Simplicial Approximation of the Hodge Laplacian Using Cauchy Sequences of Hilbert Complexes

by

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Abstract

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Discrete differential geometry arises from the use of discrete spaces such as graphs, simplicial, cubical, or polyhedral complexes for modeling geometric structures on manifolds. A common practice in this work is to transport structures on smooth manifolds to discrete counterparts in a process referred to as *discretization*. Discretizations often appear as elements of a sequence that approximates the smooth structure on the manifold through some measure of convergence. Algorithms which produce such sequences are highly sought after for computational applications but frequently ignore deeper structural relationships between successive discrete models. This thesis makes contributions to the discretization of Hodge theory through the construction of a framework that serves to axiomatize a foundational set of results in the field. The salient feature of this framework is the ability to directly measure the difference in approximation accuracy between discretizations without reference to the overarching smooth structure. This provides a Cauchy-type characterization of sequences of discretizations while opening the scope of inquiry to a much larger class of problems involving the analysis of Hodge Theory through Cauchy sequences.

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As well, in days before submitting this thesis, I discovered a critical error that put my entire thesis in jeopardy. During this time, I was incredibly fortunate to correspond with Olaf Post, whose work is intimately related. Professor Post, thank you for taking time away from your vacation to discuss the issue with me, point me towards relevant literature that I was unaware of, and provide me with feedback on my proof of Theorem 2.2.4. I will always see your input and willingness to help me, a stranger, on a moment's notice, as commendable and extremely valuable.

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Contents

Given the pace of technology, I propose we leave math to the machines and go play outside.

- Calvin, Bill Watterson's *Calvin and Hobbes*

Chapter 1

Introduction

1.1 Synopsis

The central theme of this thesis is to enable the direct study of relationships between discrete geometric structures without the use of an underlying smooth manifold. Developments in the field of Discrete Differential Geometry often provide a sequence of discrete structures that serve as an approximation scheme for a smooth structure by exhibiting some form of convergence. In many examples, the results proving convergence depend heavily on the existence of the smooth structure and do not provide an analysis of the sequence of discrete objects independently. The example that we are interested in for this thesis occurs in the landmark results that were introduced by Dodziuk in [14] and further developed in collaboration with Patodi in [15]. The goal of this thesis is to build a framework that provides such an analysis of the relationship between the discrete objects introduced in [14, 15]. In the process of doing this, we will incorporate three main fields of mathematical knowledge: Discrete Differential geometry, Computational Algebraic Topology, and Hilbert Spaces. By incorporating a diverse set of tools in the framework, we present new openings for further exploration.

Broadly speaking, the work of [14, 15] aims to approximate the Hodge theory of the differential forms on a Riemannian manifold using (finite) simplicial complexes. It begins with a compact orientable Riemannian manifold M and a *triangulation* of M, K (see Section 6.3 or Chapter IV Section 12 of [49] for details). Through a standard subdivision algorithm, K is subdivided to produce a new triangulation, $\star K$, for which each k-simplex of K has been split into 2^k k-simplices. Iterative subdivisions are performed and denoted by $\star^n K$ to construct a sequence of triangulations, $(\star^n K)_{n=1}^{\infty}$.

Consider $\Omega^\bullet(M)$ equipped with the Riemannian metric induced on forms and its metric completion $L^2\Omega^\bullet(M)$. In the work of Whitney [49], it is shown that there exists a cochain

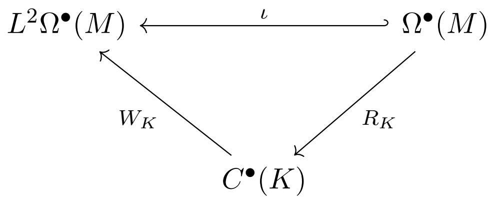


Figure 1.1: A diagram depicting the set-up of [14, 15] where ι is the inclusion map which exists because smooth forms on a compact manifold are L^2.

map $W_K \colon C^\bullet(K) \to L^2\Omega^\bullet(M)$, known as the *Whitney Cochain Embedding*, that performs a type of linear interpolation adapted to forms. On the other hand, we have a cochain map $R_K \colon \Omega^\bullet(M) \to C^\bullet(K)$ given by integration of forms over simplices. The situation is depicted in Figure 1.1. The guiding line of inquiry lies in how well the finite dimensional space $C^\bullet(K)$ approximates $\Omega^\bullet(M)$ under W_K, R_K. The first major theorem in [14] shows that for $\alpha \in \Omega^\bullet(M)$,

$$\lim_{n\to\infty} \|\alpha - W_{\star^n K} R_{\star^n K}(\alpha)\| = 0, \tag{1.1}$$

where we are regarding α as an element of $L^2\Omega^\bullet(M)$ under the inclusion.

Through $W_{\star^n K}$, we induce an inner product onto each complex $C^\bullet(\star^n K)$, $\langle , \rangle_n$, by pulling back the inner product from $L^2\Omega^\bullet(M)$. This allows us to define the codifferential map $d_n^* \colon C^\bullet(\star^n K) \to C^\bullet(\star^n K)$ and the Hodge Laplacian $\Delta_n = d_n d_n^* + d_n^* d_n = (d_n + d_n^*)^2$. We obtain a Hodge decomposition

$$C^\bullet(\star^n K) = \operatorname{im}(d_n) \oplus \ker(\Delta_n) \oplus \operatorname{im}(d_n^*). \tag{1.2}$$

This immediately introduces the question of how one may approximate the Hodge decomposition of an element $\alpha \in \Omega^\bullet(M)$. In particular, writing

$$\begin{aligned} \alpha &= d\alpha_0 + \alpha_1 + d^*\alpha_2 \\ R_{\star^n K}(\alpha) &= d_n\beta_{n,0} + \beta_{n,1} + d_n^*\beta_{n,2}, \end{aligned} \tag{1.3}$$

where $\alpha_1, \beta_{n,1}$ are harmonic in their respective complexes, it is shown that

$$\begin{aligned} \lim_{n\to\infty} \|d\alpha_0 - W_{\star^n K}(d_n\beta_{n,0})\| &= 0 \\ \lim_{n\to\infty} \|\alpha_1 - W_{\star^n K}(\beta_{n,1})\| &= 0 \\ \lim_{n\to\infty} \|d^*\alpha_2 - W_{\star^n K}(d_n^*\beta_{n,2})\| &= 0. \end{aligned} \tag{1.4}$$

Similar approximation results appear in [15] for the eigenvectors and eigenvalues of Δ and Δ_n.

Notice that the above results compare each package of data $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)$ to $\left(L^{2} \Omega^{\bullet}(M), d, \langle,\rangle\right)$ independently and make few claims about combinatorial relationships between $\left(C^{\bullet}\left(\star^{m} K\right), d_{m}, \langle,\rangle_{m}\right)$ and $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)$. Because of this, $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)_{n=1}^{\infty}$ lacks the structure of a Cauchy or convergent sequence that can be described independently of $L^{2} \Omega^{\bullet}(M)$. At a high level, we will use tools from computational algebraic topology to relate pairs of discrete cochain complexes and Hilbert space methods to provide an analysis of the relationship.

By studying $R_{\star^{n} K}, W_{\star^{n} K}$ between $L^{2} \Omega^{\bullet}(M)$ and $C^{\bullet}\left(\star^{n} K\right)$, one finds that these maps fit into larger package of maps forming a deformation retraction due to [16]. Deformation retractions have been used extensively in computational algebraic topology [26, 18, 43, 45, 11, 17] and have also been studied as a means of transferring algebraic structure in discrete differential geometry [50]. We will use a particular type of deformation retraction in our work as a structure that can relate Hodge theory between different cochain complexes in the sequence $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)_{n=1}^{\infty}$.

We will then analyze their ability to approximate by using Hilbert space methods inspired by resolvent convergence and introduced in [39] as a means of assigning an approximation accuracy value to a deformation retraction. By interpreting this approximation accuracy as a distance, this will allow us to discuss Cauchy and convergent sequences of the discrete structures $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)_{n=1}^{\infty}$. Furthermore, we will use these tools to prove that $\left(L^{2} \Omega^{\bullet}(M), d, \langle,\rangle\right)$ can be constructed using the sequence $\left(C^{\bullet}\left(\star^{n} K\right), d_{n}, \langle,\rangle_{n}\right)_{n=1}^{\infty}$.

1.2 Outline and Notation Conventions

The goal of this thesis is to explain what it means for a sequence of finite dimensional spaces to approximate Hodge theory on an infinite dimensional space. Moreover, we want to do this in such a way that elements of the sequence eventually approximate each-other. To accomplish this, we will use sequences of Hilbert spaces where each is equipped with a self-adjoint non-negative operator. The Hilbert space will be denoted by $\mathcal{H}$ and the self-adjoint non-negative operator by $\Delta\colon \mathcal{H} \rightarrow \mathcal{H}$. We will write this as a package, $(\mathcal{H}, \Delta)$, and for a sequence, $\left(\mathcal{H}_{n}, \Delta_{n}\right)_{n=1}^{\infty}$, define the properties of being Cauchy and convergent to formalize the idea of approximation. As a notation convention, in all packages with alphanumeric subscripts, $\left(\mathcal{H}_{n}, \Delta_{n}\right)$, $\mathcal{H}_{n}$ will be assumed to be finite dimensional. All manifolds considered will be smooth and without boundary.

The spectrum of Δ, $\sigma(\Delta)$, is a subset of $[0, \infty)$ which is filtered via the sequence of subsets $(\sigma(\Delta) \cap[0, B])_{B \in[0, \infty)}$. This filtration gives rise to a filtration of $\mathcal{H}$ through eigenspaces of Δ corresponding to eigenvalues contained in $\sigma(\Delta) \cap[0, B]$. The notion of a convergent (resp. Cauchy) sequence $\left(\mathcal{H}_{n}, \Delta_{n}\right)_{n=1}^{\infty}$ will be such that for any $B \in[0, \infty)$, the sequence of subsets $(\sigma(\Delta_n) \cap [0, B])_{n=1}^{\infty}$ are convergent (resp. Cauchy) in the Hausdorff metric associated to the Euclidean metric on $\mathbb{R}$. Moreover, we will see how the sequence of subspaces given by the span of eigenvectors associated to eigenvalues in $\sigma(\Delta_n) \cap [0, B]$ is Cauchy.

In Section 2.1, Definition 2.1.1, we explain the method of measuring similarity between two packages $(\mathcal{H}, \Delta)$, $(\mathcal{H}', \Delta')$ which was introduced and explored by Post in [39]. As an overview, we say that $(\mathcal{H}, \Delta)$ a *δ-retract* of $(\mathcal{H}', \Delta')$ if there exists an isometry $g\colon \mathcal{H} \to \mathcal{H}'$ that relates Δ and Δ' with error bounded above by $\delta \in \mathbb{R}$. We will think of δ as the distance between $(\mathcal{H}, \Delta)$ and $(\mathcal{H}', \Delta')$. Some results of [39] are discussed. Definition 2.1.5 comes from [39] and explains what it means for a sequence to be convergent, while Definition 2.1.7 is original (though inspired by 2.1.5) and explains what it means for a sequence to be Cauchy. Example 2.1.13 uses existing literature on Sobolev spaces to demonstrate how to use the method of δ-retracts to approximate Sobolev spaces of S^1. Proposition 2.1.14 is original and serves to narrow the scope of this work by showing that if $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}, \Delta)$, then $\sigma(\Delta)$ consists only of eigenvalues of Δ and there exists an orthonormal eigenbasis of Δ for $\mathcal{H}$.

In Section 2.2, we show how the sequence of spectra $(\sigma(\Delta_n))_{n=1}^{\infty}$ associated to a convergent sequence converges. The statement of Theorem 2.2.4 comes from [39] and says that if $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}, \Delta)$, then the sequence of spectra $(\sigma(\Delta_n))_{n=1}^{\infty}$ limits to $\sigma(\Delta)$ under the Hausdorff metric associated to the distance function

$$d(x, y) = \left| \frac{1}{x+1} - \frac{1}{y+1} \right|. \tag{1.5}$$

While the statement of this theorem is not original, the proof of it provided in [39] contains a mistake and so our proof is an original replacement.

In Chapter 3, we tackle the problem of, given a Cauchy sequence $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$, constructing a limit package $(\mathcal{H}_\infty, \Delta_\infty)$. This limit is constructed along with isometries $g_n\colon \mathcal{H}_n \to \mathcal{H}_\infty$ which show that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}_\infty, \Delta_\infty)$. We go on to show that this construction is unique up to an intertwining unitary map. In summary, this shows that every Cauchy sequence converges. Although inspired by existing work, the results of this chapter are original.

Section 4.1 consists of material that examines the properties of the sequence of spectra of a Cauchy sequence. By the preceding chapter, every Cauchy sequence converges and so we know that the sequence of spectra converges to that of the limit package by Section 2.2. Theorem 4.1.1 is a technical result that is an original correction of a claim made in [39, 38], where the initial statement and proof contain mistakes. In the remainder of the section, we make use of Theorem 4.1.1 to repair proofs of ideas contained in [39]. These ideas can be summarized as follows

1. Definition 4.1.2 and Theorem 4.1.3 show how multiplicity in the limit spectrum is approximately observed in the sequence of spectra of the Cauchy sequence.

2. Corollary 4.1.4 shows that for fixed j,

$$\lim_{n\to\infty} \lambda_j^{(n)} = \lambda_j. \tag{1.6}$$

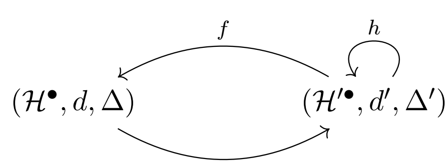
Section 4.2 consists of original work that provides a comparison between the eigenvectors of Δ_m and Δ_n through Proposition 4.2.3. This result explains that when an eigenvector of Δ_m corresponding to $\lambda_j^{(m)}$ is mapped to $\mathcal{H}_n$ using the isometry from a δ-retract, the result is close to the eigenspace of $\lambda_j^{(n)}$ where closeness is a function of δ.

In Chapter 5, we upgrade the tools developed thus far for application to the case of Hilbert complexes, a generalization of cochain complexes into the setting of Hilbert spaces. To do this, we replace the use of δ-retracts with a package of maps that we call a δ-*deformation retract* and whose definition is based on the notion from algebraic topology of a *strong deformation retraction* (see [6]).

In Section 5.1, we provide an overview of the existing work on Hilbert complexes using [8, 5, 22, 12]. For our purposes, we will work with packages of data $(\mathcal{H}^\bullet, d, \Delta)$ where $\Delta = (d + d^*)^2$ and we have the familiar orthogonal Hodge decomposition

$$\mathcal{H}^\bullet = \operatorname{im}(d) \oplus \ker(\Delta) \oplus \operatorname{im}(d^*). \tag{1.7}$$

In Section 5.2, Definition 5.2.3 defines a δ-deformation retract from a package $(\mathcal{H}'^\bullet, d', \Delta')$ onto $(\mathcal{H}^\bullet, d, \Delta)$ and serves as the Hilbert complex analogue of a δ-retract. This is an original definition but is based on [8, 6, 9, 39]. At a high level, it provides a collection of maps arranged as follows:



where g relates Δ, Δ' with error bounded above by δ and the maps f, g, h satisfy a collection of compatibility conditions including $fg = 1$ and $1 - gf = [d, h]$. Proposition 5.2.4 shows that g witnesses a δ-retract from $(\mathcal{H}'^\bullet, \Delta')$ onto $(\mathcal{H}^\bullet, \Delta)$ and so a δ-deformation retract is a generalization of a δ-retract. We use δ-deformation retracts to define the notion of a Cauchy or convergent sequence $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$.

Proposition 5.2.8 is original and shows that a Cauchy sequence $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ convergent to some $(\mathcal{H}^\bullet, d, \Delta)$ exhibits a spectral gap in the sense that

$$\lim_{n\to\infty} \min \sigma\left(\Delta_n\right) \setminus \{0\} > 0. \tag{1.8}$$

The argument for this relies on counting the dimension of eigenspaces and makes use of the isomorphism on cohomology given by the maps from a δ-deformation retract. For the remainder of the chapter, we assume $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ is a Cauchy sequence.

In Section 5.3 we use the work of [14] to show how the Hodge decomposition of elements in $(\mathcal{H}_n^\bullet, d_n, \Delta_n)$ compare to those in $(\mathcal{H}_m^\bullet, d_m, \Delta_m)$.

Section 5.4 consists of original developments that mirror Section 4.2 by showing how, for $m \leq n$, eigenvectors of Δ_m are approximately mapped to eigenvectors of Δ_n using the δ-deformation retract and while also approximately respecting the Hodge decomposition.

Section 5.5 exhibits, for a compact Riemannian manifold M, an example of approximation of $L^2\Omega^\bullet(M)$ equipped with the Hodge Laplacian, Δ. The spaces $\mathcal{H}_n^\bullet$ are given by the span of the eigenvectors of Δ corresponding to the first n eigenvalues. Work in this section is influenced by similar constructions in [35, 33].

Chapter 5 closes with Section 5.6 which adapts Chapter 3 to the context of Hilbert complexes and consists solely of original material. We exhibit the construction of a limit point, $(\mathcal{H}_\infty^\bullet, d_\infty, \Delta_\infty)$, from a Cauchy sequence, $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$. We show that $\Delta_\infty = (d_\infty + d_\infty^*)^2$ and provide δ-deformation retracts from $(\mathcal{H}_\infty^\bullet, d_\infty, \Delta_\infty)$ to $(\mathcal{H}_n^\bullet, d_n, \Delta_n)$ for each n.

Chapter 6 is dedicated to the main application of this thesis, showing that the work of [14, 15] can be described in terms of our framework. In Sections 6.1 and 6.2, we review the basics of simplicial complexes and introduce a subdivision algorithm for complexes. This review consists of material from [19, 31, 30, 1].

In Sections 6.3 and 6.4, we review material on triangulations from [49, 16, 1, 14, 15, 23]. We explain how finite simplicial complexes, K, and iterated subdivisions, $\star^n K$, can be used to model a compact orientable Riemannian manifold, M, by endowing each cochain complex $C^\bullet\left(\star^n K\right)$ with an inner product and thus a Hilbert complex structure. Due to the inner product, we obtain a codifferential d_n^* and Laplacian $\Delta_n = (d_n + d_n^*)^2$ on $C^\bullet\left(\star^n K\right)$. We show that there exists a δ-deformation retract from $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ to $\left(C^\bullet\left(\star^n K\right), d_n, \Delta_n\right)$ for some δ which uses the maps $R_{\star^n K}$, $W_{\star^n K}$ described in the previous section (Section 1.1). We also show that, for $m < n$ and some δ, there exists a δ-deformation retract from $\left(C^\bullet\left(\star^n K\right), d_n, \Delta_n\right)$ onto $\left(C^\bullet\left(\star^m K\right), d_m, \Delta_m\right)$. The remainder of the chapter is devoted to showing that $\left(C^\bullet\left(\star^n K\right), d_n, \Delta_n\right)_{n=1}^\infty$ is Cauchy and convergent to $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ using these δ-deformation retracts.

In Section 6.5, we provide background on the results of [14, 15] along with some Corollaries (6.5.4 and 6.5.5) whose proofs are original and which are later used to prove our main application.

Section 6.6 is devoted to the main proof that $(C^\bullet(\star^n K), d_n, \Delta_n)_{n=1}^\infty$ is Cauchy and convergent to $\left(L^2\Omega^\bullet(M), d, \Delta\right)$. This section consists solely of original material and the main results are as follows:

1. Theorem 6.6.3 shows that $(C^\bullet(\star^n K), d_n, \Delta_n)_{n=1}^\infty$ is convergent to $\left(L^2\Omega^\bullet(M), d, \Delta\right)$.

2. Corollary 6.6.4 uses Theorem 6.6.3 and some facts about our δ-deformation retracts to show that $(C^\bullet(\star^n K), d_n, \Delta_n)_{n=1}^\infty$ is Cauchy.

As a result, we have accomplished the main goal of constructing a framework with which to directly compare the discrete structure of the cochain complexes $(C^\bullet(\star^m K), d_m)$, $(C^\bullet(\star^n K), d_n)$ equipped with inner products. The results of this section allow us to use all the results of Chapter 5 to compare discrete Hodge decompositions and eigenvectors of the discrete Laplacians Δ_m, Δ_n.

The conclusions of Chapter 6 in the context of Section 5.6 have the interesting corollary that the sequence $(C^\bullet(\star^n K), d_n, \Delta_n)_{n=1}^\infty$ uniquely determines $\left(L^2\Omega^\bullet(M), d, \Delta\right)$. In other words, we have shown that the sequence of discretizations contains all of the data necessary to construct the space $L^2\Omega^\bullet(M)$, its inner product, and the maps d, d^*, Δ.

From this work, the author expects that the following lines of inquiry will be fruitful to explore for future work:

1. A proof of discreteness of the spectrum of the Laplacian on a compact orientable Riemannian manifold using a sequence of cochain complexes.

2. The existing literature has struggled to see convergence of the discrete codifferential d_n^* to d^* on $L^2\Omega^\bullet(M)$ [46, 3]. With the knowledge that $(C^\bullet(\star^n K), d_n, \Delta_n)_{n=1}^\infty$ contains enough information to construct d^*, it may now be possible to describe this convergence.

3. Incorporation of the Hodge star operator into our framework. Such developments would likely rely on several existing works that have defined and discussed results concerning convergence of the discrete Hodge star operator [50, 47].

4. For a strong deformation retraction (f, g, h) from $(\mathcal{H}'^\bullet, d')$ onto $(\mathcal{H}^\bullet, d)$, we obtain a Hodge-like decomposition

$$\mathcal{H}'^\bullet \cong \operatorname{im}(g) \oplus d' \ker(f) \oplus \operatorname{im}(h), \tag{1.9}$$

where we are thinking of $\mathrm{im}(g)$ as the Harmonic forms corresponding to the operator $(d+h)^2$ and h as a codifferential ([43] and Section 4.3 of [35]). The connection between this idea and the work in this thesis is not fully understood and further investigation is in order.

Chapter 2

Delta Retracts and Convergence of Spectra

2.1 Comparing Systems: Delta Retracts

We begin with the following definition which has been adapted and renamed from Definitions 4.1.1 and 4.2.3 part 2 of [39].

Definition 2.1.1. Consider $(\mathcal{H}, \Delta)$, $(\mathcal{H}', \Delta')$. For $\delta > 0$, an isometry $g\colon \mathcal{H} \to \mathcal{H}'$ is said to be a *δ-isometry* if

$$\begin{aligned} \left\| (1 - gg^*) \left(\Delta' + 1\right)^{-1} \right\| &\leq \delta \\ \left\| \left(\Delta' + 1\right)^{-1} g - g \left(\Delta + 1\right)^{-1} \right\| &\leq \delta \end{aligned} \tag{2.1}$$

When such a map exists, $(\mathcal{H}, \Delta)$ is said to be a *δ-retract* of $(\mathcal{H}', \Delta')$.

Recall that g being an isometry is equivalent to g being bounded and such that $g^*g = 1$ (but g need not be an isomorphism). This property justifies the use of the term “retract”, where $g^*\colon \mathcal{H}' \to \mathcal{H}$ is a retraction in the topological sense. In addition, note that $\mathrm{im}(g) \subseteq \mathcal{H}'$ is a closed subspace of $\mathcal{H}'$ and that $1 - gg^*$ is the orthogonal projection onto $\mathrm{im}(g)^\perp$.

It is instructive to notice that any isometry $g\colon \mathcal{H} \to \mathcal{H}'$ is a 2-isometry. To see this, first notice that by our assumption, $\|g\| \leq 1$ and due to non-negativity of Δ', Δ, we have $\left\|(\Delta' + 1)^{-1}\right\| \leq 1$, $\left\|(\Delta + 1)^{-1}\right\| \leq 1$. So,

$$\begin{aligned} \left\| (1 - gg^*) \left(\Delta' + 1\right)^{-1} \right\| &\leq \left\| \left(\Delta' + 1\right)^{-1} \right\| + \left\| gg^* \left(\Delta' + 1\right)^{-1} \right\| \\ &\leq 2 \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\left\|\left(\Delta'+1\right)^{-1} g - g\left(\Delta+1\right)^{-1}\right\| &\leq \left\|\left(\Delta'+1\right)^{-1} g\right\| + \left\|g\left(\Delta+1\right)^{-1}\right\| \\ &\leq \left\|\left(\Delta'+1\right)^{-1}\right\| + \left\|\left(\Delta+1\right)^{-1}\right\| \\ &\leq 2.\end{aligned} \tag{2.3}$$

The following comes from Proposition 4.2.4 Part 1 of [39]. The proof has been slightly expanded upon in the interest of clarity.

Lemma 2.1.2. If $(\mathcal{H}, \Delta)$ is a 0-retract of $(\mathcal{H}', \Delta')$ under the 0-isometry $g\colon \mathcal{H} \to \mathcal{H}'$, then g is unitary and intertwines $(\Delta'+1)^{-1}$ and $(\Delta+1)^{-1}$.

Proof. First note that because

$$\left\|\left(\Delta'+1\right)^{-1} g - g\left(\Delta+1\right)^{-1}\right\| = 0, \tag{2.4}$$

we have $\left(\Delta'+1\right)^{-1} g = g\left(\Delta+1\right)^{-1}$. Since Δ', Δ are self-adjoint, so are $\left(\Delta'+1\right)^{-1}, \left(\Delta+1\right)^{-1}$. Taking the adjoint of the previous equation, we additionally see that $g^*\left(\Delta'+1\right)^{-1} = \left(\Delta+1\right)^{-1} g^*$.

By definition, g is an isometry and so we need only show that g is surjective. To do this, decompose

$$\mathcal{H}' = \operatorname{im}(g) \oplus \operatorname{im}(g)^{\perp}, \tag{2.5}$$

where we get this from $\operatorname{im}(g)$ being a closed subspace. Now, fix $\alpha \in \operatorname{im}(g)^{\perp}$ and compute

$$\begin{aligned} 0 &= \left\|\left(1-gg^*\right)\left(\Delta'+1\right)^{-1}\alpha\right\| \\ &= \left\|\left(\Delta'+1\right)^{-1}\left(1-gg^*\right)\alpha\right\|, \quad \text{as shown above} \\ &= \left\|\left(\Delta'+1\right)^{-1}\alpha\right\|. \end{aligned} \tag{2.6}$$

Observe that $\left(\Delta'+1\right)^{-1}$ is an injection because it is an inverse and so the above implies that $\alpha = 0$. Thus, $\operatorname{im}(g)^{\perp} = 0$ and so g is a bijective isometry which means that g is unitary. ☐

Lemma 2.1.3. Let $(\mathcal{H}, \Delta)$ be a δ-retract of $(\mathcal{H}', \Delta')$ witnessed by the δ-isometry $g\colon \mathcal{H} \to \mathcal{H}'$ and let $\alpha \in \operatorname{dom}(\Delta')$. Then,

$$\left\|\left(1-gg^*\right)\alpha\right\| \leq \left\|\left(\Delta'+1\right)\alpha\right\| \delta. \tag{2.7}$$

Proof. Observe,

$$\begin{aligned}\|(1-gg^*)\,\alpha\| &= \left\|(1-gg^*)\left(\Delta'+1\right)^{-1}\left(\Delta'+1\right)\alpha\right\| \\ &\leq \left\|\left(\Delta'+1\right)\alpha\right\|\delta.\end{aligned} \tag{2.8}$$

□

The following will become useful in later sections.

Lemma 2.1.4. For $\alpha \in \mathcal{H}'$,

$$\inf_{\beta\in\mathcal{H}} \|\alpha - g(\beta)\| = \|(1-gg^*)(\alpha)\|. \tag{2.9}$$

Proof. Note that $1-gg^*$ is the orthogonal projection onto $\mathrm{im}(g)^{\perp}$ and $\mathrm{im}(g)$ is a closed subspace of $\mathcal{H}$ since g is an isometry. Apply the Hilbert Projection Theorem (Theorem 15.3 of [51]) to conclude. □

The following definition comes from Definition 4.2.6 of [39] with small adaptations in order to conform with our notation.

Definition 2.1.5. Given $(\mathcal{H}, \Delta)$, a sequence $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is said to *converge to* $(\mathcal{H}, \Delta)$ if there exists a sequence $(\delta_n)_{n=1}^{\infty}$, $\delta_n \geq 0$, such that $\delta_n \to 0$ and $(\mathcal{H}_n, \Delta_n)$ is a δ_n-retract of $(\mathcal{H}, \Delta)$. We will sometimes refer to $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ as an *approximation scheme* for $(\mathcal{H}, \Delta)$.

For a sequence $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ convergent to some $(\mathcal{H}, \Delta)$, we will denote the δ-isometries $g_n\colon \mathcal{H}_n \to \mathcal{H}$. To define Cauchy sequences, we need a preliminary definition.

Definition 2.1.6. Given a package $(\mathcal{H}, \Delta)$ and $I \subseteq \mathbb{R}{\geq}\,0$, define the *spectral projection map onto* I, $P_{\Delta,I}\colon \mathcal{H} \to \mathcal{H}$, as

$$P_{\Delta,I} = \mathrm{Proj}_{\bigoplus\limits_{\lambda\in\sigma(\Delta)\cap I} \mathcal{E}_\lambda}, \tag{2.10}$$

where $\mathcal{E}_\lambda$ is the λ-eigenspace of Δ. We will later see, through Proposition 2.1.14, that this definition matches the spectral projection denoted by $\mathbb{1}_I$ in [39] for our use-cases.

Definition 2.1.7. A sequence $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is said to be *Cauchy* if for every $m \leq n$, there exist isometries $g_m^n\colon \mathcal{H}_m \to \mathcal{H}_n$ such that the following hold:

1. For every $\delta > 0$, there exists $N_\delta \in \mathbb{N}$ such that for all $n > m \geq N_\delta$, g_m^n is a δ-isometry.

2. For all $k \geq n \geq m$, $g_m^k = g_n^k \circ g_m^n$.

3. For all n,

$$\lim_{B\to\infty}\lim_{k\to\infty}\left\|P_{\Delta_k,[B,\infty)}\circ g_n^k\right\|=0. \tag{2.11}$$

The last condition will not be used until Chapter 3, where we will see that it is necessary for the construction of a space for which a Cauchy sequence is an approximation scheme. We also see how convergent sequences necessarily exhibit this property.

Example 2.1.8. Let $\mathcal{H}=\ell^2(\mathbb{R})$ equipped with the standard ℓ^2 inner product. Let e_n be the sequence with nth term equal to 1 and all other terms equal to 0 so that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. Define Δ on $\mathcal{H}$ via $\Delta e_j=je_j$. Define $\mathcal{H}_n=\operatorname{span}(e_1,\dots,e_n)\subset\mathcal{H}$ with the inner product inherited from $\mathcal{H}$ and define Δ_n on $\mathcal{H}_n$ to be the restriction of Δ. The operators Δ,Δ_n are self-adjoint and non-negative. For $m<n$, let $g_m^n\colon\mathcal{H}_m\hookrightarrow\mathcal{H}_n$ be the inclusion and for all n, let $g_n\colon\mathcal{H}_n\hookrightarrow\mathcal{H}$ be the inclusion. By definition, g_m^n and g_n are isometries for all m,n.

We will show that $(\mathcal{H}_n,\Delta_n)_{n=1}^{\infty}$ is both Cauchy and convergent to $(\mathcal{H},\Delta)$. Pursuant to showing that the sequence is Cauchy, we will show that for $m<n$, $(\mathcal{H}_m,\Delta_m)$ is a δ_m-retract of $(\mathcal{H}_n,\Delta_n)$ for some $\delta_m>0$ independent of n and such that $\delta_m\to 0$ as $m\to\infty$. This will be conducted by verifying the conditions under Definition 2.1.1. Since the g_m^n are inclusions, Equation (2.11) will be automatically satisfied and so we will have shown that $(\mathcal{H}_n,\Delta_n)_{n=1}^{\infty}$ is Cauchy under Definition 2.1.7.

To show that $(\mathcal{H}_n,\Delta_n)_{n=1}^{\infty}$ is convergent to $(\mathcal{H},\Delta)$, we will show that for each n, $(\mathcal{H}_n,\Delta_n)$ is a δ_n-retract of $(\mathcal{H},\Delta)$ for δ_n such that $\delta_n\to 0$ as $n\to\infty$. This will show convergence under Definition 2.1.5.

Fix $m<n$. Since g_m^n is the inclusion map and Δ_m,Δ_n are obtained from Δ, we have that

$$\left\|(\Delta_n+1)^{-1}g_m^n-g_m^n(\Delta_m+1)^{-1}\right\|=0. \tag{2.12}$$

To verify the other condition, observe that $(1-g_m^n g_m^{n\,*})=\operatorname{Proj}_{\operatorname{span}(e_{m+1},e_{m+2},\dots,e_n)}$ and so for a generic element of $\mathcal{H}_n$, $\sum_{j=1}^n a_je_j$, we have

$$\begin{aligned}\left\|(1-g_m^ng_m^{n\,*})(\Delta_n+1)^{-1}\sum_{j=1}^n a_je_j\right\| &= \left\|(1-g_m^ng_m^{n\,*})\sum_{j=1}^n\frac{a_j}{j+1}e_j\right\|\\ &= \left\|\sum_{j=m+1}^n\frac{a_j}{j+1}e_j\right\|\\ &\le\frac{1}{m+2}\left\|\sum_{j=1}^n a_je_j\right\|.\end{aligned} \tag{2.13}$$

Thus, $(\mathcal{H}_m, \Delta_m)$ is a $\left(\frac{1}{m+2}\right)$-retract of $(\mathcal{H}_n, \Delta_n)$ and we see that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is a Cauchy sequence. Now to check that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}, \Delta)$, the same reasoning as the previous case shows

$$\left\|(\Delta+1)^{-1} g_n - g_n (\Delta_n + 1)^{-1}\right\| = 0. \tag{2.14}$$

Since $(1 - g_n g_n^*) = \mathrm{Proj}_{\mathrm{span}(e_{n+1}, e_{n+2}, \dots)}$,

$$\begin{aligned} \left\|(1 - g_n g_n^*)(\Delta+1)^{-1} \sum_{j=1}^{\infty} a_j e_j\right\| &= \left\|(1 - g_n g_n^*) \sum_{j=1}^{\infty} \frac{a_j}{j+1} e_j\right\| \\ &= \left\|\sum_{j=n+1}^{\infty} \frac{a_j}{j+1} e_j\right\| \\ &\leq \frac{1}{n+2} \left\|\sum_{j=1}^{\infty} a_j e_j\right\| . \end{aligned} \tag{2.15}$$

This tells us that $(\mathcal{H}_n, \Delta_n)$ is a $\left(\frac{1}{n+2}\right)$-retract of $(\mathcal{H}, \Delta)$ and thus $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}, \Delta)$.

Remark 2.1.9. Consider taking $\Delta = 0$ in Example 2.1.8. In this case, we still have

$$\left\|(\Delta+1)^{-1} g_n - g_n (\Delta_n + 1)^{-1}\right\| = 0. \tag{2.16}$$

However, for each n

$$\begin{aligned} \left\|(1 - g_n g_n^*)(\Delta+1)^{-1} e_{n+1}\right\| &= \|(1 - g_n g_n^*) e_{n+1}\| \\ &= \|e_{n+1}\| \\ &= 1. \end{aligned} \tag{2.17}$$

Similarly, $\left\|(1 - g_m^n g_m^{n\,*})(\Delta+1)^{-1} e_{m+1}\right\| = 1$ and so $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is neither Cauchy nor convergent to $(\mathcal{H}, \Delta)$ under the 1-isometries g_n, g_m^n.

Remark 2.1.10. Suppose that $(\mathcal{H}_1, \Delta_1), (\mathcal{H}_2, \Delta_2)$ are both δ-retracts of some $(\mathcal{H}, \Delta)$ given by the isometries $g_1 \colon \mathcal{H}_1 \to \mathcal{H}$, $g_2 \colon \mathcal{H}_2 \to \mathcal{H}$. It is not clear that we can use g_1 and g_2 to construct an isometry witnessing a δ-retract relationship between $(\mathcal{H}_1, \Delta_1)$ and $(\mathcal{H}_2, \Delta_2)$.

Proposition 2.1.11. Suppose $(\mathcal{H}_1, \Delta_1)$ is a δ_1-retract of $(\mathcal{H}, \Delta)$ witnessed by $g_1 \colon \mathcal{H}_1 \to \mathcal{H}$ and $(\mathcal{H}_2, \Delta_2)$ is a δ_2-retract of $(\mathcal{H}, \Delta)$ witnessed by $g_2 \colon \mathcal{H}_2 \to \mathcal{H}$. If $\mathrm{im}(g_1) \subseteq \mathrm{im}(g_2)$, then $g_1^2 := g_2^* g_1 \colon \mathcal{H}_1 \to \mathcal{H}_2$ is a $(\delta_1 + \delta_2)$-isometry.

