number from the right-hand side of the formula in Theorem 3.2.4.

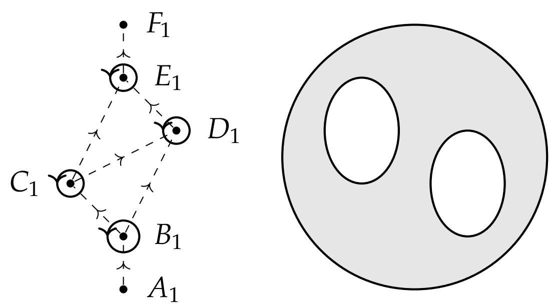
Now we can summarize the results from [15] as follows.

Theorem 3.2.6. *Let $(M, \partial M)$ be a compact oriented surface.*

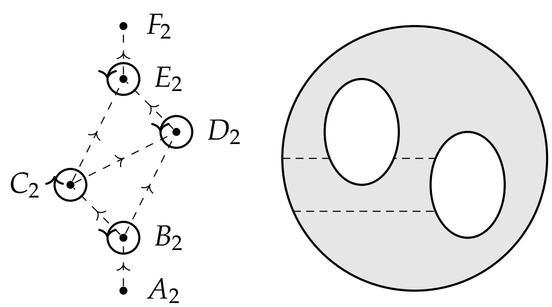
(i) Let $F, G\colon M \to \mathbb{R}$ be two simple Morse functions. Then the following conditions are equivalent:

a) There exists a diffeomorphism $\Phi : M \to M$ such that $\Phi^*F = G$.

b) Reeb graphs are isomorphic.



(i)



(ii)

Figure 3.1: An illustration to Definition 3.2.1: dashed Reeb graph with $\dim \mathrm{H}_1(\Gamma) = 2$ corresponding to both a disk with two holes (a) and torus with one hole (b). Cutting the disk drawn here along the three dashed levels and then restoring the gluings with opposite orientations, one obtains a torus with one hole. This figure is based on Figure 5 from [17].

Moreover, every isomorphism $\phi\colon (\Gamma_F, f) \to (\Gamma_G, g)$ *can be lifted to a diffeomorphism* $\Phi : M \to M$ *such that* $\Phi^* F = G$ *and the following diagram is commutative:*

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ \Gamma_F \arrow[r, "\phi"] & \Gamma_G \end{tikzcd}$$

(ii) *For each Reeb graph* (Γ, f) *such that* $g(\Gamma) = g(M)$ *there exists a simple Morse function* $F : M \to \mathbb{R}$ *such that the corresponding Reeb graph* Γ_F *is isomorphic to* (Γ, f).

3.3 MEASURED REEB GRAPHS

Now, fix an area form ω on the surface M. Then the natural projection map $\pi\colon M \to \Gamma_F$ induces a measure $\mu := \pi_* \omega$ on the graph Γ_F.

Definition 3.3.1. A measure μ on an abstract Reeb graph (Γ, f) is called *quasi-smooth* if the following conditions hold.

1. The measure μ has a C^∞-smooth non-zero density $\mathrm{d}\mu/\mathrm{d}f$ in the complement $\Gamma \setminus V(\Gamma)$.

2. In a neighbourhood of each vertex the measure μ can be expressed by the corresponding formula from Table 3.1.

Proposition 3.3.2. *Let* (M, ω) *be a compact connected symplectic surface with a boundary* ∂M*, and let* $F\colon M \to \mathbb{R}$ *be a simple Morse function. Then the measure* $\mu := \pi_* \omega$ *is quasi-smooth.*

Proof. For vertices of types VI and VII this was proved in [13, Subsection I.1.2]. The proof is based on Theorem 2.2.1, the essence of the proof is the study of the area between the non-singular level sets of the function F and a singular F-level. The proof for other types follows the same lines, with the only difference that it uses both Theorems 2.2.1 and 2.3.1. Note that or vertices of types I and VII the function ψ is uniquely (and explicitly) determined by the corresponding function λ (see Theorems 2.2.1 and 2.3.1). In other cases ψ is determined by the corresponding function λ up to a function flat at the origin, and there is no explicit expression for ψ in terms of λ (see details in Toulet's thesis [32, Subsection 2.2]). □

The above properties of the measure μ make it natural to introduce the following definition of an abstract measured Reeb graph.

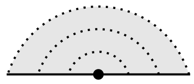
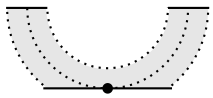
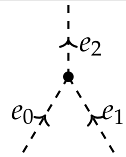
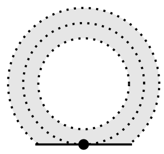
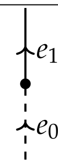
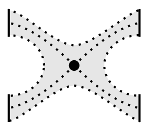
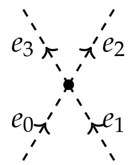
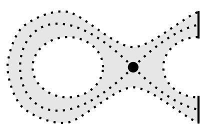
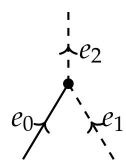
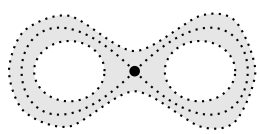
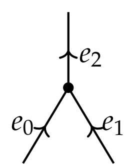
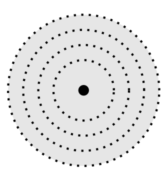
Type	Level Sets	Reeb Graph	Asymptotics
I			$\mu([v,x]) = \psi(f(x))\sqrt{\lvert f(x)\rvert}$, where $\psi(0) = 0$ and $\psi'(0) \neq 0$.
II			$\mu([v,x]) = \varepsilon_i\psi(f(x))\sqrt{\lvert f(x)\rvert} + \eta_i(f(x))$, where $\varepsilon_0 = \varepsilon_1 = -1$, $\varepsilon_2 = 2$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 = 0$.
III			$\mu([v,x]) = \varepsilon_i\psi(f(x))\sqrt{\lvert f(x)\rvert} + \eta_i(f(x))$, where $\varepsilon_0 = -1$, $\varepsilon_1 = 0$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 = 0$.
IV			$\mu([v,x]) = \varepsilon_i\psi(f(x))\ln\lvert f(x)\rvert + \eta_i(f(x))$, where $\varepsilon_0 = \varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 + \eta_3 = 0$.
V			$\mu([v,x]) = \varepsilon_i\psi(f(x))\ln\lvert f(x)\rvert + \eta_i(f(x))$, where $\varepsilon_0 = \varepsilon_1 = -1$, $\varepsilon_2 = 2$, $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 = 0$.
VI			$\mu([v,x]) = \varepsilon_i\psi(f(x))\ln\lvert f(x)\rvert + \eta_i(f(x))$, where $\varepsilon_0 = \varepsilon_1 = -1$, $\varepsilon_2 = 2$, $\psi(0) = 0$, $\psi(0) \neq 0$, and $\eta_0 + \eta_1 + \eta_2 = 0$.
VII			$\mu([v,x]) = \eta(f(x))$, where $\eta(0) = 0$ and $\eta'(0) \neq 0$.

Table 3.1: 7 types of neighborhoods of singular points with corresponding Reeb graphs and asymptotics for the measure on a Reeb graph (figures are partially taken from [17]). The notation $\mu([v,x])$ is a measure that is introduced below in Definition 3.8. In order to simplify notation we assume that $f(v) = 0$. If not, we replace f by $\tilde{f}(x) := f(x) - f(v)$.

Definition 3.3.3. A *measured Reeb graph* (Γ, f, μ) is a Reeb graph (Γ, f) equipped with a quasi-smooth measure μ.

Definition 3.3.4. Two measured Reeb graphs (Γ_1, f, μ) and (Γ_2, g, ν) are said to be *equivalent* by means of the isomorphism $\phi : \Gamma_1 \to \Gamma_2$ if the map ϕ :

(i) is an isomorphism between the Reeb graphs (Γ_1, f) and (Γ_2, g);

(ii) pushes the measure μ to the measure ν.

Definition 3.3.5. A measured Reeb graph (Γ, f, μ) is *compatible* with (M, ω) if the following conditions hold:

(i) The genus $g(\Gamma)$ of the graph Γ is equal to the genus $g(M)$ of the surface M.

(ii) The number $\sigma(\Gamma)$ of boundary cycles is equal to the number $\dim H_0(\partial M)$ of boundary components of the surface M.

(iii) The volume of Γ with respect to the measure μ is equal to the area of the surface M: $\int_\Gamma d\mu = \int_M \omega$.

We finish this section with the notion of a *smooth measure* on a Reeb graph.

Definition 3.3.6. A measure μ on an abstract Reeb graph (Γ, f) is called *smooth* if the following conditions hold.

1. The measure μ has a C^∞-smooth non-zero density $d\mu/df$ in the complement $\Gamma \setminus V(\Gamma)$.
2. In a neighbourhood of each vertex the measure μ can be expressed by the corresponding formula from Table 3.1 where instead of $\psi(\cdot)$ we plug in zero (i.e. all non-smooth terms disappear).

Proposition 3.3.7. *Let M be a compact connected orientable surface with a boundary ∂M, and let $F\colon M \to \mathbb{R}$ be a simple Morse function. Let also $\theta \in \Omega^2(M)$ be a non-negative 2-form such that the push-forward measure $\mu := \pi_*\theta$ has a non-zero density on each edge (i.e. the form θ is never identically zero along a level of F.)*

+ Then the measure μ is smooth if and only the support of θ does not include any critical points of the function F that correspond to the vertices of the following types: II, III, IV, V, VI (see Table 3.1).

Proof. It follows from the proof of Proposition 3.3.2 that non-smooth terms appear in the expression for the measure μ whenever we integrate the the form θ in a neighbourhood of a singular point (except types I and VII). □

Remark 3.3.8. It is possible to drop the conditions imposed on θ in the proposition above, but then the push-forward $\pi_*\theta$ is going to be a *signed measure* on a Reeb graph which is something we don't consider in this thesis.

3.4 CLASSIFICATION OF SIMPLE MORSE FUNCTIONS UP TO A SYMPLECTOMORPHISM

Theorem 3.4.1 ([24]). *Let M be a compact connected oriented surface with boundary ∂M. Then there is a one-to-one correspondence between simple Morse functions on M, considered up to symplectomorphism, and (isomorphism classes of) measured Reeb graphs compatible with M. In other words, the following statements hold.*

i) Let F, $G\colon M \to \mathbb{R}$ be two simple Morse functions. Then the following conditions are equivalent:

a) There exists a symplectomorphism $\Phi\colon M \to M$ such that $\Phi^* F = G$.

b) Measured Reeb graphs of F and G are isomorphic.

Moreover, every isomorphism $\phi\colon (\Gamma_F, f, \mu_F) \to (\Gamma_G, g, \mu_G)$ can be lifted to a symplectomorphism $\Phi\colon M \to M$ such that $\Phi^* F = G$.

ii) For each measured Reeb graph (Γ, f, μ) compatible with (M, ω) there exists a simple Morse function $F : M \to \mathbb{R}$ such that the corresponding measured Reeb graph Γ_F is isomorphic to (Γ, f, μ).

Remark 3.4.2. Note that the formulation of this theorem is identical to the formulation of Theorem 3.11 from [19]. The difference, of course, is that all notions in the present paper are extended to cover the case of surfaces with boundary.

Before we proceed with the proof of Theorem 3.4.1 let us formulate and prove the following.

Lemma 3.4.3. *Let M be a compact oriented surface (with or without boundary), and let $F\colon M \to \mathbb{R}$ be a simple Morse function on M. Denote by (Γ, f) the Reeb graph associated with the pair (M, F). Fix a quasi-smooth measure μ on Γ. Let v be a vertex of Γ. Then there exist a symplectic form $\omega \in \Omega^2(M)$ compatible with the orientation of M such that the push-forward measure $\mu_F = \pi_*\omega$ coincides with μ in some neighborhood of v.*

Proof. First of all, notice that we already proved this statement if v is a vertex of type I or type VII. In this case the local classification near a singular point is equivalent to the semi-local classification near a singular level. Also this statement was proved in [19] provided v is a vertex of type VI. Now, assume that v is a vertex of type III. Denote by O the singular point for $F|_{\partial M}$ such that $\pi(O) = v$. It follows from the proof of Theorem 2.3.1 that there exists a symplectic form $\omega_1 \in \Omega^2(M)$ such that $\mu - \pi_* \omega_1$ is a smooth measure in some neighborhood U of the vertex v. Indeed, near the point the point O we can use the chart (p, q) from Theorem 2.3.1 and define ω as $dp \wedge dq$. Recall that in the previous chapter we derived a formula 2.8 for the area bounded by a level curve:

$$A_{F,\omega,\ell}(\varepsilon) = \frac{4}{3}(-\lambda^{-1}(\varepsilon))^{3/2} + \text{smooth function of } \varepsilon.$$

We pick the function λ so that $\mu - \pi_* \omega_1$ is a smooth measure i.e. $\frac{4}{3}(\lambda^{-1}(\varepsilon))^{3/2} = \psi(\varepsilon)\sqrt{\varepsilon}$ for any $\varepsilon > 0$.

Fix a number $\varepsilon > 0$ that is small enough and consider two curves γ_1 and γ_2 given by the equations $\{p = \pm\varepsilon\}$ (see Figure 3.2). Denote by R_ε the part of M bounded by the curves $\gamma_{1,2}$ together with levels $\{F = \lambda(-\varepsilon^2\}$ and $\{F = \lambda(\varepsilon)\}$. The region R_ε is diffeomorphic to a closed disk. Consider a non-negative 2-form $\omega_2 \in \Omega^2(M)$ such that its support is contained in R_ε and the push-forward measure $\pi_* \omega_2$ has a smooth non-zero density $\frac{d\mu}{df}$ near v (we need this assumption to exclude ω_2 from being identically zero). Now define a 2-form $\tilde{\omega}_2$ using the formula:

$$\tilde{\omega}_2 = \frac{\frac{d(\mu - \pi_* \omega_1)}{df} \circ \pi}{\frac{d\pi_* \omega_2}{df} \circ \pi} \omega_2.$$

Notice that the compositions $\frac{d(\mu - \pi_* \omega_1)}{df} \circ \pi$ and $\frac{d\pi_* \omega_2}{df} \circ \pi$ are not necessarily smooth at the point O, but $\tilde{\omega}_2$ is smooth everywhere and its support is contained in R_ε.

It is clear that

$$\pi_* \tilde{\omega}_2 = \mu - \pi_* \omega_1,$$

or, equivalently,

$$\mu = \pi_*(\omega_1 + \tilde{\omega}_2).$$

Therefore, the form

$$\omega := \omega_1 + \tilde{\omega}_2$$

satisfies the required properties.

For other types of vertices the proof is analogous, the only difference is that instead of one region R_ε we need to consider up to four of them (for a vertex of type IV). $\square$

Proof. Let us prove the first statement. The implication $(a) \implies (b)$ is evident, so it suffices to prove the implication $(b) \implies (a)$. Let $\phi\colon \Gamma_F \to$

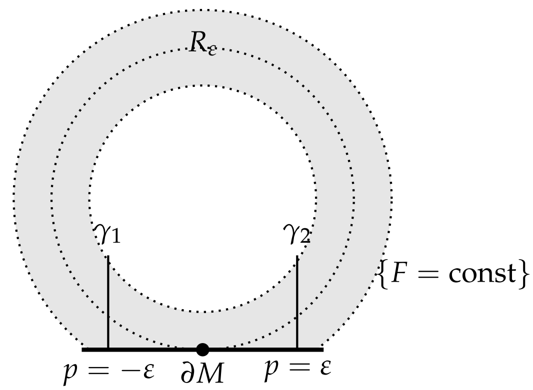


Figure 3.2: An illustration to the proof of lemma 3.4.3

Γ_G be an isomorphism of measured Reeb graphs. By π_F we denote the natural projection $M \to \Gamma_F$. We need to construct a symplectomorphism $\Phi\colon M \to M$ such that $\Phi^* F = G$ and $\pi_G \circ \Phi = \phi \circ \pi_F$.

