Directed polymers in the intermediate disorder regime and the Seppäläinen–Johansson model

by

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Abstract

In this thesis, we study two discrete models of random growth in the Kardar–Parisi–Zhang (KPZ) universality class: the directed polymer and the Seppäläinen–Johansson first-passage percolation model.

The directed polymer was introduced by Huse and Henley as a model for the domain wall in a ferromagnetic Ising model with random bond impurities. This model depends on a parameter β, the inverse temperature. We consider the intermediate disorder regime, which consists in taking β to depend on the length of the polymer $2n$, with $\beta = n^{-\alpha}$ for some $\alpha > 0$. In this regime, there is a critical phase transition that happens at $\alpha = \frac{1}{4}$. When $\alpha > \frac{1}{4}$, the fluctuations of the free energy are of order $n^{(1-4\alpha)/4}$ and converge to a Gaussian. For $\alpha < \frac{1}{4}$, it was conjectured that the polymer should fall back in the KPZ regime, and that the fluctuations should instead be of order $n^{(1-4\alpha)/3}$, and converge after rescaling to the Tracy–Widom GUE distribution. We prove this conjecture for $\frac{1}{8} < \alpha < \frac{1}{4}$ for arbitrary i.i.d weights with exponential moments.

The Seppäläinen–Johansson model was introduced by Seppäläinen as a simplified version of first-passage percolation where he was able to explicitly compute the limiting shape for Bernoulli weights. The behaviour of the fluctuations for this process were later studied by Johansson. We consider a generalization of this model, involving two families of i.i.d random variables $\{\xi_{ij}\}$ and $\{\eta_{ij}\}$ corresponding to the weights of the horizontal and vertical edges respectively. We obtain an explicit formula for the limiting shape of the first-passage distance expressed in terms of the corresponding limit shapes of the two sets of weights for the Seppäläinen–Johansson model. We also study the limiting fluctuations of this model when at least one of the sets of weights is Bernoulli distributed, and we prove that these converge to the Airy_2 process.

To my parents

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Chapter 1

Introduction

1.1 Random growth models and KPZ universality

In their seminal 1986 paper [45], Kardar, Parisi and Zhang derived their now eponymous *KPZ equation* to model the random growth of interfaces between two media:

$$\partial_t h = \nu \partial_x^2 h - \lambda(\partial_x h)^2 + \sqrt{D}\xi. \tag{1.1}$$

Here $h(t,x)$ is a height function, describing the height of the interface at time t and position x, ν, λ and D are physical constants, and $\xi(t,x)$ is space-time white noise. Thus (1.1) is a non-linear, stochastic partial differential equation.

Using renormalization group methods, they predicted that solutions h to (1.1) should have an interesting limit as $\epsilon \to 0$ under the scaling

$$h_\epsilon(t,x) = \epsilon^{1/2} h\left(\frac{t}{\epsilon^{3/2}}, \frac{x}{\epsilon}\right). \tag{1.2}$$

This means that for large times t, one expects to see non-trivial fluctuations at spatial scales $x \approx t^{2/3}$, and those fluctuations should be of order $t^{1/3}$. The scaling exponents $\zeta = \frac{2}{3}$ and $\chi = \frac{1}{3}$ are the 1-dimensional KPZ transversal and longitudinal exponents respectively, and (1.2) is the KPZ 1-2-3 scaling.

Since the original paper of Kardar, Parisi and Zhang, it has been shown that many other models of random growth, both continuous and discrete, have the same scaling exponents as above. This collection of models is now known as the *KPZ universality class* and includes directed polymers, first and last-passage percolation, interacting particle systems, random metric spaces, longest increasing subsequences of random permutations, the stochastic six vertex model, non-intersecting random walks and line ensembles, and much more. The KPZ equation has also been used in other disciplines to describe many other models, such as bacterial growth [36] and liquid crystal turbulence [71, 70]. See [23, 60] for mathematical surveys on KPZ theory, and [37] for a review from the physics literature.

The classical central limit theorem states that if S_n is the sum of n independent, identically distributed random variables with mean μ and finite, non-zero variance σ^2, then for large n,

$$S_n \approx \mu n + \sigma n^{\frac{1}{2}} X, \tag{1.3}$$

where X is a standard Gaussian random variable. There is a similar asymptotic formula that holds for models in the KPZ universality class. The main order term is also linear in n (or t for continuous models) with a coefficient representing the limiting shape of the model. However as discussed above, the size of the error term is $n^{1/3}$ as opposed to $n^{1/2}$, and instead of a normal X, there is a random variable Y following a different universal distribution, the *Tracy–Widom GUE*:

$$h_n \approx C_1 n + C_2 n^{\frac{1}{3}} Y. \tag{1.4}$$

This probability law was first discovered by Tracy and Widom in [77] in the context of random matrix theory, as the scaling limit of the top eigenvalue value in the Gaussian Unitary Ensemble. Unlike the Gaussian, the density of the Tracy–Widom GUE does not have a simple expression. The distribution function can be written as a Fredholm determinant:

$$F_{\text{GUE}}(t) = \det(I - \mathbf{Ai})_{L^2(t,\infty)},$$

where $\mathbf{Ai}$ is the Airy kernel. Alternatively, it can be written in terms of the solution of a Painlevé equation:

$$F_{\text{GUE}}(t) = \exp\left(-\int_t^\infty (x-t) q(x)^2 dx\right)$$

where q is the unique solution of the differential equation

$$q''(x) = xq(x) + 2q(x)^3$$

which satisfies the asymptotic $q(x) \sim Ai(x)$ as $x \to \infty$ (here $Ai(x)$ is the Airy function). Just like the Gaussian distribution is the basic building block for Brownian motion, the Tracy–Widom GUE describes the one-point marginals of several continuous processes conjectured to be the functional scaling limits of all the KPZ models. These processes include the Airy_2 process, the KPZ fixed point and the directed landscape.

A central open problem in this field of study is to establish *universality*. Roughly speaking, this means that the asymptotic global behaviour (i.e the scaling exponents and limiting distributions) should not depend on the local structure of the models and should be invariant under small perturbations. For discrete models, the randomness often comes from some family of i.i.d random variables; then universality in this context means that under suitable tail decay conditions, (1.4) should hold regardless of the distribution of those variables, just like the CLT holds for any choice of i.i.d non-trivial L^2 random variables.

In contrast to the CLT, for almost every model mentioned earlier, we only know how to prove belonging to the KPZ universality class for very specific choices of distribution of the noise. For example, in last-passage percolation, (1.4) is only known to hold for exponential and geometric environments. The proofs for this and other models usually exploits special algebraic properties of the laws of the random inputs which then makes these models exactly solvable. If the distributions of the variables in this environment do not have those properties, then there is little hope of obtaining exact formulas. The goal of this thesis is to obtain some universality results for two discrete models: the directed polymer and the Seppäläinen–Johansson model. We now describe these two models and state our main results.

1.2 The directed polymer

In this section, we give a brief description of the model and of the related exactly solvable model, the log-gamma polymer. We describe the intermediate disorder regime and illustrate the phase transition that occurs in this regime. We then state our main results on the universality of directed polymers in the intermediate disorder regime.

1.2.1 Description of the model

The directed polymer was introduced in the physics literature by Huse and Henley during the 80's in [40] as a model of the domain wall in a ferromagnetic Ising model with random bond impurities. Since then, its study has been taken up by both physicists and mathematicians and it has become one of the central models conjectured to be in the KPZ universality class. See [22] for a survey on the topic.

The model can be described as follows. Let $\xi_{i,j}$, with $i, j \in \mathbb{Z}_{\geq 0}$ be a collection of independent random variables, and let $\beta > 0$ be a parameter, which is commonly referred to as the *inverse temperature*. We define the *(point-to-point) partition function* by

$$Z_n(\beta) = \sum_{\pi:(0,0)\to(n,n)} \prod_{i=0}^{2n} e^{\beta \xi_{\pi(i)}}.$$

Here the sum is taken over all up-right paths π which start at $(0,0)$ and end at (n,n), that is the set of functions

$$\pi : \{0, 1, \ldots, 2n\} \to \mathbb{Z}^2_{\geq 0}$$

such that $\pi(0) = (0,0)$, $\pi(2n) = (n,n)$ and for each i, $\pi(i+1) - \pi(i) = (1,0)$ or $(0,1)$. See Figure 1.1. The *free energy* is $\log Z_n(\beta)$, and the *polymer measure* is the random probability measure $\mathbb{P}^{n,\beta}_{\text{Poly}}$ on up-right paths defined by

$$\mathbb{P}^{n,\beta}_{\text{Poly}}(\pi) = \frac{\prod_{i=0}^{2n} e^{\beta \xi_{\pi(i)}}}{Z_n(\beta)}.$$

For convenience, we will henceforth write the partition function and polymer measure in terms of Bernoulli paths. A *Bernoulli path* of length $2n$ is a function

$$\pi : \{0, 1, \ldots, 2n\} \to \mathbb{Z}_{\geq 0}$$

such that $\pi(i+1) - \pi(i) = 0$ or 1 for each i. There is an obvious bijection between the set of up-right paths from $(0,0)$ to (n,n) and the set of Bernoulli paths of length $2n$ started at 0 and ending at n, so this transformation does not change anything. This translation however will make a lot of the expressions easier to write down in terms of intersections of Bernoulli walks. In terms of Bernoulli

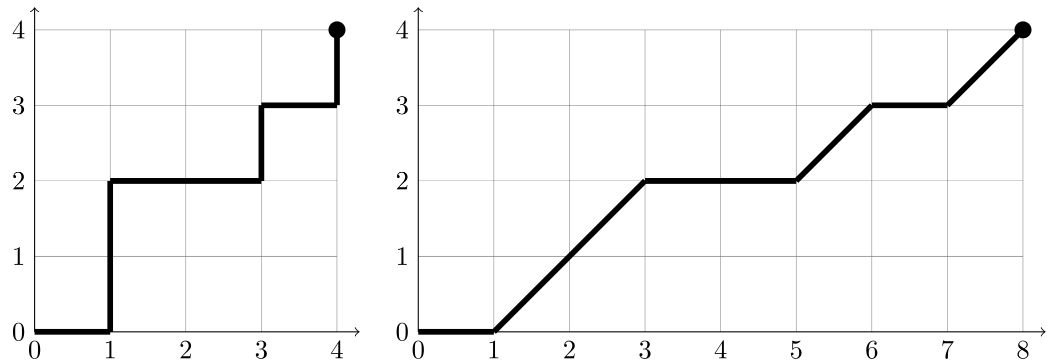

Figure 1.1: On the left: an up-right path from (0,0) to (4,4), and on the right the corresponding Bernoulli path from (0,0) to (8,4)

paths, the partition function is

$$Z_n(\beta) = \sum_{\pi:\pi(2n)=n} \prod_{i=0}^{2n} e^{\beta \xi_{i,\pi(i)}}$$

where now the sum is taken over Bernoulli paths π such that $\pi(2n) = n$, and the polymer measure is

$$\mathbb{P}^{n,\beta}_{\text{Poly}}(\pi) = \frac{\prod_{i=0}^{2n} e^{\beta \xi_{i,\pi(i)}}}{Z_n(\beta)}.$$

The central problem with the directed polymer is to understand the behaviour of the free energy and the polymer measure as a function of β and as $n \to \infty$.

The directed polymer is expected to be in the KPZ universality class. Unlike simple random walk, the (quenched) polymer measure is localized [21]. This means that there will be random locations (i, j) where paths concentrate. These locations correspond to weights $\xi_{i,j}$ which are big; indeed we can see from the definition of $\mathbb{P}^{n,\beta}_{\text{Poly}}$ that paths that visit those points get assigned higher probability. For large β, almost all of the mass of the measure will go towards the path of highest mass. It is expected that for $\beta > 0$, the midpoint distribution has variance of order 1 around a random favourite location which is situated in a window of size $n^{2/3}$.

On the other hand, the annealed measure (i.e the measure obtained after averaging over the environment) exhibits “superdiffusive” behaviour, which loosely means that the transversal fluctuations are of order n^ζ for some large fixed exponent ζ. When $\beta = 0$, the polymer measure is random walk bridge measure, and $\zeta = 1/2$. For $\beta > 0$, the conjecture is the KPZ exponent $\zeta = 2/3$. This was verified numerically in Huse and Henley’s original paper [40], but it has only been proven for the very special log-gamma polymer of Seppäläinen [13, 18, 68], and for general weights in thin rectangles [8].

As for the free energy, the KPZ predictions tell us that the fluctuations should be of order $n^{1/3}$, and the limiting distribution of the scaled fluctuations is the Tracy–Widom GUE. This has only been confirmed for the log-gamma polymer again [13, 18, 68], some other exactly solvable models like the beta polymer [12], strict weak polymer [25, 55] and inverse beta polymer [75], as well as for certain models of last passage percolation [10, 43, 44].

1.2.2 The log-gamma polymer

The *log-gamma polymer* is a similar model to the directed polymer described in terms of a collection of i.i.d weights $X_{i,j}$ following the Gamma$(\theta, 1)$ distribution, that is they have density

$$f(x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x} \quad (x > 0).$$

The partition function in this case is defined in a similar way as before as

$$Z_n(\theta) = \sum_{\pi} \prod_{i=0}^{2n} \frac{1}{X_{i,\pi(i)}},$$

with again the sum being taken over Bernoulli paths π such that $\pi(2n) = n$. The free energy is $\log Z_n(\theta)$. Here we think of θ as a positive parameter which plays a similar role to the inverse temperature β. As we will see later, by taking θ to be a function of β such that $\theta \sim c/\beta^2$ as $\beta \to 0$ for some positive constant c, the moments of the random variables $1/X_{i,j}$ will have the same asymptotic behaviour as those of $e^{\beta\xi_{i,j}}$ as $\beta \to 0$.

The name "log-gamma polymer" comes from the fact that since $X_{i,j} > 0$, we can rewrite the partition function as

$$Z_n(\theta) = \sum_{\pi} \prod_{i=0}^{2n} e^{\xi_{i,\pi(i)}(\theta)}$$

where $\xi_{i,j}(\theta) = -\log X_{i,j}$ is the log of a Gamma distributed random variable. Written this way, the partition function then has a similar form to the one for a standard polymer. However, it is important to note that the log-gamma polymer is not a special case of the directed polymer; the dependence on θ (or β if we write θ as a function of β) is non-trivial and is built into the distribution of the weights. That is, the density of $\xi_{i,j}(\theta)$ depends on θ.

The relevance of this model is that it is *exactly solvable*, in the sense that one can write down an explicit formula for the distribution of the free energy $\log Z_n(\theta)$ in terms of a Fredholm determinant, and so one can do some asymptotic analysis of these exact formulas to derive the scaling exponents and limiting distributions. For general weights, no such formula is believed to exist.

1.2.3 The intermediate disorder regime

In this thesis, we consider the *intermediate disorder regime* of the directed polymer. This consists in taking $\beta = \beta_n$ depending on n and such that $\beta_n \to 0$ as $n \to \infty$. This regime was introduced by Alberts, Khanin and Quastel in [3], and they showed in [2, 3] that for $\beta_n \sim n^{-1/4}$, the directed polymer converges to a non-trivial limit known as the *continuum directed polymer*.

If $\beta_n = n^{-\alpha}$ for some $\alpha > 1/4$, then the limiting fluctuations of the free energy are Gaussian. This can be seen by doing a Taylor expansion of $\log Z_n(\beta)$ centred at $\beta = 0$:

$$\log Z_n(\beta) = \log \binom{2n}{n} + \left(\sum_{i,j} \mathbb{P}_{\mathrm{Bri}}((i,j) \in \pi) \xi_{ij} \right) \beta + \dots,$$

for π a "Bernoulli bridge", that is π is uniformly distributed on the set of Bernoulli paths such that $\pi(2n) = n$; the notation will be made clearer in Section 2.1. One can check that the variance of the coefficient of the k-th order term is of order $n^{k/2}$. So with $\alpha > 1/4$, the terms of order 2 and higher vanish in the limit, and the order 1 term is a weighted sum of independent random variables, so it has Gaussian fluctuations by the central limit theorem. For $\alpha = 1/4$, the terms of higher order do not vanish; in fact, each term in the expansion converges in distribution individually, and Alberts, Khanin and Quastel showed that the entire sum converges.

When $\alpha < 1/4$, it was conjectured that the directed polymer should fall back in the KPZ regime and that the fluctuations are of order $\beta^{4/3}n^{1/3}$ with a Tracy–Widom GUE limit. This was proven by Krishnan and Quastel for the log-gamma polymer.

Theorem 1.1. *[46, Theorem 2.1] Let $\theta \sim c/\beta_n^2$ as $n \to \infty$, where $\beta_n = n^{-\alpha}$ for some $0 < \alpha < \frac{1}{4}$ and c is a positive constant. Then*

$$\frac{\log Z_n(\theta) + 2n\Psi(\theta/2)}{-(\Psi''(\theta/2))^{1/3}n^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}.$$

Here Ψ is the digamma function. Our goal is to extend Theorem 1.1 to the standard directed polymer in the intermediate disorder regime for arbitrary i.i.d weights $\xi_{i,j}$.

1.2.4 Main results

We consider collections of positive independent random variables $\omega_{i,j}(\beta)$, $i, j \geq 0$, which are parametrized by the inverse temperature β. The usual directed polymer corresponds to taking $\omega_{i,j}(\beta) = e^{\beta\xi_{i,j}}$ where the $\xi_{i,j}$'s are independent random variables. We allow this more complicated dependence on β in order to include the log-gamma polymer, which is not given by a standard polymer. The partition function is given by

$$Z_n(\beta) = \sum_{\pi} \prod_{i=0}^{2n} \omega_{i,\pi(i)}(\beta),$$

the free energy is $\log Z_n(\beta)$ and the polymer measure is

$$\mathbb{P}^{n,\beta}_{\mathrm{Poly}}(\pi) = \frac{\prod_{i=0}^{2n} \omega_{i,\pi(i)}(\beta)}{Z_n(\beta)}.$$

We are primarily interested in the intermediate disorder regime, which corresponds to taking $\beta = \beta_n \to 0$ as $n \to \infty$. In view of the discussion in the previous subsection, we will consider the case when $\beta = n^{-\alpha}$ for some fixed $\alpha < 1/4$. Our main result is the following.

Theorem 1.2. *Let $\frac{1}{8} < \alpha < \frac{1}{4}$, and set $\beta = n^{-\alpha}$. Let $\xi_{i,j}$ with $i, j \geq 0$ be i.i.d random variables with variance $\sigma^2 > 0$ and an exponential moment, that is $\mathbb{E}(e^{c|\xi_{1,1}|}) < \infty$ for some $c > 0$, and let $Z_n(\beta)$ be the point-to-point partition function for the corresponding directed polymer. Then there exists a deterministic sequence a_n such that*

$$\frac{\log Z_n(\beta) - a_n}{(4\sigma^4\beta^4 n)^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}.$$

We obtain an explicit expression for a_n at the end of Section 4.3. This expression depends on the distribution of $\xi_{i,j}$, but only in terms of its first six moments.

The significance of Theorem 1.2 is that it is *universal*, in the sense that the scaling exponent and limiting distribution do not depend on the choice of distribution of the weights. As discussed in Section 1.1, KPZ universality in random growth models is a major open problem, and there are very few models for which it has been shown.

The key ingredient in the proof of Theorem 1.2 is a perturbation theorem. In order to properly state it, we need to specify which sets of parametrized weights $\omega_{i,j}(\beta)$ we will be considering.

Definition 1.3. A collection of independent, parametrized random variables $\omega_{i,j}(\beta)$ where $i,j \in \mathbb{Z}_{\geq 0}$ and $\beta > 0$ is called *valid* if it satisfies the following conditions for all i and j and for all sufficiently small β:

1. $\omega_{i,j}(\beta) > 0$ almost surely.

2. $\mathbb{E}(\omega_{i,j}(\beta)) = 1$.

3. For all positive integers k, there is a constant $C_k > 0$ such that

$$\mathbb{E}(|\omega_{i,j}(\beta) - 1|^k|) \leq C_k \beta^k$$

4. There is a $p > 3$ and a constant $C > 0$ such that

$$\mathbb{E}\left(\frac{1}{\omega_{i,j}(\beta)^p}\right) \leq C.$$

Condition 1 is fairly natural and avoids the possibility of the partition function being negative. Condition 2 is mostly a convenient normalization constraint and can be achieved by simply dividing each weight by its mean if it is not satisfied already. Condition 4 is a technical condition to prevent some pathological problems of some weights taking really small values but with low probability; we will see that in the two cases that interest us (the directed polymer and the log-gamma polymer), this condition is easily verified.

The heart of Definition 1.3 is Condition 3, which gives a precise rate at which $\omega_{i,j}(\beta)$ is converging to 1 as $\beta \to 0$ uniformly in i and j. The main technique that will go into the proofs of Theorem 1.2 and Theorem 1.5 is Taylor expansion, so having precise asymptotic bounds on the centred moments of the $\omega_{i,j}(\beta)$'s will be crucial. We do not impose any restriction on how fast the C_k's grow as $k \to \infty$, only that they are independent of i, j and β.

We make one more important remark about Definition 1.3 before stating the perturbation theorem.

Remark 1.4. If $\omega_{i,j}(\beta)$ and $\omega'_{i,j}(\beta)$ are two independent collections of valid weights, then any combination of the two sets of weights is also valid. More precisely, let $G : \mathbb{Z}^2 \to \{0,1\}$ be an arbitrary function, and define

$$\omega''_{i,j}(\beta) = \begin{cases} \omega_{i,j}(\beta) & \text{if } G(i,j) = 0 \\ \omega'_{i,j}(\beta) & \text{if } G(i,j) = 1. \end{cases}$$

Then $\omega''_{i,j}(\beta)$ is valid. In fact, it is easy to see that one can pick the constants in Conditions 3 and 4 so that those inequalities work uniformly over all choices of G.

Theorem 1.5. *Let $\frac{1}{8} < \alpha < \frac{1}{4}$, and set $\beta = n^{-\alpha}$. Let $\omega_{i,j}(\beta)$ and $\omega'_{i,j}(\beta)$ be independent valid collections of weights such that for each $i, j \geq 0$, and for every sufficiently small β, $\mathbb{E}(\omega_{i,j}(\beta)^2) = \mathbb{E}(\omega'_{i,j}(\beta)^2)$. Let $Z_n(\beta)$ and $Z'_n(\beta)$ be the corresponding partition functions for these two sets of weights. Then for a probability distribution F on $\mathbb{R}$ and a sequence of real numbers (a_n), we have*

$$\frac{\log Z_n(\beta) - a_n}{\beta^{4/3} n^{1/3}} \xrightarrow{d} F$$

if and only if

$$\frac{\log Z'_n(\beta) - a_n}{\beta^{4/3} n^{1/3}} \xrightarrow{d} F.$$

The reason for which the lower exponent $\alpha > \frac{1}{8}$ comes up is due to a large deviation bound that we obtain in Section 3.2. As the calculations at the end of Section 4.1 show, Theorems 1.2 and 1.5 should hold with $\frac{2}{17} < \alpha < \frac{1}{4}$, and we conjecture that this is indeed the case. If we strengthen the assumptions of Theorem 1.5 to $\omega_{i,j}(\beta)$ and $\omega'_{i,j}(\beta)$ having the same first k moments, then Theorems 1.2 and 1.5 should hold (with different a_n's) for $\alpha > \frac{2}{3k+11}$; see Remark 4.2 at the end of Section 4.1.

1.2.5 Outline of the proofs

In Section 4.2, we will show that both the standard directed polymer and the log-gamma polymer (or at least suitably normalized versions of them) are valid. By Theorem 1.1, the fluctuations of the log-gamma polymer converge to the Tracy–Widom GUE distribution, and so Theorem 1.2 will be a consequence of Theorem 1.5.

The main tool that goes into the proof of Theorem 1.5 is the Lindeberg method. Suppose that $g_n : \mathbb{R}^n \to \mathbb{R}$ is a sequence of functions and $X_1, X_2, \ldots$ is a sequence of independent random variables for which we know $g_n(X_1, \ldots, X_n) \xrightarrow{d} F$. The Lindeberg method consists in showing that $g_n(Y_1, \ldots, Y_n)$ also converges in distribution to F for a different sequence of independent random variables by estimating the error when one changes just one of the inputs of g_n from the X sequence to the Y sequence. If the sum of all the errors after changing each input one by one is $o(1)$, then we can conclude that $g_n(Y_1, \ldots, Y_n) \xrightarrow{d} F$. This is typically done by expanding

$$\mathbb{E}(f \circ g_n(X_1, \ldots, X_{j-1}, t, Y_{j+1}, \ldots, Y_n))$$

(here f is some test function) as a Taylor series around t, evaluating the resulting expression at $t = X_j$ and $t = Y_j$ and subtracting, and then using some kind of moment matching condition to cancel certain terms. In our case, g_n will correspond to the free energy $\log Z_n(\beta)$.

The Lindeberg exchange trick was first used by Lindeberg in [49] to give an alternate proof of the central limit theorem. It is now a standard tool for proving universality results, and it has been utilized notably in random matrix theory, see for example [72, 73] on the four moment theorem for the universality of local eigenvalue statistics.

For a given (i, j), we can write

$$Z_n(\beta) = V_{i,j}(\beta) + \omega_{i,j}(\beta) W_{i,j}(\beta)$$

where

$$W_{i,j} = \sum_{\pi,(i,j)\in\pi} \prod_{\substack{\ell=0\\ \ell\neq i}}^{2n} \omega_{\ell,\pi(\ell)}, \quad V_{i,j} = \sum_{\pi,(i,j)\notin\pi} \prod_{\ell=0}^{2n} \omega_{\ell,\pi(\ell)}.$$

The summation for $W_{i,j}$ is taken over the set of paths that go through the point (i,j), and the product inside does not include the weight $\omega_{i,j}(\beta)$. The summation for $V_{i,j}$ is taken over the set of paths that do not visit the site (i,j). Let us now expand $\log(V_{i,j} + tW_{i,j})$ as a Taylor polynomial of order 3 centred at $t = 1$ (we omit the dependence on i, j and β to make the notation easier to follow):

$$\log(V + tW) \approx \log(V + W) + \frac{W}{V+W}(t-1) - \frac{W^2}{2(V+W)^2}(t-1)^2 + \frac{W^3}{3(V+W)^3}(t-1)^3.$$

We substitute $t = \omega_{i,j}$ and $t = \omega'_{i,j}$, take expectations and subtract. Since $\omega_{i,j}$ and $\omega'_{i,j}$ have the same first and second moments and V, W are independent of them, the order 0, order 1 and order 2 terms will all cancel each other out. For the third order term, we use Condition 3 of Definition 1.3 to find

$$\begin{aligned}|\mathbb{E}(\log(V + \omega_{i,j}W)) - \mathbb{E}(\log(V + \omega'_{i,j}W))| &\approx \mathbb{E}\left(\frac{W^3}{3(V+W)^3}\right) |\mathbb{E}((\omega_{i,j}-1)^3 + (\omega'_{i,j}-1)^3)| \\ &\leq C\beta^3 \mathbb{E}\left(\frac{W^3}{(V+W)^3}\right)\end{aligned}$$

for some constant C. This is the error obtained from changing the weight $\omega_{i,j}$ to $\omega'_{i,j}$; to get the total error, we do this replacement for all i and j and then sum everything to obtain

$$|\mathbb{E}(\log Z_n) - \mathbb{E}(\log Z'_n)| \leq C\beta^3 \mathbb{E}\left(\sum_{i,j} \frac{W_{i,j}^3}{(V_{i,j} + W_{i,j})^3}\right). \tag{1.5}$$

There are two technical issues here. First, we shouldn't be estimating the difference between $\log Z_n$ and $\log Z'_n$, but rather the difference between $f((\log Z_n - a_n)/(\beta^4 n)^{1/3})$ and $f((\log Z'_n - a_n)/(\beta^4 n)^{1/3})$ for f in some suitable class of test functions. This does not change the nature of the estimates that we get, but we omit the computations for this heuristic. Second, there is some ambiguity as to how $W_{i,j}$ and $V_{i,j}$ are defined; the weights that are multiplied in their expressions could be from the $(\omega_{i,j})$ polymer for some indices and from the $(\omega'_{i,j})$ polymer for other indices, depending on which stage of the replacement we have reached and the order in which we are exchanging the weights. Both of these problems are handled more carefully in Section 4.1.

Ignoring these technicalities, the double sum on the right-hand side of (1.5) has a nice interpretation in terms of the polymer measure. By Definition 1.3, we have $\omega_{i,j} \approx 1$, so

$$\frac{W_{i,j}^3}{(V_{i,j} + W_{i,j})^3} \approx \frac{(\omega_{i,j} W_{i,j})^3}{(V_{i,j} + \omega_{i,j} W_{i,j})^3} = \frac{(\omega_{i,j} W_{i,j})^3}{Z_n^3}.$$

This last expression is the probability that three independent polymer distributed paths all visit the site (i,j), and so

$$\sum_{i,j} \frac{W_{i,j}^3}{(V_{i,j} + W_{i,j})^3} \approx \sum_{i,j} \frac{(\omega_{i,j} W_{i,j})^3}{Z_n^3}$$

gives the expected number of times that three independent polymer distributed paths simultaneously intersect. That is, if π_1, π_2, π_3 are independent and distributed according to polymer measure, then it is the expected number of k's such that $\pi_1(k) = \pi_2(k) = \pi_3(k)$. For Bernoulli random walks and bridges, we will see in Section 2.4 that this expectation is of order $\log n$. However as discussed earlier, neither the quenched nor annealed polymer measure behaves like random walk, so we cannot simply substitute $\log n$ for the double sum in (1.5).

A more serious issue with the above calculations is that it supposes that the partition functions for polymers with different weights are comparable to each other, which is not the case. To see why that is, let us compute the second moment of the normalized partition function $Z_n(\beta)/\binom{2n}{n}$. We have

$$\begin{aligned}
\mathbb{E}\left(\frac{Z_n(\beta)}{\binom{2n}{n}}\right)^2 &= \mathbb{E}\left(\frac{1}{\binom{2n}{n}}\sum_{\pi}\prod_{i=0}^{2n}\omega_{i,\pi(i)}\right)^2 \\
&= \mathbb{E}\left(\frac{1}{\binom{2n}{n}^2}\sum_{\pi_1,\pi_2}\prod_{i=0}^{2n}\omega_{i,\pi_1(i)}\omega_{i,\pi_2(i)}\right) \\
&= \frac{1}{\binom{2n}{n}^2}\sum_{\pi_1,\pi_2}\prod_{i=0}^{2n}\mathbb{E}(\omega_{i,\pi_1(i)}\omega_{i,\pi_2(i)}).
\end{aligned}$$

The second and third sums are taken over pairs of paths π_1, π_2. If $i \neq j$, then the weights $\omega_{i,k}$ and $\omega_{j,\ell}$ are necessarily distinct no matter what k and ℓ are, hence they are independent which is why the expectation can be switched with the product in this way. We can further simplify the expectation of each term

$$\mathbb{E}(\omega_{i,\pi_1(i)}\omega_{i,\pi_2(i)}) = \mathbb{E}(\omega_{i,\pi_1(i)})\mathbb{E}(\omega_{i,\pi_2(i)}) = 1$$

but only if π_1 and π_2 do not intersect at time i. If they do intersect at time i, then we instead get the second moment of $\omega_{i,\pi_1(i)}$, which in view of Conditions 2 and 3 for valid sets of weights, should be roughly $1 + C\beta^2$ for some positive constant C. We therefore get

$$\mathbb{E}\left(\frac{Z_n(\beta)}{\binom{2n}{n}}\right)^2 \approx \frac{1}{\binom{2n}{n}^2}\sum_{\pi_1,\pi_2}(1+C\beta^2)^{L(\pi_1,\pi_2)},$$

where $L(\pi_1, \pi_2)$ is the *intersection local time* of π_1, π_2, i.e the number of k's such that $\pi_1(k) = \pi_2(k)$. The sum can be interpreted as an expectation over pairs of paths selected using Bernoulli random walk bridge measure, and in that case, we will see that $L(\pi_1, \pi_2) = O(\sqrt{n})$ with high probability, so that

$$\mathbb{E}\left(\frac{Z_n(\beta)}{\binom{2n}{n}}\right)^2 \approx \mathbb{E}_{\mathrm{Bri}}((1+C\beta^2)^{L(\pi_1,\pi_2)}) \approx (1+C\beta^2)^{C\sqrt{n}} \approx e^{C\beta^2\sqrt{n}}.$$

Here $\mathbb{E}_{\mathrm{Bri}}$ denotes expectation with respect to bridge measure; once again the notation will be made clearer in Section 2.1. This suggests that the partition functions are comparable to each other up to a factor of $e^{C\beta^2\sqrt{n}}$. In the regime $\beta = n^{-\alpha}$ with $\alpha < \frac{1}{4}$, this factor is way too large, and we get a worse estimate in (1.5) than if we used the trivial bound $W_{i,j}/(V_{i,j} + W_{i,j}) \leq 1$.