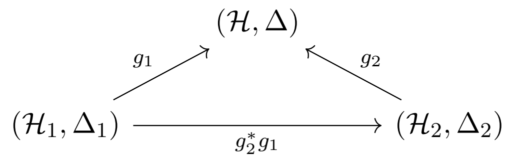


Figure 2.1: A diagram depicting the set-up of Proposition 2.1.11

Proof. Since $\operatorname{im}(g_1) \subseteq \operatorname{im}(g_2)$ and $g_2 g_2^* = \operatorname{Proj}_{\operatorname{im}(g_2)}$, we have

$$g_2 g_1^2 = \operatorname{Proj}_{\operatorname{im}(g_2)} g_1 = g_1. \tag{2.18}$$

Thus,

$$g_1^{2*} g_1^2 = g_1^* g_2 g_1^2 = g_1^* g_1 = 1. \tag{2.19}$$

Since g_1, g_2 are bounded, we have that g_1^2 is bounded and so g_1^2 is an isometry. Next,

$$\begin{aligned}
\left\|(\Delta_2+1)^{-1} g_1^2 - g_1^2 (\Delta_1+1)^{-1}\right\| &= \left\|g_2 \left((\Delta_2+1)^{-1} g_1^2 - g_1^2 (\Delta_1+1)^{-1}\right)\right\| \\
&= \left\|g_2 (\Delta_2+1)^{-1} g_1^2 - g_1 (\Delta_1+1)^{-1}\right\| \\
&\leq \left\|g_2 (\Delta_2+1)^{-1} g_1^2 - (\Delta+1)^{-1} g_1\right\| \\
&\quad + \left\|(\Delta+1)^{-1} g_1 - g_1 (\Delta_1+1)^{-1}\right\| \\
&\leq \left\|g_2 (\Delta_2+1)^{-1} g_1^2 - (\Delta+1)^{-1} g_1\right\| + \delta_1 \\
&= \left\|\left(g_2 (\Delta_2+1)^{-1} - (\Delta+1)^{-1} g_2\right) g_1^2\right\| + \delta_1 \\
&\leq \left\|g_2 (\Delta_2+1)^{-1} - (\Delta+1)^{-1} g_2\right\| + \delta_1 \\
&\leq \delta_1 + \delta_2.
\end{aligned} \tag{2.20}$$

Checking the last condition,

$$\begin{aligned}
\left\|\left(1 - g_1^2 g_1^{2*}\right)(\Delta_2+1)^{-1}\right\| &= \left\|(1 - g_2^* g_1 g_1^* g_2)(\Delta_2+1)^{-1}\right\| \\
&= \left\|g_2^* (1 - g_1 g_1^*) g_2 (\Delta_2+1)^{-1}\right\| \\
&\leq \left\|g_2^* (1 - g_1 g_1^*)(\Delta+1)^{-1} g_2\right\| \\
&\quad + \left\|g_2^* (1 - g_1 g_1^*)\left(g_2 (\Delta_2+1)^{-1} - (\Delta+1)^{-1} g_2\right)\right\| \\
&\leq \left\|g_2^* (1 - g_1 g_1^*)(\Delta+1)^{-1} g_2\right\| + \delta_2 \\
&\leq \delta_1 + \delta_2.
\end{aligned} \tag{2.21}$$

Thus, $(\mathcal{H}_1, \Delta_1)$ is a $(\delta_1+\delta_2)$-retract of $(\mathcal{H}_2, \Delta_2)$. □

Unfortunately, transitivity of δ-retracts has not yet been established. The closest results seen in this direction are taken from Proposition 4.2.5 of [39] and Section 2.4 of [41] which can be paraphrased as follows:

Proposition 2.1.12. Suppose $(\mathcal{H}_1, \Delta)$ is a δ_1-retract of $(\mathcal{H}_2, \Delta_2)$ witnessed by $g_1 \colon \mathcal{H}_1 \to \mathcal{H}_2$ and $(\mathcal{H}_2, \Delta_2)$ is a δ_2-retract of $(\mathcal{H}_3, \Delta_3)$ witnessed by $g_2 \colon \mathcal{H}_2 \to \mathcal{H}_3$. Then $g_1^3 \colon g_2 g_1 \colon \mathcal{H}_1 \to \mathcal{H}_3$ is a $(\delta_1 + 3\delta_2)$-isometry.

Example 2.1.13. This example makes use of the review of Sobolev spaces and Fourier series on S^1 presented in Chapter 8 of [21].

Consider the Hilbert space $L^2\left(S^1; \mathbb{C}\right)$ where S^1 is equipped with the flat Riemannian metric so that $\Delta = -\frac{\partial^2}{\partial \theta^2}$. For each $k \in \mathbb{N}$, define the Sobolev space

$$H^k\left(S^1\right) = \operatorname{dom}\left(\Delta^k\right). \tag{2.22}$$

Following Chapter 3.1 of [39], equip $H^k\left(S^1\right)$ with the inner product $\langle , \rangle_{H^k(S^1)}$ defined on $f, g \in H^k\left(S^1\right)$ using the inner product $\langle , \rangle$ on L^2 via

$$\langle f, g \rangle_{H^k(S^1)} = \left\langle (\Delta + 1)^k f, (\Delta + 1)^k g \right\rangle. \tag{2.23}$$

Since Δ is self-adjoint, it is closed, and thus each $H^k\left(S^1\right)$ is a Hilbert space. Denote the corresponding norm $\|\cdot\|_{H^k(S^1)}$. These spaces are related under containment like so:

$$C^\infty\left(S^1\right) \subseteq \cdots \subseteq H^{k+1}\left(S^1\right) \subseteq H^k\left(S^1\right) \subseteq \cdots \subseteq L^2\left(S^1\right). \tag{2.24}$$

We know that $C^\infty\left(S^1; \mathbb{C}\right)$ is dense in $L^2\left(S^1; \mathbb{C}\right)$ and contains an orthonormal eigenbasis for Δ, namely the functions $\varphi_j \colon S^1 \to \mathbb{C}$ for $j \in \mathbb{Z}$ defined via

$$\varphi_j(\theta) := \frac{1}{2\pi} e^{2\pi i j \theta}. \tag{2.25}$$

This implies that, for each k, $H^{k+1}\left(S^1\right)$ is dense in $H^k\left(S^1\right)$. Consequently, the operator $\Delta|_{H^k(S^1)}$ with $\operatorname{dom}\left(\Delta|_{H^k(S^1)}\right) = H^{k+1}\left(S^1\right)$ is self-adjoint and so we may consider the package $\left(H^k\left(S^1\right), \Delta|_{H^k(S^1)}\right)$ for each k. We will use the tools developed thus far to exhibit an approximation scheme for $\left(H^k\left(S^1\right), \Delta|_{H^k(S^1)}\right)$ in the same way as Example 2.1.8. The calculation is almost identical: Define

$$\begin{aligned} H_n^k &= \operatorname{span}\left(\varphi_0, \varphi_{-1}, \varphi_1, \ldots, \varphi_{-n}, \varphi_n\right) \\ \Delta_{n,k} &= \Delta|_{H_n^k}, \end{aligned} \tag{2.26}$$

where H_n^k is equipped with the inner product induced by $\langle , \rangle_{H^k(S^1)}$. Let $g_n \colon H_n^k \to H^k\left(S^1\right)$

be the inclusion. We will now show that $\left(H_n^k, \Delta_{n,k}\right)$ is a $\frac{1}{2\pi n}$-retract of $\left(H^k\left(S^1\right), \Delta\big|_{H^k(S^1)}\right)$. First, since the metric on H_n^k is induced from $H^k(S^1)$, g_n is an isometry. Similarly,

$$\left\| \left(\Delta\big|_{H^k(S^1)} + 1\right)^{-1} g_n - g_n \left(\Delta_{n,k} + 1\right)^{-1} \right\|_{H^k(S^1)} = 0. \tag{2.27}$$

For the final condition, fix $\alpha \in H^k\left(S^1\right)$ of unit norm, write $\alpha = \sum_{j\in\mathbb{Z}} a_j \varphi_j$ for $a_j \in \mathbb{C}$, and observe

$$\begin{aligned}
\left\| (1 - g_n g_n^*) \left(\Delta\big|_{H^k(S^1)} + 1\right)^{-1} \alpha \right\|_{H^k(S^1)} &= \left\| (1 - g_n g_n^*) \left(\Delta\big|_{H^k(S^1)} + 1\right)^{-1} \sum_{j\in\mathbb{Z}} a_j \varphi_j \right\|_{H^k(S^1)} \\
&= \left\| (1 - g_n g_n^*) \sum_{j\in\mathbb{Z}} \frac{a_j}{2\pi j + 1} \varphi_j \right\|_{H^k(S^1)} \\
&= \left\| \sum_{|j|>n} \frac{a_j}{2\pi j + 1} \varphi_j \right\|_{H^k(S^1)} \\
&< \frac{1}{2\pi n} \left\| \sum_{|j|>n} a_j \varphi_j \right\|_{H^k(S^1)} \\
&\le \frac{1}{2\pi n}.
\end{aligned} \tag{2.28}$$

So, $\left(H_n^k, \Delta_{n,k}\right)_{n=1}^{\infty}$ converges to $\left(H^k\left(S^1\right), \Delta\big|_{H^k(S^1)}\right)$. Moreover, for each $m \le n$, $\operatorname{im}(g_m) \subseteq \operatorname{im}(g_n)$ and so Proposition 2.1.11 implies that $\left(H_n^k, \Delta_{n,k}\right)_{n=1}^{\infty}$ satisfies all conditions of being a Cauchy sequence except for Equation (2.11). Since the g_m^n are inclusions, Equation (2.11) will be automatically satisfied and so $\left(H_n^k, \Delta_{n,k}\right)_{n=1}^{\infty}$ is Cauchy.

Proposition 2.1.14. Let $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ be a sequence convergent to $(\mathcal{H}, \Delta)$. Then, $(\Delta + 1)^{-1}$ is a compact operator. Moreover, $\sigma(\Delta)$ consists only of eigenvalues of Δ and there exists a countable eigenbasis for $\mathcal{H}$.

Proof. If $\dim(\mathcal{H})$ is finite, the result is immediate. Assume then that $\dim(\mathcal{H}) = \infty$. Since $\mathcal{H}_n$ is finite dimensional, each operator $g_n (\Delta_n + 1)^{-1} g_n^*$ has finite dimensional range and is thus compact. On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} &\left\|(\Delta+1)^{-1} - g_n (\Delta_n+1)^{-1} g_n^*\right\| \\
&\leq \lim_{n\to\infty} \left\|(1-g_n g_n^*)(\Delta+1)^{-1}\right\| + \left\|g_n\left(g_n^*(\Delta+1)^{-1} - (\Delta_n+1)^{-1} g_n^*\right)\right\| \\
&= \lim_{n\to\infty} \left\|(1-g_n g_n^*)(\Delta+1)^{-1}\right\| + \left\|g_n(\Delta_n+1)^{-1} - (\Delta+1)^{-1} g_n\right\|, \qquad (2.29)\\
&\quad \text{since } (\Delta+1)^{-1} \text{ is self-adjoint} \\
&= 0.
\end{aligned}$$

We then have that $\left(g_n(\Delta_n+1)^{-1} g_n^*\right)_{n=1}^{\infty}$ is a sequence of compact operators convergent to $(\Delta+1)^{-1}$. This is equivalent to compactness of $(\Delta+1)^{-1}$ by Theorem 1.3.25 of [13]. Theorem 11.3.13 of [13] shows that compactness of $(\Delta+1)^{-1}$ is equivalent to emptiness of the essential spectrum of Δ as well as the existence of a countable eigenbasis for $\mathcal{H}$. □

Proposition 2.1.14 represents an important bound on the scope of this work. Namely, we will only be concerned with operators Δ such that $\sigma(\Delta)$ consists of only the eigenvalues of Δ.

2.2 Convergence of the Spectra of a Convergent Sequence

We want to say something about convergence of spectra for a convergent sequence. In later sections, we will look at convergence of the spectra of a Cauchy sequence. In order to state such a result, we need to explain how to compare the spectra. Define $\mathbb{R}_+ := [0,\infty)$ and $\overline{\mathbb{R}_+} := [0,\infty]$. We will review and then use the weighted distance on $\overline{\mathbb{R}_+}$ defined in Section 4.3 of [39] to perform this comparison. Since all operators are self-adjoint and non-negative, we have that $\sigma(\Delta_n) \subseteq \overline{\mathbb{R}_+}$. Define

$$\varphi\colon \overline{\mathbb{R}_+} \to [0,1] \qquad (2.30)$$

via

$$\varphi(\lambda) = \begin{cases} \frac{1}{\lambda+1} & \text{if } \lambda < \infty, \\ 0 & \text{if } \lambda = \infty, \end{cases} \qquad (2.31)$$

and notice that φ is a homeomorphism. Define a distance function on $\overline{\mathbb{R}_+}$ via

$$d(x,y) = |\varphi(x) - \varphi(y)|. \qquad (2.32)$$

Observe that $\overline{\mathbb{R}_+}$ equipped with d is a complete compact metric space. From now on, we will assume the metric topology induced by d.

The following is a rewording of the definitions presented in Appendix A.1 of [39].

Definition 2.2.1. Given $a \in \overline{\mathbb{R}_+}$ and $B \subseteq \overline{\mathbb{R}_+}$, define

$$d(a, B) = \inf_{b \in B} d(a, b). \tag{2.33}$$

As a convention, we will say that for $B = \emptyset$, $d(a, B) = 1$. The *Hausdorff Distance* between two subsets $A, B \subseteq \overline{\mathbb{R}_+}$ is given by

$$\begin{aligned} d(A, B) &= \max\left\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\right\} \\ &= \max\left\{\sup_{a \in A} \inf_{b \in B} d(a, b),\ \sup_{b \in B} \inf_{a \in A} d(a, b)\right\}. \end{aligned} \tag{2.34}$$

We will use this Hausdorff distance to compare the spectra $\sigma(\Delta_n)$.

Proposition 2.2.2 (Proposition A.1.6 [39])**.** Let $(A_n)_{n=1}^{\infty}$ be a sequence of compact subsets $A_n \subseteq \overline{\mathbb{R}_+}$ and let $B \subseteq \overline{\mathbb{R}_+}$ be compact. Then, $A_n \to B$ in the Hausdorff distance is equivalent to satisfaction of both of the following:

- $A_n \searrow B$: For $\lambda \notin B$, there exists $\varepsilon > 0, N \in \mathbb{N}$ so that for each $n \geq N$, $B_\varepsilon(\lambda) \cap A_n = \emptyset$.
- $A_n \nearrow B$: For $\lambda \in B$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that for each $n \geq N$, $B_\varepsilon(\lambda) \cap A_n \neq \emptyset$.

Let $I \subseteq \overline{\mathbb{R}_+}$ and define

$$\partial_+ I := \overline{I} \cap \overline{\mathbb{R}_+ \setminus I}. \tag{2.35}$$

As an instructive example, for $I = [0, L)$, $\partial_+ I = \{L\}$.

We recall a crucial result, Theorem A.7 of [38], rephrased to conform with our notation. See Theorem A.0.1 in Appendix A for details.

Theorem 2.2.3. Let $\psi\colon \overline{\mathbb{R}_+} \to \mathbb{R}$ be continuous and let $\varepsilon > 0$. Then, there exists $\delta_{\psi,\varepsilon} > 0$ such that for all $\delta \leq \delta_{\psi,\varepsilon}$ and $(\mathcal{H}, \Delta)$, a δ-retract of $(\mathcal{H}', \Delta')$ with δ-isometry $g\colon \mathcal{H} \to \mathcal{H}'$,

$$\left\|\psi\left(\Delta'\right) g - g\psi\left(\Delta\right)\right\| < \varepsilon. \tag{2.36}$$

The following Theorem is a special case of Theorem 4.3.3 in [39]. During the preparation of this thesis, we discovered a mistake in the proof written in [39] and so the proof provided below is original. In writing this proof, we took inspiration from Section 1.3 of [40], Section 4.3 of [39], as well as Appendix A in [38].

Theorem 2.2.4. Suppose $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}, \Delta)$. Then,

$$d\left(\sigma(\Delta_n), \overline{\sigma(\Delta)}\right) \to 0, \tag{2.37}$$

where $\overline{\sigma(\Delta)}$ is the closure in the topology induced by d.

Proof. Apply Proposition 2.1.14 to see that $\sigma(\Delta)$ is countable and possesses no finite accumulation points. One consequence of this is that $\overline{\sigma(\Delta)}$ either coincides with $\sigma(\Delta)$ or is equal to $\sigma(\Delta) \cup \{\infty\}$. We will use Proposition 2.2.2 to determine convergence and will deal with the points of $\sigma(\Delta)$ separately from the case of $\infty \in \overline{\sigma(\Delta)}$. Let $\lambda \in \sigma(\Delta)$. We want to show that for all $\varepsilon > 0$, there exists N large enough such that for $n \geq N$, $B_\varepsilon(\lambda) \cap \sigma(\Delta_n) \neq \emptyset$. We know that $L := d(\lambda, \sigma(\Delta) \setminus \{\lambda\}) > 0$. Let $\eta = \min(\varepsilon, L)$ and construct continuous $\chi_\lambda \colon \overline{\mathbb{R}_+} \to [0,1]$ such that $\chi_\lambda(\lambda) = 1$ and $\operatorname{supp}(\chi_\lambda) \subseteq B_\eta(\lambda)$. This can be done via a piecewise linear function, for example see Figure 2.2.

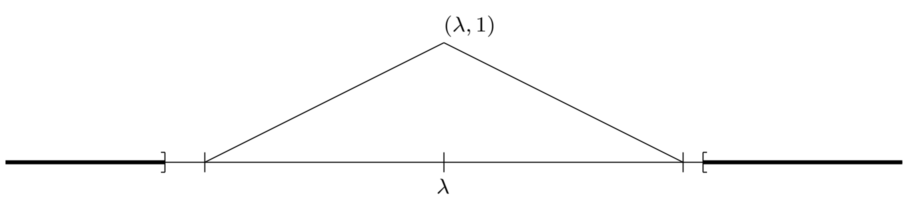


Figure 2.2: A depiction of the graph of a piecewise linear implementation of χ_λ where we can see $\chi_\lambda(\lambda) = 1$ and $\operatorname{supp}(\chi_\lambda) \subseteq B_\eta(\lambda)$. The bold region represents a closed set containing $\overline{\sigma(\Delta)} \setminus \{\lambda\}$.

Consequently, $\operatorname{im}\left(\chi_\lambda\big|_{\sigma(\Delta)}\right) \subseteq \{0,1\}$. Apply Theorem 2.2.3 and take N large enough so that for $n \geq N$, $(\mathcal{H}_n, \Delta_n)$ is a δ-retract of $(\mathcal{H}, \Delta)$ for $\delta > 0$ small enough that

$$\|\chi_\lambda(\Delta) g_n - g_n \chi_\lambda(\Delta_n)\| < \frac{1}{2} \tag{2.38}$$

and also

$$\delta \leq \frac{1}{2(\lambda+1)}. \tag{2.39}$$

Let $\beta \in \mathcal{H}$ be a unit eigenvector of Δ corresponding to λ. Then, $\chi_\lambda(\Delta)(\beta) = \beta$ by definition, and so

$$\begin{aligned} \frac{1}{2} &> \|(g_n^* \chi_\lambda(\Delta) - \chi_\lambda(\Delta_n) g_n^*)(\beta)\| \\ &= \|(1 - \chi_\lambda(\Delta_n)) g_n^*(\beta)\|. \end{aligned} \tag{2.40}$$

On the other hand, by the definition of a δ-retract, we have

$$\begin{aligned}\|g_n^*(\beta)\| &\geq \|\beta\| - \|\beta - g_n g_n^*(\beta)\| \\ &= 1 - \|(1 - g_n g_n^*)(\beta)\| \\ &= 1 - (\lambda + 1) \left\|(1 - g_n g_n^*)(\Delta + 1)^{-1}(\beta)\right\| \\ &\geq 1 - (\lambda + 1)\delta \\ &\geq \frac{1}{2},\end{aligned} \tag{2.41}$$

where the last inequality comes from our assumptions on δ. Assume, towards a contradiction, that $\chi_\lambda(\Delta_n) g_n^*(\beta) = 0$. Then equations (2.40) and (2.41) together tell us

$$\frac{1}{2} \leq \|g_n^*(\beta)\| < \frac{1}{2}, \tag{2.42}$$

a contradiction. This means that $\chi_\lambda(\Delta_n)$ is a non-zero operator and, by definition of χ_λ and spectral calculus, it means that

$$\emptyset \neq \sigma(\Delta_n) \cap \operatorname{supp}(\chi_\lambda) \subseteq \sigma(\Delta_n) \cap B_\eta(\lambda) \subseteq \sigma(\Delta_n) \cap B_\varepsilon(\lambda). \tag{2.43}$$

Now suppose that $\infty \in \overline{\sigma(\Delta)}$. Then, ∞ must be an accumulation point of $\sigma(\Delta)$ and so for every $\varepsilon > 0$, we can find $\lambda \in \sigma(\Delta) \cap B_{\frac{\varepsilon}{2}}(\infty)$. Apply the above argument to obtain $N \in \mathbb{N}$ such that for all $n \geq N$, there exists $\lambda^{(n)} \in \sigma(\Delta_n) \cap B_{\frac{\varepsilon}{2}}(\lambda)$. By the triangle inequality, $\lambda^{(n)} \in B_\varepsilon(\infty)$ and so $B_\varepsilon(\infty) \cap \sigma(\Delta_n) \neq \emptyset$.

Now fix $\lambda \notin \overline{\sigma(\Delta)}$ (note that in this case we allow for $\lambda = \infty$). We want to show that there exists $\varepsilon > 0$ and N large enough such that for all $n \geq N$, $\sigma(\Delta_n) \cap B_\varepsilon(\lambda) = \emptyset$. Using the fact that $\overline{\sigma(\Delta)}$ is closed, let $L := d\left(\lambda, \overline{\sigma(\Delta)}\right) > 0$. Construct continuous $\chi_\lambda \colon \overline{\mathbb{R}_+} \to [0,1]$ such that $\chi_\lambda|_{B_{\frac{L}{4}}(\lambda)} = 1$ and $\operatorname{supp}(\chi_\lambda) \subseteq B_{\frac{L}{2}}(\lambda)$. As in the previous argument, this be done via a piecewise linear function, for example see Figure 2.3.

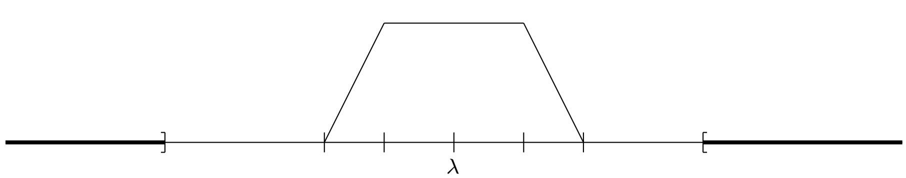


Figure 2.3: A depiction of the graph of a piecewise linear implementation of χ_λ. We can see that $\chi_\lambda = 1$ on $B_{\frac{L}{4}}(\lambda)$ and $\operatorname{supp}(\chi_\lambda) \subseteq B_{\frac{L}{2}}(\lambda)$. The bold region represents a closed set containing $\overline{\sigma(\Delta)}$.

Then, apply Theorem 2.2.3 and take N large enough so that for $n \geq N$, $(\mathcal{H}_n, \Delta_n)$ is a

δ-retract of $(\mathcal{H}, \Delta)$ for δ small enough that

$$\|\chi_\lambda(\Delta)\, g_n - g_n \chi_\lambda(\Delta_n)\| < \frac{1}{2}. \tag{2.44}$$

By definition of χ_λ, $\chi_\lambda(\Delta) = 0$ and since g_n is an isometry, we have

$$\|\chi_\lambda(\Delta_n)\| < \frac{1}{2}. \tag{2.45}$$

In particular, if we had $\alpha \in \mathcal{H}_n$ of unit norm corresponding to an eigenvalue of Δ_n lying in $B_{\frac{L}{4}}(\lambda)$, then

$$\|\chi_\lambda(\Delta_n)(\alpha)\| = \|\alpha\| = 1, \tag{2.46}$$

a contradiction. So, $B_\varepsilon(\lambda) \cap \sigma(\Delta_n) = \emptyset$ for $\varepsilon = \frac{L}{4}$ and for all $n \geq N$. □

Chapter 3

Constructing the Limit of a Cauchy Sequence

In this chapter, we describe how a Cauchy sequence, $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$, can be used to construct $(\mathcal{H}_\infty, \Delta_\infty)$ such that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is convergent to $(\mathcal{H}_\infty, \Delta_\infty)$. To the authors knowledge, no previous developments have been made to accomplish this. A somewhat similar construction was performed in [42] for convergent sequences $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty} \to (\mathcal{H}_\infty, \Delta_\infty)$ to construct a *parent* Hilbert space $\mathcal{H}$ along with injections $\iota_n \colon \mathcal{H}_n \to \mathcal{H}$ for $n \in \overline{\mathbb{N}}$. However, no such results were used in the writing of this chapter.

3.1 Convergence Implies Cauchy

We begin with a preliminary discussion of Equation (2.11) in the definition of a Cauchy sequence. Equation (2.11) may be thought of intuitively as follows: For an element $\alpha \in \mathcal{H}_n$, the eigenvectors of Δ_k appearing in $g_n^k(\alpha)$ correspond to eigenvalues which are eventually concentrated in some bounded interval of $\mathbb{R}$.

Example 3.1.1. For each n, let $\mathcal{H}_n = \mathbb{R}$ with inner product given by multiplication and let $\Delta_n(1) = n$. Let $g_m^n \colon \mathcal{H}_m \to \mathcal{H}_n$ be the identity map, which is an isometry. Then,

$$\begin{aligned} \left\| (\Delta_n + 1)^{-1} g_m^n - g_m^n (\Delta_m + 1)^{-1} \right\| &\leq \frac{1}{n+1} + \frac{1}{m+1}, \\ \left\| (1 - g_m^n g_m^{n\,*}) (\Delta_n + 1)^{-1} \right\| &\leq \frac{1}{n+1}, \end{aligned} \tag{3.1}$$

and so $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is a Cauchy sequence. On the other hand, for each $n \in \mathbb{N}$ and $B \in \mathbb{R}$,

$g_n^{\lceil B\rceil}(1) = 1$ is an eigenvector for $\Delta_{\lceil B\rceil}$ corresponding to the eigenvalue $\lceil B\rceil$ and so

$$\lim_{k\to\infty} \left\| P_{[B,\infty)} g_n^k(1) \right\| = g_n^k(1). \tag{3.2}$$

Thus, Equation (2.11) does not hold.

Proposition 3.1.2. If $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to some $(\mathcal{H}, \Delta)$ for which the δ-isometries $g_n \colon \mathcal{H}_n \to \mathcal{H}$ are such that $g_n \circ g_m^n = g_m$ for $m \leq n$, then Equation (2.11) is satisfied.

Proof. By assumption, $\mathcal{H}_n$ is finite dimensional and so for fixed n,

$$\lim_{B\to\infty} \lim_{k\to\infty} \left\| P_{[B,\infty)} \circ g_n^k \right\| = 0 \tag{3.3}$$

is equivalent to

$$\lim_{B\to\infty} \lim_{k\to\infty} \left\| P_{[B,\infty)} \left(g_n^k(\alpha) \right) \right\| = 0 \tag{3.4}$$

for each $\alpha \in \mathcal{H}_n$. Towards a contradiction, suppose that for some $n \in \mathbb{N}$, $\alpha \in \mathcal{H}_n$ of unit norm, there exists $C > 0$, arbitrarily large $B' \in \mathbb{R}$, and arbitrarily large $k' \in \mathbb{N}$ for each B' such that

$$\left\| P_{[B',\infty)} \left(g_n^{k'}(\alpha) \right) \right\| \geq C. \tag{3.5}$$

Proposition 2.1.14 implies that $\mathcal{H}$ possesses a countable eigenbasis for Δ. Thus, we may fix $B > 0$ large enough to ensure that

$$\left\| P_{[B,\infty)} \left(g_n(\alpha) \right) \right\| < \frac{C^2}{4}. \tag{3.6}$$

The main idea of the argument to follow is to show that $g_{k'}$ maps enough of $P_{[B',\infty)} \left(g_n^{k'}(\alpha) \right)$ into $P_{[0,B)}(\mathcal{H})$ to contradict convergence of $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$.

By assumption, we may take $B' \in \mathbb{R}, k' \in \mathbb{N}$ large enough so that

1. $\left\| P_{[B',\infty)} \left(g_n^{k'}(\alpha) \right) \right\| \geq C$.

2. $\frac{1}{B'+1} < \frac{C^2}{8(B+1)}$.

3. $(\mathcal{H}_{k'}, \Delta_{k'})$ is a δ-retract of $(\mathcal{H}, \Delta)$ for $\delta < \frac{C^2}{8(B+1)}$.

Define

$$\begin{aligned} \beta_{<B'} &:= P_{[0,B')} g_n^{k'}(\alpha), \\ \beta_{\geq B'} &:= P_{[B',\infty)} g_n^{k'}(\alpha), \end{aligned} \tag{3.7}$$

so that $g_n^{k'}(\alpha) = \beta_{<B'} + \beta_{\geq B'}$ is an orthogonal decomposition and so $\|\beta_{<B'}\|, \|\beta_{\geq B'}\| \leq$

$\|\alpha\| = 1$. Observe that assumption number 1. above can be written as $\|\beta_{\geq B'}\| \geq C$. Then,

$$\begin{aligned}\left\|(\Delta+1)^{-1} g_{k'}\left(\beta_{\geq B'}\right)\right\| &\leq \left\|\left((\Delta+1)^{-1} g_{k'} - g_{k'}\left(\Delta_{k'}+1\right)^{-1}\right)\left(\beta_{\geq B'}\right)\right\| \\ &\quad + \left\|g_{k'}\left(\Delta_{k'}+1\right)^{-1}\left(\beta_{\geq B'}\right)\right\| \\ &\leq \delta + \left\|g_{k'}\left(\Delta_{k'}+1\right)^{-1}\left(\beta_{\geq B'}\right)\right\|,\end{aligned} \tag{3.8}$$

by assumption number 3. By definition, we have $\left\|\left(\Delta_{k'}+1\right)^{-1}\Big|_{P_{[B',\infty)}\mathcal{H}_{k'}}\right\| \leq \frac{1}{B'+1}$ and so

$$\left\|\left(\Delta_{k'}+1\right)^{-1}\left(\beta_{\geq B'}\right)\right\| \leq \frac{1}{B'+1}. \tag{3.9}$$

Continuing our work above, we obtain

$$\begin{aligned}\left\|(\Delta+1)^{-1} g_{k'}\left(\beta_{\geq B'}\right)\right\| &\leq \delta + \frac{1}{B'+1} \\ &< \frac{C^2}{4(B+1)}.\end{aligned} \tag{3.10}$$

So,

$$\begin{aligned}\left\|P_{[0,B)} g_{k'}\left(\beta_{\geq B'}\right)\right\| &\leq (B+1)\left\|(\Delta+1)^{-1} P_{[0,B)} g_{k'}\left(\beta_{\geq B'}\right)\right\| \\ &\leq (B+1)\left\|(\Delta+1)^{-1} g_{k'}\left(\beta_{\geq B'}\right)\right\| \\ &< \frac{C^2}{4}.\end{aligned} \tag{3.11}$$

So,

$$\begin{aligned}1-\frac{C^2}{4} &< \|g_n(\alpha)\| - \left\|P_{[B,\infty)} g_n(\alpha)\right\| \\ &\leq \left\|P_{[0,B)} g_n(\alpha)\right\| \\ &= \left\|P_{[0,B)} g_{k'}\left(g_n^{k'}(\alpha)\right)\right\| \\ &\leq \left\|P_{[0,B)} g_{k'}\left(\beta_{<B'}\right)\right\| + \left\|P_{[0,B)} g_{k'}\left(\beta_{\geq B'}\right)\right\| \\ &< \|\beta_{<B'}\| + \frac{C^2}{4},\end{aligned} \tag{3.12}$$

and thus,

$$\begin{aligned}1 &= \|\alpha\|^2 \\ &= \|\beta_{<B'}\|^2 + \|\beta_{\geq B'}\|^2 \\ &> \left(1 - \frac{C^2}{2}\right)^2 + C^2 \\ &= 1 - C^2 + \frac{C^4}{4} + C^2 \\ &= 1 + \frac{C^4}{4}.\end{aligned} \tag{3.13}$$

A contradiction. □

Proposition 3.1.3. Suppose $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is a sequence convergent to $(\mathcal{H}, \Delta)$ such that for $m < n$, $\operatorname{im}(g_m) \subseteq \operatorname{im}(g_n)$. Then, $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is Cauchy and isometries witnessing this are given by $g_m^n = g_n^* \circ g_m$.

Proof. This is an immediate Corollary of Propositions 2.1.11 and 3.1.2. □

3.2 Construction of the Hilbert Space

Let

$$\widetilde{\mathcal{H}}_\infty = \left\{ (\alpha_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{H}_n \;:\; \lim_{m\to\infty} \sup_{n\geq m} \|\alpha_n - g_m^n(\alpha_m)\| = 0 \right\} \tag{3.14}$$

and define a vector space structure via

$$\begin{aligned} c(\alpha_n)_{n=1}^{\infty} &= (c\alpha_n)_{n=1}^{\infty} \quad (c \in \mathbb{R}), \\ (\alpha_n)_{n=1}^{\infty} + (\beta_n)_{n=1}^{\infty} &= (\alpha_n + \beta_n)_{n=1}^{\infty}, \end{aligned} \tag{3.15}$$

where $(c\alpha_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_\infty$ because

$$\begin{aligned} \lim_{m\to\infty} \sup_{n\geq m} \|c\alpha_n - g_m^n(c\alpha_m)\| &= |c| \lim_{m\to\infty} \sup_{n\geq m} \|\alpha_n - g_m^n(\alpha_m)\| \\ &= 0 \end{aligned} \tag{3.16}$$

and $(\alpha_n + \beta_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_\infty$ because

$$\begin{aligned} \lim_{m\to\infty} \sup_{n\geq m} \|\alpha_n + \beta_n - g_m^n(\alpha_m + \beta_m)\| &\leq \lim_{m\to\infty} \sup_{n\geq m} \|\alpha_n - g_m^n(\alpha_m)\| + \lim_{m\to\infty} \sup_{n\geq m} \|\beta_n - g_m^n(\beta_m)\| \\ &= 0. \end{aligned} \tag{3.17}$$

Lemma 3.2.1. The map $\langle , \rangle : \widetilde{\mathcal{H}}_{\infty}^{\times 2} \to \mathbb{R}$ given by

$$\langle (\alpha_n)_{n=1}^{\infty}, (\beta_n)_{n=1}^{\infty} \rangle = \lim_{n\to\infty} \langle \alpha_n, \beta_n \rangle \tag{3.18}$$

is well-defined, symmetric, and bi-linear. Moreover, the induced semi-norm

$$\|(\alpha_n)_{n=1}^{\infty}\| = \sqrt{\langle (\alpha_n)_{n=1}^{\infty}, (\alpha_n)_{n=1}^{\infty} \rangle} \tag{3.19}$$

is equal to $\lim_{n\to\infty} \|\alpha_n\|$.