Let $\ell \subset M$ be a smooth oriented curve which is transversal to the level sets of the function F, does not intersect the singular levels of the function F, and such that the function F is strictly increasing along the curve ℓ. Consider the Hamiltonian flow P_F^t corresponding to the function F. We denote by $T_F(Q)$ the time necessary to go from the curve ℓ to the point Q under the action of P^t; see Figure 3.3(i). The pair of functions (F, T_F) forms a coordinate system in some neighborhood of ℓ such that $\omega = \mathrm{d}F \wedge \mathrm{d}T_F$ (we did this computation above, see 2.4.14). The range of the function T_F along the non-critical level of F is a segment $[0, \Pi(F)]$ in the case when the F-level is a segment, and it is a half-interval $[0, \Pi(F))$ in the case when the F-level is a circle. The function Π is called a period. It follows from Stokes' theorem that $\Pi(F)$ is equal to the derivative $\frac{\mathrm{d}\mu}{\mathrm{d}f}$. In particular, this construction works for the boundary curve ∂M if we exclude all singular points for $F|_{\partial M}$; see Figure 3.3(ii).

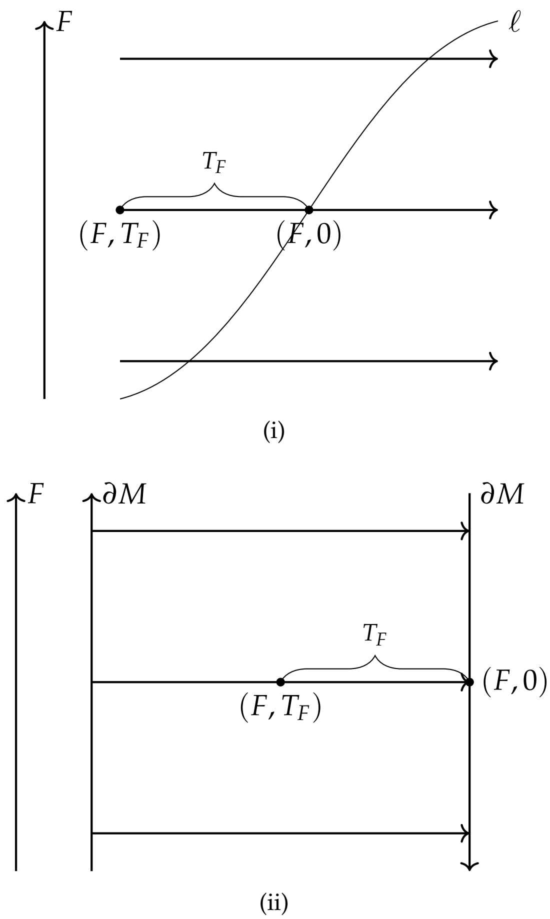


Figure 3.3: An illustration to the definition of the function T_F. Level curves are oriented by the Hamiltonian vector field v_F.

Let $e \subset \Gamma_F^d$ be a dashed edge. Let's apply the above construction in the of ℓ being the boundary curve. The formula $(F, T_F) \mapsto (G, T_G)$ defines a symplectomorphism from the interior of $\pi_F^{-1}(e)$ to the interior of $\pi_G^{-1}(\phi(e))$. The condition $\phi_* \mu_F = \mu_G$ guarantees that the periods of the functions T_F and T_G coincide and hence the symplectomorphism is well-defined. Now let $e' \subset \Gamma_F^s$ be a solid edge. Let $\ell \subset M$ be a smooth oriented curve which is transversal to the level sets of the function F. We also assume it does not intersect the singular levels of the function F; and the function F is strictly increasing along the curve ℓ. Then, as above, we obtain a symplectomorphism from the interior of $\pi_F^{-1}(e')$ to the interior of $\pi_G^{-1}(\phi(e'))$. By applying the same procedure to all edges of the graph Γ_F we obtain a symplectomorphism

$$\Phi_1 \colon \pi^{-1}[\Gamma_F \setminus V(\Gamma_F)] \to \pi^{-1}[\Gamma_G \setminus V(\Gamma_G)]$$

such that $\Phi_1^* F = G$ and $\pi_G \circ \Phi_1 = \phi \circ \pi_F$.

Now let O be a singular point for the function F or its restriction $F|_{\partial M}$. Then there is only one way to define the image of O:

$$\Phi(O) := \pi_G^{-1}(\phi(\pi_F(O))).$$

Let (p_F, q_F) (respectively, (p_G, q_G)) be a chart centered at the point O (respectively, $\Phi(O)$) as in Theorem 2.2.1 or 2.3.1. Then the condition $\phi_* \mu_F = \mu_G$ guarantees that the corresponding functions λ_F and λ_G are the same or they differ by a function flat at the origin. In the latter case it follows from Theorem 2.2.1 or 2.3.1 that we can replace the chart (p_F, q_F) with a chart $(\tilde{p}_F, \tilde{q}_F)$ such that $\tilde{\lambda}_F = \lambda_G$. So without loss of generality we may assume that $\lambda_F = \lambda_G$. Therefore, one can define Φ in some neighbourhood U_O of O by the formula

$$\Phi\colon (p_F, q_F) \mapsto (p_G, q_G).$$

This local symplectomorphism Φ extends uniquely to a semi-local symplectomorphism

$$\Phi\colon \pi_F^{-1}(\pi_F(U)) \to \pi_G^{-1}([\phi \circ \pi_F](U)).$$

Indeed, without loss of generality we may assume that $\pi_F^{-1}(\pi_F(U_O))$ is a "standard" neighbourhood of the singular level $\pi_F^{-1}(\pi_F(O))$ (see Table 3.1), i.e. it is a connected component of the set $\{P \in M\colon \left|F(P) - F(O)\right| < \varepsilon\}$ containing the point O and the number $\varepsilon > 0$ is sufficiently small so that these "standard" neighbourhoods for distinct O are pairwise disjoint. Denote by $U_{F,\varepsilon}$ the union of all these neighbourhoods. By applying the above procedure to all singular points of the function F or its restriction $F|_{\partial M}$ we obtain a symplectomorphism

$$\Phi_2\colon U_{F,\varepsilon} \to U_{G,\varepsilon}.$$

such that $\Phi_2^* F = G$ and $\pi_G \circ \Phi_2 = \phi \circ \pi_F$.

So the isomorphism $\phi\colon \Gamma_F \to \Gamma_G$ is lifted to a symplectomorphism

$$\Phi_1\colon \pi^{-1}[\Gamma_F \setminus V(\Gamma_F)] \to \pi^{-1}[\Gamma_G \setminus V(\Gamma_G)]$$

and to a symplectomorphism

$$\Phi_2\colon U_{F,\varepsilon} \to U_{G,\varepsilon}.$$

However, these two symplectomorphisms do not necessarily define a global symplectomorphism of the surface M. Let $e \subset \Gamma_F^d$ be a dashed edge. Then the intersection $U_{F,\varepsilon} \cap \pi^{-1}(e)$ is a disjoint union of two rectangles and the ratio $\Phi_2^{-1} \circ \Phi_1$ is a symplectic automorphism of this union preserving each component and also preserving the function F. The only symplectic automorphism of a fibered rectangle is the identity. So $\Phi_2^{-1} \circ \Phi_1 = \mathrm{Id}$ on $U_{F,\varepsilon} \cap \pi^{-1}(e)$ i.e. the symplectomorphisms Φ_2 and Φ_1 agree with each other on the preimage $\pi_F^{-1}(e)$ of the edge e. Now let $e \subset \Gamma_F^s$ be a solid edge. Then the intersection $U_{F,\varepsilon} \cap \pi^{-1}(e)$ is a disjoint union of two open cylinders and the ratio $\Phi_2^{-1} \circ \Phi_1$ is a symplectic automorphism of this union preserving each component and also preserving the function F. Any symplectic automorphism of a fibered cylinder is a Hamiltonian automorphism. The same holds for their union. The corresponding Hamiltonian function H extends (using a bump function) to a smooth function $\tilde{H}$ on all of M in such a way that its support is in the preimage of the edge e for all $t \in [0,1]$. Let us denote by Θ the Hamiltonian automorphism that corresponds to the function $\tilde{H}$. Notice that Θ is defined on all of M since $\tilde{H}$ is defined on all of M. Symplectomorphisms $\Phi_2 \circ \Theta$ and Φ_1 do agree with each other on the preimage $\pi_F^{-1}(e)$ of the edge e. By applying the same procedure to all solid edges of Γ_F we obtain a globally defined symplectomorphism $\Phi : M \to M$ such that $\Phi^* F = G$ and $\pi_G \circ \Phi = \phi \circ \pi_F$. This completes the proof of part i).

Now let us prove the second statement of the theorem. Given a triple (Γ, f, μ) we need to construct a quadruple $(\tilde{M}, \tilde{\pi}, \tilde{F}, \tilde{\omega})$ such that $\tilde{F} = f \circ \tilde{\pi}$ and $\tilde{\pi}_* \tilde{\omega} = \mu$. If this is done then

$$\int_{\tilde{M}} \tilde{\omega} = \int_{\Gamma} \mathrm{d}\mu = \int_M \omega$$

and it follows from Moser's theorem [27] that there is a diffeomorphism $\Phi : \tilde{M} \to M$ such that $\Phi^* \omega = \tilde{\omega}$ so that one can take $F = \tilde{F} \circ \Phi^{-1}$. It follows from [15] that there exists a surface $\tilde{M}$ with a simple Morse function $\tilde{F}$ and a projection $\tilde{\pi} : \tilde{M} \to \Gamma$ such that $\tilde{F} = f \circ \tilde{\pi}$. It remains to construct a symplectic form $\tilde{\omega}$ such that $\tilde{\pi}_* \tilde{\omega} = \mu$.

First of all, this problem is trivial if we restrict ourself to the part of the surface $\tilde{M}$ where the function $\tilde{F}$ has no critical points (e.g. a cylinder bounded by two levels of $\tilde{F}$). Indeed, we can start with any symplectic form, and then multiply it by the a ratio of two densities like we did in the proof of Lemma 3.4.3. For example, this argument works for complement of all singular levels since it is a union of cylinders and strips. Denote by W_1 the complement of all singular levels, and consider a symplectic form ω_1 defined in W_1 such that $\tilde{\pi}_* \omega_1 = \mu|_{\pi(W_1)}$. It follows from Lemma 3.4.3 that there exists a form ω_2 defined in a neighbourhood W_2 of the union of all singular levels such that $\tilde{\pi}_* \omega_2 = \mu|_{\pi(W_2)}$. Consider a partition of unity (ρ_1, ρ_2) subordinate to the open cover $\{W_1, W_2\}$ such that both functions ρ_1 and ρ_2 are constant along each level of $\tilde{F}$ (i.e. ρ_1 and ρ_2 descend to the functions on a Reeb graph). The form $\tilde{\omega} := \rho_1\omega_1 + \rho_2\omega_2$ is defined on all of $\tilde{M}$ and satisfies $\tilde{\pi}_*\tilde{\omega} = \mu$. $\square$

4 CLASSIFICATION OF GENERIC COADJOINT ORBITS OF SYMPLECTOMORPHISM GROUPS

In the present chapter we obtain a classification of generic coadjoint orbits for the group of generic coadjoint orbits for the action of symplectic (area-preserving) diffeomorphisms of compact symplectic surfaces with or without boundary.

4.1 FROM MORSE FUNCTIONS TO COADJOINT ORBITS

Throughout this section, let (M, ω) be a compact connected symplectic surface with boundary ∂M. By $\mathrm{SDiff}(M)$ we denote the Lie group[1] of all symplectomorphisms of M. Note that all elements of $\mathrm{SDiff}(M)$ map the boundary ∂M to itself, but do not necessarily preserve the boundary ∂M pointwise. The group $\mathrm{SDiff}(M)$ has the Lie algebra $\mathfrak{svect}(M)$ of divergence-free vector fields on M tangent to the boundary ∂M. The regular dual space $\mathfrak{svect}^*(M)$ can be identified with the space of cosets $\Omega^1(M)/d\Omega^0(M)$ (see Appendix). Moreover, the natural action of the group $\mathrm{SDiff}(M)$ on the space of cosets $\Omega^1(M)/\mathrm{d}\Omega^0(M)$ by means of pull-backs coincides with the coadjoint action of the group of symplectomorphisms $\mathrm{SDiff}(M)$:

$$\mathrm{Ad}^*_\Phi[\alpha] = [\Phi^*\alpha],$$

where $\Phi \in \mathrm{SDiff}(M)$ is a symplectomorphism and $\alpha \in \Omega^1(M)$ is a 1-form.

Define the exterior derivative operator d on the space of cosets $\{\alpha + \mathrm{d}f \,|\, f \in C^\infty(M)\}$ by the formula $\mathrm{d}[\alpha] := \mathrm{d}\alpha$. (This operator is well defined on cosets since $\mathrm{d}(\alpha + \mathrm{d}f) = \mathrm{d}\alpha$.) Consider the following mapping:

$$\mathrm{curl}\colon \Omega^1(M)/\mathrm{d}\Omega^0(M) \to C^\infty(M),$$

1 See [20, Chapter I, Section 1.1] for details on Lie groups and Lie algebras in an infinite-dimensional setting.

defined by taking a *vorticity function* $d\alpha/\omega =: \mathrm{curl}[\alpha]$. It is easy to see that if the boundary ∂M of the surface M is not empty then the mapping curl is a surjection. In the case of a closed surface M there is a relation:

$$\int_M \mathrm{curl}[\alpha]\omega = 0$$

and the mapping curl is surjective onto the space of zero-mean functions.

Suppose that cosets $[\alpha]$ and $[\beta]$ belong to the same coadjoint orbit of $\mathrm{SDiff}(M)$. Then by definition, there is a symplectomorphism Φ such that $[\Phi^*\beta] = [\alpha]$ and the following diagram is commutative:

$$\begin{tikzcd} [\beta] \arrow[r, "\Phi^*"] \arrow[d, "\mathrm{curl}"'] & [\alpha] \arrow[d, "\mathrm{curl}"] \\ \mathrm{curl}[\beta] \arrow[r, "\Phi^*"] & \mathrm{curl}[\alpha] \end{tikzcd}$$

Definition 4.1.1. A coset $[\alpha] \in \Omega^1(M)/d\Omega^0(M)$ is called *simple Morse* if $\mathrm{curl}[\alpha]$ is a simple Morse functions. A coadjoint orbit $\mathcal{O}$ is called *simple Morse* if some (and hence every) coset $[\alpha] \in \mathcal{O}$ is simple Morse.

With every simple Morse coset $[\alpha] \in \Omega^1(M)/d\Omega^0(M)$ one can associate a measured Reeb graph $\Gamma_{\mathrm{curl}[\alpha]}$. If two simple Morse cosets $[\alpha]$ and $[\beta]$ belong to the same coadjoint orbit, then the corresponding Reeb graphs are isomorphic.

Suppose that cosets $[\alpha]$ and $[\beta]$ have isomorphic Reeb graphs. Then it follows from Theorem 3.4.1 that there exists a symplectomorphism Φ such that $\Phi^*\mathrm{curl}[\beta] = \mathrm{curl}[\alpha]$. Therefore, the 1-form $\Phi^*[\beta] - [\alpha]$ is closed. Since this 1-form is not necessarily exact, the cosets $[\alpha]$ and $[\beta]$ do not necessarily belong to the same coadjoint orbit. Nevertheless, we conclude that the space of coadjoint orbits corresponding to the same measured Reeb graph is finite-dimensional and its dimension is at most $\dim \mathrm{H}^1(M)$. Throughout this section, unless otherwise stated, all (co)homology groups will be with coefficients in $\mathbb{R}$.

4.2 CIRCULATION FUNCTIONS ON A REEB GRAPH

In [19] the notion of a circulation function was introduced for the case of closed surfaces. In the case of a surface with boundary, we need a modification of that definition. Take a point $x \in \Gamma_F^s$ which is not a vertex. Then $\pi^{-1}(x)$ is a circle C. It is naturally oriented as the boundary of the set of smaller values of the function F. The integral of a coset $[\alpha]$ over C is well-defined. Thus, we obtain a function

$$\mathcal{C}_{[\alpha]} \colon \Gamma_F^s \setminus V(\Gamma_F^s) \to \mathbb{R},$$

defined by $\mathcal{C}_{[\alpha]}(x) = \int_{\pi^{-1}(x)} \alpha$.