The key to resolving these problems is to work *locally*. Let $[a, b]$ be some interval contained in $[0, 2n]$, and let $n_0 = b - a$. Then by factoring out the product of the weights lying outside of $[a, b] \times \mathbb{Z}$, we can rewrite the partition function as

$$Z_\mu(\beta) := \frac{Z_n(\beta)}{\mathcal{Z}} = \sum_{\pi: a\to b} \mu(\pi) \prod_{\ell\in[a,b]} \omega_{\ell,\pi(\ell)},$$

where

$$\mu(\pi) = \frac{1}{\mathcal{Z}} \sum_{\widetilde{\pi}:\, \widetilde{\pi}|_{[a,b]}=\pi} \prod_{\ell\notin[a,b]} \omega_{\ell,\widetilde{\pi}(\ell)}, \quad \mathcal{Z} = \sum_{\pi} \prod_{\ell\notin[a,b]} \omega_{\ell,\pi(\ell)}.$$

The summation for Z_μ is taken over paths on $[a, b]$, the one for $\mathcal{Z}$ is over all paths on $[0, 2n]$, and the one for μ is over paths $\widetilde{\pi}$ on $[0, 2n]$ whose restriction to $[a, b]$ is π. If we condition on the weights lying outside the strip $[a, b] \times \mathbb{Z}$, then μ is a probability measure on paths in $[a, b]$, so that $Z_\mu(\beta)$ can be interpreted as the partition function for the polymer where the paths are on $[a, b]$ and are weighted according to μ. The idea is that if n_0 is small enough, then we will have $\beta \sim n_0^{-\gamma}$ for an exponent γ that is larger than $1/4$, and so now the comparison factor $e^{C\beta^2\sqrt{n_0}}$ is $O(1)$.

There is a slight wrinkle now, which is that the paths are distributed according to μ, not Bernoulli walk measure, so that the above second moment calculation requires us to study the intersection local time of paths distributed according to μ. Furthermore, μ is random as it depends on the environment outside of $[a, b] \times \mathbb{Z}$. Showing that μ does not behave too differently from random walk measure with high probability will require a local large deviation estimate. Chapter 3 is devoted to obtaining this estimate.

To conclude the proof of Theorem 1.5, we partition the indices i and j of the double sum in (1.5) into strips of width n_0. On such a strip $[a, b] \times [0, n]$, we have

$$\sum_{j=0}^{n}\sum_{i=a}^{b} \frac{(\omega_{i,j}W_{i,j})^3}{Z_n^3} = \sum_{j=0}^{n}\sum_{i=a}^{b} \frac{(\omega_{i,j}W_{i,j}/\mathcal{Z})^3}{(Z_n/\mathcal{Z})^3} \le C \sum_{j=0}^{n}\sum_{i=a}^{b} \frac{(\omega_{i,j}W_{i,j})^3}{\mathcal{Z}^3}$$

since $Z_n/\mathcal{Z}$ is roughly constant at this scale by the above argument. This last double sum is then more or less the number of triple intersections in $[a, b]$ of three independent μ distributed paths, which should be of order $\log n_0$ if we believe that μ is close to random walk measure with high probability. There are $2n/n_0$ different strips, so this will produce an error of size $(n/n_0)\log n$ for the entire double sum over all i and j. Plugging this in (1.5) and dividing by the scaling factor $\beta^{4/3}n^{1/3}$ appearing in the statement of Theorem 1.5, we obtain the following overall error:

$$C\frac{\beta^3\left(\frac{n}{n_0}\right)\log n}{\beta^{4/3}n^{1/3}} = C\frac{\beta^{5/3}n^{2/3}\log n}{n_0}.$$

We will see that it is possible to choose n_0 so that we have $\beta = o(n_0^{-1/4})$ and also that the above expression goes to zero for $\frac{1}{8} < \alpha < \frac{1}{4}$. This will prove Theorem 1.5 (modulo the evaluation at a test function f rather than an estimate for $\log Z_n - \log Z'_n$).

1.3 The Seppäläinen–Johansson model

The second model that we study in this thesis is the Seppäläinen–Johansson first-passage percolation model, or rather a certain generalization of it. We first give a description of this model along with a brief history of some prior results on it. We then describe the limit shape phenomenon, and state our main result on how to recover the limit shape, as well as the limiting fluctuations in the Bernoulli case.

1.3.1 Description of the model

Let $B_{i,j}$ with $i, j \geq 0$ be i.i.d Bernoulli(p) random variables, and $\xi_{i,j}, \eta_{i,j}$ where $i, j \geq 0$ be families of random variables that are all independent, that is

$$\{B_{i,j} : i, j \geq 0\} \cup \{\xi_{i,j} : i, j \geq 0\} \cup \{\eta_{i,j} : i, j \geq 0\}$$

is independent (although as we will see later, the $\xi_{i,j}$'s do not need to be independent of the $\eta_{i,j}$'s). We assume that the $\xi_{i,j}$'s are non-negative, integrable and have a common distribution, and likewise for the $\eta_{i,j}$'s (though the distribution of the $\xi_{i,j}$'s can be different of that of the $\eta_{i,j}$'s). We place weights on the edges e of $\mathbb{Z}^2_{\geq 0}$ as follows.

- If e is the horizontal edge joining $(i-1, j)$ to (i, j), then e has weight $\omega_e = B_{i,j}\xi_{i,j}$.
- If e is the vertical edge joining $(i, j-1)$ to (i, j), then e has weight $\omega_e = (1 - B_{i,j})\eta_{i,j}$.

So for any vertex (i, j), one of the two "incoming" edges $(i-1, j) \to (i, j)$ or $(i, j-1) \to (i, j)$ will have weight 0. Given an up-right path π, we define its weight $S(\pi)$ as

$$S(\pi) = \sum_{e \in E(\pi)} \omega_e$$

where the sum is taken over the set $E(\pi)$ of edges that π traverses; see Figure 1.2. For two points (a, b) and (m, n) with $a \leq m$ and $b \leq n$, we define the *first-passage value* from (a, b) to (m, n) as

$$F(a, b; m, n) = \min_{\pi:(a,b)\to(m,n)} S(\pi) \tag{1.6}$$

where the minimum is taken over all up-right paths π started at (a, b) and finishing at (m, n). We write $F(m, n)$ for $F(0, 0; m, n)$. A path which achieves the minimum in (1.6) is called a *geodesic*.

The function F defines a *directed metric* on $\mathbb{Z}^2_{\geq 0} \times \mathbb{Z}^2_{\geq 0}$, in the sense that for any point (m, n), we have $F(m, n; m, n) = 0$, and F satisfies the triangle inequality, but the distance can only be measured in one direction; $F(a, b; m, n)$ is only defined when $a \leq m$ and $b \leq n$.

The special case when all the $B_{i,j}$'s are equal to one (i.e only horizontal edges have non-zero weights) is known as the Seppäläinen–Johansson (SJ) model. It was introduced by Seppäläinen in [67] as a simplified model of directed first-passage percolation, and Johansson showed in [41] that in the special case of Bernoulli weights, the model is completely solvable. Indeed, the law of the last-passage value (that is where we take maximum instead of minimum in (1.6)) is the same as that of the top point of the Krawtchouk ensemble, a discrete orthogonal polynomial ensemble.

By Kingman's subadditive ergodic theorem, there exists a deterministic function f on $\mathbb{R}^2_{\geq 0}$ (which depends on the distribution of the weights) such that for all $x, y \geq 0$,

$$\frac{F(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \to f(x, y)$$

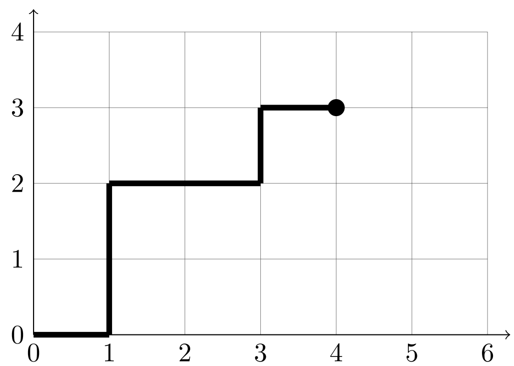

Figure 1.2: An upright path from (0,0) to (4,3). The weight of this path is $B_{1,0}\xi_{1,0}+(1-B_{1,1})\eta_{1,1}+(1-B_{1,2})\eta_{1,2}+B_{2,2}\xi_{2,2}+B_{3,2}\xi_{3,2}+(1-B_{3,3})\eta_{3,3}+B_{4,3}\xi_{4,3}$.

in L^1. If we assume that the $\xi_{i,j}$'s and $\eta_{i,j}$'s have a finite second moment, then the above convergence also holds almost surely, see Theorem 5.1. We will refer to f as the *limit shape*. It follows from the translation invariance of this model that f must be homogeneous (i.e $f(cx, cy) = cf(x, y)$ for all $c \geq 0$), and together with the triangle inequality for F, we have that f also satisfies a triangle inequality:

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2).$$

The homogeneity plus triangle inequality clearly imply that f must be convex, and so in particular it is continuous on $\mathbb{R}^2_{>0}$.

The existence of a limit shape in the general first-passage percolation model (that is with no restriction on paths going up-right) is a classical result of Cox and Durrett [26]. Despite this, there are no non-trivial examples of edge weight distributions for which the limit shape is known. The fluctuations of first-passage percolation are even less well understood. We refer the reader to the book [9] for an extensive review on first-passage percolation.

Consequently, the SJ model provides an interesting simplified version of first-passage percolation for which we can compute some limit shapes and fluctuations. Seppäläinen obtained in [67] the following explicit formula for the limit shape in the SJ model with Bernoulli(p) weights:

$$f(x, y) = \begin{cases} (\sqrt{px} - \sqrt{(1-p)y})^2 & \text{if } x \geq \frac{1-p}{p} y \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Similar formulas were obtained in the SJ model for geometric and exponential weights in [53]. The most general cases for which we can compute explicitly the limit shape are due to Martin [50]; this is when $\xi_{i,j} \overset{d}{=} B \cdot X$, where B is Bernoulli distributed, and X is a geometric or exponential independent of B. See Table 5.1 for a list of all the known limit shapes.

The generalized SJ model that we are considering first appears in [12] for $\xi_{i,j}$ and $\eta_{i,j}$ exponentially distributed as the zero temperature limit of the beta polymer. The authors show that the distribution function of a point to half-line first-passage percolation version of this model can be written down explicitly as a Fredholm determinant, and as a result, they obtain Tracy–Widom limiting fluctuations.

Much more is known about the SJ model for Bernoulli weights. Indeed, in [28], it was shown

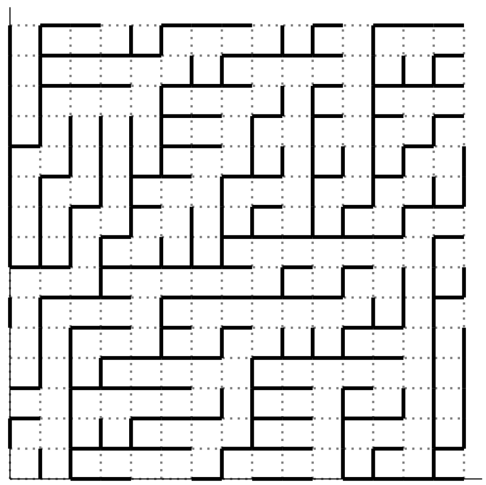

Figure 1.3: A portion of the random walk web obtained from the B_{ij}'s. If $B_{ij} = 0$, an edge is placed from $(i-1, j)$ to (i, j); otherwise, the edge is placed from $(i, j-1)$ to (i, j).

that the scaled fluctuations of the first-passage value function converges in distribution with respect to uniform convergence on compact sets to the Airy$_2$ process. Using this, we will obtain, in the Bernoulli case, the limiting distribution of the scaled fluctuations of $F(nx, ny)$ in the generalized SJ model, for points satisfying $x \neq (1-p)y/p$. It has also been shown in [30] that the SJ model converges to the directed landscape in various weak topologies. The full four-parameter uniform convergence on compact sets to the directed landscape is still open as of the time of writing this thesis.

1.3.2 The Brownian web distance

As extra motivation, we now briefly discuss a recent connection that has been made between the generalized Seppäläinen–Johansson model and a new model of random directed geometry, the *Brownian web distance*.

Consider the following subgraph of $\mathbb{Z}^2$. For each i and j, we place an edge from $(i-1, j)$ to (i, j) if the corresponding Bernoulli variable $B_{i,j}$ is 0; otherwise we place an edge from $(i, j-1)$ to (i, j). The resulting subgraph is then a system of coalescing random walks moving in the south-west direction; see Figure 1.3. We call this random graph the *random walk web*. Note that by recurrence of the one-dimensional simple random walk, the random walk web is almost surely a tree (it is connected and has no cycles).

The random walk web has a scaling limit known as the *Brownian web*. One can informally think of the Brownian web as a continuum analogue of the random walk web; each point of $\mathbb{R}^2$ has a one-dimensional Brownian motion started from it, and the Brownian motions started from distinct points behave independently until they intersect, at which point they coalesce into a single Brownian motion. The Brownian web was first constructed by Tóth and Werner in [76], using ideas from the work of Arratia [6, 7]. It was characterized as a random variable taking values in compact collections of paths under the Hausdorff distance in [34], and convergence of coalescing random walks to the Brownian web was shown in the same article. We refer to this last paper and the references therein for more details and further properties of the Brownian web.

When the $\xi_{i,j}$'s and $\eta_{i,j}$'s are all equal to 1, the generalized Seppäläinen–Johansson distance from (a, b) to (m, n) is precisely the smallest number of jumps one has to make to get from (a, b) to (m, n) while staying as much as possible on the random walk web. In [78], the authors show that when time and space are scaled diffusively, the generalized SJ distance converges to a directed metric on the Brownian web, the *Brownian web distance*. For general $\xi_{i,j}$'s and $\eta_{i,j}$'s, each empty edge of the web has a cost associated to it corresponding to the value of the weight on that edge, and the generalized SJ distance is the smallest total cost one has to pay while trying to remain on the random walk web. In this more general situation, the scaling limit should be the *Brownian net distance*. The *Brownian net* is a generalization of the Brownian web where branching is possible; see [69] for more details.

We believe that directed metrics on "random walk trees" should have their own universality class under the Brownian scaling. This is ongoing joint work of the author with Vető and Virág.

1.3.3 Main results

Let $F_H(m, n)$ be the first-passage value at (m, n) in the SJ model with weights $B_{i,j}\xi_{i,j}$ on horizontal edges $(i - 1, j) \to (i, j)$ and $F_V(m, n)$ the first-passage value at (m, n) for the SJ model on vertical edges with weights $(1 - B_{i,j})\eta_{i,j}$ (that is, we change all the weights on vertical edges to be 0 for F_H and all the weights on horizontal edges to be 0 for F_V). Let f_H and f_V be the corresponding limit shapes. Clearly, we have $F_H(m, n) \le F(m, n)$, since we can simply ignore any weights picked up along vertical edges when following a geodesic for F and this gives a valid path for F_H. Similarly, we also have $F_V(m, n) \le F(m, n)$. Passing to the limit, this implies the corresponding bound on limit shapes:

$$\max(f_H(x, y), f_V(x, y)) \le f(x, y) \tag{1.8}$$

for all $x, y \ge 0$. Our first result is that equality holds in (1.8).

Theorem 1.6. *Assume the $\xi_{i,j}$'s and $\eta_{i,j}$'s have a finite first moment. Then*

$$\max(f_H(x, y), f_V(x, y)) = f(x, y).$$

As we will see later, $f_H(x, y) = 0$ for all $x \le (1 - p)y/p$ and $f_V(x, y) = 0$ for all $x \ge (1 - p)y/p$, so Theorem 1.6 can be rewritten as

$$f(x, y) = \begin{cases} f_H(x, y) & \text{if } x \ge \frac{1-p}{p} y \\ f_V(x, y) & \text{if } x < \frac{1-p}{p} y. \end{cases}$$

Since the scaled first-passage functions converge to their limit shape, it follows that

$$F(\lfloor nx \rfloor, \lfloor ny \rfloor) = F_H(\lfloor nx \rfloor, \lfloor ny \rfloor) + o(n) \tag{1.9}$$

for any $x > (1 - p)y/p$, and a similar asymptotic holds with F_V when $x < (1 - p)y/p$. Thus the plane is divided into two regions, one where essentially only the horizontal weights matter, and one where only the vertical weights matter. Numerical simulations seem to show that the error term is actually of much smaller order than $o(n)$, and that the equality $\max(f_H(x, y), f_V(x, y)) = f(x, y)$ almost holds even in the prelimit.

Note that f_H and f_V are the limit shapes for the SJ model with weights $B_{i,j}\xi_{i,j}$ and $(1-B_{i,j})\eta_{i,j}$ respectively, and not the limit shapes with weights $\xi_{i,j}$ or $\eta_{i,j}$. This means that in order to get explicit formulas for f, we need weights with atoms at 0 for which we can compute the SJ limit shape. Bernoulli's, geometrics and Bernoulli's times geometrics/exponentials are all possible choices. See Table 5.1. Any two of the formulas from this table can be combined, except for the exponential case.

Next we consider the special case where at least one of the $\xi_{i,j}$'s or $\eta_{i,j}$'s are Bernoulli distributed (without loss of generality, we can assume it is the $\xi_{i,j}$'s). In this situation, we can show that the error in (1.9) is in fact $o(n^{1/3})$, uniformly in a window of size $n^{2/3}$. As a consequence, we obtain the following limiting fluctuations result.

Theorem 1.7. *Suppose that $B_{i,j}\xi_{i,j}$ follow a Bernoulli(p) distribution. Assume $0 < p < 1$, and let x, y be positive and satisfying $x > (1-p)y/p$. Define*

$$\tau(x,y) = 2\left[\frac{x^2}{y\sqrt{p(1-p)}}(\sqrt{px} - \sqrt{(1-p)y})(\sqrt{(1-p)x} + \sqrt{py})\right]^{1/3}$$

$$\chi(x,y) = \left[\sqrt{\frac{p(1-p)}{xy}}(\sqrt{px} - \sqrt{(1-p)y})^2(\sqrt{(1-p)x} + \sqrt{py})^2\right]^{1/3}$$

$$\rho(x,y) = p - \sqrt{\frac{p(1-p)y}{x}}$$

Then

$$\frac{F(\lfloor nx\rfloor + \lfloor \tau(x,y)n^{2/3}t\rfloor, \lfloor ny\rfloor) - nf(x,y) - \tau(x,y)n^{2/3}t\rho(x,y)}{\chi(x,y)n^{1/3}} \xrightarrow{d} -\mathcal{A}(t) \tag{1.10}$$

with respect to uniform convergence on compact sets. Here $\mathcal{A}(t)$ is the Airy$_2$ *process.*

Theorem 1.7 is known for the SJ model (with the same scaling factors τ, χ and ρ); this is Corollary 6.11 in [28]. The new part of this result is that even if we allow some vertical weights $\eta_{i,j}$ to be non-zero, we get the exact same scaling limit; the process $F(m,n)$ "does not see" the weights on the vertical edges if the angle between the line joining $(0,0)$ to the target point (m,n) and the x-axis is sufficiently small. This further reinforces this behaviour of F being almost completely determined by only one set of weights on either side of the critical line.

In the special case where we set $t = 0$, we recover [41, Theorem 5.3], again with the same scaling factors:

$$-\left(\frac{F(nx,ny) - nf(x,y)}{\chi(x,y)n^{1/3}}\right) \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}. \tag{1.11}$$

1.3.4 Outline of the proofs

As noted above, we have an easy lower bound on F:

$$F(m,n) \geq \max(F_H(m,n), F_V(m,n)),$$

and so the key will be to establish a useful upper bound on F. We will show (Proposition 6.2) that

$$F(m,n) \le F_H(m,n) + \sum_{j=1}^{\mathcal{D}(m,n)} \eta_{0,j}, \tag{1.12}$$

where $\mathcal{D}(m,n)$ is the *top-most departure point* that a geodesic for the SJ model from $(0,0)$ to (m,n) can take. That is, it is the largest k such that there is a geodesic for $F_H(m,n)$ which passes through the point $(0,k)$. Since the weights of all the vertical edges are 0 for F_H, we can also write

$$\mathcal{D}(m,n) = \max\{k \ge 0 : F_H(0,k;m,n) = F_H(m,n)\}.$$

A similar upper bound also holds in terms of F_V and the right-most departure point of a geodesic for the SJ model on vertical edges.

Inequality (1.12) is remarkable; as we will see from the proof of Proposition 6.2, it is completely deterministic and only depends on the combinatorial structure of how weights are assigned to edges. In particular, this means that the $\xi_{i,j}$'s do not have to be independent of the $\eta_{i,j}$'s. The key that makes this work of course is that for any vertex (i,j), at least one of the incoming edges must have weight 0.

In view of (1.12) and the easy lower bound, it will suffice to show that

$$\frac{\mathcal{D}(nx,ny)}{n} \to 0 \tag{1.13}$$

in probability to prove Theorem 1.6. It is more convenient to work with the *bottom-most entry point* $\mathcal{E}(m,n)$ to the line $x=m$; that is the largest k such that there is a geodesic for $F_H(m,n)$ which passes through the point $(m,n-k)$. Again, since all weights on vertical edges are 0 for the SJ model, this definition is equivalent to

$$\mathcal{E}(m,n) = \max\{k \ge 0 : F_H(m,n-k) = F_H(m,n)\}.$$

We can see that $\mathcal{E}(m,n)$ has the same law as $\mathcal{D}(m,n)$, since it corresponds exactly to the bottom-most departure point for first-passage percolation with down-left paths from (m,n) to $(0,0)$. So (1.13) is equivalent to showing that

$$\frac{\mathcal{E}(nx,ny)}{n} \to 0 \tag{1.14}$$

in probability. Now, if $\mathcal{E}(nx,ny) \ge n\epsilon$, then that means $F_H(nx,ny) = F_H(nx,ny-n\epsilon)$, and in the limit, this would imply $f_H(x,y) = f_H(x,y-\epsilon)$. Thus the limiting behaviour of $\mathcal{E}$ is connected to the local properties of the limit shape f_H, and in particular whether $y \mapsto f_H(x,y)$ is strictly decreasing or not. We will show that this function is indeed strictly decreasing on $[0,(1-q)x/q]$, where $q = \mathbb{P}(\omega_{i,j}=0)$. This will imply (1.14).

For Theorem 1.7, the idea behind the proof is as follows. Fix $\epsilon > 0$, and Taylor expand the limit shape for the SJ model $f_H(x,y)$ around y:

$$f_H\left(x, y - \frac{\epsilon}{n^{2/3}}\right) = f_H(x,y) - \frac{\partial_y f_H(x,y)\epsilon}{n^{2/3}} + O\left(\frac{1}{n^{4/3}}\right).$$

Now f_H being the limit shape tells us that $F_H(nw,nz) \approx nf_H(w,z)$ for large n, and so applying this with $w = x$, $z = y$ and also $w = x$, $z = y - \frac{\epsilon}{n^{2/3}}$ gives

$$\begin{aligned} F_H(nx, ny - \epsilon n^{1/3}) - F_H(nx, ny) &= F_H\left(nx, n\left(y - \frac{\epsilon}{n^{2/3}}\right)\right) - F_H(nx, ny) \\ &\approx n\left(f_H\left(x, y - \frac{\epsilon}{n^{2/3}}\right) - f_H(x,y)\right) \\ &= -\partial_y f_H(x,y)\epsilon n^{1/3} + O\left(\frac{1}{n^{1/3}}\right). \end{aligned} \tag{1.15}$$

Note that $\partial_y f_H(x,y)$ is strictly negative since $x > (1-p)y/p$ (recall formula (1.7) for the limit shape in the Bernoulli case; this also holds for general weights, see Lemma 6.4). It follows that for all n sufficiently large, this last expression is not 0, and therefore

$$F_H(nx, ny - \epsilon n^{1/3}) \neq F_H(nx, ny).$$

In particular, the bottom-most entry point $\mathcal{E}(nx, ny)$ must be smaller than $\epsilon n^{1/3}$, and since $\epsilon > 0$ was arbitrary, we conclude that $\mathcal{E}(nx, ny) = o(n^{1/3})$. Since $\mathcal{D}$ has the same law as $\mathcal{E}$, we also have $\mathcal{D}(nx, ny) = o(n^{1/3})$, and so combining this with the upper bound (1.12) and the fact that (1.11) is known to hold for the SJ model (that is when we replace F by F_H), we conclude that the fluctuations of F converge to the Tracy–Widom GUE. A similar argument can be used to get a uniform statement as a function of t and deduce (1.10).

Of course, the problem with this heuristic is the "$\approx$" in (1.15); there is a non-rigorous exchange of limits between $F_H(nx, ny)/n \to f_H(x,y)$ and the Taylor expansion for $f_H(x, y-\epsilon/n^{2/3})$. However for the Bernoulli case, this can be made to work by using that (1.10) is known to hold for F_H by [28, Corollary 6.11]. Looking at appropriate subsequences and using suitable changes of variables, this can essentially be rephrased as

$$\frac{F_H(nx, ny + n^{2/3}t) - nf_H(x,y) - n^{2/3}\partial_y f_H(x,y)t}{n^{1/3}} \xrightarrow{d} G(t)$$

uniformly for t in a compact set. Here G is basically the Airy$_2$ process up to some constants factors and linear rescaling in t; the details are not important for this heuristic. Since the convergence is uniform, we may take t to depend on n in the above limit. Taking first $t_n = -\epsilon/n^{1/3}$, then $t = 0$ and subtracting the expressions we get from those yields

$$\begin{aligned} \frac{F_H(nx, ny - \epsilon n^{1/3}) - F_H(nx, ny)}{n^{1/3}} &= -\partial_y f_H(x,y)\epsilon + G\left(-\frac{\epsilon}{n^{1/3}}\right) - G(0) + o(1) \\ &= -\partial_y f_H(x,y)\epsilon + o(1) \end{aligned}$$

(we use the continuity of G in the second equality) which is precisely (1.15) with a less specific error term, but fully justified.

1.4 Organization of the thesis

This thesis is divided into two independent parts. Chapters 2, 3 and 4 concern the directed polymer, while Chapters 5 and 6 deal with the Seppäläinen–Johansson model. There is no overlap between these two parts and they can be read separately. Here is a brief overview of each chapter.

In Chapter 2, we study in detail the intersection local time of Bernoulli random walks and bridges. We obtain asymptotic estimates for the moments and the moment generating function of the number of intersections of independent random walks/bridges, and we also derive an estimate for the number of triple intersections.

In Chapter 3, we obtain local large deviation estimates for the partition function and the polymer measure in the intermediate regime. We use these to find the correct local scale at which we expect polymer measure to behave like random walk measure, and we deduce the range of exponents $\frac{1}{8} < \alpha < \frac{1}{4}$ for which our main results hold.

Chapter 4 starts with the proof of Theorem 1.5. We execute the strategy laid out in Subsection 1.2.5 of comparing the free energies of two polymers using the Lindeberg replacement method, and we use the technical estimates of the two previous chapters to show that the limiting fluctuations are the same when the weights have two matching moments. Afterwards, we show that both the standard polymer and the log-gamma polymer satisfy the conditions of Definition 1.3, and we then deduce Theorem 1.2 from Theorem 1.5.

Chapter 5 is about limit shapes for the Seppäläinen–Johansson model. After a brief review of subadditive ergodic theory, we prove the existence of a limit shape for our generalized SJ model, and we show convergence in L^1 and jointly almost surely. Finally we sketch how to obtain exact formulas for the limit shape in the SJ model when the weights are a product of a Bernoulli with an independent geometric random variable.

For Chapter 6, we begin by proving the combinatorial upper bound (1.12) relating the first-passage value to the top-most departure point of a geodesic in the SJ model. We analyze this departure point by relating it to the local properties of the limit shape. We obtain Theorem 1.6 after showing that the limit shape is strictly decreasing in a certain region. Theorem 1.7 follows instead from the uniform convergence of the fluctuations in the SJ model with Bernoulli weights.

The main contributions of this thesis are Theorems 1.2 and 1.5 on the universality of directed polymers in the intermediate disorder regime, and Theorems 1.6 and 1.7 on the limit shapes and fluctuations respectively of the generalized Seppäläinen–Johansson model. These results are contained in two papers [63, 64] written by the author:

- "Universality of directed of polymers in the intermediate disorder regime" was submitted to the *Annals of Probability* and is currently under revision following a favourable report from the referee

- "Limit shape formulas for a generalized Seppäläinen–Johansson model" is now published in *Electronic Communications in Probability.*

Chapter 2

Intersections of Random walks

In this chapter, we collect several facts and estimates about the intersection local time of Bernoulli random walks and bridges. We begin by proving an estimate relating the probability of events under bridge measure to those under random walk measure. This is convenient as random walks are easier to analyze than bridges. We then obtain bounds on the moments and moment generating function of the intersection local time. Finally we consider triple intersections of random walks and bridges.

2.1 Notation and definitions

A *(Bernoulli) path* is a function $\pi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ such that $\pi(0) = 0$ and $\pi(i) - \pi(i-1) \in \{0, 1\}$ for all i. For two integers $0 \leq a < b$, a *path on* $[a, b]$ is the restriction of a path to the interval $[a, b]$. We denote the set of all paths on $[a, b]$ by $\Pi[a, b]$. Note that if $\pi \in \Pi[a, b]$, then we must have $0 \leq \pi(a) \leq a$ and $\pi(a) \leq \pi(b) \leq b$.

For $0 \leq x \leq a$ and $p \in [0, 1]$, we denote by $\mathbb{P}_{\mathrm{RW}}^{[a,b],p,x}$ the usual Bernoulli random walk measure on $\Pi[a, b]$ with mean p, started at x:

$$\mathbb{P}_{\mathrm{RW}}^{[a,b],p,x}(\pi) = \binom{b-a}{\pi(b)-x} p^{\pi(b)-x}(1-p)^{b-a-(\pi(b)-x)}.$$

If we are also given a y such that $x \leq y \leq x+b-a$, we denote by $\mathbb{P}_{\mathrm{Bri}}^{[a,b],x,y}$ the uniform measure on the sets of paths such that $\pi(a) = x$ and $\pi(b) = y$, and we refer to this measure as a *(Bernoulli) bridge*. The corresponding expectations for these measures are denoted by $\mathbb{E}_{\mathrm{RW}}^{[a,b],p,x}$ and $\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}$. We also use the same notation for the product measure and expectation on $\Pi[a, b]^k$, where the superscripts x and y are then understood to be vectors of size k corresponding to the endpoints of each of the paths, and p is also a vector of size k whose entries are the means of the k Bernoulli walks. We will usually omit the superscripts for both the Bernoulli random walk and bridge measures when it is clear from the context what $[a, b]$, p, x and y are.

If $\pi \sim \mathbb{P}_{\mathrm{RW}}^{[a,b],p,x}$, then p will be referred to as the *slope* of π, and if $\pi \sim \mathbb{P}_{\mathrm{Bri}}^{[a,b],x,y}$, the slope of π will refer to the slope of the line joining (a, x) to (b, y), namely $(y - x)/(b - a)$. When the slope is $1/2$, there is a simple bound that allows one to transfer bridge measure to random walk measure:

for a set of paths S,

$$\mathbb{P}_{\text{Bri}}^{[0,2n],0,n}(S) = \frac{|S|}{\binom{2n}{n}} \leq C\sqrt{n}\frac{|S|}{4^n} = C\sqrt{n}\mathbb{P}_{\text{RW}}^{[0,2n],\frac{1}{2},0}(S)$$

for some absolute constant C that does not depend on n. Here we used Stirling's formula. This bound is very crude but is occasionally sufficient for our needs.

Given a set of paths $\pi_1, \ldots, \pi_k$ in $\Pi[a,b]$, we let

$$v(i,j) = \text{the number of paths that visited the site } (i,j),$$

and

$$\mathcal{V}(\pi_1, \ldots, \pi_k) = \{(i,j) : v(i,j) \geq 2\}.$$

The *intersection local time* of the paths $\pi_1, \ldots, \pi_k$ is the cardinality of the set $\mathcal{V}(\pi_1, \ldots, \pi_k)$, and we denote it by $L(\pi_1, \ldots, \pi_k)$. For an interval I, we also write $L(\pi_1, \ldots, \pi_k; I)$ for the number of intersections of the paths in I, that is the number of points (i,j) in $\mathcal{V}(\pi_1, \ldots, \pi_k)$ such that $i \in I$. We have the bound

$$L(\pi_1, \ldots, \pi_k) \leq \sum_{1 \leq j < i \leq k} L(\pi_i, \pi_j), \tag{2.1}$$

since the right-hand side counts multiple times all the instances where three or more paths intersect all at once (for example, if π_1, π_2, π_3 intersect all at the same time, then the left-hand side counts this intersection once, whereas the right-hand side counts it three times). This is a simple observation, but it will be very useful in what follows since it means that we only really need to analyze the intersection local time of pairs of paths.

