# On Stable Harmonic Analysis and Stable Transfer

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy

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2024

# Abstract

Let $H$ and $G$ be connected reductive groups over a local field $F$ of characteristic zero, with $G$ quasisplit. For each tempered injective $L$-homomorphism ${}^LH \to {}^LG$, the local Langlands correspondence determines a functorial transfer map between the sets of stable tempered characters of $H$ and $G$. In 2013, Langlands posed the question of whether the functorial transfer map can be interpolated by the transpose of a continuous linear operator between the spaces of stable orbital integrals of test functions on $G(F)$ and $H(F)$. Langlands introduced these stable transfer operators as the central local ingredient of Beyond Endoscopy, the strategy proposed by Langlands in 2000 for proving the Principle of Functoriality using Arthur's stable trace formula.

For a general connected reductive group over $F$, we prove stable Paley–Wiener theorems characterising the image under the stable Fourier transform of the spaces of stable orbital integrals of test functions, $K$-finite test functions, and Harish-Chandra Schwartz functions. In the case of test functions and $K$-finite functions on quasisplit groups, the stable Paley–Wiener theorem was established by Moeglin–Waldspurger and Arthur for real and $p$-adic groups, respectively. For $p$-adic groups, we prove that stable tempered characters span a weak-$*$ dense subspace of the space of stable tempered distributions, a result established in the case of real groups by Shelstad in 1979. Using these results in stable harmonic analysis, we prove that stable transfer operators exist, preserve $K$-finiteness, and extend continuously to linear operators between spaces of stable orbital integrals of Harish-Chandra Schwartz functions. The proof involves a finiteness property of functorial transfer maps, which we establish. We provide formulas for stable transfer operators when both groups are either tori or complex groups.

To Hannah...

# Acknowledgements

First, I would like to express my gratitude to my advisor, James Arthur, for introducing me to the Langlands program and for his invaluable teachings, guidance, encouragement, and support. I thank William Casselman for serving as my external appraiser and for his comments on my thesis. I also thank Joe Repka for his role on my supervisory committee, and Stephen Kudla and Jacob Tsimerman for serving on my examination committee.

This thesis was influenced by the works of many researchers. I acknowledge the key influences of the works of James Arthur, Harish-Chandra, David Kazhdan, Robert Langlands, Colette Moeglin, Diana Shelstad, Jean-Loup Waldspurger, Sandeep Varma, and Ernest Vinberg. I also thank Clifton Cunningham, Paul Mezo, Chung Pang Mok, Jayce Getz, Yiannis Sakellaridis, Rahul Dalal, Mathilde Gerbelli-Gauthier, and Bin Xu for their interest in stable transfer and discussions I had with them.

I had the privilege of attending two edifying workshops on the Langlands program, one at the Institute for Mathematical Sciences at the National University of Singapore and the other at the IHES. I thank William Casselman, Ngô Bảo Châu, Pierre-Henri Chaudouard, Wee Teck Gan, Dihua Jiang, Tasho Kaletha, Gérard Laumon, Colette Moeglin, Yiannis Sakellaridis, Jean-Loup Waldspurger, Lei Zhang, and Chengbo Zhu for organising these events.

I am very grateful to the members of my research group at the University of Toronto: Melissa Emory, Daniel Johnstone, and my academic siblings Hannah Constantin, Malors Espinosa Lara, and Patrice Moisan-Roy. Our learning seminars, research meetings, and mathematical discussions have been invaluable. I am especially thankful for the weekly research meetings I had with Daniel and Patrice on the topic of stable transfer, which greatly contributed to my work.

I acknowledge the generous support from the Natural Sciences and Engineering Research Council of Canada (NSERC) through an Alexander Graham Bell Canada Graduate Scholarship. I express my gratitude to George A. Elliott and the estates of Elsje Mandl, Georgia Muriel Taylor, and Reginald A. Blyth for the generous support provided by the Margaret Isobel Elliott Scholarship, the Paul Mandl Scholarship, the Margaret Ronald Taylor and Thomas Paxton Taylor Scholarship, and the Blyth Fellowship.

I thank the administrative staff in the Department of Mathematics, especially Jemima Merisca, Cherylyn Stina, Diana Leonardo, Sonja Injac, Ashley Armogan, and Patrina Seepersaud for their assistance. I also thank the Fields Institute, which provided a nurturing environment during my time at the University of Toronto. I express gratitude to the McGill Physics Department, the Trottier Space Institute, Robert Brandenberger, and especially Adrian Liu and his research group for their warm reception and hospitality while I was in Montreal.

I give deep thanks to my earliest research mentors and collaborators: Young-June Kim, Stephen Tanny, Yvon Verberne, Bruce Richter, and Alan Arroyo Guevara. My ongoing collaboration with Bruce and Alan continues to bring me great joy and I have learned immensely from working with them.

I thank the professors who played an important role in my mathematical development: Edward Bierstone, John Bland, Clifton Cunningham, John Friedlander, Alfonso Gracia-Saz, Michael Groechenig, Marco Gualtieri, Florian Herzig, Joel Kamnitzer, Henry Kim, Stephen Kudla, Matilde Marcolli, Eckhard Meinrenken, Fiona Murnaghan, Kumar Murty, and Arul Shankar. I am very grateful to Henry Kim for mentoring me in a reading course on Tate's thesis.

Three teachers left a profound influence on my path over thirteen years ago. I deeply appreciate Richard Whitlock, Tony Donea, and Marcia Copping.

For camaraderie, mathematical discussions, and participation in learning seminars, I thank: Mario Apetroaie, Ali Cheraghi, Stefan Dawydiak, Vincent Girard, Debanjana Kundu, Joshua Lackman, Heejong Lee, Valia Liontou, Siddharth Mahendraker, Justin Martel, Mateusz Olechnowicz, Abhishek Oswal, David Pechersky, Waleed Qaisar, Mishty Ray, Artane Siad, Yuan Yao, and Sina Zabanfahm. Additionally, I would like to thank Petr Kosenko for being a great teammate when we taught together.

For their encouragement and support, I extend deep gratitude to my family. Hannah Fronenberg, my partner, to whom this thesis is dedicated, who inspires me and is always by my side, made this work possible.

# Contents

# 1 Introduction

## 1.1 The Principle of Functoriality

The Principle of Functoriality, introduced by Langlands in his 1967 Letter to Weil [DS15], is the central foundational conjecture of the Langlands program. Let $F$ be a number field, let $G$ be a quasisplit connected reductive groups over $F$, and let ${}^LG = G^\vee \rtimes W_F$ be the $L$-group of $G$, where $G^\vee$ is the complex Langlands dual group of $G$ and $W_F$ is the Weil group of $F$. The Principle of Functoriality asserts that if $H$ is a connected reductive group over $F$, for every $L$-homomorphism $\xi : {}^LH \to {}^LG$ there is a natural correspondence (relation) between automorphic representations of $H(\mathbb{A}_F)$ and those on $G(\mathbb{A}_F)$, where $\mathbb{A}_F$ denotes the ring of adèles of $F$. A strong form of the Principle of Functoriality is the global Langlands correspondence, which states that this correspondence is governed by a natural classification of automorphic representations. More specifically, the global Langlands correspondence is a natural surjective reciprocity map

$$\mathrm{rec}_H : \Pi_{\mathrm{aut}}(H) \longrightarrow \Phi(H)$$

from the set of equivalence classes of automorphic representations of $H$ to the set of $L$-parameters of $H$, which are $H^\vee$-conjugacy classes of relevant $L$-homomorphisms $\phi : L_F \to {}^LH$, where $L_F$ is the global Langlands group, a hypothetical locally compact extension of the Weil group $W_F$. Said differently, the $L$-parameters $\phi \in \Phi(H)$ parametrise a natural partition of $\Pi_{\mathrm{aut}}(H)$ into sets $\Pi_\phi$ called $L$-packets. The correspondence of automorphic representations conjectured by the Principle of Functoriality is the correspondence determined by this parametrisation: for every $L$-parameter $\phi : L_F \to {}^LH$, the representations in $\Pi_\phi$ and $\Pi_{\xi\circ\phi}$ are in correspondence.

The global Langlands correspondence should be compatible with the local Langlands correspondence. For each place $v$ of $F$, a global $L$-parameter $\phi : L_F \to {}^LH$ determines a local $L$-parameter $\phi_v : L_{F_v} \to {}^LH_v$ of $H_v = H_{F_v}$. Here, $L_{F_v}$ is the local

Langlands group, which is defined to be the local Weil group $W_{F_v}$ if $F_v$ is archimedean and the Weil–Deligne group $\mathrm{SL}_2(\mathbb{C}) \times W_{F_v}$ if $F_v$ is non-archimedean. The local Langlands correspondence is a natural surjective reciprocity map

$$\mathrm{rec}_{H_v} : \Pi(H_v) \to \Phi(H_v)$$

from the set of equivalence classes of irreducible admissible representations of $H_v(F_v) = H(F_v)$ to the set of $L$-parameters of $H_v$. An automorphic representation $\pi \in \Pi_{\mathrm{aut}}(H)$ decomposes as a tensor product $\pi = \bigotimes_v \pi_v$ with $\pi_v \in \Pi(H_v)$, and the compatibility between the local and global Langlands correspondences is: $\mathrm{rec}_H(\pi)_v = \mathrm{rec}_{H_v}(\pi_v)$. The local Langlands correspondence was established by Langlands in the archimedean case [Lan89] and is still partly conjectural in the non-archimedean case.

We refer the reader to [Art21; Art24; Art02a] for more background on the Principle of Functorial and the global Langlands correspondence, which we are only discussing for motivation. The local Langlands correspondence will be reviewed in detail in the body of this thesis.

## 1.2 Beyond Endoscopy

The most general tool for attacking the Principle of Functoriality is Arthur's (twisted) trace formula and its stabilisation due to Arthur and Moeglin–Waldspurger [Art02b; MW16a]. However, applications of the trace formula to Functoriality are currently limited to the cases that belong to the theory of endoscopy, in which $H$ is a (twisted) endoscopic group of $G$. These cases constitute only a small piece of the full Principle of Functoriality. In 2000, Langlands introduced a research program for using the stable trace formula to prove the Principle of Functoriality in general, including the cases that lie beyond the theory of Endoscopy [Lan] (see also [Lan04; Lan07]). This program is called Beyond Endoscopy.

The theory of endoscopy has global and local parts. The global part is mainly concerned with the stabilisation of the (twisted) trace formula. The local part is mainly concerned with the local Langlands correspondence (which is still partially conjectural for non-archimedean local fields) and the theory of endoscopic transfer. In moving beyond endoscopy, one must first have a completed theory of endoscopy for the groups under consideration. We assume this for the remainder of this introductory chapter.

Part of the local Langlands correspondence, as we shall recall below, is the assignment of a stable tempered distribution $\Theta_{\phi_v}$ on $H(F_v)$ to each local tempered $L$-parameter $\phi_v \in \Phi_{\mathrm{temp}}(H_v)$. The tempered $L$-parameters are those whose $L$-packets consist of tempered representations, and these are the representations necessary for harmonic analysis on $H(F_v)$.

In Arthur's elaboration of the strategy of Beyond Endoscopy [Art17; Art18c; Art18b], there are two main global problems. The first global problem is to develop a new trace formula for $G$, called the $r$-trace formula, for each $L$-homomorphism $r : {}^LG \to {}^L\mathrm{GL}_N$. This would be an identity $S^{G,r}_{\mathrm{geom}}(f) = S^{G,r}_{\mathrm{spec}}(f)$ of stable distributions on $G(\mathbb{A}_F)$, such that the contribution from tempered discrete automorphic representations of $G(\mathbb{A})$ to the spectral side $S^{G,r}_{\mathrm{spec}}(f)$ could be expressed as

$$\sum_{\phi \in \Phi_{2,\mathrm{temp}}(G)} m_\phi(r) m(\phi) \Theta_\phi(f)$$

if the global Langlands correspondence were known. The set $\Phi_{2,\mathrm{temp}}(G)$ is the set of tempered discrete $L$-parameters of $G$. The distribution $\Theta_\phi(f)$ is the stable tempered character attached to $\phi$, and satisfies $\Theta_\phi(f) = \prod_v \Theta_{\phi_v}(f_v)$ for $f = \prod_v f_v$ with $f_v \in C^\infty_c(G(F)v))$. The constant $m(\phi)$ is the multiplicity of $\phi$ in the spectral side of the stable trace formula for $G$ and $m_\phi(r)$ is the pole order $-\operatorname{ord}_{s=1} L(s, r \circ \phi)$ of the $L$-function $L(s, r \circ \phi)$. In the case $r = 1$, we have $m_\phi(r) = 1$ and the tempered discrete contribution to $S^{G,1}_{\mathrm{spec}}(f)$ becomes the tempered discrete contribution to the spectral side $S^G_{\mathrm{spec}}(f)$ of the stable trace formula for $G$.

The strategy for developing the $r$-trace formula is to obtain it as a limit of averages of stable trace formulas applied to functions obtained by modifying $f$ using $r$. Before one can pass to the limit, non-tempered spectral terms must be cancelled with geometric terms. Besides the above references related to Beyond Endoscopy, the problem of developing the $r$-trace formula has been studied [FLN10; Alt15; Alt17; Alt20; Lar22; Esp+; Art18a] for $\mathrm{GL}_n$ and $\mathrm{SL}_n$, mostly in the case $n = 2$.

Once the $r$-trace formula is established, the second global problem would then be to "primitivise" the $r$-trace formula for $G$ [Art17; Mok18]. The purpose of the pole orders $m_\phi(r)$ in the $r$-trace formula is that as $r$ varies, they detect whether $\phi$ factors through a proper $L$-embedding $\mathcal{H} \hookrightarrow {}^LG$. As in the case of endoscopy, one must consider groups $\mathcal{H}$ that are not $L$-groups and use the technical device of $z$-extensions. For simplicity, we ignore this complication and suppose that $\mathcal{H} = {}^LH$ is an $L$-group. A tempered discrete $L$-parameter, as well as its corresponding tempered discrete automorphic representations, is said to be primitive (also thick or hadronic [Lan11; Lan12]) if it does not factor through a proper $L$-embedding. To primitivise the $r$-trace formula is to express it as a linear combination

$$S_?^{G,r}(f) = \sum_H \iota(r,H) P_?^H(\mathcal{T}_\xi f) \quad , \quad ? \in \{\text{geom}, \text{spec}\},$$

of new "primitive" (stable) trace formulas

$$P^H_{\text{geom}} = P^H_{\text{spec}}$$

for certain groups $H$ with a tempered $L$-embedding $\xi : {}^L H \hookrightarrow {}^L G$ (ignoring technicalities involving $z$-extensions), such that $P^H_{\text{spec}}$ only contains contributions from primitive automorphic representations of $H(\mathbb{A}_F)$. A correspondence (relation) $f \mapsto \{\mathcal{T}_\xi f\}$ between $C_c^\infty(H(\mathbb{A}_F))$ and $C_c^\infty(G(\mathbb{A}_F))$ is required in the primitivisation identity above in order to pull back the stable distribution $P^H_?$ on $H(\mathbb{A}_F)$ to a stable distribution on $G(\mathbb{A}_F)$. For $f \in C_c^\infty(H(\mathbb{A}_F))$, we write $\{\mathcal{T}_\xi f\}$ for the set of elements of $C_c^\infty(G(\mathbb{A}_F))$ that are in correspondence with $f$.

The correspondence $\mathcal{T}_\xi$ is defined to be compatible with locally defined correspondences $f_v \mapsto \{\mathcal{T}_{\xi_v} f_v\}$ between $C_c^\infty(H(F_v))$ and $C_c^\infty(G(F_v))$ attached to the local tempered $L$-embeddings $\xi_v : {}^L H_v \hookrightarrow {}^L G_v$ for each place $v$ of $F$ in the sense that when $f$ decomposes as $f = \prod_v f_v$ with $f_v \in C_c^\infty(G(F_v))$, then $\{\mathcal{T}_\xi f\} = \prod_v \{\mathcal{T}_{\xi_v} f_v\}$. The temperedness condition on $\xi_v$ is to guarantee that the associated pushforward of local $L$-parameters $\xi_{v,*} : \Phi(H_v) \to \Phi(G_v)$ defined by $\xi_{v,*}(\phi_v) = \xi_v \circ \phi_v$ restricts to a map $\xi_{v,*} : \Phi_{\text{temp}}(H) \to \Phi_{\text{temp}}(G)$ of local tempered $L$-parameters. The local correspondence $f_v \mapsto \{\mathcal{T}_{\xi_v} f_v\}$ is defined as follows. For $f_v \in C_c^\infty(G(F_v))$, the set $\{\mathcal{T}_\xi f_v\}$ consists of all functions $\mathcal{T}_\xi f_v \in C_c^\infty(H(F_v))$ such that

$$\Theta_{\phi_v}(\mathcal{T}_{\xi_v} f_v) = \Theta_{\xi_v \circ \phi_v}(f_v)$$

for all tempered local $L$-parameters $\phi_v \in \Phi(H_v)$. Temperedness of the stable tempered character $\Theta_{\phi_v}$ means that it extends to a continuous linear functional of the (Harish-Chandra) Schwartz space $\mathcal{C}(H(F_v))$. Thus, the definition of the local $f_v \mapsto \{\mathcal{T}_{\xi_v} f_v\}$ extends naturally to a correspondence between the Schwartz spaces $\mathcal{C}(G(F_v))$ and $\mathcal{C}(H(F_v))$. For $f_v \in \mathcal{C}(G(F_v))$, the functions functions $\mathcal{T}_{\xi_v} f_v$ in $\{\mathcal{T}_{\xi_v} f_v\} \subseteq \mathcal{C}(H(F_v))$ are said to be stable transfers of $f_v$. Stable transfer has also been called functorial transfer [Art08] and stable-stable transfer [Tho20] in order to avoid confusion with endoscopic transfer.

An extension of Beyond Endoscopy aimed at the relative Principle of Functoriality with an accompanying notion of stable transfer was introduced by Sakellaridis [Sak13;

Sak19a; Sak19b; Sak]. Understanding stable transfer is the main local problem of Beyond Endoscopy.

## 1.3 Stable transfer

Let $F$ now denote a local field of characteristic zero. Let $H$ and $G$ be connected reductive groups over $F$ with $G$ quasisplit, and let $\xi : {}^L H \to {}^L G$ be an injective tempered $L$-homomorphism. (As remarked above, eventually a more elaborate setting will be required, but this is the essential case.) We continue to assume the local Langlands correspondence. We write $C_c^\infty(G) = C_c^\infty(G(F))$ and $\mathcal{C}(G) = \mathcal{C}(G(F))$.

The first question about the stable transfer correspondence $\mathcal{T}_\xi$ that must be addressed is whether for each $f \in C_c^\infty(G)$, a stable transfer $\mathcal{T}_\xi f \in C_c^\infty(H)$ exists. Langlands posed this as Questions A and B in [Lan13], where he initiated the study of stable transfer. Langlands showed that this question has an affirmative answer when $G = \mathrm{SL}_2$, $H$ is a maximal torus of $G$, and $\xi$ is a natural $L$-embedding [Lan13, §2.1–2.4 (p. 182–210)]. Johnstone gave an affirmative answer in the case when $G = \mathrm{SL}_\ell$ or $G = \mathrm{GL}_\ell$ with $\ell$ an odd prime, $H$ is an unramified elliptic maximal torus of $G$, and $\xi$ is a natural $L$-embedding [Joh17; Joh]. The arguments in these works involve detailed computations with explicit stable character formulas, which are not available in general. The existence of stable transfer of test functions has also been explored in [JL; Tho20]. Stable transfers in the relative extension of Beyond Endoscopy have been been shown to exist in low rank cases [Sak13; Sak19a; Sak19b; Sak22a; Sak22b]. More generally, one can ask whether for each $f \in \mathcal{C}(G)$ there exists a stable transfer $\mathcal{T}_\xi f \in \mathcal{C}(H)$.

The existence of stable transfers is useful in global applications of the stable trace formula that lie outside the theory of Endoscopy but do not belong to the program of Beyond Endoscopy. For example, in [DG] stable transfers are used to prove statistics of automorphic representations for unramified unitary groups, with applications to the Sato–Tate Conjecture, the Sarnak–Xue Conjecture, and the cohomology of locally symmetric spaces. Extensions of this work to general unitary groups and other classical groups require more general existence results and a better understanding of the properties of stable transfer. We expect that stable transfer may have other applications outside of Beyond Endoscopy, both in global and local harmonic analysis.

Although the stable Fourier transforms of $f$ and $\mathcal{T}_\xi f$ are related in a simple way, the relation between the stable orbital integrals of $f$ and $\mathcal{T}_\xi f$ is much more complicated. In order to establish the primitivisation of the geometric side of the $r$-trace formula, one will need a deep geometric understanding of stable transfer. Since little is known about the geometric side of the $r$-trace formula, we do not know what precisely will be required. Further progress on developing the $r$-trace formula will give us more guidance. In the other direction, explicit formulas for the orbital integrals of $\mathcal{T}_\xi f$ in terms of those of $f$ might help guide the development of the $r$-trace formula. Such formulas are very complicated in general. They include formulas for stable tempered characters in the case when $H = 1$. However, in some cases they are more simple. In the case when $H$ is a Levi subgroup of $G$ and $\xi$ belongs to the natural $G^\vee$-conjugacy class of $L$-embedding, then $\mathcal{T}_\xi$ is simply parabolic descent. In the case considered by Langlands in [Lan13], there is a formula for stable transfer $\mathcal{T}_\xi f$ in terms of those of $f$, which is originally due to Gelfand and Graev [GG63]. There are major hurdles to generalising the Gelfand–Graev formula (cf. [Joh17; Joh]). Sakellaridis gave a different proof of the Gelfand–Graev formula in [Sak22b] and has established formulas for various cases of the relative version stable transfer [Sak13; Sak19b; Sak22a; Sak23]. We refer the reader to [Sak] for a survey of these works.

## 1.4 Stable harmonic analysis

The natural context in which to study the stable transfer correspondence $\mathcal{T}_\xi$ is stable harmonic analysis, which is concerned with the stable Fourier transform for a (not necessarily quasisplit) connected reductive group $G$ over a local field of characteristic zero.

For $f \in \mathcal{C}(G)$, the stable Fourier transform of $f$ is the function $f^G : \Phi_{\text{temp}}(G) \to \mathbb{C}$ defined by $f^G(\phi) = \Theta_\phi(f)$ for all $\phi \in \Phi_{\text{temp}}(G)$. We define $\widehat{\mathcal{SC}}(G)$ to be the space of $f^G$ of functions $f \in \mathcal{C}(G)$, which can be regarded as the quotient of $\mathcal{C}(G)$ by the closed subspace of $f \in \mathcal{C}(G)$ with $f^G(\phi) = 0$ for all $\phi \in \Phi_{\text{temp}}(G)$. The stable Fourier transform is a continuous linear map $\mathcal{F}^{\text{st}} : \mathcal{C}(G) \to \widehat{\mathcal{SC}}(G)$. The stable Fourier transform $\phi \mapsto f^G(\phi)$ of $f \in \mathcal{C}(G)$ only depends on the normalised stable orbital integrals $f^G(\delta) = |D^G(\delta)|^{1/2} \operatorname{SO}_\delta(f)$ of $f$. We define $\mathcal{SC}(G)$ to be the space of stable orbital integrals $f^G$ of functions $f \in \mathcal{C}(G)$, which is also naturally a quotient space of $\mathcal{C}(G)$. The stable Fourier transform $\mathcal{F}^{\text{st}}$ descends to a continuous linear operator

$$\mathcal{F}^{\text{st}} : \mathcal{SC}(G) \longrightarrow \widehat{\mathcal{SC}}(G).$$

As in classical harmonic analysis, a foundational problem in stable harmonic analysis to establish that this operator $\mathcal{F}^{\text{st}}$ is an isomorphism of topological vector spaces and characterise its image $\widehat{\mathcal{SC}}(G)$. We refer to this as a stable Paley–wiener theorem for Schwartz functions, and the injectivity of $\mathcal{F}^{\mathrm{st}}$ as stable spectral density for Schwartz functions.

Replacing $\mathcal{C}(G)$ with $C_c^\infty(G)$ above, we obtain definitions of $\widehat{\mathcal{S}C_c^\infty}(G)$ and $\mathcal{S}C_c^\infty(G)$. The stable Fourier transform restricts to a continuous linear operator

$$\mathcal{F}^{\mathrm{st}} : \mathcal{S}C_c^\infty(G) \longrightarrow \widehat{\mathcal{S}C_c^\infty}(G)$$

A stable Paley–Wiener theorem for test functions asserts that this linear operator is an isomorphism of topological vector spaces and characterises its image $\widehat{\mathcal{S}C_c^\infty}(G)$. This has been established for quasisplit $G$ by Arthur when $F$ is non-archimedean [Art96] and by Moeglin and Waldspurger when $F$ is archimedean [MW16a, Ch. IV]. The space $\widehat{\mathcal{S}C_c^\infty}(G)$ is characterised as a space of "Paley–Wiener type" functions on $\Phi_{\mathrm{temp}}(G)$. Stable spectral density for test functions, i.e. the injectivity of this operator can be extended to the case of non-quasisplit groups using the results of [Var, §3.2].

Returning to the notation used in the preceding section, for $f \in \mathcal{C}(G)$ the functions $\{\mathcal{T}_\xi f\}$ are not specified directly in terms of $f$, but their stable Fourier transforms all coincide and are expressed directly in terms of the stable Fourier transform of $f$ since $(\mathcal{T}_\xi f)^H(\phi) = f^G(\xi_*\phi)$ for all $\phi \in \Phi_{\mathrm{temp}}(G)$.

## 1.5 Results

We continue with $F$ a local field of characteristic zero. In this thesis, we study stable harmonic analysis and its application to stable transfer. We work under under a hypothesis on the local Langlands correspondence in the case when $F$ is non-archimedean (Hypothesis 4.4.1), which is known in many cases. Our main result is the following.