Proposition 4.2.1 ([19]). *The function $\mathcal{C}_{[\alpha]} = \int_{\pi^{-1}(x)} \alpha$ has the following properties.*

(i) Assume that x an y are two interior points of some edge $e \subset \Gamma_F^s$, and that e is pointing from x towards y. Then $\mathcal{C}_{[\alpha]}$ satisfies the Newton-Leibniz formula

$$\mathcal{C}_{[\alpha]}(y) - \mathcal{C}_{[\alpha]}(x) = \int_x^y f d\mu$$

(ii) for all vertices of Γ^s which do not belong to Γ^d the function $\mathcal{C}_{[\alpha]}$ satisfies the Kirchhoff rule at v:

$$\sum_{e \to v} \lim_{x \xrightarrow{e} v} \mathcal{C}_{[\alpha]}(x) = \sum_{e \leftarrow v} \lim_{x \xrightarrow{e} v} \mathcal{C}_{[\alpha]}(x)\,, \tag{4.1}$$

where the notation $e \to v$ stands for the set of edges pointing at the vertex v, and $e \leftarrow v$ stands for the set of solid edges pointing away from v.

Note that the function f on the subgraph Γ_F^s can be recovered from the circulation function $\mathcal{C}$ by the formula: $f = \mathrm{d}\mathcal{C}/\mathrm{d}\mu$. It follows from Proposition 4.2.1 that the difference $\mathrm{C}_{[\alpha]} - \mathrm{C}_{[\beta]}$ is as an element of the relative homology group $\mathrm{H}_1(\Gamma_F, \Gamma_F^d)$.

The above properties of the circulation function $\mathcal{C}_{[\alpha]}$ make it natural to introduce the following definition of an abstract circulation function.

Definition 4.2.2. Let (Γ, f, μ) be a measured Reeb graph. Any function $\mathcal{C} \colon \Gamma^s \setminus V(\Gamma^s) \to \mathbb{R}$ satisfying the properties listed in Proposition 4.2.1 is called a *circulation function (an antiderivative)*.

Proposition 4.2.3. *Let (Γ, f, μ) be a measured Reeb graph.*

i) If the subgraph Γ^d is not empty, then the pair (f, μ) on Γ admits an antiderivative.

ii) If the subgraph Γ^d is empty, then the pair (f, μ) on Γ admits an antiderivative if and only if $\int_\Gamma f \mathrm{d}\mu = 0$.

iii) If the pair (f, μ) admits an antiderivative, then the set of antiderivatives of (f, μ) is an affine space whose underlying vector space is the relative homology group $\mathrm{H}_1(\Gamma, \Gamma^d)$.

4.3 AUXILIARY CLASSIFICATION RESULT

In this section we follow [16]. Let (M, ω) be a symplectic surface with boundary ∂M. Denote by $CB(M) \subset C^\infty(M)$ the space of Morse functions on M constant on the boundary ∂M, and without critical points at the boundary ∂M. Elements of $CB(M)$ are called functions of $CB-$type.

Definition 4.3.1. A coset $[\alpha] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$ is said to be of *CB-type* if $\mathrm{curl}[\alpha] \in CB(M)$. A coadjoint orbit $\mathcal{O}$ called to be of *CB-type* if some (and hence every) coset $[\alpha] \in \mathcal{O}$ is of CB-type.

All definitions from the present paper such as Reeb graph, compatibility conditions, circulation graph, etc. can be modified for the case of functions and cosets of $CB-$type, see details in [16]. The result we are interested in can be formulated as follows.

Theorem 4.3.2 ([16])**.** *Let M be a connected symplectic surface with or without boundary. Then coadjoint orbits of* $\mathrm{SDiff}(M)$ *of $CB-$type are in one-to-one correspondence with (isomorphism classes of) circulation graphs $(\Gamma, f, \mu, \mathcal{C})$ compatible with M. In other words, the following statements hold:*

i) For a symplectic surface M and cosets of $CB-$type $[\alpha], [\beta] \in \mathfrak{svect}^*(M)$ the following conditions are equivalent:

  a) $[\alpha]$ and $[\beta]$ lie in the same orbit of the $\mathrm{SDiff}(M)$ *coadjoint action;*

  b) circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to the cosets $[\alpha]$ and $[\beta]$ are isomorphic.

ii) For each circulation graph Γ which is compatible with M, there exists a generic $[\alpha] \in \mathfrak{svect}^*(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, f, \mu, \mathcal{C})$.

4.4 AUGMENTED CIRCULATION GRAPH

In the case of surfaces with boundary circulation functions do not form a complete set of invariants for coadjoint orbits, i.e. the equality $C_{[\alpha]} = C_{[\beta]}$ does not in general imply that cosets α and β belong to the same coadjoint orbit.

Example 4.4.1. Consider the disk with two holes from Figure 3.1(a). In this case there are no circulation functions since there are no solid edges in the Reeb graph. On the other hand, in this case there are no nontrivial symplectomorphisms preserving the function hence the dimension of the space of coadjoint orbits is equal to the first Betti number of the surface, i.e. it is equal to two.

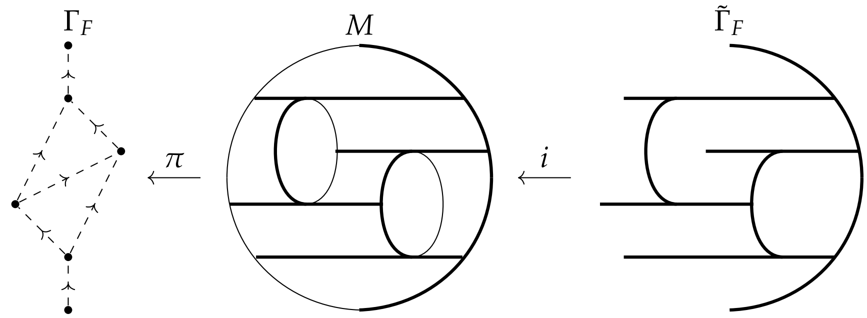


Figure 4.1: An illustration to the definition of the graph $\tilde{\Gamma}_F$.

It turns out that it is possible to define some additional invariants: integrals of cosets over certain cycles associated with the pair (M, F) in an invariant way.

There is a unique way to lift each edge $e \subset \Gamma_F^d$ to a smooth oriented (and diffeomorphic to a segment) curve $\tilde{e} \subset \partial M$ such that

i) $\pi(\tilde{e}) = e$;

ii) for each $x \in e \setminus \partial e$ the regular F-level $\pi^{-1}(x)$ is pointed in the direction of the curve $\tilde{e}$.

We define the subset $\tilde{E}_F \subset M$ to be the union

$$\tilde{E}_F := \bigcup_{e \in E(\Gamma_F^d)} \tilde{e}.$$

We also define the subset $\tilde{V}_F \subset M$ to be

$$\tilde{V}_F := \pi^{-1}(V(\Gamma_F^d) \setminus \partial\Gamma_F^d)$$

where $\partial\Gamma_F^d$ is the set of boundary vertices (i.e. vertices of types I, III, or V) of the graph Γ_F^d. And, finally, define the subset $\tilde{\Gamma}_F$ to be the union of $\tilde{E}_F$ and $\tilde{V}_F$ (see Figure 4.1). The set $\tilde{\Gamma}_F$ is a topological graph embedded into the surface M. We denote by i the inclusion $\tilde{\Gamma}_F \hookrightarrow M$.

Lemma 4.4.2. *The map $\pi \circ i \colon \tilde{\Gamma}_F \to \Gamma_F^d$ is a homotopy equivalence.*

Let $[\alpha] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$ be a coset of a one-form. There is a natural way to define the restriction $i^*[\alpha] \in \mathrm{H}^1(\tilde{\Gamma}_F)$. First, we define the restriction $i^*\alpha$ as a one-cochain such that $i^*\alpha(e) := \int_e \alpha$ for each edge $e \subset \tilde{\Gamma}_F$. Now we take $i^*[\alpha] := [i^*\alpha]$. The cohomology class $i^*[\alpha]$ is well-defined since each exact one-form $\mathrm{d}f$ restricts to the exact one-cochain $i^*\mathrm{d}f$. It follows from Lemma 4.4.2 that $i^* \circ \pi^* \colon \mathrm{H}^1(\Gamma_F) \to \mathrm{H}^1(\tilde{\Gamma}_F)$ is an isomorphism. Hence with each coset $[\alpha] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$ we can also associate an element

$\xi_{[\alpha]} \in \mathrm{H}^1(\Gamma_F)$ defined by the formula $\xi_{[\alpha]} := (i^* \circ \pi^*)^{-1}(i^*[\alpha])$. Next, we generalize the notion of a circulation graph from [19].

Definition 4.4.3. A measured Reeb graph (Γ, f, μ) endowed with a circulation function $\mathcal{C}$ and an element $\xi \in \mathrm{H}^1(\Gamma^d)$ is called a *augmented circulation graph* $(\Gamma, f, \mu, \mathcal{C}, \xi)$.

We demonstrated above that with each coset $[\alpha]$ one can associate an augmented circulation graph $\Gamma_{[\alpha]}$. Two augmented circulation graphs are isomorphic if they are isomorphic as measured Reeb graphs, and the isomorphism between them preserves all additional data. An augmented circulation graph $(\Gamma, f, \mu, \mathcal{C}, \xi)$ is *compatible* with a symplectic surface (M, ω) if the corresponding measured Reeb graph (Γ, f, μ) is compatible with (M, ω) (see Definition 3.3.5).

4.5 COADJOINT ORBITS OF SYMPLECTOMORPHISM GROUPS

Theorem 4.5.1 ([24, 19]). *Let (M, ω) be a connected symplectic surface with or without boundary. Then generic coadjoint orbits of* $\mathrm{SDiff}(M)$ *are in one-to-one correspondence with (isomorphism classes of) augmented circulation graphs $(\Gamma, f, \mu, \mathcal{C})$ compatible with M. In other words, the following statements hold:*

i) For a symplectic surface M and generic cosets $[\alpha], [\beta] \in \mathfrak{svect}^*(M)$ the following conditions are equivalent:

a) $[\alpha]$ and $[\beta]$ lie in the same orbit of the $\mathrm{SDiff}(M)$ *coadjoint action;*

b) augmented circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to the cosets $[\alpha]$ and $[\beta]$ are isomorphic.

ii) For each augmented circulation graph Γ which is compatible with M, there exists a generic $[\alpha] \in \mathfrak{svect}^*(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, f, \mu, \mathcal{C}, \xi)$.

Corollary 4.5.2. *The space of coadjoint orbits of the group* $\mathrm{SDiff}(M)$ *corresponding to the same measured Reeb graph (Γ, f, μ) is a finite-dimensional affine space and its dimension is* $\dim \mathrm{H}_1(\Gamma, \Gamma^d) + \dim \mathrm{H}_1(\Gamma^d)$.

Remark 4.5.3. It follows from the long exact sequence for the pair (Γ, Γ^d) that

$$\dim \mathrm{H}_1(\Gamma, \Gamma^d) + \dim \mathrm{H}_1(\Gamma^d) = \dim \mathrm{H}_1(\Gamma) - \dim \mathrm{H}_0(\Gamma^d) + 1.$$

Therefore, the space of coadjoint orbits of the group $\mathrm{SDiff}(M)$ corresponding to the same measured Reeb graph (Γ, f, μ) has dimension $\dim \mathrm{H}_1(\Gamma)$ in the case when the subgraph Γ^d is connected.

Example 4.5.4. Consider the torus with one boundary component from Figure 1.2 with the height function F on it, and the corresponding Reeb graph Γ_F. In this case $\mathrm{H}_1(\Gamma_F^d) = 0$ and $\mathrm{H}_1(\Gamma_F, \Gamma_F^d) = 1$. Therefore, the corresponding space of coadjoint orbits is one-dimensional.

Before we proceed with the proof of Theorem 4.5.1 let us formulate and prove two lemmas.

Lemma 4.5.5. *Let M be a connected oriented surface with non-empty boundary, and let F be a simple Morse function on M. Then*

$$\dim \mathrm{H}_1(M_F^d) = \dim \mathrm{H}_1(\Gamma_F^d) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d).$$

Proof. Let $\tilde{M}$ be the smooth surface obtained from the surface M_F^d by contracting each circle in $M_F^d \cap M_F^s$ to a point. It is clear that

$$\dim \mathrm{H}_1(M_F^d) = \dim \mathrm{H}_1(\tilde{M}) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d).$$

Let p be the canonical projection $M \to \tilde{M}$. The function F descends to a simple Morse function $\tilde{F} \colon \tilde{M} \to \mathbb{R}$ such that $F = \tilde{F} \circ p$. The Reeb graph $\Gamma_{\tilde{F}}$ consists only of dashed edges, and it is coincides with Γ_F^d. Then the surface $\tilde{M}$ is homotopy equivalent to graph $\Gamma_{\tilde{F}}$. Therefore,

$$\dim \mathrm{H}_1(M_F^d) = \dim \mathrm{H}_1(\tilde{M}) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d) = \\ \dim \mathrm{H}_1(\Gamma_{\tilde{F}}) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d) = \dim \mathrm{H}_1(\Gamma_F^d) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d).$$

□

Lemma 4.5.6. *Let M be a connected oriented surface possibly with boundary, and let F be a simple Morse function on M. Assume that $[\gamma] \in \mathrm{H}^1(M)$ is such that the integral of γ over any F-level vanishes, and $\xi_{[\gamma]}$ is a zero element in $\mathrm{H}^1(\Gamma_F^d)$. Then there exists a C^∞ function $H \colon M \to \mathbb{R}$ (with zero restriction on the surface M_F^d) such that the one-form $H\mathrm{d}F$ is closed, and its cohomology class is equal to $[\gamma]$. Moreover, H can be chosen in such a way that the ratio H/F is a smooth function.*

Proof. Denote by i_d the inclusion $M_F^d \cap M_F^s \hookrightarrow M_F^d$, and denote by π_d the restriction of the projection $\pi \colon M \to \Gamma_F$ on the surface M_F^d. Note that the homomorphism $(\pi_d)_* \colon \mathrm{H}_1(M_F^d) \to \mathrm{H}_1(\Gamma_F^d)$ is a surjection, and $\mathrm{Im}\,(i_d)_* \subset \mathrm{Ker}\,(\pi_d)_*$. It follows from Lemma 4.5.5 that

$$\dim \mathrm{H}_1(M_F^d) = \dim \mathrm{H}_1(\Gamma_F^d) + \dim \mathrm{H}_0(\Gamma_F^s \cap \Gamma_F^d).$$

Hence the image of the homomorphism $(\pi_d)_*$ coincides with the kernel of the homomorphism $(\pi_d)_*$. From above we conclude that the homomorphism $\pi_*\colon \mathrm{H}^1(\Gamma_F^d) \to \mathrm{H}^1(M_F^d)$ is an injection, and $\operatorname{Im}(\pi_d)^* = \operatorname{Ker}(i_d)^*$.

Denote by i the inclusion $M_F^d \hookrightarrow M_F$. Since the integral of $[\gamma]$ over any connected component of any closed F-level vanishes and $\xi_{[\gamma]}$ is a zero element in $\mathrm{H}^1(\Gamma_F^d)$, the cohomology class $[i^*\gamma]$ is a zero element in $\mathrm{H}^1(M_F^d)$. Consider the long exact cohomology sequence for the pair (M_F, M_F^d):

$$0 \to \mathrm{H}^0(M_F^d) \to \mathrm{H}^0(M_F) \to \mathrm{H}^1(M_F, M_F^d) \to \mathrm{H}^1(M_F) \to \mathrm{H}^1(M_F^d) \to 0.$$

The cohomology class $[\gamma]$ on M belongs to the kernel of the homomorphism $i^*\colon \mathrm{H}^1(M_F) \to \mathrm{H}^1(M_F^d)$. Hence it belongs to the image of the homomorphism $\mathrm{H}^1(M_F, M_F^d) \to \mathrm{H}^1(M_F)$, i.e. there exists a one-form $\tilde{\gamma}$ such that $[\tilde{\gamma}] = [\gamma]$ and $\tilde{\gamma}|_{M_F^d} = 0$.