Throughout all of Chapters 2, 3 and 4, we will use C to denote an unspecified constant that may change from line to line. Occasionally we will add subscripts or comments in the text to detail what the constants depend on.

2.2 Replacing bridges by random walks

We will need an estimate for $L(\pi_1, \ldots, \pi_k)$ when the π_j's are independent and distributed according to Bernoulli bridge measure. It turns out to be easier to estimate this when the π_j's are distributed according to Bernoulli walk measure, so the first step is to show that we can replace bridges by walks. The next lemma describes how to make this change.

Lemma 2.1. *There exists an n^* such that the following holds. For any interval $[a,b]$ of length $n \geq n^*$, for any π distributed according to bridge measure on $[a,b]$ with slope p, for any path $\widetilde{\pi} \in \Pi[a,(b-a)/2]$ and for any $p \in [1/4, 3/4]$,*

$$\mathbb{P}_{\text{Bri}}^{[a,b],x,y}(\pi|_{[a,(b-a)/2]} = \widetilde{\pi}) \leq 2\mathbb{P}_{\text{RW}}^{[a,(b-a)/2],p,x}(\widetilde{\pi}).$$

Here $\pi|_{[a,(b-a)/2]}$ denotes the restriction of π to the interval $[a,(b-a)/2]$. Thus Lemma 2.1 asserts that the ratio of the probabilities under either measure that π takes on a specific trajectory in the first half of $[a,b]$ is bounded by 2 (the actual constant is irrelevant, what matters is that this constant is independent of p, $[a,b]$, etc.). This implies that if F is a measurable, non-negative function on

$\Pi[a,b]^k$ which only depends on the behaviour of the paths in the first half of the interval, then

$$\mathbb{E}_{\text{Bri}}^{[a,b],x,y}(F(\pi_1,\ldots,\pi_k)) \le 2^k \mathbb{E}_{\text{RW}}^{[a,(b-a)/2],p,x}(F(\pi_1,\ldots,\pi_k)), \tag{2.2}$$

provided that all the slopes $p_i = (y_i - x_i)/(b-a)$ are in $[1/4, 3/4]$. In particular we can take F to be some moment of the intersection local time of the paths in the first half of $[a,b]$.

The main ingredient for the proof of Lemma 2.1 is the following theorem ([57, Equation 25]).

Theorem 2.2 (Platonov)**.** *Let f be the probability mass function of the binomial (n,p) distribution, and let ϕ be the density of the standard normal distribution. There exists a universal constant C, which does not depend on p or n, such that*

$$\max_{0\le k\le n}\left|\sqrt{np(1-p)}f(k) - \phi\left(\frac{k-np}{\sqrt{np(1-p)}}\right)\right| \le \frac{C}{\sqrt{np(1-p)}}.$$

Platonov's theorem gives an explicit bound for the error term in the local central limit theorem that is uniform over all possible choices of parameters of the binomial distribution.

Proof of Lemma 2.1. Without loss of generality, we can assume that $a = 0$ and the starting point x is 0. Note that $p = y/n$.

By definition of conditional probabilities and the Markov property of Bernoulli random walks, we have for a fixed given path $\widetilde{\pi}$ on $[a,(b-a)/2]$,

$$\begin{aligned}\mathbb{P}_{\text{Bri}}^{[0,n],0,y}(\pi|_{[0,n/2]} = \widetilde{\pi}) &= \frac{\mathbb{P}_{\text{RW}}^{[0,n],p,0}(\{\pi_{[0,n/2]} = \widetilde{\pi}\} \cap \{\pi(n) = y\})}{\mathbb{P}_{\text{RW}}^{[0,n],p,0}(\pi(n) = y)} \\ &= \frac{\mathbb{P}_{\text{RW}}^{[0,n/2],p,0}(\widetilde{\pi})\ \mathbb{P}_{\text{RW}}^{[n/2,n],p,\widetilde{\pi}(n/2)}(\pi(n) = y)}{\mathbb{P}_{\text{RW}}^{[0,n],p,0}(\pi(n) = y)}.\end{aligned}$$

By Platonov's theorem,

$$\begin{aligned}\mathbb{P}_{\text{RW}}^{[n/2,n],p,\widetilde{\pi}(n/2)}(\pi(n) = y) &\le \frac{1}{\sqrt{2\pi(n-\widetilde{\pi}(n/2))p(1-p)}} + \frac{C}{np(1-p)} \\ &\le \frac{1}{\sqrt{\pi np(1-p)}} + \frac{C}{np(1-p)}.\end{aligned}$$

Here we used the fact that $\phi(x) \le 1/\sqrt{2\pi}$ for all x and $\widetilde{\pi}(n/2)$ can only be at most $n/2$. Also, by Platonov's theorem again,

$$\begin{aligned}\mathbb{P}_{\text{RW}}^{[0,n],p,0}(\pi(n) = y) &\ge \frac{1}{\sqrt{np(1-p)}}\phi\left(\frac{y-np}{\sqrt{np(1-p)}}\right) - \frac{C}{np(1-p)} \\ &= \frac{1}{\sqrt{2\pi np(1-p)}} - \frac{C}{np(1-p)}\end{aligned}$$

since $y - np = 0$ and $\phi(0) = 1/\sqrt{2\pi}$. Putting these two inequalities together yields

$$\frac{\mathbb{P}_{\text{RW}}^{[n/2,n],p,\widetilde{\pi}(n/2)}(\pi(n) = y)}{\mathbb{P}_{\text{RW}}^{[0,n],p,0}(\pi(n) = y)} \le \frac{\frac{1}{\sqrt{\pi np(1-p)}} + \frac{C}{np(1-p)}}{\frac{1}{\sqrt{2\pi np(1-p)}} - \frac{C}{np(1-p)}} = \frac{\sqrt{2np(1-p)} + C}{\sqrt{np(1-p)} - C}.$$

If n is sufficiently large, then this last bound is ≤ 2 uniformly for $p \in [1/4, 3/4]$. $\square$

2.3 Estimates for the intersection local time

We now apply (2.2) with F the moments of the intersection local time in the first half of the interval. By (2.2) and the triangle inequality for L^m norms, we have that for all $m \geq 1$,

$$\begin{aligned}
&[\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(L(\pi_1,\ldots,\pi_k)^m)]^{1/m} \\
&\leq [\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(L(\pi_1,\ldots,\pi_k;[a,(b-a)/2])^m)]^{1/m} \\
&\quad + [\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(L(\pi_1,\ldots,\pi_k;[(b-a)/2,b])^m)]^{1/m} \\
&= 2[\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(L(\pi_1,\ldots,\pi_k;[a,(b-a)/2])^m)]^{1/m} \\
&\leq 2[2^k\mathbb{E}_{\mathrm{RW}}^{[a,b],p,x}(L(\pi_1,\ldots,\pi_k;[a,(b-a)/2])^m)]^{1/m} \\
&\leq 2[2^k\mathbb{E}_{\mathrm{RW}}^{[a,b],p,x}(L(\pi_1,\ldots,\pi_k)^m)]^{1/m}
\end{aligned} \tag{2.3}$$

provided that the slopes $p_i = (y_i - x_i)/(b-a)$ are all in $[1/4, 3/4]$. In the fourth line, we used the fact that the number of intersections in the first half of $[a,b]$ has the same law as the number of intersections in the second half by considering backward paths. In view of (2.1), we will only really need to consider the case $k = 2$.

We can do a similar calculation to estimate the moment generating function of the intersection local time. Since the function $x \mapsto e^{rx}$ is convex, we have again by (2.2) that for $r \geq 0$ and paths with slopes in $[1/4, 3/4]$,

$$\begin{aligned}
&\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(e^{rL(\pi_1,\ldots,\pi_k)}) \\
&\leq \frac{1}{2}\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(e^{2rL(\pi_1,\ldots,\pi_k;[a,(b-a)/2])}) + \frac{1}{2}\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(e^{2rL(\pi_1,\ldots,\pi_k;[(b-a)/2,b])}) \\
&= \mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(e^{2rL(\pi_1,\ldots,\pi_k;[a,(b-a)/2])}) \\
&\leq 2^k\mathbb{E}_{\mathrm{RW}}^{[a,b],x,p}(e^{2rL(\pi_1,\ldots,\pi_k;[a,(b-a)/2])}) \\
&\leq 2^k\mathbb{E}_{\mathrm{RW}}^{[a,b],x,p}(e^{2rL(\pi_1,\ldots,\pi_k)}).
\end{aligned} \tag{2.4}$$

So now we need to estimate the moments and moment generating function of the intersection local time of two independent Bernoulli random walks. There is yet another simplification we can make. Extend the walks to infinite Bernoulli walks, and let $G(\pi_1, \pi_2)$ be the number of times the two walks intersect before there is an interval of length at least $b-a$ where they do not intersect. Then clearly we have $L(\pi_1, \pi_2; [a,b]) \leq G(\pi_1, \pi_2)$, since any intersection that happens in $[a,b]$ is an intersection that happens before there is an interval of length at least $b-a$ with no intersection. The advantage of working with $G(\pi_1, \pi_2)$ instead is that its distribution is much simpler; it is the $\mathrm{Geo}_0(q_{b-a})$, where q_k is the probability that the two paths do not intersect before time k (we use Geo_0 to denote the geometric distribution on $\{0, 1, 2, \ldots\}$). If $X \sim \mathrm{Geo}_0(q)$, then its moments satisfy

$$\mathbb{E}(X^m) \leq \frac{m!}{q^m} \tag{2.5}$$

for all $q > 0$. One way to see this is that if $Y \sim \mathrm{Exp}(1)$, then $X = \lfloor -Y/\log(1-q) \rfloor$ follows the $\mathrm{Geo}_0(q)$ distribution, $\mathbb{E}(Y^m) = m!$ by integration by parts, and $-\log(1-q) \geq q$ for all $0 \leq q < 1$.

So all we need is a lower bound on the probability that the first intersection of two Bernoulli walks happens after time k.

Lemma 2.3. *Let π_1, π_2 be independent Bernoulli walks with slopes p_1, p_2 respectively, started at the same point, and let $T = \min\{k \geq 1 : \pi_1(k) = \pi_2(k)\}$. Then there is an absolute constant C such that for all $n \geq 1$,*

$$\mathbb{P}_{\mathrm{RW}}(T \geq n) \geq \frac{C(p_1(1-p_2) + (1-p_1)p_2)}{\sqrt{n}}.$$

Proof. Let $X(k) = \pi_1(k) - \pi_2(k)$. Then X is a "lazy" random walk, with transition probabilities given by

$$X(k+1) - X(k) = \begin{cases} 0 & \text{with probability } p_1 p_2 + (1-p_1)(1-p_2) \\ 1 & \text{with probability } p_1(1-p_2) \\ -1 & \text{with probability } (1-p_1)p_2. \end{cases}$$

The law of X is the same as a random walk Y with transition steps given by

$$Y(k+1) - Y(k) = \begin{cases} 1 & \text{with probability } \frac{p_1(1-p_2)}{p_1(1-p_2)+(1-p_1)p_2} \\ -1 & \text{with probability } \frac{(1-p_1)p_2}{p_1(1-p_2)+(1-p_1)p_2} \end{cases}$$

and where at each step, the walk stays there for a $\mathrm{Geo}_0(p_1(1-p_2) + (1-p_1)p_2)$ amount of time before jumping according to Y. Let $T' = \min\{k \geq 1 : Y(k) = 0\}$ be the first return to 0 of Y. If the walk X immediately jumps away from 0 (i.e the geometric time spent at 0 is 0) and $T' \geq n$, then we certainly have $T \geq n$. Indeed it will take longer for X to come back to 0 than Y because after each time it jumps, it needs to stay at the same position for a geometric amount of time. We thus have

$$\mathbb{P}_{\mathrm{RW}}(T \geq n) \geq \mathbb{P}_{\mathrm{RW}}(T' \geq n, X(1) \neq 0) = (p_1(1-p_2) + (1-p_1)p_2)\mathbb{P}_{\mathrm{RW}}(T' \geq n).$$

There is an explicit formula for the law of the first return time to 0 of an asymmetric simple random walk S started at 0 with probability q of going up and probability $1-q$ of going down; it is given by

$$\mathbb{P}(S(1) \neq 0, \ldots, S(2k-1) \neq 0, S(2k) = 0) = [q(1-q)]^k \frac{\binom{2k}{k}}{2k-1}.$$

Indeed, the number of such walks is given by 2 times the k^{th} Catalan number: $\binom{2k}{k}/(2k-1)$ (see for example [33, Section III.3]), and each of those walks have exactly k up steps and k down steps, so they all have probability $[q(1-q)]^k$. For any k, this quantity will clearly be the largest when $q = 1/2$, so it follows from Stirling's formula that

$$\begin{aligned} \mathbb{P}_{\mathrm{RW}}(T' \geq n) &= 1 - \sum_{k=0}^{\lfloor n/2 \rfloor} [q(1-q)]^k \frac{\binom{2k}{k}}{2k-1} \geq 1 - \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{2k}{k}}{4^k(2k-1)} \\ &= \sum_{k=\lfloor n/2 \rfloor}^{\infty} \frac{\binom{2k}{k}}{4^k(2k-1)} \geq \frac{C}{\sqrt{n}} \end{aligned}$$

for some appropriate constant C. $\square$

Lemma 2.3 thus gives a lower bound which is *uniform* over all slopes $p_1, p_2 \in [1/4, 3/4]$. Combining this with (2.5), the bound $L(\pi_1, \pi_2) \leq G(\pi_1, \pi_2)$ and (2.3), we therefore obtain the following proposition.

Proposition 2.4. *We have*

$$\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(L(\pi_1,\pi_2)^m) \leq Cm!A^m(b-a)^{m/2} \tag{2.6}$$

for constants C and A that do not depend on a or b and are uniform over all slopes $(y_1-x_1)/(b-a)$ and $(y_2-x_2)/(b-a)$ in $[1/4,3/4]$.

Next we turn our attention to the moment generating function $\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}(e^{rL(\pi_1,\pi_2)})$. Of particular interest is when $r = o(\sqrt{b-a})$.

Proposition 2.5. *Let $r,\delta > 0$, and let $n = b-a$. Then there is a constant $C_{r,\delta}$ such that*

$$\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}\left[\exp\left(rn^{-(\frac{1}{2}+\delta)}L(\pi_1,\pi_2)\right)\right] \leq C_{r,\delta},$$

for all n and for all slopes $(y_1-x_1)/(b-a)$ and $(y_2-x_2)/(b-a)$ in $[1/4,3/4]$.

Proof. By (2.4), we can replace bridge measure with random walk measure, and we can then bound $L(\pi_1,\pi_2)$ by $G(\pi_1,\pi_2)$. Then $G(\pi_1,\pi_2)$ follows the $\mathrm{Geo}_0(q)$ distribution where q is the probability that π_1 and π_2 do not intersect on an $[a,b]$, and by Lemma 2.3, $q \geq C/\sqrt{n}$ for some constant C. Since the $\mathrm{Geo}_0(q)$ stochastically dominates the $\mathrm{Geo}_0(p)$ whenever $q < p$, we may as well replace $G(\pi_1,\pi_2)$ by X where $X \sim \mathrm{Geo}_0(C/\sqrt{n})$. Thus we have

$$\mathbb{E}_{\mathrm{Bri}}^{[a,b],x,y}\left[\exp\left(rn^{-(\frac{1}{2}+\delta)}L(\pi_1,\pi_2)\right)\right] \leq \mathbb{E}(e^{2rn^{-(\frac{1}{2}+\delta)}X}).$$

We now estimate the moment generating function of X. By the inequality $(1-x) \leq e^{-x}$,

$$\begin{aligned}\mathbb{E}(e^{2rn^{-(\frac{1}{2}+\delta)}X}) &= \sum_{j=0}^{\infty} e^{2rn^{-(\frac{1}{2}+\delta)}j}\frac{C}{\sqrt{n}}\left(1-\frac{C}{\sqrt{n}}\right)^j \leq \frac{C}{\sqrt{n}}\sum_{j=0}^{\infty}\exp\left[\left(2rn^{-(\frac{1}{2}+\delta)}-\frac{C}{\sqrt{n}}\right)j\right] \\ &= \frac{C}{\sqrt{n}\left[1-\exp\left(2rn^{-(\frac{1}{2}+\delta)}-\frac{C}{\sqrt{n}}\right)\right]}.\end{aligned}$$

Note that since $n^{-(\frac{1}{2}+\delta)}$ goes to 0 faster than $1/\sqrt{n}$, the term inside the exponential will be negative for all n sufficiently large, and so the series does indeed converge. Also, by doing a Taylor expansion of the exponential, we have

$$\exp\left(2rn^{-(\frac{1}{2}+\delta)}-\frac{C}{\sqrt{n}}\right) = 1-\frac{C}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right),$$

(the error term may depend on r and δ) and so

$$\mathbb{E}(e^{2rn^{-(\frac{1}{2}+\delta)}X}) \leq \frac{C}{\sqrt{n}\left[1-\exp\left(2rn^{-(\frac{1}{2}+\delta)}-\frac{C}{\sqrt{n}}\right)\right]} = \frac{C}{C'+o(1)}.$$

This completes the proof. $\square$

2.4 A lemma on triple intersections

We finish this chapter with a somewhat technical lemma on the number of triple intersections of random walks. Given three paths π_1, π_2, π_3 on $[a,b]$, we denote by $L_3(\pi_1,\pi_2,\pi_3)$ the number of times that all three paths intersect simultaneously:

$$L_3(\pi_1,\pi_2,\pi_3) = \sum_{k=a}^{b} \mathbb{1}_{\{\pi_1(k)=\pi_2(k)=\pi_3(k)\}}.$$

By the exact same calculation as (2.3), we can bound the moments of $L_3(\pi_1,\pi_2,\pi_3)$ under bridge measure by those with respect to random walk measure, assuming that all the paths have slopes in $[1/4, 3/4]$:

$$\mathbb{E}_{\text{Bri}}^{[a,b],x,y}(L_3(\pi_1,\pi_2,\pi_3)^m) \leq 8 \cdot 2^m \mathbb{E}_{\text{RW}}^{[a,b],x,p}(L_3(\pi_1,\pi_2,\pi_3)^m). \tag{2.7}$$

For Bernoulli random walks, triple intersections are much rarer than intersections of two paths. The following makes this statement more precise.

Lemma 2.6. *Let π_1, π_2, π_3 be independent Bernoulli bridges on an interval $[a,b]$ of length n, with slopes p_1, p_2, p_3 respectively. Then there is a constant C such that for all $p_1, p_2, p_3 \in [1/4, 3/4]$,*

$$\mathbb{E}_{\text{Bri}}(L_3(\pi_2,\pi_2,\pi_3)^2) \leq C(\log n)^2.$$

Proof. By (2.7), we may consider Bernoulli walks instead of bridges, with the same slopes. Assume first that all three paths start at the same position (without loss of generality this initial position is 0 and $[a,b]=[0,n]$). Let

$$q_k = \mathbb{P}_{\text{RW}}(\pi_1(k) = \pi_2(k) = \pi_3(k)).$$

We first show that $q_k = O(1/k)$. For each i, let f_i be the probability mass function of π_i at time k. Then

$$q_k = \sum_{j=0}^{k} f_1(j) f_2(j) f_3(j). \tag{2.8}$$

By Platonov's theorem, we have, for each i,

$$f_i(j) \leq \frac{1}{\sqrt{kp_i(1-p_i)}} \phi\left(\frac{j-kp_i}{\sqrt{kp_i(1-p_i)}}\right) + \frac{C}{kp_i(1-p_i)},$$

so after expanding the product in each term of (2.8), the main order term is

$$\begin{aligned}
&\leq \frac{C}{k^{3/2}} \sum_{j=0}^{k} \phi\left(\frac{j-kp_1}{\sqrt{kp_1(1-p_1)}}\right) \phi\left(\frac{j-kp_2}{\sqrt{kp_2(1-p_2)}}\right) \phi\left(\frac{j-kp_3}{\sqrt{kp_3(1-p_3)}}\right) \\
&\leq \frac{C}{k} \int_{-\sqrt{k}}^{\sqrt{k}} \phi\left(\frac{x-p_1\sqrt{k}}{\sqrt{p_1(1-p_1)}}\right) \phi\left(\frac{x-p_2\sqrt{k}}{\sqrt{p_2(1-p_2)}}\right) \phi\left(\frac{x-p_3\sqrt{k}}{\sqrt{p_3(1-p_3)}}\right) dx \\
&\quad + O\left(\frac{1}{k^{3/2}}\right) \\
&\leq \frac{C}{k}.
\end{aligned}$$

Here we used the fact that the sum can be interpreted as a Riemann sum, and that the difference between a Riemann sum and the integral it approximates is at most a constant times the modulus of continuity of the integrand times the size of the mesh. In this case the size of the mesh is $1/\sqrt{k}$, and the modulus of continuity is bounded by a constant independent of k given that $p_1, p_2, p_3 \in [1/4, 3/4]$. The integral is easily seen to be bounded, using say Hölder's inequality. The lower order terms can be treated similarly: instead of having products of three Gaussian densities, we'll have products of two, one or zero Gaussian densities along with a higher exponent of $1/k$, and in all cases after we approximate the sum with an integral, the result will be $O(1/k^{3/2})$.

We then have for $n \geq 2$,

$$\begin{aligned}
\mathbb{E}_{\mathrm{RW}}(L_3(\pi_1,\pi_2,\pi_3)^2) &= \mathbb{E}_{\mathrm{RW}}\left(\sum_{k=1}^{n} \mathbb{1}_{\{\pi_1(k)=\pi_2(k)=\pi_3(k)\}}\right)^2 \\
&= \mathbb{E}_{\mathrm{RW}}\left(2\sum_{1\leq j<k\leq n} \mathbb{1}_{\{\pi_1(j)=\pi_2(j)=\pi_3(j)\}}\mathbb{1}_{\{\pi_1(k)=\pi_2(k)=\pi_3(k)\}}\right. \\
&\quad \left. + \sum_{k=1}^{n} \mathbb{1}_{\{\pi_1(k)=\pi_2(k)=\pi_3(k)\}}\right) \\
&= 2\sum_{1\leq j<k\leq n} q_j q_{k-j} + \sum_{k=1}^{n} q_k \\
&\leq C\left(\sum_{j=1}^{n-1}\frac{1}{j}\sum_{k=j+1}^{n}\frac{1}{k-j} + \sum_{k=1}^{n}\frac{1}{k}\right) \leq C(\log n)^2.
\end{aligned}$$

Finally, if the paths do not start at the same location, then we still have

$$L_3(\pi_1,\pi_2,\pi_3) \leq 1 + L_3(\pi_1',\pi_2',\pi_3')$$

where the π_i' are the paths after the first intersection. The above bounds then holds for $L_3(\pi_1',\pi_2',\pi_3')$ and the required inequality follows easily. $\square$

Chapter 3

Local fluctuations of directed polymers

In this chapter, we study the local behaviour of directed polymers. We first obtain an estimate on the centred moments of the partition function, which is given in terms of the moments of the intersection local time. We then obtain a large deviation bound for the polymer measure, and we derive the value of the critical exponent $\alpha = 1/8$.

3.1 Centred moments of the partition function

Let μ be a probability measure on the set of paths on $[a, b]$, and let $(\omega_{i,j}(\beta))_{i,j\geq 0}$ be a parametrized set of valid weights. We define the corresponding *partition function* as

$$Z_\mu(\beta) = \sum_{\pi\in\Pi[a,b]} \mu(\pi) \prod_{\ell\in[a,b]} \omega_{\ell,\pi(\ell)}(\beta).$$

Note that the point-to-point partition function as defined in the introduction is the special case where μ is Bernoulli bridge measure. Since the weights $\omega_{i,j}(\beta)$ are independent and have mean 1 for all β, $Z_\mu(\beta)$ clearly has mean 1.

Theorem 3.1. *Assume the $(\omega_{i,j}(\beta))_{i,j\geq 0}$ are a valid set of weights. Let μ be a probability measure on paths in $[a, b]$, $n = b - a$, and k a positive even integer. Then*

$$\mathbb{E}((Z_\mu(\beta) - 1)^k) \leq \left(\sum_{j=1}^{n} (C_k)^j \beta^j \left[\frac{\mathbb{E}_\mu(L(\pi_1, \ldots, \pi_k)^{kj/2})}{(kj/2)!} \right]^{1/k} \right)^k,$$

where $\pi_1, \ldots, \pi_k$ are independent paths distributed according to μ. The constants C_k only depend on the constants involved in the definition of valid weights.

Proof. For each i and j, let $\zeta_{i,j}(\beta) = \omega_{i,j}(\beta) - 1$. Then the $\zeta_{i,j}$ have mean 0, and by Condition 3 of valid sets of weights, $\mathbb{E}|\zeta_{i,j}(\beta)|^k \le C_k\beta^k$. We can thus rewrite $Z_\mu(\beta)-1$ as follows

$$\begin{aligned} Z_\mu(\beta)-1 &= \sum_{\pi\in\Pi[a,b]} \left[\prod_{i=a}^{b}(1+\zeta_{i,\pi(i)}(\beta))-1\right]\mu(\pi) \\ &= \sum_{\pi\in\Pi[a,b]}\mu(\pi)\sum_{S\subseteq[a,b],S\neq\emptyset}\prod_{i\in S}\zeta_{i,\pi(i)}(\beta). \end{aligned}$$

Here the inner sum is taken over non-empty subsets of $[a,b]$ (we interpret intervals in this proof as intervals of integers, so $[a,b]=\{a,a+1,\dots,b\}$). For $1\le j\le n$, let

$$Z_{\mu,j}(\beta)=\sum_{\pi\in\Pi[a,b]}\mu(\pi)\sum_{S\subseteq[a,b],|S|=j}\prod_{i\in S}\zeta_{i,\pi(i)}(\beta).$$

Then $Z_\mu(\beta)-1=Z_{\mu,1}(\beta)+\ldots+Z_{\mu,n}(\beta)$. We will estimate the k-th moment of each $Z_{\mu,j}(\beta)$.

We have

$$\mathbb{E}(Z_{\mu,j})^k=\mathbb{E}\left(\sum_{\pi_1,\dots,\pi_k}\mu(\pi_1)\cdots\mu(\pi_k)\sum_{\substack{S_1,\dots,S_k\\|S_i|=j}}\prod_{i_1\in S_1}\zeta_{i_1,\pi_1(i_1)}\cdots\prod_{i_k\in S_k}\zeta_{i_k,\pi_k(i_k)}\right). \tag{3.1}$$

Next, by Hölder's inequality and the inequalities $\mathbb{E}|\zeta_{i,j}(\beta)|^k\le C_k\beta^k$,

$$\begin{aligned} &\mathbb{E}\left|\prod_{i_1\in S_1}\zeta_{i_1,\pi_1(i_1)}\cdots\prod_{i_k\in S_k}\zeta_{i_k,\pi_k(i_k)}\right| \\ &\le\left(\mathbb{E}\left(\prod_{i_1\in S_1}\zeta^k_{i_1,\pi_1(i_1)}\right)\right)^{1/k}\cdots\left(\mathbb{E}\left(\prod_{i_k\in S_k}\zeta^k_{i_k,\pi_k(i_k)}\right)\right)^{1/k} \\ &=\left(\prod_{i_1\in S_1}\mathbb{E}(\zeta^k_{i_1,\pi_1(i_1)})\right)^{1/k}\cdots\left(\prod_{i_k\in S_k}\mathbb{E}(\zeta^k_{i_k,\pi_k(i_k)})\right)^{1/k}\le(C_k)^j\beta^{kj}. \end{aligned}$$

Now each product in the inner sum of (3.1) that has non-zero expectation is obtained by taking points $(i_1,j_1),\dots(i_\ell,j_\ell)$ in $\mathcal{V}(\pi_1,\dots,\pi_k)$ such that

$$v(i_1,j_1)+\cdots+v(i_\ell,j_\ell)=kj$$

and multiplying their corresponding weights $\zeta^{v(i_1,j_i)}_{i_1,j_1},\dots,\zeta^{v(i_\ell,j_\ell)}_{i_\ell,j_\ell}$. Since each of those $v(i,j)$ are at least 2, the number of such configurations is at most

$$\binom{L(\pi_1,\dots,\pi_k)}{\frac{kj}{2}}\le\frac{L(\pi_1,\dots,\pi_k)^{kj/2}}{(kj/2)!}.$$

Hence, the expectation of the inner sum is no larger than

$$\frac{L(\pi_1,\dots,\pi_k)^{kj/2}(C_k)^j\beta^{kj}}{(kj/2)!}$$

and therefore

$$\begin{aligned}\mathbb{E}(Z_{\mu,j})^k &\le (C_k)^j \beta^{kj} \sum_{\pi_1,\dots,\pi_k} \mu(\pi_1)\cdots\mu(\pi_k) \frac{L(\pi_1,\dots,\pi_k)^{kj/2}}{(kj/2)!} \\ &= \frac{(C_k)^j\beta^{kj}}{(kj/2)!}\mathbb{E}_\mu(L(\pi_1,\dots,\pi_k)^{kj/2}).\end{aligned}$$

Finally, by the triangle inequality for L^k norms,

$$\begin{aligned}\mathbb{E}(Z_\mu(\beta)-1)^k &= \mathbb{E}(Z_{\mu,1}(\beta)+\cdots+Z_{\mu,n}(\beta))^k \\ &\le \left((\mathbb{E}(Z_{\mu,1}(\beta))^k)^{1/k}+\cdots+(\mathbb{E}(Z_{\mu,n}(\beta))^k)^{1/k}\right)^k \\ &\le \left(\sum_{j=1}^n (C_k)^j\beta^j \left[\frac{\mathbb{E}_\mu(L(\pi_1,\dots,\pi_k)^{kj/2})}{(kj/2)!}\right]^{1/k}\right)^k.\end{aligned}$$

This concludes the proof. □

We can use Theorem 3.1 together with Proposition 2.4 on the moments of the intersection local time for bridge measure to obtain the following corollary.

Corollary 3.2. *Let $(\omega_{i,j}(\beta))_{i,j\ge 0}$ be a set of valid weights, and let μ be a convex combination of bridge measures on $[a,b]$ of slopes in $[1/4,3/4]$. Assume that $\beta = n^{-(\frac{1}{4}+\delta)}$ for some $\delta > 0$. Then for any positive even integer k, there is a positive constant C_k such that*

$$\mathbb{E}(Z_\mu(\beta)-1)^k \le \frac{C_k}{n^{k\delta}}$$

for n sufficiently large.

Proof. First off, by the bound (2.1), the fact that every term in that sum has the same law and the triangle inequality for L^m norms,

$$\mathbb{E}_\mu(L(\pi_1,\dots,\pi_k)^m) \le \binom{k}{2}^m \mathbb{E}_\mu(L(\pi_1,\pi_2)^m)$$

for each $m \ge 1$. So by Proposition 2.4,

$$\mathbb{E}_\mu(L(\pi_1,\dots,\pi_k)^m) \le \binom{k}{2}^m \mathbb{E}_\mu(L(\pi_1,\pi_2)^m) \le Cm!(Ak^2)^m n^{m/2}.$$

Indeed, this holds for any bridge measure with slope in $[1/4,3/4]$, with the same choice of C and A, so it also holds for convex combinations. Combining this with Theorem 3.1, we obtain

$$\begin{aligned}\mathbb{E}((Z_\mu(\beta)-1)^k) &\le \left(\sum_{j=1}^{2n} (C_k)^j\beta^j \left[\frac{\mathbb{E}_\mu(L(\pi_1,\dots,\pi_k)^{kj/2})}{(kj/2)!}\right]^{1/k}\right)^k \\ &\le \left(\sum_{j=1}^{2n}(C_k)^j\beta^j C^{1/k}(Ak^2)^{j/2}n^{j/4}\right)^k \le \left(\frac{C_k\beta n^{1/4}}{1-C_k\beta n^{1/4}}\right)^k.\end{aligned}$$

Since $\beta = n^{-(\frac{1}{4}+\delta)}$, it follows that for n large enough,

$$\mathbb{E}((Z_\mu(\beta)-1)^k) \le \frac{C_k}{n^{k\delta}}$$

for some appropriate constant C_k. $\square$

3.2 Local deviations and the critical exponent $\alpha = \frac{1}{8}$

Let $(\omega_{i,j}(\beta))_{i,j\ge 0}$ be a valid set of weights, and let $Z_n(\beta)$ be the corresponding point-to-point partition function:

$$Z_n(\beta) = \sum_\pi \prod_{i=0}^{2n} \omega_{i,\pi(i)}(\beta),$$

the sum being taken over paths π on $[0,2n]$ such that $\pi(0)=0$ and $\pi(2n)=n$. We consider the intermediate disorder regime, which consists in taking $\beta = n^{-\alpha}$ for some fixed $\alpha > 0$. When $\alpha > 1/4$, Corollary 3.2 implies that $Z_n(\beta)/\binom{2n}{n}$ is very close to 1 with high probability. However the case that interests us is when $\alpha < 1/4$, and in this regime, Corollary 3.2 does not hold. Nevertheless, we can still obtain a local deviations estimate.