**Theorem 1.5.1.** *Let $H$ and $G$ be connected reductive groups over $F$ with $G$ quasisplit, and let $\xi : {}^LH \to {}^LG$ be an injective tempered $L$-homomorphism. For each $f \in \mathcal{C}(G)$, a stable transfer $\mathcal{T}_\xi f \in \mathcal{C}(H)$ exists, and the stable transfer correspondence $f \mapsto \mathcal{T}_\xi f$ descends to a continuous linear operator*

$$\mathcal{T}_\xi : \mathcal{SC}(G) \longrightarrow \mathcal{SC}(H).$$

*It is uniquely characterised by $\mathcal{T}_\xi' \Theta_\phi = \Theta_{\xi_*\phi}$ for all $\phi \in \Phi_{\mathrm{temp}}(H)$. Moreover, the* *operator* $\mathcal{T}_\xi$ *restricts to a continuous linear operator*

$$\mathcal{T}_\xi : \mathcal{S}C_c^\infty(G) \longrightarrow \mathcal{S}C_c^\infty(H).$$

In particular, this answers Questions A and B of Langlands from [Lan13] in the affirmative. The above theorem is proved in Chapter 5 as part of Corollary 5.1.2. One of the main ingredients in the proof is a finiteness property of the map $\xi_* : \Phi(H) \to \Phi(G)$, which is also established in Chapter 5 (cf. Theorem 5.2.2) and may be of independent interest. The other main ingredients in the proof are stable Paley–Wiener theorems.

In Chapter 4, we give proofs of various stable Paley–Wiener theorems for (not-necessarily quasisplit) connected reductive groups $G$ groups under our hypothesis on the local Langlands correspondence when $F$ is non-archimedean. This includes a new stable Paley–Wiener theorem for Schwartz functions, characterising the image $\widehat{\mathcal{SC}}(G)$ of the stable Fourier transform as a natural space $\mathscr{S}^{\mathrm{st}}(G)$ of "Schwartz type" functions on $\Phi_{\mathrm{temp}}(G)$. In particular, we establish stable spectral density for Schwartz functions. The proof builds on Arthur's invariant (or trace) Paley–Wiener theorem for Schwartz functions, which concerns the invariant Fourier transform [Art94b]. As we discussed above, Arthur and Moeglin–Waldspurger established stable Paley–Wiener theorems for quasisplit groups. Moeglin–Waldspurger work in a more general twisted setting and Arthur establishes a more general result on collective endoscopic transfer. Our proof follows the strategy of Moeglin–Waldspurger. Their proof simplifies in the non-twisted setting since one can use stable spectral density for test functions.

Our theorem on stable transfer given above boils down to showing that pullback along $\xi_* : \Phi_{\mathrm{temp}}(H) \to \Phi_{\mathrm{temp}}(G)$ gives a well-defined continuous linear operator

$$\xi^* : \widehat{\mathcal{SC}}(G) \longrightarrow \widehat{\mathcal{SC}}(H)$$

which restricts to a continuous linear operator

$$\xi^* : \widehat{\mathcal{S}C_c^\infty}(G) \longrightarrow \widehat{\mathcal{S}C_c^\infty}(H)$$

(cf. Theorem 5.1.1). It is here that the finiteness property of $\xi_*$ mentioned above is used. Then, the operator $\mathcal{T}_\xi$ can be defined to make the following diagram commute

$$\begin{array}{ccc} \mathcal{SC}(G) & \xrightarrow{\mathcal{T}_\xi} & \mathcal{SC}(H) \\ \downarrow \mathcal{F}^{\mathrm{st}} & & \downarrow \mathcal{F}^{\mathrm{st}} \\ \widehat{\mathcal{SC}}(G) & \xrightarrow{\xi^*} & \widehat{\mathcal{SC}}(H) \end{array}$$

and it has the properties claimed. We note that this diagram is an analogue of the Fourier-slice theorem for the classical Radon transform. In this way, the stable transfer operator $\mathcal{T}_\xi$ can be thought of as an analogue in stable harmonic analysis of a Radon projection.

For applications of stable transfer operators to Beyond Endoscopy, more than just the existence and properties of stable transfer established in this thesis will be required. In Chapter 5, we establish formulas for the stable orbital integrals $\mathcal{T}_\xi f$ in terms of those of $f$ when $H$ and $G$ are either both tori or both complex groups. The formulas obtained are rather simple and reinforce the analogy with the classical Radon transform, but in general stable operators are much more complicated.

## 1.6 Guide to the reader

We have attempted as much as possible to give a leisurely exposition of background material with graduate student reader in mind. Chapter 2 contains various preliminaries that we use throughout. Chapter 3 is on invariant harmonic analysis, including Arthur's virtual tempered representations and invariant Paley–Wiener theorems. These invariant Paley–Wiener theorems are formulated in terms of abstract spaces of "Schwartz type" and "Paley–Wiener type" functions. In Chapter 3, we define these Schwartz and Paley–Wiener spaces and give basic properties of them. They are used in Chapters 4 and 5. In the first part of Chapter 4, we review the basic objects of stable harmonic analysis.

Someone more familiar with the invariant harmonic analysis and the Langlands program can skip much of the above mentioned background material. We suggest reading §3.3, where the Schwartz and Paley–Wiener spaces we use are defined; then moving to §4.4, where our hypothesis on the local Langlands correspondence for $p$-adic groups is formulated; and then proceeding from there, consulting earlier sections as necessary.

# 2 Preliminaries

## 2.1 The group

Unless otherwise stated, $F$ will denote a local field of characteristic zero. We fix an algebraic closure $\overline{F}$ of $F$. If $F$ is non-archimedean, we denote its ring of integers by $\mathcal{O}_F$, the unique maximal ideal of $\mathcal{O}_F$ by $\mathfrak{p}_F$, the characteristic of the residue field $\mathcal{O}_F/\mathfrak{p}_F$ by $p$, and the cardinality of the residue field by $q_F = |\mathcal{O}_F/\mathfrak{p}_F|$. We say that $F$ is a $p$-adic field when $F$ is non-archimedean and has residual characteristic $p$.

Let $|\cdot|_F$ denote the canonical absolute value of $F$, which is defined for all $a \in F$ by $d(ax) = |a|_F\, dx$, where $dx$ is any Haar measure of $F$. We also denote the canonical absolute value on $\overline{F}$ by $|\cdot|_F$.

Let $G$ be a connected reductive group over $F$. We will say that $G$ is a real group if $F$ is archimedean and a $p$-adic group if $F$ is a $p$-adic field. We will use $X^*(G)$ (resp. $X_*(G)$) to denote the group of algebraic characters $G \to \mathbb{G}_m$ (resp. cocharacters $\mathbb{G}_m \to G$) over $F$. Thus, the group of algebraic characters (resp. cocharacters) of the base change $G_{\overline{F}}$ will be denoted by $X^*(G_{\overline{F}})$ (resp. $X_*(G_{\overline{F}})$).

We have the real vector spaces

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R})$$

and

$$\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that we have a natural perfect pairing between $\mathfrak{a}_G$ and $\mathfrak{a}_G^*$, realising $\mathfrak{a}_G^*$ as the dual space of $\mathfrak{a}_G$. Let $A_G$ be the split component of the centre of $G$. The restriction gives an injective homomorphism $X^*(G) \to X^*(A_G)$ with finite cokernel. By functoriality, we obtain isomorphisms $\mathfrak{a}_{A_G} \xrightarrow{\sim} \mathfrak{a}_G$ and $\mathfrak{a}_G^* \xrightarrow{\sim} \mathfrak{a}_{A_G}^*$. In this way we identify $\mathfrak{a}_{A_G} = \mathfrak{a}_G$ and $\mathfrak{a}_{A_G}^* = \mathfrak{a}_G^*$.

We have the Harish-Chandra logarithm homomorphism $H_G : G(F) \to \mathfrak{a}_G$ defined by $\langle H_G(x), \chi\rangle = \log|\chi(x)|_F$ for all $x \in G(F)$ and $\chi \in X^*(G)$. Let

$$G(F)^1 = \ker H_G = \bigcap_{\chi \in X^*(G)} \ker|\chi|_F.$$

Note that $H_{A_G}$ is the restriction of $H_G$. Therefore $A_G(F)^1 = A_G(F) \cap G(F)^1$. Let

$$\mathfrak{a}_{G,F} = H_G(G(F)) = G(F)/G(F)^1$$

and

$$\widetilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{A_G,F} = H_G(A_G(F)) = A_G(F)/A_G(F)^1.$$

We define

$$\mathfrak{a}^\vee_{G,F} = \mathrm{Hom}_\mathbb{Z}(\mathfrak{a}_{G,F}, 2\pi i\mathbb{Z})$$

and similarly

$$\widetilde{\mathfrak{a}}^\vee_{G,F} = \mathrm{Hom}_\mathbb{Z}(\widetilde{\mathfrak{a}}_{G,F}, 2\pi i\mathbb{Z}).$$

We have the inclusions

$$\widetilde{\mathfrak{a}}_{G,F} \subseteq \mathfrak{a}_{G,F} \subseteq \mathfrak{a}_G.$$

If $F$ is archimedean, then $\widetilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{G,F} = \mathfrak{a}_G$ and $\widetilde{\mathfrak{a}}^\vee_{G,F} = \mathfrak{a}^\vee_{G,F} = 0$. If $F$ is non-archimedean, then $\mathfrak{a}_{G,F}$ and $\widetilde{\mathfrak{a}}_{G,F}$ are (full) lattices in $\mathfrak{a}_G$, and $\mathfrak{a}^\vee_{G,F} \subseteq \widetilde{\mathfrak{a}}^\vee_{G,F}$ are lattices in $i\mathfrak{a}^*_G$.

A (continuous) character $\chi : G(F) \to \mathbb{C}^\times$ of $G(F)$ is said to be unramified if $\chi(G(F)^1) = 1$. Thus, the unramified characters of $G(F)$ are the characters of $G(F)/G(F)^1 = \mathfrak{a}_{G,F}$. We denote the group of unramified characters of $G(F)$ by $X^{\mathrm{nr}}(G) = \mathrm{Hom}(G(F)/G(F)^1, \mathbb{C}^\times)$ and the subgroup of unitary unramified characters by $X^{\mathrm{nr}}(G)^1 = \mathrm{Hom}(G(F)/G(F)^1, \mathbb{C}^1)$.

We have a surjective homomorphism

$$\mathfrak{a}^*_{G,\mathbb{C}} \longrightarrow X^{\mathrm{nr}}(G)$$

defined by $\lambda \mapsto |\cdot|^\lambda_G$, where $|g|^\lambda_G = e^{\langle\lambda, H_G(g)\rangle}$. If $\lambda = \theta \otimes s \in \mathfrak{a}^*_{G,\mathbb{C}} = X^*(G) \otimes_\mathbb{Z} \mathbb{C}$, then $|g|^\lambda_G = |\theta(g)|^s$. The surjection $\mathfrak{a}^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G)$ descends to an isomorphism $\mathfrak{a}^*_{G,\mathbb{C}}/\mathfrak{a}^\vee_{G,F} \to X^{\mathrm{nr}}(G)$. Thus, if $F$ is archimedean, then the map $\mathfrak{a}^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G)$ is an isomorphism and $X^{\mathrm{nr}}(G)$ is a complex vector space. If $F$ is non-archimedean, then $X^{\mathrm{nr}}$ is a complex torus and the homomorphism $\mathfrak{a}^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G)$ factors through the complex torus $\mathfrak{a}^*_{G,\mathbb{C}}/\frac{2\pi i}{\log q_F}X^*(G)$ with finite kernel $\mathfrak{a}^\vee_{G,F}/\frac{2\pi i}{\log q_F}X^*(G)$.

### 2.1.1 Parabolic and Levi subgroups

We call a Levi factor of a parabolic subgroup of $G$ a Levi subgroup of $G$. Let $M$ be a Levi subgroup of $G$. We recall the following standard notation.

- $\mathcal{P}(M) = \mathcal{P}^G(M)$ is the set of parabolic subgroups of $G$ with $M$ as a Levi factor;
- $\mathcal{L}(M) = \mathcal{L}^G(M)$ is the set of Levi subgroups of $G$ containing $M$; and
- $\mathcal{F}(M) = \mathcal{F}^G(M)$ is the set of parabolic subgroups of $G$ containing $M$.

If $P$ is a parabolic subgroup of $G$, we denote the unipotent radical of $P$ by $N_P$. We refer to a pair $(P, M)$ consisting of a Levi subgroup $M$ of $G$ and a parabolic subgroup $P \in \mathcal{P}(M)$ as a parabolic pair of $G$. Any two Levi factors of a parabolic subgroup $P$ are $P(F)$-conjugate. If $M$ is a Levi factor of $P$, then $N_{N_{P(F)}}(M) = 1$ and $N_{P(F)}(M) = M(F)$. Thus, any two Levi factors of a parabolic subgroup are conjugate by a unique element of $N_P(F)$.

Two parabolic subgroups are said to be associate if they have conjugate Levi factors, and two parabolic pairs are associate if their Levi subgroups are conjugate. There is a natural bijection between the set of $G(F)$-conjugacy classes of Levi subgroups of $G$ and the set of associate classes of parabolic subgroups of $G$.

Fix a minimal parabolic pair $(P_0, M_0)$ of $G$. We will often use a subscript or superscript "0" instead of "$M_0$" to indicate dependence on $M_0$ or $P_0$. For example, we write $N_0 = N_{P_0}$ and $A_0 = A_{M_0}$. With respect to $M_0$, a parabolic subgroup $P$ of $G$ is called standard (resp. semistandard) if it contains $P_0$ (resp. $M_0$). A Levi subgroup of $G$ is said to be semistandard if it contains $M_0$.

If $P$ is a semistandard parabolic, then it has a unique semistandard Levi factor $M_P$. In this way, every semistandard parabolic $P$ determines a parabolic pair $(P, M_P)$. A Levi subgroup of $G$ is called standard if it is of the form $M_P$ for a standard parabolic $P$. Note that $\mathcal{F}^G(M_0)$ (resp. $\mathcal{L}^G(M_0)$) is the set of semistandard parabolic (resp. Levi) subgroups of $G$.

A parabolic pair $(P, M)$ is said to be semistandard if $M$ (and thus $P$) is semistandard, and is said to be standard if furthermore $P$ is standard (and thus $M = M_P$ is standard). A standard parabolic pair $(P, M)$ is uniquely determined by either $P$ or $M$. Thus, standard parabolic subgroups are in bijection with standard parabolic pairs. Every Levi subgroup is conjugate to a standard Levi subgroup. Every parabolic subgroup (resp. parabolic pair) is conjugate to a unique standard parabolic subgroup (resp. standard parabolic pair).

Recall that for every split torus $S$ in $G$, the group $C_G(S)$ is a Levi subgroup of $G$. Let $M$ be a Levi subgroup of $G$. Then $M = C_G(A_M)$. Note that if $M_1$ and $M_2$ are Levi subgroups of $G$, then $M_1 \subseteq M_2$ if and only if $A_{M_2} \subseteq A_{M_1}$. Consequently, we have that $A_0$ is a maximal split torus of $G$ and conversely the centraliser of a maximal split torus of $G$ is a minimal Levi subgroup of $G$.

Let $M \subseteq G$ be a Levi subgroup of $G$. We have $A_G \subseteq A_M$. The restriction homomorphism $X^*(G) \to X^*(M)$ is injective, so gives rise to a linear injection

$$\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathfrak{a}_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$$

and a dual linear surjection

$$\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R}) \longrightarrow \mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(M), \mathbb{R}).$$

The restriction homomorphism $X^*(A_M) \to X^*(A_G)$ is surjective, so gives rise to a linear surjection

$$\mathfrak{a}_M^* = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathfrak{a}_G^* = X^*(A_G) \otimes_{\mathbb{Z}} \mathbb{R}$$

and a dual linear injection

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(A_G), \mathbb{R}) \longrightarrow \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(X^*(A_M), \mathbb{R}).$$

Let $\mathfrak{a}_M^G = \ker(\mathfrak{a}_M \to \mathfrak{a}_G)$. The homomorphism $\mathfrak{a}_G \to \mathfrak{a}_M$ is a section of $\mathfrak{a}_M \to \mathfrak{a}_G$. Thus, we have a split short exact sequence

$$0 \longrightarrow \mathfrak{a}_M^G \longrightarrow \mathfrak{a}_M \underset{\curvearrowleft}{\longrightarrow} \mathfrak{a}_G \longrightarrow 0$$

and $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$. We also have the dual exact sequence

$$0 \longrightarrow \mathfrak{a}_G^* \underset{\curvearrowleft}{\longrightarrow} \mathfrak{a}_M^* \longrightarrow (\mathfrak{a}_M^G)^* \longrightarrow 0$$

and $\mathfrak{a}_M^* = (\mathfrak{a}_M^G)^* \oplus \mathfrak{a}_G^*$.

Let $(P, M)$ be a parabolic pair. We denote the set of simple roots of $(P, A_M)$ (or equivalently $(N_P, A_M)$) by $\Delta(P, M)$, which is a basis of $(\mathfrak{a}_M^G)^*$. There is an associated set of simple coroots $\Delta(P, M)^\vee$, which is a basis of $\mathfrak{a}_M^G$, and there is a bijection $\Delta(P, M) \to \Delta(P, M)^\vee, \alpha \mapsto \alpha^\vee$. For $(P, M)$ semistandard, we write $\Delta_P = \Delta(P, M)$ and $\Delta_P^\vee = \Delta(P, M)$. (Cf. [MW18, §1.3].)

### 2.1.2 Maximal compact subgroups

A maximal compact subgroup $K$ of $G(F)$ is said to be in good position (or admissible) relative to a Levi subgroup $M$ of $G$ if the following holds.

- When $F$ is archimedean, the Lie algebra of $K$ and $A_M(F)$ are orthogonal with respect to the Killing form of $G$.
- When $F$ is non-archimedean, $K$ is the stabiliser of a special vertex in the apartment attached to a maximal split torus of $M$.

We recall from [Art81, §1] that if $K$ is in good position relative to $M$, then

1. $G(F) = P(F)K$ for any $P \in \mathcal{P}(M)$;
2. any coset in $G(F)/M(F)$ which normalises $M$ has a representative in $K$;
3. $P(F) \cap K = (M_P(F) \cap K)(N_P(F) \cap K)$ for any $P \in \mathcal{F}(M)$; and
4. if $L \in \mathcal{L}(M)$, then $K_L \coloneqq K \cap L(F)$ is a maximal compact subgroup of $L(F)$ that is in good position relative the Levi subgroup $M$ of $L$.

If $K$ is in good position relative to $M$, then $K$ is in good position relative to every Levi subgroup containing $M$.

Fix a maximal compact subgroup $K$ of $G(F)$ that is in good position relative to our fixed minimal Levi subgroup $M_0$ of $G$. In particular, the Iwasawa decomposition $G(F) = P_0(F)K$ holds.

Let $P \in \mathcal{F}(M_0)$. We can extend $H_{M_P} : M_P(F) \to \mathfrak{a}_{M_P}$ to a homomorphism $H_P : P(F) \to \mathfrak{a}_{M_P}$ by composing with $P(F) \to M_P(F)$. We have the decompositions $G(F) = P(F)K = M_P(F)N_P(F)K$. For each $x \in G(F)$ we choose elements $p_P(x) \in P(F)$ and $k_P(x) \in K$ such that $x = p_P(x)k_P(x)$, and let $m_P(x) \in M_P(F)$ and $n_P(x) \in N_P(F)$ be the unique elements such that $p_P(x) = m_P(x)n_P(x)$. We extend $H_P$ to a function $H_P : G(F) \to \mathfrak{a}_{M_P}$ by $H_P(x) = H_P(p_P(x)) = H_{M_P}(m_P(x))$.

Let $\Delta_0 = \Delta_{P_0}$, the set of simple roots of $A_0$ associated with its action on $P_0$ (or equivalently $N_0$). Let $\mathfrak{a}_0^{\geq 0}$ be the set of $H \in \mathfrak{a}_0$ such that $\langle \alpha, H \rangle \geq 0$ for all $\alpha \in \Delta_0$, and define $M_0(F)^{\geq 0}$ to be the set of $m \in M_0(F)$ such that $H_0(m) = \mathfrak{a}_0$. Then $G(F) = KM_0(F)^{\geq 0}K$. (See [MW18, §1.1])

### 2.1.3 Weyl Groups

For $T$ a torus of $G$, we define the following Weyl groups:

- the absolute Weyl group $W(G,T) \coloneqq N_G(T)/C_G(T)$;
- the relative Weyl group $W_F(G,T) \coloneqq N_{G(F)}(T)/C_{G(F)}(T)$; and
- the stable Weyl group $W(G,T)(F) \coloneqq (N_G(T)/C_G(T))(F)$.

Note that $N_{G(F)}(T) = N_G(T)(F)$ and $C_{G(F)}(T) = C_G(T)(F)$. We have

$$W_F(G,T) \subseteq W(G,T)(F) \subseteq W(G,T).$$

By Galois descent, we can express the stable Weyl group in the following ways:

$$\begin{aligned} W(G,T)(F) &= \{g \in N_G(T)(\overline{F}) : g\sigma(g)^{-1} \in C_G(T)(\overline{F}),\ \forall \sigma \in \Gamma_F\}/C_G(T)(\overline{F}) \\ &= \{w \in W(G,T)(\overline{F}) : \mathrm{Int}(w^{-1}) : C_G(T)_{\overline{F}} \to C_G(T)_{\overline{F}} \text{ is defined over } F\} \end{aligned}$$

If $T$ is a maximal torus of $G$, then $C_G(T) = T$ and one has $W_F(G,T) = W(G(F), T(F)) \coloneqq N_{G(F)}(T(F))/C_{G(F)}(T(F))$. For a Levi subgroup $M$ of $G$, we write $W^G(M)$ for the relative Weyl group $W_F(G, A_M)$. That is,

$$W^G(M) = N_{G(F)}(A_M)/C_{G(F)}(A_M) = N_{G(F)}(M)/M(F).$$

The Weyl group $W^G(M_0)$ is the relative Weyl group of $G$, and we abbreviate it by $W_0^G$.

Let $w \in W^G(M)$ and let $\widetilde{w} \in N_{G(F)}(M)$ be a representative of $w$. Since $\widetilde{w}^{-1} \cdot A_0 = \widetilde{w}^{-1} A_0 \widetilde{w}$ is a maximal split torus of $M$, there exists $m \in M_0(F)$ such that $m \cdot A_0 = \widetilde{w}^{-1} \cdot A_0$. Then $\widetilde{w}m \cdot A_0 = A_0$. Consequently, $\widetilde{w}m \in N_{G(F)}(A_0)$. Therefore $w$ is represented by an element of $N_{G(F)}(A_0)$. It follows that

$$W^G(M) = \{\widetilde{w} \in N_{G(F)}(A_0) : \widetilde{w} \cdot M = M\}/M(F)$$

and we have a canonical isomorphism

$$W^G(M) = \{w \in W_0^G : w \cdot M = M\}/W_0^M.$$

Fix a $W_0^G$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}_0$, with associated norm denoted by $\|\cdot\|$. This restricts to a $W^G(M)$-invariant inner product on $\mathfrak{a}_M$ for each semistandard Levi subgroup $M$ of $G$, and is transported by conjugation to all non-semistandard Levi subgroups of $G$. We obtain $W^G(M)$-invariant inner product on the complexifications $\mathfrak{a}_{M,\mathbb{C}}$. We may use these inner products to identify $\mathfrak{a}_M = \mathfrak{a}_M^*$ and $\mathfrak{a}_{M,\mathbb{C}} = \mathfrak{a}_{M,\mathbb{C}}^*$. The decomposition $\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G$ is orthogonal with respect to the inner product on $\mathfrak{a}_M$.

We will fix normalisations of Haar measures as in [Art94a]. However, the precise normalisation has little import on our results.

### 2.1.4 The universal enveloping algebra

Let $\mathfrak{g}$ be a complex reductive Lie algebra. We denote the centre of $\mathfrak{U}(\mathfrak{g})$ by $\mathfrak{Z}(\mathfrak{g})$. We recall that

$$\mathfrak{Z}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})^{\mathrm{ad}(\mathfrak{g})} = \mathfrak{U}(\mathfrak{g})^{e^{\mathrm{ad}(\mathfrak{g})}} = \mathfrak{U}(\mathfrak{g})^{\mathrm{Int}(\mathfrak{g})},$$

where $\mathrm{Int}(\mathfrak{g})$ is the group of inner automorphisms of $\mathfrak{g}$, that is, the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$.

For each Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ there is a canonical Harish-Chandra homomorphism $\xi^{\mathfrak{g}}_{\mathfrak{m}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{m})$. (See Definition 6.3 in [Vog87].) The construction of $\xi^{\mathfrak{g}}_{\mathfrak{m}}$ uses a choice of a parabolic subalgebra $\mathfrak{p}$ with Levi factor $\mathfrak{m}$, but $\xi^{\mathfrak{g}}_{\mathfrak{m}}$ does not depend on this choice. The Harish-Chandra homomorphism is injective and has image $\mathfrak{Z}(\mathfrak{m})^{W(\mathfrak{g},\mathfrak{m})}$. The isomorphism $\xi^{\mathfrak{g}}_{\mathfrak{m}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{m})^{W(\mathfrak{g},\mathfrak{m})}$ is often called a Harish-Chandra isomorphism. The Harish-Chandra homomorphisms are functorial in the sense that if $\mathfrak{l} \subseteq \mathfrak{m}$ are two Levi subalgebras of $\mathfrak{g}$, then $\xi^{\mathfrak{g}}_{\mathfrak{l}} = \xi^{\mathfrak{m}}_{\mathfrak{l}} \circ \xi^{\mathfrak{g}}_{\mathfrak{m}}$. When $\mathfrak{m} = \mathfrak{t}$ is a maximal toral subalgebra, then $\mathfrak{Z}(\mathfrak{t}) = \mathfrak{U}(\mathfrak{t}) = \mathrm{Sym}(\mathfrak{t})$ and the Harish-Chandra isomorphism $\xi^{\mathfrak{g}}_{\mathfrak{t}} : \mathfrak{Z}(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{g},\mathfrak{t})}$ is the one found most often in the literature. If $\mathfrak{t}$ is contained in $\mathfrak{m}$, then it follows from functoriality of the Harish-Chandra homomorphisms that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{Z}(\mathfrak{g}) & \xrightarrow{\xi^{\mathfrak{g}}_{\mathfrak{m}}} & \mathfrak{Z}(\mathfrak{m}) \\ {\scriptstyle \xi^{\mathfrak{g}}_{\mathfrak{t}}}\downarrow & & \downarrow{\scriptstyle \xi^{\mathfrak{m}}_{\mathfrak{t}}} \\ \mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{g},\mathfrak{t})} & \hookrightarrow & \mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{m},\mathfrak{t})} \end{array}$$

That is, if we identify $\mathfrak{Z}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{g},\mathfrak{t})}$ and $\mathfrak{Z}(\mathfrak{m}) = \mathrm{Sym}^{W(\mathfrak{m},\mathfrak{t})}$ using the Harish-Chandra isomorphisms, then the Harish-Chandra homomorphism $\xi^{\mathfrak{g}}_{\mathfrak{m}} : \mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(\mathfrak{m})$ is simply the inclusion $\mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{g},\mathfrak{t})} \hookrightarrow \mathrm{Sym}(\mathfrak{t})^{W(\mathfrak{m},\mathfrak{t})}$.