Denote by π_s the restriction of the projection $\pi\colon M \to \Gamma_F$ on the surface M_F^s. The homomorphism $(\pi_s)_*\colon \mathrm{H}_1(M_F^s) \to \mathrm{H}_1(\Gamma_F^s)$ is a surjection, and its kernel is generated by those homology classes which are homologous to regular F-levels. From above we conclude that the homomorphism $(\pi_s)^*\colon \mathrm{H}^1(\Gamma_F^s) \to \mathrm{H}^1(M_F^s)$ is an injection, and $\operatorname{Im}(\pi_s)^* = \operatorname{Ann}\operatorname{Ker}(\pi_s)_*$ where

$$\operatorname{Ann}\operatorname{Ker}(\pi_s) := \left\{\omega \in \mathrm{H}^1(\Gamma_F^s) \mid \omega(c) = 0 \iff c \in \operatorname{Ker}(\pi_s)_*\right\}.$$

Therefore, there exists a one-cochain $\alpha \in \mathrm{H}^1(\Gamma_F^s)$ of the form $\alpha = \sum_{e\in E(\Gamma_F^s)} \alpha_e e^*$ such that $[\tilde{\gamma}] = (\pi_s)^*[\alpha]$.

Recall that the function f is the pushforward of the function F to the graph Γ_F. Consider a continuous function $h\colon \Gamma_F \to \mathbb{R}$ such that

i) it is a smooth function of f in a neighborhood of each point $x \in \Gamma_F$;

ii) it vanishes whenever x is sufficiently close to a vertex;

iii) $h|_{\Gamma_F^d} = 0$;

iv) for each edge e, we have

$$\alpha(e) = \int_e h \mathrm{d}f.$$

Obviously, such a function does exist. Now, lifting h to M, we obtain a smooth function H with the desired properties. □

Proof of Theorem 4.5.1. Let us prove the first statement. The implication (a) $\Rightarrow$ (b) is immediate, so it suffices to prove the implication (b) $\Rightarrow$ (a). Let $\phi\colon \Gamma_{[\alpha]} \to \Gamma_{[\beta]}$ be an isomorphism of augmented circulation graphs. By

Theorem 3.4.1, ϕ can be lifted to a symplectomorphism $\Phi\colon M \to M$ that maps the function $F = \mathrm{Diff}[\alpha]$ to the function $G = \mathrm{Diff}[\beta]$. Therefore, the 1-form γ defined by

$$\gamma = \Phi^*\beta - \alpha$$

is closed.

Assume that $\Psi\colon M \to M$ is a symplectomorphism which maps the function F to itself and is isotopic to the identity. Then the composition $\widetilde{\Phi} = \Phi \circ \Psi^{-1}$ maps F to G, and

$$[\widetilde{\Phi}^*\beta - \alpha] = [\Phi^*\beta - \Psi^*\alpha] = [\gamma] - [\Psi^*\alpha - \alpha].$$

We claim that Ψ can be chosen in such a way that $\widetilde{\Phi}^*\beta - \alpha$ is exact, i.e. one has the equality of the cohomology classes

$$[\Psi^*\alpha - \alpha] = [\gamma].$$

Moreover, we show that there exists a time-independent symplectic vector field X that preserves F and satisfies

$$[\Psi_t^*\alpha - \alpha] = t[\gamma]\,, \tag{4.2}$$

where Ψ_t is the flow of X. Differentiating (4.2) with respect to t, we get in the left-hand side

$$[\Psi_t^* L_X\alpha] = [L_X\alpha] = [i_X \mathrm{d}\alpha] = [F \cdot i_X\omega]\,,$$

since the form $L_X\alpha$ is closed and Ψ_t^* does not change its cohomology class. Thus

$$[F \cdot i_X\omega] = [\gamma]. \tag{4.3}$$

Since Φ preserves the circulation function, the integrals of γ over all connected components of F-levels vanish. In addition, $\xi_{[\Phi^*\alpha]} = \xi_{[\alpha]}$. Therefore, by Lemma 4.5.6, there exists a smooth function H such that

$$[\gamma] = [H\mathrm{d}F].$$

Now we set

$$X := \frac{H}{F}\,\omega^{-1}\mathrm{d}F.$$

It is easy to see that the vector field X is zero on M^d, symplectic, preserves the levels of F, and satisfies the equation (4.3). Therefore, its phase flow map $\Psi = \Psi_1$ has the required properties.

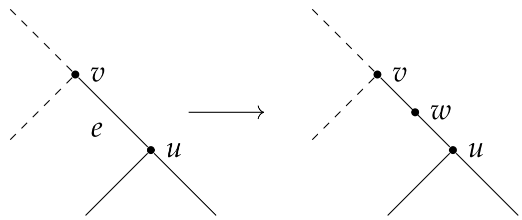


Figure 4.2: The illustration to the construction of the graph Γ'. The vertex w subdivides the edge e with endpoints $\{v, u\}$ into the edges $e_{v\to w}$ and $e_{w\to u}$.

Now let us prove the second statement. It follows from Theorem 3.4.1 that there exists a symplectic surface (M, ω) and a simple Morse function $F \colon M \to \mathbb{R}$ such that $\Gamma_F = \Gamma$. Consider the surface M^s and the restriction $F|_{M^s}$ of the function F to the surface M^s. The restriction $F|_{M^s_F}$ is a Morse function, and it is constant on the boundary ∂M^s_F since it is formed by some of closed F-levels. However, it is not necessarily a function of CB-type since it has hyperbolic critical points on the boundary whenever the graph Γ_F has vertices of type V. In order to apply the Theorem 4.3.2 we need to 'cut out' from M^s these hyperbolic critical points. Let $v \in \Gamma_F$ be a vertex of type V, let $e \subset \Gamma_F$ be the only solid edge incident to v, and also let $u \in \Gamma^s_F$ be the only other vertex adjacent to e. The edge e, with endpoints $\{v, u\}$ can be uniquely subdivided into two edges, say $e_{v\to w}$ and $e_{w\to u}$, connecting to a new vertex w such that $\mu(e_{v\to w}) = \mu(e_{w\to u})$. After that we cut out the edge $e_{v\to w}$. Denote by Γ' the (abstract) measured Reeb graph obtained by applying the above procedure to all vertices of type V in the graph Γ_F (see Figure 4.2). Denote by $M' \subset M^s_F$ the preimage $\pi^{-1}(\Gamma')$ It is clear from the above that the restriction $F|_{\tilde{M}}$ is a function of CB-type. Therefore, it follows from Theorem 4.3.2 that there exists a one-form α_0 on M' such that $\mathcal{C}_{[\alpha_0]} = \mathcal{C}|_{\Gamma'}$. It is clear that the form α_0 can be extended (along the cylinders $M^s \setminus M'$) on all of M^s in such a way that $\mathcal{C}_{[\alpha_0]} = \mathcal{C}|_{\Gamma}$. On the other hand, there exists a one-form α_1 on M such that $[(i_d)^*\alpha_1] = [(i_s)^*\alpha_0]$ and $\xi_{[\alpha]} = \xi$ since

$$\dim \mathrm{H}_1(M^d_F) = \dim \mathrm{H}_1(\Gamma^d_F) + \dim \mathrm{H}_1(M^d_F \cap M^s_F).$$

Using an appropriate partition of unity we construct a one-form α (as a combination of one-forms α_0 and α_1) such that $\mathcal{C}_{[\alpha]} = \mathcal{C}_{[\alpha_0]} = \mathcal{C}$ and $\xi_{[\alpha]} = \xi_{[\alpha_1]} = \xi$. Hence the augmented circulation graph $\Gamma_{[\alpha]}$ coincides with Γ. $\square$

4.6 REMARKS AND OPEN QUESTIONS

In this chapter we classified generic coadjoint orbits of symplectomorphism groups of surfaces. One should mention two other relevant but not overlapping with us classification results for symplectic surfaces:

1. Dufour, Molino, and Toulet classified in [13] simple Morse fibrations on closed symplectic surfaces under the action of symplectic diffeomorphisms.

2. Bolsinov [5] and Kruglikov [26] classified Hamiltonian vector fields tangent to the boundary on surfaces up to the action of arbitrary diffeomorphisms.

It would be interesting to extend those classifications for surfaces with boundary. (Note that in [26] in the contrast with the present work Hamiltonian functions are assumed to be constant on the boundary.) It also would be very interesting to classify Morse functions and Morse orbits for the action of

1. the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of a surface M;

2. the connected component $\mathrm{SDiff}_0(M)$ of the identity in the group $\mathrm{SDiff}(M)$ for the case of surfaces M with boundary.

This would generalise the corresponding results of [19] and the present work to these important subgroups of the symplectomorphism groups.

5

CLASSIFICATION OF FUNCTIONS AND ORBITS FOR NON-ORIENTABLE SURFACES

In the present chapter we generalize the results of Chapters 3 and 4 to the case of non-orientable surfaces.

This chapter is organized as follows. In Section 5.1 we discuss certain notions of differential geometry relevant for non-orientable manifolds. In Section 5.2 we present a classification of simple Morse pseudo-functions on non-orientable surfaces up to an area-preserving transformation. In Section 5.3 we obtain a classification of generic coadjoint orbits for the group of area-preserving diffeomorphisms of a non-orientable surface.

5.1 GEOMETRY OF NON-ORIENTABLE MANIFOLDS

5.1.1 *Orientation double cover and orientation bundle*

Let N be a non-orientable manifold. We define its *orientation double cover* $\widetilde{N}$ as the space of pairs $(x, \mathcal{O})$, where $x \in N$, and $\mathcal{O}$ is an orientation of the tangent space $T_x N$. Clearly, $\widetilde{N}$ is an orientable manifold. Moreover, defined in this way, the orientation double cover $\widetilde{N}$ is *canonically oriented*: the orientation of $T_{(x,\mathcal{O})}\widetilde{N}$ is defined by pulling back the orientation $\mathcal{O}$ from $T_x N$.

Let N be a non-orientable manifold, and $\widetilde{N}$ be its double cover. Then there is a fixed-point-free orientation-reversing involution $I \colon \widetilde{N} \to \widetilde{N}$ which interchanges the points in each fiber of the projection $\widetilde{N} \to N$. Conversely, let M be an oriented *connected* manifold equipped with a fixed-point-free orientation-reversing involution I. Then the quotient space $M \,/\, I$ is a non-orientable manifold whose orientation double cover is canonically diffeomorphic to M. Thus, one can go back and forth between connected non-orientable manifolds and connected oriented manifolds equipped with a fixed-point-free orientation-reversing involution. We will be using both models interchangeably throughout the paper. Note that while, in principle, everything can be done on the double cover, in some cases we found it more convenient to directly work with the non-orientable manifold.

Closely related to the orientation double cover is the concept of *orientation bundle*. Viewing the orientation double cover $\widetilde{N} \to N$ as a principal $\mathbb{Z}_2$-bundle, define the orientation bundle as the associated line bundle. In other words, given an atlas on N, the orientation bundle is given by transition functions $\operatorname{sign}(J(\phi_{\alpha\beta}))$ where $\phi_{\alpha\beta}$ are transition maps between charts, and J is the Jacobian determinant. We denote the orientation bundle of N by $o(N)$.

5.1.2 *Differential forms of even and odd type*

The language of even and odd differential forms was introduced by de Rham [10]. Let N be a non-orientable manifold. *Differential k-forms of even type* are just regular differential k-forms, i.e. sections of $\bigwedge^k T^*N$. *Differential k-forms of odd type*, also called *pseudo-forms*, are regular differential forms twisted by the orientation bundle $o(N)$, i.e. sections of $\bigwedge^k T^*N \otimes o(N)$. In more concrete terms, odd forms can be defined as follows:

Definition 5.1.1. An *odd k-form* α on a vector space V assigns to each orientation $\mathcal{O}$ of V an exterior k-form $\alpha_{\mathcal{O}}$ such that if the orientation is reversed the exterior form is replaced by its negative: $\alpha_{-\mathcal{O}} = -\alpha_{\mathcal{O}}$. An odd differential k-form on a manifold N assigns an odd k-form α to each tangent space T_xN in a smooth fashion. Non-vanishing odd forms of top degree are called *densities*. Odd 0-forms are called *pseudo-functions*.

Denote the space of even (i.e. usual) k-forms on a manifold N by $\Omega^k(N)$ and odd k-forms by $\widetilde{\Omega}^k(N)$.

Example 5.1.2. Let (N, g) be a Riemannian manifold of dimension n (orientable or not). Then we have an associated Riemannian density $\rho_g \in \widetilde{\Omega}^n(N)$, defined as the unique odd n-form that assigns to an orientation $\mathcal{O}$ of the tangent space T_pN and an orthonormal basis in T_pN the value $+1$ if the orientation of the basis agrees with $\mathcal{O}$ and -1 otherwise. In terms of local coordinates the density ρ_g has the following expression:

$$\rho_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

(Note that locally we can always write odd forms as usual differential forms, since local coordinates define a local trivialization of the orientation bundle.)

Assuming N to be non-orientable, there is a one-to-one correspondence between even (respectively, odd) differential forms on N and those differential forms on the orientation double cover $\widetilde{N}$ that are even (respectively, odd) with respect the involution $I\colon \widetilde{N} \to \widetilde{N}$:

$$\Omega^k(N) \simeq \Omega^k_{even}(\widetilde{N}) = \{\omega \in \Omega^k(\widetilde{N})\colon I^*\omega = \omega\}, \tag{5.1}$$

$$\widetilde{\Omega}^k(N) \simeq \Omega^k_{odd}(\widetilde{N}) = \{\omega \in \Omega^k(\widetilde{N})\colon I^*\omega = -\omega\}. \tag{5.2}$$

These isomorphisms are provided by pull-backs via the projection $\widetilde{N} \to N$. Note that although the pull-back of a pseudo-form on N is a pseudo-form on $\widetilde{N}$, i.e. a section of $\bigwedge^k T^*\widetilde{N} \otimes o(\widetilde{N})$, the canonical orientation on $\widetilde{N}$ gives us a non-vanishing section $\mathbf{1}$ of $o(\widetilde{N})$ (given by assigning $+1$ to every chart in a positively oriented atlas) and hence an identification between forms and pseudo-forms given by $\omega \mapsto \omega \otimes \mathbf{1}$. Since the pull-back of a pseudo-form (as well as any other object) on N to $\widetilde{N}$ is even, and the section $\mathbf{1}$ is odd, we get an identification of pseudo-forms on N with odd forms on $\widetilde{N}$.

Example 5.1.3. Any homogeneous odd-degree polynomial $P \in \mathbb{R}[x,y,z]$ is an odd function on the unit sphere S^2 and hence defines a pseudo-function on the projective plane $\mathbb{RP}^2 = S^2 \,/\, \mathbb{Z}_2$. Likewise, any homogeneous even-degree polynomial defines a regular function on $\mathbb{RP}^2$.

Note that both even and odd k-forms can be integrated over compact k-dimensional submanifolds. To integrate an even form we need the submanifold to be oriented. To integrate an odd form we need the submanifold to be *co-oriented*. In particular, any density can always be integrated over the whole manifold, assuming the manifold is compact (as opposed to a volume form which can only be integrated over an oriented manifold).

Example 5.1.4. For any (orientable or not) compact Riemannian manifold (N,g) the integral of the associated Riemannian density is well-defined (i.e. it does not depend on the orientation of N or on whether such an orientation even exists). This integral is the Riemannian volume of N and is a positive number.

Since, locally, one can identify odd and even forms, all standard local operations with even forms have odd counterparts. In particular, the differential $d\alpha$ of an odd k-form α is an odd $(k+1)$-form, so that we have a map $d\colon \widetilde{\Omega}^k(N) \to \widetilde{\Omega}^{k+1}(N)$. Likewise, for any vector field v on N we have the interior product operator $i_v\colon \widetilde{\Omega}^k(N) \to \widetilde{\Omega}^{k-1}(N)$ and the Lie derivative $L_v = di_v + i_v d\colon \widetilde{\Omega}^k(N) \to \widetilde{\Omega}^k(N)$. The exterior product of two forms of the same parity is an even form, while the product of two forms of different parity is an odd form. The latter, in particular, means that on any closed (i.e. compact without boundary) n-dimensional manifold N one has a non-degenerate pairing $\widetilde{\Omega}^k(N) \times \Omega^{n-k}(N) \to \mathbb{R}$.