Proof. First,

$$\lim_{m\to\infty} \sup_{n\geq m} |\|\alpha_n\| - \|\alpha_m\|| \leq \lim_{m\to\infty} \sup_{n\geq m} \|\alpha_n - g_m^n (\alpha_m)\| = 0, \tag{3.20}$$

which implies $(\|\alpha_n\|)_{n=1}^{\infty}$ is Cauchy and so $\lim_{n\to\infty} \|\alpha_n\|$ exists. Non-negativity is immediate. Moreover,

$$\begin{aligned} \|(\alpha_n)_{n=1}^{\infty}\| &= \sqrt{\lim_{n\to\infty} \langle \alpha_n, \alpha_n \rangle} \\ &= \sqrt{\lim_{n\to\infty} \|\alpha_n\|^2} \\ &= \lim_{n\to\infty} \|\alpha_n\|, \quad \text{by continuity.} \end{aligned} \tag{3.21}$$

To see that $\langle , \rangle$ is well-defined, we will show that the sequence $(\langle \alpha_n, \beta_n \rangle)_{n=1}^{\infty}$ is Cauchy. Fix $\varepsilon > 0$ and take k large enough that for $\ell \geq k$,

$$\begin{aligned} \left\|\alpha_\ell - g_k^\ell (\alpha_k)\right\| &< \min\left(1, \frac{\varepsilon}{4\left(\|(\beta_n)_{n=1}^{\infty}\| + 1\right)}\right), \\ \left\|\beta_\ell - g_k^\ell (\beta_k)\right\| &< \min\left(1, \frac{\varepsilon}{4\left(\|(\alpha_n)_{n=1}^{\infty}\| + 1\right)}\right). \end{aligned} \tag{3.22}$$

Observe that the above bounds imply

$$\begin{aligned} |\|\alpha_k\| - \|(\alpha_n)_{n=1}^{\infty}\|| &= \lim_{n\to\infty} |\|\alpha_k\| - \|\alpha_n\|| \\ &\leq \lim_{n\to\infty} \|g_k^n (\alpha_k) - \alpha_n\| \\ &< 1. \end{aligned} \tag{3.23}$$

Likewise, $|\|\beta_k\| - \|(\beta_n)_{n=1}^{\infty}\|| < 1$. Now observe,

$$\begin{aligned} |\langle \alpha_\ell, \beta_\ell \rangle - \langle \alpha_k, \beta_k \rangle| &= \left| \left\langle g_k^\ell(\alpha_k) - \left(g_k^\ell(\alpha_k) - \alpha_\ell \right), g_k^\ell(\beta_k) - \left(g_k^\ell(\beta_k) - \beta_\ell \right) \right\rangle - \langle \alpha_k, \beta_k \rangle \right| \\ &\le \left| \left\langle \left(g_k^\ell(\alpha_k) - \alpha_\ell \right), g_k^\ell(\beta_k) \right\rangle \right| + \left| \left\langle g_k^\ell(\alpha_k), \left(g_k^\ell(\beta_k) - \beta_\ell \right) \right\rangle \right| \\ &\quad + \left| \left\langle \left(g_k^\ell(\alpha_k) - \alpha_\ell \right), \left(g_k^\ell(\beta_k) - \beta_\ell \right) \right\rangle \right| \\ &\le \left\| g_k^\ell(\alpha_k) - \alpha_\ell \right\| \|\beta_k\| + \left\| g_k^\ell(\beta_k) - \beta_\ell \right\| \|\alpha_k\| + \left\| g_k^\ell(\alpha_k) - \alpha_\ell \right\| \left\| g_k^\ell(\beta_k) - \beta_\ell \right\| \\ &< \frac{\varepsilon}{4\left(\|(\beta_n)_{n=1}^{\infty}\| + 1\right)} \left(\|(\beta_n)_{n=1}^{\infty}\| + 1\right) + \frac{\varepsilon}{4\left(\|(\alpha_n)_{n=1}^{\infty}\| + 1\right)} \left(\|(\alpha_n)_{n=1}^{\infty}\| + 1\right) \\ &\quad + \frac{\varepsilon}{4} \\ &< \varepsilon. \end{aligned} \tag{3.24}$$

So, $(\langle \alpha_n, \beta_n \rangle)_{n=1}^{\infty}$ is Cauchy and thus convergent. Since the inner product on each $\mathcal{H}_n$ is bi-linear, so is $\langle , \rangle$. □

Define

$$K = \left\{ (\alpha_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_\infty \ : \ \|(\alpha_n)_{n=1}^{\infty}\| = 0 \right\}, \tag{3.25}$$

a subspace of $\widetilde{\mathcal{H}}_\infty$. Define the quotient vector space

$$\mathcal{H}_\infty = \widetilde{\mathcal{H}}_\infty / K. \tag{3.26}$$

Let $\pi\colon \widetilde{\mathcal{H}}_\infty \to \mathcal{H}_\infty$ be the quotient map. By the construction, $\langle , \rangle$ descends to an inner product $\langle , \rangle_{\mathcal{H}_\infty}$ on $\mathcal{H}_\infty$ where $\langle \pi(-), \pi(-) \rangle_{\mathcal{H}_\infty} = \langle , \rangle$. For the purpose of notation, we will use $\langle , \rangle$ to denote $\langle , \rangle_{\mathcal{H}_\infty}$ and when referring to $\mathcal{H}_\infty$, we will always assume the inner product space $\left(\mathcal{H}_\infty, \langle , \rangle_{\mathcal{H}_\infty}\right)$.

Lemma 3.2.2. $\mathcal{H}_\infty$ is a Hilbert space.

Proof. We need to show that $\mathcal{H}_\infty$ is complete. We will show this by fixing a Cauchy sequence, performing a diagonalization type construction to construct a limit point, and showing that said limit point is in $\mathcal{H}_\infty$.

Let $\left(\alpha^\ell\right)_{\ell=1}^{\infty}$ be a Cauchy sequence of elements in $\mathcal{H}_\infty$. For each ℓ, pick a representative $\left(\alpha_k^\ell\right)_{k=1}^{\infty} \in \pi^{-1}\left(\alpha^\ell\right) \subseteq \widetilde{\mathcal{H}}_\infty$. For each $n \in \mathbb{N}$, let ℓ_n be such that $\ell_n > \ell_{n-1}$ (for $n > 1$) and appeal to $\left(\alpha^\ell\right)_{\ell=1}^{\infty}$ being Cauchy to guarantee that for all $t \ge \ell_n$,

$$\left\| \left(\alpha_k^{\ell_n}\right)_{k=1}^{\infty} - \left(\alpha_k^t\right)_{k=1}^{\infty} \right\| = \left\| \alpha^{\ell_n} - \alpha^t \right\| < \frac{1}{2n}. \tag{3.27}$$

So, for all $m < n$, we know that

$$\left\| \left(\alpha_k^{\ell_n} \right)_{k=1}^{\infty} - \left(\alpha_k^{\ell_m} \right)_{k=1}^{\infty} \right\| < \frac{1}{2m}. \tag{3.28}$$

By the definition of $\|\cdot\|$ on $\widetilde{\mathcal{H}}_\infty$, this equation means that we can find arbitrarily large k such that

$$\left\| \alpha_k^{\ell_n} - \alpha_k^{\ell_m} \right\| < \frac{1}{2m}. \tag{3.29}$$

On the other hand, the definition of $\widetilde{\mathcal{H}}_\infty$ implies that we can find arbitrarily large k such that for $t \geq k$,

$$\left\| \alpha_t^{\ell_n} - g_{k_n}^k \left(\alpha_k^{\ell_n} \right) \right\| < \frac{1}{2n}. \tag{3.30}$$

So, let k_n be large enough that $k_n > k_{n-1}$ (for $n > 1$) and both of the following hold:

1. For $m \leq n$, $\left\| \alpha_{k_n}^{\ell_m} - \alpha_{k_n}^{\ell_n} \right\| < \frac{1}{2m}$.
2. For $k \geq k_n$, $\left\| \alpha_k^{\ell_n} - g_{k_n}^k \left(\alpha_{k_n}^{\ell_n} \right) \right\| < \frac{1}{2n}$.

Since the k_n's form an increasing sequence, we have that for every $k \geq k_1$, k lies in $[k_n, k_{n+1})$ for some n. For each $k \geq 1$, define

$$\beta_k = \begin{cases} 0 & \text{if } k < k_1, \\ \alpha_{k_n}^{\ell_n} & \text{if } k = k_n, \\ g_{k_n}^k \left(\alpha_{k_n}^{\ell_n} \right) & \text{if } k \in [k_n, k_{n+1}). \end{cases} \tag{3.31}$$

We want to show that $(\beta_k)_{k=1}^{\infty} \in \widetilde{\mathcal{H}}_\infty$ and that $\left(\alpha^\ell \right)_{\ell=1}^{\infty} \to \pi \left((\beta_k)_{k=1}^{\infty} \right)$. Pursuant to the first goal, fix $\varepsilon > 0$, let M be large enough that $\frac{1}{M} < \varepsilon$, and fix $m > k_M$. Fix $n \geq m$. We will show that

$$\| \beta_n - g_m^n (\beta_m) \| < \varepsilon. \tag{3.32}$$

Let n', m' be such that $n \in [k_{n'}, k_{n'+1})$, $m \in [k_{m'}, k_{m'+1})$. Note that $n' \geq m' \geq M$. Then,

$$\begin{aligned}
\|\beta_n - g_m^n(\beta_m)\| &= \left\| g_{k_{n'}}^n\left(\alpha_{k_{n'}}^{\ell_{n'}}\right) - g_m^n\left(g_{k_{m'}}^m\left(\alpha_{k_{m'}}^{\ell_{m'}}\right)\right)\right\| \\
&= \left\| g_{k_{n'}}^n\left(\alpha_{k_{n'}}^{\ell_{n'}} - g_m^{k_{n'}}\left(g_{k_{m'}}^m\left(\alpha_{k_{m'}}^{\ell_{m'}}\right)\right)\right)\right\| \\
&= \left\|\alpha_{k_{n'}}^{\ell_{n'}} - g_{k_{m'}}^{k_{n'}}\left(\alpha_{k_{m'}}^{\ell_{m'}}\right)\right\| \\
&\leq \left\|\alpha_{k_{n'}}^{\ell_{n'}} - \alpha_{k_{n'}}^{\ell_{m'}}\right\| + \left\|\alpha_{k_{n'}}^{\ell_{m'}} - g_{k_{m'}}^{k_{n'}}\left(\alpha_{k_{m'}}^{\ell_{m'}}\right)\right\| \qquad (3.33) \\
&< \frac{1}{m'} \\
&\leq \frac{1}{M} \\
&< \varepsilon.
\end{aligned}$$

So, $\lim_{m\to\infty}\sup_{n\geq m}\|\beta_n - g_m^n(\beta_m)\| = 0$ and thus $(\beta_k)_{k=1}^\infty \in \widetilde{\mathcal{H}}_\infty$. Next we want to show that $\left(\alpha^\ell\right)_{\ell=1}^\infty$ converges to $\pi\left((\beta_k)_{k=1}^\infty\right)$ which means showing

$$0 = \lim_{\ell\to\infty}\left\|\alpha^\ell - \pi\left((\beta_k)_{k=1}^\infty\right)\right\| = \lim_{\ell\to\infty}\lim_{k\to\infty}\left\|\alpha_k^\ell - \beta_k\right\|. \qquad (3.34)$$

We will accomplish this by showing that the sub-sequence $\left(\alpha^{\ell_n}\right)_{n=1}^\infty$ is convergent to $\pi\left((\beta_k)_{k=1}^\infty\right)$. Since $\left(\alpha^\ell\right)_{\ell=1}^\infty$ is Cauchy, this is sufficient to show convergence to $\pi\left((\beta_k)_{k=1}^\infty\right)$.

Fix $n \in \mathbb{N}$. Let $k \geq k_n$ and let $n' \in \mathbb{N}$ be such that $k \in [k_{n'}, k_{n'+1})$. Notice that $n' \geq n$. Then,

$$\begin{aligned}
\left\|\alpha_k^{\ell_n} - \beta_k\right\| &= \left\|\alpha_k^{\ell_n} - g_{k_{n'}}^k\left(\alpha_{k_{n'}}^{\ell_{n'}}\right)\right\| \\
&\leq \left\|\alpha_k^{\ell_n} - g_{k_n}^k\left(\alpha_{k_n}^{\ell_n}\right)\right\| + \left\|g_{k_n}^{k_{n'}}\left(\alpha_{k_n}^{\ell_n}\right) - \alpha_{k_{n'}}^{\ell_n}\right\| + \left\|\alpha_{k_{n'}}^{\ell_n} - \alpha_{k_{n'}}^{\ell_{n'}}\right\| \qquad (3.35) \\
&< \frac{3}{2n}.
\end{aligned}$$

This shows that

$$\lim_{n\to\infty}\left\|\left(\alpha_k^{\ell_n}\right)_{k=1}^\infty - (\beta_k)_{k=1}^\infty\right\| \leq \lim_{n\to\infty}\frac{3}{2n} = 0. \qquad (3.36)$$

So, the sub-sequence $\left(\left(\alpha_k^{\ell_n}\right)_{k=1}^\infty\right)_{n=1}^\infty$ is convergent to $(\beta_k)_{k=1}^\infty$ and we are done. □

The construction of $\mathcal{H}_\infty$ is only useful if, for all n, there exists an injection $g_n \colon \mathcal{H}_n \to \mathcal{H}_\infty$ which is an isometry. On top of this, we will want that g_n is a δ_n-isometry for $\delta_n \to 0$. The following Lemma addresses the first requirement:

Lemma 3.2.3. For each $n \in \mathbb{N}$, define $\widetilde{g}_n \colon \mathcal{H}_n \to \widetilde{\mathcal{H}}_\infty$ on $\alpha \in \mathcal{H}_n$ as follows:

$$(\widetilde{g}_n(\alpha))_k = \begin{cases} 0 & \text{if } k < n, \\ g_n^k(\alpha) & \text{if } k \geq n. \end{cases} \tag{3.37}$$

Define $g_n := \pi \circ \widetilde{g}_n$. Then, the following are true:

1. g_n is an isometry.
2. for $m \leq n$, $g_m = g_n \circ g_m^n$.
3. The sequence $(\operatorname{im}(g_n))_{n=1}^\infty$ is a filtration.
4. $\bigcup_{n \in \mathbb{N}} \operatorname{im}(g_n) \subseteq \mathcal{H}_\infty$ is dense.

Proof. We must first show that, given $\alpha \in \mathcal{H}_n$, $\widetilde{g}_n(\alpha) = \left(g_n^k(\alpha)\right)_{k=1}^\infty$ defines an element in $\widetilde{\mathcal{H}}_\infty$. To see this, observe that

$$\lim_{k \to \infty} \sup_{\ell \geq k} \left\| g_n^\ell(\alpha) - g_k^\ell\left(g_n^k(\alpha)\right) \right\| = \lim_{k \to \infty} \sup_{\ell \geq k} \left\| g_n^\ell(\alpha) - g_n^\ell(\alpha) \right\| = 0. \tag{3.38}$$

So, $\widetilde{g}_n(\alpha) \in \widetilde{\mathcal{H}}_\infty$. To verify that g_n is an isometry, consider $\alpha, \beta \in \mathcal{H}_n$,

$$\begin{aligned} \langle g_n(\alpha), g_n(\beta) \rangle &= \langle \widetilde{g}_n(\alpha), \widetilde{g}_n(\beta) \rangle \\ &= \lim_{k \to \infty} \left\langle g_n^k(\alpha), g_n^k(\beta) \right\rangle \\ &= \langle \alpha, \beta \rangle. \end{aligned} \tag{3.39}$$

This shows that each g_n is an isometry. Since $g_m^k = g_n^k \circ g_m^n$, we have $g_m = g_n \circ g_m^n$ which proves that $\operatorname{im}(g_m) \subseteq \operatorname{im}(g_n)$ and so the sequence $(\operatorname{im}(g_n))_{n=1}^\infty$ is a filtration.

To see the final claim, let $\alpha \in \mathcal{H}_\infty$, fix $(\alpha_n)_{n=1}^\infty \in \pi^{-1}(\alpha)$, and consider the sequence $(g_n(\alpha_n))_{n=1}^\infty$. We aim to show that $(g_n(\alpha_n))_{n=1}^\infty$ converges to α. Fix $\varepsilon > 0$ and take n large enough that for all $k > n$, $\left\|\alpha_k - g_n^k(\alpha_n)\right\| < \varepsilon$. Then,

$$\begin{aligned} \|\alpha - g_n(\alpha_n)\| &= \lim_{k \to \infty} \|(\alpha_n)_{n=1}^\infty - \widetilde{g}_n(\alpha)\| \\ &= \lim_{k \to \infty} \left\|\alpha_k - g_n^k(\alpha_n)\right\| \\ &< \varepsilon. \end{aligned} \tag{3.40}$$

Thus, $\overline{\bigcup_{n \in \mathbb{N}} \operatorname{im}(g_n)} = \mathcal{H}_\infty$ and we are done. ◻

3.3 Construction of the Self-Adjoint Operator

With the previous section, we have shown that $\mathcal{H}_\infty$ is a separable Hilbert space equipped with isometries $g_n \colon \mathcal{H}_n \to \mathcal{H}_\infty$. To complete the construction of a limiting package, we need to define a self-adjoint non-negative operator $\Delta_\infty \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ so that each $g_n \colon (\mathcal{H}_n, \Delta_n)_{n=1}^\infty \to (\mathcal{H}_\infty, \Delta_\infty)$ is a δ_n-isometry for each n and such that $\delta_n \to 0$. By Proposition 2.1.14, the spectrum $\sigma(\Delta_\infty)$ will consist of countably many eigenvalues with no finite accumulation points. An additional consideration is that $\sigma(\Delta_\infty)$ could be unbounded and so Δ_∞ will be defined only on a dense subspace of $\mathcal{H}_\infty$. The way we deal with this is through the construction of an operator G_∞ on $\mathcal{H}_\infty$ using the sequence of operators $G_n := (\Delta_n + 1)^{-1}$ for each $n \in \mathbb{N}$. We will see that G_∞ is bounded, everywhere defined, self-adjoint, invertible, $\operatorname{im}(G_\infty)$ is dense, and then define Δ_∞ on $\operatorname{im}(G_\infty)$ such that $G_\infty = (\Delta_\infty + 1)^{-1}$. We will then argue that Δ_∞ is self-adjoint, non-negative, and such that the isometries g_n are δ_n-isometries for $\delta_n \to 0$ as $n \to \infty$.

Lemma 3.3.1. The map $\widetilde{G}_\infty \colon \widetilde{\mathcal{H}}_\infty \to \widetilde{\mathcal{H}}_\infty$ defined via

$$(\alpha_n)_{n=1}^\infty \mapsto (G_n(\alpha_n))_{n=1}^\infty \tag{3.41}$$

is well-defined. Moreover, $\widetilde{G}_\infty$ induces a bounded (equiv. continuous) map $G_\infty \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ such that $G_\infty \circ \pi = \pi \circ \widetilde{G}_\infty$.

Proof. We must first show that for all $(\alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty$, $(G_n(\alpha_n))_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty$. Fix $\varepsilon > 0$ and let m be large enough such that for all $n \geq m$, $\|\alpha_n - g_m^n(\alpha_m)\| < \frac{\varepsilon}{2}$ and $(\mathcal{H}_m, \Delta_m)$ is an $\frac{\varepsilon}{2}$-retract of $(\mathcal{H}_n, \Delta_n)$. Then,

$$\begin{aligned}
\|G_n(\alpha_n) - g_m^n(G_m(\alpha_m))\| &= \left\|(\Delta_n + 1)^{-1}(\alpha_n) - g_m^n\left((\Delta_m + 1)^{-1}(\alpha_m)\right)\right\| \\
&\leq \left\|(\Delta_n + 1)^{-1} g_m^n(\alpha_m) - g_m^n\left((\Delta_m + 1)^{-1}(\alpha_m)\right)\right\| \\
&\quad + \left\|(\Delta_n + 1)^{-1} g_m^n(\alpha_m) - (\Delta_n + 1)^{-1}(\alpha_n)\right\| \\
&< \frac{\varepsilon}{2} + \|g_m^n(\alpha_m) - \alpha_n\| \\
&< \varepsilon.
\end{aligned} \tag{3.42}$$

So, $\widetilde{G}_\infty((\alpha_n)_{n=1}^\infty) = (G_n(\alpha_n))_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty$. Let $(\alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty$ and observe:

$$\begin{aligned}
\left\|\widetilde{G}_\infty((\alpha_n)_{n=1}^\infty)\right\| &= \lim_{n\to\infty} \|G_n(\alpha_n)\| \\
&\leq \lim_{n\to\infty} \|\alpha_n\| \\
&= \|(\alpha_n)_{n=1}^\infty\|.
\end{aligned} \tag{3.43}$$

In particular, if $(\alpha_n)_{n=1}^{\infty} \in K$, then $\left\| \widetilde{G}_{\infty} \left((\alpha_n)_{n=1}^{\infty} \right) \right\| = 0$ and so $\widetilde{G}_{\infty} \left((\alpha_n)_{n=1}^{\infty} \right) \in K$. Thus, $\widetilde{G}_{\infty}$ induces a map $G_{\infty} \colon \mathcal{H}_{\infty} \to \mathcal{H}_{\infty}$ such that $G_{\infty} \circ \pi = \pi \circ \widetilde{G}_{\infty}$. Moreover, we immediately see that G_{∞} is bounded: For $(\alpha_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_{\infty}$,

$$\left\| G_{\infty} \left(\pi \left((\alpha_n)_{n=1}^{\infty} \right) \right) \right\| = \left\| \widetilde{G}_{\infty} \left((\alpha_n)_{n=1}^{\infty} \right) \right\| \leq \left\| (\alpha_n)_{n=1}^{\infty} \right\| . \tag{3.44}$$

□

The next Lemma shows that G_{∞} is injective, which is crucial for our construction of Δ_{∞}, and uses Equation (2.11) in the proof. This is the only place in our construction that requires satisfaction of Equation (2.11).

Lemma 3.3.2. G_{∞} is injective.

Proof. G_{∞} is injective if and only if $\widetilde{G}_{\infty}^{-1} (K) \subseteq K$. So, let $(\alpha_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_{\infty}$ and suppose $\widetilde{G}_{\infty} \left((\alpha_n)_{n=1}^{\infty} \right) \in K$. Then,

$$\lim_{n \to \infty} \| G_n(\alpha_n) \| = 0. \tag{3.45}$$

We will show that $(\alpha_k)_{k=1}^{\infty} \in K$ by showing that $\|(\alpha_k)_{k=1}^{\infty}\| = 0$. Fix $\varepsilon > 0$ and take m large enough such that for all $n \geq m$,

$$\| \alpha_n - g_m^n(\alpha_m) \| < \frac{\varepsilon}{2}. \tag{3.46}$$

Let $B > 0$ large enough so that, by Equation (2.11),

$$\lim_{n \to \infty} \left\| P_{[B,\infty)} g_m^n \left(\alpha_m \right) \right\| < \frac{\varepsilon}{2}. \tag{3.47}$$

Now,

$$\begin{aligned}
\|(\alpha_k)_{k=1}^{\infty}\| &= \lim_{n \to \infty} \|\alpha_n\| \\
&\leq \lim_{n \to \infty} \left\| P_{[0,B)} \left(\alpha_n \right) \right\| + \left\| P_{[B,\infty)} \left(\alpha_n \right) \right\| \\
&\leq \lim_{n \to \infty} \left\| P_{[0,B)} \left(\alpha_n \right) \right\| + \left\| P_{[B,\infty)} \left(\alpha_n - g_m^n \left(\alpha_m \right) \right) \right\| + \left\| P_{[B,\infty)} \left(g_m^n \left(\alpha_m \right) \right) \right\| \\
&\leq \lim_{n \to \infty} \left\| P_{[0,B)} \left(\alpha_n \right) \right\| + \left\| \alpha_n - g_m^n \left(\alpha_m \right) \right\| + \left\| P_{[B,\infty)} \left(g_m^n \left(\alpha_m \right) \right) \right\| \\
&< \lim_{n \to \infty} \left\| P_{[0,B)} \left(\alpha_n \right) \right\| + \varepsilon \\
&< \left((B+1) \lim_{n \to \infty} \left\| G_n P_{[0,B)}(\alpha_n) \right\| \right) + \varepsilon \\
&\leq \left((B+1) \lim_{n \to \infty} \| G_n(\alpha_n) \| \right) + \varepsilon \\
&= \varepsilon.
\end{aligned} \tag{3.48}$$

We have shown that $\|(\alpha_k)_{k=1}^{\infty}\| < \varepsilon$ for all $\varepsilon > 0$ and so $\|(\alpha_k)_{k=1}^{\infty}\| = 0$. Thus, G_∞ is injective. □

Since $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is Cauchy, we can take $\delta_n > 0$ for each $n \in \mathbb{N}$ to be such that $(\mathcal{H}_n, \Delta_n)$ is a δ_n-retract of $(\mathcal{H}_k, \Delta_k)$ for each $k \geq n$ and such that $\lim_{n\to\infty} \delta_n = 0$.

Lemma 3.3.3.

$$\begin{aligned} \|(1 - g_n g_n^*)\, G_\infty\| &\leq \delta_n, \\ \|g_n G_n - G_\infty g_n\| &\leq \delta_n. \end{aligned} \tag{3.49}$$

Proof. Lemma 3.2.3 shows that each g_n is an isometry which implies continuity of g_n and g_n^*. On the other hand, G_∞ is continuous by Lemma 3.3.1 and so $(1 - g_n g_n^*)\, G_\infty$ is continuous. Lemma 3.2.3 also tells us that $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n) \subseteq \widetilde{\mathcal{H}}_\infty$ is dense and so it suffices to check each norm on unit norm elements of the form $g_m(\beta)$ for $\beta \in \mathcal{H}_m$. For such β,

$$\begin{aligned} \|(1 - g_n g_n^*)\, G_\infty g_m(\beta)\| &= \inf_{\alpha\in\mathcal{H}_n} \|G_\infty(g_m(\beta)) - g_n(\alpha)\|, \quad \text{by Lemma 2.1.4} \\ &\leq \|G_\infty(g_m(\beta)) - g_n G_n g_m^n(\beta)\| \\ &= \lim_{k\to\infty} \left\| G_k g_m^k(\beta) - g_n^k G_n g_m^n(\beta) \right\| \\ &= \lim_{k\to\infty} \left\| \left((\Delta_k + 1)^{-1} g_n^k - g_n^k (\Delta_n + 1)^{-1} \right) g_m^n(\beta) \right\| \\ &\leq \delta_n. \end{aligned} \tag{3.50}$$

For the second bound, suppose now that $\beta \in \mathcal{H}_n$ is a unit. Observe,

$$\begin{aligned} \|(g_n G_n - G_\infty g_n)(\beta)\| &= \left\| \left(\widetilde{g}_n G_n - \widetilde{G}_\infty \widetilde{g}_n \right)(\beta) \right\| \\ &\leq \lim_{k\to\infty} \left\| \left(g_n^k (\Delta_n + 1)^{-1} - (\Delta_k + 1)^{-1} g_n^k \right)(\beta) \right\| \\ &\leq \delta_n. \end{aligned} \tag{3.51}$$

□

Lemma 3.3.4. G_∞ is both self-adjoint and compact.

Proof. The domain of G_∞ is all of $\mathcal{H}_\infty$ and so it suffices to demonstrate that G_∞ is symmetric in order to show self-adjointness. Let $\alpha, \beta \in \mathcal{H}_\infty$. Let $(\alpha_n)_{n=1}^{\infty} \in \pi^{-1}(\alpha)$,

$(\beta_n)_{n=1}^{\infty} \in \pi^{-1}(\beta)$. Then,

$$\begin{aligned} \langle G_\infty(\alpha), \beta\rangle &= \left\langle \widetilde{G}_\infty\left((\alpha_n)_{n=1}^{\infty}\right), (\beta_n)_{n=1}^{\infty}\right\rangle \\ &= \lim_{n\to\infty} \langle G_n(\alpha_n), \beta_n\rangle \\ &= \lim_{n\to\infty} \langle \alpha_n, G_n(\beta_n)\rangle \\ &= \left\langle (\alpha_n)_{n=1}^{\infty}, \widetilde{G}_\infty\left((\beta_n)_{n=1}^{\infty}\right)\right\rangle \\ &= \langle \alpha, G_\infty(\beta)\rangle . \end{aligned} \tag{3.52}$$

Thus G_∞ is self-adjoint. In light of Lemma 3.3.3, the proof of Proposition 2.1.14 shows that the sequence $(g_n G_n g_n^*)_{n=1}^{\infty}$ converges to G_∞, implying compactness. ☐

Lemma 3.3.5. The map $\widetilde{\Delta}_\infty \colon \operatorname{im}\left(\widetilde{G}_\infty\right) \to \widetilde{\mathcal{H}}_\infty$ defined on $(\alpha_n)_{n=1}^{\infty} \in \operatorname{im}\left(\widetilde{G}_\infty\right)$ via

$$\widetilde{\Delta}_\infty\left((\alpha_n)_{n=1}^{\infty}\right) = (\Delta_n \alpha_n)_{n=1}^{\infty} . \tag{3.53}$$

is well defined and induces a map $\Delta_\infty \colon \operatorname{im}(G_\infty) \to \mathcal{H}_\infty$. Moreover, $\Delta_\infty = \left(G_\infty^{-1} - 1\right)$ and is a non-negative self-adjoint operator.

Proof. To see that $\widetilde{\Delta}_\infty$ is well-defined, let $(\alpha_n)_{n=1}^{\infty} \in \widetilde{\mathcal{H}}_\infty$ and observe:

$$\begin{aligned} &\lim_{m\to\infty} \sup_{n\geq m} \left\|\Delta_n G_n(\alpha_n) - g_m^n\left(\Delta_m G_m(\alpha_m)\right)\right\| \\ &\qquad \leq \lim_{m\to\infty} \sup_{n\geq m} \left(\left\|\alpha_n - g_m^n(\alpha_m)\right\| + \left\|G_n(\alpha_n) - g_m^n\left(G_m(\alpha_m)\right)\right\|\right) \\ &\qquad = 0. \end{aligned} \tag{3.54}$$

The last line is due to $(\alpha_n)_{n=1}^{\infty}, \widetilde{G}_\infty\left((\alpha_n)_{n=1}^{\infty}\right) \in \widetilde{\mathcal{H}}_\infty$. This shows that $\widetilde{\Delta}_\infty \widetilde{G}_\infty\left((\alpha_n)_{n=1}^{\infty}\right) \in \widetilde{\mathcal{H}}_\infty$. Now,

$$\begin{aligned} \left(\widetilde{\Delta}_\infty + 1\right) \widetilde{G}_\infty (\alpha_n)_{n=1}^{\infty} &= \left((\Delta_n + 1)(\Delta_n + 1)^{-1}(\alpha_n)\right)_{n=1}^{\infty} \\ &= (\alpha_n)_{n=1}^{\infty} . \end{aligned} \tag{3.55}$$

So, $\widetilde{\Delta}_\infty + 1$ is the one sided inverse of $\widetilde{G}_\infty$. By Lemma 3.3.2, $\widetilde{G}_\infty\left((\alpha_n)_{n=1}^{\infty}\right) \in K$ if and only if $(\alpha_n)_{n=1}^{\infty} \in K$ and so $\left(\widetilde{\Delta}_\infty + 1\right)\left(K \cap \operatorname{im}\left(\widetilde{G}_\infty\right)\right) \subseteq K$. By the reverse triangle inequality, $\widetilde{\Delta}_\infty\left(K \cap \operatorname{im}\left(\widetilde{G}_\infty\right)\right) \subseteq K$ and so $\widetilde{\Delta}_\infty$ induces a map $\Delta_\infty \colon \operatorname{im}(G_\infty) \to \mathcal{H}_\infty$ such that $(\Delta_\infty + 1) G_\infty = 1$.

By Lemma 3.3.4, G_∞ is self-adjoint and compact. Corollaries 1.6.7 and 8.1.3 of [13] then imply that there exists a countable orthonormal eigenbasis of $\mathcal{H}_\infty$ with respect to G_∞. Since G_∞ is injective, $\operatorname{im}(G_\infty)$ contains the eigenbasis and so $\operatorname{im}(G_\infty)$ is a dense subset of $\mathcal{H}_\infty$. In particular, Δ_∞ is densely defined. By Proposition 8.2, Section A of [48], $\Delta_\infty + 1$ is self-adjoint with domain $\operatorname{im}(G_\infty)$, and so Δ_∞ is self-adjoint with domain $\operatorname{im}(G_\infty)$. Non-negativity follows from the definitions of $\widetilde{\Delta}_\infty$ and the inner product. □

Theorem 3.3.6. The sequence $(\mathcal{H}_n, \Delta_n)_{n=1}^\infty$ is convergent to $(\mathcal{H}_\infty, \Delta_\infty)$ and this is witnessed by the isometries g_n.

Proof. Lemma 3.2.3 tells us that each g_n is an isometry and so we need to additionally verify that each is a δ_n-isometry. By Lemma 3.3.5, $G_\infty = (\Delta_\infty + 1)^{-1}$. Applying Lemma 3.3.3,

$$\begin{aligned} \left\| (1 - g_n g_n^*) (\Delta_\infty + 1)^{-1} \right\| &\leq \delta_n, \\ \left\| g_n (\Delta_n + 1)^{-1} - (\Delta_\infty + 1)^{-1} g_n \right\| &\leq \delta_n. \end{aligned} \tag{3.56}$$

□

Observe that the domain of Δ_∞ has been described implicitly and is not clearly witnessed by the approximation scheme $(\mathcal{H}_n, \Delta_n)_{n=1}^\infty$. We remedy this by explicitly describing $\operatorname{dom}\left(\widetilde{\Delta}_\infty\right)$ as a subspace of $\widetilde{\mathcal{H}}_\infty$.

Proposition 3.3.7. Consider the subspace

$$\begin{aligned} \mathcal{D} &:= \left\{ (\alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty \,:\, (\Delta_n \alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty \right\} \\ &= \left\{ (\alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty \,:\, \lim_{m\to\infty} \sup_{n\geq m} \left\| \Delta_n \alpha_n - g_m^n (\Delta_m \alpha_m) \right\| = 0 \right\} \\ &\subseteq \widetilde{\mathcal{H}}_\infty. \end{aligned} \tag{3.57}$$

Then,

$$\mathcal{D} = \operatorname{dom}\left(\widetilde{\Delta}_\infty\right) = \operatorname{im}\left(\widetilde{G}_\infty\right). \tag{3.58}$$

Proof. Let $(\alpha_n)_{n=1}^\infty \in \operatorname{dom}\left(\widetilde{\Delta}_\infty\right)$, then $\widetilde{\Delta}_\infty\left((\alpha_n)_{n=1}^\infty\right) = (\Delta_n \alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty$ which implies

$$\lim_{m\to\infty} \sup_{n\geq m} \left\| \Delta_n \alpha_n - g_m^n (\Delta_m \alpha_m) \right\| = 0. \tag{3.59}$$

and so $(\alpha_n)_{n=1}^\infty \in \mathcal{D}$. Now fix $(\alpha_n)_{n=1}^\infty \in \mathcal{D}$. Then,

$$\widetilde{\mathcal{H}}_\infty \ni (\Delta_n \alpha_n)_{n=1}^\infty + (\alpha_n)_{n=1}^\infty = \left(\widetilde{\Delta}_\infty + 1\right) (\alpha_n)_{n=1}^\infty. \tag{3.60}$$

So,

$$\widetilde{G}_\infty \left(\widetilde{\Delta}_\infty + 1\right) (\alpha_n)_{n=1}^\infty = (\alpha_n)_{n=1}^\infty \in \operatorname{im}\left(\widetilde{G}_\infty\right). \tag{3.61}$$

□

3.4 Uniqueness of the Construction

In this section, we want to explore the uniqueness of our construction $(\mathcal{H}_\infty, \Delta_\infty)$. We begin with Lemma 3.4.1 which shows that if we have a Hilbert space $\mathcal{H}'_\infty$ along with isometries for all n, $g'_n \colon \mathcal{H}_n \to \mathcal{H}'_\infty$ such that $g'_n g^n_m = g'_m$, then there exists an isometry $\varphi \colon \mathcal{H}_\infty \to \mathcal{H}'_\infty$. i.e. $\mathcal{H}_\infty$ is a colimit. Theorem 3.4.2 will build off this result by showing that if $\mathcal{H}'_\infty$ fits into a package $(\mathcal{H}'_\infty, \Delta'_\infty)$ such that $(\mathcal{H}_n, \Delta_n)_{n=1}^\infty$ converges to $(\mathcal{H}'_\infty, \Delta'_\infty)$, then φ is unitary and intertwining.

Lemma 3.4.1. There exists a continuous linear map $\varphi \colon \mathcal{H}_\infty \to \mathcal{H}'_\infty$ such that for $\alpha_n \in \mathcal{H}_n$,

$$\varphi(g_n(\alpha_n)) = g'_n(\alpha_n). \tag{3.62}$$

Moreover, φ is an isometry.

Proof. Define φ on $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n)$ via Equation (3.62). To see that φ is well-defined, let $\alpha_m \in \mathcal{H}_m$ and observe that

$$\begin{aligned} g'_m(\alpha_m) &= \varphi(g_m(\alpha_m)) \\ &= \varphi(g_n(g^n_m(\alpha_m))) \\ &= g'_n(g^n_m(\alpha_m)) \\ &= g'_m(\alpha_m). \end{aligned} \tag{3.63}$$

To see that φ with domain $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n)$ is an isometry, observe that for $g_n(\alpha), g_m(\beta)$ with $n \geq m$,

$$\begin{aligned} \langle g_n(\alpha), g_m(\beta)\rangle &= \langle \alpha, g^n_m(\beta)\rangle \\ &= \left\langle g'_n(\alpha), g'_m(\beta)\right\rangle \\ &= \langle \varphi(g_n(\alpha)), \varphi(g_m(\beta))\rangle. \end{aligned} \tag{3.64}$$

This implies that φ is an isometry. Lemma 3.2.3 tells us that $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n)$ is dense and so $\|\varphi\| = 1$ implies that we may continuously extend φ to an isometry with domain $\mathcal{H}_\infty$. □

Lemma 3.4.1 implies $\mathcal{H}_\infty$ is a colimit. Now assume that $\mathcal{H}'_\infty$ fits into a package $(\mathcal{H}'_\infty, \Delta'_\infty)$ such that $(\mathcal{H}_n, \Delta_n)_{n=1}^\infty$ converges to $(\mathcal{H}'_\infty, \Delta'_\infty)$.

Theorem 3.4.2. The map φ is a 0-isometry and so $(\mathcal{H}_\infty, \Delta_\infty)$ is a 0-retract of $(\mathcal{H}'_\infty, \Delta'_\infty)$. Consequently, φ is unitary and intertwines $(\Delta'_\infty + 1)^{-1}$ and $(\Delta_\infty + 1)^{-1}$.