Let $[a,b] \subset [0,2n]$ be a subinterval of length n_0, and let

$$\mathcal{Z} = \sum_\pi \prod_{i\notin[a,b]} \omega_{i,\pi(i)}(\beta).$$

Thus $\mathcal{Z}$ is the partition function where all the weights lying in the strip

$$\Delta = \{(i,j) : a \le i \le b, 0 \le j \le n\}$$

have been changed to 1. We wish to obtain an estimate for $Z_n/\mathcal{Z}$, as this quotient represents the relative change in Z_n when one changes the weights inside Δ. Now, we can factor out the weights lying in Δ from the product inside Z_n, and this allows us to write

$$Z_\mu(\beta) = \frac{Z_n}{\mathcal{Z}} = \sum_{\pi\in\Pi[a,b]} \mu(\pi) \prod_{\ell\in[a,b]} \omega_{\ell,\pi(\ell)}(\beta),$$

where μ is the random measure on $\Pi[a,b]$ defined by

$$\mu(\pi) = \frac{1}{\mathcal{Z}} \sum_{\widetilde{\pi}:\, \widetilde{\pi}|_{[a,b]}=\pi} \prod_{\ell\notin[a,b]} \omega_{\ell,\widetilde{\pi}(\ell)}(\beta). \tag{3.2}$$

Here the sum is taken over bridges $\widetilde{\pi}$ on $[0,2n]$ whose restriction to $[a,b]$ is π. Concretely, μ is simply a Bernoulli bridge measure where the endpoints are random, chosen depending on what the environment is outside of the strip Δ. Indeed, two paths on $[a,b]$ with the same endpoints have the same measure, since the paths on $[0,2n]$ with those restrictions on $[a,b]$ must agree outside of $[a,b]$, and μ only counts weights from the environment outside the strip. See Figure 3.1.

Now, the key is that if n_0, the length of $[a,b]$, is small enough, then β as a function of n_0 satisfies

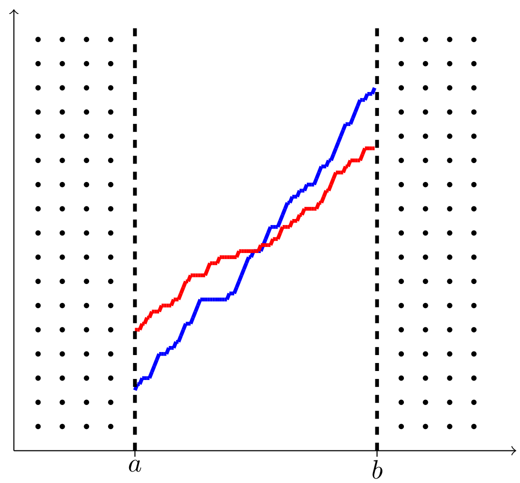

Figure 3.1: Two paths in $[a, b]$ distributed according to μ. The endpoints of the paths are random and depend on the configuration of the $\omega_{i,j}$'s outside of the strip, represented as black dots here.

$\beta \sim n_0^{-\gamma}$ for some exponent $\gamma > 1/4$. Specifically, setting

$$n_0 = 2 \left\lfloor \frac{n^{\frac{4\alpha}{1+4\delta}}}{2} \right\rfloor \tag{3.3}$$

for some $\delta > 0$ gives $\beta \sim n_0^{-(\frac{1}{4}+\delta)}$. As explained above, μ is a convex combination of bridge measures, and therefore we can apply Corollary 3.2 to obtain an estimate on $Z_n/\mathcal{Z}$ provided that there is a low probability that μ assigns high mass to paths with slopes not in $[1/4, 3/4]$. The following theorem gives a precise bound on this probability.

Theorem 3.3. *There are positive constants C, C_k such that the following hold. For all n sufficiently large, there is probability at least*

$$1 - \frac{C_k n}{n_0^{k\delta+1}}$$

that the weights $\omega_{i,j}$ not in the strip Δ yield a probability measure μ which satisfies

$$\mu(S) \leq C \exp\left(C \frac{n}{n_0} \log n\right) \mathbb{P}_{\mathrm{RW}}^{[0,2n], \frac{1}{2}, 0}(\widetilde{S})^{1/2} \tag{3.4}$$

for all sets of paths $S \subset \Pi[a, b]$. Here $\widetilde{S}$ is the set of paths on $[0, 2n]$ whose restriction to $[a, b]$ is in S.

We will take as S the set of paths whose slope on $[a, b]$ is not in $[1/4, 3/4]$:

$$S = \left\{ \pi : \frac{\pi(b) - \pi(a)}{b - a} \notin \left[\frac{1}{4}, \frac{3}{4}\right] \right\} = \left\{ \pi : \left|\pi(b) - \pi(a) - \frac{n_0}{2}\right| \geq \frac{n_0}{4} \right\}. \tag{3.5}$$

By the Markov property of random walks, $\widetilde{S}$ is independent under $\mathbb{P}_{\mathrm{RW}}$ of the behaviour of a path before time a, and therefore $\mathbb{P}_{\mathrm{RW}}(\widetilde{S})$ is simply the probability that a Bernoulli random walk with mean $1/2$, started at 0 and length n_0 ends at a position more than $n_0/4$ away from $n_0/2$. Thus by

Hoeffding's inequality,

$$\mathbb{P}_{\mathrm{RW}}(\widetilde{S}) \leq Ce^{-Cn_0}$$

for some constant $C > 0$. Plugging this in (3.4) yields

$$\mu(S) \leq C \exp\left(C\left(\frac{n}{n_0}\log n - n_0\right)\right)$$

with probability at least $1 - C_k n/n_0^{k\delta+1}$. Both n_0 and n/n_0 grow like powers of n, so as long as the exponent for n_0 is strictly larger than the one for n/n_0, this will imply that $\mu(S)$ is exponentially decreasing in some power of n. By (3.3),

$$n_0 \sim n^{\frac{4\alpha}{1+4\delta}}, \quad \frac{n}{n_0} \sim n^{1-\frac{4\alpha}{1+4\delta}},$$

so we need

$$\frac{8\alpha}{1+4\delta} > 1. \tag{3.6}$$

As long as $\alpha > 1/8$, we can find a $\delta > 0$ such that the above holds.

We henceforth assume that we have fixed some $\frac{1}{8} < \alpha < \frac{1}{4}$, and a corresponding $\delta > 0$ for which (3.6) holds. Thus when dealing with an estimate of the form $C_k/n_0^{k\delta}$, we only need to know this holds for some sufficiently large k for which $n_0^{-k\delta}$ has lower order than the main term in whatever expression we are working with. Results such as Theorem 3.3 hold for any k; the n for which "sufficiently large" starts to hold may depend on k, but this will not cause any issues as the k will be fixed depending on what α is.

Let us also mention that so far and in all of what follows, the *location* of the interval $[a, b]$ is irrelevant. All that matters is its *length*.

With these technical details out of the way, let $\mathcal{M}$ be the event that the weights outside the strip Δ yield a measure which satisfies (3.4) for all sets of paths $S \subset \Pi[a, b]$. Thus Theorem 3.3 says that

$$\mathbb{P}(\mathcal{M}) \geq 1 - \frac{C_k n}{n_0^{k\delta+1}}.$$

With S as in (3.5), the above calculations imply that on $\mathcal{M}$,

$$\mu(S) \leq Ce^{-Cn_0} \tag{3.7}$$

for some positive constant C. We then conclude the following corollary for the conditional centred moments of $Z_n/\mathcal{Z}$.

Corollary 3.4. *There are positive constants C, C_k such that for any even k, and for any n sufficiently large,*

$$\mathbb{E}\left[\left(\frac{Z_n}{\mathcal{Z}} - 1\right)^k \middle| \mathcal{M}\right] \leq \frac{C_k}{n_0^{k\delta}}.$$

Proof of Corollary 3.4. We essentially copy the proof of Corollary 3.2 but with a slight modification since μ might still assign non-zero probability to paths with slopes not in $[1/4, 3/4]$.

Let π_1 and π_2 be independent paths on $[a, b]$ distributed according to μ. We estimate the moments of the intersection local time by conditioning on the event S:

$$\begin{aligned}\mathbb{E}_\mu(L(\pi_1,\pi_2)^m) &= \mathbb{E}_\mu(L(\pi_1,\pi_2)^m|\{\pi_1 \notin S\}\cap\{\pi_2 \notin S\})\ \mu(\{\pi_1 \notin S\}\cap\{\pi_2 \notin S\}) \\ &\quad + \mathbb{E}_\mu(L(\pi_1,\pi_2)^m|\{\pi_1 \in S\}\cup\{\pi_2 \in S\})\ \mu(\{\pi_1 \in S\}\cup\{\pi_2 \in S\}) \\ &\le \mathbb{E}_\mu(L(\pi_1,\pi_2)^m|\{\pi_1 \notin S\}\cap\{\pi_2 \notin S\}) + 2n_0^m\mu(S).\end{aligned}$$

In the last line, we have used the fact that μ is a probability measure for the first term, and the trivial inequality $L(\pi_1,\pi_2) \le n_0$ plus a union bound for the second term. Conditional on S not happening, μ is a convex combination of bridge measures with slopes in $[1/4, 3/4]$, so by Proposition 2.4,

$$\mathbb{E}_\mu(L(\pi_1,\pi_2)^m|\{\pi_1 \notin S\}\cap\{\pi_2 \notin S\}) \le Cm!A^m n_0^{m/2}.$$

Next, we have

$$e^{Cn_0} = \sum_{j=0}^{\infty}\frac{(Cn_0)^j}{j!} \ge \frac{(Cn_0)^m}{m!}$$

for any m, and so after some rearranging, we have by (3.7)

$$2n_0^m\mu(S) \le Cn_0^m e^{-Cn_0} \le Cm!A^m$$

for constants $C, A > 0$ which do not depend on m. Combining these estimates thus gives

$$\mathbb{E}_\mu(L(\pi_1,\pi_2)^m) \le Cm!A^m n_0^{m/2}.$$

Then we can do exactly the same calculations done at the end of the proof of Corollary 3.2 with n replaced by n_0, and this implies the result since $\beta \sim n_0^{-(\frac{1}{4}+\delta)}$. □

It now remains to actually prove Theorem 3.3. First recall that the random measure μ is given by

$$\mu(S) = \frac{\sum_{\pi\in\widetilde{S}}\prod_{\ell\notin[a,b]}\omega_{\ell,\pi(\ell)}(\beta)}{\sum_{\pi}\prod_{\ell\notin[a,b]}\omega_{\ell,\pi(\ell)}(\beta)}$$

where for a set of paths S on $[a,b]$, we denote by $\widetilde{S}$ the set of paths on $[0,2n]$ whose restriction to $[a,b]$ is in S. The goal is to obtain an estimate for $\mu(S)$ in terms of $\mathbb{P}_{\mathrm{RW}}(\widetilde{S})$. Let us write

$$\mu(S) = \frac{\mathcal{T}}{\mathcal{B}}$$

where $\mathcal{T}$ and $\mathcal{B}$ are the normalized top and bottom parts of $\mu(S)$:

$$\mathcal{T} = \frac{1}{\binom{2n}{n}}\sum_{\pi\in\widetilde{S}}\prod_{\ell\notin[a,b]}\omega_{\ell,\pi(\ell)}(\beta), \qquad \mathcal{B} = \frac{1}{\binom{2n}{n}}\sum_{\pi}\prod_{\ell\notin[a,b]}\omega_{\ell,\pi(\ell)}(\beta).$$

We will find an upper bound for $\mathcal{T}$ and lower bound for $\mathcal{B}$ over the next two sections. Then Theorem 3.3 will result immediately from the following two lemmas. In their statements and proofs, $\mathbb{P}$ and $\mathbb{E}$ refers to probability and expectation with respect to the weights $\omega_{i,j}$ not in the strip Δ.

Lemma 3.5. *There is a positive constant C such that for all n sufficiently large, and for all sets of* *paths* $S \subset \Pi[a,b]$,

$$\mathbb{P}(\mathcal{T} \leq Ce^{C\frac{n}{n_0}}\mathbb{P}_{\mathrm{RW}}(\widetilde{S})^{1/2}) \geq 1 - C\sqrt{n}e^{-C\frac{n}{n_0}}.$$

Lemma 3.6. *There are positive constants* C, C_k *such that for all* n *sufficiently large,*

$$\mathbb{P}(\mathcal{B} \geq Ce^{-C\frac{n}{n_0}\log n}) \geq 1 - \frac{C_k n}{n_0^{k\delta+1}}.$$

3.3 Upper bound for $\mathcal{T}$

In the proofs of Lemma 3.5 and 3.6, we set $m = \lfloor 2n/n_0 \rfloor$. Thus $0 \leq 2n - mn_0 \leq n_0$.

Proof of Lemma 3.5. Using the fact that $\omega_{i,j}$ is independent of $\omega_{k,\ell}$ whenever $i \neq k$, we have

$$\mathbb{E}(\mathcal{T}^2) = \frac{1}{\binom{2n}{n}^2}\sum_{\pi_1,\pi_2}\mathbb{1}_{\{\pi_1,\pi_2\in\widetilde{S}\}}\prod_{\ell\notin[a,b]}\mathbb{E}(\omega_{\ell,\pi_1(\ell)}\omega_{\ell,\pi_2(\ell)}).$$

If $\pi_1(\ell) \neq \pi_2(\ell)$, then the inner expectation can be broken as a product of expectations by independence, and the expectation is 1. If $\pi_1(\ell) = \pi_2(\ell)$, then the expectation is

$$\mathbb{E}(\omega_{\ell,\pi_1(\ell)}^2) \leq 1 + C\beta^2$$

by Condition 3 of valid weights. Thus

$$\begin{aligned}
\mathbb{E}(\mathcal{T}^2) &\leq \frac{1}{\binom{2n}{n}^2}\sum_{\pi_1,\pi_2}\mathbb{1}_{\{\pi_1,\pi_2\in\widetilde{S}\}}(1+C\beta^2)^{L(\pi_1,\pi_2)} \\
&\leq \frac{1}{\binom{2n}{n}^2}\left(\sum_{\pi_1,\pi_2}(1+C\beta^2)^{2L(\pi_1,\pi_2)}\right)^{1/2}|\widetilde{S}| \\
&= \mathbb{P}_{\mathrm{Bri}}(\widetilde{S})\left(\mathbb{E}_{\mathrm{Bri}}(1+C\beta^2)^{2L(\pi_1,\pi_2)}\right)^{1/2},
\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the second line. By Stirling's formula,

$$\mathbb{P}_{\mathrm{Bri}}(\widetilde{S}) = \frac{|\widetilde{S}|}{\binom{2n}{n}} \leq C\sqrt{n}\frac{|\widetilde{S}|}{4^n} = C\sqrt{n}\mathbb{P}_{\mathrm{RW}}(\widetilde{S})$$

for some positive constant C, and by (2.4),

$$\mathbb{E}_{\mathrm{Bri}}(1+C\beta^2)^{2L(\pi_1,\pi_2)} \leq \mathbb{E}_{\mathrm{Bri}}(e^{2C\beta^2 L(\pi_1,\pi_2)}) \leq 4\mathbb{E}_{\mathrm{RW}}(e^{4C\beta^2 L(\pi_1,\pi_2)}).$$

Clearly, we can write

$$\begin{aligned}
L(\pi_1,\pi_2) = L(\pi_1,\pi_2;[0,n_0)) &+ L(\pi_1,\pi_2;[n_0,2n_0)) + \dots \\
&+ L(\pi_1,\pi_2;[(m-1)n_0,mn_0)) + L(\pi_1,\pi_2;[mn_0,2n]).
\end{aligned}$$

The idea is that the terms appearing in the above sum are essentially independent, and so we can use this to write the above expectation as a product of the moment generating functions of the intersection local time on each subinterval $[jn_0,(j+1)n_0)$.

For each $0 \le j \le m-1$, let

$$T_j = \min\{\ell \in [jn_0, (j+1)n_0) : \pi_1(\ell) = \pi_2(\ell)\}$$

be the first time that π_1 and π_2 intersect on the subinterval $[jn_0, (j+1)n_0)$, and similarly define

$$T_m = \min\{\ell \in [mn_0, 2n] : \pi_1(\ell) = \pi_2(\ell)\}.$$

If π_1 and π_2 do not intersect on some subinterval, then we set

$$T_j = (j+1)n_0.$$

Note that the T_j's are stopping times. Now for each $0 \le j \le m$, let ρ_1^j, ρ_2^j be independent Bernoulli(1/2) random walks started at 0, all independent of π_1 and π_2. Then define

$$\pi_1^j(\ell) = \begin{cases} \pi_1(\ell + T_j) - \pi_1(T_j) & \text{if } \ell + T_j < (j+1)n_0 \\ \pi_1((j+1)n_0) - \pi_1(T_j) + \rho_1^j(\ell) & \text{otherwise} \end{cases}$$

and similarly for π_2^j. Concretely, π_1^j and π_2^j are obtained by deleting the part of the walks before the first intersection of π_1 and π_2 in $[jn_0, (j+1)n_0)$, and then "gluing" ρ_1^j and ρ_2^j at the ends once the walks have reached time $(j+1)n_0$. In the case where π_1 and π_2 do not intersect at all in a subinterval, they are thrown away entirely and $\pi_1^j = \rho_1^j$ and $\pi_2^j = \rho_2^j$. See Figure 3.2.

By the strong Markov property of Bernoulli random walks, the π_1^j's and π_2^j's are Bernoulli(1/2) random walks, and furthermore they are all independent. Additionally, we have

$$L(\pi_1, \pi_2; [jn_0, (j+1)n_0)) \le L(\pi_1^j, \pi_2^j; [0, n_0))$$

since all the intersections of π_1 and π_2 on the subinterval have been kept in the construction of π_1^j and π_2^j. The latter two may have more intersections depending on the behaviour of ρ_1^j and ρ_2^j. Consequently, we have

$$\begin{aligned} \mathbb{E}_{\mathrm{RW}}[e^{C\beta^2 L(\pi_1,\pi_2)}] &\le \mathbb{E}_{\mathrm{RW}}[e^{C\beta^2(L(\pi_1^1,\pi_2^1;[0,n_0))+\ldots+L(\pi_1^m,\pi_2^m;[0,n_0)))}] \\ &= \prod_{j=0}^{m} \mathbb{E}_{\mathrm{RW}}[e^{C\beta^2 L(\pi_1^j,\pi_2^j;[0,n_0))}] \end{aligned}$$

Since $\beta^2 \sim n_0^{-(\frac{1}{2}+2\delta)}$, we have

$$\mathbb{E}_{\mathrm{RW}}[e^{C\beta^2 L(\pi_1^j,\pi_2^j;[0,n_0))}] \le C$$

for some $C > 1$ by Proposition 2.5 (the proposition is stated in terms of bridge measure, but the very first thing that we did in the proof is change to random walk measure, so the estimate also holds for $\mathbb{E}_{\mathrm{RW}}$). The terms in the product above are all the same since π_1^j and π_2^j have the same distribution for all j. Therefore

$$\prod_{j=0}^{m} \mathbb{E}_{\mathrm{RW}}[e^{C\beta^2 L(\pi_1^j,\pi_2^j;[0,n_0))}] \le C^{m+1} \le e^{C\frac{n}{n_0}}.$$

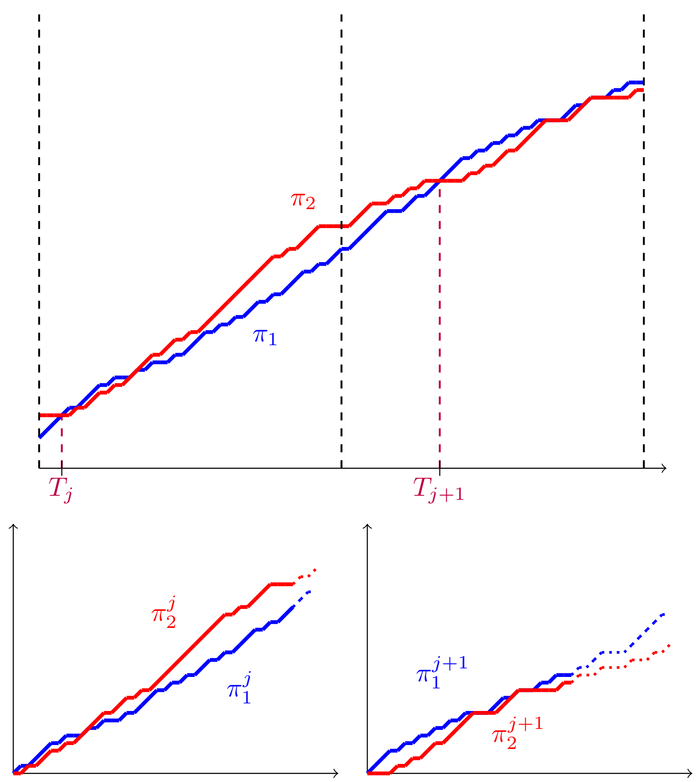

Figure 3.2: On top are the paths π_1, π_2, together with the times T_j, T_{j+1} when they first intersect on the intervals $[jn_0, (j+1)n_0)$ and $[(j+1)n_0, (j+2)n_0)$ respectively. Below are the paths $\pi_1^j, \pi_2^j, \pi_1^{j+1}$ and π_2^{j+1}. These are obtained by deleting the parts of the paths before the first intersection time on each subinterval, and gluing at the end $\rho_1^j, \rho_2^j, \rho_1^{j+1}$ and ρ_2^{j+1} (represented as dotted paths here).

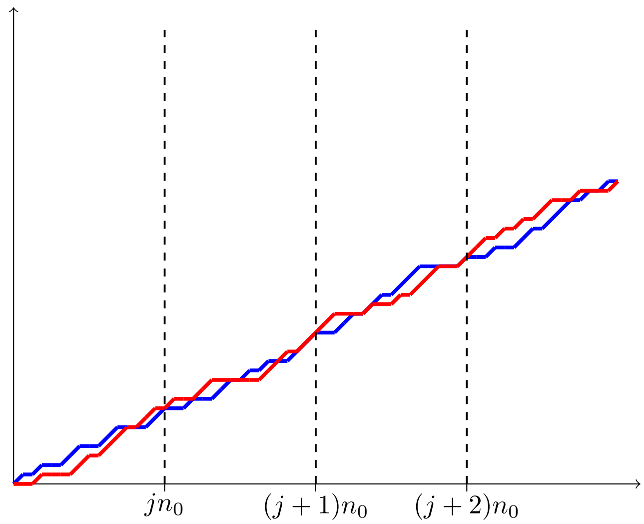

Figure 3.3: Two paths in the set G. On each subinterval $[jn_0, (j+1)n_0)$, the restriction of the paths are Bernoulli bridges of slope $1/2$.

In summary, we have obtained the bound

$$\mathbb{E}(\mathcal{T}^2) \leq C\sqrt{n}\mathbb{P}_{\mathrm{RW}}(\widetilde{S})e^{C\frac{n}{n_0}}, \tag{3.8}$$

and so by Chebyshev's inequality,

$$\mathbb{P}(\mathcal{T} \geq \lambda) \leq \frac{C\sqrt{n}\mathbb{P}_{\mathrm{RW}}(\widetilde{S})e^{C\frac{n}{n_0}}}{\lambda^2}.$$

If $\mathbb{P}_{\mathrm{RW}}(\widetilde{S})$ is 0, then the sum in the definition of $\mathcal{T}$ is empty, so in this case there is nothing to prove. So assume $\mathbb{P}_{\mathrm{RW}}(\widetilde{S}) > 0$. Then by taking $\lambda = \mathbb{P}_{\mathrm{RW}}(\widetilde{S})^{1/2}e^{C\frac{n}{n_0}}$ in (3.8) (with the same C), we thus find

$$T \leq \mathbb{P}_{\mathrm{RW}}(\widetilde{S})^{1/2}e^{C\frac{n}{n_0}}$$

with probability at least $1 - C\sqrt{n}e^{-C\frac{n}{n_0}}$. □

3.4 Lower bound for $\mathcal{B}$

Proof of Lemma 3.6. Let $G \subset \Pi[0, 2n]$ be the following set of paths

$$G = \left\{\pi : \pi(jn_0) = \frac{jn_0}{2} \text{ for } j = 0, 1, \ldots, m \text{ and } \pi(2n) = n\right\}.$$

That is, G is the set of paths which intersect the line of $y = x/2$ for all times $0, n_0, \ldots, mn_0$ and $2n$. See Figure 3.3. Clearly, by restricting the sum for $\mathcal{B}$ to only the paths in G, we obtain a lower bound on $\mathcal{B}$. The advantage of doing so is that when restricted to G, the partition function admits a special factorization.

For each j, let G_j be the set of paths on $[jn_0,(j+1)n_0)$ defined by

$$G_j = \left\{\pi : \pi(jn_0) = \frac{jn_0}{2}, \pi((j+1)n_0) = \frac{(j+1)n_0}{2}\right\}$$

and G_m the set of paths on $[mn_0, 2n)$ defined as

$$G_m = \left\{\pi : \pi(mn_0) = \frac{mn_0}{2}, \pi(2n) = n\right\}.$$

Write $\widetilde{\omega}_{i,j}(\beta) = \omega_{i,j}(\beta)$ if $i \notin [a,b]$ and $\widetilde{\omega}_{i,j}(\beta) = 1$ if $i \in [a,b]$. Finally define

$$\mathcal{Y}_j = \frac{1}{\binom{n_0}{n_0/2}} \sum_{\pi \in G_j} \prod_{\ell = jn_0}^{(j+1)n_0 - 1} \widetilde{\omega}_{\ell,\pi(\ell)}$$

for $j = 0, 1, \ldots, m-1$ and

$$\mathcal{Y}_m = \frac{1}{\binom{2n-mn_0}{n-\frac{mn_0}{2}}} \sum_{\pi \in G_m} \prod_{\ell=m}^{2n} \widetilde{\omega}_{\ell,\pi(\ell)}$$

(if $2n = mn_0$, i.e n_0 divides $2n$, then we set $\mathcal{Y}_m = 1$). Then we have

$$\mathcal{B} \geq \frac{1}{\binom{2n}{n}} \sum_{\pi \in G} \prod_{\ell \notin [a,b]} \omega_{\ell,\pi(\ell)} = \frac{\binom{n_0}{n_0/2}^m}{\binom{2n}{n}} \binom{2n - mn_0}{n - \frac{mn_0}{2}} \prod_{j=0}^{m} \mathcal{Y}_j. \tag{3.9}$$

This last equality is because the set G is precisely the set of paths obtained by gluing together paths from $G_0, G_1, \ldots, G_m$. Now, by Stirling's formula and the bound $2n - mn_0 \leq n_0$, we have when $2n \neq mn_0$,

$$\begin{aligned}\binom{n_0}{n_0/2}^m \binom{2n - mn_0}{n - \frac{mn_0}{2}} &\geq \left(\frac{C}{\sqrt{n_0}} 2^{n_0}\right)^m C \frac{4^{n - \frac{mn_0}{2}}}{\sqrt{n - \frac{mn_0}{2}}} \\ &\geq C 4^n C^m e^{-Cm \log(\sqrt{n_0})} \geq C 4^n e^{-C \frac{n}{n_0} \log n}\end{aligned}$$

for some positive constant C. When $2n = mn_0$, then $\mathcal{Y}_m$ and the corresponding binomial coefficient do not appear in (3.9), and instead we just get

$$\binom{n_0}{n_0/2}^m \geq \left(\frac{C}{\sqrt{n_0}} 2^{n_0}\right)^m = \frac{C^m 4^n}{n_0^{n/n_0}} \geq C 4^n e^{-C \frac{n}{n_0} \log n}.$$

We also have the trivial bound

$$\binom{2n}{n} \leq 2^{2n} = 4^n.$$

Plugging these estimates in (3.9) gives in all cases that

$$\mathcal{B} \geq C e^{-C \frac{n}{n_0} \log n} \prod_{j=0}^{m} \mathcal{Y}_j. \tag{3.10}$$

Each $\mathcal{Y}_j$ has the law of a normalized point-to-point partition function for a valid set of weights on an interval of length n_0 (or $2n - mn_0$ in the case $j = m$). Since $\beta \sim n_0^{-(\frac{1}{4}+\delta)}$, it follows from Corollary

3.2 that for any positive even integer k, there is a constant C_k such that

$$\mathbb{E}(\mathcal{Y}_j - 1)^k \leq \frac{C_k}{n_0^{k\delta}}.$$

The C_k's are the same for each j, since the $\mathcal{Y}_j$'s are partition functions for the same set of valid weights. By Chebyshev's inequality, we therefore have

$$\mathbb{P}(\mathcal{Y}_j \leq 1/2) \leq \mathbb{P}(|\mathcal{Y}_j - 1| \geq 1/2) \leq \frac{2^k C_k}{n_0^{k\delta}}.$$

Hence, with the same C as in (3.10),

$$\begin{aligned}
\mathbb{P}(\mathcal{B} \geq Ce^{-C\frac{n}{n_0}\log n} 2^{-(m+1)}) &\geq \mathbb{P}\left(\prod_{j=0}^{m} \mathcal{Y}_j \geq 2^{-(m+1)}\right) \\
&\geq \mathbb{P}\left(\bigcap_{j=0}^{m} \{\mathcal{Y}_j \geq 1/2\}\right) \\
&\geq 1 - \sum_{j=0}^{m} \mathbb{P}(\mathcal{Y}_j < 1/2) \\
&\geq 1 - \frac{2^k (m+1) C_k}{n_0^{k\delta}} \geq 1 - \frac{C_k n}{n_0^{k\delta+1}}.
\end{aligned}$$

The result follows since $2^{-(m+1)} \geq e^{-C\frac{n}{n_0}\log n}$ for n sufficiently large. □

Chapter 4

Universality of directed polymers in the intermediate regime

We prove our two main theorems on the directed polymer in the intermediate disorder regime. We first prove Theorem 1.5, using the Lindeberg replacement strategy. We then show that both the standard polymer and the log-gamma polymer (suitably renormalized) are valid sets of weights, and we thus conclude Theorem 1.2 from Theorem 1.5.

4.1 Proof of Theorem 1.5

Having established the prerequisite estimates on the intersection local time of random walks and the local fluctuations of polymers, we are finally ready to complete the proof of Theorem 1.5. Before we begin, let us briefly recall the setup; we are given two parametrized valid (see Definition 1.3) sets of weights $(\omega_{i,j}(\beta))_{i,j\geq 0}$ and $(\omega'_{i,j}(\beta))_{i,j\geq 0}$ whose second moment coincide for all β sufficiently small:

$$\mathbb{E}(\omega_{i,j}(\beta)^2) = \mathbb{E}(\omega'_{i,j}(\beta)^2).$$

The corresponding point-to-point partition functions $Z_n(\beta)$ and $Z'_n(\beta)$ are as follows

$$Z_n(\beta) = \sum_{\pi} \prod_{i=0}^{2n} \omega_{i,\pi(i)}(\beta), \quad Z'_n(\beta) = \sum_{\pi} \prod_{i=0}^{2n} \omega'_{i,\pi(i)}(\beta).$$

We consider the intermediate disorder regime; so β depends on the length of the polymer via $\beta = n^{-\alpha}$ for some $\alpha > 0$. We assume that $\frac{1}{8} < \alpha < \frac{1}{4}$ is fixed, and $\delta > 0$ is chosen so that

$$\frac{8\alpha}{1+4\delta} > 1$$

and

$$(2 - 17\alpha) + 8\delta - 20\alpha\delta < 0. \tag{4.1}$$

The first of these conditions is simply (3.6), and we have already established in Section 3.2 that this needs to hold. On the other hand, (4.1) looks like a bizarre assumption to make, but this will come

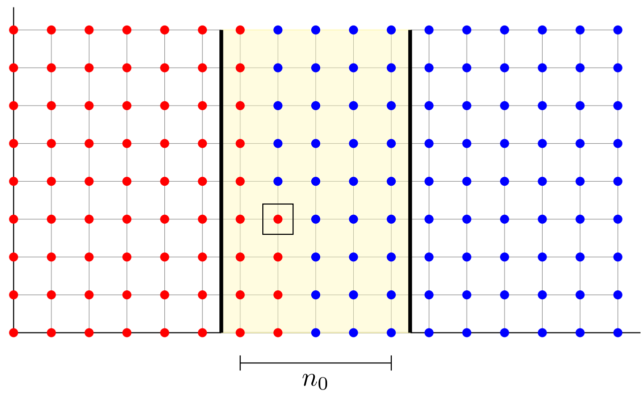

Figure 4.1: The Lindeberg method in action. The blue dots are the weights from the $(\omega_{i,j}(\beta))$ polymer, and these are gradually replaced by weights from the $(\omega'_{i,j}(\beta))$ polymer, represented by red dots. The vertex inside the square has just been replaced. We partition the whole range into strips and estimate the errors accumulated from each replacement over a strip.

out of the calculations at the end of the proof. Note that such a $\delta > 0$ exists for (4.1) as long as $\alpha > 2/17$, which is indeed our case.