The annihilator of the space of exact (respectively, closed) k-forms of any given parity with respect to that pairing is the space of closed (respectively, exact) $(n-k)$-forms of opposite parity. This gives *twisted Poincaré duality*

$$\widetilde{H}^k(N;\mathbb{R}) \simeq H^{n-k}(N;\mathbb{R})^*,$$

where $\widetilde{H}^\bullet(N;\mathbb{R})$ is the cohomology of the cochain complex $(\widetilde{\Omega}^\bullet(N), d)$. One also has identifications

$$H^k(N;\mathbb{R}) \simeq H^k_{even}(\widetilde{N};\mathbb{R}), \quad \widetilde{H}^k(N;\mathbb{R}) \simeq H^k_{odd}(\widetilde{N};\mathbb{R}),$$

where $H^k_{even}(\widetilde{N};\mathbb{R})$ and $H^k_{odd}(\widetilde{N};\mathbb{R})$ are, respectively $+1$ and -1 eigenspaces for the action of the orientation-reversing involution on the cohomology $H^k(\widetilde{N};\mathbb{R})$ of the orientation double cover. In those terms twisted Poincaré duality rewrites as

$$H^k_{even}(\widetilde{N};\mathbb{R}) \simeq H^{n-k}_{odd}(\widetilde{N};\mathbb{R})^*.$$

5.1.3 *The group of measure-preserving diffeomorphisms, its Lie algebra, and the dual of the Lie algebra*

Let N be a closed (i.e. compact without boundary) non-orientable manifold with density (i.e. a non-vanishing top-degree pseudo-form) ρ. The group $\mathrm{Diff}_\rho(N)$ consists of measure-preserving diffeomorphisms:

$$\mathrm{Diff}_\rho(N) := \{\Phi \in \mathrm{Diff}(N) \mid \Phi^*\rho = \rho\}.$$

Let $\widetilde{N}$ be the orientation double cover of N. Then $\mathrm{Diff}_\rho(N)$ can be identified with the subgroup of $\mathrm{Diff}_\mu(\widetilde{N})$ consisting of volume-preserving diffeomorphisms of the orientation double cover that commute with the orientation-reversing involution (the volume form μ on $\widetilde{N}$ is constructed as a pull-back of the density ρ).

The Lie algebra $\mathrm{Vect}_\rho(N)$ of the group $\mathrm{Diff}_\rho(N)$ consists of divergence-free vector fields:

$$\mathrm{Vect}_\rho(N) := \{v \in \mathfrak{vect}(N) \mid L_v\rho = 0\}.$$

With any (even) 1-form $\alpha \in \Omega^1(N)$ we can associate a linear functional ℓ_α on the Lie algebra $\mathrm{Vect}_\rho(N)$ given by

$$\ell_\alpha(v) := \int_N (i_v\alpha)\rho$$

(recall that the integral of a density over the whole manifold is well-defined).

Definition 5.1.5. A linear functional $\ell\colon \mathrm{Vect}_\rho(N) \to \mathbb{R}$ is called *regular* if there exists $\alpha \in \Omega^1(N)$ such that $\ell = \ell_\alpha$. Denote the space of regular functionals by $\mathrm{Vect}^*_\rho(N)$.

Proposition 5.1.6. *The space of regular functionals* $\mathrm{Vect}^*_\rho(N)$ *is isomorphic to the space of cosets* $\Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$ *of* 1*-forms modulo exact* 1*-forms.*

Proof. By definition of a regular functional, we have a surjective vector space homomorphism $\Omega^1(N) \to \mathrm{Vect}^*_\rho(N)$ given by $\alpha \mapsto \ell_\alpha$. We need to show that its kernel is the space of exact (even) 1-forms. To that end, rewrite $\ell_\alpha(v)$ as

$$\ell_\alpha(v) = \int_N \alpha \wedge i_v\rho.$$

Since v is an arbitrary divergence-free vector field, $i_v\rho$ is an arbitrary closed odd $(n-1)$-form. So, the kernel of the map $\alpha \mapsto \ell_\alpha$ is the annihilator of closed odd $(n-1)$-forms under the pairing $\Omega^1(N) \times \widetilde{\Omega}^{n-1}(N) \to \mathbb{R}$, i.e. the space of exact even 1-forms, as needed. □

The coadjoint action of the group $\mathrm{Diff}_\rho(N)$ on the regular dual $\mathrm{Vect}_\rho(N)^* = \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$ coincides with the natural action of diffeomorphisms on (cosets of) 1-forms:

$$\mathrm{Ad}^*_\Phi[\alpha] = [\Phi^*\alpha],$$

where $\Phi \in \mathrm{Diff}_\rho(N)$ is a measure-preserving diffeomorphism and $\alpha \in \Omega^1(N)$ is a 1-form. The focus of the present paper is on classifying all generic orbits of that action. Note that those orbits can also be interpreted as symplectic leaves of the Lie-Poisson structure on $\mathrm{Vect}^*_\rho(N)$.

5.2 CLASSIFICATION OF GENERIC PSEUDO-FUNCTIONS IN 2D

5.2.1 *Simple Morse pseudo-functions*

We begin with recalling standard definitions of a Morse function and a simple Morse function.

Definition 5.2.1. Let N be a smooth manifold. A smooth function $F\colon N \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate. A Morse function $F\colon N \to \mathbb{R}$ is *simple* if all its critical values are distinct.

Below we formulate analogous notions for pseudo-functions.

Definition 5.2.2. Let N be a non-orientable manifold.

- A pseudo-function $F \in \widetilde{\Omega}^0(N)$ is *Morse* if its lift $\widetilde{F} \in \Omega^0_{odd}(\widetilde{N})$ to the orientation double cover $\widetilde{N}$ is a Morse function.
- A pseudo-function $F \in \widetilde{\Omega}^0(N)$ is *simple Morse* if its lift $\widetilde{F} \in \Omega^0_{odd}(\widetilde{N})$ is simple Morse. The latter in particular implies that 0 is not a critical value of $\widetilde{F}$ (if there was a critical point where $\widetilde{F} = 0$, then its image under the orientation-reversing involution would give another critical point at the same zero level).

The space of smooth pseudo-functions $\tilde{\Omega}^0(N)$ can be identified with the space $\Omega^0_{odd}(\widetilde{N})$ of odd smooth functions $\widetilde{N} \to \mathbb{R}$. Consider the C^2-topology on the space $\widetilde{\Omega}^0(N) \simeq \Omega^0_{odd}(\widetilde{N})$ induced by C^2-topology on the space $\Omega^0(\widetilde{N})$ of all smooth functions on $\widetilde{N}$.

Theorem 5.2.3. *Simple Morse pseudo-functions on N form an open and dense subset in the space of smooth pseudo-functions in the C^2-topology.*

The proof is based on the following version of Whitney's embedding theorem:

Lemma 5.2.4 (On the realization of a fixed-point-free involution as the antipodal map)**.** *Let M be a compact smooth n-dimensional manifold equipped with a fixed-point-free involution $I\colon M \to M$. Then there exists an embedding $\Phi\colon M \to V$, where V is a vector space of dimension $2n(2n+1)$, such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & V \\ \big\downarrow{\scriptstyle I} & & \big\downarrow{\scriptstyle -\mathrm{id}} \\ M & \xrightarrow{\Phi} & V, \end{array} \tag{5.3}$$

where $-\mathrm{id}\colon V \to V$ is the antipodal map $x \mapsto -x$. In other words, any fixed-point-free involution of a compact manifold can be realized as restriction of the antipodal map.

Remark 5.2.5. Although the exact dimension of the vector space V is irrelevant for our purposes, we note that composing the embedding given by Lemma 5.2.4 with a projection onto a suitable subspace, one can bring down the dimension of V to $2n+1$ (as in weak Whitney's theorem).

Proof of Lemma 5.2.4. By Whitney's theorem there exists an embedding $\Phi_1\colon M \to W$, where W is a vector space of dimension $2n$. We extend it to an embedding

$$\Phi_2\colon M \to W \oplus W, \quad \Phi_2(x) := (\Phi_1(x), \Phi_1(I(x))).$$

Then the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\Phi_2"] \arrow[d, "I"] & W \oplus W \arrow[d, "P"] \\ M \arrow[r, "\Phi_2"] & W \oplus W \end{tikzcd} \tag{5.4}$$

where the involution P on $W \oplus W$ is $(x, y) \mapsto (y, x)$. Furthermore, since the involution I is fixed-point-free, the image $\Phi_2(M)$ does not intersect the diagonal $\Delta \subset W \oplus W$. Now, consider the map

$$\Psi\colon W \oplus W \to W \oplus (W \otimes W), \quad \Psi(x, y) := (x - y, (x - y) \otimes (x + y)).$$

It is easy to see that Ψ is an injective immersion away from the diagonal Δ. Therefore, since M is compact, the map $\Psi \circ \Phi_2\colon M \to W \oplus (W \otimes W)$ is an embedding. Furthermore, we have $\Psi \circ P = -\Psi$, so the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\Phi_2"] \arrow[d, "I"] & W \oplus W \arrow[r, "\Psi"] \arrow[d, "P"] & W \oplus (W \otimes W) \arrow[d, "-\mathrm{id}"] \\ M \arrow[r, "\Phi_2"] & W \oplus W \arrow[r, "\Psi"] & W \oplus (W \otimes W). \end{tikzcd} \tag{5.5}$$

Taking $V := W \oplus (W \otimes W)$ and $\Phi := \Psi \circ \Phi_2$ completes the proof. □

Proof of Theorem 5.2.3. It is clear that simple Morse pseudo-functions on N form an open subset in the space of all smooth pseudo-functions, so we only need to show that it is dense. First we show that the set of all (not necessarily simple) Morse pseudo-functions is dense. That is equivalent to showing that any odd function $f \in \Omega^0_{odd}(\widetilde{N})$ can be approximated by an odd Morse function. Thanks to Lemma 5.2.4, we can assume that $\widetilde{N}$ is embedded in a vector space V, with the orientation-reversing involution given by the antipodal map $x \mapsto -x$. Take an odd function $f \in \Omega^0_{odd}(\widetilde{N})$, and consider the family of functions $F_\ell := f + \ell$ where $\ell \in V^*$. Notice that all functions in this family are odd with respect to the antipodal map. Furthermore, almost all functions in such a family are Morse [31, Theorem 8.1.1], so one can indeed find an odd Morse function arbitrarily close to f.

The second step is to show that simple Morse pseudo-functions on N are dense in the space of all Morse pseudo-functions. To that end, notice that the corresponding argument for functions [28, Proposition 1.2.12] is *local*: one can perturb a given Morse function in the neighborhood of a critical point to slightly change its critical value. But locally one can trivialize the orientation bundle $o(N)$ and thus identify pseudo-functions with functions. Therefore, the same argument works for pseudo-functions, proving the theorem. □

5.2.2 *Reeb graphs with involution*

Below we classify simple Morse pseudo-functions on closed connected non-orientable 2D surfaces up to area-preserving diffeomorphisms. This is equivalent to symplectic classification of quadruples (M, I, ω, F) where

- M is a closed connected surface with symplectic form ω;
- $I \colon M \to M$ is a fixed-point-free anti-symplectic involution;
- $F \in \Omega^0_{odd}(M)$ is a simple Morse function anti-symmetric under the action of I.

Consider for a moment an arbitrary simple Morse function F on a closed connected orientable surface M. Define an *F-level* as a connected component of a level set of F. Non-critical F-levels are diffeomorphic to circles. The surface M is a union of F-levels, which form a foliation with singularities. The base space of that foliation with the quotient topology is homeomorphic to a finite connected graph Γ_F (see Figure 1.1 for a Morse function F on a torus) whose vertices correspond to critical values of F. This graph Γ_F is called the *Reeb graph* of the function F (also called the *Kronrod graph* [30, 1]). By π we denote the projection $M \to \Gamma_F$. The function F on M descends to a function f on the Reeb graph Γ_F. It is also convenient to assume that Γ_F is oriented: edges are oriented in the direction of increasing f. One can recover the topology of a closed connected orientable surface from its Reeb graph by the following formula: $b_1(M) = 2b_1(\Gamma_F)$, where $b_1(.) := \dim H_1(.)$ is the first Betti number.

Now return to consideration of a simple Morse function F which is anti-symmetric under the involution I. In that setting I descends to an involution $\iota \colon \Gamma_F \to \Gamma_F$ such that $\iota^* f = -f$ (see Figure 1.3). Note that the involution ι is not necessarily fixed-point-free.

Definition 5.2.6. A *Reeb graph with involution* (Γ, ι, f) is an oriented connected finite graph Γ equipped with an involution $\iota \colon \Gamma \to \Gamma$ and a continuous function $f : \Gamma \to \mathbb{R}$, with the following properties:

(i) Each vertex of Γ is either 1-valent or 3-valent;

(ii) The function f is anti-invariant under the action of ι i.e. $\iota^* f = -f$;

(iii) f is strictly increasing along each edge of Γ;

(iv) f takes distinct non-zero values at vertices of Γ.

5.2.3 *Measured Reeb graphs with involution*

In this section we discuss the notion of a *measured* Reeb graph with involution. Recall that we consider quadruples (M, I, ω, F) where (M, ω) is a closed connected symplectic surface, $I\colon M \to M$ is a fixed-point-free anti-symplectic involution, and $F \in \Omega^0_{odd}(M)$ is a simple Morse function anti-symmetric under the action of I.

The natural projection map $\pi\colon M \to \Gamma_F$ induces a measure $\mu := \pi_*\omega$ on the graph Γ_F. According to [19, Proposition 3.4], measure μ is *log-smooth* in the sense of [16, Definition 3.5] (in a nutshell, this means that the measure is smooth at interior points of all edges as well as at 1-valent vertices, while at 3-valent vertices it has logarithmic singularities). Furthermore, this measure is invariant with respect to the involution $\iota\colon \Gamma_F \to \Gamma_F$.

Definition 5.2.7. A *measured Reeb graph with involution* (Γ, ι, f, μ) is a Reeb graph (Γ, ι, f) with involution equipped with a log-smooth measure μ invariant under the involution ι.

Definition 5.2.8. A map $\phi : \Gamma_1 \to \Gamma_2$ between two measured Reeb graphs with involution $(\Gamma_1, \iota_1, f_1, \mu_1)$ and $(\Gamma_2, \iota_2, f_2, \mu_2)$ is an *isomorphism* if it is an isomorphism of topological graphs which maps all objects in Γ_1 to the corresponding objects in Γ_2, i.e. $\phi \circ \iota_1 = \iota_2 \circ \phi$, $\phi^* f_2 = f_1$, and $\phi_*\mu_1 = \mu_2$.

The following definition from [19] makes sense regardless of the presence of an involution:

Definition 5.2.9. A measured Reeb graph (Γ, ι, f, μ) is *compatible* with a symplectic surface (M, ω) if

$$2b_1(\Gamma) = b_1(M), \quad \int_\Gamma \mathrm{d}\mu = \int_M \omega$$

(where $b_1(.)$ stands for the first Betti number).

For a non-orientable surface N with density $\rho \in \widetilde{\Omega}^2(N)$, we say that a measured Reeb graph (Γ, ι, f, μ) is *compatible* with (N, ρ) if it is compatible with $(\widetilde{N}, I, \omega)$, where $\widetilde{N}$ is the orientation double cover of N, the symplectic form $\omega \in \Omega^2(\widetilde{N})$ is the pull-back of ρ, and I is involution on $\widetilde{N}$ such that $\widetilde{N} \,/\, I = N$. More explicitly, the compatibility conditions can be stated as follows:

Definition 5.2.10. A measured Reeb graph (Γ, ι, f, μ) is *compatible* with a non-orientable surface N equipped with a density ρ if

$$b_1(\Gamma) = b_1(N), \quad \int_\Gamma \mathrm{d}\mu = 2\int_N \rho.$$

5.2.4 *Proof of the classification theorem for pseudo-functions*

Here we prove the following classification result for pseudo-functions on non-orientable surfaces:

Theorem 5.2.11. *Let N be a closed connected non-orientable surface equipped with a density (non-degenerate pseudo-form of top degree) ρ. Then there is a one-to-one correspondence between simple Morse pseudo-functions on N, considered up to area-preserving diffeomorphisms, and isomorphism classes of measured Reeb graphs with involution compatible with (N, ρ).*

Equivalently, and in more detail, this theorem can be formulated as follows.