Proof. We must show that φ is a 0-isometry. By Lemma 3.4.1, φ is an isometry and so we just need to check that

$$
\begin{aligned}
&\left\|\varphi\left(\Delta_{\infty}+1\right)^{-1}-\left(\Delta_{\infty}^{\prime}+1\right)^{-1} \varphi\right\|=0 \\
&\left\|\left(1-\varphi \varphi^{*}\right)\left(\Delta_{\infty}^{\prime}+1\right)^{-1}\right\|=0 .
\end{aligned} \tag{3.65}
$$

It is sufficient to check both equations on $\bigcup_{n \in \mathbb{N}} \operatorname{im} g_n$ due to continuity. Let $\alpha_m \in \mathcal{H}_m$. Then,

$$
\begin{aligned}
&\left\|\left(\varphi\left(\Delta_{\infty}+1\right)^{-1}-\left(\Delta_{\infty}^{\prime}+1\right)^{-1} \varphi\right) g_m\left(\alpha_m\right)\right\| \\
&\quad=\lim_{n \rightarrow \infty}\left\|\left(\varphi\left(\Delta_{\infty}+1\right)^{-1}-\left(\Delta_{\infty}^{\prime}+1\right)^{-1} \varphi\right) g_n g_m^n\left(\alpha_m\right)\right\| \\
&\quad=\lim_{n \rightarrow \infty}\left\|\left(\varphi\left(\Delta_{\infty}+1\right)^{-1} g_n-\left(\Delta_{\infty}^{\prime}+1\right)^{-1} g_n^{\prime}\right) g_m^n\left(\alpha_m\right)\right\| \\
&\quad\leq \lim_{n \rightarrow \infty}\left\|\left(g_n^{\prime}\left(\Delta_n+1\right)^{-1}-\left(\Delta_{\infty}^{\prime}+1\right)^{-1} g_n^{\prime}\right) g_m^n\left(\alpha_m\right)\right\| \\
&\qquad+\left\|\varphi\left(\left(\Delta_{\infty}+1\right)^{-1} g_n-g_n\left(\Delta_n+1\right)^{-1}\right) g_m^n\left(\alpha_m\right)\right\| \\
&\quad\leq \lim_{n \rightarrow \infty}\left(\delta_n^{\prime}+\delta_n\right)\left\|\alpha_m\right\|, \text { since }\|\varphi\|=1 \\
&\quad=0 .
\end{aligned} \tag{3.66}
$$

For the second condition, observe,

$$
\begin{aligned}
\left\|\left(1-\varphi \varphi^{*}\right)\left(\Delta_{\infty}^{\prime}+1\right)^{-1} g_m^{\prime}(\alpha_m)\right\| &=\lim_{n \rightarrow \infty}\left\|\left(1-\varphi \varphi^{*}\right)\left(\Delta_{\infty}^{\prime}+1\right)^{-1} g_n^{\prime} g_m^n(\alpha_m)\right\| \\
&=\lim_{n \rightarrow \infty}\left\|\left(1-\varphi \varphi^{*}\right) g_n^{\prime}\left(\Delta_n+1\right)^{-1} g_m^n(\alpha_m)\right\| \\
&\quad+\left\|\left(1-\varphi \varphi^{*}\right)\left(\left(\Delta_{\infty}^{\prime}+1\right)^{-1} g_n^{\prime}-g_n^{\prime}\left(\Delta_n+1\right)^{-1}\right) g_m^n(\alpha_m)\right\| \\
&\leq \lim_{n \rightarrow \infty}\left\|\left(1-\varphi \varphi^{*}\right) g_n^{\prime}\left(\Delta_n+1\right)^{-1} g_m^n(\alpha_m)\right\|+\delta_n^{\prime}\left\|\alpha_m\right\| \\
&=\lim_{n \rightarrow \infty}\left\|\left(1-\varphi \varphi^{*}\right) \varphi g_n\left(\Delta_n+1\right)^{-1} g_m^n(\alpha_m)\right\| \\
&=0 .
\end{aligned} \tag{3.67}
$$

By continuity, we have that $\left(1-\varphi \varphi^{*}\right)\left(\Delta_{\infty}^{\prime}+1\right)^{-1}$ vanishes on the closure

$$
\overline{\bigcup_{n \in \mathbb{N}} \operatorname{im}\left(g_n^{\prime}\right)} \subseteq \mathcal{H}_{\infty}^{\prime} . \tag{3.68}
$$

Suppose $\alpha \in \left(\overline{\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n')}\right)^{\perp}$. Then

$$\begin{aligned}
0 &= \lim_{n\to\infty} \delta_n' \\
&= \lim_{n\to\infty} \left\| \left(1 - g_n' {g_n'}^*\right) \left(\Delta_\infty' + 1\right)^{-1} \alpha \right\| \\
&\leq \lim_{n\to\infty} \left\| \left(\left(\Delta_\infty' + 1\right)^{-1} - g_n' \left(\Delta_n + 1\right)^{-1} {g_n'}^* \right) \alpha \right\| \\
&\quad + \left\| \left(\left(\Delta_n + 1\right)^{-1} {g_n'}^* - {g_n'}^* \left(\Delta_\infty' + 1\right)^{-1} \right) \alpha \right\| \\
&\leq \lim_{n\to\infty} \left\| \left(\Delta_\infty' + 1\right)^{-1} \left(1 - g_n' {g_n'}^*\right) \alpha \right\| \\
&\quad + \left\| \left(\left(\Delta_\infty' + 1\right)^{-1} g_n' - g_n' \left(\Delta_n + 1\right)^{-1} \right) {g_n'}^*(\alpha) \right\| \\
&\quad + \left\| \left(\left(\Delta_n + 1\right)^{-1} {g_n'}^* - {g_n'}^* \left(\Delta_\infty' + 1\right)^{-1} \right) \alpha \right\| \\
&= \lim_{n\to\infty} \left\| \left(\Delta_\infty' + 1\right)^{-1} \left(1 - g_n' {g_n'}^*\right) \alpha \right\| \\
&= \lim_{n\to\infty} \left\| \left(\Delta_\infty' + 1\right)^{-1} \alpha \right\| \\
&= \left\| \left(\Delta_\infty' + 1\right)^{-1} \alpha \right\|.
\end{aligned} \tag{3.69}$$

This implies that $\alpha \in \ker\left(\Delta_\infty' + 1\right)^{-1} = 0$ and so $\alpha = 0$. Thus,

$$\overline{\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n')} = \mathcal{H}_\infty' \tag{3.70}$$

and so our equations hold. By Lemma 2.1.2, φ is unitary. □

Chapter 4

Properties of Limit Spectra

4.1 Properties of the Limit Spectrum

For this section, we will assume that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ is a Cauchy sequence and $(\mathcal{H}_\infty, \Delta_\infty)$ is the limit point constructed in Chapter 3. We begin by stating a result whose idea is based on Proposition 4.3.1 from [39]. Due to an error in the proof of this result that we discovered when preparing this thesis, Proposition 4.3.1 is false as stated but the following represents a statement that can be salvaged from it. The proof is written under Theorem A.0.2 in Appendix A.

Theorem 4.1.1. For each finite closed interval I with

$$\partial_+ I \cap \sigma(\Delta_\infty) = \emptyset, \tag{4.1}$$

there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dim \operatorname{im}(P_{\Delta_n, I}) = \dim \operatorname{im}(P_{\Delta_\infty, I}). \tag{4.2}$$

The remainder of this section is based on ideas from the statement and proof of Theorem 4.3.5 of [39]. The proof presented in [39] relies on Proposition 4.3.1 of [39] and so we exhibit similar proofs that use Theorem 4.1.1 instead.

Definition 4.1.2. Define $\mu \colon \overline{\mathbb{R}_+} \to \mathbb{R}_+$ via

$$\mu(\lambda) := \begin{cases} \lim\limits_{n\to\infty} \dim \operatorname{im}\left(P_{\Delta_n, \overline{B_{\frac{d(\lambda, \sigma(\Delta_\infty)\setminus\{\lambda\})}{2}}}}\right) & \text{if } \lambda \neq \infty, \\ 0 & \text{if } \lambda = \infty. \end{cases} \tag{4.3}$$

By Theorem 4.1.1, since each $\mathcal{H}_n$ is finite dimensional, μ is finite and well-defined.

Theorem 4.1.3. The multiplicity of each $\lambda \in \sigma(\Delta_\infty)$ is equal to $\mu(\lambda)$.

Proof. This follows from direct application of Theorem 4.1.1. $\square$

As a consequence of Proposition 2.1.14, we can enumerate $\sigma(\Delta_\infty)$ as $(\lambda_j)_{j=1}^\infty$ where $\lambda_j \leq \lambda_{j+1}$ and elements are repeated according to multiplicity. i.e. λ_j appears $\mu(\lambda_j)$-times in the enumeration. Since each $\mathcal{H}_n$ is finite dimensional, we can similarly enumerate:

$$\sigma(\Delta_n) = \left(\lambda_j^{(n)}\right)_{j=1}^{\dim \mathcal{H}_n}. \tag{4.4}$$

Corollary 4.1.4. For each j,

$$\lim_{n\to\infty} \lambda_j^{(n)} = \lambda_j. \tag{4.5}$$

where $\lambda_j^{(n)} = 0$ for n such that $j > \dim \mathcal{H}_n$.

Proof. Let $\varepsilon > 0$. We will show that for large enough n, $d\left(\lambda_j^{(n)}, \lambda_j\right) < \varepsilon$. Define

$$\varepsilon' = \min\left(\varepsilon, \min_{k\leq j} \frac{d\left(\lambda_k, \sigma\left(\Delta_\infty\right) \setminus \{\lambda_k\}\right)}{2}\right) > 0. \tag{4.6}$$

By definition, for $k, \ell \leq j$,

$$B_{\varepsilon'}(\lambda_k) \cap B_{\varepsilon'}(\lambda_\ell) \neq \emptyset \quad \text{if and only if} \quad \lambda_k = \lambda_\ell. \tag{4.7}$$

and there exists $N \in \mathbb{N}$ such that for $n \geq N$,

1. $\left\{\lambda_1^{(n)}, \ldots, \lambda_j^{(n)}\right\} \subseteq \bigcup_{k=1}^j B_{\varepsilon'}(\lambda_k)$.
2. For each $k \leq j$,
$$\dim \operatorname{im}\left(P_{\Delta_n, \overline{B_{\varepsilon'}(\lambda_k)}}\right) = \mu(\lambda_k). \tag{4.8}$$

This implies that each $\lambda_k^{(n)} \in B_{\varepsilon'}(\lambda_k)$. Namely, $\lambda_j^{(n)} \in B_{\varepsilon'}(\lambda_j)$ and so

$$d\left(\lambda_j^{(n)}, \lambda_j\right) < \varepsilon' \leq \varepsilon. \tag{4.9}$$

$\square$

The idea behind Corollary 4.1.4 is depicted pictorially in Figure 4.1.

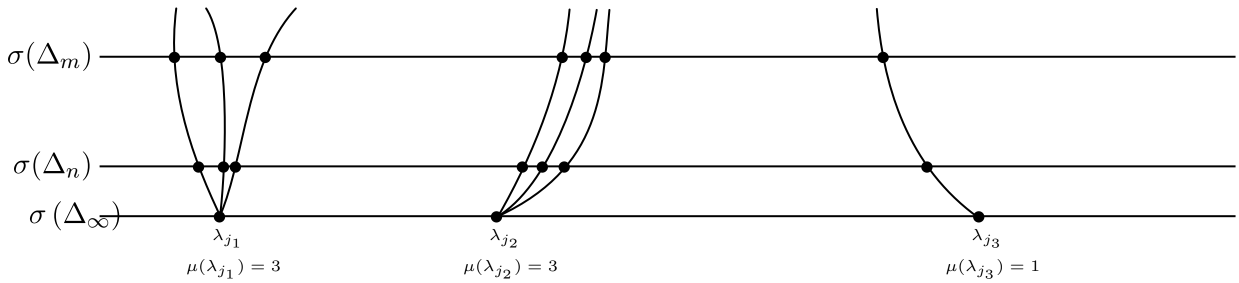


Figure 4.1: A depiction of the convergence of spectra where $m < n$ and the curves are visual aids meant to signify relationships between eigenvalues.

4.2 Behavior of Eigenvectors

Definition 4.2.1. For each n, define $\rho_n \colon \mathbb{R}_+ \to \mathcal{P}(\sigma(\Delta_n))$ via

$$\rho_n(\lambda) = \left\{ \lambda_j^{(n)} \in \sigma(\Delta_n) \,:\, \lambda_j = \lambda \right\}. \tag{4.10}$$

For a subset $I \subseteq \mathbb{R}_+$,

$$\rho_n(I) = \bigcup_{\lambda \in I} \left\{ \lambda_j^{(n)} \in \sigma(\Delta_n) \,:\, \lambda_j \in I \right\}. \tag{4.11}$$

Notice that for $\lambda \in \mathbb{R}_+ \setminus \sigma(\Delta_\infty)$, $\rho_n(\lambda) = \emptyset$.

Definition 4.2.2. Define $\kappa \colon \mathbb{R}_+ \to \mathcal{P}(\mathbb{N})$ via

$$\kappa(\lambda) = \{ j \in \mathbb{N} \,:\, \lambda_j = \lambda \}. \tag{4.12}$$

For a subset $I \subseteq \mathbb{R}_+$,

$$\kappa(I) = \{ j \in \mathbb{N} \,:\, \lambda_j \in I \}. \tag{4.13}$$

Note that $|\kappa(\lambda)| = \mu(\lambda)$.

Notation. In the work to follow, we will make heavy use of the spectral projection maps $P_{\Delta_n, I}$ in settings where the domain is clear. Such examples will include expressions of the form $P_{\Delta_n, I} \circ g_m^n$ or $P_{\Delta_n, \rho_n(\lambda)}$. In such cases where there is no risk of confusion of domain, denote $P_{\Delta_n, I}$ by P_I.

Proposition 4.2.3. Let $\lambda \in \sigma(\Delta_\infty)$ and let $L = d(\lambda, \sigma(\Delta_\infty) \setminus \{\lambda\}) > 0$. Then, there exists N such that for all $n > m \geq N$,

$$\left\| P_{\sigma(\Delta_n) \setminus \rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)} \right\| \leq \frac{2\mu(\lambda)}{L} \delta_m, \tag{4.14}$$

where δ_m is such that $(\mathcal{H}_m, \Delta_m)$ is a δ_m-retract of $(\mathcal{H}_n, \Delta_n)$.

Proof. Apply Theorem 2.2.4 to obtain N such that for all $n \geq N$,

1. For $\lambda^{(n)} \in \rho_n(\lambda)$,
$$d\left(\lambda^{(n)}, \lambda\right) < \frac{L}{4}. \tag{4.15}$$

2. For $\lambda^{(n)} \in \sigma(\Delta_n) \setminus \rho_n(\lambda)$,
$$d\left(\lambda^{(n)}, \lambda\right) > \frac{3L}{4}. \tag{4.16}$$

Let $n > m \geq N$. Let $\left(\alpha_k^{(m)}\right)_{k=1}^{\dim(\mathcal{H}_m)}, \left(\alpha_k^{(n)}\right)_{k=1}^{\dim(\mathcal{H}_n)}$ be orthonormal eigenbases for $\mathcal{H}_m, \mathcal{H}_n$ respectively such that $\Delta_n \alpha_k^{(n)} = \lambda_k^{(n)} \alpha_k^{(n)}$ and likewise for m. For each j, write

$$g_m^n\left(\alpha_j^{(m)}\right) = \sum_{k=1}^{\dim(\mathcal{H}_n)} a_{j,k}\alpha_k^{(n)}, \tag{4.17}$$

where each $a_{j,k} \in \mathbb{R}$. Then, for each $j \in \kappa(\lambda)$,

$$\begin{aligned}
\delta_m &\geq \left\| \left((\Delta_n + 1)^{-1} g_m^n - g_m^n (\Delta_m + 1)^{-1} \right) \left(\alpha_j^{(m)}\right) \right\| \\
&= \left\| \sum_{k=1}^{\dim(\mathcal{H}_n)} a_{j,k} \left(\frac{1}{\lambda_k^{(n)} + 1} - \frac{1}{\lambda_j^{(m)} + 1} \right) \alpha_k^{(n)} \right\| \\
&= \left(\sum_{k=1}^{\dim(\mathcal{H}_n)} a_{j,k}^2 d\left(\lambda_k^{(n)}, \lambda_j^{(m)}\right)^2 \right)^{\frac{1}{2}} \\
&\geq \left(\sum_{k \notin \kappa(\lambda)} a_{j,k}^2 d\left(\lambda_k^{(n)}, \lambda_j^{(m)}\right)^2 \right)^{\frac{1}{2}} \\
&\geq \left(\min_{k \notin \kappa(\lambda)} d\left(\lambda_k^{(n)}, \lambda_j^{(m)}\right) \right) \left\| \sum_{k \notin \kappa(\lambda)} a_{j,k}\alpha_k^{(n)} \right\| \\
&\geq \left(\min_{k \notin \kappa(\lambda)} d\left(\lambda_k^{(n)}, \lambda_j^{(m)}\right) \right) \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)} \left(g_m^n \left(\alpha_j^{(m)}\right) \right) \right\|.
\end{aligned} \tag{4.18}$$

Observe that

$$\begin{aligned}
\min_{k \notin \kappa(\lambda)} d\left(\lambda_k^{(n)}, \lambda_j^{(m)}\right) &\geq \min_{k \notin \kappa(\lambda)} d\left(\lambda_k^{(n)}, \lambda\right) - d\left(\lambda, \lambda_j^{(m)}\right) \\
&> \frac{3L}{4} - \frac{L}{4} \\
&= \frac{L}{2}.
\end{aligned} \tag{4.19}$$

So,

$$\left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)}\left(g_m^n\left(\alpha_j^{(m)}\right)\right)\right\| < \frac{2}{L}\delta_m. \tag{4.20}$$

Now fix $\alpha^{(m)} \in P_{\rho_m(\lambda)}(\mathcal{H}_m)$ of unit norm. Write

$$\alpha^{(m)} = \sum_{j\in\kappa(\lambda)} b_j \alpha_j^{(m)}, \tag{4.21}$$

where each $b_j \in [-1, 1]$ by assumption. Then,

$$\begin{aligned}\left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)}\left(g_m^n\left(\alpha^{(m)}\right)\right)\right\| &\leq \sum_{j\in\kappa(\lambda)} |b_j| \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)}\left(g_m^n\left(\alpha_j^{(m)}\right)\right)\right\| \\ &< \frac{2\,|\kappa(\lambda)|}{L}\delta_m \\ &= \frac{2\mu(\lambda)}{L}\delta_m.\end{aligned} \tag{4.22}$$

□

Assuming the set-up for Proposition 4.2.3, we get

Corollary 4.2.4. For large enough n, m,

$$P_{\rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)} \tag{4.23}$$

is an isomorphism.

Proof. By Proposition 4.2.3, $P_{\rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)}$ is an injection for large enough m. By Theorem 4.1.3, for large enough $n \geq m$, we see that $|\rho_n(\lambda)| = |\rho_m(\lambda)|$ and so $P_{\rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)}$ is an isomorphism. □

Corollary 4.2.5. Let $\Lambda \subseteq \sigma(\Delta_\infty)$ be a finite set. Let $L = \min_{\lambda\in\Lambda} d(\lambda, \sigma(\Delta_\infty) \setminus \{\lambda\})$. Then, there exists N such that for all $n > m \geq N$,

$$\left\| P_{\sigma(\Delta_n)\setminus\rho_n(\Lambda)} \circ g_m^n \circ P_{\rho_m(\Lambda)}\right\| \leq \frac{2\sum_{\lambda\in\Lambda}\mu(\lambda)}{L}\delta_n, \tag{4.24}$$

where δ_n is such that $(\mathcal{H}_m, \Delta_m)$ is a δ_n-retract of $(\mathcal{H}_n, \Delta_n)$.

Proof. Apply Proposition 4.2.3 with the triangle inequality. □

Chapter 5

Sequences of Hilbert Complexes with the Hodge Laplacian

In this section, we specialize our tools for use on *Hilbert complexes* [8], an analogue of cochain complexes in the setting of Hilbert spaces. We aim to generalize phenomena involving $\big(L^2\Omega^\bullet(M), d\big)$, which is not a cochain complex in general because $\mathrm{dom}(d) \neq L^2\Omega^\bullet(M)$. So, our theory cannot strictly rely on the traditional notion of a cochain complex. We will see, however, that our approximation schemes consist of cochain complexes due to finite dimensionality of each element.

We explain how, in the setting of Hilbert complexes, there exists a notion of Hodge decomposition and a Hodge Laplacian. We aim to compare the structure of differing Hilbert complexes along Cauchy sequences. To perform this study meaningfully, the definition of a δ-retract alone will be insufficient and we will instead work with a more elaborate mapping structure inspired by *strong deformation retractions* (also known as *contractions*, or *special deformation retractions*) in the context of cochain complexes. Strong deformation retractions have a rich history in the field of Homological Algebra. In particular, Homological Perturbation Theory [6] and Mathematical Physics through A_∞ algebras [28, 29] where they are used in name or implicitly for the transfer of an A_∞ structure [32, 9, 33]. In line with the context of discretizations, techniques related to strong deformation retractions are used in the field of Computational Algebraic Topology [11, 26] and in the Mathematical Physics literature through the formation of discrete physical models based on triangulations of manifolds [36]. One bridge between Topological applications and Geometric developments of particular interest emerged from Discrete Morse Theory [18] in attempts to import the geometric notions of Morse Theory into the algebraic setting [43, 45]. The majority of the mathematical applications in this field have followed a similar motive of realizing geometric structures as examples from the perspective of algebraic topology [24, 17], while the potential geometric utility of the algebraic techniques used in the discrete setting are less well understood. Developments to this effect are beginning [10] and this chapter aims to add to this emerging body of literature.

5.1 Background on Hilbert Complexes

We begin by reviewing basic material on Hilbert Complexes as outlined in Sections 1 and 2 of [8], Sections 3 and 4 of [5], and [22]. Both [12, 25] were consulted for a review on L^2 cohomology.

Definition 5.1.1 (Definition 4.1 [5], Definition 3.1 [22])**.** Let $\left(\mathcal{H}^i\right)_{i\in\mathbb{Z}}$ be a sequence of Hilbert spaces and, for each $i \in \mathbb{Z}$, let $d_i\colon \mathcal{H}^i \to \mathcal{H}^{i+1}$ be a closed, densely defined map such that $\operatorname{im}(d_i) \subseteq \ker(d_{i+1})$. The resulting structure is referred to as a *Hilbert complex.*

$$\dots \xrightarrow{d_{-1}} \mathcal{H}^0 \xrightarrow{d_0} \mathcal{H}^1 \xrightarrow{d_1} \mathcal{H}^2 \xrightarrow{d_2} \dots$$

We will denote Hilbert complexes using the notation $(\mathcal{H}^\bullet, d)$, refer to each $\mathcal{H}^i$ as possessing degree i, and omit the subscripts on the d_i maps when convenient. Notice that

$$\mathcal{H}^\bullet := \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i \tag{5.1}$$

is a Hilbert space under the inner product induced from each $\mathcal{H}^i$ such that $\mathcal{H}^i \perp \mathcal{H}^j$ for $i \neq j$.

The associated cohomology at degree i is given by

$$\ker(d_i) \,/\, \operatorname{im}(d_{i-1}). \tag{5.2}$$

Note that there is an underlying cochain complex given by

$$\dots \xrightarrow{d_{-1}} \operatorname{dom}(d_0) \xrightarrow{d_0} \operatorname{dom}(d_1) \xrightarrow{d_1} \operatorname{dom}(d_2) \xrightarrow{d_2} \dots$$

We will denote this complex by $(\operatorname{dom}(d)^\bullet, d)$ where $\operatorname{dom}(d)^k = \operatorname{dom}(d_k)$. In the case that each $\mathcal{H}^i$ is finite dimensional,

$$(\operatorname{dom}(d)^\bullet, d) = (\mathcal{H}^\bullet, d). \tag{5.3}$$

By taking the adjoint of d, d^*, we obtain the complex

$$\dots \xleftarrow{d^*_{-1}} \mathcal{H}^0 \xleftarrow{d^*_0} \mathcal{H}^1 \xleftarrow{d^*_1} \mathcal{H}^2 \xleftarrow{d^*_2} \dots.$$

Since d is closed and densely defined, so is d^* (Theorem 1.8 (i) [44]).

Definition 5.1.2 (Section 2 [8], Section 4.1.1 [5])**.** A *Fredholm complex*, $(\mathcal{H}^\bullet, d)$, is a Hilbert complex such that the degree i cohomology is finite dimensional for all i.

By Theorem 2.4 of [8], if $(\mathcal{H}^\bullet, d)$ is a Fredholm complex, then $\mathrm{im}(d_i)$ is closed for all i. Moreover, Theorem 5.13 from [27] implies that $\mathrm{im}(d^*)$ is closed.

Example 5.1.3. Following Section 1 of [12], if M is a compact Riemannian manifold, then the cohomology of $\left(L^2\Omega^\bullet(M), d\right)$ is isomorphic to the cohomology of $(\Omega^\bullet(M), d)$. Since the latter cohomology is finite dimensional, $\left(L^2\Omega^\bullet(M), d\right)$ is a Fredholm complex.

We will only work in the case that $(\mathcal{H}^\bullet, d)$ is a Fredholm complex. Following the work of Section 2 of [8], observe that d, d^* being closed implies

$$\begin{aligned} \mathrm{im}(d)^\perp &= \ker\left(d^*\right), \\ \mathrm{im}\left(d^*\right) &= \ker\left(d\right)^\perp, \\ \mathcal{H}^\bullet &= \mathrm{im}(d) \oplus \mathrm{im}(d)^\perp \end{aligned} \tag{5.4}$$

and so we obtain the following sequence of orthogonal decompositions:

$$\begin{aligned} \mathcal{H}^\bullet &= \ker(d) \oplus \ker(d)^\perp \\ &= \mathrm{im}(d) \oplus \left(\ker(d) \cap \mathrm{im}(d)^\perp\right) \oplus \ker(d)^\perp \\ &= \mathrm{im}(d) \oplus \left(\ker(d) \cap \ker\left(d^*\right)\right) \oplus \mathrm{im}\left(d^*\right). \end{aligned} \tag{5.5}$$

Refer to $\mathrm{im}(d)$ as the *exact* elements, $\mathrm{im}(d^*)$ as *coexact*, $\ker(d) \cap \ker(d^*)$ as *harmonic*, and this decomposition as the *Hodge decomposition*. For the harmonic forms, we saw in the above equations that

$$\ker(d) = \mathrm{im}(d) \oplus \left(\ker(d) \cap \ker\left(d^*\right)\right) \tag{5.6}$$

and so $\ker(d) \cap \ker(d^*)$ is isomorphic to the cohomology of $(\mathcal{H}^\bullet, d)$.

Define $\Delta = (d + d^*)^2$, which we will refer to as the *Hodge Laplacian*. The operator Δ is non-negative and, by Section 2 of [20] or Lemma 3.11 of [7], self-adjoint. Notice that $\ker(\Delta) = \ker(d) \cap \ker(d^*)$ and so we obtain the familiar Hodge decomposition:

$$\mathcal{H}^\bullet = \mathrm{im}(d) \oplus \ker(\Delta) \oplus \mathrm{im}\left(d^*\right). \tag{5.7}$$

Immediately, we have that $\ker(\Delta)$ is isomorphic to the cohomology of $(\mathcal{H}^\bullet, d)$. In the case where $(\mathcal{H}^\bullet, d) = \left(L^2\Omega^\bullet(M), d\right)$ for some Riemannian manifold M, the property is sometimes known as satisfaction of the *Strong Hodge Theorem* (see Section 1.3 of [12]).

Definition 5.1.4. Given Hilbert complexes $\left(\mathcal{H}'^\bullet, d'\right)$ and $(\mathcal{H}^\bullet, d)$, a *Hilbert cochain map*, $f\colon \mathcal{H}'^\bullet \to \mathcal{H}^\bullet$, is a (densely defined) linear map such that $(\mathrm{dom}^\bullet(f), d')$ is a subcomplex of

$(\mathrm{dom}^\bullet(d'), d')$ and, for $\alpha \in \mathrm{dom}(f)$, $f(\alpha) \in \mathrm{dom}(d)$ and

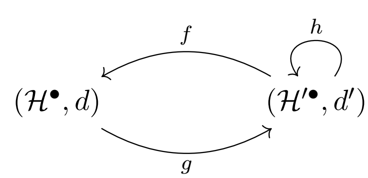
$$f\left(d'\alpha\right) = df\left(\alpha\right). \tag{5.8}$$

In other words, f is a cochain map from $(\mathrm{dom}^\bullet(f), d')$ to $(\mathrm{dom}(d)^\bullet, d)$.

In all subsequent work, we will work with packages of data $(\mathcal{H}^\bullet, d, \Delta)$ where $\Delta = dd^* + d^*d$. Note that this notation is a strict extension of our previous notation, since $\mathcal{H}^\bullet$ is a Hilbert space and Δ is a non-negative self-adjoint operator on it. In line with previous sections, we will assume that all Hilbert complexes with subscripts, denoted $\mathcal{H}_n^\bullet$, are finite dimensional as Hilbert spaces. In other words, $\mathcal{H}_n^i = 0$ for all but finitely many i and each such $\mathcal{H}_n^i$ is finite dimensional. Of course, in this case, $\mathcal{H}_n^\bullet$ is a cochain complex where d is defined everywhere and $(\mathcal{H}_n^\bullet, d)$ is Fredholm.

5.2 Comparing Hilbert Complexes: Delta Deformation Retracts

Definition 5.2.1 (adapted from [6] and Definition 1 of [9])**.** Given Hilbert complexes $\left(\mathcal{H}'^\bullet, d'\right)$ and $(\mathcal{H}^\bullet, d)$, a *strong deformation retraction* from $\left(\mathcal{H}'^\bullet, d'\right)$ and $(\mathcal{H}^\bullet, d)$ is a package of linear maps $f\colon \mathcal{H}'^\bullet \to \mathcal{H}^\bullet$, $g\colon \mathcal{H}^\bullet \to \mathcal{H}'^\bullet$, $h\colon \mathcal{H}'^\bullet \to \mathcal{H}'^\bullet$, summarized in the following diagram:



such that

1. f, g are Hilbert cochain maps (degree 0).
2. $\mathrm{dom}(h) = \mathrm{dom}(f) \supseteq \mathrm{im}(g)$ and $\mathrm{im}(h) \subseteq \ker(h)$.
3. The cohomology in each degree of $(\mathrm{dom}^\bullet(f), d')$ is isomorphic to that of $\left(\mathcal{H}'^\bullet, d'\right)$.
4. $\mathrm{dom}(g) = \mathcal{H}^\bullet$.
5. h has degree -1.
6. $fh = hg = h^2 = 0$.
7. $fg = 1$.

8. $1 - gf = d'h + hd'$.

We will denote this relationship using the following notation:

$$(f, g, h) \colon \left(\mathcal{H}'^{\bullet}, d'\right) \to \left(\mathcal{H}^{\bullet}, d\right). \tag{5.9}$$

We will sometimes omit the distinction between d', d due to f and g being cochain maps. Note that a deformation retraction induces an isomorphism between the cohomology of $\left(\mathcal{H}'^{\bullet}, d'\right)$ and of $\left(\mathcal{H}^{\bullet}, d\right)$.

Remark 5.2.2. Conditions 3 and 4 in Definition 5.2.1 are slightly unusual but exist because of the unruly nature of Hilbert complex cohomology. Condition 3 is necessary to capture the intended use of deformation retractions in Algebraic Topology. Condition 4 implies that $\mathrm{dom}(d) = \mathcal{H}^{\bullet}$ and so $\left(\mathcal{H}^{\bullet}, d\right)$ is a cochain complex. This condition will not hinder any work to follow because $\mathcal{H}^{\bullet}$ will always be a finite dimensional Hilbert space and thus a cochain complex by default.

Definition 5.2.3. A *δ-deformation retraction* from $\left(\mathcal{H}'^{\bullet}, d', \Delta'\right)$ to $\left(\mathcal{H}^{\bullet}, d, \Delta\right)$ is a strong deformation retraction $(f, g, h) : \left(\mathcal{H}'^{\bullet}, d'\right) \to \left(\mathcal{H}^{\bullet}, d\right)$ where g is an isometry and such that:

$$\begin{aligned} \left\| \left(\Delta' + 1\right)^{-\frac{1}{2}} g - g \left(\Delta + 1\right)^{-\frac{1}{2}} \right\| &< \frac{1}{2}\delta, \\ \left\| \left(1 - gg^*\right) \left(\Delta' + 1\right)^{-\frac{1}{2}} \right\| &< \frac{1}{2}\delta. \end{aligned} \tag{5.10}$$

We will say that $\left(\mathcal{H}^{\bullet}, d, \Delta\right)$ is a *δ-deformation retract* of $\left(\mathcal{H}'^{\bullet}, d', \Delta'\right)$ if there exists a δ-deformation retraction $(f, g, h) : \left(\mathcal{H}'^{\bullet}, d', \Delta'\right) \to \left(\mathcal{H}^{\bullet}, d, \Delta\right)$.

Notice that in the above definition, the conditions (5.10) on g closely resemble the conditions required for a δ-isometry in Definition 2.1.1. We will see that Definition 5.2.3 implies g is indeed a δ-isometry. We impose Equation (5.10) on g instead of those from Definition 2.1.1 for reasons that will become more clear in Section 5.6.

Proposition 5.2.4. If $(f, g, h) : \left(\mathcal{H}'^{\bullet}, d, \Delta'\right) \to \left(\mathcal{H}^{\bullet}, d, \Delta\right)$ is a δ-deformation retraction, then $\left(\mathcal{H}^{\bullet}, \Delta\right)$ is a δ-retract of $\left(\mathcal{H}'^{\bullet}, \Delta'\right)$ and g is a δ-isometry.

Proof. This amounts to verification of the properties listed in 2.1.1. Observe:

$$\begin{aligned} \left\| \left(\Delta' + 1\right)^{-1} g - g \left(\Delta + 1\right)^{-1} \right\| &\leq \left\| \left(\Delta' + 1\right)^{-\frac{1}{2}} \left(\left(\Delta' + 1\right)^{-\frac{1}{2}} g - g \left(\Delta + 1\right)^{-\frac{1}{2}} \right) \right\| \\ &\quad + \left\| \left(\left(\Delta' + 1\right)^{-\frac{1}{2}} g - g \left(\Delta + 1\right)^{-\frac{1}{2}} \right) \left(\Delta + 1\right)^{-\frac{1}{2}} \right\| \\ &\leq 2 \left\| \left(\Delta' + 1\right)^{-\frac{1}{2}} g - g \left(\Delta + 1\right)^{-\frac{1}{2}} \right\| \\ &< \delta \end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\left\|(1-gg^*)\left(\Delta'+1\right)^{-1}\right\| &= \left\|(1-gg^*)\left(\Delta'+1\right)^{-\frac{1}{2}}\left(\Delta'+1\right)^{-\frac{1}{2}}\right\| \\ &= \left\|(1-gg^*)\left(\Delta'+1\right)^{-\frac{1}{2}}\right\| \\ &< \delta.\end{aligned} \tag{5.12}$$

□

Proposition 5.2.4 tells us that the spectrum convergence results of Chapter 2 applies in the context Hilbert complexes and δ-deformation retractions. The following Proposition illustrates how Proposition 2.1.11 is adapted to the context of Equation (5.10).

Proposition 5.2.5. Suppose $g_1\colon \mathcal{H}_1 \to \mathcal{H}$ satisfies Equation (5.10) for $\delta_1 > 0$ and $g_2\colon \mathcal{H}_2 \to \mathcal{H}$ satisfies Equation (5.10) for $\delta_2 > 0$. Suppose as well that $\operatorname{im}(g_1) \subseteq \operatorname{im}(g_2)$. Then, $g_1^2 := g_2^* g_1\colon \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry which satisfies Equation (5.10) with $\delta_1 + \delta_2$.

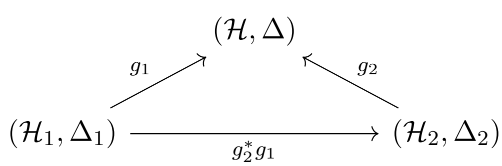


Figure 5.1: A diagram depicting the set-up of Proposition 5.2.5

Proof. The proof is identical to that of Proposition 2.1.11, only with $(\Delta+1)^{-1}$, $(\Delta_1+1)^{-1}$, $(\Delta_2+1)^{-1}$ replaced with $(\Delta+1)^{-\frac{1}{2}}$, $(\Delta_1+1)^{-\frac{1}{2}}$, $(\Delta_2+1)^{-\frac{1}{2}}$. □

Proposition 5.2.6. Suppose $(f,g,h) : \left(\mathcal{H}'^{\bullet}, d', \Delta'\right) \to \left(\mathcal{H}^{\bullet}, d, \Delta\right)$ is a δ-deformation retraction with bounded f. Then, for $\alpha \in \operatorname{dom}(f)$,

$$\|(1-gf)\,\alpha\| < \delta\left(\|f\|+1\right)\left\|\left(\Delta'+1\right)^{\frac{1}{2}}\alpha\right\|. \tag{5.13}$$

Proof. This is a calculation that uses an idea used in the proof of Theorem 5.6 in [4] for the first equality:

$$\begin{aligned}\|(1-gf)\,\alpha\| &= \inf_{\beta\in\mathcal{H}^{\bullet}} \|(1-gf)\left(\alpha - g(\beta)\right)\|, && \text{Since } (1-gf)\big|_{\operatorname{im}(g)} = 0 \\ &\le \left(\|f\|+1\right)\inf_{\beta\in\mathcal{H}^{\bullet}} \|\alpha - g(\beta)\| \\ &= \left(\|f\|+1\right)\|(1-gg^*)(\alpha)\|, && \text{by Lemma 2.1.4} \\ &< \delta\left(\|f\|+1\right)\left\|\left(\Delta'+1\right)^{\frac{1}{2}}\alpha\right\|.\end{aligned} \tag{5.14}$$

□

Definition 5.2.7. Definitions 2.1.5 and 2.1.7 of convergent and Cauchy sequences carry over directly: A sequence $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ is said to be...