Our main hypothesis is that the free energy $\log Z'_n(\beta)$ has a scaling limit:

$$\frac{\log Z'_n(\beta) - a_n}{\sigma_n} \xrightarrow{d} F$$

for some deterministic sequence a_n, and a probability distribution F. Here σ_n is the scaling factor

$$\sigma_n = \beta^{\frac{4}{3}} n^{\frac{1}{3}}.$$

Note that since $\alpha < 1/4$, we have $\sigma_n \to \infty$. Our goal is to show that we also have

$$\frac{\log Z_n(\beta) - a_n}{\sigma_n} \xrightarrow{d} F$$

for the same choice of a_n and F. By the Portmanteau theorem [15, Theorem 2.1] and a standard density argument, it is enough to show that

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(\frac{\log Z_n(\beta) - a_n}{\sigma_n}\right)\right] = \lim_{n\to\infty} \mathbb{E}\left[f\left(\frac{\log Z'_n(\beta) - a_n}{\sigma_n}\right)\right]$$

for all C^3 functions f whose derivatives are all bounded.

To estimate the difference of these two expectations, we will use the Lindeberg method, which consists in replacing each weight of the $(\omega_{i,j}(\beta))_{i,j\geq 0}$ polymer one by one to a weight from the $(\omega'_{i,j}(\beta))_{i,j\geq 0}$ polymer and estimating the resulting error from each step by a Taylor expansion. The order in which we replace the weights will be along vertical lines: we change $\omega_{i,0}$ to $\omega'_{i,0}$, then $\omega_{i,1}$ to $\omega'_{i,1}$ and so on until we reach (i, n), at which point we move over to the $(i+1)^{\text{th}}$ vertical line and then replace those moving upwards. Eventually we will be partitioning the points into vertical strips of width n_0, so doing things this way ensures that the random measure μ introduced in (3.2) stays the same at each stage of the replacement on a given strip. See Figure 4.1.

Let

$$W_{i,j}(\beta) = \sum_{\pi,(i,j)\in\pi} \prod_{\ell=0}^{2n} \omega^{(i,j)}_{\ell,\pi(\ell)}(\beta)$$
$$V_{i,j}(\beta) = \sum_{\pi,(i,j)\notin\pi} \prod_{\ell=0}^{2n} \omega^{(i,j)}_{\ell,\pi(\ell)}(\beta).$$

Here the sum for $W_{i,j}(\beta)$ is taken over the paths which go through the point (i,j), the sum for $V_{i,j}(\beta)$ is over the paths which do not go through the point (i,j), and

$$\omega^{(i,j)}_{k,\ell}(\beta) = \begin{cases} \omega_{k,\ell}(\beta) & \text{if } k > i \text{ or } (k = i \text{ and } \ell > j) \\ \omega'_{k,\ell}(\beta) & \text{if } k < i \text{ or } (k = i \text{ and } \ell < j) \\ 1 & \text{if } i = k \text{ and } j = \ell. \end{cases}$$

Thus the $(\omega^{(i,j)}_{k,\ell}(\beta))_{k,\ell\geq 0}$ are the weights in the polymer at the $(i,j)^{\text{th}}$ step of the replacement. We have that $V_{i,j} + \omega_{i,j}W_{i,j}$ is the partition function if we choose the weight at (i,j) to be $\omega_{i,j}$, and $V_{i,j} + \omega'_{i,j}W_{i,j}$ is the partition function if we instead choose this weight to be $\omega'_{i,j}$. To ease the notation, we will usually omit to write β, i or j when it is clear that we are working with some specific point (i,j). Likewise, we will also generally omit the superscript (i,j) from $\omega^{(i,j)}_{k,\ell}$, and it should then be understood that the set of weights for this particular polymer is the one at a particular phase of the replacement. The estimates that will follow will only depend on the weights in terms of the constants in Definition 1.3, and as already noted in Remark 1.4, those can be chosen to be uniform over any combination of the weights. Hence, this abuse of notation should not cause any issues.

Observe that $V_{0,0} + \omega_{0,0}W_{0,0} = Z_n$ and $V_{2n,n} + \omega'_{2n,n}W_{2n,n} = Z'_n$, so we have the following telescoping sum formula:

$$\begin{aligned} &\mathbb{E}\left[f\left(\frac{\log Z_n(\beta) - a_n}{\sigma_n}\right) - f\left(\frac{\log Z'_n(\beta) - a_n}{\sigma_n}\right)\right] \\ &= \sum_{i=0}^{2n}\sum_{j=0}^{n} \mathbb{E}\left[f\left(\frac{\log(V_{i,j} + \omega_{i,j}W_{i,j}) - a_n}{\sigma_n}\right) - f\left(\frac{\log(V_{i,j} + \omega'_{i,j}W_{i,j}) - a_n}{\sigma_n}\right)\right]. \end{aligned} \tag{4.2}$$

Note that some of the points (i,j) for $0 \leq i \leq 2n$ and $0 \leq j \leq n$ do not get visited by any path, since paths are constrained to finish at $(2n,n)$. So $W_{i,j} = 0$ for those points, and the corresponding term in the sum is 0. We choose to keep these terms written in the sum for notational convenience.

By Taylor's theorem,

$$
\begin{aligned}
f\left(\frac{\log(V+\omega_{i,j}W)-a_n}{\sigma_n}\right) &= f\left(\frac{\log(V+W)-a_n}{\sigma_n}\right) \\
&+ f'\left(\frac{\log(V+W)-a_n}{\sigma_n}\right)\frac{W}{\sigma_n(V+W)}(\omega_{i,j}-1) \\
&- \frac{1}{2}\left[f'\left(\frac{\log(V+W)-a_n}{\sigma_n}\right)\right. \\
&\quad \left. -\sigma_n^{-1}f''\left(\frac{\log(V+W)-a_n}{\sigma_n}\right)\right]\frac{W^2}{\sigma_n(V+W)^2}(\omega_{i,j}-1)^2 \\
&+ \frac{1}{6}\left[2f'\left(\frac{\log(V+\eta W)-a_n}{\sigma_n}\right)\right. \\
&\quad - 3\sigma_n^{-1}f''\left(\frac{\log(V+\eta W)-a_n}{\sigma_n}\right) \\
&\quad \left. +\sigma_n^{-2}f'''\left(\frac{\log(V+\eta W)-a_n}{\sigma_n}\right)\right]\frac{W^3}{\sigma_n(V+\eta W)^3}(\omega_{i,j}-1)^3
\end{aligned}
$$

for some η between 1 and $\omega_{i,j}$. The same expansion holds as well with $\omega'_{i,j}$, but with instead some η' between 1 and $\omega'_{i,j}$.

We now substitute these into (4.2). Since $\omega_{i,j}(\beta)$ and $\omega'_{i,j}(\beta)$ have the same first and second moments for all sufficiently small β, and $V_{i,j}$ and $W_{i,j}$ are independent of $\omega_{i,j}$ and $\omega'_{i,j}$, the order 0, order 1 and order 2 terms all cancel each other out. For the third order terms, we can use the fact that f has bounded derivatives and $\sigma_n \to \infty$ to get an upper bound. In summary, we have

$$
\begin{aligned}
&\left|\mathbb{E}\left[f\left(\frac{\log(V+\omega_{i,j}W)-a_n}{\sigma_n}\right) - f\left(\frac{\log(V+\omega'_{i,j}W)-a_n}{\sigma_n}\right)\right]\right| \\
&\le \frac{C}{\sigma_n}\mathbb{E}\left[\frac{W^3}{\sigma_n(V+\eta W)^3}|\omega_{i,j}-1|^3 + \frac{W^3}{\sigma_n(V+\eta' W)^3}|\omega'_{i,j}-1|^3\right].
\end{aligned}
$$

We can get rid of the dependence on η and η' by proceeding as follows. Consider the events $\{\omega \le 1/2\}$ and $\{\omega > 1/2\}$. On the second event, we have $\eta > 1/2$ (since η is between ω and 1), so

$$
\begin{aligned}
\mathbb{E}\left|\frac{W^3}{(V+\eta W)^3}|\omega-1|^3 \mathbb{1}_{\{\omega>1/2\}}\right| &\le \mathbb{E}\left(\frac{W^3}{(V+\frac{1}{2}W)^3}|\omega-1|^3\right) \\
&\le 8\mathbb{E}\left(\frac{W^3}{(V+W)^3}\right)\mathbb{E}|\omega-1|^3 \\
&\le C\mathbb{E}\left(\frac{W^3}{(V+W)^3}\right)\beta^3.
\end{aligned}
$$

Here the breaking up of the expectation is justified because W and V are independent of ω. On the other event, we have by Hölder's inequality that

$$\begin{aligned}\mathbb{E}\left|\frac{W^3}{(V+\eta W)^3}|\omega-1|^3\mathbb{1}_{\{\omega\le 1/2\}}\right| &\le \mathbb{E}\left(\frac{W^3}{(V+\omega W)^3}\mathbb{1}_{\{\omega\le 1/2\}}\right)\\ &\le \mathbb{E}\left(\frac{1}{\omega^3}\mathbb{1}_{\{\omega\le 1/2\}}\right)\\ &\le \mathbb{E}\left(\frac{1}{\omega^p}\right)^{3/p}\mathbb{P}(\omega\le 1/2)^{1-3/p}\\ &\le C\mathbb{P}(|\omega-1|\ge 1/2)^{1-3/p}\end{aligned}$$

where $p>3$ is the exponent appearing in Condition 4 of Definition 1.3. By Chebyshev's inequality,

$$\mathbb{P}(|\omega-1|\ge 1/2)^{1-3/p}\le (2^k\mathbb{E}|\omega-1|^k)^{1-3/p}\le C_k\beta^{k(1-3/p)}=\frac{C_k}{n^{k(1-3/p)\alpha}}$$

for any positive integer k. A corresponding error term holds for ω', and in fact for each i,j in the sum (4.2) with the same C_k's (those only depend on the constants in the definition of valid weights). There are $O(n^2)$ terms in the sum, and this produces an error of size

$$\frac{C_k n^2}{n^{k(1-3/p)\alpha}}.$$

We are free to choose k as large as we want, so by choosing it sufficiently large, this term will go to zero as $n\to\infty$, and we do not need to worry about it in the rest. All of the above calculations hold with ω', and so we have

$$\begin{aligned}&\left|\mathbb{E}\left[f\left(\frac{\log(V+\omega_{i,j}W)-a_n}{\sigma_n}\right)-f\left(\frac{\log(V+\omega'_{i,j}W)-a_n}{\sigma_n}\right)\right]\right|\\ &\le\frac{C\beta^3}{\sigma_n}\mathbb{E}\left(\frac{W^3}{(V+W)^3}\right)+o\left(\frac{1}{n^2}\right).\end{aligned}$$

The constant C and the $o(1/n^2)$ term are uniform over all i and j as already mentioned, so substituting this back in (4.2) yields

$$\left|\mathbb{E}\left[f\left(\frac{\log Z_n(\beta)-a_n}{\sigma_n}\right)-f\left(\frac{\log Z'_n(\beta)-a_n}{\sigma_n}\right)\right]\right|\le\frac{C\beta^3}{\sigma_n}\sum_{i=0}^{2n}\sum_{j=0}^{n}\mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\right)+o(1).\quad(4.3)$$

Let $[a,b]\subset[0,2n]$ be a subinterval of length n_0, where

$$n_0=2\left\lfloor\frac{n^{\frac{4\alpha}{1+4\delta}}}{2}\right\rfloor$$

as in (3.3), and let Δ (represented as the shaded yellow region in Figure 4.1) be the strip

$$\Delta=\{(i,j):a\le i\le b, 0\le j\le n\}.$$

We wish to obtain an estimate for

$$\sum_{(i,j)\in\Delta} \mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\right). \tag{4.4}$$

Let μ be the random measure on $\Pi[a,b]$ introduced in (3.2):

$$\mu(\pi) = \frac{1}{\mathcal{Z}} \sum_{\widetilde{\pi}:\, \widetilde{\pi}|_{[a,b]}=\pi} \prod_{\ell\notin[a,b]} \omega_{\ell,\widetilde{\pi}(\ell)}(\beta)$$

where $\mathcal{Z}$ is the normalizing factor

$$\mathcal{Z} = \sum_{\pi} \prod_{\ell\notin[a,b]} \omega_{\ell,\pi(\ell)}(\beta).$$

As explained earlier, our choice of the order in which the weights are changed means that μ and $\mathcal{Z}$ are the same at each stage of the replacement for $(i,j)\in\Delta$. Technically, in the definition of μ and $\mathcal{Z}$, we should use the weights $\omega'_{\ell,\pi(\ell)}$ when $\ell < a$ and $\omega_{\ell,\pi(\ell)}$ when $\ell > b$, but we will omit to introduce more notation and will simply write ω everywhere using our convention described before.

By Theorem 3.3 and the discussion that follows it, there are positive constants C, C_k such that if $\mathcal{M}$ is the event

$$\mathcal{M} = \{\mu(S) \le Ce^{-Cn_0}\}$$

where $S \subset \Pi[a,b]$ as defined in (3.5) is the set of paths with slopes not in $[1/4, 3/4]$, then

$$\mathbb{P}(\mathcal{M}) \ge 1 - \frac{C_k n}{n_0^{k\delta+1}}.$$

By conditioning on $\mathcal{M}$, we then have

$$\begin{aligned}\mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\right) &= \mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\,\middle|\,\mathcal{M}\right)\mathbb{P}(\mathcal{M}) + \mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\,\middle|\,\mathcal{M}^c\right)\mathbb{P}(\mathcal{M}^c)\\ &\le \mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\,\middle|\,\mathcal{M}\right) + \mathbb{P}(\mathcal{M}^c).\end{aligned}$$

When we sum over $(i,j)\in\Delta$, the second term above will produce an error of order

$$\sum_{(i,j)\in\Delta} \mathbb{P}(\mathcal{M}^c) \le \frac{C_k n^2}{n_0^{k\delta}},$$

and again we can make this term go to 0 as fast as any polynomial that we want by choosing k large enough, so we may ignore it in all that follows. For the first term, we have by Corollary 3.4

$$\begin{aligned}
&\mathbb{E}\left(\frac{W^3}{(V+W)^3}\,\middle|\,\mathcal{M}\right)\\
&=\mathbb{E}\left(\frac{W^3}{(V+W)^3}\mathbb{1}_{\{(V+W)/\mathcal{Z}\le 1/2\}}\,\middle|\,\mathcal{M}\right)+\mathbb{E}\left(\frac{(W/\mathcal{Z})^3}{((V+W)/\mathcal{Z})^3}\mathbb{1}_{\{(V+W)/\mathcal{Z}>1/2\}}\,\middle|\,\mathcal{M}\right)\\
&\le \mathbb{P}\left(\frac{V+W}{\mathcal{Z}}\le\frac{1}{2}\,\middle|\,\mathcal{M}\right)+8\mathbb{E}\left(\frac{W^3}{\mathcal{Z}^3}\,\middle|\,\mathcal{M}\right).
\end{aligned}$$

Now $V_{i,j}+W_{i,j}$ is the partition function where the weight at (i,j) has been changed to 1, so by Chebyshev's inequality and Corollary 3.4,

$$\begin{aligned}
\mathbb{P}\left(\frac{V+W}{\mathcal{Z}}\le\frac{1}{2}\,\middle|\,\mathcal{M}\right)\le\mathbb{P}\left(\left|\frac{V+W}{\mathcal{Z}}-1\right|\ge\frac{1}{2}\,\middle|\,\mathcal{M}\right)&\le 2^k\mathbb{E}\left[\left(\frac{V+W}{\mathcal{Z}}-1\right)^k\,\middle|\,\mathcal{M}\right]\\
&\le\frac{2^kC_k}{n_0^{k\delta}}
\end{aligned}$$

for any positive even integer k. Once again, we are free to pick k as large as we want, so we can again ignore this term in the rest of the calculations by the same reasoning as above.

So all that remains now is to estimate

$$\mathbb{E}\left(\frac{W^3}{\mathcal{Z}^3}\,\middle|\,\mathcal{M}\right).$$

Expanding out $W^3/\mathcal{Z}^3$ gives

$$\begin{aligned}
\frac{W^3}{\mathcal{Z}^3}&=\left(\sum_{\pi\in\Pi[a,b],(i,j)\in\pi}\mu(\pi)\prod_{\ell\in[a,b]\setminus\{i\}}\omega_{\ell,\pi(\ell)}\right)^3\\
&=\sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]}\mu(\pi_1)\mu(\pi_2)\mu(\pi_3)\mathbb{1}_{\{(i,j)\in\pi_1,\pi_2,\pi_3\}}\prod_{\ell\in[a,b]\setminus\{i\}}\omega_{\ell,\pi_1(\ell)}\omega_{\ell,\pi_2(\ell)}\omega_{\ell,\pi_3(\ell)}
\end{aligned}$$

where the sum is taken over triplets of paths that all visit the point (i,j). The product inside does not include $\ell=i$, since for $W_{i,j}$, the weight at (i,j) has been replaced by one. Let us now take the conditional expectation of this quantity given the weights outside of Δ. Since the weights in Δ are independent of those outside, and the measure μ only depends on the weights outside of Δ, we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{W^3}{\mathcal{Z}^3}\,\middle|\,(\omega_{k,\ell}):(k,\ell)\notin\Delta\right]\\
&=\sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]}\mu(\pi_1)\mu(\pi_2)\mu(\pi_3)\mathbb{1}_{\{(i,j)\in\pi_1,\pi_2,\pi_3\}}\prod_{\ell\in[a,b]\setminus\{i\}}\mathbb{E}(\omega_{\ell,\pi_1(\ell)}\omega_{\ell,\pi_2(\ell)}\omega_{\ell,\pi_3(\ell)}).
\end{aligned}$$

Now, for the expectation inside the product, there are three possible cases. It will be 1 if the paths π_1,π_2 and π_3 are all at different locations at time ℓ; it will be

$$\mathbb{E}(\omega_{\ell,k}(\beta)^2)=\mathbb{E}((\omega_{\ell,k}(\beta)-1)^2)+1\le 1+C_2\beta^2$$

if exactly two paths intersect at time ℓ (at location k) and it will be

$$\mathbb{E}(\omega_{\ell,k}(\beta)^3) = \mathbb{E}((\omega_{\ell,k}(\beta)-1)^3 + 3(\omega_{\ell,k}-1)^2 + 1) \le 1 + 3C_2\beta^2 + C_3\beta^3$$

if all three paths intersect at time ℓ (at location k). Here C_2 and C_3 are the constants appearing in the Definition 1.3. If n is sufficiently large so that $\beta < 1$, then in both of those last two cases, we get an upper bound of $1 + C\beta^2$ for some positive constant C. To summarize, the expectation is one when the paths do not intersect and it is at most $1 + C\beta^2$ when at least two paths intersect, so that

$$\begin{aligned}
&\sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]} \mu(\pi_1)\mu(\pi_2)\mu(\pi_3)\mathbb{1}_{\{(i,j)\in\pi_1,\pi_2,\pi_3\}} \prod_{\ell\in[a,b]\setminus\{i\}} \mathbb{E}(\omega_{\ell,\pi_1(\ell)}\omega_{\ell,\pi_2(\ell)}\omega_{\ell,\pi_3(\ell)}) \\
&\le \sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]} \mu(\pi_1)\mu(\pi_2)\mu(\pi_3)\mathbb{1}_{\{(i,j)\in\pi_1,\pi_2,\pi_3\}}(1+C\beta^2)^{L(\pi_1,\pi_2,\pi_3)} \\
&\le \sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]} \mu(\pi_1)\mu(\pi_2)\mu(\pi_3)\mathbb{1}_{\{(i,j)\in\pi_1,\pi_2,\pi_3\}} e^{C\beta^2 L(\pi_1,\pi_2,\pi_3)}.
\end{aligned}$$

The above bound is the same for any $(i,j) \in \Delta$ (that is C is the same for any (i,j)). Since the random measure μ stays the same throughout the replacement of the weights in Δ, we can add this bound over $(i,j) \in \Delta$, and this gives

$$\begin{aligned}
\sum_{(i,j)\in\Delta} \mathbb{E}\left[\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\middle| \mathcal{M}\right] &= \mathbb{E}\left[\sum_{(i,j)\in\Delta} \mathbb{E}\left[\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\middle| (\omega_{i,j}) : (i,j)\notin\Delta\right]\middle| \mathcal{M}\right] \\
&\le \mathbb{E}\left[\sum_{\pi_1,\pi_2,\pi_3\in\Pi[a,b]} \mu(\pi_1)\mu(\pi_2)\mu(\pi_3)L_3(\pi_1,\pi_2,\pi_3)e^{C\beta^2 L(\pi_1,\pi_2,\pi_3)}\middle| \mathcal{M}\right] \\
&= \mathbb{E}\left[\mathbb{E}_\mu(L_3(\pi_1,\pi_2,\pi_3)e^{C\beta^2 L(\pi_1,\pi_2,\pi_3)})\middle| \mathcal{M}\right]
\end{aligned}$$

where we recall that $L_3(\pi_1,\pi_2,\pi_3)$ denotes the number of triple intersections of the walks. It is now that our work on triple intersections from Section 2.4 comes to fruition.

Lemma 4.1. *There is a positive constant C such that for any configuration of weights in $\mathcal{M}$,*

$$\mathbb{E}_\mu(L_3(\pi_1,\pi_2,\pi_3)e^{C\beta^2 L(\pi_1,\pi_2,\pi_3)}) \le C\log n.$$

Furthermore, C does not depend on the location of the interval $[a,b]$.

Proof. Using the bound $L(\pi_1,\pi_2,\pi_3) \le L(\pi_1,\pi_2)+L(\pi_1,\pi_3)+L(\pi_2,\pi_3)$ (inequality (2.1)), that each of these have the same law and Hölder's inequality, we get

$$\begin{aligned}
&\mathbb{E}_\mu\left(L_3(\pi_1,\pi_2,\pi_3)e^{C\beta^2 L(\pi_1,\pi_2,\pi_3)}\right) \\
&\le \mathbb{E}_\mu\left(L_3(\pi_1,\pi_2,\pi_3)e^{C\beta^2(L(\pi_1,\pi_2)+L(\pi_1,\pi_3)+L(\pi_2,\pi_3))}\right) \\
&\le \left(\mathbb{E}_\mu(L_3(\pi_1,\pi_2,\pi_3)^2)\right)^{1/2}\left(\mathbb{E}_\mu(e^{6C\beta^2 L(\pi_1,\pi_2)})\right)^{1/2}.
\end{aligned}$$

Both expectations are estimated in a similar way to how the moments of $L(\pi_1,\pi_2)$ were calculated in the proof of Corollary 3.4. Let S be the set of paths on $[a,b]$ with slopes not in $[1/4,3/4]$. Then by definition of $\mathcal{M}$, there is a positive constant C such that for any configuration in $\mathcal{M}$, we have $\mu(S) \le Ce^{-Cn_0}$ (this constant arose from Theorem 3.3 and did not depend on the location of $[a,b]$). Thus

$$\mathbb{E}_\mu(L_3(\pi_1,\pi_2,\pi_3)^2) \le \mathbb{E}_\mu((L_3(\pi_1,\pi_2,\pi_3)^2)|\pi_1 \notin S, \pi_2 \notin S, \pi_3 \notin S) + 3n_0^2\mu(S).$$

The second term is

$$3n_0^2\mu(S) \le Cn_0^2 e^{-Cn_0} = O(1).$$

For the first term, if none of π_1, π_2 or π_3 are in S, then they are all Bernoulli bridges with slopes in $[1/4, 3/4]$, so by Lemma 2.6,

$$\mathbb{E}_\mu((L_3(\pi_1,\pi_2,\pi_3)^2)|\pi_1 \notin S, \pi_2 \notin S, \pi_3 \notin S) \le C(\log n_0)^2.$$

Next,

$$\mathbb{E}_\mu(e^{6C\beta^2 L(\pi_1,\pi_2)}) \le \mathbb{E}_\mu(e^{6C\beta^2 L(\pi_1,\pi_2)}|\pi_1 \notin S, \pi_2 \notin S) + 2e^{C\beta^2 n_0}\mu(S).$$

Now, n_0 was specifically chosen so that $\beta^2 \sim n_0^{-(\frac{1}{2}+2\delta)}$. Thus the second term above will be

$$2e^{C\beta^2 n_0}\mu(S) \le Ce^{Cn_0^{\frac{1}{2}}} e^{-Cn_0} = O(1),$$

and the first term is also $O(1)$ by Proposition 2.5. $\square$

We are finally ready to conclude. What we have shown is that we have the following estimate for (4.4):

$$\sum_{(i,j)\in\Delta} \mathbb{E}\left(\frac{W_{i,j}^3}{(V_{i,j}+W_{i,j})^3}\right) \le C\log n,$$

plus some lower order terms that we can make smaller than any power of n that we need. We reiterate again the crucial fact that this constant C does not depend on n or the location of $[a,b]$; it may depend on α, δ and the constants in Definition 1.3, but all of those were fixed at the start. Therefore, we can partition the double sum in (4.3) into sums over strips of width n_0, and each such sum will be at most $C\log n$ for the same C. There are $2n/n_0$ different strips, so this gives the bound

$$\left|\mathbb{E}\left[f\left(\frac{\log Z_n(\beta)-a_n}{\sigma_n}\right) - f\left(\frac{\log Z_n'(\beta)-a_n}{\sigma_n}\right)\right]\right| \le \frac{C\beta^3 n\log n}{\sigma_n n_0}. \tag{4.5}$$

Note that if $2n/n_0$ is not an integer, then we can just add some overlapping strips to cover the whole of $[0,2n]$. Some points may be contained in two different strips, but this just means we add the errors for those points twice, and so all this does is add an extra factor of 2 in the above bound.

We now express β, n_0 and σ_n in terms of n to conclude. We had $\beta = n^{-\alpha}$, $n_0 = n^{4\alpha/(1+4\delta)}$ and $\sigma_n = \beta^{4/3}n^{1/3}$. So then

$$\frac{\beta^3 n\log n}{\sigma_n n_0} = \frac{\beta^3 n\log n}{\beta^{4/3}n^{1/3}n_0} = \frac{\beta^{5/3}n^{2/3}\log n}{n^{4\alpha/(1+4\delta)}} = n^\lambda \log n$$

where

$$\lambda = -\frac{5\alpha}{3} + \frac{2}{3} - \frac{4\alpha}{1+4\delta} = \frac{(2-17\alpha)+8\delta-20\alpha\delta}{3(1+4\delta)}.$$

By (4.1), we have $\lambda < 0$, and so our bound goes to 0 as $n \to \infty$. This concludes the proof of Theorem 1.5!

Remark 4.2. The above calculation seems to indicate that Theorem 1.5 should hold for any $\alpha > \frac{2}{17}$. In fact, if we strengthen our hypotheses and assume that the weights $\omega_{i,j}(\beta)$ and $\omega'_{i,j}(\beta)$ have the same first k moments for all β sufficiently small, then we can do all the calculations of this section with a Taylor expansion of order $k+1$ instead. The end result will be the same estimate as (4.5) but with β^3 replaced by β^{k+1}. After substituting the values of β and n_0 in terms of n in this expression and simplifying, one obtains a bound which goes to 0 provided that

$$\alpha > \frac{2}{3k+11}.$$

This for now is only a conjecture, since we are at the moment limited to $\alpha > \frac{1}{8}$ by our work from Chapter 3.

4.2 Valid sets of weights

We now wish to apply Theorem 1.5 together with Krishnan and Quastel's Theorem 1.1 to show that the fluctuations of the directed polymer converge to the Tracy–Widom GUE distribution in the intermediate regime. In order for this to work, there are two more tasks left to complete. First, we need to show that appropriately normalized versions of the standard directed polymer and the log-gamma polymer are both valid sets weights. Second, we need to show that it is possible to reparametrize the log-gamma polymer so that the first and second moments of each weight matches with those of the standard polymer for all β sufficiently small. We deal with the first problem in this section and the second in the next section.

Proposition 4.3. *The following two parametrizations of weights are valid.*

1. $\omega_{i,j}(\beta) = \psi_{i,j}(\beta)^{-1}e^{\beta\xi_{i,j}}$, *where the* $\xi_{i,j}$ *are independent, have a uniform exponential tail and* $\psi_{i,j}$ *is the moment generating function of* $\xi_{i,j}$.

2. $\omega_{i,j}(\beta) = (\theta-1)/X_{i,j}$, *where* $\theta = \theta(\beta) \sim c/\beta^2$ *for some positive constant* c *as* $\beta \to 0$ *and* $X_{i,j}$ *are i.i.d with* $X_{i,j} \sim \mathrm{Gamma}(\theta, 1)$.

Proof. 1. Properties 1 and 2 trivially hold. Here "uniform exponential tail" means there are constants $C, c > 0$ such that for all i, j and all $\lambda \geq 0$,

$$\mathbb{P}(|\xi_{i,j}| \geq \lambda) \leq Ce^{-c\lambda}.$$

Therefore, for all i, j,

$$\begin{aligned}\psi_{i,j}(\beta) &= \mathbb{E}(e^{\beta\xi_{i,j}}) \leq \mathbb{E}(e^{\beta|\xi_{i,j}|}) = \int_0^\infty \mathbb{P}(e^{|\beta\xi_{i,j}|} > t)dt \\ &= 1 + \int_1^\infty \mathbb{P}\left(|\xi_{i,j}| > \frac{\log t}{\beta}\right)dt \leq 1 + \int_1^\infty Ce^{-c\frac{\log t}{\beta}}dt \\ &= 1 + \frac{C\beta}{c-\beta},\end{aligned}$$

and

$$\begin{aligned}\psi_{i,j}(\beta) &\geq \mathbb{E}(e^{-\beta|\xi_{i,j}|}) = \int_0^\infty \mathbb{P}(e^{-|\beta\xi_{i,j}|} > t)dt \\ &= 1 - \int_0^1 \mathbb{P}\left(|\xi_{i,j}| \geq -\frac{\log t}{\beta}\right) dt \geq 1 - \int_0^1 Ce^{c\frac{\log t}{\beta}} dt \\ &= 1 - \frac{C\beta}{c+\beta}.\end{aligned}$$

These two inequalities imply that there is a positive constant C such that for all i, j and all sufficiently small β,

$$|\psi_{i,j}(\beta) - 1| \leq C\beta. \tag{4.6}$$

In particular, for sufficiently small β, we have $\psi_{i,j}(\beta) \geq 1/2$ for all i, j. Therefore, by the triangle inequality for L^k norms,

$$\begin{aligned}\mathbb{E}(|\omega_{i,j} - 1|^k) &= \frac{\mathbb{E}|e^{\beta\xi_{i,j}} - \psi_{i,j}(\beta)|^k}{\psi_{i,j}(\beta)^k} \leq 2^k \mathbb{E}|e^{\beta\xi_{i,j}} - \psi_{i,j}(\beta)|^k \\ &\leq 2^k \left((\mathbb{E}|e^{\beta\xi_{i,j}} - 1|^k)^{1/k} + (\mathbb{E}|\psi_{i,j}(\beta) - 1|^k)^{1/k}\right)^k \\ &\leq 2^k \left((\mathbb{E}|e^{\beta\xi_{i,j}} - 1|^k)^{1/k} + C\beta\right)^k.\end{aligned} \tag{4.7}$$

By the mean value theorem, $|e^{\beta\xi_{i,j}} - 1| \leq \beta|\xi_{i,j}|e^{\beta|\xi_{i,j}|}$. Since

$$|\xi_{i,j}| \leq \frac{2k}{c} e^{\frac{c|\xi_{i,j}|}{2k}},$$

it follows that for $\beta < c/2k$, we have

$$\mathbb{E}|e^{\beta\xi_{i,j}} - 1|^k \leq \beta^k \mathbb{E}(|\xi_{i,j}|^k e^{\beta k|\xi_{i,j}|}) \leq \frac{2^k k^k}{c^k} \beta^k \mathbb{E}(e^{c|\xi_{i,j}|}) \leq C_k \beta^k$$

where C_k depends on k but not i or j. Plugging this back in (4.7) then gives Property 3.

Finally for Property 4, any $p > 3$ will do; indeed by (4.6)

$$\mathbb{E}\left(\frac{1}{\omega_{i,j}(\beta)^p}\right) = \mathbb{E}(e^{-p\beta\xi_{i,j}})\psi_{i,j}(\beta)^p = \psi_{i,j}(-p\beta)\psi_{i,j}(\beta)^p \leq C$$

for all i, j and for all sufficiently small β.

2. Property 1 holds provided we only consider β small enough. Using well-known formulas for the moments of the Gamma distribution, we have $\mathbb{E}(X^{-1}) = (\theta - 1)^{-1}$, so Property 2 holds as well.