Theorem 5.2.12. *Let (M, ω) be a closed connected symplectic surface together with a fixed-point-free anti-symplectic involution $I \colon M \to M$. Then there is a one-to-one correspondence between odd (i.e. anti-symmetric under I) simple Morse functions on M, considered up to symplectic diffeomorphisms which commute with I, and isomorphism classes of measured Reeb graphs with involution compatible with M. In other words, the following statements hold.*

i) Let F, $G \in \Omega^0_{odd}(M)$ be two odd simple Morse functions. Then the following conditions are equivalent:

a) There exists a symplectic diffeomorphism $\Phi \colon M \to M$ such that $\Phi \circ I = I \circ \Phi$ and $\Phi_* F = G$.

b) Measured Reeb graphs with involution associated with F and G are isomorphic.

Moreover, any isomorphism between measured Reeb graphs with involution associated with F and G can be lifted to a symplectic diffeomorphism $\Phi \colon M \to M$ such that $\Phi \circ I = I \circ \Phi$ and $\Phi_* F = G$.

ii) For each measured Reeb graph with involution (Γ, ι, f, μ) compatible with (M, ω) there exists an odd simple Morse function $F \in \Omega^0_{odd}(M)$ such that the corresponding measured Reeb graph with involution is isomorphic to (Γ, ι, f, μ).

Proof. We begin with part (i). The implication (a) $\Rightarrow$ (b) is immediate from the definition of a measured Reeb graph of a function. To prove (b) $\Rightarrow$ (a) we will show that any isomorphism between measured Reeb graphs with involution associated with F and G can be lifted to a symplectic diffeomorphism $\Phi \colon M \to M$ such that $\Phi \circ I = I \circ \Phi$ and $\Phi_* F = G$. Let $\phi \colon \Gamma_F \to \Gamma_G$ be an isomorphism of measured Reeb graphs with involution. Then, by [19, Theorem 3.11(i)], it can be lifted to a symplectic diffeomorphism $\Phi \colon M \to M$ such that $\Phi_* F = G$. We need to show that Φ can be chosen so that it commutes with the involution I. The idea is to pick an arbitrary Φ and then compose it on the right with a suitable "shear flow". The required shear flow on every level of F will be simply a shift along the Hamiltonian vector field $\omega^{-1}dF$ by an amount depending on the level. To find the needed magnitude of the shift on every level, consider the commutator

$$\Psi := [I, \Phi^{-1}] = I \circ \Phi^{-1} \circ I \circ \Phi. \tag{5.6}$$

This map Ψ is symplectic and preserves each F-level. Let E be the union of open edges of Γ_F containing points where $f = 0$. Let also π be the projection $M \to \Gamma_F$. Then $M_0 := \pi^{-1}(E)$ is a union of open cylinders foliated into regular F-levels. For any smooth function ψ on E (where we define charts of E by using the function f) denote by $S_\psi \colon M_0 \to M_0$ a symplectomorphism defined as follows: every point $x \in M_0$ moves for time $\psi(\pi(x))$ along trajectories of the Hamiltonian vector field $\omega^{-1}dF$.

Lemma 5.2.13. *There exists a smooth function $\psi \colon E \to \mathbb{R}$ which is odd (i.e. $\iota^*\psi = -\psi$) and satisfies $\Psi|_{M_0} = S_\psi$, where $\Psi|_{M_0}$ is the restriction of the commutator map (5.6) to M_0.*

Proof. Since Ψ is symplectic and preserves each F-level, there exists some (a priori not necessarily odd) smooth function $\psi \colon E \to \mathbb{R}$ such that $\Psi|_{M_0} = S_\psi$. Further, observe that since the Hamiltonian vector field $\omega^{-1}dF$ is even, we have

$$S_\psi \circ I = I \circ S_{\iota^*\psi}. \tag{5.7}$$

Furthermore, from the definition of the map Ψ we get $I \circ \Psi = \Psi^{-1} \circ I$, so, since $\Psi|_{M_0} = S_\psi$, we obtain

$$I \circ S_\psi = S_{-\psi} \circ I = I \circ S_{-\iota^*\psi},$$

meaning that $S_{\psi+\iota^*\psi}$ is the identity map. Now, let $t \colon E \to \mathbb{R}$ be a smooth function defined as follows: for every $x \in E$, it is equal to the period of the Hamiltonian vector field $\omega^{-1}dF$ on the circle $\pi^{-1}(x)$. Then, since $S_{\psi+\iota^*\psi} = \mathrm{id}$, we have that

$$\lambda := \frac{\psi + \iota^*\psi}{t}$$

is a continuous integer-valued function on E, i.e. an element of $H^0(E; \mathbb{Z})$. Furthermore, since the period function t is even, the same is true for the function λ, i.e. $\lambda \in H^0_{even}(E; \mathbb{Z})$. Consider the map $H^0(E; \mathbb{Z}) \to H^0_{even}(E; \mathbb{Z})$ given by $\eta \mapsto \eta + \iota^*\eta$. The image of that map consists of those integral cochains $\eta \in H^0_{even}(E; \mathbb{Z})$ which take even values on all edges in E fixed by ι. We claim that λ has that property and hence belongs to the image. Indeed, every edge e fixed by ι has a point x_e whose preimage under the projection $\pi\colon M \to \Gamma_F$ is an F-level γ_e fixed by I (that x_e is the unique point on e where $F = 0$). Since I has no fixed points, the restriction of I to that level must be a half-period shift along the vector field $\omega^{-1}dF$. Likewise, the restriction of I to the corresponding G-level $\Phi(\gamma_e)$ is half-period shift along $\omega^{-1}dG$. But since the diffeomorphism Φ maps the Hamiltonian vector field $\omega^{-1}dF$ to the Hamiltonian field $\omega^{-1}dG$, it follows that the commutator Ψ on the F-level γ_e is the identity. Therefore, $\psi(x_e)$ must be an integer multiple of $T(x_e)$, which forces $\lambda(e)$ to be an even number. So indeed we have $\lambda = \eta + \iota^*\eta$ for some $\eta \in H^0(E;\mathbb{Z})$, and replacing ψ with $\psi - \eta t$ we get a function with desired properties. $\square$

Back to the proof of the theorem, let $m < 0$ be a real number such that $\inf_{x\in e} f(x) < m$ for any edge $e \in E$. Let also $\zeta\colon \mathbb{R} \to \mathbb{R}$ be an odd smooth function which is equal to -1 for $x < m$. Define a smooth function $\xi\colon E \to \mathbb{R}$ by $\xi := \frac{1}{2}(1 + \zeta \circ f)\psi$. Then ξ is equal to 0 near lower (i.e. where $f < 0$) endpoints of edges in E and satisfies

$$\xi - \iota^*\xi = \psi \tag{5.8}$$

everywhere in E. Let $\widetilde{\Phi} := \Phi \circ S_{-\xi}$. That is a symplectomorphism $M_0 \to M_0$ pushing F to G. Moreover,

$$[I, \widetilde{\Phi}^{-1}] = I \circ \widetilde{\Phi}^{-1} \circ I \circ \widetilde{\Phi} = I \circ S_\xi \circ \Phi^{-1} \circ I \circ \Phi \circ S_{-\xi} \tag{5.9}$$

$$= S_{\iota^*\xi} \circ I \circ \Phi^{-1} \circ I \circ \Phi \circ S_{-\xi} = S_{\iota^*\xi} \circ S_\psi \circ S_{-\xi} = \mathrm{id}, \tag{5.10}$$

where the third equality follows from (5.7) and the last one from (5.8). So, $\widetilde{\Phi}$ commutes with I. Also, since $\xi = 0$ near lower endpoints of edges in E, one can smoothly extend $\widetilde{\Phi}$ to $M_- := (M \setminus M_0) \cap \{F < 0\}$ by setting $\widetilde{\Phi} := \Phi$ in M_-. Furthermore, since Φ commutes with I in M_0, it can be smoothly extended to $M_+ := (M \setminus M_0) \cap \{F > 0\}$ by setting $\widetilde{\Phi} := I\widetilde{\Phi}I$ in M_+. That way we get an extension of $\widetilde{\Phi}$ to all of M. It is symplectic, commutes with I, and maps F to G, as needed. Thus, part (i) of the theorem is proved.

Now, let us prove part (ii). The proof consists of four steps: (1) we construct a function F on a symplectic surface which has a given measured Reeb graph and is anti-invariant under an anti-symplectic map I_1 (which is not necessarily an involution); (2) we modify I_1 so that it becomes an involution, which we call I_2; (3) by composing I_2 with appropriate Dehn twists we turn it into a fixed-point-free involution I_3; (4) we find a symplectomorphism conjugating I_3 and I, which yields a function with desired properties.

Step 1. By [19, Theorem 3.11(ii)], there exists a (not necessarily odd) simple Morse function $F\colon M \to \mathbb{R}$ whose measured Reeb graph (without involution) is (Γ, f, μ). Consider also $-F$ as a simple Morse function on the symplectic surface $(M, -\omega)$. The measured Reeb graph of the latter function is $(\Gamma, -f, \mu)$. So, the involution ι is an isomorphism of measured Reeb graphs of F and $-F$ and hence it lifts, by [19, Theorem 3.11(i)], to a diffeomorphism $I_1\colon M \to M$ such that $I_1^* F = -F$ and $I_1^*\omega = -\omega$.

Step 2. Consider the restriction of I_1^2 to the set M_0 defined in the proof of part (i). It is a symplectic diffeomorphism preserving F-levels and hence can be written as S_j for a suitable smooth function $j\colon E \to \mathbb{R}$. Furthermore, we claim that j can be chosen to be even. The proof is similar to that of Lemma 5.2.13: since the map I_1^2 commutes with I_1, which is a lift of ι, we must have

$$\lambda := \frac{j - \iota^* j}{t} \in H^0(E; \mathbb{Z}).$$

Furthermore, λ is odd and hence can be written as $\eta - \iota^*\eta$ for some $\eta \in H^0(E;\mathbb{Z})$ (in contrast to the map $H^0(E;\mathbb{Z}) \to H^0_{even}(E;\mathbb{Z})$ given by $\eta \mapsto \eta + \iota^*\eta$, the map $H^0(E;\mathbb{Z}) \to H^0_{odd}(E;\mathbb{Z})$ given by $\eta \mapsto \eta - \iota^*\eta$ is surjective). So, one can make the function j even by replacing it with $j - \eta t$.

Consider now the function $\xi\colon E \to \mathbb{R}$ defined by $\xi := \frac{1}{2}(1 + \zeta \circ f) j$, where $\zeta\colon \mathbb{R} \to \mathbb{R}$ is a function from the proof of part (i). Then $\xi + \iota^*\xi = \psi$, and thus $I_2 := I_1 \circ S_{-\xi}$ is an involution $M_0 \to M_0$:

$$I_2^2 = I_1 \circ S_{-\xi} \circ I_1 \circ S_{-\xi} = S_{-\iota^*\xi} \circ I_1^2 \circ S_{-\xi} = S_{-\iota^*\xi} \circ S_\psi \circ S_{-\xi} = \mathrm{id}.$$

Furthermore, I_2 is anti-symplectic and takes F to $-F$. Also observe that I_2 coincides with I_1 near preimages of lower endpoints of edges in E and thus can be extended to M_- by setting $I_2 := I_1$ in that domain. Similarly, in M_+ we set $I_2 := I_1^{-1}$. That way we get an anti-symplectic involution $I_2\colon M \to M$ which takes F to $-F$.

Step 3. The set of fixed points of I_2 is a union of F-levels, each of which projects to a fixed point of ι. So the fixed point set of I_2 is a union of finitely many circles. To show that such fixed points can be removed, it suffices to prove that we can get rid of one fixed circle. This is done by composing I_2 with a Dehn twist about that circle. Specifically, to get rid of a fixed circle $\pi^{-1}(x_0)$, where $x_0 \in \Gamma$ is a fixed point of ι, and $\pi\colon M \to \Gamma_f$ is the projection, consider the edge e of Γ containing x_0. Let $t\colon e \to \mathbb{R}$ be the period function defined as in Lemma 5.2.13: for every $x \in e$, it is equal to the period of the the Hamiltonian vector field $\omega^{-1}dF$ on the circle $\pi^{-1}(x)$. Since the involution I_2 preserves the Hamiltonian vector field $\omega^{-1}dF$, the period function t is even: $\iota^* t = t$. Define a function $\eta\colon e \to \mathbb{R}$

by $\eta := \frac{1}{2}(1 + \zeta \circ f)t$, where $\zeta \colon \mathbb{R} \to \mathbb{R}$ is as above. Then $\eta + \iota^*\eta = t$, and so the map $\widetilde{I}_2 := I_2 \circ S_\eta$, extended to the whole M by setting $\widetilde{I}_2 :=$ id away from $\pi^{-1}(e)$, is again an involution:

$$\widetilde{I}_2^2 = I_2 \circ S_\eta \circ I_2 \circ S_\eta = S_{\iota^*\eta + \eta} = S_t = \mathrm{id}.$$

Furthermore, just like I_2, the involution $\widetilde{I}_2$ is anti-symplectic and maps F to $-F$. In addition to that, it has less fixed circles than I_2. Proceeding in this fashion, we finally get an anti-symplectic fixed-point-free involution I_3 such that $I_3^* F = -F$.

Step 4. It follows from Moser's theorem for non-orientable surfaces [9, p. 4894] that $I_3 = \Phi I \Phi^{-1}$ for some symplectic diffeomorphism Φ. But then $\Phi^* F$ is a function anti–invariant under I whose measured Reeb graph with involution is the given one. Thus, Theorem 5.2.12 (and hence Theorem 5.2.11) is proved. □

Remark 5.2.14. The above description of invariants of pseudo-functions under the action of area-preserving diffeomorphisms extends to the case of non-orientable surfaces N with boundary, cf. [24]. Recall that in the boundary case *simple Morse functions* F have to satisfy the following conditions: a) all critical points of F are non-degenerate; b) F does not have critical points on the boundary ∂N; c) the restriction of F to the boundary ∂N is a Morse function; and d) all critical values of F and of its restriction $F|_{\partial N}$ are distinct.

A *pseudo-function* F on a non-orientable surface N with boundary is *simple Morse* if its lift $\widetilde{F}$ to the orientation double cover $\widetilde{N}$ is simple Morse. The Reeb graph of such pseudo-function F (defined as the Reeb graph of $\widetilde{F}$) contains both solid edges (corresponding to $\widetilde{F}$-levels diffeomorphic to a circle) and dashed edges (corresponding to $\widetilde{F}$-levels diffeomorphic to a segment). That graph is equipped with an involution induced by the involution of $\widetilde{N}$. The involution on the graph cannot have fixed points on dashed edges, since the corresponding fixed-point-free involution on the surface $\widetilde{N}$ cannot map a segment to itself.

5.3 CLASSIFICATION OF COADJOINT ORBITS IN 2D

5.3.1 *Coadjoint orbits and pseudo-functions*

Let $\mathrm{Diff}_\rho(N)$ be the group of area-preserving diffeomorphisms of a non-orientable surface N endowed with a density ρ. In this section we classify generic orbits of the coadjoint action of $\mathrm{Diff}_\rho(N)$ on its regular dual space

$\mathrm{Vect}^*_\rho(N) = \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$. Recall that this action coincides with the natural action by pull-backs.