If $G$ is a real Lie group, then

$$\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}(G^\circ)} = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}(\exp(\mathfrak{g}))} = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{e^{\mathrm{ad}(\mathfrak{g})}} = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{ad}(\mathfrak{g})} = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{ad}(\mathfrak{g}_{\mathbb{C}})} = \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}).$$

Consequently, we have $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}(G)} \subseteq \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$. If $\mathrm{Ad}(G) \subseteq \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$, then

$$\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})} \subseteq \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}(G)}$$

and $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})^{\mathrm{Ad}(G)}$. If $G$ belongs to the Harish-Chandra class, in particular, if $G$ is the group of $\mathbb{R}$-points of a connected reductive group over $\mathbb{R}$, then $\mathrm{Ad}(G) \subseteq \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$.

We will need a "norm" on the set $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C})$ of infinitesimal characters of $G$. Choose a maximal torus $T$ of $G_{\mathbb{C}}$. By the Harish-Chandra isomorphism, we may identify $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C} = \mathfrak{t}^*/W(G_{\mathbb{C}}, T)$. We fix a $W(G_{\mathbb{C}}, T)$-invariant inner product on $\mathfrak{t}^*$. Then, for $\mu \in \mathfrak{t}^*/W(G_{\mathbb{C}}, T)$ the resulting norm $\|\mu\| = \|\mu\|_G$ is well-defined. Different choices of inner products or maximal tori result in a function $\|\cdot\|'$ on the set of infinitesimal characters of $G$ with $\|\cdot\|' \asymp \|\cdot\|$. Since we will only use $\|\cdot\|$ in estimates, definitions made using $\|\cdot\|$ will not depend on the choices made to define it. Note that if $M$ is a Levi subgroup of $G$, then $\|\cdot\|_M \asymp \|\cdot\|_G$.

### 2.1.5 Group norms

We recall the notion of norms on affine varieties over local fields from [Kot05, §18], which are used to capture polynomial growth and decay. Let $X$ be a set. An abstract norm on $X$ is a function $\|\cdot\| : X \to \mathbb{R}_{\geq 1}$. Let $\|\cdot\|_1, \|\cdot\|_2$ be abstract norms on $X$. We write $\|\cdot\|_1 \preceq \|\cdot\|_2$ if there exist $M > 0$ such that $\|\cdot\|_1 \ll \|\cdot\|_2^M$. The abstract norms $\|\cdot\|_1, \|\cdot\|_2$ are said to be equivalent and we write $\|\cdot\|_1 \approx \|\cdot\|_2$ if $\|\cdot\|_1 \preceq \|\cdot\|_2$ and $\|\cdot\|_2 \preceq \|\cdot\|_1$.

Let $X$ be an affine scheme of finite type over $F$. There is a canonical equivalence class of abstract norms on $X(F)$, defined as follows. For any set of generators $f_1, \ldots, f_m$ of the $F$-algebra $\mathcal{O}_X(X)$, of regular functions on $X$, we have an abstract norm $\|\cdot\|$ on $X(F)$ defined by

$$\|x\| \coloneqq \sup\{1, |f_1(x)|_F, \ldots, |f_m(x)|_F\}.$$

The equivalence class of $\|\cdot\|$ does not depend on the choice of $f_1, \ldots, f_m$. We call any abstract norm in the equivalence class of $\|\cdot\|$ a norm on $X(F)$. For the definition of norms on $X(F)$ when $X$ is a non-affine scheme of finite type over $F$, see [Kot05, §18.5].

Let $G$ be a linear algebraic group over $F$. By [Kot05, Proposition 18.1 (7)], if $\Omega$ is a bounded (relatively compact) subspace of $G(F)$, for every norm $\|\cdot\|$ on $G(F)$ there exist $C, M > 0$ such that $\|\omega_1 g \omega_2\| \leq C\|g\|^M$ for all $\omega_1, \omega_2 \in \Omega$ and $g \in G(F)$.

We may construct a norm on $G(F)$ as follows. Let $\iota : G \to \mathrm{GL}(V)$ be a faithful algebraic representation of $G$ on a finite-dimensional vector space $V$ over $F$. Choose a vector space norm $\|\cdot\|$ on $V$ that is compatible with the canonical absolute value $|\cdot|_F$ on $F$. For example, if $e_1, \ldots, e_n$ is a basis of $V$ and $v = \sum_{i=1}^n a_i e_i$, we can take $\|v\| = \max_{i=1,\ldots,n}\{|a_i|_F\}$. We equip $\mathrm{End}(V)$ with the corresponding operator norm, which we also denote by $\|\cdot\|$. Define $\|g\| = \max(\|\iota(g)\|, \|\iota(g^{-1})\|)$ for all $g \in G(F)$. Then $\|\cdot\| : G \to \mathbb{R}_{\geq 1}$ is a norm on $G(F)$ that satisfies the following properties:

1. $\|\cdot\|$ is continuous;
2. $\|g_1 g_2\| \leq \|g_1\|\|g_2\|$ for all $g_1, g_2 \in G(F)$;
3. $\|g^{-1}\| = \|g\| \geq 1$ for all $g \in G(F)$;
4. for all $R \geq 0$, the subspace $B_R = \{g \in G(F) : \|g\| \leq R\}$ is compact.

Since $\|\cdot\|$ is a norm on $G(F)$, changing $\iota$ or the choice of vector space norm on $V$ results in an equivalent norm on $G(F)$. We call the norms $\|\cdot\|$ on $G(F)$ that are obtained from the above construction group norms.

If $\Omega_1, \Omega_2$ are bounded subsets of $G(F)$, then $\|\omega_1 x \omega_2\| \asymp \|x\|$ for $x \in G(F)$ and $\omega_i \in \Omega_i$. This follows from

$$\|x\| = \|y_1^{-1} y_1 x y_2 y_2^{-1}\| \leq \|y_1^{-1}\|\|y_1 x y_2\|\|y_2^{-1}\|$$

and $\|y_1 x y_2\| \leq \|y_1\|\|x\|\|y_2\|$ for all $x, y_1, y_2 \in G(F)$.

Now suppose that $G$ is our connected reductive group. We fix a group norm $\|\cdot\| = \|\cdot\|_G$ on $G(F)$ and define the non-negative function $\sigma = \sigma_G \coloneqq \log\|\cdot\|$. For a closed central subgroup $\mathcal{Z}$ of $G(F)$, we define $\sigma^{\mathcal{Z}} = \sigma_G^{\mathcal{Z}} : G(F) \to \mathbb{R}_{\geq 0}$ by $\sigma^{\mathcal{Z}}(g) = \inf_{z \in \mathcal{Z}} \sigma(zg)$. We will use this to define growth properties modulo $\mathcal{Z}$. If $\mathcal{Z}$ is of the form $\mathcal{Z} = Z(F)$ for a subgroup $Z \subseteq Z_G$, we will write $\sigma^Z = \sigma_G^Z$ instead of $\sigma^{\mathcal{Z}} = \sigma_G^{\mathcal{Z}}$ to simplify the notation.

Recall that we have fixed a $W_0^G$-invariant inner product on $\mathfrak{a}_0$. There exist constants $C_1, C_2 > 0$ such that

$$C_1(1 + \|H_0(m_0)\|) \leq 1 + \sigma(m_0) \leq C_2(1 + \|H_0(m_0)\|)$$

for all $m_0 \in M_0(F)$. That is, $1 + \sigma(m_0) \asymp 1 + \|H_0(m_0)\|$ for $m_0 \in M_0(F)$.

## 2.2 Functions and distributions

Let $M$ be a (second countable) smooth manifold. We assume the reader is familiar with the Fréchet space $C^\infty(M)$, and the LF-space $C_c^\infty(M)$.

For each locally compact (Hausdorff) group $G$, Bruhat defined locally convex space $C_c^\infty(G)$ and $C^\infty(G)$ in a way that extends the definition for Lie groups. These constructions behave best when $G$ is second countable. We recall that for a locally compact Hausdorff space $X$, the following are equivalent: (i) $X$ is second countable; (ii) $X$ is metrisable and countable at infinity; (iii) $X$ is Polish (separable and completely metrisable) and countable at infinity.

We define a td space to be a second countable totally disconnected locally compact Hausdorff space. We define a td group to be a topological group that is a td space. Besides the case of Lie groups, we will only need Bruhat's definition of $C_c^\infty(G)$ and $C^\infty(G)$ for td groups. More generally, we can define $C^\infty(X)$ and $C_c^\infty(X)$ for a td space $X$. First, we recall that for any set $S$ a function $f : X \to S$ is said to be smooth if it is locally constant. If $X$ is a td group, we say that $f : G \to S$ is uniformly smooth if there exists an open neighbourhood $U$ of the identity such that $f$ is constant on $UgU$ for all $g \in G$; one may take $U$ to be a compact open subgroup. The space $C^\infty(X)$ is the space of locally constant $\mathbb{C}$-valued functions on $X$ and $C_c^\infty(X)$ is the subspace of compactly supported smooth functions. The space $C_c^\infty(X)$ has a countable basis. The natural topology on $C_c^\infty(X)$ is the finest locally convex topology.

We recall that the finest locally convex topology on a complex vector space $V$ is the topology defined by all seminorms on $V$, and it is a Hausdorff topology. The finest locally convex topology on $V$ can also be characterised as the topology that makes every linear map from $V$ to a locally convex space continuous. Colimits of finite-dimensional locally convex spaces in the category of locally convex spaces have the finest locally convex topology. If $V$ has the finest locally convex topology, then it is the colimit in the category of locally convex spaces of its finite-dimensional subspaces. If moreover $V$ has at most countable dimension, or equivalently is an increasing union of a sequence of finite-dimensional spaces, then then $V$ is a nuclear strict LF-space.

Since $C_c^\infty(X)$ has the finest locally convex topology, we have $C_c^\infty(X)' = C_c^\infty(X)^*$ as vector spaces (i.e. continuous linear functionals are the same as linear functionals on $C_c^\infty(X)$). For this reason, many authors state that $C_c^\infty(X)$ is not given a topology. However, the natural topology is important for some purposes. Indeed, since $C_c^\infty(X)$ is reflexive, the (standard) strong dual topology on $C_c^\infty(X)'$ of uniform convergence on bounded subsets is not the finest locally convex topology unless $C_c^\infty(X)$ is finite-dimensional.

We have $C^\infty(X) = \lim_C C_c^\infty(C)$ in the category of vector spaces, where $C$ runs over the compact subspaces of $X$ and the structure maps are given by restriction. We give $C^\infty(X)$ the locally convex limit topology so that the above limit is in the category of locally convex spaces. We can take $C$ to run over the compact open subspaces of $X$ in the above limits. Note that $C^\infty(X) = C_c^\infty(X)$ if $X$ is compact.

### 2.2.1 Equivariant test functions and distributions

For this and the next subsection, we let $G$ be a Lie group or a td group. Let $\mathcal{Z}$ denote a closed central subgroup of $G$ equipped with a choice of Haar measure and let $\zeta : \mathcal{Z} \to \mathbb{C}^\times$ be a character of $\mathcal{Z}$. The pair $(\mathcal{Z}, \zeta)$ is called a central character datum or simply a central datum of $G$. We say that a central datum $(\mathcal{Z}, \zeta)$ is unitary if $\zeta$ is unitary. The trivial central datum is the one with $\mathcal{Z} = 1$. Later, we will only use unitary central data and only consider the cases $\mathcal{Z} = 1$ or $\mathcal{Z}$ is the split component of the centre of the $F$-points of a connected reductive group over $F$.

We define $C_c^\infty(G, \zeta)$ to be the space of all functions $f : G \to \mathbb{C}$ that are $\zeta^{-1}$-equivariant (i.e. $f(zg) = \zeta(z)^{-1} f(g)$ for all $z \in \mathcal{Z}$ and $g \in G$) and whose support is compact modulo $\mathcal{Z}$. The space $C_c^\infty(G, \zeta)$ has a natural topology.

Suppose that $G$ is a Lie group. Let $B$ be a closed subspace of $G$ that is $\mathcal{Z}$-stable and compact modulo $\mathcal{Z}$. Define $C_B^\infty(G, \zeta)$ to be the subspace of functions in $C_c^\infty(G, \zeta)$ whose support is contained in $B$. We give $C_B^\infty(G, \zeta)$ the Fréchet space topology defined by the family of seminorms

$$\|f\|_D \coloneqq \sup_{g \in G} |Df(g)|$$

for $D$ a $\mathcal{Z}$-invariant differential operator on $D$. We have $C_c^\infty(G, \zeta) = \bigcup_B C_B^\infty(G, \zeta)$ and give $C_c^\infty(G, \zeta)$ the inductive limit topology in the category of locally convex spaces, making it a strict LF-space.

If $G$ is a td group, we give $C_c^\infty(G, \zeta)$ the finest locally convex topology. We explain why this is natural in terms of the structure of $C_c^\infty(G, \zeta)$. Since $\mathcal{Z}$ is a td group, the continuous character $\zeta : \mathcal{Z} \to \mathbb{C}^\times$ is automatically smooth. For each compact open subgroup $K_0$ of $G$, let $C_c^\infty(K_0 \backslash G / K_0, \zeta)$ be the subspace of left and right $K_0$-invariant functions in $C_c^\infty(G, \zeta)$. Note that $C_c^\infty(K_0 \backslash G / K_0, \zeta) = 0$ unless $\zeta(\mathcal{Z} \cap K_0) = 1$. Choose a compact open subgroup $K_0$ of $G$ that is sufficiently small so that $\zeta(\mathcal{Z} \cap K_0) = 1$. Let $K_1 \supseteq K_2 \supseteq \cdots$ be a deceasing sequence of compact open subgroups of $K_0$ with $\bigcap_{i=1}^\infty K_i = 1$. Then $C_c^\infty(G, \zeta)$ is the increasing union $C_c^\infty(G, \zeta) =$

$\bigcup_{i=1}^{\infty} C_c^\infty(K_i\backslash G/K_i,\zeta)$. For each $i$, let $B_{i,1} \subseteq B_{i,2} \subseteq \cdots$ be an increasing sequence of subspaces of $G$ that are left and right $K_i$-invariant, $\mathcal{Z}$-invariant, compact modulo $\mathcal{Z}$, and such that $G = \bigcup_{j=1}^{\infty} B_{i,j}$. Define $C_{B_{i,j}}^\infty(K_i\backslash G/K_i,\zeta)$ to be the finite-dimensional subspace of functions in $C_c^\infty(K_i\backslash G/K_i,\zeta)$ whose support is contained in $B_{i,j}$. Then $C_c^\infty(K_i\backslash G/K_i,\zeta)$ is the increasing union $C_c^\infty(K_i\backslash G/K_i,\zeta) = \bigcup_{j=1}^{\infty} C_{B_{i,j}}^\infty(K_i\backslash G/K_i,\zeta)$. Since $C_{B_{i,j}}^\infty(K_i\backslash G/K_i,\zeta)$ is finite-dimensional, it has a unique locally convex topology. We give $C_c^\infty(K_i\backslash G/K_i,\zeta)$ and $C_c^\infty(G,\zeta)$ the locally convex topologies that make the increasing unions above colimits in the category of locally convex spaces. Thus, the at most countable dimensional spaces $C_c^\infty(K_i\backslash G/K_i,\zeta)$ and $C_c^\infty(G,\zeta)$ are both given finest locally convex topologies.

We define the space of $\zeta$-equivariant distributions on $G(F)$ to be $C_c^\infty(G,\zeta)'$, the continuous dual with the (standard) strong dual topology. The transpose of the continuous surjection $C_c^\infty(G) \to C_c^\infty(G,\zeta)$ is a continuous injection $C_c^\infty(G,\zeta)' \to C_c^\infty(G)'$, which allows us to identify $\zeta$-equivariant distributions with distributions on $G(F)$ that are $\zeta$-equivariant with respect to a natural action of $\mathcal{Z}$.

For $f \in C_c(G)$, define $f^\zeta : G \to \mathbb{C}$ by

$$f^\zeta(g) = \int_{\mathcal{Z}} \zeta(z) f(zg)\, dz.$$

Then $f^\zeta$ lies in $C_c(G,\zeta)$, the space of continuous $\zeta^{-1}$-equivariant functions that are compactly supported modulo $\mathcal{Z}$. The linear map $C_c^\infty(G) \to C_c^\infty(G,\zeta)$ is a surjection. It is continuous with respect to the natural strict LF-space topologies on $C_c^\infty(G)$ and $C_c^\infty(G,\zeta)$, but we will not need this fact. Further more, it restricts to a continuous linear surjection $C_c^\infty(G) \to C_c^\infty(G,\zeta)$.

### 2.2.2 $K$-finite functions

We continue with $G$ a Lie group or a td group and $(\mathcal{Z},\zeta)$ a central character of $G$ for the remainder of this subsection. Let $K$ be a compact subgroup of $G$. We let $C_c^\infty(G,\zeta,K)$ denote the subspace of left and right $K$-finite functions in $C_c^\infty(G)$, i.e. $(K \times K)$-finite vectors for the natural representation of $K \times K$ on $C_c^\infty(G)$. If $G$ is a td group, then $C_c^\infty(G,\zeta,K) = C_c^\infty(G,\zeta)$.

Suppose that $G$ is a Lie group. Then $C_c^\infty(G,\zeta,K)$ is a dense subspace of $C_c^\infty(G,\zeta)$. The space $C_c^\infty(G,\zeta,K)$ naturally colimit [Art89, §11]. Indeed, let $\Gamma \subseteq \widehat{K}$ be a finite set of irreducible representations of $K$. We define $C_c^\infty(G,\zeta,K)_\Gamma$ to be the subspace of functions in $C_c^\infty(G,\zeta,K)$ that transform on each side under $K$ according to a finite direct sum of representations in $\Gamma$. For each compact subspace $B$ of $G$, we define $C_B^\infty(G,\zeta,K)_\Gamma = C_B^\infty(G,\zeta) \cap C_c^\infty(G,\zeta,K)_\Gamma$, which is a closed subspace of $C_B^\infty(G,\zeta)$ and thus a Fréchet space. We have $C_c^\infty(G,\zeta,K)_\Gamma = \operatorname{colim}_B C_B^\infty(G,\zeta,K)_\Gamma$ and $C_c^\infty(G,\zeta,K) = \operatorname{colim}_\Gamma C_c^\infty(G,\zeta,K)_\Gamma$, and we give each the locally convex colimit topology, which makes them strict LF-spaces. The inclusion $C_c^\infty(G,\zeta,K) \to C_c^\infty(G,\zeta)$ is continuous.

### 2.2.3 Harish-Chandra Schwartz functions

We return to our general context of $G$ being a connected reductive group over $F$. Recall that we have fixed a minimal parabolic $P_0$ of $G$, a Levi factor $M_0$ of $P_0$, and maximal compact subgroup $K$ of $G(F)$ in good position relative to $M_0$. Thus, the Iwasawa decomposition $G(F) = P_0(F)K$ holds. Let $e : G(F) \to \mathbb{R}_{>0}$ be the unique function satisfying $e(K) = 1$ and $e(p_0 g) = \delta_{P_0}^{1/2}(p_0)e(g)$ for all $p_0 \in P_0(F)$ and $g \in G(F)$. That is, $e$ is the unique smooth vector of the parabolically induced representation $I_{M_0,P_0}^G(1_{M_0})$ satisfying $e(K) = 1$. (See below for our notation and conventions on parabolic induction.) Note that $e$ is $Z_G(F)$-invariant and right $K$-invariant. Although $e$ depends on the choice of $P_0$ and $K$, it does not depend on the choice of $M_0$. We define $\Xi = \Xi^G : G(F) \to \mathbb{R}_{>0}$ by

$$\Xi(g) = \int_K e(kg)\, dk$$

for all $g \in G(F)$. (That is, $\Xi(g) = \langle I_{M_0,P_0}^G(1,g)e, e\rangle_K$ in our notation introduced below.) Note that $\Xi$ is $Z_G(F)$-invariant and bi-$K$-invariant. Recall that for $x \in G(F)$, we choose elements $p_0(x) = p_{P_0}(x) \in P_0(F)$ and $k_0(x) = k_{P_0}(x) \in K$ such that $x = p_0(x)k_0(x)$. We have $\Xi(g) = \int_K \delta_{P_0}^{1/2}(p_0(kg))\, dk$.

Although $\Xi$ depends on the choices of $P_0$ and $K$, different choices result in a Xi-function $\Xi'$ with $\Xi \asymp \Xi'$. Since we will only use $\Xi$ in estimates, the choices made in the definition of $\Xi$ will have no impact on definitions made in terms of it.

Let $\mathcal{Z}$ be a closed central subgroup of $G(F)$. A continuous function $f : G(F) \to \mathbb{C}$ is said to be rapidly decreasing modulo $\mathcal{Z}$ if $|f|$ is $\mathcal{Z}$-invariant and it satisfies one of the following equivalent conditions [Vig94, §5].

1. For all $N \in \mathbb{N}$, we have

$$\|f\|_N \coloneqq \sup_{g \in G(F)} |f(g)|(1+\sigma^{\mathcal{Z}}(g))^N \Xi^{-1}(g) < \infty.$$

2. For all $N \in \mathbb{N}$, we have $f \in L^2(G(F)/\mathcal{Z}, (1+\sigma^{\mathcal{Z}}(g))^N\, dg)$.

We now define the Harish-Chandra Schwartz space of $\mathcal{C}(G, \zeta)$ for a unitary central datum $(\mathcal{Z}, \zeta)$ of $G(F)$.

Suppose that $F$ is non-archimedean, the Harish-Chandra Schwartz space $\mathcal{C}(G, \zeta)$ is the space of uniformly smooth complex-valued functions on $G(F)$ that are $\zeta^{-1}$-equivariant and rapidly decreasing modulo $\mathcal{Z}$ (cf. [Sil79, §4.4] and [Vig94, §5]). Let $K_0$ be a compact open subgroup of $G(F)$ that is sufficiently small so that $\zeta(\mathcal{Z} \cap K_0) = 1$. For each compact open subgroup $K$ of $K_0$, let $\mathcal{C}_K(G, \zeta)$ denote the subspace of $\mathcal{C}(G, \zeta)$ that are left and right $K$-invariant. Then $\mathcal{C}(G, \zeta) = \bigcup_K \mathcal{C}_K(G, \zeta)$. The spaces $\mathcal{C}_K(G, \zeta)$ are Fréchet spaces when given the topology determined by the seminorms $\|\cdot\|_N(f) = \sup_{g \in G(F)} |f(g)|(1+\sigma^{\mathcal{Z}}(g))^N \Xi(g)^{-1}$ for $N \in \mathbb{N}$. The space $\mathcal{C}(G)$ is given the inductive limit topology in the category of locally convex Hausdorff topological vector spaces and is an LF-space.

Suppose that $F$ is archimedean. The Harish-Chandra Schwartz space $\mathcal{C}(G, \zeta)$ is defined to be the space of all smooth functions $f$ on $G(F)$ such that all derivatives $ufv$ for $u, v \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ are rapidly decreasing modulo $\mathcal{Z}$. Given $u, v \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ and $N \in \mathbb{N}$, for each $f \in C^\infty(G(F))$, we define

$$\|f\|_{u,v,N} = \|ufv\|_N = \sup_{g \in G(F)} |ufv(g)|\Xi(g)^{-1}(1+\sigma^{Z}(g))^N.$$

Then $\mathcal{C}(G)$ is a Fréchet space when given the topology determined by the seminorms $\|\cdot\|_{u,v,N}$.

There are other Schwartz spaces of $G(F)$ in the literature, but since we will only use the Harish-Chandra Schwartz space we will often refer to it as simply the Schwartz space and its elements as Schwartz functions.

The other characterisation given above of a function being rapidly decreasing modulo $\mathcal{Z}$, leads to a different way of defining the topology on the Schwartz space $\mathcal{C}(G, \zeta)$. Denote the $L^2$-norm on $L^2(G(F)/\mathcal{Z}, (1+\sigma^{\mathcal{Z}}(g))^N\, dg)$ by $\|\cdot\|_{2,N}$. Replacing $\|\cdot\|_N$ with $\|\cdot\|_{2,N}$ in the definition of the topology on $\mathcal{C}(G, \zeta)$ given above results in the same topology.

We will write $\mathcal{C}_c(G, \zeta) = C_c^\infty(G, \zeta)$ and we will write $\mathcal{C}_{(c)}(G, \zeta)$ as a short hand for $\mathcal{C}(G, \zeta)$ (resp. $\mathcal{C}_c(G, \zeta)$) so that we can make statements about both spaces at the same time.