Next, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\left|\frac{\theta - 1}{X} - 1\right|^k = \mathbb{E}\left|\frac{\theta - 1 - X}{X}\right|^k \leq (\mathbb{E}(\theta - 1 - X)^{2k})^{1/2} \left(\mathbb{E}\left(\frac{1}{X^{2k}}\right)\right)^{1/2}.$$

The second expectation is, for θ sufficiently large,

$$\left(\mathbb{E}\left(\frac{1}{X^{2k}}\right)\right)^{1/2} = \left(\frac{1}{(\theta - 1)\cdots(\theta - 2k)}\right)^{1/2} \leq \frac{C_k}{\theta^k}.$$

Let X_t be a Gamma process, i.e X_t is a Lévy process with independent increments given by $X_t - X_s \sim \text{Gamma}(t-s, 1)$. Then $t - X_t$ is a square-integrable martingale, with quadratic variation $[X]_t = t$, so by the Burkholder–Davis–Gundy inequality ([5, Theorem 4.4.20]),

$$\begin{aligned}(\mathbb{E}(\theta-1-X)^{2k})^{1/2} &\leq ((\mathbb{E}(\theta-X)^{2k})^{1/2k}+1)^k \leq (C_k\mathbb{E}([X]_\theta^k)^{1/2k}+1)^k \\ &= (C_k\theta^{1/2}+1)^k \leq C_k\theta^{k/2}.\end{aligned}$$

Combining these 2 inequalities, we obtain Property 3:

$$\mathbb{E}\left|\frac{\theta-1}{X}-1\right|^k \leq C_k\theta^{-k/2} \leq C_k\beta^k.$$

Finally, Property 4 holds with $p=4$ for example:

$$\frac{1}{(\theta-1)^4}\mathbb{E}(X^4) = \frac{(\theta+3)(\theta+2)(\theta+1)\theta}{(\theta-1)^4} \leq 2$$

for all θ sufficiently large (or in terms of β, for all β sufficiently small). □

4.3 Proof of Theorem 1.2

We are ready to complete the proof of Theorem 1.2. Recall that the log-gamma polymer is the directed polymer with weights $1/X_{i,j}$, where the $X_{i,j}$ are i.i.d with the Gamma$(\theta,1)$ distribution. The partition function is given by

$$F_n(\theta) = \sum_\pi \prod_{i=0}^{2n} \frac{1}{X_{i,\pi(i)}},$$

and the free energy is $\log F_n(\theta)$. By Krishnan and Quastel's Theorem 1.1, if we take $\theta \sim c/\beta^2$ as $\beta \to 0$ with $\beta = n^{-\alpha}$ for some $0 < \alpha < 1/4$, then

$$\frac{\log F_n(\theta) + 2n\Psi(\theta/2)}{-(\Psi''(\theta/2))^{1/3}n^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}} \tag{4.8}$$

where Ψ is the digamma function. Let $Z_n'(\theta)$ be the corresponding normalized partition function:

$$Z_n'(\theta) = \log\left(\sum_\pi \prod_{i=0}^{2n} \frac{\theta-1}{X_{i,\pi(i)}}\right).$$

By factoring a $(\theta-1)^{2n+1}$ from each term in the sum, we can rewrite the free energy of the log-gamma polymer as

$$\log\left(\sum_\pi \prod_{i=0}^{2n} \frac{1}{X_{i,\pi(i)}}\right) = \log\left(\sum_\pi \prod_{i=0}^{2n} \frac{\theta-1}{X_{i,\pi(i)}}\right) - (2n+1)\log(\theta-1),$$

and plugging this back in (4.8) gives

$$\frac{\log Z_n'(\theta) + 2n(\Psi(\theta/2) - \log(\theta-1))}{-(\Psi''(\theta/2))^{1/3}n^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}. \tag{4.9}$$

The second moment of the set of weights $\omega'_{i,j}(\beta) = (\theta-1)/X_{i,j}$ is

$$\mathbb{E}\left[\left(\frac{\theta-1}{X_{i,j}}\right)^2\right] = \frac{\theta-1}{\theta-2},$$

and for the weights $\omega_{i,j}(\beta) = e^{\beta\xi_{i,j}}/\psi(\beta)$,

$$\mathbb{E}\left[\left(\frac{e^{\beta\xi_{i,j}}}{\psi(\beta)}\right)^2\right] = \frac{\psi(2\beta)}{\psi(\beta)^2}.$$

Solving

$$\frac{\theta-1}{\theta-2} = \frac{\psi(2\beta)}{\psi(\beta)^2}$$

yields

$$\theta = 2 + \frac{\psi(\beta)^2}{\psi(2\beta)-\psi(\beta)^2},$$

and so with this choice of θ, the two sets of weights have the same first and second moments. Note that by expanding $\psi(\beta)$ and $\psi(2\beta)$ as Taylor series, we can see after some calculations that

$$\theta = 2 + \frac{1+2\mathbb{E}(\xi)\beta + O(\beta^2)}{\mathrm{Var}(\xi)\beta^2 + O(\beta^3)}, \tag{4.10}$$

and so $\theta \sim 1/\sigma^2\beta^2$ as $\beta \to 0$, where σ^2 is the variance of the $\xi_{i,j}$'s. We are thus in the context of the Krishnan–Quastel theorem and Proposition 4.3.

The digamma function satisfies the following asymptotics:

$$\Psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^4}\right), \qquad -\Psi''(x) \sim \frac{1}{x^2} \tag{4.11}$$

as $x \to \infty$ (see [1, Chapter 6]). Therefore, the denominator of (4.9) is

$$-(\Psi''(\theta/2))^{1/3} n^{1/3} \sim \frac{4^{1/3}n^{1/3}}{\theta^{2/3}} \sim (4\sigma^4\beta^4 n)^{1/3}$$

as $n \to \infty$. It then follows from Theorem 1.5 that if $\beta = n^{-\alpha}$ for some $\frac{1}{8} < \alpha < \frac{1}{4}$,

$$\frac{\log \widetilde{Z}_n(\beta) + 2n(\Psi(\theta/2) - \log(\theta-1))}{(4\sigma^4\beta^4 n)^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}$$

where $\widetilde{Z}_n(\beta)$ is the normalized partition function of the directed polymer with weights $\xi_{i,j}$

$$\widetilde{Z}_n(\beta) = \sum_{\pi} \prod_{i=0}^{2n} \frac{e^{\beta\xi_{i,j}}}{\psi(\beta)}.$$

To get the limiting fluctuations for the unnormalized free energy, we can factor out a $\psi(\beta)^{-(2n+1)}$ from each term in the sum of $\widetilde{Z}_n(\beta)$, and this finally gives

$$\frac{\log Z_n(\beta) - a_n}{(4\sigma^4\beta^4 n)^{1/3}} \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}}$$

with

$$a_n = 2n(-\Psi(\theta/2) + \log(\theta - 1) + \log(\psi(\beta)).$$

We can clean up the expression for a_n and get an asymptotic expansion as $\beta \to 0$. Using (4.11) and a Taylor expansion for $\log(1 - 1/\theta)$,

$$\begin{aligned} a_n &= 2n\left[\left(-\log\theta + \log 2 + \frac{1}{\theta} + \frac{1}{3\theta^2}\right) + \left(\log\theta - \frac{1}{\theta} - \frac{1}{2\theta^2} - \frac{1}{3\theta^3}\right) + \log(\psi(\beta))\right] \\ &= 2n\left[\log 2 - \frac{1}{6\theta^2} - \frac{1}{3\theta^3} + \log(\psi(\beta))\right]. \end{aligned}$$

The terms of order $O(1/\theta^4)$ can be thrown out; indeed after multiplying by $2n$ and dividing by $(\beta^4 n)^{1/3}$, this will result in a term of order

$$\frac{n\theta^{-4}}{\beta^{4/3}n^{1/3}} \sim \frac{n^{2/3}\sigma^8\beta^8}{\beta^{4/3}} = \sigma^8 n^{\frac{2-20\alpha}{3}}$$

which goes to 0 as $n \to \infty$ since $\alpha > \frac{1}{8} > \frac{1}{10}$. Finally, substituting the expression for θ in terms of β in the above and expanding the whole thing as a Taylor series, one obtains after a lot of work that

$$\begin{aligned} a_n = 2n\Bigg[& \log 2 + \kappa_1\beta + \frac{\kappa_2}{2}\beta^2 + \frac{\kappa_3}{6}\beta^3 + \left(\frac{\kappa_4}{24} - \frac{\kappa_2^2}{6}\right)\beta^4 + \left(\frac{\kappa_5}{120} - \frac{\kappa_2\kappa_3}{3}\right)\beta^5 \\ & + \left(\frac{\kappa_6}{720} + \frac{\kappa_2^3}{6} - \frac{7\kappa_2\kappa_4}{36} - \frac{\kappa_3^2}{6}\right)\beta^6 + O(\beta^7)\Bigg] \end{aligned}$$

where κ_j denotes the j-th cumulant of the $\xi_{i,j}$'s. Once again, we do not need the terms of order $O(\beta^7)$ since

$$\frac{n\beta^7}{\beta^{4/3}n^{1/3}} = n^{2/3}\beta^{17/3} = n^{\frac{2-17\alpha}{3}}$$

which goes to 0 since $\alpha > \frac{1}{8} > \frac{2}{17}$. This concludes the proof of Theorem 1.2!

Chapter 5

Limit shapes for the Seppäläinen–Johansson model

In this chapter, we prove the existence of the limit shape for the generalized Seppäläinen–Johansson model described in Subsection 1.3.1. We then look at some special cases of weights distributions for which we can explicitly compute the limit shape, and we present a sketch of a proof of how to obtain these formulas.

5.1 Existence of the limit shape

Recall the generalized SJ model described in Subsection 1.3.1: we are given three arrays of independent random variables $B_{i,j}, \xi_{i,j}$ and $\eta_{i,j}$ for $i, j \geq 0$. The $B_{i,j}$'s are Bernoulli(p) distributed, the $\xi_{i,j}$'s have the same law, similarly for the $\eta_{i,j}$'s (we allow the distribution of the $\xi_{i,j}$'s to be different of those of the $\eta_{i,j}$'s). Each edge of $\mathbb{Z}^2_{\geq 0}$ is then assigned a weight depending on whether it is horizontal or vertical: the horizontal edge joining $(i-1, j)$ to (i, j) has weight $\omega_{i,j} = B_{i,j}\xi_{i,j}$ and the vertical edge joining $(i, j-1)$ to (i, j) has weight $\widetilde{\omega}_{i,j} = (1 - B_{i,j})\eta_{i,j}$. The first-passage value $F(m, n)$ at (m, n) is the minimum over all up-right paths from $(0, 0)$ to (m, n) of the sum of the weights along that path. We refer again to Figure 1.2 for an example of a path and its corresponding total weight.

In this section, we give a proof of the following theorem.

Theorem 5.1. *Assume that the $\xi_{i,j}$'s and $\eta_{i,j}$'s have a finite first moment. Then there exists a deterministic function $f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ such that for all $x, y \geq 0$,*

$$\lim_{n\to\infty} \frac{F(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} = \lim_{n\to\infty} \frac{\mathbb{E}(F(\lfloor nx \rfloor, \lfloor ny \rfloor))}{n} = f(x, y) \tag{5.1}$$

in L^1. If the $\xi_{i,j}$'s and $\eta_{i,j}$'s have a finite second moment, then the above convergence also holds with probability 1. Additionally, the function f satisfies the following two properties

1. *$f(cx, cy) = cf(x, y)$ for all $c \geq 0$*

2. *$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2)$.*

The above two properties of f clearly imply that f is convex, and therefore continuous on $\mathbb{R}^2_{>0}$. Of course a convex function on $\mathbb{R}^2_{\geq 0}$ may not be continuous on the axes. Whether f is continuous up to the boundary in our case is thus a more subtle question. We refer the reader to [51] for some results in this direction for certain models of directed first-passage and last-passage percolation.

Limit shape theorems of this type go back to Richardson [65] who proved a version of this for the Eden model [32]. Cox and Durrett proved the general version for (undirected) first-passage percolation in [26]. The Cox–Durrett theorem is usually stated in terms of the growing cluster

$$C(t) = \{\mathbf{x} \in \mathbb{R}^d : F(\lfloor \mathbf{x} \rfloor) \leq t\}$$

of points reachable within distance t "converging" to the unit ball

$$\mathcal{B} = \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \leq 1\}$$

(we reuse F and f to denote the (undirected) first-passage value and corresponding limit shape in d dimensions respectively for this brief digression) in the following sense: for every $\epsilon > 0$,

$$\mathbb{P}\left((1-\epsilon)\mathcal{B} \subset \frac{C(t)}{t} \subset (1+\epsilon)\mathcal{B} \text{ for all sufficiently large } t\right) = 1.$$

Under certain conditions on the moments and mass of the atom at 0 of the weights in the environment, this formulation is essentially equivalent to the claim that for all $\mathbf{x} \in \mathbb{R}^d$, $F(\lfloor n\mathbf{x} \rfloor)/n \to f(\mathbf{x})$ almost surely. We refer to [9, Section 2.3] for a more complete discussion on this topic.

The main ingredient that goes into the proof of Theorem 5.1 is Kingman's subadditive ergodic theorem. We will use the following "random variable" version of the theorem, which can be found in [47].

Theorem 5.2 (Kingman's subadditive ergodic theorem). *Let $(X_{m,n})_{0\leq m<n}$ be a collection of random variables such that*

1. $X_{0,n} \leq X_{0,m} + X_{m,n}$ *for all* $0 \leq m < n$,
2. *for each* $k \geq 1$, $(X_{nk,(n+1)k})_{n\geq 1}$ *is an i.i.d. sequence,*
3. *for any* $m \geq 1$, $(X_{0,k})_{k\geq 1}$ *and* $(X_{m,m+k})_{k\geq 1}$ *have the same joint distribution,*
4. $\mathbb{E}|X_{0,1}| < \infty$ *and there is a constant* $M > 0$ *such that for all* $n \geq 1$, $\mathbb{E}(X_{0,n}) \geq -Mn$.

Then

1. $\gamma := \lim_{n\to\infty} \frac{\mathbb{E}(X_{0,n})}{n}$ *exists and* $\gamma = \inf_{n\geq 1} \frac{\mathbb{E}(X_{0,n})}{n}$,
2. $\frac{X_{0,n}}{n} \to \gamma$ *almost surely and in* L^1.

The overall plan is as follows. We will apply the subadditive ergodic theorem to the random variables

$$X_{m,n} = F(mx, my; nx, ny)$$

where x, y are fixed non-negative integers. Conditions 1 to 4 are easy to verify in this case, and this will imply the existence of $f(x, y)$ for integer x and y. Property 1 of Theorem 5.1 will follow in this case from the invariance of this process under translations, and Property 2 follows from the triangle inequality. It is then relatively straightforward to extend this to all rational x and y by looking along the subsequence of n's which are multiples of q, where q is some large enough integer for which qx and qy are both integers. The final more technical step is to extend this to all $x, y \geq 0$. We do this by showing that f, at this point only defined on the rationals, admits a unique continuous extension to $\mathbb{R}^2_{>0}$, and we then use this along with a law of large numbers for triangular arrays to obtain the convergence for any x and y.

Before we begin the proof, we need two preliminary results that will justify this last step.

Lemma 5.3. *Let $f : \mathbb{Q}^2_{\geq 0} \to \mathbb{R}$ be a non-negative function which satisfies Properties 1 and 2 in the conclusion of Theorem 5.1 for rational c and rational inputs x, y, x_1, x_2, y_1, y_2. Then f is locally Lipschitz on $\mathbb{Q}^2_{>0}$. In particular, f admits a unique continuous extension to $\mathbb{R}^2_{>0}$, and this extension satisfies Properties 1 and 2.*

Proof. We will work throughout this proof with the ℓ^1 norm. Of course, all norms on $\mathbb{R}^2$ are equivalent, so this does not make any difference, but it has the advantage that the norm of a vector with rational coordinates is rational.

First, it follows easily from Properties 1 and 2 that

$$f(x, y) \leq f(x, 0) + f(0, y) = xf(1, 0) + yf(0, 1) \leq \|(x, y)\|_1(f(1, 0) + f(0, 1)),$$

and so f is bounded on bounded subsets of $\mathbb{Q}_{\geq 0}$.

Second, fix $(x_0, y_0) \in \mathbb{Q}^2_{>0}$ and choose $r > 0$ rational sufficiently small so that $B((x_0, y_0), r)$ (the ℓ^1 ball of radius r) is contained in $\mathbb{Q}^2_{>0}$. Let $M = \sup_{B((x_0,y_0),r)} f(x, y)$ (M is finite by the above). We claim that

$$|f(x_1, y_1) - f(x_2, y_2)| \leq \frac{2M}{r}\|(x_1, y_1) - (x_2, y_2)\|_1 \tag{5.2}$$

for all $(x_1, y_1), (x_2, y_2) \in B((x_0, y_0), r/4)$. Given two distinct such points, let

$$(x_3, y_3) = (x_1, y_1) + \frac{r}{2\|(x_1, y_1) - (x_2, y_2)\|_1}(x_2 - x_1, y_2 - y_1)$$
$$(x_4, y_4) = (x_2, y_2) + \frac{r}{2\|(x_1, y_1) - (x_2, y_2)\|_1}(x_1 - x_2, y_1 - y_2).$$

Then (x_3, y_3) and (x_4, y_4) have rational coordinates, and

$$\|(x_3, y_3) - (x_0, y_0)\|_1 \leq \|(x_1, y_1) - (x_0, y_0)\|_1 + \frac{r}{2} < \frac{3r}{4}$$

and similarly for (x_4, y_4). So both of these points are in the ball $B((x_0, y_0), r)$. Next, define $t = 2\|(x_1, y_1) - (x_2, y_2)\|_1/r$. Then t is rational, positive and

$$t \leq \frac{2(\|(x_1, y_1) - (x_0, y_0)\|_1 + \|(x_0, y_0) - (x_2, y_2)\|_1)}{r} < 1.$$

Note that we have $(x_3, y_3) = (x_1, y_1) + \frac{1}{t}(x_2 - x_1, y_2 - y_1)$, so after some rearranging this gives

$$(x_2, y_2) = t(x_3, y_3) + (1 - t)(x_1, y_1),$$

and therefore by Properties 1 and 2,

$$f(x_2, y_2) \leq t f(x_3, y_3) + (1-t) f(x_1, y_1).$$

After some more rearranging, we find

$$f(x_2, y_2) - f(x_1, y_1) \leq t(f(x_3, y_3) - f(x_1, y_1)) \leq tM = \frac{2M}{r} \|(x_1, y_1) - (x_2, y_2)\|_1.$$

On the other hand, we have $(x_4, y_4) = (x_2, y_2) + \frac{1}{t}(x_1 - x_2, y_1 - y_2)$, and after doing similar calculations to the above, this yields

$$f(x_1, y_1) - f(x_2, y_2) \leq t(f(x_4, y_4) - f(x_2, y_2)) \leq tM = \frac{2M}{r} \|(x_1, y_1) - (x_2, y_2)\|_1.$$

These two inequalities together give (5.2). Then f has a unique continuous extension to $\mathbb{R}^2_{>0}$ by a standard fact from calculus, and Properties 1 and 2 for real c and real inputs for f follow from approximating with rationals and using the continuity of f. $\square$

The next result is an old theorem of Hsu and Robbins which gives a strengthening of the law of large numbers when the random variables have a finite second moment. See [39].

Theorem 5.4. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. mean 0 random variables with a finite second moment, and let $S_n = X_1 + \ldots + X_n$. Then for every $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n| > n\epsilon) < \infty.$$

Together with the Borel–Cantelli lemma, this theorem immediately implies the following strong law of large numbers for triangular arrays.

Corollary 5.5. *Let $(X_{i,n})_{1 \leq i \leq n}$ be a collection of row-wise independent, identically distributed random variables with mean μ and a finite second moment. That is, we assume that for each fixed n, $X_{1,n}, \ldots, X_{n,n}$ are independent, but we do not require independence between different rows. Let $S_n = X_{1,n} + \ldots + X_{n,n}$. Then*

$$\frac{S_n}{n} \to \mu$$

almost surely.

It is worth noting that Corollary 5.5 is false without the assumption of a finite second moment. For example, consider $(X_{i,n})_{1 \leq i \leq n}$ all i.i.d (including across different rows) with distribution function

$$F(x) = 1 - \frac{1}{x^2 + 1}$$

for $x \geq 0$, and $F(x) = 0$ for $x < 0$. Note that this distribution has a finite $(2-\delta)^{\text{th}}$ moment for every $\delta > 0$, but does not have a finite second moment. We have, for $M > 0$,

$$\begin{aligned}\mathbb{P}(S_n \geq nM) &\geq \mathbb{P}(X_{i,n} \geq nM \text{ for some } 1 \leq i \leq n) = 1 - (1 - \mathbb{P}(X_{1,1} \geq nM))^n \\ &\geq 1 - \left(1 - \frac{C}{n^2}\right)^n \geq 1 - e^{-C/n} \geq \frac{C}{n}\end{aligned}$$

for some positive constant C depending on M but not n. Since the $(S_n)_{n\geq 1}$ are independent, it follows from the second Borel–Cantelli lemma that $S_n \geq nM$ infinitely often almost surely, and because M was arbitrary, we conclude that S_n/n is unbounded almost surely. Thus a finite second moment is the borderline condition needed.

Proof of Theorem 5.1. We first fix non-negative integers x and y. As described earlier, we will verify that the random variables

$$X_{m,n} = F(mx, my; nx, ny)$$

satisfy the conditions of the subadditive ergodic theorem. First, if π_1 is the optimal path from $(0,0)$ to (mx, my) and π_2 is the optimal path from (mx, my) to (nx, ny), then concatenating these two paths together gives a possible path from $(0,0)$ to (nx, ny), and so

$$X_{0,n} = F(0,0; nx, ny) \leq F(0,0; mx, my) + F(mx, my; nx, ny) = X_{0,m} + X_{m,n}.$$

Second, for each k, we have

$$X_{nk,(n+1)k} = F(nkx, nky; (n+1)kx, (n+1)ky).$$

The grid whose bottom left corner is (nkx, nky) and whose top right corner is $((n+1)kx, (n+1)ky)$ has the same size for any n, and they are composed of edges with weights which are independent from the other grids, that is the grids for different values of n are independent. So for every k, $(X_{nk,(n+1)k})$ is an i.i.d. sequence. Third, we have

$$X_{0,k} = F(0,0; kx, ky)$$

and

$$X_{m,m+k} = F(mx, my; mx + kx, my + ky).$$

These have the same distribution for any m since $X_{0,k}$ and $X_{m,m+k}$ describe the distance traveled in grids of the same size, with edge weights that have the same distribution. Finally,

$$\mathbb{E}|X_{0,1}| = \mathbb{E}(F(x,y)) \leq \mathbb{E}\left(\sum_{i=1}^{x} \omega_{i,0} + \sum_{j=1}^{y} \widetilde{\omega}_{x,j}\right) < \infty$$

since the path that only goes right on the x-axis until $(x, 0)$ and then goes right until (x, y) is a valid path from $(0,0)$ to (x,y). Also $X_{0,n}$ is nonnegative. So by the subadditive ergodic theorem,

$$\lim_{n\to\infty} \frac{F(nx, ny)}{n} = \lim_{n\to\infty} \frac{\mathbb{E}(F(nx, ny))}{n}$$

exists almost surely and in L^1. We call $f(x,y)$ this limit.

By the triangle inequality for F,

$$\frac{F(n(x_1+x_2), n(y_1+y_2))}{n} \leq \frac{F(nx_1, ny_1)}{n} + \frac{F(nx_1, ny_1; n(x_1+x_2), n(y_1+y_2))}{n}. \tag{5.3}$$

The first term converges almost surely to $f(x_1, y_1)$. The second term has the same distribution as

$F(nx_2, ny_2)/n$, and so it converges in probability to $f(x_2, y_2)$. And so by looking at (5.3) along an appropriate subsequence and taking limits, we conclude that

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2). \tag{5.4}$$

If $c \in \mathbb{N}$, then

$$f(cx, cy) = \lim_{n\to\infty} \frac{F(ncx, ncy)}{n} = c \lim_{n\to\infty} \frac{F(cnx, cny)}{cn} = cf(x, y),$$

which proves homogeneity of f on the integers, for integer scalars.

Next, we extend the definition of f to the rationals. Let x and y be nonnegative rationals, and write them under a common denominator, say $x = s/q$ and $y = t/q$. Define

$$f(x, y) := \frac{f(qx, qy)}{q}.$$

This definition is independent of the representation of x and y as quotients of integers by homogeneity for integer scalars. This new f is clearly homogeneous for c rational and also satisfies the inequality (5.4) for rational inputs (simply multiply x_1, x_2, y_1, y_2 by an appropriate integer to make them integers, and apply (5.4) in the integer case along with homogeneity). To show the convergence (5.1) holds for rational x and y, we proceed as follows. Again write $x = s/q$ and $y = t/q$. We have for all n that

$$\left\lfloor \frac{n}{q} \right\rfloor qx \leq nx.$$

Since the left-hand side is an integer and $\lfloor nx \rfloor$ is the largest integer less than or equal to nx, it follows that

$$\left\lfloor \frac{n}{q} \right\rfloor qx \leq \lfloor nx \rfloor.$$

Also, we have

$$\left\lfloor \frac{n}{q} \right\rfloor qx \geq \left(\frac{n}{q} - 1\right) qx = nx - qx \geq \lfloor nx \rfloor - qx$$

which implies that there are at most $qx = s$ integers between $\lfloor nx \rfloor$ and $\lfloor \frac{n}{q} \rfloor qx$ for any n. A similar reasoning also shows that there are at most t integers between $\lfloor ny \rfloor$ and $\lfloor \frac{n}{q} \rfloor qy$ for any n. Now, a possible path from $(0, 0)$ to $(\lfloor nx \rfloor, \lfloor ny \rfloor)$ is to take the optimal path from $(0, 0)$ to $(\lfloor \frac{n}{q} \rfloor qx, \lfloor ny \rfloor)$ then move right until $(\lfloor nx \rfloor, \lfloor ny \rfloor)$, while picking up at most s horizontal weights along the way. This gives the bound

$$F(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq F\left(\left\lfloor \frac{n}{q} \right\rfloor qx, \lfloor ny \rfloor\right) + \omega_{\lfloor \frac{n}{q} \rfloor qx+1, \lfloor ny \rfloor} + \ldots + \omega_{\lfloor \frac{n}{q} \rfloor qx+s, \lfloor ny \rfloor}.$$

Similarly, a possible path from $(0, 0)$ to $(\lfloor \frac{n}{q} \rfloor qx, \lfloor ny \rfloor)$ is to take the optimal path to $(\lfloor \frac{n}{q} \rfloor qx, \lfloor \frac{n}{q} \rfloor qy)$ and then move up until $(\lfloor \frac{n}{q} \rfloor qx, \lfloor ny \rfloor)$, while picking up at most t vertical weights along the way. Combining this with the above bound then yields

$$\begin{aligned} F(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq F\left(\left\lfloor \frac{n}{q} \right\rfloor qx, \left\lfloor \frac{n}{q} \right\rfloor qy\right) &+ \omega_{\lfloor n/q \rfloor qx+1, \lfloor ny \rfloor} + \ldots + \omega_{\lfloor \frac{n}{q} \rfloor qx+s, \lfloor ny \rfloor} \\ &+ \widetilde{\omega}_{\lfloor \frac{n}{q} \rfloor qx, \lfloor \frac{n}{q} qy \rfloor+1} + \ldots + \widetilde{\omega}_{\lfloor \frac{n}{q} \rfloor qx, \lfloor \frac{n}{q} qy \rfloor+t}. \end{aligned} \tag{5.5}$$

We then divide both sides by n and let $n \to \infty$. By the convergence to the limit shape shown for integer inputs, we have

$$\frac{F\left(\left\lfloor \frac{n}{q} \right\rfloor qx, \left\lfloor \frac{n}{q} \right\rfloor qy\right)}{n} = \frac{\left\lfloor \frac{n}{q} \right\rfloor}{n} \frac{F\left(\left\lfloor \frac{n}{q} \right\rfloor qx, \left\lfloor \frac{n}{q} \right\rfloor qy\right)}{\left\lfloor \frac{n}{q} \right\rfloor} \to \frac{f(qx, qy)}{q} = f(x, y)$$

almost surely and in L^1. To deal with the remaining terms, we observe the following: if i_n and j_n are sequences of distinct integers, then

$$\frac{\omega_{i_n, j_n}}{n} \to 0$$

almost surely and in L^1. Indeed, this just follows from the law of large of numbers:

$$\frac{\omega_{i_n,j_n}}{n} = \frac{\omega_{i_1,j_1} + \ldots + \omega_{i_n,j_n}}{n} - \left(\frac{n-1}{n}\right) \frac{\omega_{i_1,j_1} + \ldots + \omega_{i_{n-1},j_{n-1}}}{n-1} \to \mathbb{E}(\omega_{1,1}) - \mathbb{E}(\omega_{1,1}) = 0,$$

and likewise for $\widetilde{\omega}_{i_n,j_n}$. The indices of the weights appearing in (5.5) are not necessarily distinct, but they repeat at most $\max(s,t)$ times, and the total number of weights appearing is $s + t$ which is constant in n, so on the whole these terms converge to 0 almost surely and in L^1 after being divided by n. We can use a very similar argument to obtain an analogous lower bound to (5.5) for $F(\lfloor nx \rfloor, \lfloor ny \rfloor)/n$ which also converges to $f(x,y)$, and so we conclude that

$$\frac{F(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \to f(x, y)$$

almost surely and in L^1 for all rational x and y.

Finally we generalize to real x and y. By Lemma 5.3, f has a unique continuous extension to $\mathbb{R}^2_{>0}$ and this f satisfies Properties 1 and 2. Fix $x, y > 0$, and let p_1, p_2, q_1, q_2 be rationals such that $p_1 < x < p_2$ and $q_1 < y < q_2$. Then by the same reasoning used to derive (5.5), we have

$$\begin{aligned} F(\lfloor nx \rfloor, \lfloor ny \rfloor) &\le F(\lfloor np_1 \rfloor, \lfloor ny \rfloor) + \omega_{\lfloor np_1 \rfloor+1, \lfloor ny \rfloor} + \ldots + \omega_{\lfloor np_2 \rfloor, \lfloor ny \rfloor} \\ &\le F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor) + \omega_{\lfloor np_1 \rfloor+1, \lfloor ny \rfloor} + \ldots + \omega_{\lfloor np_2 \rfloor, \lfloor ny \rfloor} \\ &\quad + \widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor+1} + \ldots + \widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_2 \rfloor}. \end{aligned}$$

Divide by n and let n go to infinity. By what we have already shown for rational inputs, we have

$$\frac{F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor)}{n} \to f(p_1, q_1)$$

almost surely and in L^1. For the other terms, we have

$$\begin{aligned} &\frac{\omega_{\lfloor np_1 \rfloor+1, \lfloor ny \rfloor} + \ldots + \omega_{\lfloor np_2 \rfloor, \lfloor ny \rfloor}}{n} + \frac{\widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor+1} + \ldots + \widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_2 \rfloor}}{n} \\ &\to (p_2 - p_1)\mathbb{E}(\omega_{1,1}) + (q_2 - q_1)\mathbb{E}(\widetilde{\omega}_{1,1}) \end{aligned}$$

in L^1 but not necessarily almost surely. This is because the above are partial sums in a triangular array of random variables and the indices are changing with n. The L^1 convergence follows from the law of large numbers and the fact that these have the same law as a regular partial sum in a sequence of i.i.d. random variables. If the weights have a finite second moment, then by Corollary 5.5, we do have almost sure convergence, but as discussed after this corollary, almost sure convergence may not hold if the $\omega_{i,j}$'s or $\widetilde{\omega}_{i,j}$'s are not in L^2. We can similarly get a lower bound on $F(\lfloor nx\rfloor, \lfloor ny\rfloor)$:

$$F(\lfloor nx\rfloor, \lfloor ny\rfloor) \geq F(\lfloor np_2\rfloor, \lfloor nq_2\rfloor) - \omega_{\lfloor np_1\rfloor+1,\lfloor ny\rfloor} - \ldots - \omega_{\lfloor np_2\rfloor,\lfloor ny\rfloor}$$
$$- \widetilde{\omega}_{\lfloor np_2\rfloor,\lfloor nq_1\rfloor+1} - \ldots - \widetilde{\omega}_{\lfloor np_2\rfloor,\lfloor nq_2\rfloor},$$

and the right-hand side, after being divided by n, converges to

$$f(p_2, q_2) - (p_2 - p_1)\mathbb{E}(\omega_{1,1}) - (q_2 - q_1)\mathbb{E}(\widetilde{\omega}_{1,1})$$

almost surely for L^2 weights, and in L^1 for general weights. Letting $p_1 \nearrow x$, $p_2 \searrow x$ and $q_1 \nearrow y$, $q_2 \searrow y$ and using the continuity of f on $\mathbb{R}^2_{>0}$, we obtain the desired conclusion of the theorem. The only remaining case is when $x = 0, y > 0$ or $x > 0, y = 0$, however in this situation, things are much easier. Indeed, there is only one path from $(0, 0)$ to $(\lfloor nx\rfloor, 0)$, namely the path that always goes right, so in this situation the convergence just follows directly from the law of large numbers:

$$\frac{F(\lfloor nx\rfloor, 0)}{n} = \frac{\omega_{1,0} + \ldots + \omega_{\lfloor nx\rfloor,0}}{n} \to x\mathbb{E}(\omega_{1,1})$$

almost surely and in L^1. The case $x > 0, y = 0$ is handled the same way. □

5.2 Joint almost sure convergence

Theorem 5.1 shows that in the L^2 case,

$$\frac{F(\lfloor nx\rfloor, \lfloor ny\rfloor)}{n} \to f(x, y)$$

for each $x, y \geq 0$ almost surely, however the set of probability 1 on which this holds may a priori depend on x and y. We now show that under the assumption of Theorem 1.6, we can get the convergence to hold jointly for all x and y with probability 1. The precise statement is as follows.