- *convergent* to some $(\mathcal{H}^\bullet, d, \Delta)$ if there exists a sequence $\delta_n \to 0$ such that $(\mathcal{H}_n^\bullet, d_n, \Delta_n)$ is a δ_n-deformation retract of $(\mathcal{H}^\bullet, d, \Delta)$. In this case, and as in Definition 2.1.5, we will sometimes refer to $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ as an *approximation scheme* for $(\mathcal{H}^\bullet, d, \Delta)$.

- *Cauchy* if for every $m \leq n$, there exist strong deformation retractions $(f_n^m, g_m^n, h_{n,m}) : (\mathcal{H}_n^\bullet, d_n) \to (\mathcal{H}_m^\bullet, d_m)$ such that the following hold:

  1. For every $\delta > 0$, there exists $N_\delta \in \mathbb{N}$ such that for all $n > m \geq N_\delta$, $(f_n^m, g_m^n, h_{n,m})$ is a δ-deformation retraction.
  2. For all $k \geq n \geq m$, $g_m^k = g_n^k \circ g_m^n$, $f_k^m = f_n^m \circ f_k^n$.
  3. For all n,
$$\lim_{B\to\infty} \lim_{k\to\infty} \left\| P_{[B,\infty)} \circ g_n^k \right\| = 0. \tag{5.15}$$

From the definition of Cauchy, Proposition 5.2.4, and Chapter 3, for each Cauchy sequence $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ there exists $(\mathcal{H}_\infty^\bullet, \Delta_\infty)$ such that $(\mathcal{H}_n^\bullet, \Delta_n)_{n=1}^\infty$ converges to $(\mathcal{H}_\infty^\bullet, \Delta_\infty)$. We do not yet know that there exists $(\mathcal{H}_\infty^\bullet, d_\infty, \Delta_\infty)$ such that $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ converges to $(\mathcal{H}_\infty^\bullet, d_\infty, \Delta_\infty)$. Section 5.6 will be devoted to this.

The following Proposition explains how the topological implications of a strong deformation retraction provide utility in the geometric setting by showing the existence of a uniform spectral gap between 0 and the first non-zero eigenvalue of $\sigma(\Delta_n)$. Existing results concerning spectral gaps in the context of convergent Hilbert spaces appear in [39] (see Theorem 1.5.1) but address the slightly different problem of showing that for intervals (a,b) such that $(a,b) \cap \sigma(\Delta) = \emptyset$ and $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ large enough such that $(a+\varepsilon, b-\varepsilon) \cap \sigma(\Delta_n) = \emptyset$ for $n \geq n_\varepsilon$. Such results are not applicable to bounding the first non-zero eigenvalue uniformly in the sequence of spectra $\sigma(\Delta_n)$ and thus, the following result is new in this regard.

Proposition 5.2.8. Suppose $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ is a Cauchy sequence which is convergent to $(\mathcal{H}_\infty^\bullet, d_\infty, \Delta_\infty)$ and let $L = \frac{d(0, \sigma(\Delta_\infty)\setminus\{0\})}{2}$. Then, for large enough n, $\min(\sigma(\Delta_n) \setminus \{0\}) \geq L$.

Proof. By Proposition 2.1.14, $L > 0$. Using Theorem 4.1.1, let N be large enough so that

$$\dim \operatorname{im}\left(P_{\Delta_n, \overline{B_L(0)}}\right) = \dim \operatorname{im}\left(P_{\Delta_\infty, \overline{B_L(0)}}\right) \tag{5.16}$$

for $n \geq N$. By the definition of L, $\operatorname{im}\left(P_{\Delta_\infty, \overline{B_L(0)}}\right) = \ker(\Delta_\infty)$ and that the latter subspace is isomorphic to cohomology. On the other hand, the data of a strong deformation retraction $(f_n, g_n, h_n) : (\mathcal{H}^\bullet_\infty, d_\infty, \Delta_\infty) \to (\mathcal{H}^\bullet_n, d_n, \Delta_n)$ tells us that the cohomology of the Hilbert complexes are isomorphic and so $\dim\ker(\Delta_\infty) = \dim\ker(\Delta_n)$. Consequently,

$$\dim \operatorname{im}\left(P_{\Delta_n,\overline{B_L(0)}}\right) = \dim\ker(\Delta_n) \tag{5.17}$$

and since $\ker(\Delta_n) \subseteq \operatorname{im}\left(P_{\Delta_n,\overline{B_L(0)}}\right)$, we have

$$\operatorname{im}\left(P_{\Delta_n,\overline{B_L(0)}}\right) = \dim\ker(\Delta_n). \tag{5.18}$$

It must then be that $B_L(0) \cap \sigma(\Delta_n) = \{0\}$ and thus $\min(\sigma(\Delta_n) \setminus \{0\}) \geq L$. □

For the remainder of this section, assume that $(\mathcal{H}^\bullet_n, d_n, \Delta_n)_{n=1}^\infty$ is a Cauchy sequence which is observed by strong deformation retractions $(f_n^m, g_m^n, h_{n,m}) : (\mathcal{H}^\bullet_n, d_n, \Delta_n) \to (\mathcal{H}^\bullet_m, d_m, \Delta_m)$. For all n, we have previously assumed that $\dim \mathcal{H}^\bullet_n < \infty$ and so, by definition, we have $\operatorname{dom}(f_n^m) = \mathcal{H}^\bullet_n$.

5.3 Comparing the Hodge Decomposition of Elements

Theorem 5.3.1. Let $\alpha \in \mathcal{H}^\bullet_n$ be of unit norm and write

$$\begin{aligned} \alpha &= \alpha_0 + d_n^*\alpha_1 + d_n\alpha_2, \\ f_n^m(\alpha) &= \beta_0 + d_m^*\beta_1 + d_m\beta_2, \end{aligned} \tag{5.19}$$

according to the Hodge decompositions. Then,

$$\begin{aligned} \|\alpha_0 - g_m^n(\beta_0)\| &\leq 3\delta_{m,n}(\|f_n^m\| + 1)\left\|(\Delta_n + 1)^{\frac{1}{2}}\alpha\right\|, \\ \|d_n^*\alpha_1 - g_m^n(d_m^*\beta_1)\| &\leq 3\delta_{m,n}(\|f_n^m\| + 1)\left\|(\Delta_n + 1)^{\frac{1}{2}}\alpha\right\|, \\ \|d_n\alpha_2 - g_m^n(d_m\beta_2)\| &\leq 3\delta_{m,n}(\|f_n^m\| + 1)\left\|(\Delta_n + 1)^{\frac{1}{2}}\alpha\right\|. \end{aligned} \tag{5.20}$$

Proof. Proposition 5.2.6 allows the direct use of the proof of Theorem 4.9 in [14]. □

One should view the previous Theorem as a description and bound on the change of Hodge decomposition when mapping down $\mathcal{H}^\bullet_n \to \mathcal{H}^\bullet_m$. This bound involves the chosen element in $\mathcal{H}^\bullet_n$ along with Δ_n.

Corollary 5.3.2. Let $\beta \in \mathcal{H}^\bullet_m$ and write

$$\begin{aligned} \beta &= \beta_0 + d_m^*\beta_1 + d_m\beta_2, \\ g_m^n(\beta) &= \alpha_0 + d_n^*\alpha_1 + d_n\alpha_2. \end{aligned} \tag{5.21}$$

Then,

$$\begin{aligned}\left\|\alpha_0 - g_m^n\left(\beta_0\right)\right\| &\leq 3\delta_{m,n}\left(\left\|f_n^m\right\|+1\right)\left\|\left(\Delta_n+1\right)^{\frac{1}{2}} g_m^n\left(\beta\right)\right\|, \\ \left\|d_n^*\alpha_1 - g_m^n\left(d_m^*\beta_1\right)\right\| &\leq 3\delta_{m,n}\left(\left\|f_n^m\right\|+1\right)\left\|\left(\Delta_n+1\right)^{\frac{1}{2}} g_m^n\left(\beta\right)\right\|, \\ \left\|d_n\alpha_2 - g_m^n\left(d_m\beta_2\right)\right\| &\leq 3\delta_{m,n}\left(\left\|f_n^m\right\|+1\right)\left\|\left(\Delta_n+1\right)^{\frac{1}{2}} g_m^n\left(\beta\right)\right\|.\end{aligned} \tag{5.22}$$

Proof. Apply Theorem 5.3.1 with $\alpha = g_m^n(\beta)$, noting that $f_n^m(\alpha) = f_n^m g_m^n(\beta) = \beta$. □

Proposition 5.3.3. Suppose $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ is a Cauchy sequence which is convergent to $(\mathcal{H}^\bullet, d, \Delta)$ under $(f_n, g_n, h_n) : (\mathcal{H}^\bullet, d, \Delta) \to (\mathcal{H}_n^\bullet, d_n, \Delta_n)$. Let $\alpha \in \ker(\Delta)$. Then, for $n \in \mathbb{N}$, define

$$\alpha_n = \mathrm{Proj}_{\ker(\Delta_n)}(f_n(\alpha)). \tag{5.23}$$

Then, for $n \geq m$,

$$\alpha_n = \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n(\alpha_m)\right). \tag{5.24}$$

As a result, α_n represents the same cohomology class as α for all n.

Proof. This amounts to computation. Observe,

$$\begin{aligned}\alpha_n &= \mathrm{Proj}_{\ker(\Delta_n)}\left(f_n(\alpha)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(\left(1 - dh_{n,m} - h_{n,m}d\right) f_n(\alpha)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n f_n^m f_n(\alpha)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n f_m(\alpha)\right).\end{aligned} \tag{5.25}$$

At this step, notice that $f_m(\alpha) = \beta_0 + d\beta_1$ for $\beta_0 \in \ker(\Delta_m)$ and $\beta_1 \in \mathcal{H}_m^\bullet$. Since g_m^n is a cochain map,

$$\begin{aligned}\mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n f_m(\alpha)\right) &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n\left(\beta_0 + d\beta_1\right)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n\left(\beta_0\right)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n \mathrm{Proj}_{\ker(\Delta_m)} f_m(\alpha)\right) \\ &= \mathrm{Proj}_{\ker(\Delta_n)}\left(g_m^n\left(\alpha_m\right)\right).\end{aligned} \tag{5.26}$$

□

5.4 Behavior of Eigenvectors

In this section, we study the behavior of eigenvectors of Δ_m when mapped by g_m^n. Observe first that for each n, Δ_n restricts to an endomorphism of each summand of the Hodge decomposition

$$\operatorname{im}(d_n) \oplus \ker(\Delta_n) \oplus \operatorname{im}(d_n^*)\,. \tag{5.27}$$

Thus, the eigenvectors of Δ_n arise from eigenvectors of each summand above. Let

- $\gamma_1^{(n)}, \ldots, \gamma_{\dim(\ker(\Delta_n))}^{(n)}$ be an orthonormal basis of $\ker(\Delta_n)$.
- $\alpha_{r_1}^{(n)}, \ldots, \alpha_{r_{\dim(\operatorname{im}(d_n^*))}}^{(n)}$ be an orthonormal eigenbasis of $\operatorname{im}(d_n^*)$ such that $\Delta_n \alpha_{r_k}^{(n)} = \lambda_{r_k}^{(n)} \alpha_{r_k}^{(n)}$.
- $\beta_{s_1}^{(n)}, \ldots, \beta_{s_{\dim(\operatorname{im}(d_n))}}^{(n)}$ be an orthonormal eigenbasis of $\operatorname{im}(d_n)$ such that $\Delta_n \beta_{s_k}^{(n)} = \lambda_{s_k}^{(n)} \beta_{s_k}^{(n)}$.

For each k, there exists t such that $d\alpha_{r_k}^{(n)} = \sqrt{\lambda_{r_k}^{(n)}}\beta_{s_t}^{(n)}$ and $d^*\beta_{s_t}^{(n)} = \sqrt{\lambda_{r_k}^{(n)}}\alpha_{r_k}^{(n)}$ and so we see that d, d^* are endomorphisms of eigenspaces of Δ_n. Conversely, the Hodge decomposition tells us that each eigenspace decomposes into harmonic, exact, and coexact forms. We will measure how well g_m^n respects this decomposition. Indeed, this splitting of eigenspaces according to the Hodge decomposition distinguishes this section from that which was studied in Section 4.2.

Proposition 5.4.1. Fix $\varepsilon > 0$. Then, there exists N large enough such that for $n > m \geq N$,

$$\left\| P_{(0,\infty)} \circ g_m^n \circ P_{\{0\}} \right\| < \varepsilon. \tag{5.28}$$

Proof. This is a direct application of Proposition 4.2.3. ☐

Lemma 5.4.2. For each n, j, $P_{\rho_n(\lambda_j)} \colon \mathcal{H}_n^\bullet \to \mathcal{H}_n^\bullet$ is a cochain map.

Proof. This is immediate, since d is an endomorphism of each eigenspace. ☐

Notation. For a subset $I \subseteq \mathbb{R}_+$ and fixed $n \in \mathbb{N}$, write

$$\begin{aligned}
P_d &= \operatorname{Proj}_{\operatorname{im}(d)}, \\
P_{d^*} &= \operatorname{Proj}_{\operatorname{im}(d^*)}, \\
P_{d,I} &= P_d \circ P_I = P_I \circ P_d = \operatorname{Proj}_{\operatorname{im}(d) \cap \bigoplus_{\lambda \in \rho_n(I)} \mathcal{E}_\lambda^{(n)}}, \\
P_{d^*,I} &= P_{d^*} \circ P_I = P_I \circ P_{d^*} = \operatorname{Proj}_{\operatorname{im}(d^*) \cap \bigoplus_{\lambda \in \rho_n(I)} \mathcal{E}_\lambda^{(n)}}.
\end{aligned} \tag{5.29}$$

Similarly, define $\kappa_d, \kappa_{d^*} : \mathbb{R}_+ \to \mathbb{N}$ via

$$\begin{aligned}
\kappa_d(\lambda) &= \left\{ j \in \mathbb{N} \,:\, \lambda_{s_j} = \lambda \right\}, \\
\kappa_{d^*}(\lambda) &= \left\{ j \in \mathbb{N} \,:\, \lambda_{r_j} = \lambda \right\}.
\end{aligned} \tag{5.30}$$

Proposition 5.4.3. Fix $\varepsilon > 0$ and let $\lambda \in \sigma(\Delta_\infty)$, There exists N large enough such that for $n > m \geq N$,

$$\left\| \left(1 - P_{d,\rho_n(\lambda)}\right) \circ g_m^n \circ P_{d,\rho_m(\lambda)} \right\| < \varepsilon. \tag{5.31}$$

Proof. Since g_m^n is a cochain map,

$$P_{d,\rho_n(\lambda)} \circ g_m^n \circ P_{d,\rho_m(\lambda)} = P_{\rho_n(\lambda)} \circ g_m^n \circ P_{d,\rho_m(\lambda)}. \tag{5.32}$$

This implies

$$\begin{aligned} \left\| \left(1 - P_{d,\rho_n(\lambda)}\right) \circ g_m^n \circ P_{d,\rho_m(\lambda)} \right\| &= \left\| \left(1 - P_{\rho_n(\lambda)}\right) \circ g_m^n \circ P_{d,\rho_m(\lambda)} \right\| \\ &= \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)} \circ g_m^n \circ P_{d,\rho_m(\lambda)} \right\| \\ &= \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)} \circ P_d \right\| \\ &\leq \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)} \circ g_m^n \circ P_{\rho_m(\lambda)} \right\|. \end{aligned} \tag{5.33}$$

Proposition 4.2.3 yields the result. □

Proposition 5.4.4. Let $\lambda \in \sigma(\Delta_\infty) \setminus \{0\}$ and fix $\varepsilon > 0$. Then, there exists N large enough such that for $n > m \geq N$,

$$\left\| \left(1 - P_{d^*,\rho_n(\lambda)}\right) \circ g_m^n \circ P_{d^*,\rho_m(\lambda)} \right\| < \varepsilon. \tag{5.34}$$

Proof. Let $j \in \kappa_{d^*}(\lambda)$ and write

$$g_m^n\left(\alpha_{r_j}^{(m)}\right) = \sum_{i=1}^{\dim(\ker(\Delta_n))} c_i \gamma_i^{(n)} + \sum_{i=1}^{\dim(\mathrm{im}(d_n^*))} a_i \alpha_{r_i}^{(n)} + \sum_{i=1}^{\dim(\mathrm{im}(d_n))} b_i \beta_{s_i}^{(n)}. \tag{5.35}$$

Observe,

$$\left\| d g_m^n\left(\alpha_{r_j}^{(m)}\right) \right\| = \left\| d\alpha_{r_j}^{(m)} \right\| = \sqrt{\lambda_{r_j}^{(m)}}. \tag{5.36}$$

For large enough N, Proposition 4.2.3 ensures that for all $n \geq N$,

$$\begin{aligned} \frac{2\mu(\lambda)\sqrt{\lambda_{r_j}^{(m)}}}{d\left(\lambda, \sigma\left(\Delta_\infty\right) \setminus \{\lambda\}\right)} \delta_n &\geq \left\| P_{\sigma(\Delta_n)\setminus\rho_n(\lambda)}\left(g_m^n\left(d\alpha_{r_j}^{(m)}\right)\right) \right\| \\ &= \left\| dP_{\sigma(\Delta_n)\setminus\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right) \right\|, \quad \text{Lemma 5.4.2} \\ &\geq \left| \left\| d g_m^n\left(\alpha_{r_k}^{(m)}\right) \right\| - \left\| dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right) \right\| \right| \\ &= \left| \sqrt{\lambda_{r_j}^{(m)}} - \left\| dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right) \right\| \right|. \end{aligned} \tag{5.37}$$

By Corollary 4.1.4 and since $\delta_n \to 0$, we may take n large enough so that

$$\left|1 - \frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|}{\sqrt{\lambda_{r_j}^{(m)}}}\right| \leq \frac{\varepsilon^2}{4\mu(\lambda)^2}. \tag{5.38}$$

Note that

$$\left|1 + \frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|}{\sqrt{\lambda_{r_j}^{(m)}}}\right| \leq 2 \tag{5.39}$$

and so

$$\left|1 - \frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2}{\lambda_{r_j}^{(m)}}\right| \leq \frac{\varepsilon^2}{2\mu(\lambda)^2}. \tag{5.40}$$

On the other hand, by Corollary 4.1.4, we can take n large enough so that for all $i \in \kappa_{d^*}(\lambda)$,

$$\left|\frac{\lambda_{r_i}^{(n)}}{\lambda_{r_j}^{(m)}} - 1\right| < \frac{\varepsilon^2}{2\mu(\lambda)^2}. \tag{5.41}$$

Then,

$$\begin{aligned}
\left|\frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2}{\lambda_{r_j}^{(m)}} - \left\|P_{d^*,\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2\right| &= \left|\sum_{i\in\kappa_{d^*}(\lambda)} a_i^2\left(\frac{\lambda_{r_i}^{(n)}}{\lambda_{r_j}^{(m)}} - 1\right)\right| \\
&\leq \sum_{i\in\kappa_{d^*}(\lambda)} a_i^2\left|\frac{\lambda_{r_i}^{(n)}}{\lambda_{r_j}^{(m)}} - 1\right| \\
&< \frac{\varepsilon^2}{2\mu(\lambda)^2}.
\end{aligned} \tag{5.42}$$

The last line follows from $\left\|\alpha_{r_j}^{(m)}\right\| = 1$. Combining this with (5.40), we get

$$\begin{aligned}
\left|1 - \left\|P_{d^*,\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2\right| &\leq \left|\frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2}{\lambda_{r_j}^{(m)}} - \left\|P_{d^*,\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2\right| \\
&\quad + \left|1 - \frac{\left\|dP_{\rho_n(\lambda)}\left(g_m^n\left(\alpha_{r_j}^{(m)}\right)\right)\right\|^2}{\lambda_{r_j}^{(m)}}\right| \\
&< \frac{\varepsilon^2}{\mu(\lambda)^2}.
\end{aligned} \tag{5.43}$$

Observe,

$$
\begin{aligned}
\left\| \left(1 - P_{d^*,\rho_n(\lambda)}\right) \left(g_m^n \left(\alpha_{r_j}^{(m)}\right)\right) \right\| &= \left(\sum_{i=1}^{\dim(\ker(\Delta_n))} c_i^2 + \sum_{i \notin \kappa_{d_n^*}(\lambda)} a_i^2 + \sum_{i=1}^{\dim(\mathrm{im}(d_n))} b_i^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{\dim(\ker(\Delta_n))} c_i^2 + \sum_{i=1}^{\dim(\mathrm{im}(d_n^*))} a_i^2 + \sum_{i=1}^{\dim(\mathrm{im}(d_n))} b_i^2 - \sum_{i \in \kappa_{d_n^*}(\lambda)} a_i^2 \right)^{\frac{1}{2}} \\
&= \left(1 - \left\| P_{d^*,\rho_n(\lambda)} \left(g_m^n \left(\alpha_{r_j}^{(m)}\right)\right) \right\|^2 \right)^{\frac{1}{2}} \\
&< \frac{\varepsilon}{\mu(\lambda)}.
\end{aligned}
\tag{5.44}
$$

Now fix $\alpha^{(m)} \in P_{d^*,\rho_m(\lambda)}(\mathcal{H}_m^\bullet)$ of unit norm and write

$$
\sum_{j \in \kappa_{d^*}(\lambda)} a_j \alpha_{r_j}^{(m)} \tag{5.45}
$$

where each $a_j \in [-1, 1]$. We may then apply what has been shown to each summand of $\alpha^{(m)}$ using the triangle inequality:

$$
\begin{aligned}
\left\| \left(1 - P_{d^*,\rho_n(\lambda)}\right) \left(g_m^n \left(\alpha^{(m)}\right)\right) \right\| &\leq \sum_{j \in \kappa_{d^*}(\lambda)} |a_j| \left\| \left(1 - P_{d^*,\rho_n(\lambda)}\right) \left(g_m^n \left(\alpha_{r_j}^{(m)}\right)\right) \right\| \\
&< \sum_{j \in \kappa_{d^*}(\lambda)} |a_j| \frac{\varepsilon}{\mu(\lambda)} \\
&\leq \sum_{j \in \kappa_{d^*}(\lambda)} \frac{\varepsilon}{\mu(\lambda)} \\
&\leq \varepsilon.
\end{aligned}
\tag{5.46}
$$

The last line follows from the previous by noting that $|\kappa_{d^*}(\lambda)| \leq \mu(\lambda)$. □

5.5 Example: Truncation of the Spectrum of the Smooth Laplacian

In this section, we outline a basic example inspired by ideas from renormalization and effective actions in Quantum Field Theory [36, 35]. The strong deformation retraction constructed here is similar to and takes cues from Section 7.1 of [35] and Chapter 4 of [33].

Consider a compact orientable Riemmanian manifold M and n large enough so that $\lambda_n > 0$. Let $\mathcal{H}^\bullet = L^2\Omega^\bullet(M)$ with inner product coming from the Riemmanian metric.

and let $\mathcal{H}_n^\bullet = \bigoplus_{j=0}^n \mathcal{E}_{\lambda_j}$. Let $G\colon L^2\Omega^\bullet(M) \to L^2\Omega^\bullet(M)$ be the Green's operator. Note the following properties:

1. For $m \leq n$, $\mathcal{H}_m^\bullet \subseteq \mathcal{H}_n^\bullet \subseteq \mathcal{H}^\bullet$. Let $g_m^n\colon \mathcal{H}_m^\bullet \to \mathcal{H}_n^\bullet$, $g_n\colon \mathcal{H}_n^\bullet \to \mathcal{H}^\bullet$ be the respective inclusions. Note that these maps are isometries.

2. d, d^*, Δ, G restrict to endomorphisms of $\mathcal{H}_n^\bullet$. Define $\Delta_n = \Delta\big|_{\mathcal{H}_n^\bullet}$.

3. $G\big|_{\mathcal{H}_n^\bullet}$ restricts to an endomorphism of $(g_m^n \mathcal{H}_m^\bullet)^\perp$.

We aim to show that $(\mathcal{H}_n^\bullet, d, \Delta_n)_{n=1}^\infty$ is Cauchy and convergent to $(\mathcal{H}^\bullet, d, \Delta)$. For each n, define

$$\begin{aligned} f_n &= \mathrm{Proj}_{\mathcal{H}_n^\bullet}, \\ h_n &= \mathrm{Proj}_{\bigoplus_{j=n+1}^\infty \mathcal{E}_{\lambda_j}} G d^* = (1 - g_n f_n)\, G d^*. \end{aligned} \tag{5.47}$$

The triple $(f_n, g_n, h_n) : (\mathcal{H}^\bullet, d) \to (\mathcal{H}_n^\bullet, d)$ defines a strong deformation retraction. Observe that

$$\begin{aligned} \left\| (1 - g_n g_n^*)\,(\Delta + 1)^{-\frac{1}{2}} \right\| &\leq \frac{1}{\sqrt{\lambda_{n+1} + 1}}, \\ \left\| (\Delta + 1)^{-\frac{1}{2}} g_n - g_n(\Delta_n + 1)^{-\frac{1}{2}} \right\| &= 0. \end{aligned} \tag{5.48}$$

We know that $\lambda_{n+1} \to \infty$ as $n \to \infty$ and so $\frac{1}{\sqrt{\lambda_{n+1}+1}} \to 0$ as $n \to \infty$. This shows that $(\mathcal{H}_n, d, \Delta_n)_{n=1}^\infty$ is convergent to $(\mathcal{H}, d, \Delta)$. For $m \leq n$, define

$$\begin{aligned} f_n^m &= \mathrm{Proj}_{\mathcal{H}_m^\bullet}\Big|_{\mathcal{H}_n^\bullet}, \\ h_{n,m} &= \mathrm{Proj}_{\bigoplus_{j=m+1}^n \mathcal{E}_{\lambda_j}} G d^*\big|_{\mathcal{H}_n^\bullet} = (1 - g_m^n f_n^m)\, G d^*\big|_{\mathcal{H}_n^\bullet}. \end{aligned} \tag{5.49}$$

The triple $(f_n^m, g_m^n, h_{n,m}) : (\mathcal{H}_n^\bullet, d) \to (\mathcal{H}_m^\bullet, d)$ defines a strong deformation retraction. Moreover, $\mathrm{im}\,(g_m) \subseteq \mathrm{im}\,(g_n)$ and $g_n^* g_m = g_m^n$. So, by Proposition 5.2.5, $(f_n^m, g_m^n, h_{n,m})$ is a $\left(\frac{1}{\sqrt{\lambda_{m+1}+1}} + \frac{1}{\sqrt{\lambda_{n+1}+1}}\right)$-deformation retraction. Since each g_m^n is an inclusion, Equation (5.15) is trivially satisfied and so we see that the sequence is Cauchy.

5.6 Constructing the limit of a Cauchy Sequence of Hilbert Complexes

In this section, we apply the ideas developed in Chapter 3 to the setting of Hilbert complexes. This is a non-trivial task due to the extra structure of a grading, differential, codifferential, the Hodge Laplacian being built from these operators, and the Hodge decomposition. We want the ability to construct d, d^*, Δ_∞ in the limit such that $\Delta_\infty = (d + d^*)^2$, and aim to describe $\mathrm{dom}(d)$, $\mathrm{dom}\,(d^*)$ in such a way that $\mathrm{dom}\,(\Delta_\infty) \subseteq \mathrm{dom}\,(d) \cap \mathrm{dom}\,(d^*)$.

Let $(\mathcal{H}_n^\bullet, d_n, \Delta_n)_{n=1}^\infty$ be a Cauchy sequence. Since elements of differing degree are orthogonal, apply Section 3.2 to each sequence $\left(\mathcal{H}_n^k\right)_{n=1}^\infty$ to obtain a limiting Hilbert space for each degree, $\mathcal{H}_\infty^k$. Let

$$\begin{aligned} \widetilde{\mathcal{H}}_\infty^\bullet &= \bigoplus_{k\in\mathbb{Z}} \widetilde{\mathcal{H}}_\infty^k, \\ \mathcal{H}_\infty^\bullet &= \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_\infty^k. \end{aligned} \tag{5.50}$$

For each n, take the direct sum of the inclusion maps $g_{n,k} \colon \mathcal{H}_n^k \to \mathcal{H}_\infty^k$ constructed in Lemma 3.2.3 to define a map $g_n \colon \mathcal{H}_n^\bullet \to \mathcal{H}_\infty^\bullet$. I.e. define

$$g_n := \bigoplus_{k\in\mathbb{Z}} g_{n,k} \colon \mathcal{H}_n^\bullet \to \mathcal{H}_\infty^\bullet, \tag{5.51}$$

where g_n is an isometry because each of the constituent maps $g_{n,k}$ is an isometry. Apply the results of Section 3.3 with $G_n^{\frac{1}{2}}$ in place of G_n to construct an operator $\widetilde{G}_\infty^{\frac{1}{2}}$ on $\widetilde{\mathcal{H}}_\infty^\bullet$ defined on $(\alpha_n)_{n=1}^\infty \in \widetilde{\mathcal{H}}_\infty^\bullet$ via

$$\widetilde{G}_\infty^{\frac{1}{2}}\left((\alpha_n)_{n=1}^\infty\right) = \left(G_n^{\frac{1}{2}}(\alpha_n)\right)_{n=1}^\infty. \tag{5.52}$$

The arguments of Section 3.3 apply to show that $\widetilde{G}_\infty^{\frac{1}{2}}$ induces an operator $G_\infty^{\frac{1}{2}}$ on $\mathcal{H}_\infty^\bullet$ such that $G_\infty^{\frac{1}{2}} \circ \pi = \pi \widetilde{G}_\infty^{\frac{1}{2}}$ and

$$\begin{aligned} \left\| G_\infty^{\frac{1}{2}} g_n - g_n G_n^{\frac{1}{2}} \right\| &< \frac{\delta_n}{2}, \\ \left\| \left(1 - g_n g_n^*\right) G_\infty^{\frac{1}{2}} \right\| &< \frac{\delta_n}{2}. \end{aligned} \tag{5.53}$$

By definition of $\widetilde{G}_\infty, \widetilde{G}_\infty^{\frac{1}{2}}$, we have $\left(G_\infty^{\frac{1}{2}}\right)^2 = G_\infty$. Following the argument of Lemma 3.3.5, construct the self-adjoint operator $(\Delta_\infty + 1)^{\frac{1}{2}}$ on $\mathcal{H}_\infty^\bullet$ with $\operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right) = \operatorname{im}\left(G_\infty^{\frac{1}{2}}\right)$ such that $\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)^2 = \Delta_\infty + 1$.

A tempting approach for the definition of d is to define it on $\widetilde{\mathcal{H}}_\infty^\bullet$ via

$$(\alpha_n)_{n=1}^\infty \mapsto (d\alpha_n)_{n=1}^\infty. \tag{5.54}$$

Without restricting the domain of d, this approach will fail since it is not clear that if

$$\lim_{m\to\infty} \sup_{n\geq m} \left\| \alpha_n - g_m^n(\alpha_m) \right\| = 0 \tag{5.55}$$

then,

$$\lim_{m\to\infty}\sup_{n\geq m}\left\|d\alpha_n - g_m^n\left(d\alpha_m\right)\right\| = 0. \tag{5.56}$$

Moreover, $dK \not\subseteq K$ in general. We elect to take a slightly more obscure path towards defining d which will provide a concise domain of definition and result in the same map as above on its domain. As an overview, we define $D_n = dG_n^{\frac{1}{2}}$ for each n, show that this induces a well-defined degree $+1$ map $\widetilde{D}_\infty$ on $\widetilde{\mathcal{H}}_\infty^\bullet$ which induces a map D_∞ on $\mathcal{H}_\infty^\bullet$, and finally define $d = D_\infty\left(\Delta_\infty+1\right)^{\frac{1}{2}}$ on $\operatorname{dom}\left(\left(\Delta_\infty+1\right)^{\frac{1}{2}}\right) \subseteq \mathcal{H}_\infty^\bullet$.

Lemma 5.6.1. For all n, $\|D_n\| \leq 1$.

Proof. Let $\alpha \in \mathcal{H}_n^\bullet$ be a unit and write

$$\alpha = \sum_{i=1}^{\ker(\Delta_n)} c_i\gamma_i^{(n)} + \sum_{i=1}^{\dim(\operatorname{im}(d_n^*))} a_i\alpha_{r_i}^{(n)} + \sum_{i=1}^{\dim(\operatorname{im}(d_n))} b_i\beta_{s_i}^{(n)}. \tag{5.57}$$

We know that d maps each α_{r_i} to $\sqrt{\lambda_{r_i}^{(n)}}\beta_{s_j}$ for some j and also that $G_n^{\frac{1}{2}}\alpha_{r_i} = \frac{1}{\sqrt{1+\lambda_{r_i}^{(n)}}}\alpha_{r_i}$. So,

$$\begin{aligned}
\|D_n(\alpha)\|^2 &= \left\|D_n\left(\sum_{i=1}^{\ker(\Delta_n)} c_i\gamma_i^{(n)} + \sum_{i=1}^{\dim(\operatorname{im}(d_n^*))} a_i\alpha_{r_i}^{(n)} + \sum_{i=1}^{\dim(\operatorname{im}(d_n))} b_i\beta_{s_i}^{(n)}\right)\right\|^2 \\
&= \sum_{i=1}^{\dim(\operatorname{im}(d_n^*))} a_i^2\frac{\sqrt{\lambda_{r_i}}}{\sqrt{1+\lambda_{r_i}}} \\
&\leq \sum_{i=1}^{\dim(\operatorname{im}(d_n^*))} a_i^2 \\
&\leq \|\alpha\|^2.
\end{aligned} \tag{5.58}$$

So, $\|D_n\| \leq 1$. ☐

Lemma 5.6.2. The map $\widetilde{D}_\infty\colon \bigcup_{n\in\mathbb{N}}\operatorname{im}\left(\widetilde{g}_n\right) \subseteq \widetilde{\mathcal{H}}_\infty^\bullet \to \widetilde{\mathcal{H}}_\infty^\bullet$ given by

$$\widetilde{D}_\infty\left(\alpha_n\right)_{n=1}^\infty = \left(D_n(\alpha_n)\right)_{n=1}^\infty \tag{5.59}$$

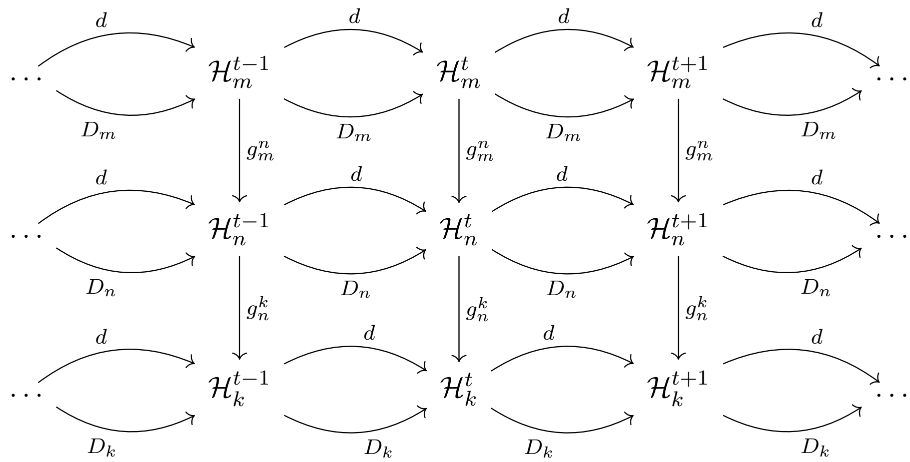
is a well-defined degree $+1$ map. Moreover,

$$\widetilde{D}_\infty\left(\bigcup_{n\in\mathbb{N}}\operatorname{im}\left(\widetilde{g}_n\right)\cap K\right) \subseteq K. \tag{5.60}$$

Proof. Fix $m \in \mathbb{N}$ and let $\alpha \in \mathcal{H}_m^\bullet$ with unit norm. We want to show that $\widetilde{D}_\infty$ is well defined on $\widetilde{g}_m(\alpha) = (g_m^n(\alpha))_{n=m}^{\infty} \in \widetilde{\mathcal{H}}_{\infty}^{\bullet}$. To do this, we will show that for every $\varepsilon > 0$, and $n \in \mathbb{N}$ sufficiently large,

$$\sup_{k \geq n} \left\| D_k g_n^k \left(g_m^n(\alpha)\right) - g_n^k \left(D_n g_m^n(\alpha)\right)\right\| < \varepsilon. \tag{5.61}$$

The setting is summarized in the following diagram:



By Equation (5.15) and discreteness of $\sigma(\Delta_\infty)$, we may select $B \in \mathbb{R}_+ \setminus \sigma(\Delta_\infty)$ and $n \in \mathbb{N}$ large enough so that for all $k \geq n$,

$$\left\| P_{[B,\infty)} g_m^k \right\| < \frac{\varepsilon}{3}. \tag{5.62}$$

By taking n larger if necessary, we can ensure that for all $k \geq n$,

1. $\left\| G_k^{\frac{1}{2}} g_n^k - g_n^k G_n^{\frac{1}{2}} \right\| < \frac{\varepsilon}{3\sqrt{B}(1+|\kappa([0,B))|)}$. This is valid because $|\kappa([0,B))| < \infty$ by Theorem 4.1.3.