We define the space of tempered $\zeta$-equivariant distributions to be $\mathcal{C}(G, \zeta)'$. Since we have a the inclusion $C_c^\infty(G, \zeta) \to \mathcal{C}(G, \zeta)$ is continuous with dense image, we have a continuous injection $\mathcal{C}(G, \zeta)' \to C_c^\infty(G, \zeta)'$. Thus, we may identify tempered distributions with distributions, and a distribution $\Theta \in C_c^\infty(G,\zeta)'$ is tempered if and only if it extends to a continuous linear functional $\Theta : \mathcal{C}(G,\zeta) \to \mathbb{C}$.

## 2.3 Conjugacy classes

We begin by recalling some basic facts about conjugacy classes from [Kot05]. Let $x \in G(F)$. We denote the centraliser of $x$ in $G$ by $G^x = C_G(x)$ and we denote the connected centraliser $(G^x)^\circ = C_G(x)^\circ$ by $G_x$. We denote the conjugacy class of $x$ in $G$ by $x^G$ and the conjugacy class of $x$ in $G(F)$ by $x^{G(F)}$. We have that $x^G$ is a locally closed subset of $G$ in the Zariski topology, it is defined over $F$, and it is naturally isomorphic to $G^x\backslash G$. The $F$-points $x^G(F) \cong (G^x\backslash G)(F)$ form an $F$-analytic manifold. Moreover, we have a natural isomorphism of locally compact Hausdorff spaces $x^{G(F)} \cong G^x(F)\backslash G(F)$. It follows that $x^{G(F)}$ is an $F$-analytic submanifold of $G(F)$ (see [Ser06, Part II, Ch. IV, §5]). Moreover, $x^G(F)$ is a union of $G(F)$-conjugacy classes $y^{G(F)}$. It follows from the exact sequence of pointed sets

$$1 \longrightarrow G^x(F) \longrightarrow G(F) \longrightarrow (G^x\backslash G)(F) \longrightarrow H^1(\Gamma_F, G^x(\overline{F})) \longrightarrow H^1(\Gamma_F, G(\overline{F}))$$

that we have a bijection from the set of $G(F)$-conjugacy classes in $x^G(F)$ to

$$\ker[H^1(\Gamma_F, G^x(\overline{F})) \to H^1(\Gamma_F, G(\overline{F}))].$$

By [Ser02, §4.3, Theorem 4] we have that $H^1(\Gamma_F, G^x(\overline{F}))$ is finite, so there are finitely many $G(F)$-conjugacy classes in $x^G(F)$. It follows that the $G(F)$-conjugacy classes in $x^G(F)$ are all open and thus all closed in $x^G(F)$. Moreover, $x^G(F)$ and the $G(F)$-conjugacy classes in $x^G(F)$ are locally closed in $G(F)$, and thus locally compact. When $x$ is semisimple, then $x^G$ is closed in $G$ [Bor91, Theorem 9.2], and consequently $x^G(F)$ the $G(F)$-conjugacy classes $y^{G(F)} \subseteq x^G(F)$ are closed in $G(F)$.

The group $G^x(F)$, and thus $G_x(F)$, is unimodular. This is easily established when $x$ is semisimple, for then $G^x$ is reductive (but not necessarily connected), and thus $G^x(F)$ is unimodular. It was proved in general by Springer and Steinberg in [SS70, Chapter E, Remark 3.27 (b)], even when $F$ is allowed to have positive characteristic. They remark that Harish-Chandra proved it for characteristic zero. See also [Kot05, §3.3, §17.3].

### 2.3.1 Orbital integrals

Fix a non-zero right $G(F)$-invariant Radon measure on $G^x(F)\backslash G(F)$ (that is, a quotient of Haar measures). The pushfoward of the measure on $G^x(F)\backslash G(F)$ to $G(F)$ is a Borel measure on $G(F)$ supported on $x^{G(F)}$. Ranga Rao and Deligne proved that it is a Radon measure, that is, for each $f \in C_c(G(F))$, the integral

$$\int_{G^x(F)\backslash G(F)} f(g^{-1}xg)\, dg$$

converges absolutely [Ran72; DKV84]. This is immediate in the case when $x$ is semisimple, since then $x^{G(F)}$ is closed and the support of $f$ meets $x^{G(F)}$ in a compact subset. Consequently, the integral

$$O_x(f) = \int_{G_x(F)\backslash G(F)} f(g^{-1}xg)\, dg = |G^x(F)/G_x(F)| \int_{G^x(F)\backslash G(F)} f(g^{-1}xg)\, dg$$

converges absolutely for all $f \in C_c(G(F))$. This integral is called the orbital integral of $f$ at $x$ with respect to the chosen measure. If $f \in C_c(G(F), \zeta)$, the space of $\zeta^{-1}$-equivariant continuous functions on $G(F)$ that are compactly supported modulo $\mathcal{Z}$, then the integral defining $O_x(f)$ converges absolutely and defines a $\zeta$-equivariant distribution $O_x \in C_c^\infty(G, \zeta)'$. This can be proved using the continuous surjection $C_c(G) \to C_c(G, \zeta)$ as in [DKV84, A.2.b].

We will mainly be concerned with orbital integrals at regular semisimple elements, and even strongly regular (semisimple) elements. Recall that an element $x \in G$ is regular if its centraliser $G^x$ has least possible dimension, which is the rank $r$ of $G$. The set of regular elements of $G$ forms an open subvariety $G_{\mathrm{reg}}$. An element $x \in G$ is semisimple if and only if $x$ lies in maximal torus of $G$, and is regular semisimple if and only if it lies in a unique maximal torus of $G$ (namely $G_x$). The Weyl discriminant is the regular function $D^G$ on $G$ defined by

$$\det(t - (\mathrm{Ad}(g)) - 1) \in D^G(g) t^{\mathrm{rk}(G)} + t^{r+1}\mathcal{O}_G(G)[t].$$

Elements of the open subvariety $G' = G_{\mathrm{rs}} = \{g \in G : D^G(g) \neq 0\}$ are the regular semisimple elements of $G$. In older literature they are often simply referred to as regular elements elements of $G$, and the elements of the closed subvariety $G_{\mathrm{sing}} = G \backslash G_{\mathrm{rs}}$ are said to be singular. We will still refer to the elements of $G \backslash G_{\mathrm{rs}}$ as singular. If $T \subseteq G$ is a maximal torus of $G$ and $t \in T$, then $D^G(t) = \det(1 - \mathrm{Ad}(t)|_{\mathfrak{g}/\mathfrak{t}})$. An element $x \in G$ is strongly regular if $G^x$ is a maximal torus of $G$. Note that strongly regular elements are regular semisimple. We denote the open subvariety of strongly regular elements of $G$ by $G_{\mathrm{sr}} \subseteq G_{\mathrm{rs}}$.

If $H \subseteq G$, we say that an element is $G$-regular (resp. $G$-singular, etc.) if it is regular (resp. singular, etc) as an element of $G$. We will use subscripts $G$-reg (resp. $G$-sing, etc.) to indicate that we are restricting to $G$-regular (resp. $G$-singular) elements.

If $x \in G_{\mathrm{rs}}(F)$, then our fixed a Haar measures on the maximal torus $G_x(F)$ and $G(F)$ determine a $G(F)$-invariant Radon measure on $G_x(F)\backslash G(F)$, which we use to define $O_x$.

For $x$ regular semisimple, the integral defining $O_x(f)$ even converges absolutely for $f \in \mathcal{C}(G, \zeta)$ and defines a $\zeta$-equivariant tempered distribution. This was proved by Harish-Chandra. (See [Var77, Part II, §12, Theorem 6] for the case of real groups and [Har73] for $p$-adic groups). For $p$-adic groups, this was extended to arbitrary $x \in G(F)$ by Clozel [Clo91, Theorem 2] using the Howe conjecture and the Shalika germ expansion.

We have

$$G_{\mathrm{rs}}(F) = \coprod_{T} T_{G\text{-reg}}(F)^{G(F)}$$

where the disjoint union is over some choice of representatives of the $G(F)$-conjugacy classes of maximal tori of $G$. Moreover, For each maximal torus $T$ of $G$, the map

$$\begin{aligned} T_{G\text{-reg}}(F) \times T(F)\backslash G(F) &\longrightarrow T_{G\text{-reg}}(F)^{G(F)} \\ (t, g) &\longmapsto g^{-1}tg \end{aligned}$$

is a surjective local diffeomorphism, and thus $T_{G\text{-reg}}(F)^{G(F)}$ is an open subset of $G(F)$. Its fibres have cardinality $|W_F(G, T)|$ and the change of variables formula results in the Weyl integration formula

$$\int_{G(F)} f(x)\, dx = \sum_{T} |W_F(G, T)|^{-1} \int_{T(F)} |D^G(t)| O_t(f)\, dt,$$

whenever one side and hence the other converges. We write $\Gamma(G)$ for the set of conjugacy classes of $G(F)$. We write

$$\Gamma_{\mathrm{ss}}(G) \supseteq \Gamma_{\mathrm{rs}}(G) \supseteq \Gamma_{\mathrm{sr}}(G)$$

for the sets of semisimple, regular semisimple, and strongly regular conjugacy classes in $G(F)$.

We give $\Gamma(G)$ the quotient topology from $G(F)$. This makes $\Gamma_{\mathrm{rs}}(G)$ and $\Gamma_{\mathrm{sr}}(G)$ open dense locally compact Hausdorff subspaces of $\Gamma(G)$, and $\Gamma_{\mathrm{sr}}(G)$ is naturally an $F$-analytic manifold. For $f \in C_c(G(F), \zeta)$ or $f \in \mathcal{C}(G, \zeta)$, the function $\gamma \mapsto O_\gamma(f)$ is a continuous on $\Gamma_{\mathrm{rs}}(G)$. The Weyl integration formula makes it natural to define a Radon measure $d\gamma$ on $\Gamma_{\mathrm{rs}}(G)$ by

$$\int_{\Gamma_{\mathrm{rs}}(G)} \varphi(\gamma)\, d\gamma = \sum_T |W(G,T)|^{-1} \int_{T(F)} \varphi(t)\, dt,$$

for all $\varphi \in C_c(\Gamma_{\mathrm{rs}}(G))$, where the sum runs over as set of representatives of the conjugacy classes of maximal tori of $G$. The Weyl integration formula then becomes

$$\int_{G(F)} f(x)\, dx = \int_{\Gamma_{\mathrm{rs}}(G)} |D^G(\gamma)| O_\gamma(f)\, d\gamma.$$

Note that $\Gamma_{\mathrm{sr}}(G)$ has comeasure zero in $\Gamma_{\mathrm{rs}}(G)$.

A maximal torus $T$ of $G$ is said to be elliptic if $T/A_G$ is anisotropic, or equivalently $A_T = A_G$. A semisimple element $x \in G(F)$ is said to be elliptic if $x \in T(F)$ for some elliptic maximal torus $T$ of $G$. We define $G(F)_{\mathrm{rs,ell}}$ (resp. $G(F)_{\mathrm{sr,ell}}$) to be the open subsets of regular semisimple (resp. strongly regular) elliptic elements of $G(F)$, and we define $\Gamma_{\mathrm{rs,ell}}(G)$ (resp. $\Gamma_{\mathrm{sr,ell}}(G)$) to be their open images in $\Gamma(G)$. The Radon measure $d\gamma$ restricts to a Radon measure on $\Gamma_{\mathrm{rs,ell}}(G)$ satisfying

$$\int_{\Gamma_{\mathrm{rs,ell}}(G)} \varphi(\gamma)\, d\gamma = \sum_{T \text{ ell}} |W(G,T)|^{-1} \int_{T(F)} \varphi(t)\, dt$$

for all $\varphi \in C_c(\Gamma_{\mathrm{rs,ell}}(G))$, where the sum is over a set of representatives for the $G(F)$-conjugacy classes of elliptic maximal tori. Using the decomposition

$$\Gamma(G) = \coprod_{M \in \mathcal{L}^G(M_0)/W_0^G} \Gamma_{G\text{-rs,ell}}(M)/W^G(M)$$

one obtains

$$\int_{\Gamma_{\mathrm{rs}}(g)} \varphi(\gamma)\, d\gamma = \sum_{M \in \mathcal{L}^G(M_0)/W_0^G} |W^G(M)|^{-1} \int_{\Gamma_{\mathrm{rs,ell}}(M)} \varphi(\gamma)\, d\gamma.$$

One can thus express the Weyl integration formula in the form:

$$\int_{G(F)} f(x)\,dx = \sum_{M \in \mathcal{L}^G(M_0)/W_0^G} |W^G(M)|^{-1} \int_{\Gamma_{\mathrm{rs,ell}}(M)} |D^G(\gamma)| O_\gamma(f)\,d\gamma.$$

The appearance of the Weyl discriminant in the Weyl integration formula makes it natural to introduce the normalised orbital integrals $f_G(\gamma) = |D^G(\gamma)|^{1/2} O_\gamma(f)$. For $f \in \mathcal{C}(G,\zeta)$, the normalised orbital integrals of $f$ define a continuous function $f_G : \Gamma_{\mathrm{rs}}(G) \to \mathbb{C}$, which is smooth on $\Gamma_{\mathrm{sr}}$, and locally bounded. We define the spaces of orbital integrals $\mathcal{I}(G,\zeta)$ and $\mathcal{I}_c(G,\zeta)$ by

$$\begin{aligned}\mathcal{I}_{(c)}(G,\zeta) &= \{f_G : \Gamma_{\mathrm{sr}}(G) \to \mathbb{C} : f \in \mathcal{C}_{(c)}(G,\zeta)\} \\ &= \mathcal{C}_{(c)}(G,\zeta)/\operatorname{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{O_\gamma : \gamma \in \Gamma_{\mathrm{sr}}(G)\})\end{aligned}$$

with their natural quotient topologies. We define the space $\mathcal{I}_c(G,\zeta,K)$ of orbital integrals of elements of $C_c^\infty(G,\zeta,K)$ in an analogous way. This space does not depend on the choice of $K$. For non-archimedean $F$, this is because $C_c^\infty(G,\zeta,K) = C_c^\infty(G,\zeta)$. For archimedean $F$, it follows from the fact that all maximal compact subgroups of $G(F)$ are $G(F)$-conjugate. Thus, we will write $\mathcal{I}_f(G,\zeta) = \mathcal{I}_c(G,\zeta,K)$.

We recall that a function $f \in \mathcal{C}(G,\zeta)$ is said to be cuspidal if $f_G(\gamma) = 0$ for all $\gamma \in \Gamma_{\mathrm{sr}}(G) \setminus \Gamma_{\mathrm{ell}}(G)$, or equivalently for all $\gamma \in \Gamma_{\mathrm{rs}}(G) \setminus \Gamma_{\mathrm{ell}}(G)$. We denote the subspace of cuspidal functions in $\mathcal{C}_{(c)}(G,\zeta)$ by $\mathcal{C}_{(c),\mathrm{cusp}}(G,\zeta)$, and we denote its image in $\mathcal{I}_{(c)}(G,\zeta)$ by $\mathcal{I}_{(c),\mathrm{cusp}}(G,\zeta)$. We denote the subspace of cuspidal functions in $C_c^\infty(G,\zeta,K)$ by $C_{c,\mathrm{cusp}}^\infty(G,\zeta,K)$, and its image in $\mathcal{I}_f(G,\zeta)$ by $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$.

We refer to elements of $\mathcal{I}_c(G,\zeta)'$ (resp. $\mathcal{I}(G,\zeta)'$) as invariant $\zeta$-equivariant distributions (resp. invariant tempered $\zeta$-equivariant distributions). Note that we have a continuous linear injection $\mathcal{I}_c(G,\zeta) \to \mathcal{I}(G,\zeta)$ with dense image. Its transpose is a continuous linear injection $\mathcal{I}(G,\zeta)' \to \mathcal{I}_c(G,\zeta)'$, which enables us to identify each tempered $\zeta$-equivariant invariant distribution with a $\zeta$-equivariant invariant distribution.

We may identify $\mathcal{I}_c(G,\zeta)'$ with a subspace of $\mathcal{C}_{(c)}(G,\zeta)'$ via the transpose of the quotient map $\mathcal{C}_{(c)}(G,\zeta) \to \mathcal{I}_{(c)}(G,\zeta)$. As vector spaces, we have

$$\begin{aligned}\mathcal{I}_{(c)}(G,\zeta)' &= \operatorname{Ann}_{\mathcal{C}_{(c)}(G,\zeta)'}(\operatorname{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{O_\gamma : \gamma \in \Gamma_{\mathrm{sr}}(G)\})) \\ &= \mathrm{cl}_{\mathcal{C}_{(c)}(G,\zeta)',\text{weak-}*}(\{O_\gamma : \gamma \in \Gamma_{\mathrm{sr}}(G)\}).\end{aligned}$$

That is, an distribution in $\mathcal{C}_{(c)}(G,\zeta)'$ belongs to $\mathcal{I}_{(c)}(G,\zeta)'$ if and only if it lies in the weak-$*$ closure in $\mathcal{C}_{(c)}(G,\zeta)'$ of the linear span of the set of strongly regular orbital integrals of $G$. A locally integrable function $\Theta$ on $G(F)$ that is continuous on $G_{\mathrm{sr}}(F)$ defines an invariant distribution if and only if $\Theta$ is conjugation invariant on $G_{\mathrm{sr}}(F)$.

One can also define the notion of conjugation invariant distributions. For $y \in G(F)$ and $f \in \mathcal{C}_{(c)}(G,\zeta)$, we define ${}^y f \in \mathcal{C}_{(c)}(G,\zeta)$ by ${}^y f(x) = f(y^{-1}xy)$. This defines a left action of $G(F)$ on $\mathcal{C}_{(c)}(G,\zeta)$. The left action of $G(F)$ on $\mathcal{C}_{(c)}(G,\zeta)$ in turn leads to a right action of $G(F)$ on the associated space of distributions: for all $y \in G(F)$ and $u \in \mathcal{C}_{(c)}(G,\zeta)'$, we define $u^y \in \mathcal{C}_{(c)}(G,\zeta)'$ by $u^y(f) = u({}^y f)$.

Let $\mathcal{C}_{(c)}(G,\zeta)_G$ denote the quotient of $\mathcal{C}_{(c)}(G,\zeta)$ by the smallest closed subspace of $\mathcal{C}_{(c)}(G,\zeta)$ containing all functions of the form ${}^y f - f$ for $y \in G(F)$ and $f \in \mathcal{C}_{(c)}(G,\zeta)$. We define $(\mathcal{C}_c(G,\zeta)_G)'$ (resp. $(\mathcal{C}(G,\zeta)_G)'$) to be the space of conjugation invariant distributions (resp. tempered distributions). We have a natural injection $(\mathcal{C}_{(c)}(G,\zeta)_G)' \to \mathcal{C}_{(c)}(G,\zeta)'$, and its image is the subspace of distributions in $\mathcal{C}_{(c)}(G,\zeta)'$ such that $u^y = u$.

Every orbital integral is conjugation invariant. Thus, if $f \in \mathcal{C}_{(c)}(G,\zeta)$ is annihilated by all conjugation invariant functions, then it is annihilated by all orbital integrals. It follows that the quotient $\mathcal{C}_{(c)}(G,\zeta) \to \mathcal{I}_{(c)}(G,\zeta)$ descends to a quotient $\mathcal{C}_{(c)}(G,\zeta)_G \to \mathcal{I}_{(c)}(G,\zeta)$. It is known that for $f \in \mathcal{C}_c(G)$, if $f$ is annihilated by all strongly regular orbital integrals, then $f$ is annihilated by all conjugation invariant distributions. This was proved for real groups by Bouaziz [Bou94, Theorem 3.2.1] and for $p$-adic groups by Harish-Chandra [Har99, Theorem 10]. Thus, $\mathcal{C}_c(G)_G = \mathcal{I}_c(G)$, and conjugation invariant distributions are the same as invariant distributions.

## 2.4 Representations

Our representations $\pi = (\pi, V_\pi)$ of $G(F)$ will be assumed to be continuous and complex unless indicated otherwise. We use the term module instead of representation instead to emphasise that we are talking about an algebraic object whose morphisms have no continuity condition. We recall that a representation $\pi$ has an underlying infinitesimal module, which belongs to a purely algebraic category in the sense that the morphisms have no continuity condition imposed on them. For archimedean $F$, the underlying infinitesimal module of $\pi$ is the $(\mathfrak{g}, K)$-module (or Harish-Chandra module) $(\pi^{\infty,(K)}, V_\pi^{\infty,(K)})$ of smooth $K$-finite vectors of $\pi$. For $F$ non-archimedean, the underlying infinitesimal module of $\pi$ is the $G(F)$-module of smooth (hence $K$-finite) vectors $(\pi^\infty, V_\pi^\infty)$ of $\pi$. Passing to underlying infinitesimal modules is a functor. Two admissible representations are said to be equivalent (or infinitesimally equivalent) if their underlying infinitesimal modules are isomorphic. This equivalence relation is coarser than isomorphism and is the one used in the classification of irreducible admissible representations. Every admissible representation of finite length is equivalent to a Hilbert space representation. If $\pi$ has a central character, we denote it by $\zeta_\pi$. If $F$ is archimedean and $\pi$ has an infinitesimal character, we denote it by $\mu_\pi$. For $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, we denote by $\pi_\lambda = \pi \otimes |\cdot|_G^\lambda$ the unramified twist of $\pi$ by $\lambda$.

Let $(\mathcal{Z}, \zeta)$ be a central datum of $G(F)$. We denote by $\Pi(G, \zeta)$ the set of equivalence classes of irreducible admissible representations $\pi$ of $G(F)$ with $\mathcal{Z}$-character $\zeta_\pi|_{\mathcal{Z}} = \zeta$. We have the subsets

$$\Pi_u(G, \zeta) \supseteq \Pi_{\mathrm{temp}}(G, \zeta) \supseteq \Pi_2(G, \zeta)$$

of equivalence classes of irreducible unitary, tempered, and square-integrable representations, respectively. (These are only non-empty if $\zeta$ is unitary.) If the central datum $(\mathcal{Z}, \zeta)$ is trivial in the sense that $\mathcal{Z} = 1$, we may omit $(\mathcal{Z}, \zeta)$ from notation and write $\Pi(G) = \Pi(G, \zeta)$. We call the elements of $\Pi_2(G)$ discrete series representations of $G(F)$. The space of virtual representations of $G(F)$ with $\mathcal{Z}$-character $\zeta$ is the complex vector space $D_{\mathrm{spec}}(G, \zeta) = \mathbb{C}\Pi(G, \zeta)$ with basis $\Pi(G, \zeta)$. Inside of it sits the space $D_{\mathrm{temp}}(G, \zeta) = \mathbb{C}\Pi_{\mathrm{temp}}(G, \zeta)$ of virtual tempered representations of $G(F)$ with $\mathcal{Z}$-character $\zeta$.

### 2.4.1 Characters

Let $(\pi, V)$ be a finite-length admissible representation of $G(F)$ with $\mathcal{Z}$-character $\zeta$. For each $f \in C_c^\infty(G, \zeta)$, we can form the operator

$$\pi(f) = \int_{G(F)/\mathcal{Z}} f(g)\pi(g)\,dg.$$

Note that this depends on the measure on $G(F)/\mathcal{Z}$, thus on the Haar measures on $G(F)$ and $\mathcal{Z}$, which we have fixed. Suppose that $f \in C_c^\infty(G, \zeta, K)$. Then $f \in C_c^\infty(G, \zeta, K)_\Gamma$ for some finite set $\Gamma \subseteq \widehat{K}$ and the image of $\pi(f)$ is contained in $V_\Gamma = \sum_{\delta \in \Gamma} V_\delta$. Thus, $\pi(f)$ has finite rank and $\Theta_\pi(f) \coloneqq \operatorname{tr} \pi(f)$ is well-defined. The linear functional $\Theta_\pi = \operatorname{tr} \pi : C_c^\infty(G, \zeta, K) \to \mathbb{C}$ is continuous and is called the character of $\pi$. Equivalent admissible representations of finite length have the same character.

If $F$ is non-archimedean, then $C_c^\infty(G, \zeta, K) = C_c^\infty(G, \zeta)$, so $\Theta_\pi$ is a $\zeta$-equivariant distribution. Suppose that $F$ is archimedean. Then $\Theta_\pi$ has a (unique) continuous extension to a $\zeta$-equivariant distribution. We may assume that $\pi$ is a Hilbert space representation by replacing it with an equivalent representation. Harish-Chandra proved that for all $f \in C_c^\infty(G)$, the operator $\pi(f)$ is of trace class and the linear functional $\Theta_\pi = \operatorname{tr}\pi : C_c^\infty(G) \to \mathbb{C}$ defined by $\Theta_\pi(f) = \operatorname{tr}\pi(f)$ is continuous. The proof of this uses the Harish-Chandra subquotient theorem to show that $K$ is uniformly large in $G(F)$ in the sense of [GV88]. For all $f \in C_c^\infty(G)$, we have $\pi(f^\zeta) = \pi(f)$. Since the map $f \mapsto f^\zeta$ is a continuous surjection $C_c^\infty(G) \to C_c^\infty(G,\zeta)$, it follows that for all $f \in C_c^\infty(G,\zeta)$, the operator $\pi(f)$ is of trace class and the linear functional $\Theta_\pi = \operatorname{tr}\pi : C_c^\infty(G,\zeta) \to \mathbb{C}$ is continuous, hence a $\zeta$-equivariant distribution.

The map $\pi \mapsto \Theta_\pi$ is exact in the sense that it is additive on short exact sequences. Consequently, the character of $\pi$ only depends on its semisimplification. Harish-Chandra proved that the characters of the infinitesimal equivalence classes of irreducible admissible representations are linearly independent. (See [Har54, §7] for real groups and [Sil79, Lemma 1.13.1] for $p$-adic groups). Consequently, the character of a finite length admissible representation determines its semisimplification. We often identify semisimple representations with their characters.