Consider the mapping

$$\mathrm{curl}\colon \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N) \to \widetilde{\Omega}^0(N),$$

defined by taking the *vorticity* pseudo-function

$$\mathrm{curl}[\alpha] := \frac{\mathrm{d}\alpha}{\rho}.$$

This mapping is well-defined on cosets since $\mathrm{d}(\alpha + \mathrm{d}f) = \mathrm{d}\alpha$. Furthermore, the mapping curl is a surjection onto the space $\widetilde{\Omega}^0(N)$ of pseudo-functions, since $H^2(N;\mathbb{R}) = 0$. Finally, observe that the mapping curl is $\mathrm{Diff}_\rho(N)$-equivariant. In other words, the following diagram commutes for any $\Phi \in \mathrm{Diff}_\rho(N)$:

$$\begin{tikzcd} \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N) \arrow[r, "\Phi^*"] \arrow[d, "\mathrm{curl}"] & \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N) \arrow[d, "\mathrm{curl}"] \\ \widetilde{\Omega}^0(N) \arrow[r, "\Phi^*"] & \widetilde{\Omega}^0(N). \end{tikzcd} \tag{5.11}$$

Definition 5.3.1. A coset $[\alpha] \in \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$ is called *simple Morse* if $\mathrm{curl}[\alpha]$ is a simple Morse function. A coadjoint orbit $\mathcal{O}$ is called *simple Morse* if some (and hence every) coset $[\alpha] \in \mathcal{O}$ is simple Morse.

With every simple Morse coset $[\alpha] \in \Omega^1(N) \,/\, \mathrm{d}\Omega^0(N)$ one can associate a measured Reeb graph $\Gamma_{\mathrm{curl}[\alpha]}$ with involution. If two simple Morse cosets $[\alpha]$ and $[\beta]$ belong to the same coadjoint orbit then the corresponding Reeb graphs are isomorphic.

The converse statement is more subtle. Indeed, suppose that cosets $[\alpha]$ and $[\beta]$ have isomorphic Reeb graphs. Then it follows from Theorem 5.2.11 that there exists an area-preserving diffeomorphism Φ such that $\Phi^*\mathrm{curl}[\beta] = \mathrm{curl}[\alpha]$. Therefore, the 1-form $\Phi^*\beta - \alpha$ is closed. Since this 1-form is not necessarily exact, the cosets $[\alpha]$ and $[\beta]$ do not necessarily belong to the same coadjoint orbit. Nevertheless, we conclude that the space of coadjoint orbits corresponding to the same measured Reeb graph with involution is finite-dimensional and its dimension is at most $\dim \mathrm{H}^1(N;\mathbb{R})$.

5.3.2 *Even circulation functions on Reeb graphs with involution*

In order to obtain a complete set of invariants of simple coadjoint orbits for the group $\mathrm{Diff}_\rho(N)$, we lift all objects discussed in the previous section to the orientation double cover $M = \widetilde{N}$ of N. That orientation double cover is a symplectic surface with a symplectic form ω and a fixed-point-free anti-symplectic involution I. Our aim is to classify simple Morse cosets $\Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$ up to even (i.e. commuting with I) symplectic diffeomorphisms. To that end we employ the notion of a circulation function introduced in [19].

Consider a simple Morse coset $[\alpha] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$. Then $F := \mathrm{d}\alpha/\omega$ is an odd simple Morse function on M. Let Γ be the set of F-levels. Recall that this set has a structure of a measured Reeb graph with involution ι, and such graphs classify pseudo-functions up to area-preserving diffeomorphisms. To obtain classification of orbits, we define an additional structure on Γ.

Let $\pi\colon M \to \Gamma$ be the natural projection. Take any point x lying in the interior of some edge e of Γ. Then $\pi^{-1}(x)$ is a closed curve C in M. It is naturally oriented by the Hamiltonian vector field $\omega^{-1}dF$. The integral of α over C does not change if we change α by a function differential. Thus, we obtain a function $\mathcal{C}\colon \Gamma \setminus V(\Gamma) \to \mathbb{R}$ given by

$$\mathcal{C}(x) = \oint\limits_{\pi^{-1}(x)} \alpha$$

and defined outside of the set of vertices $V(\Gamma)$ of the graph Γ.

Proposition 5.3.2. *The function $\mathcal{C}$ has the following properties.*

i) Assume that x, y are two interior points of an edge e of Γ. Then

$$\mathcal{C}(y) - \mathcal{C}(x) = \int_x^y f \,\mathrm{d}\mu. \tag{5.12}$$

ii) Let v be a vertex of Γ. Then $\mathcal{C}$ satisfies the Kirchhoff rule at v:

$$\sum_{e \to v} \lim_{x \xrightarrow{e} v} \mathcal{C}(x) = \sum_{e \leftarrow v} \lim_{x \xrightarrow{e} v} \mathcal{C}(x)\,, \tag{5.13}$$

where $\sum_{e\to v}$ stands for summation over edges pointing at the vertex v, $\sum_{e\leftarrow v}$ stands for summation over edges pointing away from v, and $x \xrightarrow{e} v$ means $x \in \Gamma \setminus V(\Gamma)$ tends to v along e.

iii) The function $\mathcal{C}$ is even with respect to the involution ι on Γ.

Proof. The first two properties hold regardless of the presence of involution [19]. The last property holds because the form α and the vector field $\omega^{-1}dF$ are both even. $\square$

Definition 5.3.3. Let (Γ, ι, f, μ) be a measured Reeb graph with involution. Any function $\mathcal{C}\colon \Gamma \setminus V(\Gamma) \to \mathbb{R}$ satisfying properties listed in Proposition 5.3.2 is called an *even circulation function*. A measured Reeb graph with involution endowed with an even circulation function is called a *circulation graph with involution* $(\Gamma, \iota, f, \mu, \mathcal{C})$.

Two circulation graphs with involution are isomorphic if they are isomorphic as measured Reeb graphs with involution, and the isomorphism between them preserves the circulation function.

Above we associated a circulation graph with involution $\Gamma_{[\alpha]} := (\Gamma, \iota, f, \mu, \mathcal{C})$ to any simple Morse coset $[\alpha] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$.

Remark 5.3.4. Note that the function f on a circulation graph can be recovered from the circulation function $\mathcal{C}$, as (5.12) implies $f = \mathrm{d}\mathcal{C}/\mathrm{d}\mu$.

The following result describes the space of even circulation functions on a given measured Reeb graph with involution.

Proposition 5.3.5. *The space of even circulation functions on a measured Reeb graph with involution* (Γ, ι, f, μ) *is an affine space whose associated vector space is* $\mathrm{H}_1^{odd}(\Gamma; \mathbb{R}) := \{\lambda \in \mathrm{H}_1(\Gamma; \mathbb{R}) \mid \iota_* \lambda = -\lambda\}$.

Proof. By definition, a function $\mathcal{C}\colon \Gamma \setminus V(\Gamma) \to \mathbb{R}$ is an even circulation function if it satisfies certain inhomogeneous linear equations. So, the set of even circulation functions on Γ is indeed an affine space. Let us first show that it is non-empty. To that end, observe that since f is odd, we have $\int_\Gamma f \mathrm{d}\mu = 0$, so by [19, Propoisition 4.5(i)] the measured Reeb graph (Γ, f, μ) admits a circulation function $\mathcal{C}$. Furthermore, the latter can be made even by considering the averaged function $\frac{1}{2}(\mathcal{C} + \iota^* \mathcal{C})$. So, the space of even circulation functions is a solution space of a consistent inhomogeneous linear system, which means that the corresponding vector space is the solution space of the associated homogeneous system. That solution space consists of even functions $\xi\colon \Gamma \setminus V(\Gamma) \to \mathbb{R}$ which are constant on each edge and satisfy Kirchhoff's rule at each vertex. For each element ξ of that solution space, consider a 1-chain on Γ given by $\lambda(\xi) := \sum \xi|_e \cdot e$, where the sum is over all edges of Γ. Then Kirchhoff's equations on ξ are equivalent to $\lambda(\xi)$ being a cycle, i.e. $\lambda(\xi) \in \mathrm{H}_1(\Gamma; \mathbb{R})$. Furthermore, since the involution ι reverses orientation of edges, ξ is even if and only if $\lambda(\xi)$ is odd. So, the vector space associated with the affine space of even circulation functions on Γ is indeed $\mathrm{H}_1^{odd}(\Gamma; \mathbb{R})$, as claimed. $\square$

Corollary 5.3.6. *The dimension d of the space of even circulation functions on a measured Reeb graph with involution (Γ, ι, f, μ) is given by*

$$d = \dim \mathrm{H}_1^{odd}(\Gamma; \mathbb{R}) = \frac{1}{2}(\#\mathrm{Fix}(\iota) + b_1(\Gamma) - 1), \tag{5.14}$$

where $\#\mathrm{Fix}(\iota)$ is the number of fixed points of ι, and $b_1(\Gamma) = \dim H_1(\Gamma, \mathbb{R})$ is the first Betti number of Γ. In particular,

$$\frac{1}{2}(b_1(\Gamma) - 1) \le d \le b_1(\Gamma). \tag{5.15}$$

Proof. By the Hopf trace formula we have

$$\#\mathrm{Fix}(\iota) = 1 - \dim \mathrm{H}_1^{even}(\Gamma; \mathbb{R}) + \dim \mathrm{H}_1^{odd}(\Gamma; \mathbb{R}) \tag{5.16}$$
$$= 1 - \dim \mathrm{H}_1(\Gamma; \mathbb{R}) + 2 \dim \mathrm{H}_1^{odd}(\Gamma; \mathbb{R}), \tag{5.17}$$

hence the result. □

Remark 5.3.7. Another way to express this dimension is $d = b_1(\Gamma) - b_1(\Gamma/\iota)$. Indeed, this follows from the fact that odd classes form the kernel of the projection $\mathrm{H}_1(\Gamma; \mathbb{R}) \to \mathrm{H}_1(\Gamma/\iota, \mathbb{R})$.

Remark 5.3.8. The inequality $d \le b_1(\Gamma)$ holds since the space $\mathrm{H}_1^{odd}(\Gamma; \mathbb{R})$ is a subspace of $\mathrm{H}_1(\Gamma; \mathbb{R})$. That inequality is also equivalent to $\#\mathrm{Fix}(\iota) \le b_1(\Gamma) + 1$. The latter is true since the set $\{f = 0\}$ splits Γ into two connected components and hence consists of at most $b_1(\Gamma) + 1$ points, while the fixed point set $\mathrm{Fix}(\iota)$ is a subset of $\{f = 0\}$.

It is easy to see that there are no other restrictions on the number $\#\mathrm{Fix}(\iota)$ in addition to $0 \le \#\mathrm{Fix}(\iota) \le b_1(\Gamma) + 1$ and $\#\mathrm{Fix}(\iota) \equiv b_1(\Gamma) + 1 \mod 2$, so that all integer dimensions d satisfying (5.15) can occur.

Example 5.3.9. Assume that the graph Γ is a tree. Then $\dim \mathrm{H}_1^{odd}(\Gamma; \mathbb{R}) = 0$, so there is a unique even circulation function on Γ.

Example 5.3.10. Assume that $b_1(\Gamma) = 1$. Then inequalities (5.15) imply that the dimension d of the space of even circulation functions on Γ is 0 or 1. Furthermore, by formula (5.14) we have that $d = 0$ if and only if the involution ι on Γ has no fixed points. An example of such an involution is shown in Figure 1.3 in the introduction. As for the case $d = 1$, that corresponds to two fixed points, see Figure 1.4.

5.3.3 *Proof of the classification theorem for orbits*

The main result of this section is the following classification of generic coadjoint orbits for the group of measure-preserving diffeomorphisms of a non-orientable surface:

Theorem 5.3.11. *Let N be a closed connected non-orientable surface equipped with a density ρ. Then simple Morse coadjoint orbits of $\mathrm{Diff}_\rho(N)$ are in one-to-one correspondence with isomorphism classes of circulation graphs compatible with (N, ρ).*

Compatibility of a graph and non-orientable surface is understood as in Definition 5.2.10. Since a circulation graph is also a measured Reeb graph (with an additional structure), the definition applies.

An equivalent and more detailed form of this classification, which we are going to prove, can be formulated in terms of the corresponding orientation double cover:

Theorem 5.3.12. *Let (M, I, ω) be a closed connected symplectic surface equipped with a fixed-point-free anti-symplectic involution I. Then generic orbits of the action of even (i.e. commuting with I) symplectomorphisms of M on the coset space $[\alpha] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$ are in one-to-one correspondence with (isomorphism classes of) circulation graphs compatible with M (in the sense of Definition 5.2.10). In other words, the following statements hold:*

i) For two simple Morse cosets $[\alpha], [\beta] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$, the following conditions are equivalent:

a) $\Phi_*[\alpha] = [\beta]$ for some even symplectomorphism Φ;

b) circulation graphs $\Gamma_{[\alpha]}$ and $\Gamma_{[\beta]}$ corresponding to the cosets $[\alpha]$ and $[\beta]$ are isomorphic.

ii) For each circulation graph $(\Gamma, \iota, f, \mu, \mathcal{C})$ which is compatible with (M, ω), there exists a simple Morse coset $[\alpha] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$ such that $\Gamma_{[\alpha]} = (\Gamma, \iota, f, \mu, \mathcal{C})$.

Proof. We first prove part (i). The implication (a) $\Rightarrow$ (b) is by construction, so we only need to prove (b) $\Rightarrow$ (a). In view of Theorem 5.2.12, it suffices to consider the case $\mathrm{d}[\alpha] = \mathrm{d}[\beta] = F\omega$ and prove that if the circulation functions on the graph Γ of F given by cosets $[\alpha], [\beta]$ are the same, then there is an even symplectic diffeomorphism $\Phi \in \mathrm{Diff}_\omega(M)$ such that $\Phi^*[\beta] = [\alpha]$. Consider $\xi := [\alpha] - [\beta]$. Then $\xi \in \mathrm{H}^1_{even}(M)$. Furthermore, since the circulation functions of $[\alpha]$ and $[\beta]$ coincide, it follows that the class ξ has zero periods over F-levels. Therefore, by [19, Lemma 4.8], there exists a smooth function $G \in \Omega^0(M)$ such that the 1-form $GFdF$ is closed and its cohomology class is ξ. (The lemma says that there is $H \in \Omega^0(M)$ such that HdF is closed and its class is ξ. Furthermore, that H is divisible by F in $\Omega^0(M)$, so we just set $G := H/F$.) Furthermore, since the class ξ is even, without loss of generality we can assume that G is even as well, i.e. $G \in \Omega^0_{even}(M)$ (if not, we replace G with $\frac{1}{2}(G + I^*G)$).

Consider even symplectic vector field $X := G\omega^{-1}dF$. Then, for the flow Φ_t of X, we have

$$\frac{d}{dt}\Phi_t^*[\beta] = \Phi_t^* L_X[\beta] = \Phi_t^*[i_X F\omega] = [GFdF] = \xi.$$

In particular, the time-1 flow Φ_1 of V takes $[\beta]$ to $[\alpha]$, as needed.

We now prove part (ii). By Theorem 5.2.12, there exists an odd simple Morse function $F \in \Omega^0_{odd}(M)$ whose measured Reeb graph with involution is (Γ, ι, f, μ). We need to show that the map from the affine space of cosets $[\alpha] \in \Omega^1_{even}(M) \,/\, \mathrm{d}\Omega^0_{even}(M)$ such that $\mathrm{d}\alpha = F\omega$ to the space of even circulation functions on Γ, given by mapping a coset $[\alpha]$ to the associated circulation function $\mathcal{C}_{[\alpha]}$, is surjective. To that end consider the associated map of vector spaces $\mathrm{H}^1_{even}(M;\mathbb{R}) \to \mathrm{H}^{odd}_1(\Gamma;\mathbb{R})$ which takes a class $[\beta] \in \mathrm{H}^1_{even}(M;\mathbb{R})$ to a chain $\sum \beta(e)e$ where $\beta(e)$ is the integral of β over the preimage of any interior point of e under the projection $\pi\colon M \to \Gamma$. Upon identification $\mathrm{H}^1_{even}(M;\mathbb{R}) \simeq \mathrm{H}^{odd}_1(M;\mathbb{R})$ given by (twisted) Poincaré duality, that vector space map becomes the projection $\pi_*\colon \mathrm{H}^{odd}_1(M;\mathbb{R}) \to \mathrm{H}^{odd}_1(\Gamma;\mathbb{R})$, which is surjective. Therefore, the map $[\alpha] \mapsto \mathcal{C}_{[\alpha]}$ between affine spaces is surjective as well. Thus, the theorem is proved. □

Corollary 5.3.13. *Let N be a closed connected non-orientable surface equipped with a density ρ. Then the space of coadjoint orbits of the group $\mathrm{Diff}_\rho(N)$ corresponding to the same measured Reeb graph (Γ, ι, f, μ) is an affine space of dimension*

$$d = \dim \mathrm{H}^{odd}_1(\Gamma;\mathbb{R}) = \frac{1}{2}(\#\mathrm{Fix}(\iota) + b_1(N) - 1),$$

where $\#\mathrm{Fix}(\iota)$ is the number of fixed points of ι, and $b_1(N) = \dim H_1(N,\mathbb{R})$ is the first Betti number of N. In particular,

$$\frac{1}{2}(b_1(N) - 1) \le d \le b_1(N). \tag{5.18}$$

Note that for an orientable surface M the corresponding dimension d is always $\frac{1}{2}b_1(M)$, i.e. the genus of M.