2. $\rho_k([0,B)) \subseteq [0,B)$ and $\rho_k([B,\infty)) \cap [0,B] = \emptyset$. We can do this by Theorem 4.1.1, Corollary 4.1.4, and our assumption that $B \notin \sigma(\Delta_\infty)$.

For each $k \geq m$, write

$$g_m^k(\alpha) = \sum_{i=1}^{\ker(\Delta_n)} c_i^{(k)} \gamma_i^{(k)} + \sum_{i=1}^{\dim(\mathrm{im}(d_n^*))} a_i^{(k)} \alpha_{r_i}^{(k)} + \sum_{i=1}^{\dim(\mathrm{im}(d_n))} b_i^{(k)} \beta_{s_i}^{(k)}. \tag{5.63}$$

Notice that $D_k P_{[B,\infty)} = P_{[B,\infty)} D_k$ since $D_k = dG_k^{\frac{1}{2}}$ has the same eigenspaces as Δ_k. Now observe

$$\begin{aligned}
\sup_{k\geq n}\left\|D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right\| &\leq \sup_{k\geq n}\left\|P_{[0,B)}\left(D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right)\right\| \\
&\quad + \left\|P_{[B,\infty)}D_k g_m^k(\alpha)\right\| + \left\|P_{[B,\infty)}D_n g_m^n(\alpha)\right\| \\
&= \sup_{k\geq n}\left\|P_{[0,B)}\left(D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right)\right\| \\
&\quad + \left\|D_k P_{[B,\infty)} g_m^k(\alpha)\right\| + \left\|D_n P_{[B,\infty)} g_m^n(\alpha)\right\| \\
&\leq \sup_{k\geq n}\left\|P_{[0,B)}\left(D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right)\right\| + \frac{2}{3}\varepsilon,
\end{aligned} \tag{5.64}$$

where the last line follows from Lemma 5.6.1 and Equation (5.62). Now, by assumption 2 above,

$$P_{[0,B)}\left(\mathcal{H}_k^\bullet\right) \subseteq \bigoplus_{\lambda_n^{(k)}\in\rho_k([0,B))} \mathcal{E}_{\lambda_i^{(k)}} = \bigoplus_{i\in\kappa([0,B))} \mathcal{E}_{\lambda_i^{(k)}} \tag{5.65}$$

and

$$\max \rho_k\left([0,B)\right) \leq B. \tag{5.66}$$

So,

$$\begin{aligned}
\sup_{k\geq n}\left\|P_{[0,B)}\left(D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right)\right\| &\leq \sup_{k\geq n}\left\|\sum_{i\in\kappa([0,B))}\left(G_k^{\frac{1}{2}} g_n^k - g_n^k G_n^{\frac{1}{2}}\right)\left(a_i^{(n)} d\alpha_{r_i}^{(n)}\right)\right\| \\
&\leq \sup_{k\geq n}\left\|G_k^{\frac{1}{2}} g_n^k - g_n^k G_n^{\frac{1}{2}}\right\| \sum_{i\in\kappa([0,B))}\left|a_i^{(n)}\right|\sqrt{\lambda_i^{(n)}} \\
&< \frac{\varepsilon\sqrt{B}|\kappa([0,B))|}{3\sqrt{B}(1+|\kappa([0,B))|)}, \quad \text{since each } \left|a_i^{(n)}\right| \leq 1 \\
&< \frac{\varepsilon}{3}.
\end{aligned} \tag{5.67}$$

Thus,

$$\sup_{k\geq n}\left\|D_k g_m^k(\alpha) - g_n^k\left(D_n g_m^n(\alpha)\right)\right\| < \varepsilon \tag{5.68}$$

and so $\widetilde{D}_\infty$ is well defined on $\bigcup_{n\in\mathbb{N}} \operatorname{im}\left(\widetilde{g}_n\right)$. Now, for

$$\alpha \in \bigcup_{n\in\mathbb{N}} \operatorname{im}\left(\widetilde{g}_n\right) \cap K \tag{5.69}$$

we have $\left\|\widetilde{D}_\infty(\alpha)\right\| \le \|\alpha\| = 0$ by Lemma 5.6.1 and so

$$\widetilde{D}_\infty(\alpha) \in K. \tag{5.70}$$

□

Lemma 5.6.3. $\widetilde{D}_\infty$ induces a continuous map D_∞ on $\mathcal{H}^\bullet_\infty$ with $\operatorname{dom}(D_\infty) = \mathcal{H}^\bullet_\infty$.

Proof. Lemma 5.6.2 implies that $\widetilde{D}_\infty$ induces a map D_∞ on $\mathcal{H}^\bullet_\infty$ with

$$\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n) \subseteq \operatorname{dom}(D_\infty) \tag{5.71}$$

and so D_∞ is densely defined. Lemma 5.6.1 implies that $\|D_\infty\| \le 1$ and thus D_∞ is continuous. Consequently, we may continuously extend D_∞ to be well defined on all of $\mathcal{H}^\bullet_\infty$. □

Definition 5.6.4. Define $d\colon \mathcal{H}^\bullet_\infty \to \mathcal{H}^\bullet_\infty$ by

$$d = (\Delta_\infty + 1)^{\frac{1}{2}} D_\infty \tag{5.72}$$

and observe that

$$d^* = D^*_\infty (\Delta_\infty + 1)^{\frac{1}{2}}. \tag{5.73}$$

Lemma 5.6.5. The following are true of d:

1. $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_n) \subseteq \operatorname{dom}(d)$.
2. $D_\infty G_\infty^{\frac{1}{2}} = G_\infty^{\frac{1}{2}} D_\infty$.
3. $\operatorname{dom}(\Delta_\infty) \subseteq \operatorname{dom}(d)$.
4. $d(\operatorname{dom}(\Delta_\infty)) \subseteq \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right) \subseteq \operatorname{dom}(d)$.
5. $\operatorname{im}(d) \subseteq \ker(d)$.

Proof. We verify the claims in order. Let $\alpha \in \mathcal{H}_n^\bullet$, then

$$\begin{aligned}
D_\infty g_n(\alpha) &= \pi\left(\widetilde{D}_\infty \widetilde{g}_n(\alpha)\right) \\
&= \pi\left(\left(D_k g_n^k(\alpha)\right)_{k=n}^\infty\right) \\
&= \pi\left(\left(dG_k^{\frac{1}{2}} g_n^k(\alpha)\right)_{k=n}^\infty\right) \\
&= \pi\left(\left(G_k^{\frac{1}{2}} g_n^k(d\alpha)\right)_{k=n}^\infty\right) \\
&= \pi\left(\widetilde{G}_\infty^{\frac{1}{2}}\left(\widetilde{g}_n(d\alpha)\right)\right) \\
&= G_\infty^{\frac{1}{2}}\left(g_n(d\alpha)\right).
\end{aligned} \tag{5.74}$$

We know that $\operatorname{im}\left(G_\infty^{\frac{1}{2}}\right) = \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$ and so $D_\infty g_n(\alpha) \in \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$, implying that $g_n(\alpha) \in \operatorname{dom}(d)$.

Now let $\alpha \in \mathcal{H}_\infty^\bullet$, pick $(\alpha_n)_{n=1}^\infty \in \pi^{-1}(\alpha)$. Then,

$$\begin{aligned}
D_\infty G_\infty^{\frac{1}{2}} \alpha &= \pi\left(\left(D_n G_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^\infty\right) \\
&= \pi\left(\left(dG_n^{\frac{1}{2}} G_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^\infty\right) \\
&= \pi\left(\left(G_n^{\frac{1}{2}} dG_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^\infty\right) \\
&= G_\infty^{\frac{1}{2}} D_\infty \alpha.
\end{aligned} \tag{5.75}$$

Since $\operatorname{dom}(\Delta_\infty) = \operatorname{im}(G_\infty)$, the above shows that $D_\infty(\operatorname{dom}(\Delta_\infty)) \subseteq \operatorname{dom}(\Delta_\infty)$. Since $\operatorname{dom}(\Delta_\infty) \subseteq \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$, we have that $\operatorname{dom}(\Delta_\infty) \subseteq \operatorname{dom}(d)$. Moreover,

$$(\Delta_\infty + 1)^{\frac{1}{2}}(\operatorname{dom}(\Delta_\infty)) \subseteq \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right) \tag{5.76}$$

and so

$$d(\operatorname{dom}(\Delta_\infty)) \subseteq \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right). \tag{5.77}$$

For $\alpha \in \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$, $(\alpha_n)_{n=1}^{\infty} \in \pi^{-1}(\alpha) \cap \operatorname{dom}\left(\left(\widetilde{\Delta}_\infty + 1\right)^{\frac{1}{2}}\right)$, we have

$$\begin{aligned} D_\infty (\Delta_\infty + 1)^{\frac{1}{2}} \alpha &= \pi\left(\left(D_n (\Delta_n + 1)^{\frac{1}{2}} \alpha_n\right)_{n=1}^{\infty}\right) \\ &= \pi\left(\left(dG_n^{\frac{1}{2}} (\Delta_n + 1)^{\frac{1}{2}} \alpha_n\right)_{n=1}^{\infty}\right) \\ &= \pi\left(\left((\Delta_n + 1)^{\frac{1}{2}} dG_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^{\infty}\right) \\ &= (\Delta_\infty + 1)^{\frac{1}{2}} D_\infty \alpha \\ &= d\alpha. \end{aligned} \tag{5.78}$$

This implies that $\operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right) \subseteq \operatorname{dom}(d)$.

For the final claim, let $\alpha \in \operatorname{dom}(d)$, $(\alpha_n)_{n=1}^{\infty} \in \pi^{-1}(\alpha) \cap \operatorname{dom}\left(\left(\widetilde{\Delta}_\infty + 1\right)^{\frac{1}{2}} \widetilde{D}_\infty\right)$. Then,

$$\begin{aligned} D_\infty d\alpha &= D_\infty (\Delta_\infty + 1)^{\frac{1}{2}} D_\infty \alpha \\ &= \pi\left(\left(dG_n^{\frac{1}{2}} (\Delta_n + 1)^{\frac{1}{2}} dG_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^{\infty}\right) \\ &= \pi\left(\left(d^2 G_n^{\frac{1}{2}} \alpha_n\right)_{n=1}^{\infty}\right) \\ &= 0 \end{aligned} \tag{5.79}$$

and so $D_\infty d\alpha = 0 \in \operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$, implying that $\operatorname{im}(d) \subseteq \ker(d)$. ☐

Lemma 5.6.6. *d is a closed operator.*

Proof. Recall that $(\Delta_\infty + 1)^{\frac{1}{2}}$ is self-adjoint, thus closed while D_∞ is continuous and so $d = (\Delta_\infty + 1)^{\frac{1}{2}} D_\infty$ is closed. ☐

The above allows us to conclude that $(\mathcal{H}_\infty^\bullet, d)$ is a Hilbert complex and we turn our attention to the codifferential map d^*. Because D_∞ is bounded, the adjoint operator D_∞^* is bounded and defined everywhere. Thus, d^* is well defined on $\operatorname{dom}\left((\Delta_\infty + 1)^{\frac{1}{2}}\right)$. In particular, by Lemma 5.6.5,

$$\operatorname{dom}(\Delta_\infty) \subseteq \operatorname{dom}(d) \cap \operatorname{dom}(d^*) \tag{5.80}$$

Moreover, $G_\infty^{\frac{1}{2}} D_\infty^* = D_\infty^* G_\infty^{\frac{1}{2}}$ and so, for $\alpha \in \mathcal{H}_\infty^\bullet$,

$$\begin{aligned} d^* G_\infty(\alpha) &= D_\infty^* \left(\Delta_\infty + 1\right)^{\frac{1}{2}} G_\infty(\alpha) \\ &= D_\infty^* G_\infty^{\frac{1}{2}}(\alpha) \\ &= G_\infty^{\frac{1}{2}} D_\infty^*(\alpha) \end{aligned} \tag{5.81}$$

This implies that

$$d^* \left(\operatorname{dom}\left(\Delta_\infty\right)\right) \subseteq \operatorname{dom}\left(\left(\Delta_\infty + 1\right)^{\frac{1}{2}}\right) \subseteq \operatorname{dom}(d). \tag{5.82}$$

On the other hand, Lemma 5.6.5 tells us

$$d \left(\operatorname{dom}\left(\Delta_\infty\right)\right) \subseteq \operatorname{dom}\left(\left(\Delta_\infty + 1\right)^{\frac{1}{2}}\right) \subseteq \operatorname{dom}\left(d^*\right). \tag{5.83}$$

So, $(d + d^*)^2$ is defined on all of $\operatorname{dom}(\Delta_\infty)$. By Lemma 3.3.5 and the remarks above, we have $(d + d^*)^2 = \Delta_\infty$ as expected.

Lemma 5.6.7. We have the following orthogonal decomposition:

$$\mathcal{H}_\infty^\bullet = \ker\left(\Delta_\infty\right) \oplus \operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right), \tag{5.84}$$

where $\operatorname{im}(d)$ and $\operatorname{im}(d^*)$ are closed. Moreover,

$$\ker\left(\Delta_\infty\right) = \ker(d) \cap \ker\left(d^*\right). \tag{5.85}$$

Proof. Suppose $\alpha \in \ker(\Delta_\infty)$. Then,

$$\begin{aligned} 0 &= \langle \Delta_\infty \alpha, \alpha \rangle \\ &= \langle d d^* \alpha, \alpha \rangle + \langle d^* d\alpha, \alpha \rangle \\ &= \|d^* \alpha\|^2 + \|d\alpha\|^2 . \end{aligned} \tag{5.86}$$

This implies $\ker(\Delta_\infty) = \ker(d) \cap \ker(d^*)$. Moreover, for all $\beta \in \operatorname{dom}(d)$,

$$\langle \alpha, d\beta \rangle = \langle d^* \alpha, \beta \rangle = 0. \tag{5.87}$$

So $\ker(\Delta_\infty)$ is orthogonal to $\operatorname{im}(d)$ and similarly, $\ker(\Delta_\infty)$ is orthogonal to $\operatorname{im}(d^*)$. In summary,

$$\operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right) \subseteq \ker\left(\Delta_\infty\right)^\perp . \tag{5.88}$$

Now suppose $\alpha \in \mathcal{H}^\bullet_\infty$ is an eigenvector corresponding to eigenvalue $\lambda > 0$. Then,

$$\alpha = \frac{1}{\lambda}\left(dd^*\alpha + d^*d\alpha\right) \in \operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right). \tag{5.89}$$

So,

$$\operatorname{im}\left(\Delta_\infty\right) \subseteq \operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right). \tag{5.90}$$

Fix $\alpha \in \ker\left(\Delta_\infty\right)^\perp$ and write

$$\alpha = \sum_{j=1}^{\infty} a_j \varphi_j, \tag{5.91}$$

where each $a_j \in \mathbb{R}$ and the collection $(\varphi_j)_{j=1}^{\infty}$ form an orthonormal eigenbasis for $\operatorname{dom}\left(\Delta_\infty\right) \cap \ker\left(\Delta_\infty\right)^\perp$ (corresponding to the non-zero eigenvalues). Theorem 4.1.3 implies that the number of eigenvalues (with multiplicity) in $\sigma\left(\Delta_\infty\right) \cap (0,1)$ is finite and so the following definition is valid:

$$\alpha' = \sum_{j=1}^{\infty} \frac{a_j}{\lambda_j} \varphi_j \in \mathcal{H}^\bullet_\infty. \tag{5.92}$$

For each n, define

$$\alpha_n = \sum_{j=1}^{n} \frac{a_j}{\lambda_j} \varphi_j \in \operatorname{dom}\left(\Delta_\infty\right). \tag{5.93}$$

Observe that

$$\alpha_n \to \alpha', \tag{5.94}$$

since the λ_j are written in non-decreasing order, while

$$\Delta_\infty(\alpha_n) = \sum_{j=1}^{n} a_j \varphi_j \to \alpha. \tag{5.95}$$

Recall that Δ_∞ is a closed operator and so the above implies $\alpha' \in \operatorname{dom}\left(\Delta_\infty\right)$ and $\Delta_\infty(\alpha') = \alpha$. This implies that

$$\ker\left(\Delta_\infty\right)^\perp \subseteq \operatorname{im}\left(\Delta_\infty\right). \tag{5.96}$$

On the other hand, for $\beta \in \ker\left(\Delta_\infty\right)$ and $\alpha \in \operatorname{dom}\left(\Delta_\infty\right)$,

$$\langle \beta, \Delta_\infty \alpha \rangle = \langle \Delta_\infty \beta, \alpha \rangle = 0. \tag{5.97}$$

So,

$$\ker\left(\Delta_\infty\right)^\perp = \operatorname{im}\left(\Delta_\infty\right) = \operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right) \tag{5.98}$$

which implies that $\operatorname{im}(d) \oplus \operatorname{im}\left(d^*\right)$ is closed. Since $\operatorname{im}(d), \operatorname{im}\left(d^*\right)$ are orthogonal, we have that each subspace is closed. In conclusion,

$$\begin{aligned}\mathcal{H}_\infty^\bullet &= \ker(\Delta_\infty) \oplus \ker(\Delta_\infty)^\perp \\ &= \ker(\Delta_\infty) \oplus \operatorname{im}(d) \oplus \operatorname{im}(d^*)\,.\end{aligned} \tag{5.99}$$

□

Note that $\mu(0) < \infty$ and so Lemma 5.6.7 tells us that $(\mathcal{H}_\infty^\bullet, d)$ is a Fredholm complex.

We aim to construct strong deformation retractions $(f_m, g_m, h_m) : (\mathcal{H}_\infty^\bullet, d, \Delta_\infty) \to (\mathcal{H}_m^\bullet, d, \Delta_m)$ for each n to show that $(\mathcal{H}_n^\bullet, d, \Delta_n)_{n=1}^\infty$ is convergent to $(\mathcal{H}_\infty^\bullet, d, \Delta_\infty)$. We will proceed by defining each f_m, h_m on $\bigcup_{n\in\mathbb{N}} \operatorname{im}(g_m)$. Let $\alpha \in \bigcup_{n\in\mathbb{N}} \operatorname{im}(g_m)$ and write $\alpha = g_\ell(\beta)$ for $\beta \in \mathcal{H}_\ell^\bullet$, $\ell \in \mathbb{N}$. Define

$$f_m(\alpha) = \begin{cases} f_\ell^m(\beta) & \text{if } \ell > m, \\ g_\ell^m(\beta) & \text{if } \ell \leq m, \end{cases} \tag{5.100}$$

and

$$h_m(\alpha) = \begin{cases} g_\ell(h_{\ell,m}(\beta)) & \text{if } \ell > m, \\ 0 & \text{if } \ell \leq m. \end{cases} \tag{5.101}$$

By Proposition 5.2.8, if the sequence $(\mathcal{H}_n^\bullet, d, \Delta_n)_{n=1}^\infty$ is convergent to $(\mathcal{H}_\infty^\bullet, d, \Delta_\infty)$, then it must be that there exists an n-independent spectral gap in each $\sigma(\Delta_n)$. So, in order for (f_m, g_m, h_m) to be a strong deformation retraction, we must assume this property of the sequence.

Proposition 5.6.8. If the sequence of spectra $\sigma(\Delta_n)$ possesses a spectral gap in the sense that

$$\lim_{n\to\infty} \min \sigma(\Delta_n) \setminus \{0\} > 0 \tag{5.102}$$

then, $(f_m, g_m, h_m) : (\mathcal{H}_\infty^\bullet, d, \Delta_\infty) \to (\mathcal{H}_m^\bullet, d, \Delta_m)$ is a strong deformation retraction.

Proof. We must verify the numbered conditions under 5.2.1. We already know that g_m is a cochain map with $\operatorname{dom}(g_m) = \mathcal{H}_m^\bullet$ and that $\operatorname{im}(g_m) \subseteq \operatorname{dom}(f_m)$. To see that f_m is a cochain map, let $\alpha \in \bigcup_{n\in\mathbb{N}} \operatorname{im}(g_m)$ where $\alpha = g_\ell(\beta)$. First assume $\ell > m$, then

$$\begin{aligned} df_m(\alpha) &= df_\ell^m(\beta) \\ &= f_\ell^m(d\beta) \\ &= f_m(g_\ell(d\beta)) \\ &= f_m(dg_\ell(\beta)) \\ &= f_m(d\alpha) \end{aligned} \tag{5.103}$$

and so f_m is a cochain map. Since g_m is a cochain map, we have that for $\ell > m$,

$$\begin{aligned} (dh_m + h_m d)(\alpha) &= dg_\ell(h_{\ell,m}(\beta)) + h_m dg_\ell(\beta) \\ &= g_\ell(dh_{\ell,m}(\beta)) + g_\ell(h_{\ell,m}(d\beta)) \\ &= g_\ell(dh_{\ell,m} + h_{\ell,m}d)(\beta) \\ &= g_\ell\left(1 - g_m^\ell f_\ell^m\right)(\beta) \\ &= (1 - g_m f_m)(\alpha). \end{aligned} \tag{5.104}$$

For $\ell \leq m$,

$$\begin{aligned} (dh_m + h_m d)(\alpha) &= 0 \\ &= \alpha - g_\ell(\beta) \\ &= \alpha - g_m g_\ell^m(\beta) \\ &= (1 - g_m f_m)(\alpha). \end{aligned} \tag{5.105}$$

The relation $f_m g_m = 1$ is immediate from the definition as is $h_m g_m = 0$. To see that $f_m h_m = 0$, let $\alpha \in \bigcup_{n\in\mathbb{N}} \operatorname{im}(g_m)$ where $\alpha = g_\ell(\beta)$ and suppose first that $\ell > m$. Then,

$$\begin{aligned} f_m h_m(\alpha) &= f_m g_\ell(h_{\ell,m}(\beta)) \\ &= f_\ell^m h_{\ell,m}(\beta) \\ &= 0. \end{aligned} \tag{5.106}$$

If $\ell \leq m$, then $h_m(\alpha) = 0$ and so $f_m h_m(\alpha) = 0$.

To see that $(\operatorname{dom}(f_m), d)$ has cohomology isomorphic to $(\mathcal{H}_\infty^\bullet, d)$, note first that the existence of a strong deformation retraction from $(\operatorname{dom}(f_m), d)$ to $(\mathcal{H}_m^\bullet, d)$ implies an isomorphism on the level of cohomology and so it suffices to show that the cohomology of $(\mathcal{H}_\infty^\bullet, d)$ is isomorphic to that of $(\mathcal{H}_m^\bullet, d)$ for any m. We will do this by showing that the dimension of the cohomology groups are equal in each degree.

For each degree i, we know that the dimension of the cohomology in degree i is equal to the dimension of the harmonic forms in degree i. In particular, the dimension of the cohomology in degree i of $(\mathcal{H}^\bullet_\infty, d)$ is equal to the multiplicity of 0 as an eigenvalue of $\Delta_\infty\big|_{\mathcal{H}^i_\infty}$ and likewise for $(\mathcal{H}^\bullet_m, d)$. Let

$$L = \frac{1}{2}\min\Big(d\left(\sigma\left(\Delta_\infty\right)\setminus\{0\},0\right), \lim_{m\to\infty} d\left(\sigma\left(\Delta_m\right)\setminus\{0\},0\right)\Big). \tag{5.107}$$

By our assumption on the existence of a spectral gap and for m large enough,

$$\sigma\left(\Delta_m\right)\cap\overline{B_L(0)} = \{0\}. \tag{5.108}$$

We know that $\left(\mathcal{H}^i_n, \Delta_n\big|_{\mathcal{H}^i_n}\right)_{n=1}^\infty$ is Cauchy and converges to $\left(\mathcal{H}^i_\infty, \Delta_\infty\big|_{\mathcal{H}^i_\infty}\right)$. Theorem 4.1.1 tells us that, by taking m large enough, we obtain

$$\dim\operatorname{im}\left(P_{\Delta_m\big|_{\mathcal{H}^i_m},\overline{B_L(0)}}\right) = \dim\operatorname{im}\left(P_{\Delta_\infty\big|_{\mathcal{H}^i_\infty},\overline{B_L(0)}}\right). \tag{5.109}$$

The left hand side corresponds to the multiplicity of 0 as an eigenvalue of $\Delta_m\big|_{\mathcal{H}^i_m}$ while the right hand side corresponds to the multiplicity of 0 as an eigenvalue of $\Delta_\infty\big|_{\mathcal{H}^i_\infty}$. This implies that the cohomology groups of $\left(\mathcal{H}^i_\infty, d\right)$ and $\left(\mathcal{H}^i_m, d\right)$ are isomorphic and so (f_m, g_m, h_m) is a strong deformation retraction. ☐

Due to the remarks at the beginning of this section, the arguments from Chapter 3 along with Proposition 5.6.8 show that the existence of a spectral gap implies that $(\mathcal{H}^\bullet_n, d_n, \Delta_n)_{n=1}^\infty$ converges to $(\mathcal{H}^\bullet_\infty, d, \Delta_\infty)$.

Chapter 6

Application: Simplicial Approximation of the Hodge Laplacian

Let M be a compact orientable Riemannian manifold without boundary. Consider the Hilbert complex $\left(L^2\Omega^\bullet(M), d\right)$ where the inner product is induced by the Riemannian metric and d is the closure of the exterior derivative on smooth forms. In this chapter, we will exhibit an approximation scheme $(\mathcal{H}_n^\bullet, d, \Delta_n)_{n=1}^\infty$ of $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ which exhibits convergence and is also Cauchy. The Hilbert spaces of this sequence will be given by cochain complexes of triangulations of M (to be defined momentarily) equipped with inner products and should be thought of as discrete approximations of $\Omega^\bullet(M)$. The work in this section serves to add structure that provides computational insight into the results of [14, 15, 50].

A method similar to the notion of convergence defined in Section 2.1 was used to study convergence of the discrete Hodge-Dirac operator on the square lattice $h\mathbb{Z}^n$ as $h \to 0$ in [34]. In addition, in Chapter 5 of [37], Müller shows that the resolvent of the discrete Green's operator converges to that of the Green's operator on $L^2\Omega^\bullet(M)$. While these references were not explicitly used in the preparation of this section, they were useful as indicators of the validity of our approach.

6.1 Background and Notation for Simplicial Complexes

We begin by recalling the basics of simplicial complexes from Chapter 2 of [19] and Chapter 5 of [31]. This section mostly serves to establish notation.

Definition 6.1.1 (Section 2.1 [19])**.** For $n \in \mathbb{N}$, an *n-simplex*, alternatively a *simplex of di-*

mension n, is the convex hull of a collection of $n+1$ distinct points $v_0, \ldots, v_n$ in a Euclidean space. We impose the further constraint that the collection $v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0$ is linearly independent. The points $v_0, \ldots, v_n$ are referred to as the *vertices* of the simplex and we denote the corresponding n-simplex by $[v_0, \ldots, v_n]$.

For $0 \leq k < n$, a *k-face* of an n-simplex is the convex hull of a size $k+1$ subset of its vertices.

The *standard n-simplex*, denoted Δ^n, is the n-simplex given by $[0, e_1, \ldots, e_n]$ where e_i is the ith standard basis vector in $\mathbb{R}^n$.

Definition 6.1.2 (Chapter 5 [31])**.** Given an n-simplex $[v_0, \ldots, v_n]$ and $x \in [v_0, \ldots, v_n]$, x can be written as

$$x = \sum_{i=0}^{n} t_i v_i, \tag{6.1}$$

where each $t_i \in [0,1]$ and are such that

$$\sum_{i=0}^{n} t_i = 1. \tag{6.2}$$

We refer to the numbers t_i as the *barycentric coordinates* of x. The last n barycentric coordinates yield a homeomorphism $[v_0, \ldots, v_n] \cong \Delta^n$ and a diffeomorphism from the interior of $[v_0, \ldots, v_n]$ to the interior of Δ^n.

Definition 6.1.3. A *simplicial complex*, K, in $\mathbb{R}^N$ is a finite set of simplices such that

1. Each simplex is contained in $\mathbb{R}^N$.
2. Every face of every simplex in K is contained in K.
3. The intersection of any two simplices of K is either empty or a face of both.

We will denote the set of n-simplices of K by K_n.

Notice that a simplicial complex determines a subset of $\mathbb{R}^N$ via the union over all simplices. We will freely regard simplicial complexes in this way in order to avoid the notions of *geometric realization* or *abstract simplicial complexes*. We will also assume that K_0 possesses a linear order.

Definition 6.1.4. The associated chain complex on K is the linear span of simplices with coefficients in $\mathbb{R}$, $C_\bullet(K)$, equipped with grading given by simplex dimension and the differential ∂ defined via

$$\partial[v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_{i-1}, \hat{v_i}, v_{i+1}, \dots, v_n]. \tag{6.3}$$

We will also account for orientation of simplices by dictating that for any permutation $\sigma \in S_n$,

$$[v_0, \dots, v_n] = \mathrm{sign}(\sigma) \cdot [v_{\sigma(0)}, v_{\sigma(1)}, \dots, v_{\sigma(n)}]. \tag{6.4}$$

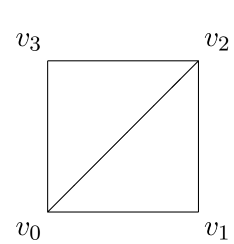
The cochain complex $(C^\bullet(K), d)$ is given by the dual of $(C_\bullet(K), \partial)$.

6.2 Subdivision Algorithms for a Simplicial Complex

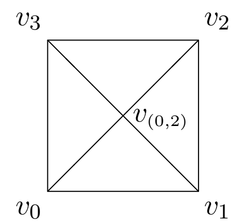
Definition 6.2.1 (Adapted from Definition 2.22 [30])**.** Given a simplicial complex K and simplex $\sigma \in K$, the *stellar subdivision* of K with respect to σ, $\star_\sigma K$, is the simplicial complex produced by the following algorithm:

1. Start with $\star_\sigma K = \emptyset$.
2. Let v_σ be the barycenter of σ and let $(\star_\sigma K)_0 = K_0 \cup \{v_\sigma\}$. If σ is itself a vertex, then $v_\sigma = \sigma$, and we terminate the algorithm with $\star_\sigma K = K$.
3. For each $\tau = [v_{i_0}, \dots, v_{i_k}] \in K_k$,
   (a) If σ is not a face of τ:
      i. Add τ to $(\star_\sigma K)_k$.
      ii. If there exists $\beta \in K$ such that τ and σ are faces of β, then add $[v_\sigma, v_{i_0}, \dots, v_{i_k}]$ to $(\star_\sigma K)_k$.
   (b) If σ is a face of τ, do nothing.

Example 6.2.2. Let K be the following simplicial complex:



where $[v_0, v_1, v_2]$ and $[v_0, v_2, v_3]$ are present in K. The stellar subdivision of K with respect to $\sigma := [v_0, v_2]$ yields the following complex:



The definitions to follow and proofs of their properties are paraphrased from [1]. Enumerate the vertices of K, $v_0, v_1, \ldots$.

Definition 6.2.3 (Definition 4.0.2 [1])**.** Fix a k-simplex $\sigma = [v_{i_0}, v_{i_1}, \ldots, v_{i_k}] \in K$. Define the *stellar inclusion* map $i_\sigma \colon C_\bullet(K) \to C_\bullet(\star_\sigma K)$ by

$$i_\sigma([v_{j_0}, \ldots, v_{j_\ell}]) = \begin{cases} [v_{j_0}, \ldots, v_{j_\ell}] & \text{if } \{j_0, \ldots, j_\ell\} \not\supseteq \{i_0, \ldots, i_k\}, \\ \sum_{j_i \in \{i_0, \ldots, i_k\}} (-1)^i [v_\star, v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_\ell}] & \text{else.} \end{cases} \tag{6.5}$$

To see that i_σ is a chain map, if $\{j_0, \ldots, j_\ell\} \not\supset \{i_0, \ldots, i_k\}$, then no subset of $\{j_0, \ldots, j_\ell\}$ contains $\{i_0, \ldots, i_k\}$ and so

$$\partial i_\sigma\left([v_{j_0}, \ldots, v_{j_\ell}]\right) = i_\sigma\left(\partial [v_{j_0}, \ldots, v_{j_\ell}]\right). \tag{6.6}$$

Now suppose $\{j_0, \dots, j_\ell\} \supseteq \{i_0, \dots, i_k\}$ and behold,

$$
\begin{aligned}
\partial i_\sigma\left([v_{j_0}, \dots, v_{j_\ell}]\right) &= \partial \sum_{j_i \in \{i_0,\dots,i_k\}} (-1)^i [v_\star, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&= \sum_{j_i \in \{i_0,\dots,i_k\}} (-1)^i [v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \in \{i_0,\dots,i_k\} \\ i<i'}} (-1)^{i+i'} [v_\star, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, \hat{v}_{j_{i'}}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \in \{i_0,\dots,i_k\} \\ i>i'}} (-1)^{i+i'-1} [v_\star, v_{j_0}, \dots, \hat{v}_{j_{i'}}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \notin \{i_0,\dots,i_k\} \\ i<i'}} (-1)^{i+i'} [v_\star, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, \hat{v}_{j_{i'}}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \notin \{i_0,\dots,i_k\} \\ i>i'}} (-1)^{i+i'-1} [v_\star, v_{j_0}, \dots, \hat{v}_{j_{i'}}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&= \sum_{j_i \in \{i_0,\dots,i_k\}} (-1)^i [v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \notin \{i_0,\dots,i_k\} \\ i<i'}} (-1)^{i+i'} [v_\star, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, \hat{v}_{j_{i'}}, \dots, v_{j_\ell}] \\
&\quad - \sum_{j_i \in \{i_0,\dots,i_k\}} \sum_{\substack{j_{i'} \notin \{i_0,\dots,i_k\} \\ i>i'}} (-1)^{i+i'-1} [v_\star, v_{j_0}, \dots, \hat{v}_{j_{i'}}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\
&= i_\sigma \left(\sum_{j_i \in \{i_0,\dots,i_k\}} (-1)^i [v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \right) \\
&\quad + i_\sigma \left(\sum_{j_{i'} \notin \{i_0,\dots,i_k\}} (-1)^{i'} [v_\star, v_{j_0}, \dots, \hat{v}_{j_{i'}}, \dots, v_{j_\ell}] \right) \\
&= i_\sigma\left(\partial[v_{j_0}, \dots, v_{j_\ell}]\right).
\end{aligned}
\tag{6.7}
$$

Definition 6.2.4 (Definition 4.0.3 [1]). Define the *stellar projection* map $p_\sigma \colon C_\bullet(\star_\sigma \Delta^n) \to C_\bullet(\Delta^n)$ by

$$
\begin{aligned}
p_\sigma([v_{j_0}, \dots, v_{j_\ell}]) &= [v_{j_0}, \dots, v_{j_\ell}], \\
p_\sigma([v_\star, v_{j_0}, \dots, v_{j_\ell}]) &= \frac{1}{k+1} \sum_{\alpha \in \{i_0,\dots,i_k\} \setminus \{j_0,\dots,j_\ell\}} [v_\alpha, v_{j_0}, \dots, v_{j_\ell}].
\end{aligned}
\tag{6.8}
$$

To see that p is a chain map, we turn our attention to the second case of the definition, since the verification of the first is trivial. Observe,

$$
\begin{aligned}
\partial p_\sigma\left([v_\star, v_{j_0}, \dots, v_{j_\ell}]\right) &= \frac{1}{k+1}\partial \sum_{\alpha\in\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}} [v_\alpha, v_{j_0}, \dots, v_{j_\ell}] \\
&= \frac{|\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}|}{k+1}[v_{j_0},\dots,v_{j_\ell}] \\
&\quad - \frac{1}{k+1}\sum_{\alpha\in\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}}\sum_{t=0}^{\ell}(-1)^t[v_\alpha, v_{j_0},\dots,\hat{v}_{j_t},\dots,v_{j_\ell}] \\
&= \frac{|\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}| + |\{i_0,\dots,i_k\}\cap\{j_0,\dots,j_\ell\}|}{k+1}[v_{j_0},\dots,v_{j_\ell}] \\
&\quad - \frac{1}{k+1}\sum_{t=0}^{\ell}\sum_{\alpha\in\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_{t-1},j_{t+1},\dots,j_\ell\}}(-1)^t[v_\alpha, v_{j_0},\dots,\hat{v}_{j_t},\dots,v_{j_\ell}] \\
&= [v_{j_0},\dots,v_{j_\ell}] \\
&\quad - \frac{1}{k+1}\sum_{t=0}^{\ell}\sum_{\alpha\in\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_{t-1},j_{t+1},\dots,j_\ell\}}(-1)^t[v_\alpha, v_{j_0},\dots,\hat{v}_{j_t},\dots,v_{j_\ell}] \\
&= p_\sigma\left(\partial[v_\star, v_{j_0},\dots,v_{j_\ell}]\right).
\end{aligned}
\tag{6.9}
$$

Definition 6.2.5 (Definition 4.0.4 [1])**.** Define the map $a_\sigma\colon C_\bullet(\star_\sigma\Delta^n)\to C_{\bullet+1}(\star_\sigma\Delta^n)$ by

$$
\begin{aligned}
a_\sigma([v_{j_0},\dots,v_{j_\ell}]) &= 0, \\
a_\sigma([v_\star, v_{j_0},\dots,v_{j_\ell}]) &= \begin{cases} 0 & \text{if } |\,\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}\,| = 1, \\ -\frac{1}{k+1}\sum_{\alpha\in\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}}[v_\star, v_\alpha, v_{j_0},\dots,v_{j_\ell}] & \text{else.}\end{cases}
\end{aligned}
\tag{6.10}
$$

Theorem 6.2.6 (Theorem 4.0.1 [1])**.** The following relations hold:

1. $a_\sigma^2 = 0$.