We define the character of a virtual representation $\pi = \sum_{i=1}^m c_i\pi_i$ to be $\Theta_\pi = \sum_{i=1}^m c_i\Theta_{\pi_i}$, and we call it a virtual character. We may and do identify virtual representations with their characters. Thus, the space $D_{\mathrm{spec}}(G,\zeta)$ is identified with a subspace of the space of $\zeta$-equivariant distributions on $G(F)$. For $\pi \in \Pi_{\mathrm{temp}}(G,\zeta)$ its $\Theta_\pi$ is a $\zeta$-equivariant tempered distribution. Thus, the space $D_{\mathrm{temp}}(G,\zeta)$ of virtual tempered representations sits inside the space of $\zeta$-equivariant tempered distributions on $G(F)$. We call the characters of virtual tempered representations virtual tempered characters.

Let $\pi$ be a finite-length admissible representation of $G(F)$ with $\mathcal{Z}$-character $\zeta$. Harish-Chandra's regularity theorem tells us that $\Theta_\pi$ is a smooth—even analytic if $F$ is archimedean—conjugation invariant $\zeta$-equivariant function on $G_{\mathrm{rs}}(F)$; that $\Theta_\pi \in L^1_{\mathrm{loc}}(G(F))$; and that the normalised character $I_G(\pi,x) = |D^G(x)|^{1/2}\Theta_\pi(x)$ is locally bounded on $G(F)$. The following lemma is an immediate consequence of the Harish-Chandra regularity theorem and the Weyl integration formula.

**Lemma 2.4.1.** *If $f \in \mathcal{C}_c(G,\zeta)$ with $f_G(\gamma) = 0$ for all $\gamma \in \Gamma_{\mathrm{sr}}(G)$, then $f_G(\pi) = 0$ for all $\pi \in \Pi(G,\zeta)$. Consequently, $\Theta_\pi \in \mathcal{I}_c(G,\zeta)'$ for all $\pi \in \Pi(G,\zeta)$*

*If $\zeta$ is unitary and $f \in \mathcal{C}(G,\zeta)$ with $f_G(\gamma) = 0$ for all $\gamma \in \Gamma_{\mathrm{sr}}(G)$, then $f_G(\pi) = 0$ for all $\pi \in \Pi_{\mathrm{temp}}(G,\zeta)$. Consequently, $\Theta_\pi \in \mathcal{I}_c(G,\zeta)'$ for all $\pi \in \Pi_{\mathrm{temp}}(G,\zeta)$.*

Now suppose that $\zeta$ is unitary. Let $f \in \mathcal{C}(G,\zeta)$. The operator-valued Fourier transform of $f$ is the section of $\coprod_{\pi\in\Pi_{\mathrm{temp}}(G,\zeta)} \operatorname{End}(V_\pi) \to \Pi_{\mathrm{temp}}(G,\zeta)$ defined by $\pi \mapsto$

$\pi(f)$. It is the subject of harmonic analysis on $G(F)$, upon which invariant harmonic analysis is built. Invariant harmonic analysis is the study of the invariant (or scalar-valued) Fourier transform of $f$ is the function $f_G : \Pi_{\mathrm{temp}}(G,\zeta) \to \mathbb{C}$ defined by $f_G(\pi) = \operatorname{tr}\pi(f)$.

We have the space of invariant Fourier transforms

$$\begin{aligned}\widehat{\mathcal{I}_{(c)}}(G,\zeta) &= \{f_G : f \in \mathcal{C}_{(c)}(G,\zeta)\} \\ &= \mathcal{C}_{(c)}(G,\zeta)/\operatorname{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{\Theta_\pi : \pi \in \Pi_{\mathrm{temp}}(G,\zeta)\})\end{aligned}$$

with its natural quotient topology. We define the space $\widehat{\mathcal{I}_{c,f}}(G,\zeta) = \widehat{\mathcal{I}}_c(G,\zeta,K)$ of invariant Fourier transforms of elements of $C_c^\infty(G,\zeta,K)$ in an analogous way. The space $\widehat{\mathcal{I}_{c,f}}(G,\zeta)$ does not depend on the choice of $K$ and $\widehat{\mathcal{I}_{c,f}}(G,\zeta) = \widehat{\mathcal{I}}_c(G,\zeta)$ if $F$ is non-archimedean.

The invariant Fourier transform

$$\mathcal{C}_{(c)}(G,\zeta) \longrightarrow \widehat{\mathcal{C}_{(c)}}(G,\zeta)$$

is a surjective linear map by definition of $\widehat{\mathcal{C}}(G,\zeta)$. Since tempered characters are invariant, it descends to a continuous surjective linear map

$$\mathcal{F} : \mathcal{I}_{(c)}(G,\zeta) \longrightarrow \widehat{\mathcal{I}_{(c)}}(G,\zeta).$$

This restricts to a continuous surjective linear map

$$\mathcal{F} : \mathcal{I}_{c,f}(G,\zeta) \longrightarrow \widehat{\mathcal{I}_{c,f}}(G,\zeta).$$

In the next chapter we will give a reformulation of the invariant Fourier transform due to Arthur and recall the results from invariant harmonic analysis that we need. Before turning to that, we will recall some background on parabolic induction.

## 2.5 Parabolic induction

Let $(P,M)$ be a parabolic pair of $G$. We have the (normalised) continuous parabolic induction functor $I_{M,P}^G$ defined on continuous representations $(\sigma, V_\sigma)$ of $M(F)$ by

$$\begin{aligned}I_{M,P}^G(V_\sigma) = \{f \in C(G(F),V_\pi) :& f(mng) = \sigma(m)\delta_P^{1/2}(m)f(g), \\ & \forall m \in M(F), n \in N(F) g \in G(F)\}\end{aligned}$$

with $G(F)$ acting by right translation: $I^G_{M,P}(\sigma, g_0)f(g) = f(gg_0)$ for all $g_0, g \in G(F)$ and $f \in I^G_{M,P}(V_\sigma)$. (If we extend $\sigma$ to $P(F)$ using the quotient homomorphism $P(F) \to P(F)/N(F) \cong M(F)$, then we can write $f(pg) = \sigma(p)\delta_P(p)^{1/2}f(g)$ for all $p \in P(F)$ and $g \in G(F)$ in the definition.) The vector space $I^G_{M,P}(V_\pi)$ inherits a topology from $C(G(F), V_\sigma)$, which is equipped with the topology of uniform convergence on compact subspaces.

Parabolic induction is exact, preserves admissibility, temperedness, and preserves finite length. For all $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$ there is a natural isomorphism $I^G_{M,P}(\sigma)_\lambda \cong I^G_{M,P}(\sigma_\lambda)$ defined by $f \mapsto |\cdot|^\lambda_G f$. If $\sigma$ has a central character $\zeta_\sigma$, then the central character of $I^G_{M,P}$ is $\zeta_\pi = \zeta_\sigma|_{Z_G(F)}$. If $F$ is archimedean and $\sigma$ has an infinitesimal character $\mu_\sigma$, then the infinitesimal character of $\pi$ is $\mu_\sigma \circ \xi^G_M$, where $\xi^G_M : \mathfrak{Z}(\mathfrak{g}_\mathbb{C}) \to \mathfrak{Z}(\mathfrak{m}_\mathbb{C})$ is the Harish-Chandra homomorphism. (See [Vog87, Prop. 6.7] and [KV95, Prop. 11.43].) Let $T$ be a maximal torus of $M$. Then we have Harish-Chandra isomorphisms $\mathfrak{Z}(\mathfrak{g}_\mathbb{C}) \cong \mathrm{Sym}(\mathfrak{t}_\mathbb{C})^{W(G_\mathbb{C},T_\mathbb{C})}$ and $\mathfrak{Z}(\mathfrak{m}_\mathbb{C}) \cong \mathrm{Sym}(\mathfrak{t}_\mathbb{C})^{W(M_\mathbb{C},T_\mathbb{C})}$. Identifying infinitesimal characters of $G$ and $M$ with elements of $\mathfrak{t}^*_\mathbb{C}/W(G_\mathbb{C}, T_\mathbb{C})$ and $\mathfrak{t}^*_\mathbb{C}/W(M_\mathbb{C}, T_\mathbb{C})$ respectively, we have that $\mu_\pi \in \mathfrak{t}^*_\mathbb{C}/W(G_\mathbb{C}, T_\mathbb{C})$ is the image of $\mu_\sigma \in \mathfrak{t}^*_\mathbb{C}/W(M_\mathbb{C}, T_\mathbb{C})$ under the natural quotient map $\mathfrak{t}^*_\mathbb{C}/W(M_\mathbb{C}, T_\mathbb{C}) \to \mathfrak{t}^*_\mathbb{C}/W(G_\mathbb{C}, T_\mathbb{C})$.

The smooth vectors $I^G_{M,P}(\sigma)^\infty$ of $I^G_{M,P}(\sigma)$ can be obtained by applying the (normalised) smooth parabolic induction functor $I^{G,\infty}_{M,P}$ to the smooth vectors $\sigma^\infty$ of $\sigma$. (See [BW00, §7.11] for the case of real groups.) If $F$ is archimedean, the underlying $(\mathfrak{g}, K)$-module $I^G_{M,P}(\sigma)^{\infty,(K)}$ of smooth $K$-finite vectors of $I^G_{M,P}(\sigma)$ can be described in terms of the underlying $(\mathfrak{m}, M(F) \cap K)$-module $\sigma^{\infty,(M(F)\cap K)}$ of smooth $(M(F) \cap K)$-finite vectors of $\sigma$ using the (normalised) parabolic induction functor $I^{(\mathfrak{g},K)}_{(\mathfrak{m},M(F)\cap K),\mathfrak{p}}$ of Harish-Chandra modules [KV95, Ch. XI, §2]. It follows that equivalence class of $I^G_{M,P}(\sigma)$ only depends on the equivalence class of $\sigma$.

Let $K$ be any maximal compact subgroup of $G(F)$ with $G(F) = P(F)K$ and fix the normalised Haar measure on $K$. We have the (normalised) Hilbert space induction functor $I^G_{M,P,K}$. If $\sigma$ is a Hilbert space representation of $G$, then $I^G_{M,P,K}(V_\sigma)$ is defined to be the completion of $I^G_{M,P}(V_\sigma)$ with respect to the inner product

$$\langle f_1, f_2\rangle_K := \int_K \langle f_1(k), f_2(k)\rangle \, dk.$$

Alternatively, $I^G_{M,P,K}(V_\sigma)$ is the space of equivalence classes of measurable functions $f : G(F) \to V_\sigma$ such that $f(pg) = \sigma(p)\delta_P(p)^{1/2}f(g)$ for all $p \in P(F)$ and $g \in G(F)$, and

$$\|f\|_K := \int_K \|f(k)\| \, dk < \infty.$$

The space $I^G_{M,P,K}(V_\sigma)$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_K$. The representation $I^G_{M,P,K}(\sigma)$ of $G(F)$ on $I^G_{M,P,K}(V_\sigma)$ by right translation is a continuous representation and its restriction to $K$ is unitary. Moreover, $I^G_{M,P,K}(\sigma)$ is unitary if $\sigma$ is. We have $I^G_{M,P,K}(\sigma)^\infty = I^G_{M,P}(\sigma)^\infty$. Consequently, $I^G_{M,P,K}(\sigma)$ is equivalent to $I^G_{M,P}(\sigma)$, and the equivalence class of $I^G_{M,P,K}(\sigma)$ only depends on the infinitesimal equivalence class of $\sigma$. In particular, the infinitesimal equivalence class of $I^G_{M,P,K}(\sigma)$ does not depend on $K$.

We also have the so-called "compact picture" or "compact model" of $I^G_{M,P,K}(\sigma)$. Let $K_P = P(F)\cap K$. Let $I^G_{M,P,K}(V_\sigma)|_K$ be the space of all equivalence classes of measurable functions $f : K \to V_\sigma$ such that $f(pk) = \sigma(p)f(k)$ for all $p \in K_P$ and $k \in K$, and $\|f\|_K < \infty$. Then $I^G_{M,P,K}(V_\sigma)|_K$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_K$. It is a representation of $G(F)$ with the action $I^G_{M,P,K}(\sigma, g_0)|_K f(k) = (kk_P(g_0))$. Moreover, restriction gives a unitary isomorphism $I^G_{M,P,K}(V_\sigma) \to I^G_{M,P,K}(V_\sigma)|_K, f \mapsto f_{K_P}$. Transporting the representation $I^G_{M,P,K}(\sigma)$ along this unitary isomorphism gives an isomorphic representation $I^G_{M,P,K}(\sigma)_K$ on $I^G_{M,P,K}(V_\sigma)_K$ defined by

$$(I^G_{M,P,K}(\sigma, g_0)_K f)(k) = \sigma(m_P(kg_0))f(k_P(kg_0)).$$

Note that if $\chi : M(F) \to \mathbb{C}$ is a multiplicative character that is trivial on $K_M = M(F)\cap K$, then $I^G_{M,P,K}(V_\sigma)_K = I^G_{M,P,K}(V_{\sigma\otimes\chi})_K$. (Note that this does not say that $I^G_{M,P,K}(\sigma)_K = I^G_{M,P,K}(\sigma\otimes\chi)_K$). In particular, we have $I^G_{M,P,K}(V_\sigma)_K = I^G_{M,P,K}(V_{\sigma_\lambda})_K$ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Thus, using the compact picture, the parabolically induced representations $I^G_{M,P,K}(\sigma_\lambda)_K$ are all realised on the same Hilbert space $I^G_{M,P,K}(V_\sigma)_K$, the unramified twist by $\lambda$ only affects the action of $G(F)$. This is important for the construction of standard intertwining operators.

### 2.5.1 Standard intertwining operators

Let $M$ be a Levi subgroup of $G$, let $\sigma \in \Pi_{\text{temp}}(M)$, and let $P, Q \in \mathcal{P}(M)$. Recall that $\Delta(P, A_M)$ the set of simple roots associated with $(P, A_M)$. For $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with $Re\langle\lambda, \alpha^\vee\rangle$ sufficiently large for all $\alpha \in \Delta(P, A_M)$, the integral

$$(J_{Q|P}(\sigma_\lambda)f)(g) = \int_{N_P(F)\cap N_Q(F)\backslash N_Q(F)} f(ng)\,dn$$

is absolutely convergent for all $f \in I^G_{M,P}(\sigma_\lambda)$ and $g \in G(F)$, and defines a (continuous) intertwining operator

$$J_{Q|P}(\sigma_\lambda) : I^G_{M,P,K}(\sigma_\lambda) \longrightarrow I^G_{M,Q,K}(\sigma_\lambda).$$

We may transport this intertwining operator to an operator on the compact models

$$J_{Q|P}(\sigma_\lambda) : I^G_{M,P,K}(V_\sigma)_K \longrightarrow I^G_{M,Q,K}(V_\sigma)_K$$

which intertwines $I^G_{M,P,K}(\sigma_\lambda)_K$ with $I^G_{M,Q,K}(\sigma_\lambda)_K$. The resulting operator-valued function $\lambda \mapsto J_{Q|P}(\sigma_\lambda)$ has meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$. (This is [Wal92, Theorem 10.1.6] for real groups and [Wal03, p. IV.1.1] for $p$-adic groups.)

There exists a scalar meromorphic function $\lambda \mapsto r_{Q|P}(\sigma_\lambda)$ on $\mathfrak{a}^*_{M,\mathbb{C}}$, called a scalar normalising factor, such that the function

$$\lambda \longmapsto R_{Q|P}(\sigma_\lambda) \coloneqq r_{Q|P}(\sigma_\lambda)^{-1} J_{Q|P}(\sigma_\lambda)$$

is a holomorphic operator valued function on $\mathfrak{a}^*_{M,\mathbb{C}}$. Scalar normalising factors are not unique. They can be chosen to satisfy several conditions, including the conditions (R1)–(R8) of [Art94a, Theorem 2.1]. We will make use of some of these properties below and will recall them there.

We will often simply use the notation $I^G_{M,P}$ to refer to any of the parabolic induction functors, allowing context to disambiguate them. Moreover, this causes no harm especially since we are for the most part only concerned with infinitesimal equivalence classes of representations.

## 2.5.2 Parabolic descent

Let $(P, M)$ be a parabolic pair. For $f \in \mathcal{C}_{(c)}(G, \zeta)$ (assuming $\zeta$ is unitary in the case of Schwartz functions), one defines $f^{(P)} : M(F) \to \mathbb{C}$ by

$$f^{(P)}(m) = \delta_P(m)^{1/2} \int_{N_P} \int_K f(k^{-1}mnk)\, dn\, dk.$$

Then $f^{(P)} \in \mathcal{C}_{(c)}(M, \zeta)$. Note that $f^{(P)}$ depends on the choice of $K$. This results in a continuous linear operator $\mathcal{C}_{(c)}(G, \zeta) \to \mathcal{C}_{(c)}(M, \zeta)$ called parabolic descent.

If $\sigma$ is a finite-length admissible representation of $M(F)$ with $\mathcal{Z}$-character $\zeta$, then

$$\langle \Theta_\sigma, f^{(P)} \rangle = \langle \Theta_{I^G_{M,P}\sigma}, f \rangle$$

for all $f \in C_c^\infty(G,\zeta)$. If $\sigma$ is tempered, then this holds for all $f \in \mathcal{C}(G,\zeta)$.

The parabolic descent map $f \mapsto f^{(P)}$ descends to a continuous map $\mathcal{I}_{(c)}(G,\zeta) \to \mathcal{I}_{(c)}(M,\zeta)^{W^G(M)}$, which we write as $f_G \mapsto f_M$ and also call parabolic descent. Let $\Gamma_{G\text{-rs}}(M)$ denote the set of $G$-regular semisimple conjugacy classes in $M(F)$. For all $\gamma \in \Gamma_{G\text{-rs}}(M)$ we have $f_M(\gamma) = f_G(\gamma)$. Consequently, although $f^{(P)}$ depends on the choice of $P$ and $K$, the function $f_M$ does not. For $f \in \mathcal{C}(G,\zeta)$, we have that $f$ is cuspidal if and only if $f_M = 0$ for all $M \in \mathcal{L}^G(M_0)$ with $M \neq G$.

The transpose of parabolic descent $\mathcal{I}_{(c)}(G,\zeta) \to \mathcal{I}_{(c)}(M,\zeta)^{W^G(M)}$ is a continuous linear map $I_M^G : \mathcal{I}_{(c)}(M,\zeta)'/W^G(M) \to \mathcal{I}_{(c)}(G,\zeta)'$, which we call parabolic induction since it extends the parabolic induction of characters. If $f \in \mathcal{C}_{\text{cusp}}(G,\zeta)$, then every invariant distribution that is parabolically induced from a proper Levi of $G$ annihilates $f$.

### 2.5.3 The Langlands and Harish-Chandra classifications

The Langlands classification gives a classification of $\Pi(G)$ in terms of the sets $\Pi_{\text{temp}}(M)$ for $M$ a Levi subgroup of $G$. For a parabolic pair $(P,M)$, we define $(\mathfrak{a}_M^*)^{P,+} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle > 0, \forall \alpha \in \Delta(P, A_M)\}$. A Langlands datum is defined to be a triple $((P,M),\sigma,\sigma,\lambda)$, where $(P,M)$ is a parabolic pair, $\sigma \in \Pi_{\text{temp}}(M)$, and $\lambda \in (\mathfrak{a}_M^*)^{P,+}$. Given a Langlands datum $((P,M),\sigma,\lambda)$, the equivalence class of $I_{M,P}^G(\sigma_\lambda)$ is uniquely determined and called a standard representation of $G(F)$. The Langlands classification is the following theorem, due to Langlands for real groups [Lan89] and Silberger, Borel–Wallach, and Konno for $p$-adic groups [Kon03].

**Theorem 2.5.1.** *1. For each Langlands datum $((P,M),\sigma,\lambda)$, the induced representation $I_{M,P}^G(\sigma_\lambda)$ is indecomposable and has a unique irreducible quotient $J((P,M),\sigma,\lambda) \in \Pi(G)$, called the Langlands quotient.*

*2. The map $((P,M),\sigma,\lambda) \mapsto J((P,M),\sigma,\lambda)$ is a bijection from the set of $G(F)$-conjugacy classes of Langlands data to $\Pi(G)$.*

The Harish-Chandra classification is a classification of $\Pi_{\text{temp}}(G)$ in terms of the sets $\Pi_2(M)$ for $M$ a Levi subgroup of $G$. Let $M$ be a Levi subgroup of $G$ and let $P \in \mathcal{P}(M)$. Let $\sigma$ be a finite length admissible representation of $M(F)$. Then $I_{M,P}^G(\sigma)$ is a finite length admissible representation of $M(F)$. Suppose that $\sigma$ is unitary. Then so is $I_{M,P}^G(\sigma)$ and it is thus a finite direct sum of irreducible unitary representations of $G(F)$. Its isomorphism class does not depend on $P$ and we will thus omit it from the notation. Moreover, the isomorphism class of $I_M^G(\sigma)$ only depends on the

$G(F)$-conjugacy class of the pair $(M, \sigma)$. Let $\Pi_{(M,\sigma)}(G)$ denote the set of (equivalence classes of) irreducible subrepresentations of $I_M^G(\sigma)$. Then $\Pi_{(M,\sigma)}(G) = \Pi_{g\cdot(M,\sigma)}(G)$ for all $g \in G(F)$, and we may write this set as $\Pi_{G(F)\cdot(M,\sigma)}(G)$. If $\sigma \in \Pi_{\text{temp}}(M)$, then $\mathcal{I}_M^G(\sigma)$ is tempered and therefore all of its subrepresentations are tempered. Harish-Chandra and Langlands proved the following.

**Theorem 2.5.2.** *We have the disjoint union*

$$\Pi_{\text{temp}}(G) = \coprod_{\substack{G(F)\cdot(M,\sigma),\\ \sigma\in\Pi_2(M)}} \Pi_{G(F)\cdot(M,\sigma)}(G).$$

Harish-Chandra proved that the union exhausts $\Pi_{\text{temp}}(G)$ and Langlands proved that the union is disjoint using Harish-Chandra's asymptotic estimates for matrix coefficients. We refer to this decomposition as Harish-Chandra's classification of tempered representations. (See [Art93, §1] for pointers to the literature.)

We may describe this decomposition using only semistandard Levi subgroups and orbits of pairs under the Weyl group $W_0^G$. Indeed, each Levi is conjugate to a semistandard Levi. The group $N_{G(F)}(M_0)$ acts on the set of pairs $(M, \sigma)$ consisting of a semistandard Levi subgroup $M$ of $G$ and $\sigma \in \Pi(M)$ by conjugation and descends to an action of $W_0^G = N_{G(F)}(M_0)/M_0(F)$. Suppose that $(M, \sigma), (M', \sigma')$ are two $G(F)$-conjugate pairs. and let $g \in G(F)$ with $(M', \sigma') = g \cdot (M, \sigma)$. Then there exists $m \in M(F)$ such that $gm \in N_{G(F)}(M_0)$. Since $m \cdot (M, \sigma) = (M, \sigma)$, we have $(M', \sigma') = (gm) \cdot (M, \sigma)$. Let $w \in W_0^G$ be the element represented by $(gm)$. Then $(M', \sigma') = w \cdot (M, \sigma)$. It follows that we have the disjoint union

$$\Pi_{\text{temp}}(G) = \coprod_{\substack{W_0^G\cdot(M,\sigma),\\ M\in\mathcal{L}^G(M_0),\, \sigma\in\Pi_2(M)}} \Pi_{W_0^G\cdot(M,\sigma)}(G).$$

The parametrisation of the elements of $\Pi_{\text{temp}}(G)$ thus boils down to parametrisation of the elements of $\Pi_2(M)$ and the elements of the finite sets $\Pi_{G(F)\cdot(M,\sigma)}(G)$. The latter problem is solved by the theory of R-groups (reducibility groups) introduced by Knapp and Stein, and these play a crucial role in invariant harmonic analysis.

# 3 Invariant Harmonic Analysis

In this Chapter we recall the invariant Paley–Wiener theorems that we will stabilise. These invariant Paley–Wiener theorems are formulated in terms of Arthur's virtual tempered representations, which are in turn based on the theory of the $R$-group.

## 3.1 Theory of the $R$-group

Recall that we have fixed a maximal compact subgroup $K$ of $G(F)$ that is in good position relative to $M_0$. For $M \in \mathcal{L}^G(M_0)$ and $\sigma$ a Hilbert space representation of $M(F)$, we use $K$ to form compact models of the Hilbert spaces for the (normalised) parabolically induced representations $I^G_{M,P}(\sigma) = I^G_{M,P,K}(\sigma)$. Since $M$ is uniquely determined by any $P \in \mathcal{P}(M_0)$ and since we are only considering parabolic induction to $G(F)$, we will write $I_P = I^G_{M,P}$ for brevity.

Let $\sigma$ be a discrete series representation of $M(F)$. Then the representation $I_P(\sigma)$ decomposes into a finite direct sum of irreducible tempered representations of $G(F)$ and the theory of the $R$-group introduced by Knapp and Stein gives a parametrisation of the irreducible summands of $I_P(\sigma)$. We follow the presentation in [MW18, §1.9–1.11] with some minor modifications.

We write $\widetilde{w} \in N_{G(F)}(M)$ for an element with image $w \in W^G(M)$. We have a unitary intertwining operator

$$\widetilde{w} : I_P(\sigma) \longrightarrow I_{w \cdot P}(\widetilde{w} \cdot \sigma)$$

defined by $\phi \mapsto \phi(\widetilde{w}^{-1} \cdot)$. Let $N_{G(F)}(M)_\sigma = \{\widetilde{w} \in N_{G(F)}(M) : \widetilde{w} \cdot \sigma \cong \sigma\}$. Suppose that $\widetilde{w} \in N_{G(F)}(M)_\sigma$. For each unitary intertwining operator $A : \widetilde{w} \cdot \sigma \to \sigma$, we have a unitary intertwining operator

$$I_{w \cdot P}(A) : I_{w \cdot P}(\widetilde{w} \cdot \sigma) \longrightarrow I_{w \cdot P}(\sigma)$$

since $I_{w\cdot P}$ is a functor.