Theorem 5.6. *Assume that the $\omega_{i,j}$'s and $\widetilde{\omega}_{i,j}$'s have a finite second moment. Then with probability 1, we have*

$$\max(f_H(x, y), f_V(x, y)) \leq \liminf_{n\to\infty} \frac{F(\lfloor nx\rfloor, \lfloor ny\rfloor)}{n} \leq \limsup_{n\to\infty} \frac{F(\lfloor nx\rfloor, \lfloor ny\rfloor)}{n} \leq f(x, y)$$

simultaneously for all $x, y \geq 0$.

Let us mention that Theorem 5.6 is not used in the proof of Theorem 1.6, so there is no circular argument going on here.

Proof. Recall that $F_H(m, n)$ denotes the Seppäläinen–Johansson first-passage value where the horizontal weights are the $\omega_{i,j}$'s and the vertical weights are replaced by zero, while $F_V(m, n)$ is the first-passage value where the vertical weights are the $\widetilde{\omega}_{i,j}$'s and the horizontal weights are replaced by 0. Then we have the trivial inequality

$$F(m, n) \geq \max(F_H(m, n), F_V(m, n)).$$

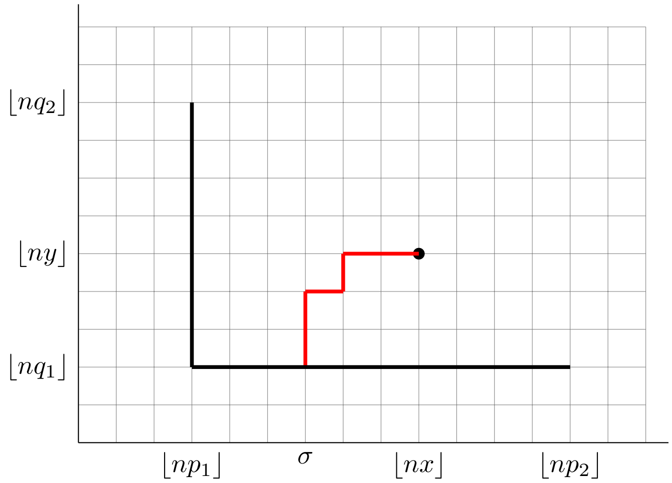

Figure 5.1: The down-left path of weight 0 started from $(\lfloor nx \rfloor, \lfloor ny \rfloor)$ is represented in red. The sum of the weights on the edges between $(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor)$ and $(\sigma, \lfloor nq_1 \rfloor)$ is then an upper bound for $F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor; \lfloor nx \rfloor, \lfloor ny \rfloor)$. The sum of those weights is certainly no larger than the sum of all the weights on the black "L shape"

Now, in the SJ case, the joint almost sure convergence is much easier to establish since the first-passage function is monotonic in this situation. Indeed, for fixed n, the function $m \mapsto F_H(m,n)$ is non-decreasing, and for fixed m, the function $n \mapsto F_H(m,n)$ is non-increasing. So if p_1, p_2, q_1, q_2 are rationals such that $p_1 < x < p_2$ and $q_1 < y < q_2$, then

$$\frac{F_H(\lfloor np_1 \rfloor, \lfloor nq_2 \rfloor)}{n} \leq \frac{F_H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \leq \frac{F_H(\lfloor np_2 \rfloor, \lfloor nq_1 \rfloor)}{n}.$$

It follows from the almost sure convergence along the rationals that

$$f_H(p_1, q_2) \leq \liminf_{n\to\infty} \frac{F_H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \leq \limsup_{n\to\infty} \frac{F_H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \leq f_H(p_2, q_1)$$

almost surely, and by continuity of f_H, we get upon sending p_1, p_2 to x and q_1, q_2 to y along the rationals that

$$\frac{F_H(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \to f_H(x,y).$$

The above holds with probability 1 simultaneously for all x and y, since this only depends on having the limit hold for all rationals. The same idea works for F_V as well, and so we get

$$\liminf_{n\to\infty} \frac{F(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \geq \max(f_H(x,y), f_V(x,y))$$

almost surely for all $x, y \geq 0$.

For the upper bound, we have by the triangle inequality that

$$F(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor) + F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor; \lfloor nx \rfloor, \lfloor ny \rfloor). \tag{5.6}$$

To estimate the second term in the right-hand side above, we note the following. For each (i,j), at least one of the weights from the two incoming edges $(i-1,j) \to (i,j)$ or $(i,j-1) \to (i,j)$ is

0. This means that from each point (i, j), there exists a down-left path started from (i, j) which only visits edges of weight 0. This path that starts at $(\lfloor nx \rfloor, \lfloor ny \rfloor)$ will eventually hit a point of the form $(\lfloor np_1 \rfloor, \tau)$ or $(\sigma, \lfloor nq_1 \rfloor)$ for some $\lfloor nq_1 \rfloor \leq \tau \leq \lfloor ny \rfloor$ or $\lfloor np_1 \rfloor \leq \sigma \leq \lfloor nx \rfloor$. See Figure 5.1. The path that moves vertically or horizontally until it hits $(\lfloor np_1 \rfloor, \tau)$ or $(\sigma, \lfloor nq_1 \rfloor)$ and then follows the down-left path of weight 0 gives a valid path from $(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor)$ to $(\lfloor nx \rfloor, \lfloor ny \rfloor)$, so this yields an upper bound for $F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor; \lfloor nx \rfloor, \lfloor ny \rfloor)$. We could figure out the distribution of τ and σ; this is the hitting location of a random walk to a wedge, but this is completely unnecessary. Instead, we can simply use the crude bound

$$\begin{aligned} F(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor; \lfloor nx \rfloor, \lfloor ny \rfloor) \leq\ & \omega_{\lfloor np_1 \rfloor + 1, \lfloor nq_1 \rfloor} + \ldots + \omega_{\lfloor np_2 \rfloor, \lfloor nq_1 \rfloor} \\ & + \widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor + 1} + \ldots + \widetilde{\omega}_{\lfloor np_1 \rfloor, \lfloor nq_2 \rfloor} \end{aligned}$$

which is obtained by adding all the weights in the "L shape" started from $(\lfloor np_1 \rfloor, \lfloor nq_1 \rfloor)$; see Figure 5.1 again. Plugging this in (5.6), it follows from the convergence shown along the rationals and Corollary 5.5 that

$$\limsup_{n \to \infty} \frac{F(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} \leq f(p_1, q_1) + (p_2 - p_1)\mathbb{E}(\omega_{1,1}) + (q_2 - q_1)\mathbb{E}(\widetilde{\omega}_{1,1})$$

almost surely. Letting $p_1 \nearrow x$, $p_2 \searrow x$ and $q_1 \nearrow y$, $q_2 \searrow y$ and using the continuity of f gives the required upper bound; this bounds holds with probability 1 simultaneously for all x and y since the above limits only depend on them holding along all rational directions. $\square$

5.3 Exact limit shape formulas

In this section, we describe how to obtain some exact formulas for the limiting shape in certain special cases of weights. Since these formulas were already found by Martin in [50], we will omit the technical calculations and only sketch the main ideas that go into obtaining these formulas. Martin uses the language of queuing theory to obtain these expressions. We will follow the more geometric approach used in [66, Section 4.6] in the context of last-passage percolation, as well as [74].

Throughout this section, we consider the SJ model with i.i.d weights $\omega_{i,j}$ on the horizontal edges $(i-1, j) \to (i, j)$, with distribution given by $\omega_{i,j} \sim \mathrm{Ber}(q)\mathrm{Geo}(p)$, that is their distribution is the product of a Bernoulli with an independent geometric. Thus we have for k a non-negative integer,

$$\mathbb{P}(\omega_{i,j} = k) = \begin{cases} (1-q) & \text{if } k = 0 \\ qp(1-p)^{k-1} & \text{if } k \geq 1. \end{cases}$$

Note that this includes both the Bernoulli distribution (take $p = 1$) and the Geometric distribution (take $q = 1 - p$) as special cases.

What makes this distribution special is that there exists in this case a set of boundary conditions that turn the SJ model into a stationary process in the sense described in Proposition 5.8 below. Let $0 \leq \alpha < 1$ be a parameter. We change the weights lying on the x-axis to be $\omega^{\alpha}_{i,0}$, and we also

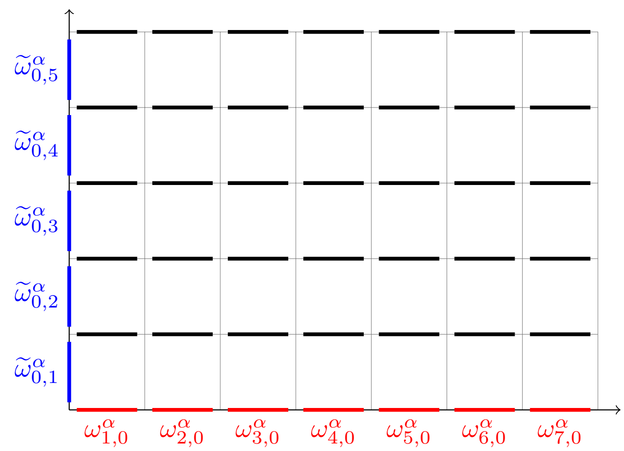

Figure 5.2: The special boundary conditions. The bulk horizontal edges (represented as bold black lines) have weights $\omega_{i,j}$ distributed according to a Ber(q)Geo(p). The bulk vertical edges have weight 0. The laws of the boundary edges are different and given by (5.7) and (5.8).

introduce weights on the vertical edges of the y-axis $\widetilde{\omega}^{\alpha}_{0,j}$ with distributions

$$\omega^{\alpha}_{i,0} \sim \mathrm{Ber}\left(\frac{pq\alpha}{p\alpha + (1-q)(1-\alpha)}\right)\mathrm{Geo}(1-\alpha(1-p)), \tag{5.7}$$

$$-\widetilde{\omega}^{\alpha}_{0,j} \sim \mathrm{Ber}\left(\frac{p\alpha}{1-\alpha(1-p)}\right)\mathrm{Geo}(1-\alpha) \tag{5.8}$$

(the $\widetilde{\omega}^{\alpha}_{0,j}$ weights are negative). All weights are assumed to be independent. We will denote by $F^{\alpha}(m,n)$ the first-passage values with the above special boundary conditions. See Figure 5.2.

Next we consider the increments of first-passage values in this model. Let

$$\begin{aligned} X_{i,j} &= F^{\alpha}(i,j) - F^{\alpha}(i-1,j) \\ Y_{i,j} &= F^{\alpha}(i,j-1) - F^{\alpha}(i,j) \end{aligned}$$

so the $X_{i,j}$ are the horizontal increments, and the $Y_{i,j}$ are the vertical increments. Notice that the order in which the subtraction occurs in these definitions is different; this is to ensure that the $X_{i,j}$ and the $Y_{i,j}$ are always non-negative. The SJ first-passage values are non-decreasing in the x direction, and non-increasing in the y direction. We have of course $X_{i,0} = \omega^{\alpha}_{i,0}$ and $Y_{0,j} = -\widetilde{\omega}^{\alpha}_{0,j}$.

The first-passage values F^{α} satisfy the recursions

$$F^{\alpha}(i,j) = \min(F^{\alpha}(i-1,j) + \omega_{i,j}, F^{\alpha}(i,j-1))$$

and so we can obtain from this recursions for the $X_{i,j}$'s and $Y_{i,j}$'s:

$$\begin{aligned} X_{i,j} &= \min(F^{\alpha}(i-1,j) + \omega_{i,j}, F^{\alpha}(i,j-1)) - F^{\alpha}(i-1,j) \\ &= \min(\omega_{i,j}, X_{i,j-1} + Y_{i-1,j}) \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} Y_{i,j} &= F^\alpha(i,j-1) - \min(F^\alpha(i-1,j)+\omega_{i,j}, F^\alpha(i,j-1)) \\ &= \max(X_{i,j-1}+Y_{i-1,j}-\omega_{i,j},0). \end{aligned} \tag{5.10}$$

The reason for these choices of boundary conditions is the following lemma.

Lemma 5.7. *Let X, Y and ω be independent random variables with the following distributions*

$$X \sim \mathrm{Ber}\left(\frac{pq\alpha}{p\alpha+(1-q)(1-\alpha)}\right)\mathrm{Geo}(1-\alpha(1-p)),$$

$$Y \sim \mathrm{Ber}\left(\frac{p\alpha}{1-\alpha(1-p)}\right)\mathrm{Geo}(1-\alpha)$$

$$\omega \sim \mathrm{Ber}(q)\mathrm{Geo}(p).$$

Then

$$(\min(X+Y,\omega), \max(X+Y-\omega,0)) \stackrel{d}{=} (X,Y).$$

The proof of Lemma 5.7 is only a long and tedious (but not difficult) computation so we will skip it. One first finds the law of the sum $S = X+Y$, either directly by calculating a convolution or using characteristic functions. This distribution is also a $\mathrm{Ber}(q')\mathrm{Geo}(p')$ for certain q' and p'. It is then easy to show that $\min(S,\omega)$ has the same distribution as X, and $\max(S-\omega,0)$ has the same distribution as Y. To check independence, note that if $\max(S-\omega,0)=j$ for a non-zero j, it means that we must have $S \geq \omega$, so

$$\mathbb{P}(\min(S,\omega)=k, \max(S-\omega,0)=j) = \mathbb{P}(\omega=k, S=k+j) = \mathbb{P}(\omega=k)\mathbb{P}(S=k+j)$$

and then one easily verifies that this last expression equals $\mathbb{P}(X=k)\mathbb{P}(Y=j)$. A similar computation can be done when $j=0$.

The next step is to show that the distributions chosen for the boundary variables $X_{i,0}$ and $Y_{0,j}$ will "propagate" the right way throughout the lattice to get a stationary process. For this, we look at edges of the lattice contained in down-right paths.

A *down-right path* is a two-sided infinite sequence of vertices in the lattice $\mathbb{Z}^2_{\geq 0}$, say $\pi = (\dots, v_{-1}, v_0, v_1, \dots)$, such that for any $n \in \mathbb{Z}$,

$$v_n - v_{n-1} = (1,0) \text{ or } (0,-1).$$

We let $N(\pi)$ denote the number of vertices in the upper right quadrant of $\mathbb{Z}^2$ which are strictly below and to the left of π, that is, $N(\pi)$ is the cardinality of the set

$$B(\pi) = \{(i,j) : i \geq 0, j \geq 0, \text{ there is } (m,n) \in \pi \text{ such that } i<m, j<n\}.$$

See Figure 5.3. Note that $N(\pi)$ can be infinite. The set of edges that connect two consecutive vertices in π is denoted by $E(\pi)$. For each edge $e \in E(\pi)$, we define a random variable Z_e which is

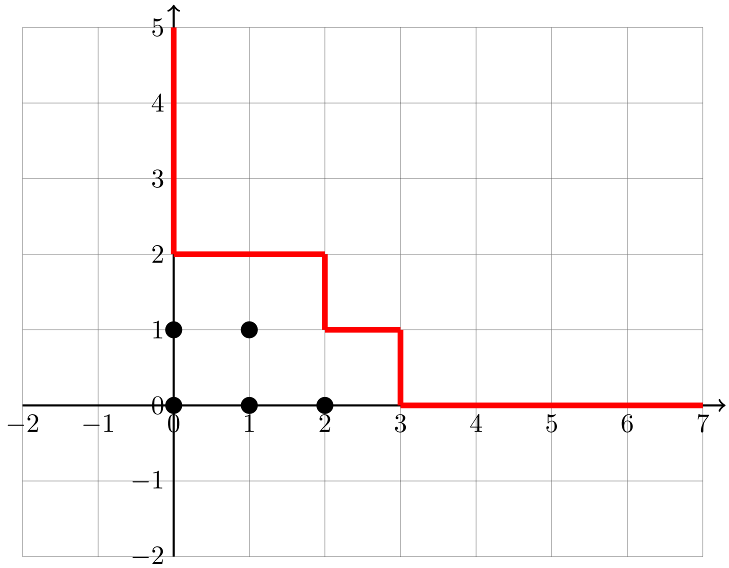

Figure 5.3: A down-right path. Here $B(\pi)$ is represented by the black dots, so $N(\pi) = 5$.

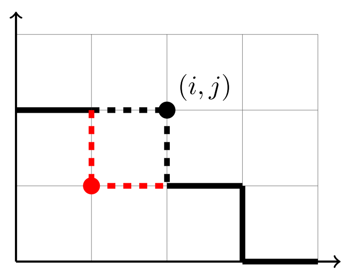

Figure 5.4: The black line is π. To get $\widetilde{\pi}$, we replace the black dashed line by the red dashed line.

the increment $X_{i,j}$ or $Y_{i,j}$ associated to that edge:

$$Z_e = \begin{cases} X_{i,j} & \text{if } e \text{ is the edge } (i-1, j) \to (i, j) \\ Y_{i,j} & \text{if } e \text{ is the edge } (i, j) \to (i, j-1). \end{cases}$$

Proposition 5.8. *Let π be a down-right path. Then the random variables $\{Z_e : e \in E(\pi)\}$ are independent. If e is horizontal, then Z_e has the distribution (5.7), and if e is vertical, then Z_e has the distribution (5.8).*

Proof. We first prove this in the case $N(\pi) < \infty$, using induction on $N(\pi)$. If $N(\pi) = 0$, this means that π is precisely the path that goes down along the y-axis, and then turns to the right at the origin and continues on the x-axis. The random variables corresponding to those edges are independent and have the right distributions by definition of the boundary conditions (5.7) and (5.8).

Now suppose that $0 < N(\pi) < \infty$, and that the result has been proven for paths π' with $N(\pi') < N(\pi)$. Consider a *corner vertex* of π: this is a vertex (i, j) such that $(i-1, j), (i, j)$ and $(i, j-1)$ all belong to π, and $i \geq 1$, $j \geq 1$. Such a vertex must exist since $0 < N(\pi) < \infty$. We now define a new down-right path $\widetilde{\pi}$: it has the same vertices as π, except that (i, j) is replaced by

$(i-1, j-1)$. See Figure 5.4.

Now, $N(\widetilde{\pi}) = N(\pi) - 1$, so by induction, the random variables $\{Z_e, e \in E(\widetilde{\pi})\}$ are independent with the correct distributions. The families $\{Z_e, e \in E(\widetilde{\pi})\}$ and $\{Z_e, e \in E(\pi)\}$ are the same, except that $X_{i,j-1}$ and $Y_{i-1,j}$ are in the first family, and they are replaced by $X_{i,j}$ and $Y_{i,j}$ in the second family. By Lemma 5.7 and the recursions (5.9) and (5.10), $X_{i,j}$ and $Y_{i,j}$ are independent and have the right distributions. Furthermore, these recursions mean they only depend on $X_{i,j-1}, Y_{i-1,j}$ and $\omega_{i,j}$, and all three of these are independent of the Z_e's for all the other edges $e \in E(\pi)$. This completes the induction and proves the proposition when $N(\pi) < \infty$.

Finally if $N(\pi) = \infty$, then at least one of the following must happen:

- There are $i \geq 0$ and $j \geq 1$ such that eventually, π becomes $\pi = (\ldots, (i,j), (i+1,j), (i+2,j), \ldots)$ (so π ends up horizontal).
- There are $i \geq 1$ and $j \geq 0$ such that π starts off as $\pi = (\ldots, (i,j+2), (i,j+1), (i,j), \ldots)$ (so π starts off vertical).

By definition, independence of $\{Z_e, e \in E(\pi)\}$ means that every finite sub-collection of these random variables are independent. Given any finite sub-collection $A \subset E(\pi)$, it is easy to see that in both of the above cases, one can find a path $\widetilde{\pi}$ which passes through all the edges of A and $N(\widetilde{\pi}) < \infty$. The result follows from applying the finite case to $\widetilde{\pi}$. □

Vertical and horizontal lines in $\mathbb{Z}^2$ are special cases of down-right paths, and so we immediately deduce the following.

Corollary 5.9. *The following hold*

- *For any fixed $n \geq 0$, the random variables $X_{1,n}, X_{2,n}, X_{3,n}, \ldots$ are i.i.d with the distribution* (5.7).
- *For any fixed $m \geq 0$, the random variables $Y_{m,1}, Y_{m,2}, Y_{m,3}, \ldots$ are i.i.d with the distribution* (5.8).

We can rewrite F^α as a sum/difference of increments

$$F^\alpha(\lfloor nx \rfloor, \lfloor ny \rfloor) = \sum_{i=1}^{\lfloor nx \rfloor} X_{i,0} - \sum_{j=1}^{\lfloor ny \rfloor} Y_{\lfloor nx \rfloor, j}.$$

We divide by n and let $n \to \infty$. The first term converges almost surely to $x\mathbb{E}(X_{1,0})$ by the law of large numbers, and the second terms converges almost surely to $-y\mathbb{E}(Y_{0,1})$ by Corollary 5.5 (or more simply we can just use Markov's inequality and the Borel–Cantelli lemma directly since the $Y_{i,j}$'s have exponential tails). Thus

$$\begin{aligned}\frac{F^\alpha(\lfloor nx \rfloor, \lfloor ny \rfloor)}{n} &\to x\mathbb{E}(X_{1,0}) - y\mathbb{E}(Y_{0,1}) \\ &= \frac{pq\alpha x}{(1-(1-p)\alpha)(p\alpha + (1-q)(1-\alpha))} - \frac{p\alpha y}{(1-(1-p)\alpha)(1-\alpha)}\end{aligned}$$

almost surely. On the other hand, we can compute this limit in a different way. Any path from $(0,0)$ to $(\lfloor nx \rfloor, \lfloor ny \rfloor)$ will spend some time on one of the axes before it leaves, and then it will only pick up weights that are also in the non-stationary model F. So we find

$$\begin{aligned}F^\alpha(\lfloor nx \rfloor, \lfloor ny \rfloor) = \min\Bigg[&\min_{1\le k\le \lfloor nx\rfloor}\left(\sum_{i=1}^{k}\omega^\alpha_{i,0} + F(k,1;\lfloor nx\rfloor,\lfloor ny\rfloor)\right),\\ &\min_{1\le k\le \lfloor ny\rfloor}\left(-\sum_{j=1}^{k}\widetilde{\omega}^\alpha_{0,j} + \omega_{1,k} + F(1,k;\lfloor nx\rfloor,\lfloor ny\rfloor)\right)\Bigg]\end{aligned} \tag{5.11}$$

After dividing by n, (5.11) converges to a similar continuum expression and so we have

$$\begin{aligned}&\frac{pq\alpha x}{(1-(1-p)\alpha)(p\alpha+(1-q)(1-\alpha))} - \frac{p\alpha y}{(1-(1-p)\alpha)(1-\alpha)} =\\ &\min\left[\inf_{0\le t\le x}\left(\frac{pq\alpha t}{(1-(1-p)\alpha)(p\alpha+(1-q)(1-\alpha))} + f(x-t,y)\right),\right.\\ &\qquad\left.\inf_{0\le t\le y}\left(-\frac{p\alpha t}{(1-(1-p)\alpha)(1-\alpha)} + f(x,y-t)\right)\right].\end{aligned} \tag{5.12}$$

The equation (5.12) has the form of a *Legendre transform*. Since the limit shape f is convex, it can therefore be inverted to find a formula for $f(x,y)$. We omit the details; see [74, Section VI B] for more on how to solve equations like (5.12). The end result is

$$f(x,y) = \sup_{0\le\alpha<1}\left(\frac{pq\alpha x}{(1-(1-p)\alpha)(p\alpha+(1-q)(1-\alpha))} - \frac{p\alpha y}{(1-(1-p)\alpha)(1-\alpha)}\right). \tag{5.13}$$

If $h(\alpha)$ denotes the expression inside the sup above, then one can solve $h'(\alpha) = 0$ and plug this answer back in h to get a more explicit formula. The equation $h'(\alpha) = 0$ is a quartic equation, and so in principle there is a formula for the solution. However it is extremely messy and it does not appear to be simplifiable all that much, so we instead keep the form (5.13). In the special case of Bernoulli weights ($p = 1$) or geometric weights ($q = 1-p$), then $h'(\alpha) = 0$ is a quadratic equation, and (5.13) does simplify to a more compact expression.

One can also use (5.13) to derive a formula for the limit shape when the weights follow a Ber(q)Exp(1) distribution (that is they are the product of a Bernoulli with an independent exponential random variable). To obtain this formula, we first reparametrize (5.13) in terms of a new variable γ, where we set $\gamma = (1-\alpha)/p$. After some calculations, this gives

$$f(x,y) = \frac{1}{p}\sup_{0<\gamma\le\frac{1}{p}}\left(\frac{q(1-p\gamma)x}{(\gamma+1-p\gamma)(1+(1-q)\gamma-p\gamma)} - \frac{(1-p\gamma)y}{(\gamma+1-p\gamma)\gamma}\right).$$

Now it is a well-known fact of probability that if $X_p \sim \mathrm{Geo}(p)$, then $pX_p \xrightarrow{d} \mathrm{Exp}(1)$ as $p \to 0$. Multiplying the above by p, we obtain the limit shape for the weights $p\omega_{i,j}$ where $\omega_{i,j} \sim \mathrm{Ber}(q)\mathrm{Geo}(p)$, and so letting $p \to 0$ will give the limit shape for weights with the distribution Ber(q)Exp(1):

$$f(x,y) = \sup_{0<\gamma<\infty}\left(\frac{qx}{(\gamma+1)(1+(1-q)\gamma)} - \frac{y}{(\gamma+1)\gamma}\right). \tag{5.14}$$

There are several technicalities that we are sweeping under the rug here, for instance the fact that the map which associates a probability distribution to the corresponding limit shape is continuous under some tail bound conditions, and the $0 < \gamma \le 1/p$ which achieves the sup depends on p and

Distribution of the $\omega_{i,j}$'s	Limit shape $f(x,y)$
$\mathrm{Ber}(q)$	$\begin{cases} (\sqrt{qx}-\sqrt{(1-q)y})^2 & \text{if } x \geq \frac{1-q}{q}y \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{Geo}_0(p)$	$\begin{cases} \frac{(\sqrt{(1-p)(x+y)}-\sqrt{y})^2}{p} & \text{if } x > \frac{p}{1-p}y \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{Exp}(1)$	$(\sqrt{x+y}-\sqrt{y})^2$
$\mathrm{Ber}(q)\mathrm{Geo}(p)$	$\sup_{0\leq\alpha<1}\left(\frac{1}{1-(1-p)\alpha}\left[\frac{pq\alpha x}{p\alpha+(1-q)(1-\alpha)}-\frac{p\alpha y}{1-\alpha}\right]\right)$
$\mathrm{Ber}(q)\mathrm{Exp}(1)$	$\sup_{0<\gamma<\infty}\left(\frac{qx}{(\gamma+1)(1+(1-q)\gamma)}-\frac{y}{(\gamma+1)\gamma}\right)$

Table 5.1: A list of all the known limit shapes for the Seppäläinen–Johansson model. Although the Bernoulli, geometric and exponential are special cases of the other two, we still include these in this table since they have nice and simple expressions.

could a priori go to ∞ as $p \to 0$, but we will ignore these. It is possible to start from scratch and instead find stationary boundary conditions similar to (5.7) and (5.8). Those will be of the form $\mathrm{Ber}(q')\mathrm{Exp}(\lambda)$ for certain choices of q' and λ, and then one can reprove Lemma 5.7 and Proposition 5.8 in this setup and obtain a formula like (5.12) which can be inverted as well. The final result will be exactly (5.14).

Computing the sup in (5.14) also comes down to solving a quartic expression for which it does not seem possible to simplify. When $q = 1$ (that is when the weights are Exp(1)), then we instead get a quadratic which can be more easily solved and this gives a nice and clean expression for $f(x, y)$. We summarize all the distributions of weights for which the limit shape can be computed explicitly in Table 5.1.

It is not expected that there are other formulas that one can explicitly compute for other distributions. Indeed, the existence of an X and Y for which independence in Lemma 5.7 holds is essentially equivalent to a kind of lack of memory property for ω, so we can only expect this to work for weights that are exponentially or geometrically distributed, possibly with an extra delta mass at 0.

In [20], the authors characterize all positive temperature models which admit versions of Lemma 5.7 and Proposition 5.8. It is expected that there should be a similar classification for zero temperature models; see [27] for an extensive discussion on this topic.

Chapter 6

The generalized Seppäläinen–Johansson model

In this chapter, we prove Theorems 1.6 and 1.7. We first show a combinatorial identity that relates $F(m,n)$ to the top-most exit point of a geodesic $\mathcal{D}(m,n)$ in the SJ model. The limiting behaviour of $\mathcal{D}(m,n)$ is related to the local properties of the SJ limit shape f_H. We will show that in the general case, $\mathcal{D}(nx,ny)$ is $o(n)$ in probability which will prove Theorem 1.6. In the Bernoulli case, $\mathcal{D}(nx,ny+\tau n^{2/3}t)$ is $o(n^{1/3})$ in probability, uniformly for t in a compact set, which will imply Theorem 1.7.

6.1 The boundary condition lemma

Throughout this chapter, we will usually simplify notation by omitting the floor function when evaluating F or F_H at non-integer points. It is then understood that if m and n are not integers, then $F(m,n)$ is defined to be $F(\lfloor m\rfloor,\lfloor n\rfloor)$. The main ingredient which goes in the proofs of Theorems 1.6 and 1.7 is the following curious identity.

Lemma 6.1. *Let $B_{i,j}\in\{0,1\}$, and let $\xi_{i,j},\eta_{i,j}$ be collections of real numbers. Let $F(m,n)$ be the first passage value from $(0,0)$ to (m,n) with weights $\omega_{i,j}=B_{i,j}\xi_{i,j}$ on horizontal edges $(i-1,j)\to(i,j)$ and weights $\widetilde{\omega}_{i,j}=(1-B_{i,j})\eta_{i,j}$ on vertical edges $(i,j-1)\to(i,j)$, and let $F_H(m,n)$ be the first passage value from $(0,0)$ to (m,n) where the weights on the horizontal edges are $\omega_{i,j}$ and the weights on the vertical edges are 0 except for the weights on the y-axis which are $\widetilde{\omega}_{0,j}$. Suppose that*

$$\xi_{i,j}\geq 0 \quad \textit{for all } i,j\geq 0,$$

$$\eta_{i,j}\geq 0 \quad \textit{for all } i\geq 1, j\geq 0,$$

$$\eta_{0,j}\leq 0 \quad \textit{for all } j\geq 0,$$

that is the $\xi_{i,j}$'s and $\eta_{i,j}$'s are all non-negative except for the $\eta_{i,j}$'s lying on the y-axis which are all non-positive. Then for all m,n,

$$F(m,n)=F_H(m,n).$$

Thus if one replaces the weights on the y-axis with non-positive weights, then the first-passage value corresponds exactly to the first-passage value on horizontal edges. That is, we can completely ignore any vertical edges not on the y-axis; given a geodesic from $(0,0)$ to (m,n), it cannot pass through a vertical edge of non-zero weight except for edges on the y-axis. Lemma 6.1 is deterministic and holds for arbitrary collections of numbers $\xi_{i,j}$, $\eta_{i,j}$ and $B_{i,j}$ satisfying the conditions in the lemma. A similar result holds with weights on the x-axis changed to being non-positive and first-passage on vertical edges.