2. $p_\sigma\circ a_\sigma = a_\sigma\circ i_\sigma = 0$.

3. $p_\sigma\circ i_\sigma = 1$.

4. $\partial a_\sigma + a_\sigma\partial = 1 - i_\sigma\circ p_\sigma$.

Proof. We will verify the properties in the order listed above. To see that $a_\sigma^2 = 0$, consider $\{j_0,\dots,j_\ell\}$ such that

$$
|\{i_0,\dots,i_k\}\setminus\{j_0,\dots,j_\ell\}| \geq 2. \tag{6.11}
$$

In all other cases, $a_\sigma([j_0, \ldots, j_\ell]) = 0$ and so the result is trivial. So,

$$\begin{aligned}
a_\sigma^2\left([v_\star, v_{j_0}, \ldots, v_{j_\ell}]\right) &= -\frac{1}{k+1} \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} a\left([v_\star, v_\alpha, v_{j_0}, \ldots, v_{j_\ell}]\right) \\
&= \frac{1}{(k+1)^2} \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} \sum_{\alpha' \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell,\alpha\}} [v_\star, v_{\alpha'}, v_\alpha, v_{j_0}, \ldots, v_{j_\ell}] \\
&= 0.
\end{aligned} \tag{6.12}$$

Next, we will show that $p_\sigma \circ a_\sigma = 0$. Continue with the assumption that

$$|\{i_0, \ldots, i_k\} \setminus \{j_0, \ldots, j_\ell\}| \geq 2. \tag{6.13}$$

Then,

$$\begin{aligned}
p_\sigma\left(a_\sigma\left([v_\star, v_{j_0}, \ldots, v_{j_\ell}]\right)\right) &= -\frac{1}{k+1} \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} p\left([v_\star, v_\alpha, v_{j_0}, \ldots, v_{j_\ell}]\right) \\
&= -\frac{1}{(k+1)^2} \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} \sum_{\alpha' \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell,\alpha\}} [v_{\alpha'}, v_\alpha, v_{j_0}, \ldots, v_{j_\ell}] \\
&= 0.
\end{aligned} \tag{6.14}$$

To see that $a_\sigma \circ i_\sigma = 0$, consider $\{j_0, \ldots, j_\ell\} \supseteq \{i_0, \ldots, i_k\}$ since in the other case, the result is obvious. Observe,

$$\begin{aligned}
a_\sigma\left(i_\sigma\left([j_0, \ldots, j_\ell]\right)\right) &= \sum_{j_i \in \{i_0,\ldots,i_k\}} (-1)^i a_\sigma\left([v_\star, v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_\ell}]\right) \\
&= 0.
\end{aligned} \tag{6.15}$$

Next, we check that $p_\sigma i_\sigma = 1$. For $\{j_0, \ldots, j_\ell\} \not\supseteq \{i_0, \ldots, i_k\}$, it is clear that

$$p_\sigma i_\sigma\left([v_{j_0}, \ldots, v_{j_\ell}]\right) = [v_{j_0}, \ldots, v_{j_\ell}]. \tag{6.16}$$

For $\{j_0, \ldots, j_\ell\} \supseteq \{i_0, \ldots, i_k\}$,

$$\begin{aligned}
p_\sigma\left(i_\sigma\left([v_{j_0}, \ldots, v_{j_\ell}]\right)\right) &= p_\sigma\left(\sum_{j_i \in \{i_0,\ldots,i_k\}} (-1)^i [v_\star, v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_\ell}]\right) \\
&= \frac{1}{k+1} \sum_{j_i \in \{i_0,\ldots,i_k\}} (-1)^i \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_{i-1},j_{i+1},\ldots,j_\ell\}} [v_\alpha, v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_\ell}] \\
&= [v_{j_0}, \ldots, v_{j_\ell}].
\end{aligned} \tag{6.17}$$

Finally, we verify the homotopy relation $\partial a_\sigma + a_\sigma \partial = 1 - i_\sigma \circ p_\sigma$. For $[v_{j_0}, \dots, v_{j_\ell}] \in \star_\sigma K$, $i_\sigma p_\sigma([v_{j_0}, \dots, v_{j_\ell}]) = [v_{j_0}, \dots, v_{j_\ell}]$ and, by what we have shown above, $(\partial a_\sigma + a_\sigma \partial)[v_{j_0}, \dots, v_{j_\ell}] = 0$. It remains to check the relation on $[v_\star, v_{j_0}, \dots, v_{j_\ell}]$. Suppose first that $\{i_0, \dots, i_k\} \setminus \{j_0, \dots, j_\ell\} = \{i_m\}$. Then,

$$\begin{aligned} i_\sigma p_\sigma([v_\star, v_{j_0}, \dots, v_{j_\ell}]) &= \frac{1}{k+1} \sum_{\alpha \in \{i_0, \dots, i_k\} \setminus \{j_0, \dots, j_\ell\}} i_\sigma([v_\alpha, v_{j_0}, \dots, v_{j_\ell}]) \\ &= \frac{1}{k+1} i_\sigma([v_{i_m}, v_{j_0}, \dots, v_{j_\ell}]) \\ &= \frac{1}{k+1} \left([v_\star, v_{j_0}, \dots, v_{j_\ell}] - \sum_{j_i \in \{i_0, \dots, i_k\}} (-1)^i [v_\star, v_{i_m}, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \right). \end{aligned} \tag{6.18}$$

On the other hand,

$$\begin{aligned} (\partial a_\sigma + a_\sigma \partial)[v_\star, v_{j_0}, \dots, v_{j_\ell}] &= a_\sigma \partial [v_\star, v_{j_0}, \dots, v_{j_\ell}] \\ &= a_\sigma \left([v_{j_0}, \dots, v_{j_\ell}] - \sum_{t=0}^{\ell} (-1)^t a([v_\star, v_{j_0}, \dots, \hat{v}_{j_t}, \dots, v_{j_\ell}]) \right) \\ &= - \sum_{j_i \in \{i_0, \dots, i_k\}} (-1)^i a([v_\star, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}]) \\ &= \frac{1}{k+1} \sum_{j_i \in \{i_0, \dots, i_k\}} (-1)^i [v_\star, v_{i_m}, v_{j_0}, \dots, \hat{v}_{j_i}, \dots, v_{j_\ell}] \\ &\quad + \frac{k}{k+1} [v_\star, v_{j_0}, \dots, v_{j_\ell}] \\ &= (1 - i_\sigma p_\sigma)[v_\star, v_{j_0}, \dots, v_{j_\ell}]. \end{aligned} \tag{6.19}$$

Now suppose that $|\{i_0, \dots, i_k\} \setminus \{j_0, \dots, j_\ell\}| \geq 2$ and observe,

$$\begin{aligned} i_\sigma p_\sigma([v_\star, v_{j_0}, \dots, v_{j_\ell}]) &= \frac{1}{k+1} \sum_{\alpha \in \{i_0, \dots, i_k\} \setminus \{j_0, \dots, j_\ell\}} i_\sigma([v_\alpha, v_{j_0}, \dots, v_{j_\ell}]) \\ &= \frac{1}{k+1} \sum_{\alpha \in \{i_0, \dots, i_k\} \setminus \{j_0, \dots, j_\ell\}} [v_\alpha, v_{j_0}, \dots, v_{j_\ell}] \end{aligned} \tag{6.20}$$

and on the other hand

$$
\begin{aligned}
&(\partial a_\sigma + a_\sigma \partial)\left([v_\star, v_{j_0}, \ldots, v_{j_\ell}]\right) \\
&\quad = -\frac{1}{k+1} \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} \partial[v_\star, v_\alpha, v_{j_0}, \ldots, v_{j_\ell}] \\
&\qquad - \sum_{t=0}^{\ell} (-1)^t a_\sigma\left([v_\star, v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_\ell}]\right) \\
&\quad = -i_\sigma p_\sigma([v_\star, v_{j_0}, \ldots, v_{j_\ell}]) + \frac{|\{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}|}{k+1}[v_\star, v_{j_0}, \ldots, v_{j_\ell}] \\
&\qquad - \frac{1}{k+1} \sum_{t=0}^{\ell} (-1)^t \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_\ell\}} [v_\star, v_\alpha, v_{j_0}, \ldots, \hat{v}_{j_t}, \ldots, v_{j_\ell}] \\
&\qquad + \frac{1}{k+1} \sum_{t=0}^{\ell} (-1)^t \sum_{\alpha \in \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,j_{t-1},j_{t+1},j_\ell\}} [v_\star, v_\alpha, v_{j_0}, \ldots, \hat{v}_{j_t}, \ldots, v_{j_\ell}] \\
&\quad = (1 - i_\sigma p_\sigma)\left([v_\star, v_{j_0}, \ldots, v_{j_\ell}]\right).
\end{aligned} \tag{6.21}
$$

□

To define maps on cochains, take duals. Define $g_\sigma := p_\sigma^*$, $f_\sigma := i_\sigma^*$, $h_\sigma := a_\sigma^*$. These maps are given explicitly below and come from Definitions 4.0.5, 4.0.6, 4.0.7 of [1].

$$
\begin{aligned}
f_\sigma\left(\delta_{[v_{j_0},\ldots,v_{j_\ell}]}\right) &= \delta_{[v_{j_0},\ldots,v_{j_\ell}]}, \\
f_\sigma\left(\delta_{[v_\star,v_{j_0},\ldots,v_{j_\ell}]}\right) &= \begin{cases} \delta_{[v_{i_m},v_{j_0},\ldots,v_{j_\ell}]} & \text{if } \{i_0,\ldots,i_k\} \setminus \{j_0,\ldots,v_{j_\ell}\} = \{i_m\}, \\ 0 \quad \text{else.} \end{cases}
\end{aligned} \tag{6.22}
$$

$$
g_\sigma\left(\delta_{[v_{j_0},\ldots,v_{j_\ell}]}\right) = \begin{cases} \delta_{[v_{j_0},\ldots,v_{j_\ell}]} + \frac{1}{k+1}\sum_{j_i \in \{i_0,\ldots,i_k\}} (-1)^i \delta_{[v_\star,v_{j_0},\ldots,\hat{v}_{j_i},\ldots,v_{j_\ell}]} \\ \qquad \text{if } \{v_{j_0},\ldots,v_{j_\ell}\} \not\supset \{i_0,\ldots,i_k\}, \\ \frac{1}{k+1}\sum_{j_i \in \{i_0,\ldots,i_k\}} (-1)^i \delta_{[v_\star,v_{j_0},\ldots,\hat{v}_{j_i},\ldots,v_{j_\ell}]} \quad \text{else.} \end{cases} \tag{6.23}
$$

$$
\begin{aligned}
h_\sigma\left(\delta_{[v_{j_0},\ldots,v_{j_\ell}]}\right) &= 0, \\
h_\sigma\left(\delta_{[v_\star,v_{j_0},\ldots,v_{j_\ell}]}\right) &= -\frac{1}{k+1} \sum_{j_t \in \{i_0,\ldots,i_k\} \cap \{j_0,\ldots,j_\ell\}} (-1)^t \delta_{[v_\star,v_{j_0},\ldots,\hat{v}_{j_t},\ldots,v_{j_\ell}]}.
\end{aligned} \tag{6.24}
$$

Corollary 6.2.7 (Theorem 4.0.2 [1])**.** The package of maps $(f_\sigma, g_\sigma, h_\sigma)$ is a strong deformation retraction $(C^\bullet(\star_\sigma K), d) \to (C^\bullet(K), d)$.

6.3 Triangulations of Manifolds

Definition 6.3.1 (Chapter IV Section 12 [49]). A *smooth triangulation of M* is a pair (K, ϕ) where K is a simplicial complex and $\phi\colon K \to M$ is a homeomorphism with the property that for each $\sigma \in K$, there is a chart (χ_σ, U_σ) such that $\phi(\sigma) \subseteq U_\sigma \subseteq M$ and $(\chi_\sigma \phi)\big|_\sigma$ is affine. We will refer to such a chart as a *simplicial chart*.

Theorem 6.3.2 (Chapter IV Section 12 Theorem 12A [49]). Every smooth manifold has a smooth triangulation.

Definition 6.3.3 (Chapter 2 [16]). Given a simplicial complex K, a *simplicial differential k-form on K* is a family $\{\varphi_\sigma\}_{\sigma \in K}$ of forms $\varphi_\sigma \in \Omega^k\left(\Delta^{\dim(\sigma)}\right)$ such that for a face of $\tau \leq \sigma$ with inclusion map $\iota\colon \tau \to \sigma$, we have $\iota^*(\varphi_\sigma) = \varphi_\tau$. Define $\Omega^\bullet(K)$ to be the collection of simplicial forms. Equipped with the de Rham differential d, $(\Omega^\bullet(K), d)$ is a cochain complex.

Let K be a smooth triangulation of M. We will adopt a relaxed approach to notation where each $\sigma \in K$ is regarded as a subset of M when required. Each simplicial differential form on K defines an L^2 form on M and conversely each smooth form on M defines a simplicial differential form on K. Furthermore, for the subdivision $\star_\sigma K$, each simplicial form on K is also a simplicial form on $\star_\sigma K$. In other words, we may identify the spaces like so:

$$\Omega^\bullet(M) \subseteq \Omega^\bullet(K) \subseteq \Omega^\bullet(\star_\sigma K) \subseteq L^2\Omega^\bullet(M). \tag{6.25}$$

Since $\Omega^\bullet(M)$ is dense in $L^2\Omega^\bullet(M)$, so is $\Omega^\bullet(K)$. The cohomology of $\Omega^\bullet(M)$ is isomorphic to the cohomology of $L^2\Omega^\bullet(M)$ and (Chapter 9.5 [23]) isomorphic to the cohomology of $\Omega^\bullet(K)$.

Define the integration map $R_K\colon \Omega^\bullet(K) \to C^\bullet(K)$ for $\alpha \in \Omega^\bullet(K)$ and $\sigma \in K$ via

$$R_K(\alpha)(\sigma) = \int_\sigma \alpha. \tag{6.26}$$

Note that R_K is densely defined, is a cochain map by Stokes' Theorem, and is also a surjection. Barycentric coordinates with non-zero restrictions to a face τ of σ restrict to barycentric coordinates of τ. This observation allows us to construct the Whitney cochain embedding map $W_K\colon C^\bullet(K) \to \Omega^\bullet(K)$ defined for $\sigma \in K$ of dimension k as follows:

$$W_K(\delta_\sigma) = k! \sum_{j=0}^{k} (-1)^j t_j dt_0 \wedge \cdots \wedge dt_{j-1} \wedge dt_{j+1} \wedge \cdots \wedge dt_k. \tag{6.27}$$

Note that W_K is an injective cochain map such that $R_K W_K = 1$. Moreover, there exists $s_K \in \mathrm{End}^{-1}(\Omega^\bullet(K))$, known as the Dupont homotopy [16], such that

$$(R_K, W_K, s_K) : (\Omega^\bullet(K), d) \to (C^\bullet(K), d) \tag{6.28}$$

is a strong deformation retraction of Hilbert complexes. Several technical details of this are exhibited in [9].

6.4 Subdivision of a Triangulation

We now connect the ideas of Sections 6.2 and 6.3 by describing the affect of subdivision on the triangulation K.

Theorem 6.4.1 (Theorem 6.0.1 [1])**.** For $\sigma \in K$, $W_{\star_\sigma K} \circ g_{\star\sigma} = W_K$ and $f_{\star\sigma} \circ R_{\star_\sigma K}\big|_{\Omega^\bullet(K)} = R_K$.

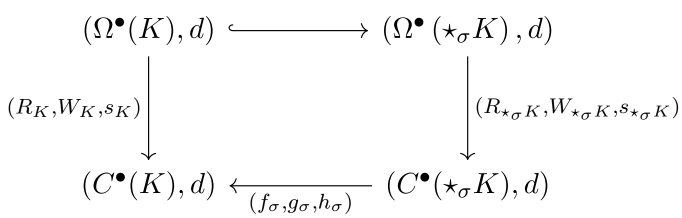


Figure 6.1: A diagram depicting the setting of Theorem 6.4.1, where the arrow at the top is the inclusion given by Equation 6.25.

It should be noted here that $s_K \neq (W_{\star_\sigma K} h R_{\star_\sigma K} + s_{\star_\sigma K})\big|_{\Omega^\bullet(K)}$ (see Theorem 6.0.2 [1]).

The papers [14, 15, 50] use a subdivision algorithm, called *Whitney standard subdivision* (Appendix II of [49]), that is slightly more sophisticated than stellar subdivision.

Definition 6.4.2 (Section 4, Appendix II of [49], Section 2 [14])**.** Let $K = \Delta^n$, the simplicial complex on the standard n-simplex. Suppose $K \subseteq \mathbb{R}^N$. The *Whitney Standard Subdivision* of K is the simplicial complex $\star K$ formed as follows:

1. Enumerate the vertices of Δ^n, $v_0, \dots, v_n \in \mathbb{R}^N$.

2. For each $i \leq j$, define $v_{i,j} = \frac{v_i + v_j}{2}$. Notice that $v_{i,i} = v_i$.

3. Define a partial ordering on the set of $v_{i,j}$ as follows:

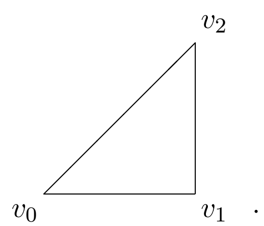
$$v_{i,j} \leq v_{h,k}, \quad \text{if } h \leq i \text{ and } j \leq k. \tag{6.29}$$

4. Define

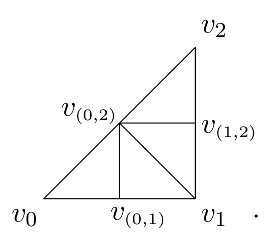
$$\star K = \left\{ [v_{i_0,j_0}, v_{i_1,j_1}, \ldots, v_{i_\ell,j_\ell}] \subseteq \mathbb{R}^N \ : v_{i_0,j_0} < v_{i_1,j_1} < \cdots < v_{i_\ell,j_\ell} \right\} \tag{6.30}$$

Note that for a k-simplex $\sigma \in \Delta^n$, the k-faces of $\star\Delta^n$ contained in σ form $\star\sigma$. By construction, there are 2^k such faces. This allows us to define the Whitney Standard Subdivision of an arbitrary simplicial complex K by applying Whitney Standard Subdivision to each simplex of K.

Example 6.4.3. Let $K = \Delta^2$:



The Whitney standard subdivision of K, $\star K$, is given by



Denote by $\star^n K$, the simplicial complex formed by applying n iterations of Whitney standard subdivision. Proposition B.0.1 in Appendix B shows that Whitney standard subdivision decomposes as a sequence of stellar subdivisions. Since strong deformation retractions are composable, the stellar subdivision strong deformation retractions provide strong deformation retractions between iterations $m < n$ of Whitney subdivision:

$$(f_n^m, g_m^n, h_{n,m}) : (C^\bullet(\star^n K), d) \to (C^\bullet(\star^m K), d), \tag{6.31}$$

with the consistency conditions $W_{\star^n K} g_m^n = W_{\star^m K}$ and $f_n^m R_{\star^n K}\big|_{\Omega^\bullet(\star^m K)} = R_{\star^m K}$ given by Theorem 6.4.1. For ease of notation, write (R_n, W_n, s_n) to signify $(R_{\star^n K}, W_{\star^n K}, s_{\star^n K})$.

Pull back the inner product on $L^2\Omega^\bullet(M)$ to $C^\bullet(\star^n K)$ via W_n to endow $C^\bullet(\star^n K)$ with a Hilbert space structure and define the *discrete Hodge Laplacian* $\Delta_n := (d + d^*)^2$ on $C^\bullet(\star^n K)$. The aim of the remainder of this chapter is to show that the sequence $(C^\bullet(\star^n K), d, \Delta_n)_{n=1}^\infty$ is convergent to $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ using the strong deformation retractions $(R_n, W_n, s_n) : (\Omega^\bullet(\star^n K), d) \to (C^\bullet(\star^n K), d)$ and that $(C^\bullet(\star^n K), \Delta_n)_{n=1}^\infty$ is

Cauchy using the strong deformation retractions $(f_n^m, g_m^n, h_{n,m}) : (C^\bullet(\star^n K), d, \Delta_n) \to (C^\bullet(\star^m K), d, \Delta_m)$. The remainder of this section is devoted to showing some properties of triangulation subdivision which will be useful after our main proofs.

Definition 6.4.4. The *mesh* of $\star^n K$, η_n, is defined to be

$$\eta_n := \sup_{[v_{i_0}, v_{i_1}] \in (\star^n K)_1} r(v_{i_0}, v_{i_1}), \tag{6.32}$$

where $r(v_{i_0}, v_{i_1})$ is the geodesic distance between v_{i_0} and v_{i_1} in M.

By definition, the Euclidean distance between each v_{i_0} and v_{i_1} in $\star^n K$ is halved at each step of Whitney subdivision. For each $\sigma \in K$, the simplicial charts and equivalence of norms in Euclidean space tells us that $\eta_n \sim 2^{-n}$.

Lemma 6.4.5 (Lemma 7.22 [15])**.** There exist constants $c_1, c_2 > 0$ independent of K and n such that for $\alpha \in C^q(\star^n K)$,

$$c_1 \eta_n^{2q-N} \|W_n \alpha\|^2 \leq |||\alpha|||^2 \leq c_2 \eta_n^{2q-N} \|W_n \alpha\|^2. \tag{6.33}$$

where $|||\cdot|||$ is the norm arising from the standard L^2 inner product on cochains i.e. from

$$\langle\langle \delta_\tau, \delta_\gamma \rangle\rangle = \begin{cases} 1 & \text{if } \tau = \gamma, \\ 0 & \text{else.} \end{cases} \tag{6.34}$$

Lemma 6.4.6. There exists $C \in \mathbb{R}$ such that for each $n \in \mathbb{N}$,

$$\left\| f_n^{n-1}|_{C^q(\star^n K)} \right\| < \sqrt{C 2^{N-q}}. \tag{6.35}$$

Proof. Let $\alpha \in C^q(\star^n K)$ have unit norm and write

$$\alpha = \sum_{\sigma \in (\star^n K)_q} a_\sigma \delta_\sigma. \tag{6.36}$$

Observe,

$$
\begin{aligned}
\left\| f_n^{n-1}(\alpha) \right\|^2 &= \left\| \sum_{\sigma \in (\star^n K)_q} a_\sigma f_n^{n-1}(\delta_\sigma) \right\|^2 \\
&= \left\| \sum_{\sigma' \in (\star^{n-1} K)_q} \sum_{\substack{\sigma \in (\star^n K)_q \\ f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'}}} a_\sigma \delta_{\sigma'} \right\|^2 \\
&\leq \frac{\eta_{n-1}^{N-2q}}{c_1} \left|\left|\left| \sum_{\sigma' \in (\star^{n-1} K)_q} \sum_{\substack{\sigma \in (\star^n K)_q \\ f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'}}} a_\sigma \delta_{\sigma'} \right|\right|\right|^2, \quad \text{by Lemma 6.4.5} \qquad (6.37) \\
&\leq \frac{\eta_{n-1}^{N-2q}}{c_1} \sum_{\sigma' \in (\star^{n-1} K)_q} \left(\sum_{\substack{\sigma \in (\star^n K)_q \\ f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'}}} a_\sigma \right)^2 \\
&\leq \frac{\eta_{n-1}^{N-2q}}{c_1} \sum_{\sigma' \in (\star^{n-1} K)_q} \left| \left\{ \sigma \in (\star^n K)_q \ : f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'} \right\} \right| \sum_{\substack{\sigma \in (\star^n K)_q \\ f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'}}} a_\sigma^2 .
\end{aligned}
$$

For $\sigma' \in \left(\star^{n-1} K\right)_q$, the set $\left\{ \sigma \in (\star^n K)_q \ : f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'} \right\}$ consists of $\sigma \in (\star^n K)_q$ such that $\sigma \subset \sigma'$. By the definition of Whitney standard subdivision, there are 2^q such σ and thus

$$
\left| \left\{ \sigma \in (\star^n K)_q \ : f_n^{n-1}(\delta_\sigma) = \delta_{\sigma'} \right\} \right| = 2^q. \qquad (6.38)
$$

Proceeding with our computation, we have

$$\begin{aligned}
\left\|f_n^{n-1}(\alpha)\right\|^2 &= \frac{\eta_{n-1}^{N-2q}}{c_1} 2^q \sum_{\sigma \in (\star^n K)_q} a_\sigma^2 \\
&= \frac{\eta_{n-1}^{N-2q}}{c_1} 2^q \sum_{\sigma \in (\star^n K)_q} \left\|\left|a_\sigma \delta_\sigma\right|\right\|^2 \\
&= \frac{\eta_{n-1}^{N-2q}}{c_1} 2^q \left\|\left|\sum_{\sigma \in (\star^n K)_q} a_\sigma \delta_\sigma\right|\right\|^2 \\
&\leq \frac{c_2}{c_1} \left(\frac{\eta_{n-1}}{\eta_n}\right)^{N-2q} 2^q \left\|\sum_{\sigma \in (\star^n K)_q} a_\sigma \delta_\sigma\right\|^2, \quad \text{by Lemma 6.4.5} \\
&= \frac{c_2}{c_1} \left(\frac{\eta_{n-1}}{\eta_n}\right)^{N-2q} 2^q \|\alpha\|^2 \\
&\leq C \cdot 2^{N-q}.
\end{aligned} \tag{6.39}$$

The last line follows from $\eta_n \sim 2^{-n}$. □

6.5 Simplicial Approximation of Hodge Theory Through Triangulation Subdivision

We will need a few facts from [14, 15] in order to show convergence. Enumerate $\sigma(\Delta_n)$ in non-decreasing order, $\left(\lambda_j^{(n)}\right)_{j \in \mathbb{N}}$, and do the same for $\sigma(\Delta)$, $(\lambda_j)_{j \in \mathbb{N}}$.

Theorem 6.5.1 (Corollary 3.27 [14])**.** For $\alpha \in \Omega^\bullet(M)$,

$$\lim_{n \to \infty} \|(1 - W_n R_n)\alpha\| = 0. \tag{6.40}$$

Theorem 6.5.2 (Theorem 3.7 [15])**.** For fixed j,

$$\lim_{n \to \infty} \lambda_j^{(n)} = \lambda_j. \tag{6.41}$$

Theorem 6.5.3. Fix an eigenvalue λ of Δ and recall from Definition 4.2.1 that

$$\rho_n(\lambda) = \left\{\lambda_j^{(n)} \in \sigma(\Delta_n) : \lambda_j = \lambda\right\}. \tag{6.42}$$

For each $\alpha \in \mathcal{E}_\lambda$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, there exists

$\alpha^{(n)} \in \bigoplus_{\lambda^{(n)} \in \rho_n(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$ with

$$\left\| \alpha - W_n\left(\alpha^{(n)}\right)\right\| < \varepsilon. \tag{6.43}$$

Proof. This is an immediate corollary of Theorem 4.4 of [15] by Remark 4.8 of [15]. □

Corollary 6.5.4. Fix an orthonormal basis $\alpha_1, \ldots, \alpha_{\mu(\lambda)} \in \mathcal{E}_\lambda$. Then, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there is an orthonormal basis $\alpha_1^{(n)}, \ldots, \alpha_{\mu(\lambda)}^{(n)} \in \bigoplus_{\lambda^{(n)} \in \rho_n(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$ with

$$\left\| \alpha_j - W_n\left(\alpha_j^{(n)}\right)\right\| < \varepsilon. \tag{6.44}$$

Proof. Let $\delta = \min\left(\frac{\varepsilon}{2(1+6(\mu(\lambda)-1))}, \frac{1}{2}\right)$. For each $1 \leq j \leq \mu(\lambda)$, apply Theorem 6.5.3 to obtain $N_j \in \mathbb{N}$ and ${\alpha'}_j^{(n)}$ with $\left\|\alpha_j - W_n\left({\alpha'}_j^{(n)}\right)\right\| < \delta$ for each $n \geq N_j$. Since $\delta < 1$, we know that each ${\alpha'}_j^{(n)} \neq 0$. Let $N = \max_{j=1,\ldots,\mu(\lambda)} N_j$ and take $n \geq N$. We will proceed using the Gram-Schmidt algorithm to produce an orthogonal collection ${\alpha''}_1^{(n)}, \ldots, {\alpha''}_{\mu(\lambda)}^{(n)}$, then normalize to obtain an orthonormal basis $\alpha_1^{(n)}, \ldots, \alpha_{\mu(\lambda)}^{(n)}$ and argue the desired bound. Let

$$ {\alpha''}_1^{(n)} = {\alpha'}_1^{(n)} - \sum_{j=2}^{\mu(\lambda)} \frac{\left\langle {\alpha'}_1^{(n)}, {\alpha'}_j^{(n)}\right\rangle}{\left\|{\alpha'}_j^{(n)}\right\|^2} {\alpha'}_j^{(n)}. \tag{6.45}$$

Observe that for $j > 1$,

$$\begin{aligned}
\left|\left\langle {\alpha'}_1^{(n)}, {\alpha'}_j^{(n)}\right\rangle\right| &= \left|\left\langle {\alpha'}_1^{(n)}, {\alpha'}_j^{(n)}\right\rangle\right| - \left|\left\langle W_n\left({\alpha'}_1^{(n)}\right), \alpha_j\right\rangle\right| + \left|\left\langle W_n\left({\alpha'}_1^{(n)}\right) - \alpha_1, \alpha_j\right\rangle\right| \\
&< \left|\left\langle W_n\left({\alpha'}_1^{(n)}\right), W_n\left({\alpha'}_j^{(n)}\right)\right\rangle\right| - \left|\left\langle W_n\left({\alpha'}_1^{(n)}\right), \alpha_j\right\rangle\right| + \delta \\
&\leq \left|\left\langle W_n\left({\alpha'}_1^{(n)}\right), W_n\left({\alpha'}_j^{(n)}\right) - \alpha_j\right\rangle\right| + \delta \\
&\leq \left\|W_n\left({\alpha'}_1^{(n)}\right)\right\| \left\|W_n\left({\alpha'}_j^{(n)}\right) - \alpha_j\right\| + \delta \\
&< \delta\left(\left\|W_n\left({\alpha'}_1^{(n)}\right)\right\| + 1\right) \\
&\leq \delta\left(\left\|W_n\left({\alpha'}_1^{(n)}\right) - \alpha_1\right\| + \|\alpha_1\| + 1\right) \\
&< \delta\left(\delta + 2\right).
\end{aligned} \tag{6.46}$$

Then,

$$\begin{aligned}\left\|\sum_{j=2}^{\mu(\lambda)} \frac{\left\langle {\alpha'_1}^{(n)}, {\alpha'_j}^{(n)}\right\rangle}{\left\|{\alpha'_j}^{(n)}\right\|^2} {\alpha'_j}^{(n)}\right\| &\leq \sum_{j=2}^{\mu(\lambda)} \frac{\delta\left(\delta+2\right)}{\left\|{\alpha'_j}^{(n)}\right\|} \\ &\leq \sum_{j=2}^{\mu(\lambda)} \frac{\delta\left(\delta+2\right)}{\left\|\alpha_j\right\| - \left\|\alpha_j - W_n\left({\alpha'_j}^{(n)}\right)\right\|} \\ &\leq \sum_{j=2}^{\mu(\lambda)} \frac{\delta\left(\delta+2\right)}{1-\frac{1}{2}} \\ &= 2\delta\left(\delta+2\right)\left(\mu(\lambda)-1\right),\end{aligned} \tag{6.47}$$

and so

$$\begin{aligned}\left\|\alpha_1 - W_n\left(\alpha''^{(n)}_1\right)\right\| &\leq \left\|\alpha_1 - W_n\left({\alpha'_1}^{(n)}\right)\right\| + \left\|\sum_{j=2}^{\mu(\lambda)} \frac{\left\langle {\alpha'_1}^{(n)}, {\alpha'_j}^{(n)}\right\rangle}{\left\|{\alpha'_j}^{(n)}\right\|^2} {\alpha'_j}^{(n)}\right\| \\ &< \delta + 2\delta\left(\delta+2\right)\left(\mu(\lambda)-1\right) \\ &< \frac{\varepsilon}{2}.\end{aligned} \tag{6.48}$$

Now define $\alpha^{(n)}_1 = \frac{\alpha''^{(n)}_1}{\left\|\alpha''^{(n)}_1\right\|}$ and observe:

$$\begin{aligned}\left\|\alpha_1 - W_n\left(\alpha^{(n)}_1\right)\right\| &\leq \left\|\alpha_1 - W_n\left(\alpha''^{(n)}_1\right)\right\| + \left\|\alpha''^{(n)}_1 - \alpha^{(n)}_1\right\| \\ &< \frac{\varepsilon}{2} + \left|1 - \left\|\alpha''^{(n)}_1\right\|\right| \\ &= \frac{\varepsilon}{2} + \left|\left\|\alpha_1\right\| - \left\|\alpha''^{(n)}_1\right\|\right| \\ &\leq \frac{\varepsilon}{2} + \left\|\alpha_1 - W_n\left(\alpha''^{(n)}_1\right)\right\| \\ &< \varepsilon.\end{aligned} \tag{6.49}$$

Proceed with the Gram-Schmidt algorithm, repeating the above process for ${\alpha'_2}^{(n)}, \ldots, {\alpha'_{\mu(\lambda)}}^{(n)}$. The proof that $\left\|\alpha_1 - W_n\left(\alpha^{(n)}_1\right)\right\| < \varepsilon$ applies to show that $\left\|\alpha_j - W_n\left(\alpha^{(n)}_j\right)\right\| < \varepsilon$ for each j. □

It is important to notice that in Corollary 6.5.4, it is not necessarily the case that each $\alpha^{(n)}_j$ is an eigenvector of Δ_n. This property will be important in the work to follow and so the following result addresses this deficiency while providing a similar approximation result. There is a key difference in the set-up: Informally, Corollary 6.5.4 takes an orthonormal basis for $\mathcal{E}_\lambda$ as input and outputs an orthonormal basis $\bigoplus_{\lambda^{(n)}\in\rho(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$ with the given approximation property, while the next result takes as input an orthonormal eigenbasis of $\bigoplus_{\lambda^{(n)} \in \rho(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$ and outputs an orthonormal basis for $\mathcal{E}_\lambda$ satisfying the same approximation property. The idea for this Corollary comes from Theorem 1.3 (b) of [2] where a similar claim is made, albeit less precisely, and the proof is not provided.