We define the unitary intertwining operator $R_P(A,\widetilde{w}) : I_P(\sigma) \to I_P(\sigma)$ to be the composition $R_P(A,\widetilde{w}) = R_{P|w\cdot P}(\sigma) \circ I_{w\cdot P}(A) \circ \widetilde{w}$:

$$R_P(A,\widetilde{w}) : I_P(\sigma) \xrightarrow{\widetilde{w}} I_{w\cdot P}(\widetilde{w}\cdot\sigma) \xrightarrow{I_{w\cdot P}(A)} I_{w\cdot P}(\sigma) \xrightarrow{R_{P|w\cdot P}(\sigma)} I_P(\sigma).$$

Recall that $R_{P|w\cdot P}(\sigma) : I_{w\cdot P}(\sigma) \to I_P(\sigma)$ is a normalised standard intertwining operator, which depends on a choice of scalar normalising factors $r_{Q|P}(\sigma)$.

Define $\mathcal{N}^G(\sigma)$ to be the set of pairs $(A,\widetilde{w})$ where $\widetilde{w} \in N_{G(F)}(M)_\sigma$ and $A : \widetilde{w}\cdot\sigma \to \sigma$ is a unitary intertwining operator. The set $\mathcal{N}^G(\sigma)$ is naturally a group with multiplication defined by $(A_1,\widetilde{w_1})(A_2,\widetilde{w_2}) = (A_1A_2,\widetilde{w_1}\widetilde{w_2})$. The map $M(F) \to \mathcal{N}^G(\sigma), m \mapsto (\sigma(m), m)$ is an injective homomorphism with normal image, and we denote the resulting quotient group by $\mathcal{W}^G(\sigma) = \mathcal{N}^G(\sigma)/M(F)$. Note that for each $(A,\widetilde{w}) \in \mathcal{N}^G(\sigma)$, the intertwining operator $I_{w\cdot P}(A) \circ \widetilde{w} : I_P(\sigma) \to I_{w\cdot P}(\sigma)$, and thus also $R_P(A,\widetilde{w})$, only depends on the image of $(A,\widetilde{w})$ in $\mathcal{W}^G(\sigma)$. The map $R_P$ from $\mathcal{W}^G(\sigma)$ to the group of unitary intertwining operators on $I_P(\sigma)$ is a homomorphism. This follows from the following facts.

1. For each $(A,\widetilde{w}) \in \mathcal{N}^G(\sigma)$ and $P, Q \in \mathcal{P}^G(M)$, we have

$$\widetilde{w} \circ R_{Q|P}(\sigma) = R_{w\cdot Q|w\cdot P}(\widetilde{w}\cdot\sigma) \circ \widetilde{w}.$$

2. For each unitary intertwining operator $A : \sigma' \to \sigma$ unitary representations $\sigma', \sigma$ of $M(F)$ and for each $P, Q \in \mathcal{P}^G(M)$, we have

$$I_Q(A) \circ R_{Q|P}(\sigma') = R_{Q|P}(\sigma) \circ I_P(A).$$

3. For each $(A_1,\widetilde{w_1}), (A_2,\widetilde{w_2}) \in \mathcal{W}^G(\sigma)$, we have

$$(I_{w_1w_2\cdot P}(A_1) \circ \widetilde{w_1}) \circ (I_{w_2\cdot P}(A_2) \circ \widetilde{w_2}) = I_{w_1w_2\cdot P}(A_1A_2) \circ \widetilde{w_1}\widetilde{w_2}.$$

4. For each $P, Q, S \in \mathcal{P}^G(M)$, we have

$$R_{S|Q}(\sigma) \circ R_{Q|P}(\sigma) = R_{S|P}(\sigma).$$

Items 1, 2, and 4 rely on the choice of scalar normalising factors used to define the normalised standard intertwining operators. Applying these facts in turn, we obtain for all $(A_1, \widetilde{w_1}), (A_2, \widetilde{w_2}) \in \mathcal{W}^G(\sigma)$ that

$$\begin{aligned}
&R_P(A_1, \widetilde{w_2}) \circ R_P(A_2, \widetilde{w_2}) \\
&= R_{P|w_1 \cdot P}(\sigma) \circ I_{w_1 \cdot P}(A_1) \circ \widetilde{w_1} \circ R_{P|w_2 \cdot P}(\sigma) \circ I_{w_2 \cdot P}(A_2) \circ \widetilde{w_2} \\
&= R_{P|w_1 \cdot P}(\sigma) \circ I_{w_1 \cdot P}(A_1) \circ R_{w_1 \cdot P|w_1 w_2 \cdot P}(\widetilde{w_1} \cdot \sigma) \circ \widetilde{w_1} \circ I_{w_2 \cdot P}(A_2) \circ \widetilde{w_2} \\
&= R_{P|w_1 \cdot P}(\sigma) \circ R_{w_1 \cdot P|w_1 w_2 \cdot P}(\sigma) \circ I_{w_1 w_2 \cdot P}(A_1) \circ \widetilde{w_1} \circ I_{w_2 \cdot P}(A_2) \circ \widetilde{w_2} \\
&= R_{P|w_1 w_2 \cdot P}(\sigma) \circ I_{w_1 w_2 \cdot P}(A_1 A_2) \circ \widetilde{w_1}\widetilde{w_2} \\
&= R_P(A_1 A_2, \widetilde{w_1}\widetilde{w_2}).
\end{aligned}$$

Thus, $R_P$ is indeed a homomorphism from $\mathcal{W}^G(\sigma)$ to the group of unitary intertwining operators on $I_P(\sigma)$.

For each $z \in \mathbb{C}^1$, we have that $(z, 1)$ lies in the centre of $\mathcal{N}^G(\sigma)$. Note that the maps $\mathbb{C}^1 \to \mathcal{N}^G(\sigma)$ and $\mathbb{C}^1 \to \mathcal{W}^G(\sigma)$ are injective homomorphisms. Let

$$\begin{aligned}
W^G(\sigma) &= \{w \in W^G(M) : w \cdot \sigma \cong \sigma\} \\
&\cong \{w \in W_0^G : w \cdot M = M, w \cdot \sigma \cong \sigma\}/W_0^M.
\end{aligned}$$

We have a canonical surjective homomorphism $\mathcal{W}^G(\sigma) \to W^G(\sigma)$ with kernel $\mathbb{C}^1$. Thus, we have the central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathcal{W}^G(\sigma) \longrightarrow W^G(\sigma) \longrightarrow 1.$$

Consequently, $\mathcal{W}^G(\sigma)$ is naturally a compact group. Moreover, $R_P$ is a (continuous) unitary representation of $\mathcal{W}^G(\sigma)$ on $I_P(V_\sigma)$ through which $\mathbb{C}^1$ acts by multiplication.

The definition of $R_P$ depends in a simple manner on the choice of normalising factors and the parabolic subgroup $P \in \mathcal{P}^G(M)$. Consider a different choice of normalising factors and the resulting a representation $\underline{R}_P$ of $\mathcal{W}^G(\sigma)$. There exists a unitary character $\chi$ of $W^G(\sigma)$ such that for all $(A, \widetilde{w}) \in \mathcal{W}^G(\sigma)$ we have $\underline{R}_P(A, \widetilde{w}) = \chi(A, \widetilde{w}) R_P(A, \widetilde{w})$. We can define an automorphism $\alpha_\chi$ of $\mathcal{W}^G(\sigma)$ by $\alpha_\chi(A, \widetilde{w}) = (\chi(A, \widetilde{w})A, \widetilde{w})$. Then $\underline{R}_P = R_P \circ \alpha_\chi$. If $P' \in \mathcal{P}^G(M)$ is another choice of parabolic, then $R_{P|P'}(\sigma) \circ R_{P'}(A, \widetilde{w}) = R_P(A, \widetilde{w}) \circ R_{P|P'}(\sigma)$ for all $w \in \mathcal{W}^G(\sigma)$, so $R_{P'}$ is unitarily equivalent to $R_P$. Consequently, the kernel $W^G(\sigma)_0$ of $R_P$ is independent of the choice of normalising factors and parabolic $P \in \mathcal{P}^G(M)$.

The subgroup $W^G(\sigma)_0$ of $\mathcal{W}^G(\sigma)$ injects into $W^G(\sigma)$, and we also denote its image by $W^G(\sigma)_0$. We define $\mathcal{R}^G(\sigma) = \mathcal{W}^G(\sigma)/W^G(\sigma)_0$. The Knapp–Stein $R$-group of $\sigma$ is defined to be $R^G(\sigma) = W^G(\sigma)/W^G(\sigma)_0$. We thus have a central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathcal{R}^G(\sigma) \longrightarrow R^G(\sigma) \longrightarrow 1.$$

Since $\mathbb{C}^1$ acts by multiplication through $R_P$, the representation $R_P$ of $\mathcal{W}^G(\sigma)$ descends to a representation $R_P$ of $\mathcal{R}^G(\sigma)$ through which $\mathbb{C}^1$ acts by multiplication.

We denote by $\Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})$ the set of irreducible representations of $\mathcal{R}^G(\sigma)$ through which $\mathbb{C}^1$ acts by multiplication. Let $\mathcal{R}_P$ denote the representation of $\mathcal{R}^G(\sigma) \times G(F)$ on $I_P^G(V_\sigma)$ defined by $\mathcal{R}_P(\widetilde{r}, x) = R_P(\widetilde{r}) I_P(\sigma, x)$.

Harish-Chandra's commuting algebra theorem ([Har76, Theorem 38.1] for real groups and [Sil79, Theorem 5.5.3.2] for $p$-adic groups) states that the operators $\{R_P(\widetilde{r}) : \widetilde{r} \in \mathcal{R}^G(\sigma)\}$ span $\mathrm{End}(I_P(\sigma))$. The dimension theorem (due to Knapp–Stein for real groups [KS75, Theorem 2], [KS80, Theorem 13.4] and Silberger for $p$-adic groups [Sil78]) states that the dimension of $\mathrm{End}(I_P(\sigma))$ is the cardinality of $\mathcal{R}^G(\sigma)/\mathbb{C}^1 = R^G(\sigma)$, and thus

$$\mathrm{End}(I_P(\sigma)) = \bigoplus_{\mathbb{C}^1\widetilde{r} \in \mathcal{R}^G(\sigma)/\mathbb{C}^1} \mathbb{C} R_P(\widetilde{r}).$$

As a corollary, one obtains the main theorem of the theory of $R$-groups, namely that that there is a bijection $\rho \mapsto \pi_\rho$ between $\Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})$ and the set $\Pi_\sigma(G)$ of irreducible summands of $I_P^G(\sigma)$ characterised by the decomposition

$$\mathcal{R} \cong \bigoplus_{\rho \in \Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})} \rho \boxtimes \pi_\rho.$$

If $P' \in \mathcal{P}^G(M)$, then $R_{P|P'}(\sigma)$ intertwines the representation $\mathcal{R}_{P'}$ with $\mathcal{R}_P$, so the bijection $\rho \mapsto \pi_\rho$ is independent of the choice of $P \in \mathcal{P}^G(M)$. Consider a different choice of normalising factors and the resulting representation $\underline{R}_P$, which is of the form $\underline{R}_P = \chi R_P$ for a unitary character $\chi$ of $W^G(\sigma)$ as above. The resulting representation $\underline{\mathcal{R}}_P \coloneqq \underline{R}_P I_P(\sigma) = \chi \mathcal{R}_P$ of $\mathcal{R}^G(\sigma) \times G(F)$ decomposes as

$$\bigoplus_{\rho \in \Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})} \chi\rho \boxtimes \pi_\rho,$$

and therefore determines the bijection $\rho \mapsto \pi_{\chi^{-1}\rho}$.

The group $\mathcal{N}^G(\sigma)$ and the related objects depend on the representation $\sigma$ itself, not just its isomorphism class. This point is often glossed over in the literature. However, suppose that $T : \sigma \to \sigma'$ is an isomorphism of unitary representations.

Then we obtain an isomorphism

$$\begin{aligned} T : \mathcal{N}^G(\sigma) &\longrightarrow \mathcal{N}^G(\sigma') \\ (A, \widetilde{w}) &\longmapsto (T \circ A \circ T^{-1}, \widetilde{w}) \end{aligned}$$

that induces isomorphisms $\mathcal{W}^G(\sigma) \to \mathcal{W}^G(\sigma')$, $W^G(\sigma)_0 \to W^G(\sigma')_0$, $\mathcal{R}^G(\sigma) \to \mathcal{R}^G(\sigma')$, and $R^G(\sigma) \to R^G(\sigma')$, which we also denote by $T$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}^G(\sigma) & \xrightarrow{\;T\;} & \mathcal{R}^G(\sigma') \\ \downarrow{\scriptstyle R_P} & & \downarrow{\scriptstyle R_P} \\ I_P(\sigma) & \xrightarrow{\;I_P(T)\;} & I_P(\sigma') \end{array}$$

## 3.2 Arthur's virtual tempered representations

Fix a unitary central datum $(\mathcal{Z}, \zeta)$ and let $f \in \mathcal{C}(G, \zeta)$. The invariant Fourier transform $f_G : \Pi_{\text{temp}}(G, \zeta) \to \mathbb{C}$ of $f$ has a unique extension to a $\mathbb{C}$-linear form $f_G : D_{\text{temp}}(G, \zeta) \to \mathbb{C}$ on the space of virtual tempered representations $D_{\text{temp}}(G, \zeta)$. The invariant Fourier transform is just the representation of this linear form $f_G$ with respect to the basis $\Pi_{\text{temp}}(G, \zeta)$ of $D_{\text{temp}}(G, \zeta)$. In [Art93], Arthur defines a set of virtual tempered representations $\{\pi_\tau\}_{\tau \in T_{\text{temp}}(G,\zeta)}$ parametrised by a set $T_{\text{temp}}(G, \zeta)$ of $W_0^G$-orbits of equivalence classes of certain triples. These virtual representations arise naturally in the theory of the $R$-group. There is a natural action of $\mathbb{C}^1$ on $T_{\text{temp}}(G, \zeta)$, and the assignment $\tau \mapsto \pi_\tau$ is equivariant with-respect to the action. That is, $\pi_{z\tau} = z\pi_\tau$ for all $z \in \mathbb{C}^1$ and $\tau \in T_{\text{temp}}(G, \zeta)$. A choice of representative $\tau$ for each $\mathbb{C}^1$-orbit $\mathbb{C}^1\tau$ in $T_{\text{temp}}(G, \zeta)/\mathbb{C}^1$ gives a basis $\{\pi_\tau\}_{\mathbb{C}^1\tau \in T_{\text{temp}}(G,\zeta)/\mathbb{C}^1}$ of $\mathbb{C}\Pi_{\text{temp}}(G)$. For some purposes, the virtual tempered representations $\{\pi_\tau\}_{\tau \in \widetilde{T}_{\text{temp}}(G,\zeta)}$ are more natural than $\Pi_{\text{temp}}(G, \zeta)$ and can be regarded as the spectral objects that play the role dual to conjugacy classes in invariant harmonic analysis (cf. [Art94a]). It is useful to view the Fourier transform of $f$ as the $\mathbb{C}^1$-equivariant function $f_G : T_{\text{temp}}(G, \zeta) \to \mathbb{C}$ defined by $f_G(\tau) = f_G(\pi_\tau)$. This representation of the Fourier transform allows for more natural formulations of Paley–Wiener theorems.

We will review what we need to know about Arthur's virtual tempered representations. The original reference is [Art93] and other useful references in the work of Arthur are [Art94b; Art94a; Art96]. We follow the presentation in [MW18, §1.9–1.11] with some minor modifications.

Consider the set of triples $\tau = (M, \sigma, \widetilde{r})$, where $M \in \mathcal{L}^G(M_0)$, $\sigma$ is a discrete series representation of $M(F)$, and $\widetilde{r} \in \mathcal{R}^G(\sigma)$. We define two triples $(M, \sigma, \widetilde{r})$ and $(M, \sigma', \widetilde{r}')$ to be equivalent if there exists a unitary isomorphism $T : \sigma \to \sigma'$ such that $T(\widetilde{r}) = \widetilde{r}'$.

Let $g \in N_{G(F)}(M_0)$ and consider $g{\cdot}M \in \mathcal{L}^G(M_0)$ and $g{\cdot}\sigma$. We have an isomorphism

$$\begin{aligned} \mathcal{N}^G(\sigma) &\longrightarrow \mathcal{N}^G(g \cdot \sigma) \\ (A, \widetilde{w}) &\longmapsto g \cdot (A, \widetilde{w}) = (A, g\widetilde{w}g^{-1}) \end{aligned}$$

and this determines isomorphisms $\mathcal{W}^G(\sigma) \to \mathcal{W}^G(g \cdot \sigma)$, $W^G(\sigma) \to W^G(g \cdot \sigma)$, $\mathcal{R}^G(\sigma) \to \mathcal{R}^G(g \cdot \sigma)$, and $R^G(\sigma) \to R^G(g \cdot \sigma)$. Thus, the group $N_{G(F)}(M_0)$ acts on the set of triples $\tau = (M, \sigma, \widetilde{r})$ by $g \cdot \tau = (g \cdot M, g \cdot \sigma, g \cdot \widetilde{r})$. This descends to an action on the set of equivalence classes of triples through which $M_0(F)$ acts trivially. Thus, we obtain an action of $W_0^G$ on the set of equivalence classes of triples.

Let $\tau = (M, \sigma, \widetilde{r})$ be a triple and recall the representation $\mathcal{R}$ from the previous section, with its decomposition

$$\mathcal{R} \cong \bigoplus_{\rho \in \Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})} \rho \boxtimes \pi_\rho.$$

Define the virtual tempered representation

$$\pi_\tau = \sum_{\rho \in \Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})} \mathrm{tr}(\rho(\widetilde{r}))\pi_\rho.$$

The invariant tempered distribution character of $\pi_\tau$ is given by

$$\Theta_\tau(f) = \mathrm{tr}(R_P(\widetilde{r})\mathcal{I}_P(f)) = \sum_{\rho \in \Pi(\mathcal{R}^G(\sigma), \mathrm{id}_{\mathbb{C}^1})} \mathrm{tr}(\rho(\widetilde{r}))\Theta_{\pi_\rho}(f), \quad f \in \mathcal{C}(G).$$

The irreducible tempered representations $\pi_\rho$ all have the central character $\zeta_\sigma$. We write $\zeta_\tau = \zeta_\sigma$ and call it the central character of $\tau$. In particular, $\tau$ has a well-defined $\mathcal{Z}$-character, namely the restriction $\zeta_\tau|_{\mathcal{Z}}$, and $\pi_\tau \in D_{\mathrm{temp}}(G, \zeta_\tau|_{\mathcal{Z}})$.

In the archimedean case, the representation $I_P(\sigma)$ has an infinitesimal character obtained from $\mu_\sigma$ in the manner described in our review of parabolic induction. Thus, the irreducible tempered representations $\pi_\rho$ have all have the same infinitesimal character, which we call the infinitesimal character of $\tau$ and denote by $\mu_\tau$.

The virtual tempered representation $\pi_\tau$ only depends on the $W_0^G$-orbit of the equivalence class of $\tau$. Moreover, $\pi_\tau$ does not depend on the choice of parabolic $P \in \mathcal{P}(M)$, but it does depend on the choice of normalising factors. A different choice of normalising factors results in the virtual tempered representation $\chi(\widetilde{r})\pi_\tau$ for some unitary character $\chi$ of $W^G(\sigma)$. Thus, $\pi_\tau$ is uniquely determined up to multiplication by an element of $\mathbb{C}^1$.

The group $\mathbb{C}^1$ acts on the set of triples by $z(M, \sigma, z\widetilde{r}) = (M, \sigma, z\widetilde{r})$, and this descends to an action on equivalence classes that commutes with the action of $W_0^G$. The map $\tau \mapsto \pi_\tau$ is equivariant with respect to the action of $\mathbb{C}^1$ action: $\pi_{z\tau} = z\pi_\tau$.

Consider an equivalence class of triples $\tau$. It can happen that there exists a nontrivial $z \in \mathbb{C}^1$ such that $z\tau = w \cdot \tau$ for some $w \in W_0^G$. When this happens, we have $z\pi_\tau = \pi_{z\tau} = \pi_\tau$, and thus $\pi_\tau = 0$. One says that $\tau$ is essential if this does not happen. The set of essential equivalence classes of triples is stable under the actions of $W_0^G$ and $\mathbb{C}^1$. We define $\widetilde{T}_{\text{temp}}(G, \zeta)$ to be the set of essential equivalence classes of triples $(M, \sigma, \widetilde{r})$ with $\mathcal{Z}$-character $\zeta$, and we define $T_{\text{temp}}(G, \zeta) = \widetilde{T}_{\text{temp}}(G, \zeta)/W_0^G$. When $\mathcal{Z}$ is trivial, we omit $\zeta$ from the notation. Arthur proved the following (cf. [MW18, Proposition 2.9]).

**Proposition 3.2.1.** *For each $\tau \in T_{\text{temp}}(G, \zeta)$, we have $\pi_\tau \neq 0$, and the space of virtual tempered representations of $G(F)$ is*

$$D_{\text{temp}}(G, \zeta) = \bigoplus_{\mathbb{C}^1\tau \in T_{\text{temp}}(G,\zeta)/\mathbb{C}^1} \mathbb{C}\pi_\tau.$$

In particular, the $\mathbb{C}^1$-equivariant map $T_{\text{temp}}(G) \to D_{\text{temp}}(G), \tau \mapsto \pi_\tau$ is injective. We will identify $\tau$ with $\pi_\tau$, and thus also $\Theta_\tau$.

Suppose that $L \in \mathcal{L}^G(M_0)$. We can relate $T_{\text{temp}}(L, \zeta)$ and $T_{\text{temp}}(G, \zeta)$ for $L \in \mathcal{L}^G(M_0)$ in the following way. If $M \in \mathcal{L}^L(M_0)$ and $\sigma$ be a discrete series representation of $M(F)$, then there is a natural injective homomorphism $\mathcal{R}^L(\sigma) \to \mathcal{R}^G(\sigma)$. This gives rise to a embedding

$$\iota_L^G : \widetilde{T}_{\text{temp}}(L, \zeta) \longrightarrow \widetilde{T}_{\text{temp}}(G, \zeta)$$

that is equivariant with respect to the action of $\mathbb{C}^1$ and the action of $W_0^L \subseteq W_0^G$. This map descends to a $\mathbb{C}^1$ equivariant map

$$\iota_L^G : T_{\text{temp}}(L, \zeta) \longrightarrow T_{\text{temp}}(G, \zeta).$$

The group $W^G(L)$ acts on $T_{\text{temp}}(L, \zeta)$ and this map is the quotient for this action. This map is compatible with parabolic induction in the following sense. If $\tau \in T_{\text{temp}}(L, \zeta)$, then

$$I_L^G(\pi_\tau) = \pi_{\iota_L^G(\tau)}.$$

### 3.2.1 Arthur's elliptic virtual tempered representations

We say that $\tau \in \widetilde{T}_{\mathrm{temp}}(G,\zeta)$ is elliptic if it does not lie in the image of $\iota_L^G : \widetilde{T}_{\mathrm{temp}}(L,\zeta) \to \widetilde{T}_{\mathrm{temp}}(G,\zeta)$ for any $L \neq G$. Let $\widetilde{T}_{\mathrm{ell}}(G,\zeta)$ denote the set of elliptic elements in $\widetilde{T}_{\mathrm{temp}}(G,\zeta)$ and let $T_{\mathrm{ell}}(G,\zeta) = \widetilde{T}_{\mathrm{ell}}(G,\zeta)/W_0^G$. We can also define elliptic triples as follows. Let $\tau = (M,\sigma,\widetilde{r}) \in \widetilde{T}_{\mathrm{temp}}(G,\zeta)$ and let $r \in R^G(\sigma)$ be the image of $\widetilde{r}$ in $R^G(\sigma)$. We define $W^G_{\mathrm{reg}}(\sigma)$ to be the set of $w \in W^G(\sigma)$ such that the space $\mathfrak{a}_M^w$ of $w$-invariants in $\mathfrak{a}_M$ is equal to $\mathfrak{a}_G$. Then $\tau$ is elliptic if and only if $W^G(\sigma)_0 = 1$ (in which case $R^G(\sigma) = W^G(\sigma)$, so $r \in W^G(\sigma)$) and $r \in W^G_{\mathrm{reg}}(\sigma)$ (i.e. $\mathfrak{a}_M^r = \mathfrak{a}_G$).

We have

$$\widetilde{T}_{\mathrm{temp}}(G,\zeta) = \coprod_{L \in \mathcal{L}^G(M_0)} \widetilde{T}_{\mathrm{ell}}(L,\zeta)$$

and

$$T_{\mathrm{temp}}(G,\zeta) = \left( \coprod_{L \in \mathcal{L}^G(M_0)} T_{\mathrm{ell}}(L,\zeta) \right)^{W_0^G} = \coprod_{L \in \mathcal{L}^G(M_0)/W_0^G} T_{\mathrm{ell}}(L,\zeta)/W^G(L).$$

We define $D_{\mathrm{ell}}(G,\zeta)$ to be the subspace of $D_{\mathrm{temp}}(G,\zeta)$ generated by $T_{\mathrm{ell}}(G,\zeta)$, that is,

$$D_{\mathrm{ell}}(G,\zeta) = \bigoplus_{\mathbb{C}^1\tau \in T_{\mathrm{ell}}(G,\zeta)/\mathbb{C}^1} \mathbb{C}\tau.$$

### 3.2.2 The space of Arthur's virtual tempered characters

For $\lambda \in i\mathfrak{a}_G^*$, the map

$$\begin{aligned} \mathcal{N}^G(\sigma) &\longrightarrow \mathcal{N}^G(\sigma_\lambda) \\ (A,n) &\longmapsto (Ae^{\langle \lambda, H_G(n)\rangle}, n) \end{aligned}$$

is an isomorphism. It descends to an isomorphism $\mathcal{R}^G(\sigma) \to \mathcal{R}^G(\sigma_\lambda), \widetilde{r} \mapsto \widetilde{r}_\lambda$ compatible with $R(\sigma) = R(\sigma_\lambda)$.