Proof. As was done in Section 5.3, let $X_{i,j}$, $Y_{i,j}$ be the horizontal and vertical increments for F:

$$\begin{aligned} X_{i,j} &= F(i,j) - F(i-1,j) \\ Y_{i,j} &= F(i,j-1) - F(i,j) \end{aligned}$$

and define $X^H_{i,j}$ and $Y^H_{i,j}$ similarly for F_H. If we're given all the increments of a model, we can deduce what the first-passage values are by just adding/subtracting the increments:

$$F(m,n) = \sum_{i=1}^{m} X_{i,0} - \sum_{j=1}^{n} Y_{m,j}$$

(and similarly for F_H). It is therefore enough to show that $X_{i,j} = X^H_{i,j}$ and $Y_{i,j} = Y^H_{i,j}$ for all i,j. Since any path from $(0,0)$ to (i,j) must pass through exactly one of the vertices $(i-1,j)$ or $(i,j-1)$, it is easy to see that F and F_H satisfy the recursions

$$\begin{aligned} F(i,j) &= \min(F(i-1,j) + \omega_{i,j}, F(i,j-1) + \widetilde{\omega}_{i,j}) \\ F_H(i,j) &= \min(F_H(i-1,j) + \omega_{i,j}, F_H(i,j-1)). \end{aligned} \tag{6.1}$$

Using (6.1), we obtain recursions for $X_{i,j}$ and $Y_{i,j}$:

$$\begin{aligned} X_{i,j} &= \min(F(i-1,j) + \omega_{i,j}, F(i,j-1) + \widetilde{\omega}_{i,j}) - F(i-1,j) \\ &= \min(\omega_{i,j}, X_{i,j-1} + Y_{i-1,j} + \widetilde{\omega}_{i,j}) \end{aligned}$$

and

$$\begin{aligned} Y_{i,j} &= F(i,j-1) - \min(F(i-1,j) + \omega_{i,j}, F(i,j-1) + \widetilde{\omega}_{i,j}) \\ &= \max(X_{i,j-1} + Y_{i-1,j} - \omega_{i,j}, -\widetilde{\omega}_{i,j}). \end{aligned}$$

Now, suppose that $X_{i,j-1}$ and $Y_{i-1,j}$ are both non-negative. By definition of the model, at least one of $\omega_{i,j}$ or $\widetilde{\omega}_{i,j}$ must be 0. If $\widetilde{\omega}_{i,j} = 0$, then $Y_{i,j} = \max(X_{i,j-1} + Y_{i-1,j} - \omega_{i,j}, 0)$. If $\omega_{i,j} = 0$, then because $X_{i,j-1} + Y_{i-1,j}$ and $\widetilde{\omega}_{i,j}$ are non-negative,

$$\begin{aligned} Y_{i,j} &= \max(X_{i,j-1} + Y_{i-1,j}, -\widetilde{\omega}_{i,j}) = X_{i,j-1} + Y_{i-1,j} \\ &= \max(X_{i,j-1} + Y_{i-1,j} - \omega_{i,j}, 0). \end{aligned}$$

Likewise, if both $X_{i,j-1}$ and $Y_{i-1,j}$ are non-negative and $\widetilde{\omega}_{i,j} = 0$, then

$$X_{i,j} = \min(\omega_{i,j}, X_{i,j-1} + Y_{i-1,j}),$$

and if instead $\omega_{i,j} = 0$, then

$$X_{i,j} = \min(0, X_{i,j-1} + Y_{i-1,j} + \widetilde{\omega}_{i,j}) = 0 = \min(\omega_{i,j}, X_{i,j-1} + Y_{i-1,j}).$$

Note also from these recursions that $X_{i,j}$ and $Y_{i,j}$ are then both non-negative. Using (6.1) for F_H (which we had already established in Section 5.3) and the fact that vertical edges have weight 0 in this model, we find

$$X^H_{i,j} = \min(X^H_{i,j-1} + Y^H_{i-1,j}, \omega_{i,j}).$$

and

$$Y^H_{i,j} = \max(0, X^H_{i,j-1} + Y^H_{i-1,j} - \omega_{i,j})$$

So the increments for F and F_H satisfy the exact same recursion provided the increments for F are non-negative. The increments are indeed non-negative; this is clear when $i = 0$ or $j = 0$ since $\xi_{i,j} \geq 0$ and $\eta_{i,0} \leq 0$ for all i and j and for a point (i, j) on one of the axes, there is only one path joining $(0, 0)$ to (i, j), namely the straight line that always stays on the axis. For general i and j, this follows by an induction similar to the one done in the proof of Proposition 5.8 using what we have just shown about the increments of F and F_H. Finally the boundary conditions are the same since the weights on the edges of both axes are the same. □

We now return to the case we are interested in, which is when all the $\xi_{i,j}$'s and $\eta_{i,j}$'s are non-negative. Applying Lemma 6.1 to the special case where the weights on the y-axis are zero will then yield the upper bound (6.2). Recall that the top-most departure point $\mathcal{D}(m, n)$ was defined as

$$\mathcal{D}(m, n) = \max\{k \geq 0 : F_H(0, k; m, n) = F_H(m, n)\}.$$

Proposition 6.2. *Let $\mathcal{D}(m, n)$ be the top-most departure point of a geodesic for $F_H(m, n)$, as defined above. Then*

$$F(m, n) \leq F_H(m, n) + \sum_{j=1}^{\mathcal{D}(m,n)} \eta_{0,j}. \tag{6.2}$$

Proof. By Lemma 6.1, the first-passage value $F(m, n)$ is the same as $F_H(m, n)$ when we change all the weights on the y-axis to be zero. So there must be a geodesic π for $F_H(m, n)$ which does not pass through any vertical edge of non-zero weight except possibly on the y-axis. We obviously have that the length of this path $S(\pi)$ (where we include the weights on the y-axis) satisfies $S(\pi) \geq F(m, n)$, since $F(m, n)$ is the length of the shortest path. The only difference between these two are the extra weights picked up by π along the y-axis:

$$F(m, n) \leq S(\pi) = F_H(m, n) + \sum_{j=1}^{Z} (1 - B_{0,j})\eta_{0,j}$$

where $Z = \min\{k \geq 0 : (1, k) \in \pi\}$ is the position where π exits the y-axis (when $Z = 0$, we interpret the sum as being 0). Since π is a geodesic for $F_H(m, n)$, we have $Z \leq \mathcal{D}(m, n)$, and $1 - B_{0,j} \leq 1$

because $B_{0,j}$ is either 0 or 1. This concludes the proof. □

6.2 Proof of Theorem 1.6

We now begin the proof of Theorem 1.6. As explained in Subsection 1.3.4, our strategy will be to show that the function $y \mapsto f_H(x, y)$ is strictly decreasing on $[0, (1-q)x/q]$, where $q = \mathbb{P}(\omega_{i,j} = 0)$. This is done via a sequence of lemmas.

Lemma 6.3. *Let $\omega_{i,j}$ be i.i.d and non-negative, and let f_H be the limit shape for the SJ model on horizontal edges with weights $\omega_{i,j}$. Let $q = \mathbb{P}(\omega_{i,j} = 0)$. Then $f_H(x, y) > 0$ if and only if $x > \frac{q}{1-q}y$.*

Proof. First assume that $0 < x < qy/(1-q)$ (the case where $x = 0$ is trivial, since $F_H(0, n) = 0$ for all n). Note in particular that this implies $q > 0$ in that case. In order for $F_H(m, n)$ to be 0, there has to be a path from the origin to (m, n) which only visits edges of weight 0. Every time the path sees a horizontal edge of weight 0, it will take it, otherwise it will keep moving up until it sees an edge of weight 0. The number of up steps it needs to take before it sees such an edge has the geometric distribution on $\{0, 1, \dots\}$ with probability of success q, and it needs to take m right steps. So

$$\mathbb{P}(F_H(m, n) = 0) = \mathbb{P}(Z_1 + \cdots + Z_m \le n)$$

where $Z_1, \dots, Z_m$ are i.i.d. $\mathrm{Geo}_0(q)$ random variables. Take $\lfloor nx \rfloor$ and $\lfloor ny \rfloor$ instead of m and n. Then for $\theta > 0$, we have, by Markov's inequality,

$$\begin{aligned}
\mathbb{P}(F_H(\lfloor nx \rfloor, \lfloor ny \rfloor) \neq 0) &= \mathbb{P}(Z_1 + \cdots + Z_{\lfloor nx \rfloor} > \lfloor ny \rfloor) \\
&= \mathbb{P}(e^{\theta(Z_1 + \cdots + Z_{\lfloor nx \rfloor})} > e^{\theta \lfloor ny \rfloor}) \le \frac{\mathbb{E}(e^{\theta Z_1})^{\lfloor nx \rfloor}}{e^{\theta \lfloor ny \rfloor}} \\
&= \exp(\lfloor nx \rfloor \log q - \lfloor nx \rfloor \log(1 - (1-q)e^{\theta}) - \theta \lfloor ny \rfloor).
\end{aligned}$$

There is some $0 < \epsilon < 1$ such that $x = (1-\epsilon)qy/(1-q)$. Take

$$\theta := \log\left(\frac{y}{(1-q)(x+y)}\right).$$

By this condition on x and y, we have $\theta = -\log(1-\epsilon q) > 0$. Substituting θ in the above, we then find

$$\begin{aligned}
&\mathbb{P}(F_H(\lfloor nx \rfloor, \lfloor ny \rfloor) \neq 0) \\
&\le \exp\left(\lfloor nx \rfloor \log q - \lfloor nx \rfloor \log\left(1 - \frac{(1-q)}{1-\epsilon q}\right) + \lfloor ny \rfloor \log(1-\epsilon q)\right) \\
&= \exp((\lfloor nx \rfloor + \lfloor ny \rfloor) \log(1-\epsilon q) - \lfloor nx \rfloor \log(1-\epsilon)) \\
&\le \exp((nx + ny - 2) \log(1-\epsilon q) - nx \log(1-\epsilon)) \\
&= \left(\frac{1}{1-\epsilon q}\right)^2 \exp\left(nx\left[\frac{1-\epsilon q}{p(1-\epsilon)} \log(1-\epsilon q) - \log(1-\epsilon)\right]\right).
\end{aligned}$$

Let $g(\epsilon)$ be the expression in square brackets above. Then

$$g'(\epsilon) = \frac{(1-q)\log(1-\epsilon q)}{q(1-\epsilon)^2} < 0$$

so g is strictly decreasing. Since $g(0) = 0$, it follows that $g(\epsilon) < 0$ for every $0 < \epsilon < 1$, and therefore the above probabilities are summable in n. By the Borel–Cantelli lemma, it follows that $F_H(\lfloor nx \rfloor, \lfloor ny \rfloor)$ is 0 for all but finitely many n almost surely, and thus the limit shape must satisfy $f_H(x,y) = 0$. This proves that $f_H(x,y) = 0$ for all $x < qy/(1-q)$, and by continuity, we obtain this for $x = qy/(1-q)$ as well.

Finally assume $x > qy/(1-q)$, and choose $r > q$ such that $x > ry/(1-r)$. Since

$$1 - q = \mathbb{P}(\omega_{i,j} > 0) = \lim_{s \downarrow 0} \mathbb{P}(\omega_{i,j} > s),$$

there is some $c > 0$ such that $\mathbb{P}(\omega_{i,j} > c) > 1 - r$, or equivalently that $\mathbb{P}(\omega_{i,j} \le c) < r$. Define new weights $\widehat{\omega}_{i,j}$ as follows:

$$\widehat{\omega}_{i,j} = \begin{cases} 0 & \text{if } \omega_{i,j} \le c \\ c & \text{if } \omega_{i,j} > c. \end{cases}$$

Then we have $\omega_{i,j} \ge \widehat{\omega}_{i,j}$ for all i,j, and $\widehat{\omega}_{i,j}$ is c times a Bernoulli$(1-s)$ for some $0 \le s < r$. With $\widehat{f}_H$ the limit shape of the $\widehat{\omega}_{i,j}$'s (which we can compute explicitly; see Table 5.1), we then have

$$f_H(x,y) \ge \widehat{f}_H(x,y) = c(\sqrt{(1-s)x} - \sqrt{sy})^2 > 0$$

which finishes the proof. □

Lemma 6.4. *With the hypotheses of the previous lemma and the assumption that $q > 0$, we have for any fixed x that the function $y \mapsto f_H(x,y)$ is strictly decreasing on $[0,(1-q)x/q]$.*

Proof. Let $0 \le y_1 < y_2 < (1-q)x/q$, and pick any $z > (1-q)x/q$. Then there is a $t \in (0,1)$ such that $y_2 = (1-t)y_1 + tz$. By Lemma 6.3, $f_H(x,y_1) > 0$ and $f_H(x,z) = 0$. Since $y \mapsto f_H(x,y)$ is convex, it follows that

$$\begin{aligned} f_H(x,y_2) &= f_H(x,(1-t)y_1 + tz) \le (1-t)f_H(x,y_1) + tf_H(x,z) \\ &= (1-t)f_H(x,y_1) < f_H(x,y_1). \end{aligned}$$

Thus $y \mapsto f_H(x,y)$ is strictly decreasing on $[0,(1-q)x/q]$. □

In view of (6.2), what we actually need to show is that

$$\frac{1}{n} \sum_{j=1}^{\mathcal{D}(nx,ny)} \eta_{0,j} \to 0$$

in probability rather than $\mathcal{D}(nx,ny)/n \to 0$ in probability. The next lemma handles this issue.

Lemma 6.5. *Let $(N_n)_{n \ge 1}$ be a sequence of non-negative integer-valued random variables, and*

$(X_n)_{n\geq 1}$ *a sequence of integrable random variables with a uniformly bounded absolute first moment:*

$$M = \sup_{n\geq 1} \mathbb{E}|X_n| < \infty.$$

Suppose that $(a_n)_{n\geq 1}$ *is a deterministic sequence of positive real numbers such that* $a_n \to \infty$ *as* $n \to \infty$ *and* $N_n/a_n \to 0$ *in probability. Then*

$$\frac{1}{a_n} \sum_{i=1}^{N_n} X_i \to 0$$

in probability.

Proof. Let $\epsilon > 0$ and $\delta > 0$. Then

$$\mathbb{P}\left(\left| \sum_{i=1}^{N_n} X_i \right| \geq \epsilon a_n \right) \leq \mathbb{P}(N_n \geq \delta\epsilon a_n) + \mathbb{P}\left(\left\{ \left| \sum_{i=1}^{N_n} X_i \right| \geq \epsilon a_n \right\} \cap \{N_n \leq \delta\epsilon a_n\} \right).$$

The first term on the right converges to 0 since $N_n/a_n \to 0$ in probability. For the second term, we have by Markov's inequality

$$\begin{aligned} \mathbb{P}\left(\left\{ \left| \sum_{i=1}^{N_n} X_i \right| \geq \epsilon a_n \right\} \cap \{N_n \leq \delta\epsilon a_n\} \right) &\leq \mathbb{P}\left(\sum_{i=1}^{\lceil \delta\epsilon a_n \rceil} |X_i| \geq \epsilon a_n \right) \leq \frac{1}{\epsilon a_n} \mathbb{E}\left(\sum_{i=1}^{\lceil \delta\epsilon a_n \rceil} |X_i| \right) \\ &\leq \frac{M\lceil \delta\epsilon a_n \rceil}{\epsilon a_n}. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \mathbb{P}\left(\left| \sum_{i=1}^{N_n} X_i \right| \geq \epsilon a_n \right) \leq M\delta.$$

Since $\delta > 0$ was arbitrary, we obtain what we wanted upon sending $\delta \to 0$. □

We are now ready to finish the proof of Theorem 1.6. Let $q = \mathbb{P}(B_{i,j}\xi_{i,j} = 0)$, and recall that the bottom-most entry point $\mathcal{E}(m,n)$ is defined as

$$\mathcal{E}(m,n) = \max\{k \geq 0 : F_H(m, n-k) = F_H(m,n)\}.$$

Then for a point (x,y) satisfying $x > qy/(1-q)$, we have that the function $z \mapsto f_H(x,z)$ is strictly decreasing in a neighbourhood of y by Lemma 6.4. So for all $\epsilon > 0$ small enough, $f_H(x, y-\epsilon) > f_H(x,y)$, and because

$$\frac{F_H(nx,ny)}{n} \to f_H(x,y), \qquad \frac{F_H(nx, n(y-\epsilon))}{n} \to f_H(x, y-\epsilon)$$

in probability, it follows that

$$\mathbb{P}(\mathcal{E}(nx,ny) \geq n\epsilon) = \mathbb{P}(F_H(nx,ny) = F_H(nx, n(y-\epsilon))) \to 0.$$

The random variables $\mathcal{D}(nx,ny)$ and $\mathcal{E}(nx,ny)$ have the same distribution, so $\mathcal{D}(nx,ny)/n \to 0$ in

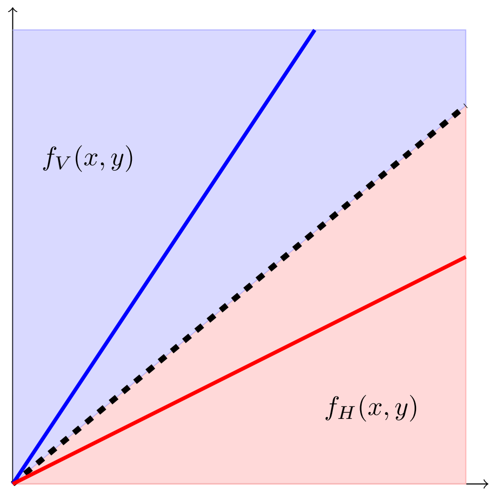

Figure 6.1: The limit shape f in terms of f_H and f_V. The dashed black line is $x = \frac{1-p}{p}y$, the red line is $x = \frac{q}{1-q}y$ and the blue line is $x = \frac{1-r}{r}y$. Everywhere above the red line, we have $f_H(x,y) = 0$, and everywhere below the blue line, $f_V(x,y) = 0$. The limit shape f is obtained by "gluing" f_H and f_V along the black dashed line.

probability, and by Lemma 6.5, we deduce that

$$\frac{1}{n}\sum_{j=1}^{\mathcal{D}(nx,ny)} \eta_{0,j} \to 0$$

in probability. By Proposition 6.2 and the lower bound $F(nx,ny) \geq F_H(nx,ny)$, we infer that $f(x,y) = f_H(x,y)$ for $x > qy/(1-q)$.

We can also get Theorem 1.6 for $x < (1-r)y/r$ where $r = \mathbb{P}((1-B_{ij})\eta_{ij} = 0)$ by using the same arguments employed in the last two sections but with first-passage percolation on vertical edges instead and by considering the right-most departure from the x-axis and left-most entry to the line $y = n$ to derive a similar inequality as in (6.2). The proof for this case is exactly the same so we will not write it down.

So far we have shown that $f(x,y) = f_H(x,y)$ for $x > qy/(1-q)$ and $f(x,y) = f_V(x,y)$ for $x < (1-r)y/r$. By continuity of the limit shape and Lemma 6.3, we also have $f(x,y) = f_H(x,y) = 0$ for $x = qy/(1-q)$ and $f(x,y) = f_V(x,y) = 0$ for $x = (1-r)y/r$.

In the case where both ξ_{ij} and η_{ij} are positive almost surely, we are done (since then we have $q = 1-p$ and $r = p$). If not, then we still have to deal with points in between the lines $x = qy/(1-q)$ and $x = (1-r)y/r$. However on those lines, we have $f(x,y) = 0$, and f is non-negative and convex. So f must also be zero in between those lines, and therefore equals both f_H and f_V there (see Figure 6.1). This shows that $f(x,y) = \max(f_H(x,y), f_V(x,y))$ in all cases and concludes the proof of Theorem 1.6!

6.3 Proof of Theorem 1.7

We now prove Theorem 1.7. As defined in the statement the theorem, we let

$$\tau(x,y) = 2\left[\frac{x^2}{y\sqrt{p(1-p)}}(\sqrt{px}-\sqrt{(1-p)y})(\sqrt{(1-p)x}+\sqrt{py})\right]^{1/3}$$

$$\chi(x,y) = \left[\sqrt{\frac{p(1-p)}{xy}}(\sqrt{px}-\sqrt{(1-p)y})^2(\sqrt{(1-p)x}+\sqrt{py})^2\right]^{1/3}$$

$$\rho(x,y) = p - \sqrt{\frac{p(1-p)y}{x}}$$

where $0 < p < 1$, and x, y are positive and satisfy $x > (1-p)y/p$. We will abbreviate things by omitting the dependence on x and y for τ, χ and ρ.

In [28, Corollary 6.11], it was shown that

$$\frac{F_H(\lfloor nx\rfloor + \lfloor \tau n^{2/3}t\rfloor, \lfloor ny\rfloor) - nf_H(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}} \xrightarrow{d} -\mathcal{A}(t) \tag{6.3}$$

uniformly on compact sets, where $\mathcal{A}(t)$ is the Airy_2 process. Their result is actually stated in a slightly different way. The authors consider the Seppäläinen–Johansson last-passage percolation value instead of first-passage. However if $(L(m,n))_{m,n\geq 0}$ denotes the array of last-passage values from $(0,0)$ with Bernoulli(p) weights, then it is easy to see that the array $(m - L(m,n))_{m,n\geq 0}$ is an array of first-passage values from $(0,0)$ with Bernoulli($1-p$) weights. Thus [28, Corollary 6.11] can be rewritten in terms of first-passage. The other difference is that the authors only state their corollary for $y = 1$, but since f_H is homogenous, it is easy to recover (6.3) for arbitrary y by looking at an appropriate subsequence. We omit the calculations of the scaling factors τ, χ and ρ that we obtain when we consider arbitrary y. Our goal is to show that (6.3) also holds when we replace F_H by the generalized SJ model F. Note that for this choice of x, y and p, we have $f(x,y) = f_H(x,y)$ by Theorem 1.6. We let $\mathcal{F}_n(t)$ denote the left-hand side of (6.3).

Before we continue, we need to address a technical point, which is how to make sense of (6.3). This is saying that a sequence of probability measures on a certain space of functions converges weakly to minus the Airy_2 process. What should this space of functions be? The function F_H is only defined at integer coordinates. One possible approach is to extend F_H to all of $\mathbb{R}$ by making it a step function. Then $\mathcal{F}_n(t)$ is a càdlàg function, and the limit is in the sense of uniform convergence on compact sets. This is essentially what we have been doing so far by using the floor function. The other approach is to interpolate linearly between the values of $t \mapsto F_H(\lfloor nx\rfloor + \tau n^{2/3}t, \lfloor ny\rfloor)$. Then $\mathcal{F}_n(t)$ is a continuous function, and again the convergence is in the sense of uniform convergence on compact sets. It does not really make a difference which method we use, as it is an easy fact from analysis to check that for a sequence of functions h_n defined on a sequence of finer and finer partitions and a continuous function h, the step function version of the h_n's converge uniformly to h if and only if the linear interpolation version of the h_n's converge uniformly to h. We will therefore stick with the step function version for convenience.

Fix $t \in \mathbb{R}$ and $\epsilon > 0$. By the Skorokhod representation theorem [15, Theorem 6.7], the $\mathcal{F}_n$ and $\mathcal{A}$ can be coupled together on the same probability space such that $\mathcal{F}_n \to -\mathcal{A}$ uniformly on compact sets, almost surely. We henceforth work with this particular coupling. Define

$$t_n = \frac{\frac{\epsilon}{y}n^{1/3}x + \tau n^{2/3}t}{(n - \frac{\epsilon}{y}n^{1/3})^{2/3}\tau}.$$

Then $t_n \to t$, and

$$\mathcal{F}_{n-\frac{\epsilon}{y}n^{1/3}}(t_n) \to -\mathcal{A}(t),$$

(we again use the convention that $\mathcal{F}_k = \mathcal{F}_{\lfloor k \rfloor}$ when k is not an integer). Indeed, let K be a compact subset of $\mathbb{R}$ which contains all the t_n's. Then

$$|\mathcal{F}_{n-\frac{\epsilon}{y}n^{1/3}}(t_n) + \mathcal{A}(t)| \le \sup_{s\in K} |\mathcal{F}_{n-\frac{\epsilon}{y}n^{1/3}}(s) + \mathcal{A}(s)| + |\mathcal{A}(t_n) - \mathcal{A}(t)|.$$

The first term on the right-hand side above converges to 0 since $\mathcal{F}_n \to -\mathcal{A}$ uniformly on K, and the second term converges to 0 because $\mathcal{A}$ is continuous. We have

$$\mathcal{F}_{n-\frac{\epsilon}{y}n^{1/3}}(t_n) = \frac{F_H(nx + \tau n^{2/3}t, ny - \epsilon n^{1/3}) - (n - \frac{\epsilon}{y}n^{1/3})f(x,y) - (\frac{\epsilon}{y}xn^{1/3} + \tau n^{2/3}t)\rho}{(n - \frac{\epsilon}{y}n^{1/3})^{1/3}\chi},$$

and so

$$\begin{aligned} &F_H(nx + \tau n^{2/3}t, ny - \epsilon n^{1/3}) - (n - \frac{\epsilon}{y}n^{1/3})f(x,y) - \frac{\epsilon}{y}x\rho n^{1/3} - \tau n^{2/3}t\rho \\ &= -\chi\mathcal{A}(t)n^{1/3} + o(n^{1/3}). \end{aligned} \tag{6.4}$$

We also have $\mathcal{F}_n(t) \to -\mathcal{A}(t)$, and this gives

$$F_H(nx + \tau n^{2/3}t, ny) - nf(x,y) - \tau n^{2/3}t\rho = -\chi\mathcal{A}(t)n^{1/3} + o(n^{1/3}). \tag{6.5}$$

Now subtract (6.5) from (6.4) and divide by $n^{1/3}$. After some rearranging, this yields

$$\frac{F_H(nx + \tau n^{2/3}t, ny - \epsilon n^{1/3}) - F_H(nx + \tau n^{2/3}t, ny)}{n^{1/3}} = \frac{\epsilon}{y}(x\rho - f(x,y)) + o(1). \tag{6.6}$$

Since we are in the Bernoulli case, $f(x,y)$ is given by the formula in Table 5.1, and so we find

$$\begin{aligned} x\rho(x,y) - f(x,y) &= px - \sqrt{p(1-p)xy} - (\sqrt{px} - \sqrt{(1-p)y})^2 \\ &= \sqrt{p(1-p)xy} - (1-p)y > 0 \end{aligned}$$

by our assumption on x, y and p. This together with (6.6) implies that for all sufficiently large n, we must have

$$F_H(nx + \tau n^{2/3}t, ny) \neq F_H(nx + \tau n^{2/3}t, ny - \epsilon n^{1/3}).$$

Consequently, the bottom-most entry point $\mathcal{E}(nx + \tau n^{2/3}t, ny)$ is at most $\epsilon n^{1/3}$, and this implies

$$\limsup_{n\to\infty} \frac{\mathcal{E}(nx + \tau n^{2/3}t, ny)}{n^{1/3}} \le \epsilon$$

almost surely. Since ϵ was arbitrary, it follows that

$$\frac{\mathcal{E}(nx+\tau n^{2/3}t, ny)}{n^{1/3}} \to 0 \tag{6.7}$$

almost surely, and because $\mathcal{D}(nx+\tau n^{2/3}t, ny)$ has the same distribution as $\mathcal{E}(nx+\tau n^{2/3}t, ny)$, we deduce that

$$\frac{\mathcal{D}(nx+\tau n^{2/3}t, ny)}{n^{1/3}} \to 0 \tag{6.8}$$

in probability. By Lemma 6.5 applied to the sequence $a_n = n^{1/3}$, it then follows that

$$\frac{1}{n^{1/3}} \sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t, ny)} \eta_{0j} \to 0 \tag{6.9}$$

in probability.

It is fairly straightforward to generalize the above argument to obtain that (6.7) holds uniformly for t in a compact set almost surely. However, while it is true that $\mathcal{E}(m,n)$ and $\mathcal{D}(m,n)$ have the same distribution for a *fixed* endpoint (m,n), it is not the case that the joint laws of $\{\mathcal{E}(m,k) : k \in S\}$ and $\{\mathcal{D}(m,k) : k \in S\}$ are the same for k varying in some set of integers S. However, we can still obtain (6.8) and (6.9) uniformly for t in a compact set since $\mathcal{D}$ is monotonic.

Lemma 6.6. *For each $n \geq 0$, the function $m \mapsto \mathcal{D}(m,n)$ is non-increasing.*

Proof. We proceed by induction on n. When $n = 0$, the result is clear; there is only one path from $(0,0)$ to $(m,0)$, so the departure point is 0:

$$\mathcal{D}(m,0) = 0.$$

Assume we have proven that $m \mapsto \mathcal{D}(m,n-1)$ is non-increasing, and let $m \geq 1$. Let π be a geodesic from $(0,0)$ to (m,n) whose departure point is $\mathcal{D}(m,n)$. Then π necessarily passes through exactly one of $(m-1,n)$ or $(m,n-1)$.

Case 1: The geodesic π passes through $(m-1,n)$. In this case, π must also be a geodesic for $(m-1,n)$ which maximizes the departure point. If π were not a geodesic, then there is some other path π' from $(0,0)$ to $(m-1,n)$ such that $S(\pi') < S(\pi)$ (not counting the weight on the edge $(m-1,n) \to (m,n)$). But then we can follow π' and then go from $(m-1,n)$ to (m,n), and this will give a path of smaller weight than π, contradicting the fact that π is a geodesic for (m,n). So π is a geodesic for $(m-1,n)$. Likewise, the same logic implies that π maximizes the departure point for $(m-1,n)$, for if not, then there is some geodesic π' from the origin to $(m-1,n)$ with a higher departure point, and by extending this path to (m,n), we a get a geodesic with a higher departure point than π, another contradiction. Thus in this case, we have $\mathcal{D}(m,n) = \mathcal{D}(m-1,n)$.

Case 2: The geodesic π passes through $(m,n-1)$. By the exact same reasoning as above, π must then also be a geodesic to $(m,n-1)$ which maximizes the departure point, and so we have $\mathcal{D}(m,n) = \mathcal{D}(m,n-1)$. Now let π^* be a geodesic from $(0,0)$ to $(m-1,n)$ whose departure point is $\mathcal{D}(m-1,n)$. Then π^* necessarily visits a point of the form $(j,n-1)$ for some $0 \leq j \leq m-1$. By the same argument as above once again, π^* is then also a geodesic to $(j,n-1)$ which maximizes the departure point, so we have $\mathcal{D}(m-1,n) = \mathcal{D}(j,n-1)$. Therefore by the induction hypothesis, we have

$$\mathcal{D}(m,n) = \mathcal{D}(m,n-1) \leq \mathcal{D}(j,n-1) = \mathcal{D}(m-1,n).$$

So in both cases, we have $\mathcal{D}(m,n) \leq \mathcal{D}(m-1,n)$. $\square$

Thus if $K \subset \mathbb{R}$ is compact and t_0 is such that $t_0 \leq t$ for all $t \in K$, then

$$\sup_{t\in K}\left(\frac{1}{n^{1/3}}\sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t,ny)}\eta_{0j}\right) \leq \frac{1}{n^{1/3}}\sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t_0,ny)}\eta_{0j} \to 0 \tag{6.10}$$

in probability.

Let us now conclude. First let S be the following metric which metrizes uniform convergence on compact sets

$$S(\phi,\psi) = \sum_{n=1}^{\infty}\frac{1}{2^n}\min\left(\sup_{t\in[-n,n]}|\phi(t)-\psi(t)|,\ 1\right).$$

Note that with this choice, if $\phi(t) \leq \sigma(t) \leq \psi(t)$ for all $t \in \mathbb{R}$, then

$$S(\phi,\sigma) \leq S(\phi,\psi). \tag{6.11}$$

Let G be a bounded, real-valued, uniformly continuous function on the space of càdlàg functions with respect to uniform convergence on compact sets, and let $\epsilon > 0$. Then there is a $\delta > 0$ such that $|G(\phi) - G(\psi)| < \epsilon$ whenever $S(\phi,\psi) < \delta$. By Proposition 6.2,

$$\begin{aligned}&\frac{F_H(nx+\tau n^{2/3}t,ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}} \leq \frac{F(nx+\tau n^{2/3}t,ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}}\\ &\leq \frac{F_H(nx+\tau n^{2/3}t,ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}} + \frac{1}{\chi n^{1/3}}\sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t,ny)}\eta_{0j}.\end{aligned}$$

Thus if the scaled fluctuations of F and F_H are at least δ apart from each other (with respect to the S metric), then by (6.11),

$$S\left(\frac{1}{\chi n^{1/3}}\sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t,ny)}\eta_{0j},\ 0\right) \geq \delta.$$

Hence,

$$\begin{aligned}&\mathbb{E}\left|G\left(\frac{F_H(nx+\tau n^{2/3}t,ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}}\right)\right.\\ &\quad\left.-G\left(\frac{F(nx+\tau n^{2/3}t,ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}}\right)\right|\\ &\leq 2\sup|G|\,\mathbb{P}\left[S\left(\frac{1}{\chi n^{1/3}}\sum_{j=1}^{\mathcal{D}(nx+\tau n^{2/3}t,ny)}\eta_{0j},\ 0\right) \geq \delta\right] + \epsilon \to \epsilon\end{aligned}$$

by (6.10). Since ϵ was arbitrary, it then follows along with (6.3) that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[G\left(\frac{F(nx+\tau n^{2/3}t, ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}}\right)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[G\left(\frac{F_H(nx+\tau n^{2/3}t, ny) - nf(x,y) - \tau n^{2/3}t\rho}{\chi n^{1/3}}\right)\right] \\
&= \int G(\phi)\, d\mu_{-\mathrm{Airy}_2}(\phi)
\end{aligned}$$

where $\mu_{-\mathrm{Airy}_2}$ is the law of minus the Airy_2 process. This concludes the proof of Theorem 1.7!

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