Corollary 6.5.5. Fix $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that for $n \geq N$ and any orthonormal eigenbasis $\alpha_1^{(n)}, \ldots, \alpha_{\mu(\lambda)}^{(n)} \in \bigoplus_{\lambda^{(n)} \in \rho_n(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$, there exists an orthonormal basis $\alpha_1, \ldots, \alpha_{\mu(\lambda)} \in \mathcal{E}_\lambda$ such that for all $j = 1, \ldots, \mu(\lambda)$,

$$\left\| \alpha_j - W_n \left(\alpha_j^{(n)} \right) \right\| < \varepsilon. \tag{6.50}$$

Proof. Fix an orthonormal basis $\varphi_1, \ldots, \varphi_{\mu(\lambda)} \in \mathcal{E}_\lambda$. Apply Corollary 6.5.4 to obtain $N \in \mathbb{N}$ and an orthonormal basis ${\alpha'}_1^{(n)}, \ldots, {\alpha'}_{\mu(\lambda)}^{(n)} \in \bigoplus_{\lambda^{(n)} \in \rho_n(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$ for each $n \geq N$ such that for all $j = 1, \ldots, \mu(\lambda)$,

$$\left\| \varphi_j - W_n \left({\alpha'}_j^{(n)} \right) \right\| < \frac{\varepsilon}{\mu(\lambda)}. \tag{6.51}$$

Fix $n \geq N$ and any orthonormal eigenbasis $\alpha_1^{(n)}, \ldots, \alpha_{\mu(\lambda)}^{(n)} \in \bigoplus_{\lambda^{(n)} \in \rho_n(\lambda)} \mathcal{E}^{(n)}_{\lambda^{(n)}}$. For each $j = 1, \ldots, \mu(\lambda)$, write

$$\alpha_j^{(n)} = \sum_{k=1}^{\mu(\lambda)} b_{j,k} {\alpha'}_k^{(n)}, \tag{6.52}$$

where each $b_{j,k} \in [-1, 1]$, since $\alpha_j^{(n)}$ has unit norm. Now, define

$$\alpha_j = \sum_{k=1}^{\mu(\lambda)} b_{j,k} \varphi_k. \tag{6.53}$$

Notice that for $1 \leq i, j \leq \mu(\lambda)$,

$$\begin{aligned} \langle \alpha_i, \alpha_j \rangle &= \sum_{k=1}^{\mu(\lambda)} b_{i,k} b_{j,k} \\ &= \left\langle \alpha_i^{(n)}, \alpha_j^{(n)} \right\rangle \\ &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{6.54}$$

which proves that $\alpha_1, \dots, \alpha_{\mu(\lambda)}$ is an orthonormal basis for $\mathcal{E}_\lambda$. Finally,

$$\begin{aligned}\left\|\alpha_j - W_n\left(\alpha_j^{(n)}\right)\right\| &= \left\|\sum_{k=1}^{\mu(\lambda)} b_{j,k}\left(\varphi_k - W_n\left({\alpha'}_k^{(n)}\right)\right)\right\| \\ &\leq \sum_{k=1}^{\mu(\lambda)} |b_{j,k}| \left\|\varphi_k - W_n\left({\alpha'}_k^{(n)}\right)\right\| \\ &< \sum_{k=1}^{\mu(\lambda)} |b_{j,k}| \frac{\varepsilon}{\mu(\lambda)} \\ &\leq \varepsilon.\end{aligned} \tag{6.55}$$

□

6.6 Simplicial Approximation as a Cauchy Sequence

We will first show that the sequence $(C^\bullet(\star^n K), d, \Delta_n)_{n=1}^\infty$ is convergent to $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ under the strong deformation retractions $(R_n, W_n, h_n) : \left(L^2\Omega^\bullet(M), d, \Delta\right) \to (C^\bullet(\star^n K), d, \Delta_n)$. Notice that by definition, each W_n is an isometry. It then remains to be shown that each W_n is a δ_n-isometry for $\delta_n \to 0$ as $n \to \infty$.

Lemma 6.6.1. Fix $\varepsilon > 0$, then there exists N large enough so that for $n \geq N$,

$$\left\|(1 - W_n W_n^*)(\Delta + 1)^{-\frac{1}{2}}\right\| < \varepsilon. \tag{6.56}$$

Proof. Let $\alpha_1, \alpha_2, \dots \in \Omega^\bullet(M)$ be an orthornormal eigenbasis of Δ for $L^2\Omega^\bullet(M)$ where α_j is an eigenvector corresponding to eigenvalue λ_j. Fix $\varepsilon > 0$. Because $(\lambda_j)_{j=1}^\infty$ is increasing and unbounded, we may take $T \in \mathbb{N}$ large enough such that for all $j \geq T$, $\frac{1}{\sqrt{\lambda_j + 1}} < \frac{\varepsilon}{2}$. Then, using Theorem 6.5.1, take $N \in \mathbb{N}$ large enough so that for each $n \geq N$ and $j < T$,

$$\|(1 - W_n R_n)\alpha_j\| < \frac{\varepsilon}{2T} \tag{6.57}$$

Let $\beta \in L^2\Omega^\bullet(M)$ have unit norm. Write $\beta = \sum_{j\geq 1} a_j\alpha_j$. Observe,

$$\begin{aligned}
\left\|\left(1 - W_nW_n^*\right)(\Delta+1)^{-\frac{1}{2}}\beta\right\| &= \left\|\left(1 - W_nW_n^*\right)\sum_{j\geq 1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| \\
&\leq \left\|\left(1 - W_nW_n^*\right)\sum_{j=1}^{T-1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| + \left\|\left(1 - W_nW_n^*\right)\sum_{j=T}^{\infty}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| \\
&< \left\|\left(1 - W_nW_n^*\right)\sum_{j=1}^{T-1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| + \frac{\varepsilon}{2}\left\|\sum_{j=T}^{\infty}a_j\alpha_j\right\| \\
&\leq \left\|\left(1 - W_nW_n^*\right)\sum_{j=1}^{T-1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| + \frac{\varepsilon}{2}.
\end{aligned} \tag{6.58}$$

By Lemma 2.1.4, we see that

$$\begin{aligned}
\left\|\left(1 - W_nW_n^*\right)\sum_{j=1}^{T-1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| &\leq \left\|\left(1 - W_nR_n\right)\sum_{j=1}^{T-1}\frac{a_j}{\sqrt{\lambda_j+1}}\alpha_j\right\| \\
&\leq \sum_{j=1}^{T-1}\frac{|a_j|}{\sqrt{\lambda_j+1}}\left\|\left(1 - W_nR_n\right)\alpha_j\right\|.
\end{aligned} \tag{6.59}$$

Since β is a unit, we have each $|a_j| \leq 1$ and since $\frac{1}{\sqrt{\lambda_j+1}} \leq 1$, and so the above quantity can be further bounded:

$$\begin{aligned}
\sum_{j=1}^{T-1}\frac{|a_j|}{\sqrt{\lambda_j+1}}\left\|\left(1 - W_nR_n\right)\alpha_j\right\| &\leq \sum_{j=1}^{T-1}\left\|\left(1 - W_nR_n\right)\alpha_j\right\| \\
&\leq \frac{\varepsilon}{2}.
\end{aligned} \tag{6.60}$$

In summary, we have shown that for any $\beta \in L^2\Omega^\bullet(M)$ of unit norm,

$$\left\|\left(1 - W_nW_n^*\right)(\Delta+1)^{-\frac{1}{2}}\beta\right\| < \varepsilon \tag{6.61}$$

and so we are done. □

Lemma 6.6.2. Fix $\varepsilon > 0$, then there exists N large enough so that for $n \geq N$,

$$\left\|(\Delta+1)^{-\frac{1}{2}}W_n - W_n\left(\Delta_n+1\right)^{-\frac{1}{2}}\right\| < \varepsilon. \tag{6.62}$$

Proof. Pick $T \in \mathbb{N}$ large enough such that

$$\frac{1}{\sqrt{\lambda_T + 1}} < \frac{\varepsilon}{8} \tag{6.63}$$

and $\lambda_T < \lambda_{T+1}$. Apply Theorem 6.5.2 to obtain N large enough such that for all $j \leq T$ and $n \geq N$,

$$\left| \frac{1}{\sqrt{\lambda_j + 1}} - \frac{1}{\sqrt{\lambda_j^{(n)} + 1}} \right| < \frac{\varepsilon}{8T}. \tag{6.64}$$

Apply Corollary 6.5.5 to each of the distinct eigenvalues in $(\lambda_j)_{j=1}^T$ and take N larger if necessary so that for fixed $n \geq N$, we obtain an orthonormal eigenbasis $\alpha_1, \ldots, \alpha_T$ for $\bigoplus_{j=1}^T \mathcal{E}_{\lambda_j}$ of Δ and an orthonormal eigenbasis $\alpha_1^{(n)}, \ldots, \alpha_T^{(n)}$ for $\bigoplus_{j=1}^T \mathcal{E}_{\lambda_j^{(n)}}$ of Δ_n such that for each $j = 1, \ldots T$,

$$\left\| \alpha_j - W_n \left(\alpha_j^{(n)} \right) \right\| < \frac{\varepsilon}{8T}. \tag{6.65}$$

Extend $(\alpha_1, \ldots, \alpha_T)$ to an eigenbasis $(\alpha_j)_{j=1}^\infty$ for $L^2\Omega^\bullet(M)$ of Δ and extend $\left(\alpha_1^{(n)}, \ldots, \alpha_T^{(n)}\right)$ to an eigenbasis $\left(\alpha_j^{(n)}\right)_{j=1}^\infty$ for $C^\bullet(\star^n K)$ of Δ_n. Let $\beta \in C^\bullet(\star^n K)$ have unit norm. Write $\beta = \sum_{j \geq 1} a_j^{(n)} \alpha_j^{(n)}$ where $a_j^{(n)} \in \mathbb{R}$ and also write $\beta = \beta_{\leq T} + \beta_{>T}$ where

$$\beta_{\leq T} = \sum_{j \leq T} a_j^{(n)} \alpha_j^{(n)}, \quad \beta_{>T} = \sum_{j > T} a_j^{(n)} \alpha_j^{(n)}. \tag{6.66}$$

Use this decomposition to split the quantity that we aim to bound:

$$\begin{aligned} \left\| \left((\Delta + 1)^{-\frac{1}{2}} W_n - W_n (\Delta_n + 1)^{-\frac{1}{2}} \right) (\beta) \right\| &\leq \left\| \left((\Delta + 1)^{-\frac{1}{2}} W_n - W_n (\Delta_n + 1)^{-\frac{1}{2}} \right) (\beta_{\leq T}) \right\| \\ &\quad + \left\| \left((\Delta + 1)^{-\frac{1}{2}} W_n - W_n (\Delta_n + 1)^{-\frac{1}{2}} \right) (\beta_{>T}) \right\|. \end{aligned} \tag{6.67}$$

We will bound each summand separately. First,

$$\begin{aligned} &\left\| \left((\Delta + 1)^{-\frac{1}{2}} W_n - W_n (\Delta_n + 1)^{-\frac{1}{2}} \right) (\beta_{\leq T}) \right\| \\ &\quad \leq \left\| (\Delta + 1)^{-\frac{1}{2}} \left(\left(\sum_{j \leq T} a_j^{(n)} \alpha_j \right) - W_n (\beta_{\leq T}) \right) \right\| \\ &\qquad + \left\| (\Delta + 1)^{-\frac{1}{2}} \left(\sum_{j \leq T} a_j^{(n)} \alpha_j \right) - W_n (\Delta_n + 1)^{-\frac{1}{2}} (\beta_{\leq T}) \right\| \\ &\quad \leq \left\| \sum_{j \leq T} a_j^{(n)} \left(\alpha_j - W_n \left(\alpha_j^{(n)} \right) \right) \right\| + \left\| \sum_{j \leq T} a_j^{(n)} \left(\frac{\alpha_j}{\sqrt{\lambda_j + 1}} - \frac{W_n \left(\alpha_j^{(n)} \right)}{\sqrt{\lambda_j^{(n)} + 1}} \right) \right\|. \end{aligned} \tag{6.68}$$

Notice here that $\|\beta\| = 1$ implies each $\left|a_j^{(n)}\right| \leq 1$ and so we get

$$\begin{aligned}
&\left\|\sum_{j\leq T} a_j^{(n)}\left(\alpha_j - W_n\left(\alpha_j^{(n)}\right)\right)\right\| + \left\|\sum_{j\leq T} a_j^{(n)}\left(\frac{\alpha_j}{\sqrt{\lambda_j+1}} - \frac{W_n\left(\alpha_j^{(n)}\right)}{\sqrt{\lambda_j^{(n)}+1}}\right)\right\| \\
&\quad \leq \sum_{j\leq T}\left\|\alpha_j - W_n\left(\alpha_j^{(n)}\right)\right\| + \sum_{j\leq T}\left\|\frac{\alpha_j}{\sqrt{\lambda_j+1}} - \frac{W_n\left(\alpha_j^{(n)}\right)}{\sqrt{\lambda_j^{(n)}+1}}\right\| \\
&\qquad < \frac{\varepsilon}{8} + \sum_{j\leq T}\left\|\frac{\alpha_j}{\sqrt{\lambda_j+1}} - \frac{W_n\left(\alpha_j^{(n)}\right)}{\sqrt{\lambda_j^{(n)}+1}}\right\| \qquad (6.69)\\
&\qquad = \frac{\varepsilon}{8} + \sum_{j\leq T}\left|\frac{1}{\sqrt{\lambda_j+1}} - \frac{1}{\sqrt{\lambda_j^{(n)}+1}}\right| + \frac{1}{\sqrt{\lambda_j^{(n)}+1}}\left\|\alpha_j - W_n\left(\alpha_j^{(n)}\right)\right\| \\
&\qquad < \frac{\varepsilon}{4} + \sum_{j\leq T}\left\|\alpha_j - W_n\left(\alpha_j^{(n)}\right)\right\| \\
&\qquad < \frac{\varepsilon}{2}.
\end{aligned}$$

For the second summand,

$$\begin{aligned}
\left\|\left((\Delta+1)^{-\frac{1}{2}} W_n - W_n(\Delta_n+1)^{-\frac{1}{2}}\right)(\beta_{>T})\right\| &\leq \left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| \\
&\quad + \left\|W_n(\Delta_n+1)^{-\frac{1}{2}}(\beta_{>T})\right\| \\
&\leq \left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| \\
&\quad + \frac{1}{\sqrt{\lambda_T^{(n)}+1}}\|\beta_{>T}\| \qquad (6.70)\\
&\leq \left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| + \frac{1}{\sqrt{\lambda_T+1}} \\
&\quad + \left|\frac{1}{\sqrt{\lambda_T+1}} - \frac{1}{\sqrt{\lambda_T^{(n)}+1}}\right| \\
&< \left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| + \frac{\varepsilon}{4}.
\end{aligned}$$

We aim to show that $\left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| < \frac{\varepsilon}{4}$. Write $W_n(\beta_{>T}) = \sum_{t\geq 1} b_t\alpha_t$ and observe that since $\left\|\sum_{t\geq 1} b_t\alpha_t\right\| = \|W_n(\beta_{>T})\| \leq 1$, we have

$$\begin{aligned}
\left\|(\Delta+1)^{-\frac{1}{2}} W_n(\beta_{>T})\right\| &= \left\|(\Delta+1)^{-\frac{1}{2}}\left(\sum_{t\geq 1} b_t\alpha_t\right)\right\| \\
&\leq \left\|\sum_{t\leq T} b_t\alpha_t\right\| + \frac{1}{\sqrt{\lambda_T+1}}\left\|\sum_{t>T} b_t\alpha_t\right\| \\
&< \left\|\sum_{t\leq T} b_t\alpha_t\right\| + \frac{\varepsilon}{8}.
\end{aligned} \tag{6.71}$$

To bound $\left\|\sum_{t\leq T} b_t\alpha_t\right\|$, observe that for $t \leq T$,

$$\begin{aligned}
|b_t| &= \|\langle W_n(\beta_{>T}), \alpha_t\rangle\| \\
&= \left\|\left\langle W_n(\beta_{>T}), \alpha_t - W_n\left(\alpha_t^{(n)}\right)\right\rangle\right\|, \quad \text{since } t \leq T \\
&\leq \left\|\alpha_t - W_n\left(\alpha_t^{(n)}\right)\right\|, \quad \text{by Cauchy-Schwarz} \\
&< \frac{\varepsilon}{8T}.
\end{aligned} \tag{6.72}$$

So,

$$\left\|\sum_{t\leq T} b_t\alpha_t\right\| < \left(\sum_{t\leq T}\left(\frac{\varepsilon}{8T}\right)^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{4}. \tag{6.73}$$

□

In summary, we have proven the following:

Theorem 6.6.3. The sequence $(C^\bullet(\star^n K), d, \Delta_n)_{n=1}^\infty$ is convergent to $\big(L^2\Omega^\bullet(M), d, \Delta\big)$.

Corollary 6.6.4. The sequence $(C^\bullet(\star^n K), d, \Delta_n)_{n=1}^\infty$ is Cauchy and strong deformation retractions witnessing this are given by

$$(f_n^m, g_m^n, h_{n,m}) : (C^\bullet(\star^n K), d, \Delta_n) \to (C^\bullet(\star^m K), d, \Delta_m). \tag{6.74}$$

Proof. Fix $\varepsilon > 0$. Theorem 6.6.3 tells us that we can take $N \in \mathbb{N}$ large enough so that for $n \geq m \geq N$, (R_n, W_n, s_n) and (R_m, W_m, s_m) are $\frac{\varepsilon}{2}$-deformation retractions. We know that $(f_n^m, g_m^n, h_{n,m})$ is a strong deformation retraction and that $W_m = W_n g_m^n$, implying $\operatorname{im}(W_m) \subseteq \operatorname{im}(W_n)$. Moreover,

$$W_n^* W_m = W_n^* W_n g_m^n = g_m^n. \tag{6.75}$$

By Proposition 5.2.5, g_m^n satisfies Equation (5.10) for ε and so $(f_n^m, g_m^n, h_{n,m})$ is a ε-deformation retraction. By applying Proposition 3.1.2, we see that $\left(C^\bullet\left(\star^n K\right), \Delta_n\right)_{n=1}^\infty$ is Cauchy through the strong deformation retractions given by

$$(f_n^m, g_m^n, h_{n,m}) : \left(C^\bullet\left(\star^n K\right), d, \Delta_n\right) \to \left(C^\bullet\left(\star^m K\right), d, \Delta_m\right). \tag{6.76}$$

□

Corollary 6.6.4 and Lemma 6.4.6, gives us access to the Hodge decomposition approximation results of Section 5.3. Moreover, Section 5.6 implies that $\left(L^2\Omega^\bullet(M), d, \Delta\right)$ can be constructed using $(C^\bullet(M), d, \Delta_n)_{n=1}^\infty$. This is interesting in light of previous works investigating the consistency of the codifferential map d^* [46, 3]. Namely, for $\alpha \in \Omega^\bullet(M)$, one might ask if the following quantity is bounded above by a function that is decreasing in n:

$$\|d^*\alpha - W_n d^* R_n(\alpha)\|. \tag{6.77}$$

It is shown in [3], through experimental techniques, that this is not the case in general. However, the results in this thesis show that the data necessary to construct the continuum codifferential are present in the sequence of discretizations $(C^\bullet(M), d, \Delta_n)_{n=1}^\infty$.

Appendix

A Spectral Projections

In this section, we prove Theorems 2.2.3 and 4.1.1. Theorem 2.2.3 and its proof are essentially restatements of Theorem A.7 and its proof in [38], with cosmetic modifications in order to conform with our notation.

Theorem 4.1.1 is based on ideas from the statements of Proposition 4.3.1 in [39] and, equivalently, Theorem A.11 in [38]. As remarked in Section 4.1, the proofs of this result in [39, 38] contain mistakes and so Theorem 4.1.1 is our best attempt at salvaging the result. To do this, we used ideas from the statements and proofs of Theorems 4.1.5, 4.2.9, 4.2.10, Proposition 4.3.1 in [39] and Theorem A.7, A.8, A.11 in [38].

Theorem A.0.1 (Theorem A.7 [38])**.** Let $\psi\colon \overline{\mathbb{R}_+} \to \mathbb{R}$ be continuous and let $\varepsilon > 0$. Then, there exists $\delta_{\psi,\varepsilon} > 0$ such that for all $\delta \leq \delta_{\psi,\varepsilon}$, $(\mathcal{H}, \Delta)$, $(\mathcal{H}', \Delta')$, $g\colon \mathcal{H} \to \mathcal{H}'$ with $\|g\| = 1$ and $\left\|(\Delta' + 1)^{-1} g - g(\Delta + 1)^{-1}\right\| < \delta$,

$$\left\|\psi\left(\Delta'\right) g - g\psi\left(\Delta\right)\right\| < \varepsilon. \tag{78}$$

Proof. Note that $\psi \circ \varphi^{-1} \in C^0([0, 1])$ (where φ comes from Equation (2.31)). Apply the Stone-Weierstrass theorem to $\psi \circ \varphi^{-1}$, obtaining a polynomial

$$p(\lambda) = \sum_{k=0}^{n} a_k \lambda^k \tag{79}$$

such that $\left\|p-\psi\circ\varphi^{-1}\right\|_{\infty}<\frac{\varepsilon}{4}$. This implies that $\|p\circ\varphi-\psi\|_{\infty}<\frac{\varepsilon}{4}$ and so

$$\begin{aligned}\left\|\psi\left(\Delta'\right)g-g\psi\left(\Delta\right)\right\| &\leq \left\|\left(\psi-p\circ\varphi\right)\left(\Delta'\right)g-g\left(\psi-p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\quad+\left\|\left(p\circ\varphi\right)\left(\Delta'\right)g-g\left(p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\leq \left\|\left(\psi-p\circ\varphi\right)\left(\Delta'\right)g\right\| \\ &\quad+\left\|g\left(\psi-p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\quad+\left\|\left(p\circ\varphi\right)\left(\Delta'\right)g-g\left(p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\leq \left\|\left(\psi-p\circ\varphi\right)\left(\Delta'\right)\right\| \\ &\quad+\left\|\left(\psi-p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\quad+\left\|\left(p\circ\varphi\right)\left(\Delta'\right)g-g\left(p\circ\varphi\right)\left(\Delta\right)\right\| \\ &\leq \frac{\varepsilon}{2}+\left\|\left(p\circ\varphi\right)\left(\Delta'\right)g-g\left(p\circ\varphi\right)\left(\Delta\right)\right\|.\end{aligned}\tag{80}$$

We turn our attention to bounding the second summand and employ the proof of Lemma 4.2.8 in [39] in noting that for $k\in\mathbb{N}$

$$\left(\Delta'+1\right)^{-k}g-g\left(\Delta+1\right)^{-k}=\sum_{j=1}^{k}\left(\Delta'+1\right)^{-k+j}\left(\left(\Delta'+1\right)^{-1}g-g\left(\Delta+1\right)^{-1}\right)\left(\Delta+1\right)^{1-j}.\tag{81}$$

Hence,

$$\begin{aligned}\left\|\left(p\circ\varphi\right)\left(\Delta'\right)g-g\left(p\circ\varphi\right)\left(\Delta\right)\right\| &= \left\|\sum_{k=0}^{n}a_k\left(\left(\Delta'+1\right)^{-k}g-g\left(\Delta+1\right)^{-k}\right)\right\| \\ &\leq \sum_{k=0}^{n}|a_k|\left\|\left(\Delta'+1\right)^{-k}g-g\left(\Delta+1\right)^{-k}\right\| \\ &= \sum_{k=0}^{n}|a_k|\sum_{j=1}^{k}\left\|\left(\Delta'+1\right)^{-k+j}\right\|\left\|\left(\Delta'+1\right)^{-1}g-g\left(\Delta+1\right)^{-1}\right\|\left\|\left(\Delta+1\right)^{1-j}\right\| \\ &\leq \sum_{k=0}^{n}|a_k|k\delta.\end{aligned}\tag{82}$$

Suppose

$$0<\delta<\frac{\varepsilon}{2\sum_{k=1}^{n}|a_k|k}.\tag{83}$$

Notice that the denominator of the upper bound on δ is dependent on ε but not δ, $(\mathcal{H},\Delta)$, or $(\mathcal{H}',\Delta')$. Then,

$$\left\|\psi\left(\Delta'\right)g-g\psi\left(\Delta\right)\right\|<\varepsilon.\tag{84}$$

□

Using this Theorem, we were able to prove Theorem 2.2.4 which says that if $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}_\infty, \Delta_\infty)$, then $d(\sigma(\Delta_n), \sigma(\Delta_\infty)) \to 0$.

Theorem A.0.2. Suppose that $(\mathcal{H}_n, \Delta_n)_{n=1}^{\infty}$ converges to $(\mathcal{H}_\infty, \Delta_\infty)$. Then, for each finite closed interval I with

$$\partial_+ I \cap \sigma(\Delta_\infty) = \emptyset, \tag{85}$$

there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dim \operatorname{im}(P_{\Delta_n, I}) = \dim \operatorname{im}(P_{\Delta_\infty, I}). \tag{86}$$

Proof. Since $\partial_+ I \cap \sigma(\Delta_\infty) = \emptyset$ and $\partial_+ I, \sigma(\Delta_\infty)$ are closed, we have

$$d(\sigma(\Delta_\infty), \partial_+ I) > 0. \tag{87}$$

Let $\chi_I \colon \overline{\mathbb{R}_+} \to [0,1]$ be a continuous function such that $\chi_I\big|_I = 1$ and for $\lambda \in \overline{\mathbb{R}_+} \setminus I$ such that

$$d(\lambda, I) \geq \frac{d(\sigma(\Delta_\infty), \partial_+ I)}{2}, \tag{88}$$

we have $\chi_I(\lambda) = 0$. Notice that $\operatorname{im}\left(\chi_I\big|_{\sigma(\Delta_\infty)}\right) \subseteq \{0,1\}$ and so $\chi_I(\Delta_\infty) = P_{\Delta_\infty, I}$. By Theorem 2.2.4, $\sigma(\Delta_n) \to \sigma(\Delta_\infty)$, so we have that for large enough N_1, $n \geq N_1$, $\operatorname{im}\left(\chi_I\big|_{\sigma(\Delta_n)}\right) \subseteq \{0,1\}$ and so $\chi_I(\Delta_n) = P_{\Delta_n, I}$. Now notice

$$\begin{aligned} \|(\Delta_\infty + 1)\chi_I(\Delta_\infty)\| &\leq \sup_{\lambda \in I \cap \sigma(\Delta_\infty)} |\lambda + 1| \\ &\leq \sup_{\lambda \in I} |\lambda + 1| \\ &< \infty, \quad \text{since } I \text{ is a finite interval.} \end{aligned} \tag{89}$$

Let $C_1 = \frac{1}{2\|(\Delta_\infty + 1)\chi_I(\Delta_\infty)\|} > 0$. Apply Theorem A.0.1 with $\varepsilon = \frac{1}{2}$ to obtain $\delta_{\chi_I, I} > 0$ such that for all $\delta < \delta_{\chi_I, I}$, if $(\mathcal{H}_n, \Delta_n)$ a δ-retract of $(\mathcal{H}_\infty, \Delta_\infty)$ seen via g_n,

$$\|\chi_I(\Delta_\infty) g_n - g_n \chi_I(\Delta_n)\| < \frac{1}{2}. \tag{90}$$

Let $\delta_I = \min(\delta_{\chi_I, I}, C_1)$ and take $N \geq N_1$ large enough such that for all $n \geq N$, $(\mathcal{H}_n, \Delta_n)$ a δ-retract of $(\mathcal{H}_\infty, \Delta_\infty)$ for $\delta < \delta_I$. Let $\alpha \in \operatorname{im}(P_{\Delta_n, I})$ be a unit. Then,

$$\begin{aligned} \|P_{\Delta_\infty, I} g_n(\alpha)\| &\geq \|g_n P_{\Delta_n, I}(\alpha)\| - \|(P_{\Delta_\infty, I} g - g P_{\Delta_n, I})(\alpha)\| \\ &= 1 - \|(\chi_I(\Delta_\infty) g_n - g_n \chi_I(\Delta_n))(\alpha)\| \\ &> \frac{1}{2}, \end{aligned} \tag{91}$$

which proves that $P_{\Delta_\infty,I}g_n\big|_{\mathrm{im}\left(P_{\Delta_n,I}\right)}$ is injective. Now let $\beta \in \mathrm{im}\left(P_{\Delta_\infty,I}\right)$ be a unit. Observe,

$$\begin{aligned}
\|P_{\Delta_n,I}g_n^*(\beta)\| &\geq \|g_n^* P_{\Delta_\infty,I}(\beta)\| - \|\left(g_n^* P_{\Delta_\infty,I} - P_{\Delta_n,I}g_n^*\right)(\beta)\| \\
&\geq \|P_{\Delta_\infty,I}(\beta)\| - \|\left(1 - g_n g_n^*\right) P_{\Delta_\infty,I}(\beta)\| - \|\left(g_n^* P_{\Delta_\infty,I} - P_{\Delta_n,I}g_n^*\right)(\beta)\| \\
&= \|P_{\Delta_\infty,I}(\beta)\| - \|\left(1 - g_n g_n^*\right) P_{\Delta_\infty,I}(\beta)\| - \|\left(g_n^* \chi_I\left(\Delta_\infty\right) - \chi_I\left(\Delta_n\right) g_n^*\right)(\beta)\| \\
&> 1 - \|\left(1 - g_n g_n^*\right) P_{\Delta_\infty,I}(\beta)\| - \frac{1}{2} \\
&\geq \frac{1}{2} - \delta \|\left(\Delta_\infty + 1\right) P_{\Delta_\infty,I}(\beta)\|, \quad \text{by Lemma 2.1.3} \\
&= \frac{1}{2} - \delta \|\left(\Delta_\infty + 1\right) \chi_I\left(\Delta_\infty\right)(\beta)\| \\
&\geq 0,
\end{aligned} \tag{92}$$

which proves that $P_{\Delta_n,I}g^*\big|_{\mathrm{im}\left(P_{\Delta_\infty,I}\right)}$ is injective and so we are done. □

B Whitney Standard Subdivision

Results on simplicial approximation of the smooth differential forms hinge on the existence of the Whitney standard subdivision algorithm (Definition 6.4.2). In this section, we prove that the algorithm decomposes as a sequence of stellar subdivison operations (Definition 6.2.1).

Proposition B.0.1. Given a simplicial complex K, there exists an ordering of the edges (1-simplices) $\{e_0, \ldots, e_L\} = K_1$ such that the iterated stellar subdivision $\star_{e_0} \star_{e_1} \cdots \star_{e_L} K$ is equal to the Whitney Standard Subdivision $\star K$.

Proof. Suppose first that $K = \Delta^n$. The case of $n = 1$ is obvious, so assume $n \geq 2$. Enumerate the vertices of K, $v_0, \ldots, v_n$. Define a total order on edges like so:

$$[v_i, v_j] \preceq [v_h, v_k] \text{ if } h < i \text{ or } h = i \text{ and } j \leq k. \tag{93}$$

Enumerate the edges of K in increasing order according to $\preceq$, $e_0, \ldots, e_L$. Consider the iterated stellar subdivision

$$K' := \star_{e_0} \star_{e_1} \cdots \star_{e_L} K. \tag{94}$$

We will show that $K' = \star K$ by proving the following claims:

1. For each $\ell \leq L$, each 1-simplex in $\left(\star_{e_\ell}, \star_{e_{\ell-1}} \cdots \star_{e_L} K\right)_1 \setminus \left(\star_{e_{\ell-1}} \cdots \star_{e_L} K\right)_1$ consists of two comparable vertices with respect to the partial order $\leq$ defined in the Whitney Standard Subdivision algorithm 6.4.2.

2. All 1-simplices of K' consist of comparable vertices.

3. All simplices of K' consist of comparable vertices and thus $K' \subseteq \star K$.

4. All simplices consisting of comparable vertices occur in K', i.e. $\star K \subseteq K'$.

Proceed with the first claim by induction. For $\ell = L$, we must examine the 1-simplices that get added in $\star_{e_L} K$. Through our definition of $\preceq$, $e_L = [v_0, v_n]$, and so the 1-simplices that get added to $\star_{e_L} K$ come from 2-simplices $[v_0, v_i, v_n]$ for $0 < i < n$ and are given by $[v_i, v_{0,n}]$. By definition, this 1-simplex is consists of comparable vertices according to $\leq$.

Now fix ℓ and suppose that for all $t > \ell$, each 1-simplex in $\left(\star_{e_t}, \star_{e_{t-1}} \cdots \star_{e_L} K\right)_1 \setminus \left(\star_{e_{t-1}} \cdots \star_{e_L} K\right)_1$ consists of comparable vertices. Write $e_\ell = [v_{i_\ell}, v_{j_\ell}]$. Similar to the base-case, each 1-simplex that gets added through stellar subdivision by e_ℓ has the form $[v_{h,k}, v_{i_\ell, j_\ell}]$ where $[v_{h,k}, v_{i_\ell}, v_{j_\ell}] \in \star_{e_{\ell-1}} \cdots \star_{e_L} K$. There are 2 cases for $v_{h,k}$, $h = k$ and $h < k$. If $h = k$, then neither of $[v_h, v_{i_\ell}], [v_h, v_{j_\ell}]$ were subdivided in previous iterations of stellar subdivision and so it must be that both $[v_h, v_{i_\ell}], [v_h, v_{j_\ell}] \preceq [v_{i_\ell}, v_{j_\ell}]$. Thus, $i_\ell \leq h \leq j_\ell$ which means that $v_h \leq v_{i_\ell, j_\ell}$ and so $[v_{h,k}, v_{i_\ell, j_\ell}]$ consists of comparable vertices. Now suppose $h < k$. Since $[v_{h,k}, v_{i_\ell}, v_{j_\ell}] \in \star_{e_{\ell-1}} \cdots \star_{e_L} K$, $[v_{h,k}, v_{i_\ell}], [v_{h,k}, v_{j_\ell}] \in \star_{e_{\ell-1}} \cdots \star_{e_L} K$, and the inductive assumption implies that

$$\begin{aligned} &v_{h,k} \leq v_{j_\ell} \text{ or } v_{j_\ell} \leq v_{h,k}, \\ &v_{h,k} \leq v_{i_\ell} \text{ or } v_{i_\ell} \leq v_{h,k} \end{aligned} \tag{95}$$

are both true. Suppose that $v_{h,k} \leq v_{j_\ell}$, then $j_\ell \leq h < k \leq j_\ell$, a contradiction. Similarly, we cannot have $v_{h,k} \leq v_{i_\ell}$. So, $v_{j_\ell} \leq v_{h,k}$ and $v_{i_\ell} \leq v_{h,k}$. This means that

$$h \leq i_\ell < j_\ell \leq k, \tag{96}$$

which is equivalent to $v_{i_\ell, j_\ell} \leq v_{h,k}$ and so $[v_{i_\ell, j_\ell}, v_{h,k}]$ consists of comparable vertices.

To see that all 1-simplices of K' consist of comparable vertices, it suffices to show that each 1-simplex was added during a subdivision. Equivalently, we must show that K' and K share no 1-simplices. This is immediate, however, since each 1-simplex of K appears as some e_ℓ and is subdivided.

We now aim to show that all simplices of K' consist of comparable vertices. Fix a k-simplex $[v_{\alpha_0,\beta_0}, v_{\alpha_1,\beta_1}, \ldots, v_{\alpha_k,\beta_k}] \in K'$. For $h < k$, the face $[v_{\alpha_h,\beta_h}, v_{\alpha_k,\beta_k}]$ gets added to K' and by the previous claims, we must have that either $v_{\alpha_h,\beta_h} \leq v_{\alpha_k,\beta_k}$ or $v_{\alpha_k,\beta_k} \leq v_{\alpha_h,\beta_h}$. Thus, the vertices $v_{\alpha_0,\beta_0}, v_{\alpha_1,\beta_1}, \ldots, v_{\alpha_k,\beta_k}$ can be sorted into increasing order and we have established the third claim.

To address the fourth claim, fix in increasing sequence of vertices $v_{\alpha_0,\beta_0} < v_{\alpha_1,\beta_1} <$

$\cdots < v_{\alpha_k,\beta_k} \in K'$. We will show that

$$[v_{\alpha_0,\beta_0}, v_{\alpha_1,\beta_1}, \ldots, v_{\alpha_k,\beta_k}] \in K'. \tag{97}$$

By the Whitney partial ordering,

$$\alpha_k \le \alpha_{k-1} \le \cdots \le \alpha_1 \le \alpha_0 \le \beta_0 \le \beta_1 \le \cdots \le \beta_{k-1} \le \beta_k. \tag{98}$$

This shows that we can have at most one vertex from K, v_{α_0,β_0}. By the total ordering, $[v_{\alpha_\ell}, v_{\beta_\ell}] \prec [v_{\alpha_{\ell+1}}, v_{\beta_{\ell+1}}]$ for each ℓ. So, we can define an increasing sequence $(i_\ell)_{\ell=0}^k$ where i_ℓ is the index such that $e_{i_\ell} = [v_{\alpha_\ell}, v_{\beta_\ell}]$ and in the case that $v_{\alpha_0,\beta_0} \in K$, $i_0 = -1$. Now, let $\sigma_{k+1} \in K$ be the simplex spanned by

$$v_{\alpha_k}, v_{\alpha_{k-1}}, \ldots, v_{\alpha_1}, v_{\alpha_0}, v_{\beta_0}, v_{\beta_1}, \ldots, v_{\beta_{k-1}}, v_{\beta_k}. \tag{99}$$

For each $\ell = 0, \ldots, k$, let σ_ℓ be the simplex spanned by

$$v_{\alpha_k,\beta_k}, v_{\alpha_{k-1},\beta_{k-1}}, \ldots, v_{\alpha_\ell,\beta_\ell}, v_{\alpha_{\ell-1}}, v_{\alpha_{\ell-2}}, \ldots, v_{\alpha_0}, v_{\beta_0}, \ldots, v_{\beta_{\ell-2}}, v_{\beta_{\ell-1}}. \tag{100}$$

Notice that for $t > i_\ell$, the edge e_t is not a face of σ_ℓ due to the ordering (98). By induction, we get that for all ℓ,

$$\sigma_\ell \in \star_{e_{i_\ell}} \star_{e_{i_\ell+1}} \cdots \star_{e_L} K. \tag{101}$$

In the case that $v_{\alpha_0,\beta_0} \in K$, we have $\sigma_1 = [v_{\alpha_0,\beta_0}, v_{\alpha_1,\beta_1}, \ldots, v_{\alpha_k,\beta_k}] \in K'$ and we are done. If $v_{\alpha_0,\beta_0} \notin K$, then $\sigma_0 = [v_{\alpha_0,\beta_0}, v_{\alpha_1,\beta_1}, \ldots, v_{\alpha_k,\beta_k}] \in K'$. Consequently, $K' = \star K$.

For an arbitrary simplicial complex K, order the vertices of K and apply the same stellar subdivision procedure on the edges ordered according to $\preceq$, noting that the restriction of this process to each simplex yields the process described above. □

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