We obtain an action of $i\mathfrak{a}_G^*$ on the set of triples $\tau = (M,\sigma,\widetilde{r})$ defined by

$$(\lambda,\tau) \longmapsto \tau_\lambda = (M,\sigma_\lambda,\widetilde{r}_\lambda),$$

for all $\lambda \in i\mathfrak{a}_G^*$. This gives rise to an action of $i\mathfrak{a}_G^*$ on the set $\widetilde{T}_{\text{temp}}(G)$ of equivalence classes of triples, which preserves $\widetilde{T}_{\text{ell}}(G)$.

We have encountered three actions on $\widetilde{T}_{\text{ell}}(G)$, namely the actions of $i\mathfrak{a}_G^*$, $W_0^G$, and $\mathbb{C}^1$. Let us summarise these here. If $\tau = [(M, \sigma, (A, n))] \in \widetilde{T}_{\text{ell}}(G)$, $\lambda \in i\mathfrak{a}_G^*$, $w \in W_0^G$, and $z \in \mathbb{C}^1$, then

$$\begin{aligned} \tau_\lambda &= [(M, |\cdot|_M^\lambda \sigma, (|n|_G^\lambda A, n))] \\ w \cdot \tau &= [(wMw^{-1}, \sigma \circ \operatorname{Int}(w^{-1}), (A, wnw^{-1}))] \\ z \cdot \tau &= [(M, \sigma, (zA, n))]. \end{aligned}$$

Each pair of the above three actions commute with each other. We thus, have commuting actions of $i\mathfrak{a}_G^*$ and $\mathbb{C}^1$ on $T_{\text{ell}}(G)$.

We define

$$\widetilde{T}_{\text{ell}}(G)_{\mathbb{C}} = \widetilde{T}_{\text{ell}}(G) \times_{i\mathfrak{a}_G^*} \mathfrak{a}_{G,\mathbb{C}}^* = \widetilde{T}_{\text{ell}}(G) \times \mathfrak{a}_{G,\mathbb{C}}^* / \sim,$$

where $\sim$ is the equivalence relation generated by $(\tau_0, \lambda_0 + \lambda_1) \sim ((\tau_0)_{\lambda_0}, \lambda_1)$ for all $\tau_0 \in \widetilde{T}_{\text{ell}}(G)$, $\lambda_0 \in i\mathfrak{a}_G^*$, and $\lambda_1 \in \mathfrak{a}_{G,\mathbb{C}}^*$. For $\tau = (\tau_0, \lambda_1) \in \widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}$ and $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$, we define $\tau_\lambda = (\tau_0, \lambda_1 + \lambda)$. This defines an action of $\mathfrak{a}_{G,\mathbb{C}}^*$ on $\widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}$. The commuting actions of $W_0^G$ and $\mathbb{C}^1$ on $\widetilde{T}_{\text{ell}}(G)$ determine commuting actions on $\widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}$, which also commute with the action of $\mathfrak{a}_{G,\mathbb{C}}^*$. The injection $\widetilde{T}_{\text{ell}}(G) \to \widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}, \tau \mapsto (\tau, 0)$ is equivariant with respect to the actions of $i\mathfrak{a}_G^*$, $W_0^G$, and $\mathbb{C}^1$. We identify $\widetilde{T}_{\text{ell}}(G)$ with its image in $\widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}$. We define

$$T_{\text{ell}}(G)_{\mathbb{C}} = \widetilde{T}_{\text{ell}}(G)_{\mathbb{C}} / W_0^G = T_{\text{ell}}(G) \times_{i\mathfrak{a}_G^*} \mathfrak{a}_{G,\mathbb{C}}^*,$$

which inherits commuting actions of $\mathfrak{a}_{G,\mathbb{C}}^*$ and $\mathbb{C}^1$. The injection $\widetilde{T}_{\text{ell}}(G) \to \widetilde{T}_{\text{ell}}(G)_{\mathbb{C}}$ descends to an injection $T_{\text{ell}}(G) \to T_{\text{ell}}(G)_{\mathbb{C}}$, which is equivariant with respect to the actions of $i\mathfrak{a}_G^*$ and $\mathbb{C}^1$.

The action of $\mathfrak{a}_{G,\mathbb{C}}^*$ on $\Pi(G)$ extends to a linear action of $D_{\text{spec}}(G)$, and the action of $i\mathfrak{a}_G^*$ preserves $D_{\text{temp}}(G)$. The injection $T_{\text{ell}}(G) \to D_{\text{ell}}(G), \tau \mapsto \pi_\tau$ is equivariant with respect to the actions of $i\mathfrak{a}_G^*$ and $\mathbb{C}^1$. To extend this to $T_{\text{ell}}(G)_{\mathbb{C}}$, for each $\tau = (\tau_0, \lambda_1) \in T_{\text{ell}}(G)_{\mathbb{C}}$, we define the virtual representation $\pi_\tau \in D_{\text{spec}}(G)$ by $\pi_\tau = (\pi_{\tau_0})_{\lambda_1}$. The distribution character of $\tau$ is defined to be $\Theta_\tau = \Theta_{\pi_\tau}$, and thus $\Theta_\tau(x) = \Theta_{\tau_0}(x) e^{\langle \lambda_1, H_G(x) \rangle}$. The map $T_{\text{ell}}(G)_{\mathbb{C}} \to D_{\text{spec}}(G)_{\mathbb{C}}, \tau \mapsto \pi_\tau$ is injective and equivariant with respect to the actions of $\mathfrak{a}_{G,\mathbb{C}}^*$ and $\mathbb{C}^1$.

Let $\tau \in T_{\mathrm{ell}}(L)_{\mathbb{C}}$. We denote the isotropy subgroup of $\tau$ in $\mathfrak{a}^*_{L,\mathbb{C}}$ by $\mathfrak{a}^\vee_{L,\tau}$. We have

$$\mathfrak{a}^\vee_{L,F} \subseteq \mathfrak{a}^\vee_{L,\tau} \subseteq \widetilde{\mathfrak{a}}^\vee_{L,F}\,.$$

Thus, if $F$ is archimedean we have $\mathfrak{a}^\vee_{L,\tau} = 0$, and if $F$ is non-archimedean we have that $\mathfrak{a}^\vee_{L,\tau}$ is a full lattice in $i\mathfrak{a}^*_L$. The set $T_{\mathrm{ell}}(L)$ is naturally a smooth manifold with uncountably many connected components $i\mathfrak{a}^*_L \cdot \tau = i\mathfrak{a}^*_L/\mathfrak{a}^\vee_{L,\tau}$, which are Euclidean spaces if $F$ is archimedean and compact tori if $F$ is non-archimedean. The manifold $T_{\mathrm{ell}}(L)$ has uncountably many components because of the action of $\mathbb{C}^1$. Moreover, $T_{\mathrm{ell}}(L)_{\mathbb{C}}$ is the complexification of $T_{\mathrm{ell}}(L)$ with connected components $\mathfrak{a}^*_{L,\mathbb{C}} \cdot \tau = \mathfrak{a}^*_{L,\mathbb{C}}/\mathfrak{a}^\vee_{L,\tau}$. Since

$$T_{\mathrm{temp}}(G) = \coprod_{L \in \mathcal{L}^G(M_0)/W_0^G} T_{\mathrm{ell}}(L)/W^G(L),$$

we have that $T_{\mathrm{temp}}(G)$ is naturally a topological space. We say that a function on $T_{\mathrm{temp}}(G)$ is smooth if its pullback to each $T_{\mathrm{ell}}(L)$ is smooth.

### 3.2.3 The elliptic inner product

We recall the elliptic inner product from [MW18, p. 7.3]. For each $\gamma \in \Gamma_{\mathrm{sr,ell}}(G)$, define $m(\gamma) = \mathrm{vol}(G_\gamma(F)/A_G(F))$. There is a canonical measure on $\Gamma_{\mathrm{sr,ell}}(G)/A_G(F)$ defined by

$$\int_{\Gamma_{\mathrm{sr,ell}}(G)/A_G(F)} m(\gamma)^{-1} a(\gamma)\, d\gamma = \sum_T |W_F(G,T)|^{-1}\, \mathrm{vol}(T(F)/A_G(F)) \int_{T(F)/A_G(F)} a(t)\, dt$$

for all $a \in C_c(\Gamma_{\mathrm{sr,ell}}(G)/A_G(F))$, where $T$ runs over the conjugacy classes of elliptic maximal tori of $G$.

Let $\zeta$ be a unitary character of $A_G(F)$. For $\zeta$-equivariant functions $a, b : \Gamma_{\mathrm{ell}}(G) \to \mathbb{C}$, we define

$$\langle a, b\rangle_{\mathrm{ell}} = \int_{\Gamma_{\mathrm{sr,ell}}(G)/A_G(F)} a(\gamma)\overline{b(\gamma)}\, d\gamma,$$

provided the integral converges. For $\pi, \pi' \in \mathbb{C}\Pi_{\mathrm{ell}}(G,\zeta)$, we define

$$\langle \pi, \pi'\rangle_{\mathrm{ell}} = (|D^G|^{1/2}\Theta_\pi, |D^G|^{1/2}\Theta_{\pi'}).$$

For $\tau = (M, \sigma, \widetilde{r}) \in T_{\mathrm{ell}}(G)$, one defines $\iota(\tau) = |\det(1 - \widetilde{r})|_{\mathfrak{a}_M^G}|^{-1}$. We have the following orthogonality relations due to Arthur [MW18, Theorem 7.3].

**Theorem 3.2.2.** *Let $\tau, \tau' \in T_{\mathrm{ell}}(G,\zeta)$. We have*

$$\langle \tau, \tau' \rangle_{\mathrm{ell}} = |W^G(\tau)| \iota(\tau)^{-1} \delta_{\tau,\tau'}.$$

It follows that $\langle \cdot, \cdot \rangle_{\mathrm{ell}}$ extends to an inner product (positive-definite Hermitian) on $D_{\mathrm{ell}}(G,\zeta)$ and that the decomposition

$$D_{\mathrm{ell}}(G,\zeta) = \bigoplus_{\mathbb{C}^1\tau \in T_{\mathrm{ell}}(G,\zeta)/\mathbb{C}^1} \mathbb{C}\tau$$

is an orthogonal direct sum. It follows from the above theorem that $\{\|\tau\|_{\mathrm{ell}} : \tau \in T_{\mathrm{ell}}(G)\}$ is bounded. Note that when $\tau = \pi \in \Pi_2(G)$, that is, $\tau = (G, \pi, 1)$, we have $\|\pi\|_{\mathrm{ell}} = 1$.

We have $D_{\mathrm{ell}}(G) = \bigoplus_\zeta D_{\mathrm{ell}}(G,\zeta)$. Thus, we obtain an inner product $\langle \cdot, \cdot \rangle_{\mathrm{ell}}$ on $D_{\mathrm{ell}}(G)$ as the orthogonal direct sum of the inner products on the summands. This inner product is called the elliptic inner product.

## 3.3 Paley–Wiener and Schwartz spaces

In this section we define abstract Paley–Wiener spaces and Schwartz spaces of the sort needed for the invariant and stable harmonic analysis. See [MW16a, Ch. IV] for a similar presentation of abstract Paley–Wiener spaces suitable for invariant and stable harmonic analysis on a real group.

Let $V$ be a Euclidean space and let $\Lambda = \{\Lambda_e\}_{e \in E}$ be a countable family of one of the following types:

1. archimedean type: for each $e \in E$, $\Lambda_e = iV^*$ equipped with a non-negative real number $\|e\|$.

2. non-archimedean type: for each $e \in E$, $\Lambda_e = iV^*/\Gamma_e^\vee$, where $\Gamma_e$ is a lattice in $V$ and $\Gamma_e^\vee = \mathrm{Hom}(\Gamma_e, 2\pi i\mathbb{Z})$. Thus, $\Lambda_e$ is a compact torus of dimension $\dim V$.

The terminology comes from the types of spaces that appear in Paley–Wiener theorems for $G$ when $F$ is archimedean or non-archimedean. We will also use $\Lambda$ denote the disjoint union $\Lambda = \coprod_{e \in E} \Lambda_e$. The function spaces we define will be spaces of certain smooth functions on $\Lambda$.

Note that for each $e \in E$, the space $\Lambda_e$ has a natural complexification $\Lambda_{e,\mathbb{C}}$, which is $V_{\mathbb{C}}^*$ in the archimedean case and $V_{\mathbb{C}}^*/\Gamma_e^\vee$ in the non-archimedean case. Thus, the space $\Lambda$ has a natural complexification $\Lambda_{\mathbb{C}} = \coprod_{e \in E} \Lambda_{e,\mathbb{C}}$. For each $e \in E$, we extend the inner product on $V$ to a Hermitian inner product on $V_{\mathbb{C}}$. Note that a function $\varphi : \Lambda \to \mathbb{C}$ can be identified with a family of functions $\{\varphi_e\}_{e\in E}$.

We define the Paley–Wiener space $PW(\Lambda)$ on $\Lambda$ to be the vector space of smooth functions $\varphi : \Lambda \to \mathbb{C}$ such that the following hold.

1. In the non-archimedean case we require that $\varphi$ is supported on finitely many connected components $\Lambda_e$.

2. $\varphi$ extends to an entire function on $\Lambda_{\mathbb{C}}$.

3. $\varphi$ satisfies a growth condition:

   (a) in the archimedean case we require that there exists $r > 0$ such that for all $N \in \mathbb{N}$ we have that

   $$\|\varphi\|_{r,N} := \sup_{e\in E, \lambda\in\Lambda_{e,\mathbb{C}}} |\varphi(\lambda)|(1 + \|e\| + \|\lambda\|)^N e^{-r\|Re(\lambda)\|}$$

   is finite;

   (b) In the non-archimedean case we require that there exists $r > 0$ such that

   $$\|\varphi\|_r := \sup_{\lambda\in\Lambda_{\mathbb{C}}} |\varphi(\lambda)|e^{-r\|Re(\lambda)\|}$$

   is finite.

We now define a locally convex topology on $PW(\Lambda)$. Suppose first that we are in the archimedean case. For each $r > 0$, we define $PW^r(\Lambda)$ to be the subspace of all $\varphi \in PW(\Lambda)$ such that $\|\varphi\|_{r,N} < \infty$ for all $N \in \mathbb{N}$, and we give $PW^r(\Lambda)$ the topology defined by the family of norms $\|\cdot\|_{r,N}$ with $N \in \mathbb{N}$. It is a Fréchet space. We give $PW(\Lambda) = \bigcup_r PW^r(\Lambda)$ the inductive limit topology in the category of locally convex spaces, making it a strict LF-space.

In general, we define $PW_f(\Lambda)$ to be the linear subspace of $PW(\Lambda)$ consisting of all $\varphi \in PW(\Lambda)$ that are supported on finitely many components $\Lambda_e$. Note that in the non-archimedean case we have $PW_f(\Lambda) = PW(\Lambda)$. We topologise $PW_f$ as follows. First, for each finite set $E_0 \subseteq E$, define $PW_{E_0}(\Lambda)$ to be the subspace of all $\varphi \in PW(\Lambda)$ that are supported on $\coprod_{e\in E_0} \Lambda_e$. For each $r > 0$, define $PW^r_{E_0}(\Lambda)$ to be the subspace of all $\varphi \in PW_{E_0}(\Lambda)$ such that $\|\varphi\|_r < \infty$ in the non-archimedean case and $\|\varphi\|_{r,N} < \infty$ for all $N \in \mathbb{N}$ in the archimedean case. We give $PW^r_{E_0}(\Lambda)$ the Banach space topology defined by the norm $\|\cdot\|_r$ in the non-archimedean case and the Fréchet space topology defined by the family of norms $\|\cdot\|_{r,N}$ in the archimedean case. We give $PW_{E_0}(\Lambda) = \bigcup_r PW^r_{E_0}(\Lambda)$ the inductive limit topology in the category of locally convex spaces, making it a strict LF-space. Finally, we give $PW_f(\Lambda) = \bigcup_{E_0} PW_{E_0}(\Lambda)$ the inductive limit topology in the category of locally convex spaces, making it also a strict LF-space. In the archimedean case, the topology on $PW_f(\Lambda)$ is at least as fine as the subspace topology inherited from $PW(\Lambda)$, that is, the injection $PW_f(\Lambda) \to PW(\Lambda)$ is continuous.

We have defined the topology on $PW_f(\Lambda)$ in an analogous way to how the topology is defined on $C_c^\infty(G, K)$. A simpler way of describing the topology on $PW_f(\Lambda)$ is as follows. We write $PW_e(\Lambda) = PW_{\{e\}}(\Lambda)$. The subspace $PW_e(\Lambda)$ is the classical Paley–Wiener space $PW(\Lambda_e)$ on $\Lambda_e$. Observe that we have the locally convex direct sum decomposition $PW_{E_0}(\Lambda) = \bigoplus_{e\in E_0} PW_e(\Lambda)$, where $PW_e(\Lambda) = PW_{\{e\}}(\Lambda)$. This can be shown by checking that $PW_{E_0}(\Lambda)$ satisfies the correct universal property. Alternatively, this follows from the fact that $\bigoplus_{e\in E_0} PW_e(\Lambda)$ is a Fréchet space (even a Banach space in the non-archimedean case) and applying the a suitable version of the open mapping theorem to the continuous bijection $\bigoplus_{e\in E_0} PW_e(\Lambda) \to PW_{E_0}(\Lambda)$. Choosing an enumeration $e_1, e_2, \ldots$ of $E_0$, we have that

$$PW_f(\Lambda) = \bigcup_{n=1}^{\infty} PW_{\{e_1,\ldots,e_n\}}(\Lambda) = \bigcup_{n=1}^{\infty} \bigoplus_{i=1}^{n} PW_{e_i}(\Lambda)$$

with the inductive limit topology in the category of locally convex spaces, and consequently we have the locally convex direct sum decomposition

$$PW_f(\Lambda) = \bigoplus_{e\in E} PW_e(\Lambda).$$

To define the Schwartz space $\mathscr{S}(\Lambda)$ on $\Lambda$, in the archimedean case we need to make use of certain constant coefficient differential operators on $\Lambda$. Recall in the archimedean case we have an identification of each $\Lambda_e$ with a fixed vector space $iV^*$. Thus, we have a notion of a differential operator on $\Lambda$ that is the same on almost all (all but finitely many) components $\Lambda_e$.

We define the Schwartz space $\mathscr{S}(\Lambda)$ on $\Lambda$ to be the space of smooth functions $\varphi : \Lambda \to \mathbb{C}$ such that the following holds.

1. In the non-archimedean case we require that $\varphi$ is supported on finitely many connected components $\Lambda_e$.

2. $\varphi$ satisfies a decay condition:

(a) In the archimedean case, we require that for each differential operator $D$ on $\Lambda$ that is the same on almost all components and for each $N \in \mathbb{N}$, we have that

$$\|\varphi\|_{D,N} \coloneqq \sup_{e\in E, \lambda\in\Lambda_e} |D\varphi(\lambda)|(1 + \|e\| + \|\lambda\|)^N$$

is finite.

(b) In the non-archimedean case, we require that for each differential operator $D$ on $\Lambda$, we have that

$$\|\varphi\|_D \coloneqq \sup_{\lambda\in\Lambda} |D\varphi(\lambda)|$$

is finite.

In the archimedean case, $\mathscr{S}(\Lambda)$ is a Fréchet space with the topology defined by the family of seminorms $\|\cdot\|_{D,N}$. Note that in the definition of $\mathscr{S}(\Lambda)$ in the archimedean case, we could have restricted attention to differential operators $D$ that are the same on all components of $\Lambda$; this is how such Schwartz spaces are often defined in the literature.

For each finite set $E_0 \subseteq E$ we define $\mathscr{S}_{E_0}(\Lambda)$ to be the subspace of all $\varphi \in \mathscr{S}(\Lambda)$ that are supported on $\coprod_{e\in E_0} \Lambda_e$. In the archimedean case, $\mathscr{S}_{E_0}(\Lambda)$ is closed subspace of $\mathscr{S}(\Lambda)$ and is thus a Fréchet space with respect to the topology defined by the family of seminorms $\|\cdot\|_{D,N}$. In the non-archimedean case, we give $\mathscr{S}_{E_0}(\Lambda)$ the topology defined by the family of seminorms $\|\cdot\|_D$. It is a Fréchet space. We define $\mathscr{S}_e(\Lambda) = \mathscr{S}_{\{e\}}(\Lambda)$. Note that $\mathscr{S}_e(\Lambda)$ is the classical Schwartz space $\mathscr{S}(\Lambda_e)$ on $\Lambda_e$.

In the non-archimedean case, we give $\mathscr{S}(\Lambda) = \bigcup_{E_0} \mathscr{S}_{E_0}(\Lambda)$ the inductive limit topology in the category of locally convex spaces, making it a strict LF-space. We have $\mathscr{S}(\Lambda) = C_c^\infty(\Lambda)$.

We could also define a space $\mathscr{S}_f(\Lambda)$ in the archimedean case in a manner similar to how we defined $PW_f(\Lambda)$, although we will not make use of this space. Such Schwartz spaces would presumably be used in an invariant Paley–Wiener theorem for $K$-finite Schwartz functions, but to our knowledge such a theorem has not yet been established.

### 3.3.1 The Fourier transform

Fix a space $\Lambda$ as above. It is naturally dual to the space $X = \coprod_{e\in E} X_e$, where $X_e = V$ in the archimedean case and $X_e = \Gamma_e$ in the non-archimedean case.

We will define function spaces $C_c^\infty(X)$ and $\mathscr{S}(X)$ and a Fourier transform that gives isomorphisms of topological vector spaces $C_c^\infty(X) \to PW(\Lambda)$ and $\mathscr{S}(X) \to$

$\mathscr{S}(\Lambda)$. We remark that in the archimedean case, $C_c^\infty(X)$ will not be the space of compactly supported smooth functions on $X$ if $X$ has infinitely many components.

First, we define the spaces $C_c^\infty(X)$ and $\mathscr{S}(X)$ in the non-archimedean case. We define $\mathscr{S}(X) = \bigoplus_{e\in E} \mathscr{S}(\Gamma_e)$, where $\mathscr{S}(\Gamma_e)$ is the space of functions $\phi : \Gamma_e \to \mathbb{C}$ that are rapidly decreasing, i.e. for all $N \in \mathbb{N}$ we have that

$$\|\phi\|_N \coloneqq \sup_{x\in X} |\phi(x)|(1 + \|x\|)^N < \infty.$$

The space $\mathscr{S}(\Gamma_e)$ is a Fréchet space with respect to the topology defined by the seminorms $\|\cdot\|_N$, and $\mathscr{S}(X)$ is a strict LF-space.

We define $C_c^\infty(X) = \bigoplus_{e\in E} C_c^\infty(\Gamma_e)$. Here, $C_c^\infty(\Gamma_e)$ is the space of compactly supported (i.e. finitely supported) functions on $\Gamma_e$ equipped with the finest locally convex topology. Thus, $C_c^\infty(X)$ also has the finest locally convex topology. For $r > 0$, we define $C_r^\infty(\Gamma_e)$ to be the space of functions on $\Gamma_e$ with support contained in $\{x \in \Gamma_e : \|x\| \le r\}$. It is a finite-dimensional space, and we give it the locally convex topology. Then $C_r^\infty(\Gamma_e)$ is a closed subspace of $C_c^\infty(\Gamma_e)$. Moreover, we have an increasing union $C_c^\infty(\Gamma_e) = \bigcup_{r>0} C_r^\infty(\Gamma_e)$ and $C_c^\infty(\Gamma_e)$ is the inductive limit of the $C_r^\infty(\Gamma_e)$ in the category of locally convex spaces. It is a strict LF-space. We define $C_r^\infty(X) = \bigoplus_{e\in E} C_r^\infty(\Gamma_e)$. Again, we have that $C_c^\infty(X)$ is the increasing union $C_c^\infty(X) = \bigcup_{r>0} C_r^\infty(X)$, it is the inductive limit of the $C_r^\infty(X)$ in the category of locally convex spaces, and it is a strict LF-space.

Now, we treat the archimedean case. We may view $V = i(iV^*)^*$, and thus we have a definition of $\mathscr{S}(X)$ (and also $PW(X)$, but we will not need it). It remains for us to define $C_c^\infty(X)$ in the archimedean case. For $r > 0$, we define $C_r^\infty(X)$ to be the space of smooth functions $\phi : X \to \mathbb{C}$ such that:

1. the support of $\phi_e$ is contained in the closed ball of radius $r$ in $X_e$;
2. for all $N > 0$ and all differential operators $D$ that are the same on almost all components $X_e$, we have that

$$\|\phi\|_{D,N} \coloneqq \sup_{e\in E, x\in X_e} \|D\phi_e(x)\|(1 + \|e\|)^N$$

   is finite.

We give $C_r^\infty(X)$ the topology defined by the family of seminorms $\|\cdot\|_{D,N}$, which makes $C_r^\infty(X)$ a Fréchet space. (Note that in the definition of $C_r^\infty(X)$ and its topology, it suffices to use differential operators that are the same on all components of $X$.) We define $C_c^\infty(X)$ to be the increasing union $C_c^\infty(X) = \bigcup_{r>0} C_r^\infty(X)$ and we give it the inductive limit topology in the category of locally convex spaces, making it a strict LF-space.

Note that $C_c^\infty(X)$ is a dense subspace of $\mathscr{S}(X)$ in all cases and the inclusion map $C_c^\infty(X) \to \mathscr{S}(X)$ is continuous.

We now fix measures on $X$ and $\Lambda$. We fix dual measures on $V$ and $iV^*$. These determine dual measures on $X_e$ and $\Lambda_e$ in the archimedean case. In the non-archimedean case, we give each $X_e = \Gamma_e$ the counting measure and $\Lambda_e = iV^*/\Gamma_e^\vee$ the dual measure.

For $\phi \in \mathscr{S}(X)$, we define its Fourier transform $\widehat{\phi} = (\widehat{\phi_e})_{e\in E} : \Lambda \to \mathbb{C}$ by taking the Fourier transform on each component $X_e$, that is, $\widehat{\phi_e}(\lambda) = \int_{X_e} \phi_e(x) e^{-2\pi\langle x,\lambda\rangle}\, dx$.