Example 5.3.14. Consider the projective plane $\mathbb{RP}^2$. The first homology group $\mathrm{H}_1(\mathbb{RP}^2;\mathbb{R})$ is trivial. Therefore, in this case there is a one-to one correspondence between generic coadjoint orbits and measured Reeb graphs with involution, in agreement with Example 5.3.9.

Example 5.3.15. Here we elaborate on Example 1.3.12 from the introduction. The function on a torus shown in Figure 1.3 defines a pseudo-function on the Klein bottle K^2. One has $b_1(K^2) = 1$, while the involution ι has no fixed points. Therefore, in this case the space of coadjoint obits corresponding to the given measured Reeb graphs with involution is 0-dimensional, i.e. the graph completely determines the orbit, just like in Example 5.3.14.

Now consider a donut lying on a horizontal table, and let F be the height function on its surface, normalized so that the center of symmetry of the donut is at height 0. Then F is odd with respect to the central symmetry. Furthermore, even though F is not a Morse function (its critical points are degenerate and form two circles), we can still consider the corresponding graph Γ_F defined as the set of F-levels with quotient topology, and that graph is equipped with an involution ι induced by central symmetry of the donut. Topologically, the graph Γ_F is a circle, while the involution ι is given by axial symmetry and has two fixed points. Now consider a small odd Morse perturbation of F (e.g. consider the height function for a donut on a slightly inclined table). Then each critical circle of F will fall apart into two Morse critical points, and the resulting graph with involution will be as shown in Figure 1.4: by continuity the involution on the graph still has two fixed points. The so-obtained function on the torus can again be thought as a pseudo-function on the Klein bottle K^2. By Corollary 5.3.13, the space of coadjoint orbits of $\mathrm{Diff}_\rho(K^2)$ corresponding to such a function is 1-dimensional, as opposed to the height function on a "standing torus" where the dimension of the orbit space is 0. Note that 0 and 1 are the only possible dimensions of the orbit space for the Klein bottle, see Example 5.3.10.

6 CASIMIR INVARIANTS OF THE 2D EULER EQUATIONS

6.1 MOMENTS OF FUNCTIONS

Having classified coadjoint orbits of the group SDiff(N) in terms of graphs with involution and certain additional structures, now one can extract the following list of numerical invariants of the coadjoint action, i.e., Casimir functions. Recall first the description of such invariants for functions on symplectic surfaces.

Let (M, μ) be a closed connected oriented symplectic surface, and let F be a simple Morse function on M. As invariants of the coadjoint action of SDiff(M), one usually considers total moments

$$m_i(F) = \int_M F^i \mu \ = \int_\Gamma f^i \,\mathrm{d}\rho\,, \quad i = 0, 1, 2, \dots$$

for the vorticity function $F = \mathrm{Diff}[\alpha]$, where (Γ, f, ρ) is the measured Reeb graph of F. However, the latter moments do not form a complete set of invariants even in the case of a sphere.

Remark 6.1.1. Consider, for example, the measured Reeb graph (Γ, f, ρ) depicted in Figure 6.1. Define a new measure $\tilde{\rho}$ on Γ by "moving some density from one branch to another", from I_1 to I_2. Then $(\Gamma, f, \tilde{\rho})$ is again a measured Reeb graph. Moreover, for all i we have the equality of total moments:

$$\int_\Gamma f^i \,\mathrm{d}\rho = \int_\Gamma f^i \,\mathrm{d}\tilde{\rho}\,.$$

However, the measured graphs (Γ, f, μ) and $(\Gamma, f, \tilde{\mu})$ are not isomorphic and thus correspond to two different coadjoint orbits of SDiff(S^2).

This is why one needs to refine the measure to each edge of the measured Reeb graph. With each edge e of the graph $\Gamma_F = (\Gamma, f, \rho)$, one can associate an infinite sequence of moments

$$m_{i,e}(F) = \int_e f^i \,\mathrm{d}\rho = \int_{M_e} F^i \mu\,,$$

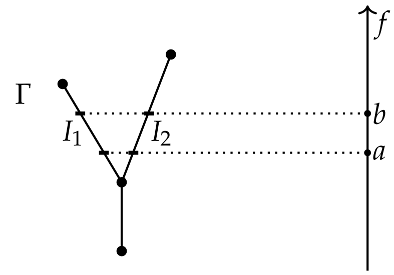


Figure 6.1: Modifying the measure on the edges.

where $i = 0, 1, 2, \dots$, and $M_e = \pi^{-1}(e)$ for the natural projection $\pi\colon M \to \Gamma$. (The corresponding moments $m_{i,e}(F)$ for vorticity F are natural to call *generalized enstrophies*.) Obviously, the moments $m_{i,e}(F)$ are invariant under the action of SDiff(M) on simple Morse functions. Moreover, they form a complete set of invariants in the sense that an isomorphism of abstract directed graphs which preserves moments of two functions on all edges implies that the measured Reeb graphs of those functions are also isomorphic, and hence the corresponding functions on the surface lie in the same coadjoint SDiff(M)-orbit, see [17].

6.2 MOMENTS OF PSEUDO-FUNCTIONS

For pseudo-functions on a non-oriented surface N one has to consider the Reeb graphs Γ_F of their lifts to the oriented double cover $M = \widetilde{N}$ together with involution.

Theorem 6.2.1 ([18]). *Given a graph Γ_F with involution ι, the set of moments $m_{i,e}(F)$ for all edges is a complete set of invariants for the* SDiff(N)*-action on a pseudo-function F. Namely, an isomorphism of abstract directed graphs preserving moments of two pseudo-functions on all edges implies that the corresponding pseudo-functions on N lie in the same* SDiff(N)*-orbit.*

Proof. The first part of the argument repeats that in the orientable case. Namely, for any edge $e = [v, w] \in \Gamma_F$ pushing forward the measure ρ on e by means of the homeomorphism $f\colon e \to [f(v), f(w)] \subset \mathbb{R}$, we obtain a measure ρ_f on the interval $I_f = [f(v), f(w)]$, whose moments coincide with the moments of ρ at e. All the moments of ρ_f define the measure ρ_f on I_f uniquely by the uniqueness theorem for the Hausdorff moment problem (see Remark 6.2.2).

Taking into account the involution ι on the graph Γ_F, one can consider the moments for only "half" of the edges, mapped to each other by the involution. For the edges that are anti-invariant under the involution one can confine to only even moments ρ_f. This gives a complete (although not necessarily minimal) set of the $\mathrm{SDiff}(N)$-invariants for pseudo-functions. □

Remark 6.2.2. The Hausdorff moment problem gives the following necessary and sufficient condition: a sequence of numbers m_k can be the set of moments $m_k(\lambda) = \int_0^1 \lambda^k \, d\rho(\lambda)$ of some Borel measure ρ supported on the interval $[0,1]$ if and only if it satisfies the so-called monotonicity conditions. The latter are linear inequalities on m_k, which can be derived from the relations $\int_0^1 \lambda^k (1-\lambda)^n \, d\rho(\lambda) \geq 0$ for all integer $k, n \geq 0$, where the left-hand side is expressed in terms of m_k. (For instance, $m_5 - 2m_6 + m_7 = \int_0^1 \lambda^5 (1-\lambda)^2 \, d\rho(\lambda) \geq 0$.) In our case, replacing λ by the parameter f we only employ the statement that the measure $\rho(f)$ is fully determined by the set $\{m_k, k = 0, 1, 2, \ldots\}$. Note that under certain regularity conditions the measure ρ can be found in a constructive way from the moment sequence $\{m_k\}$, as a normalized jump across the cut in the real axis for a function defined by the Laurent series $\sum_{k \geq 0} m_k / \lambda^{k+1}$.

6.3 GENERALIZED ENSTROPHIES AND CIRCULATIONS

The above Theorem 6.2.1 allows one to describe Casimirs of the 2D Euler equation on a sphere and projective plane, where the corresponding Reeb graph is a tree. In those cases the vorticity (pseudo-)function fully describes the corresponding fluid velocity u and hence the corresponding coadjoint orbit. The full set of Casimirs is formed by the corresponding generalized enstrophies, i.e. moments $m_{i,e}(F)$ for the Morse vorticity function F.

For the surfaces of higher genus (or the Reeb graphs Γ with nontrivial $H_1(\Gamma, \mathbb{R})$) the above Casimirs must be supplemented by several circulations. For instance, for a torus one needs to fix the value of one circulation (and, more generally, one value for each handle of an orientable surface). In the example in Figure 1.1, one can fix a value of the circulation function, e.g., at the lower boundary of domain M_e, corresponding to the bottom of edge e. (In particular, one can set circulations over all critical levels of F, which contain the one above, and several other circulations which are dependent on it.) Overall, one needs to fix d independent circulations, where the value of d is given by Corollary 5.3.13.

Theorem 6.3.1 ([18])**.** *A complete set of Casimirs for the 2D Euler equation on a non-orientable surface N in a neighborhood of a Morse-type coadjoint orbit is given by the moments $m_{i,e}$ for each edge $e \in \Gamma$ of the graph Γ with involution, and by circulations of the velocity v over certain d independent cycles on N, where $d = \frac{1}{2}(\#\mathrm{Fix}(\iota) + \dim \mathrm{H}_1(N; \mathbb{R}) - 1)$. Here $\#\mathrm{Fix}(\iota)$ is the number of fixed points of the involution ι on the Reeb graph Γ and $i = 0, 1, 2, \ldots$.*

Similarly to the above discussion, the set of all generalized enstrophies and circulations described in this theorem is not a "minimal set" of Casimirs, as the Hausdorff moment problem does not claim the minimality.

BIBLIOGRAPHY

[1] G. Adelson-Welsky and A. Kronrode. Sur les lignes de niveau des fonctions continues possédant des dérivées partielles. *Comptes rendus (Doklady) de l'Académie des sciences de l'URSS*, 49(4):235–237, 1945.

[2] V. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*. Springer, 1999.

[3] N. Balabanova. A Hamiltonian approach for point vortices on non-orientable surfaces II: the Klein bottle. *arXiv:2202.06175*, 2022.

[4] N. Balabanova and J. Montaldi. Hamiltonian approach for point vortices on non-orientable surfaces I: the Mobius band. *arXiv:2202.06160*, 2022.

[5] A. Bolsinov. A smooth trajectory classification of integrable Hamiltonian systems with two degrees of freedom. *Sbornik: Mathematics*, 186(1):1, 1995.

[6] A. Bolsinov and A. Fomenko. *Integrable Hamiltonian Systems: Geometry, Topology, Classification*. CRC Press, 2004.

[7] A. Bolsinov and A. Oshemkov. Singularities of integrable Hamiltonian systems. *Topological methods in the theory of integrable systems*, pages 1–67, 2006.

[8] M. Borodzik, A. Némethi, and A. Ranicki. Morse theory for manifolds with boundary. *Algebr. Geom. Topol.*, 16(2):971–1023, 2016.

[9] M. Bruveris, P. Michor, A. Parusiński, and A. Rainer. Moser's theorem on manifolds with corners. *Proceedings of the American Mathematical Society*, 146(11):4889–4897, 2018.

[10] G. De Rham. *Differentiable manifolds: forms, currents, harmonic forms*, volume 266. Springer Science & Business Media, 2012.

[11] Y. Colin De Verdière and J. Vey. Le lemme de Morse isochore. *Topology*, 18(4):283–293, 1979.

[12] W. Donnelly, L. Freidel, S.F. Moosavian, and A.J. Speranza. Gravitational edge modes, coadjoint orbits, and hydrodynamics. *Journal of High Energy Physics*, 2021(9):1–63, 2021.

[13] J.P. Dufour, P. Molino, and A Toulet. Classification des systèmes intègrables en dimension 2 et invariants des modèles de Fomenko. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 318(10):949–952, 1994.

[14] L.H. Eliasson. Normal forms for Hamiltonian systems with poisson commuting integrals—elliptic case. *Comment. Math. Helv.*, 65(1):4–35, 1990.

[15] B. Hladysh and A. Prishlyak. Simple Morse functions on an oriented surface with boundary. *Zh. Mat. Fiz. Anal. Geom.*, 15(3):354–368, 2019.

[16] A. Izosimov and B. Khesin. Characterization of steady solutions to the 2D Euler equation. *Int. Math. Res. Not. IMRN*, 2017(24):7459–7503, 2017.

[17] A. Izosimov and B. Khesin. Classification of Casimirs in 2D hydrodynamics. *Moscow Mathematical Journal*, 17(4):699–716, 2017.

[18] A. Izosimov, B. Khesin, and I. Kirillov. Coadjoint orbits of area-preserving diffeomorphisms of non-orientable surfaces. *arXiv:2304.09354*, 2023.

[19] A. Izosimov, B. Khesin, and M. Mousavi. Coadjoint orbits of symplectic diffeomorphisms of surfaces and ideal hydrodynamics. *Annales de l'Institut Fourier*, 66(6):2385–2433, 2016.

[20] B. Khesin and R. Wendt. *The Geometry of Infinite-Dimensional Groups*. Springer, 2008.

[21] A. Kirillov. Orbits of the group of diffeomorphisms of a circle and local lie superalgebras. *Funct. Anal. Appl.*, 15(2):135–137, 1981.

[22] A. Kirillov. The orbit method, II: Infinite-dimensional Lie groups and Lie algebras. *Contemp. Math.*, 145:33–33, 1993.

[23] I. Kirillov. Morse-Darboux lemma for surfaces with boundary. *J. Geom. Phys.*, 129:34–40, 2018.

[24] I. Kirillov. Classification of coadjoint orbits for symplectomorphism groups of surfaces. *International Mathematics Research Notices*, 2023(7):6219–6251, 2023.

[25] K. Kourliouros. Local diffeomorphisms in symplectic space and Hamiltonian systems with constraints. *J. Geom. Phys.*, 138:206–214, 2019.

[26] B. Kruglikov. Exact smooth classification of Hamiltonian vector fields on two-dimensional manifolds. *Math. Notes*, 61(2):146–163, 1997.

[27] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.*, 120(2):286–294, 1965.

[28] L. Nicolaescu. *An invitation to Morse theory*. Springer, 2007.

[29] R. Penna. SDiff(S^2) and the orbit method. *J. Math. Phys.*, 61:012301, 2020.

[30] G. Reeb. Sur les points singuliers dúne forme de Pfaff complètement intégrable ou d'une fonction numérique. *Comptes Rendus de l'Académie des Sciences. Paris*, 222:847–849, 1946.

[31] A.R. Shastri. *Elements of differential topology*. CRC Press, 2011.

[32] A. Toulet. Classification des systèmes intégrables en dimension 2. *PhD thesis, Université de Montpellier 2*, 1996.

[33] J. Vanneste. Vortex dynamics on a Möbius strip. *Journal of Fluid Mechanics*, 923:A12, 2021.

COLOPHON

This thesis was typeset using the typographical look-and-feel `classicthesis` developed by André Miede and Ivo Pletikosić.

The style was inspired by Robert Bringhurst's seminal book on typography *"The Elements of Typographic Style"*.