**Lemma 3.3.1.** *The Fourier transform gives isomorphisms of topological vector spaces $\mathscr{S}(X) \to \mathscr{S}(\Lambda)$, $C_{r/2\pi}^\infty(X) \to PW^r(\Lambda)$, and $C_c^\infty(X) \to PW(\Lambda)$.*

This follows easily from the classical version of this lemma when $E$ is a singleton, together with the following basic inequalities: $(1+x+y) \leq (1+x)(1+y) \leq (1+x+y)^2$ for $x, y \geq 0$; and for $N \in \mathbb{N}$, we have $(1+x)^N \ll (1+x^2)^N \ll (1+x)^{2N}$ for $x \geq 0$.

**Corollary 3.3.2.** *$PW(\Lambda)$ is a dense subspace of $\mathscr{S}(\Lambda)$ and the inclusion map*

$$PW(\Lambda) \longrightarrow \mathscr{S}(\Lambda)$$

*is continuous.*

### 3.3.2 Pullback mappings

Let $V_1$ and $V_2$ be Euclidean spaces and let $T : iV_1^* \to iV_2^*$ be an injective linear map. Then pullback defines continuous linear maps $T^* : \mathscr{S}(iV_2^*) \to \mathscr{S}(iV_1^*)$ and $T^* : PW(iV_2^*) \to PW(iV_1^*)$. Similarly, if $\Gamma_1$ and $\Gamma_2$ are lattices in $V_1$ and $V_2$, respectively, and $T : iV_1^*/\Gamma_1^\vee \to iV_2^*/\Gamma_2^\vee$ is a smooth homomorphism, then pullback along $T$ defines continuous linear maps $T^* : \mathscr{S}(iV_2^*/\Gamma_2^\vee) \to \mathscr{S}(iV_1^*/\Gamma_1^\vee)$ and $T^* : PW(iV_2^*/\Gamma_2^\vee) \to PW(iV_1^*/\Gamma_1^\vee)$.

There is a simple generalisation of this to the Schwartz and Paley–Wiener spaces $\mathscr{S}(\Lambda)$, $PW(\Lambda)$, and $PW_f(\Lambda)$ defined above.

**Lemma 3.3.3.** *For $i = 1, 2$, let $\Lambda_i = \coprod_{e\in E_i} \Lambda_e$ be a space as defined above in relation to $V_i$.*

*Let $T_E : E_1 \to E_2$ be a partially defined map and let $T_V : iV_1^* \to iV_2^*$ be a linear map. In the archimedean case, assume that $\|T_E(e_1)\| \gg \|e_1\|$ and that $T_V$ is injective.*

*In the non-archimedean case, assume that $T_E$ has finite fibres and that $T_V(\Gamma_1^\vee) \subseteq \Gamma_2^\vee$ so that $T_V$ descends to a smooth homomorphisms $T_V : iV_1^*/\Gamma_1^\vee \to iV_2^*/\Gamma_2^\vee$.*

*Let $\varphi = (\varphi_{e_2})_{e_2 \in E_2} \in \mathscr{S}(\Lambda_2)$. Define $T^*\varphi = ((T^*\varphi)_{e_1})_{e_1 \in E_1}$ by $(T^*\varphi)_{e_1} = 0$ if $T_E(e_1)$ is undefined and $(T^*\varphi)_{e_1} = T_V^*\varphi_{T_E(e_1)}$ otherwise. To simplify notation, we write $\varphi_{T_E(e_1)} = 0$ if $T_E(e_1)$ is not defined. Then $(T^*\varphi)_{e_1} = T_V^*\varphi_{T_E(e_1)}$ for all $e_1 \in E_1$.*

*For all $\varphi \in \mathscr{S}(\Lambda_2)$, we have $T^*\varphi \in \mathscr{S}(\Lambda_1)$ and*

$$T^* : \mathscr{S}(\Lambda_2) \longrightarrow \mathscr{S}(\Lambda_1)$$

*is a continuous linear map. Moreover, $T^*$ restricts to a continuous linear map*

$$T^* : PW(\Lambda_2) \longrightarrow PW(\Lambda_1).$$

*In the archimedean case, if $T_E$ in addition has finite fibres, then $T^*$ restricts further to a continuous linear map*

$$T^* : PW_f(\Lambda_2) \longrightarrow PW_f(\Lambda_1).$$

*Proof.* The non-archimedean case and the last claim is straightforward. We prove the first claim of in the archimedean case, the second being similar. Let $D$ be an invariant differential operator on $iV_1^*$ and let $N \in \mathbb{N}$. We will also use $D$ to denote the extension of $D$ along $T_V$ to an invariant differential operator on $iV_2^*$. Let $\varphi \in \mathscr{S}(\Lambda_2)$. We have

$$\|T^*\varphi\|_{D,N} = \sup_{\substack{e_1 \in E_1 \\ \lambda \in \Lambda_{e_1}}} |D\varphi_{T_E(e_1)}(T_V\lambda)|(1 + \|e_1\| + \|\lambda\|)^N.$$

For all $N' \in \mathbb{N}$, we have

$$|D\varphi_{T_E(e_1)}(T_V\lambda)| \leq \|\varphi\|_{D,N'}(1 + \|T_E e_1\| + \|T_V\lambda\|)^{-N'}.$$

Since $T_V : iV_1^* \to iV_2^*$ is injective, we have that $\lambda \mapsto \|T_V\lambda\|$ is a norm on $iV_1^*$ and thus $\|T_V\lambda\| \asymp \|\lambda\|$. Since $\|T_E e_1\| \gg \|e_1\|$, we have that

$$(1 + \|T_E e_1\| + \|T_V\lambda\|)^{-N'} \ll (1 + \|e_1\| + \|\lambda\|)^N$$

for $N'$ sufficiently large. Thus,

$$\|T^*\varphi\|_{D,N} \ll \|\varphi\|_{D,N'}$$

for $N'$ sufficiently large. Thus, we have $T^*\varphi \in \mathscr{S}(\Lambda_1)$ and $T^*$ defines a continuous linear map $T^* : \mathscr{S}(\Lambda_2) \to \mathscr{S}(\Lambda_1)$. That $T^*$ restricts to a continuous linear map $T^* : PW(\Lambda_2) \to PW(\Lambda_1)$ follows in a similar way. □

The proof of Theorem 5.1.1, from which the existence of stable transfer follows, makes use of this lemma.

## 3.4 Invariant Paley-Wiener theorems

As remarked above, we view the invariant Fourier transform of $f \in \mathcal{C}(G)$ as the $\mathbb{C}^1$-equivariant function $f_G : T_{\text{temp}}(G) \to \mathbb{C}$ defined by $f_G(\tau) = f_G(\pi_\tau)$. We thus view the space of invariant Fourier transforms of elements of $\mathcal{I}_{(c)}(G)$ defined above as

$$\begin{aligned}\widehat{\mathcal{I}_{(c)}}(G) &= \{f_G : T_{\text{temp}}(G) \to \mathbb{C} : f \in \mathcal{C}_{(c)}(G)\} \\ &= \mathcal{C}_{(c)}(G)/\operatorname{Ann}_{\mathcal{C}_{(c)}(G)}(T_{\text{temp}}(G)).\end{aligned}$$

The invariant Paley–Wiener theorem for $\mathcal{I}_{(c)}(G)$ asserts that the invariant Fourier transform $\mathcal{F} : \mathcal{I}_{(c)}(G) \to \widehat{\mathcal{I}_{(c)}}(G)$ is an isomorphism of topological vector spaces and gives a characterisation of $\widehat{\mathcal{I}_{(c)}}(G)$ as a Schwartz (resp. Paley–Wiener) space. In this section we review this and other invariant Paley–Wiener theorems that we will stabilise.

Let $L \in \mathcal{L}^G(M_0)$. Recall that if $F$ is archimedean we have define a "norm" on the set of infinitesimal characters of $L$. We define $\|\tau\| = \|\mu_\tau\|$ for all $\tau \in \widetilde{T}_{\text{ell}}(L)$.

For any countable set $E \subseteq T_{\text{ell}}(L)$, we have a Schwartz space $\mathscr{S}(\Lambda)$ and a Paley–Wiener space $PW(\Lambda)$ defined on the space $\Lambda = \coprod_{\tau\in E} \Lambda_\tau$ with $\Lambda_\tau = i\mathfrak{a}_L^*/\mathfrak{a}_{L,\tau}^\vee$.

We define $\mathscr{S}_{\text{ell}}(L)$ to be the space of smooth $\mathbb{C}^1$-equivariant functions $\varphi : T_{\text{ell}}(L) \to \mathbb{C}$ such that for some (and hence any) choice of representatives $E_{\text{ell}}(L) \subseteq T_{\text{ell}}(L)$ for the connected components of $T_{\text{ell}}(L)/\mathbb{C}^1$, we have

$$\varphi \in \mathscr{S}\left(\coprod_{\tau\in E_{\text{ell}}(L)} i\mathfrak{a}_L^*/\mathfrak{a}_{L,\tau}^\vee\right).$$

We define $PW_{\text{ell}}(L)$ (resp. $PW_{\text{ell},f}(L)$) in the same way as $\mathscr{S}_{\text{ell}}(L)$, except that we replace $\mathscr{S}(\cdot)$ by $PW(\cdot)$ (resp. $PW_f(\cdot)$). We define

$$\mathscr{S}(G) = \left(\bigoplus_{L\in\mathcal{L}^G(M_0)} PW_{\text{ell}}(L)\right)^{W_0^G} = \bigoplus_{L\in\mathcal{L}^G(M_0)/W_0^G} PW_{\text{ell}}(L)^{W^G(L)}$$

and similarly we define $PW(G)$ and $PW_f(G)$. It follows from the decomposition of $T_{\text{temp}}(G)$ in terms of the $T_{\text{ell}}(L)$, that $\mathscr{S}(G)$, $PW(G)$, and $PW_f(G)$ are naturally spaces of smooth $\mathbb{C}^1$-equivariant functions on $T_{\text{temp}}(G)$. We have the following invariant Paley–Wiener theorems.

**Theorem 3.4.1.** *The invariant Fourier transform is an isomorphism of topological vector spaces*

$$\mathcal{I}(G) \longrightarrow \mathscr{S}(G).$$

*It restricts to isomorphism of topological vector spaces*

$$\mathcal{I}_c(G) \longrightarrow PW(G).$$

*This further restricts to an isomorphism of topological vector spaces*

$$\mathcal{I}_f(G) \longrightarrow PW_f(G)$$

*Thus, we have* $\widehat{\mathcal{I}}(G) = \mathscr{S}(G)$, $\widehat{\mathcal{I}}_c(G) = PW(G)$, *and* $\widehat{\mathcal{I}}_f(G) = PW_f(G)$.

Injectivity of the first map is equivalent to the assertion that for $f \in \mathcal{C}(G)$, if if $f_G$ vanishes on $D_{\text{temp}}(G)$, then $f_G(\gamma) = 0$ for all $\gamma \in \Gamma_{\text{rs}}(G)$. This is called spectral density. Kazhdan proved spectral density follows for $p$-adic groups [Kaz86, Theorem J.(a)]. Spectral density for both real and $p$-adic groups can also be seen as a corollary of Arthur's Fourier inversion theorem for orbital integrals [Art94a, Theorem 4.1 with $M = G$], which we will recall in a moment. The first map in the theorem was proved to be an open continuous surjection by Arthur in [Art94b]. For real groups, the second statement in the theorem is proved in a more general twisted form in [MW16a] using a twisted invariant Paley–Wiener theorem due Renard, which generalises a theorem due to Bouaziz in the non-twisted case. For $p$-adic groups, the second statement is proved in [Art93]. The third statement is only different from the second in the case of real groups, and it is also proved in [Art93]. (See also [MW18, §6.2].)

To state Arthur's Fourier inversion theorem for orbital integrals, we need a measure on $T_{\text{temp}}(G)/\mathbb{C}^1$. First, we define a measure on $T_{\text{ell}}(L)/\mathbb{C}^1$ for each $L \in \mathcal{L}^G(M_0)$ by

$$\int_{T_{\text{ell}}(L)/\mathbb{C}^1} \alpha(\tau)\, d\tau = \sum_{\tau \in T_{\text{ell}}(L)/(\mathbb{C}^1 \times i\mathfrak{a}_L^*)} \int_{i\mathfrak{a}_L^*/\mathfrak{a}_{L,\tau}^\vee} \alpha(\tau_\lambda)\, d\lambda,$$

for all $\alpha \in C_c(T_{\text{ell}}(L)/\mathbb{C}^1)$, where the measure $d\lambda$ is determined by a choice of Haar measure on $i\mathfrak{a}_L^*$ and the counting measure on $\mathfrak{a}_{L,\tau}^\vee$. Then, we define a measure on

$T_{\text{temp}}(G)/\mathbb{C}^1$ by

$$\int_{T_{\text{temp}}(G)/\mathbb{C}^1} \alpha(\tau)\, d\tau = \sum_{L \in \mathcal{L}^G(M_0)/W_0^G} |W^G(L)|^{-1} \int_{T_{\text{ell}}(L)/\mathbb{C}^1} \alpha(\tau)\, d\tau$$

for all $\alpha \in C_c(T_{\text{temp}}(G)/\mathbb{C}^1)$. The following is Arthur's Fourier inversion theorem for orbital integrals.

**Theorem 3.4.2.** *Let $f \in \mathcal{C}(G)$. There exists a smooth function $I_G(\gamma, \tau)$ on $\Gamma_{\text{sr}}(G) \times T_{\text{temp}}(G)$, which satisfies $I_G(\gamma, z\tau) = z^{-1} I_G(\gamma, \tau)$ for all $z \in \mathbb{C}^1$, and satisfies*

$$f_G(\gamma) = \int_{T_{\text{temp}}(G)/\mathbb{C}^1} I_G(\gamma, \tau) f_G(\tau)\, d\tau$$

*for all $\gamma \in \Gamma_{\text{sr}}(G)$.*

This is [Art94a, Theorem 4.1 with $M = G$]. See the discussion before Lemma 6.3 in [Art96], where this specialisation of [Art94a, Theorem 4.1] is discussed.

Let $\mathcal{I}_{\text{cusp}}(G)$ (resp. $\mathcal{I}_{c,\text{cusp}}(G)$, $\mathcal{I}_{f,\text{cusp}}(G)$) denote the image of $\mathcal{C}_{\text{cusp}}(G)$ (resp. $C^\infty_{c,\text{cusp}}(G)$, $C^\infty_{c,\text{cusp}}(G,K)$) in $\mathcal{I}(G)$ (resp. $\mathcal{I}_c(G)$, $\mathcal{I}_c(G,K)$).

For $f \in \mathcal{C}(G)$ it follows from spectral density that $f \in \mathcal{C}_{\text{cusp}}(G)$ if and only if its invariant Fourier transform is supported on $T_{\text{ell}}(G)$. We obtain the following invariant Paley–Wiener theorems for cuspidal functions as a corollary of the above invariant Paley–Wiener theorem.

**Corollary 3.4.3.** *The invariant Fourier transform is an isomorphism of topological vector spaces*

$$\mathcal{I}_{\text{cusp}}(G) \longrightarrow \mathscr{S}_{\text{ell}}(G).$$

*It restricts to isomorphisms of topological vector spaces*

$$\mathcal{I}_{c,\text{cusp}}(G) \longrightarrow PW_{\text{ell}}(G)$$

*and*

$$\mathcal{I}_{f,\text{cusp}}(G) \longrightarrow PW_{\text{ell},f}(G).$$

### 3.4.1 Pseudocoefficients

Let $\zeta$ be a unitary character of $A_G(F)$. The finite group $\widetilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,F}$ acts on $T_{\text{ell}}(G,\zeta)$ and the $i\mathfrak{a}^*_G$-orbits in $T_{\text{ell}}(G)$ meet $T_{\text{ell}}(G,\zeta)$ in $(\widetilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,F})$-orbits. Thus, it is natural to regard $T_{\mathrm{ell}}(G,\zeta)$ as a discrete space. Define $PW_{\mathrm{ell},f}(G,\zeta)$ to be the space of all $\mathbb{C}^1$-equivariant functions $\varphi : T_{\mathrm{ell}}(G,\zeta) \to \mathbb{C}$ that are supported on finitely many $\mathbb{C}^1$-orbits. Let $E_{\mathrm{ell}}(G,\zeta) \subseteq T_{\mathrm{ell}}(G,\zeta)$ be a set of representatives for the countable set $T_{\mathrm{ell}}(G,\zeta)/\mathbb{C}^1$. Then we have an identification

$$PW_{\mathrm{ell},f}(G,\zeta) = \bigoplus_{\tau \in E_{\mathrm{ell}}(G,\zeta)} \mathbb{C},$$

defined by mapping $\varphi$ to $(\varphi(\tau))_{\tau\in E_{\mathrm{ell}}(G,\zeta)}$. The direct sum $\bigoplus_{\tau\in E_{\mathrm{ell}}(G,\zeta)} \mathbb{C}$ can be thought of as the $PW_f$-space on the countable 0-dimensional manifold $E_{\mathrm{ell}}(G,\zeta)$. Recall that $\mathbb{C}^1\Pi_2(G,\zeta) \subseteq T_{\mathrm{ell}}(G,\zeta)$. Thus, we may and will assume that $\Pi_2(G,\zeta) \subseteq E_{\mathrm{ell}}(G,\zeta)$.

There is a Paley–Wiener theorem for the invariant Fourier transform on $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$, which we now recall from [MW18, §7.2].

**Theorem 3.4.4.** *The invariant Fourier transform gives an isomorphism of topological vector spaces*

$$\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta) \longrightarrow PW_{\mathrm{ell},f}(G,\zeta).$$

A consequence is that for each $\tau \in T_{\mathrm{ell}}(G,\zeta)$, there exists a unique function $f[\tau]_G \in \mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$ such that $f[\tau]_G(z\tau) = z\|\tau\|_{\mathrm{ell}}$ for $z \in \mathbb{C}^1$, and $f[\tau]_G(\tau') = 0$ for $\tau' \in T_{\mathrm{ell}}(G,\zeta)$ with $\tau' \notin \mathbb{C}^1\tau$. The function $f[\tau]_G$ is called the pseudocoefficient of $\tau$ in $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$. Note that for all $\tau \in T_{\mathrm{ell}}(G,\zeta)$ and $z \in \mathbb{C}^1$, we have $f[z\tau]_G = z^{-1}f[\tau]_G$. Moreover, the set $\{f[\tau]_G\}_{\tau\in E_{\mathrm{ell}}(G,\zeta)}$ is a basis of $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$. It follows from the simple form of the local trace formula that for $\gamma \in \Gamma_{\mathrm{sr,ell}}(G)$ we have

$$f[\tau]_G(\gamma) = m(\gamma)^{-1}|D^G(\gamma)|^{1/2}\overline{\Theta_\tau(\gamma)}.$$

Recall that $m(\gamma) = \mathrm{vol}(G_\gamma(F)/A_G(F))$. (See [MW18, §7.2].)

Since $\{f[\tau]_G\}_{\tau\in E_{\mathrm{ell}}(G,\zeta)}$ is a basis of $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$, there is a unique conjugate-linear isomorphism

$$D_{\mathrm{ell}}(G,\zeta) \longrightarrow \mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$$

given on the basis $E_{\mathrm{ell}}(G,\zeta)$ by $\tau \mapsto f[\tau]_G$. By conjugate-linearity, for all $z \in \mathbb{C}^1$ we have $z\tau \mapsto z^{-1}f[\tau]_G = f[z\tau]_G$. Consequently, the isomorphism does not depend on the choice of representatives $E_{\mathrm{ell}}(G,\zeta)$. For all $\pi \in D_{\mathrm{ell}}(G,\zeta)$ we write $f[\pi]_G \in \mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$ for the corresponding element under the above isomorphism. We call $f[\pi]_G$ the pseudocoefficient of $\pi$ in $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$.

For all $\pi \in D_{\mathrm{ell}}(G,\zeta)$, we have $\langle mf_G, mf[\pi]_G\rangle_{\mathrm{ell}} = f_G(\pi)$. Indeed, it suffices to check this for $\pi = \tau \in T_{\mathrm{ell}}(G, \zeta)$, and this in turn follows from the formula for $f[\tau]_G$ given above. Note that the map $\pi \mapsto mf[\pi]_G$ is unitary with respect to the elliptic inner product in the sense that for all $\pi_1, \pi_2 \in D_{\mathrm{ell}}(G, \zeta)$, we have $\langle \pi_1, \pi_2 \rangle_{\mathrm{ell}} = \langle mf[\pi_1]_G, mf[\pi_2]_G \rangle_{\mathrm{ell}}$. If $B$ is an orthogonal basis of $D_{\mathrm{ell}}(G, \zeta)$ (with respect to the elliptic inner product), then for all $b, b' \in B$, we have

$$f[b]_G(b') = \langle mf[b]_G, mf[b']_G \rangle_{\mathrm{ell}} = \langle b, b' \rangle_{\mathrm{ell}} = \|b\|_{\mathrm{ell}} \delta_b(b').$$

# 4 Stable Harmonic Analysis

The local version of the Principle of Functoriality is concerned with the relationship between irreducible admissible representations of different groups $H$ and $G$ over $F$ whose $L$-groups are related by an $L$-homomorphism ${}^L H \to {}^L G$ (notions that we will define below). In this case, one has (conjecturally for non-archimedean $F$) a correspondence between irreducible admissible representations of $H$ and $G$. This correspondence is more directly expressed as a map from the set of $L$-packets of $H$ to the set of $L$-packets of $G$. An $L$-packet is the set of irreducible admissible representations that are $L$-indistinguishable in the sense that they have the same $L$-parameter, an arithmetic invariant that determines the local $L$-function of an irreducible admissible representation. The packaging of admissible representations into $L$-packets determined by their $L$-parameters is the (conjectural for non-archimedean $F$) local Langlands correspondence. Part of the local Langlands correspondence also attaches virtual representations with positive coefficients to $L$-packets of tempered representations, and the associated virtual tempered characters are called stable tempered characters. They are stable in the sense that they are constant not just on regular semisimple conjugacy classes but on certain finite unions of regular semisimple conjugacy classes called regular semisimple stable conjugacy classes. This leads to the notion of stable orbital integrals and a stable Fourier transform defined in terms of stable tempered characters. This is the subject of stable harmonic analysis, which plays a central role in the Langlands program.

We will recall the foundations of stable harmonic analysis and then prove stable Paley–Wiener theorems. These will be used to establish the existence and basic properties of stable transfer in the next chapter.

## 4.1 Stability

We begin by recalling stable conjugacy. For a reference, see [Kot82, §3]. Let $x, x' \in G(F)$. If there exists $g \in G(\overline{F})$ such that $x' = g^{-1}xg$, then $g\sigma(g)^{-1} \in G^{x_s}(\overline{F})$, where

$x_s$ is the semisimple component of $x$ in its Jordan decomposition. The elements $x, x' \in G(F)$ are said to be stably conjugate if there exists $g \in G(\overline{F})$ such that $x' = g^{-1}xg$ and $g\sigma(g)^{-1} \in G_{x_s}(\overline{F})$ for all $\sigma \in \Gamma_F$. Stable conjugacy is an equivalence relation on $G(F)$, and each stable conjugacy class is a finite union of $G(F)$-conjugacy classes. Note that for complex groups, stable conjugacy is the same as conjugacy. The set of stable conjugacy classes in $G(F)$ is usually denoted by $\Delta(G)$. We write

$$\Delta_{\mathrm{ss}}(G) \supset \Delta_{\mathrm{rs}}(G) \supset \Delta_{\mathrm{sr}}(G)$$

for the sets of semisimple, regular semisimple, and strongly regular stable conjugacy classes in $G(F)$. The relation of stable conjugacy is intermediate in strength between $G(F)$-conjugacy and $G(\overline{F})$-conjugacy. Two strongly regular elements of $G(F)$ are stably conjugate if and only if they are $G(\overline{F})$-conjugate, and thus the stable conjugacy class of a strongly regular element $x \in G(F)$ is simply the intersection $x^G \cap G(F)$ of its $G(\overline{F})$-conjugacy class with $G(F)$. The Weyl discriminant $D^G$ is stably invariant. If $T$ is a maximal torus of $G$, then two $G$-regular elements of $T(F)$ are conjugate if and only if they lie in the same orbit of the stable Weyl group $W(G,T)(F)$ of $T$. A function on a subset of $G(F)$ is said to be stably invariant if it is constant on all stable conjugacy classes in its domain.

The stable orbital integral at $\delta \in \Delta_{\mathrm{ss}}(G)$ is defined in [Kot86, §5.2] by

$$SO_\delta = \sum_{\gamma \in \Gamma(G)} c_{\delta,\gamma} O_\gamma,$$

where $c_{\delta,\gamma} = 0$ unless $\gamma \subseteq \delta$, in which case

$$c_{\delta,\gamma} = e(G_\gamma^\circ) | \ker[H^1(F, G_\gamma^\circ, G_\gamma)]|,$$

where $e(G_\gamma^\circ)$ is the Kottwitz sign of $G_\gamma^\circ$ (see [Kot83]). Note that $SO_\delta(f)$ converges for $f \in C_c(G,\zeta)$ and $f \in \mathcal{C}(G,\zeta)$ and defines a tempered distribution. The normalised stable orbital integral of $f \in \mathcal{C}(G,\zeta)$ is defined by

$$f^G(\delta) = |D^G(\delta)|^{1/2} SO_\delta(f) = \sum_{\gamma \in \Gamma(G)} c_{\delta,\gamma} f_G(\gamma).$$

If $\delta \in \Delta_{\mathrm{rs}}(G)$, then $c_{\delta,\gamma} = 1$ if $\gamma \subseteq \delta$, and thus

$$SO_\delta = \sum_{\gamma \subseteq \delta} O_\gamma,$$

where the sum is over all conjugacy classes $\gamma$ in $\delta$. Therefore

$$f^G(\delta) = \sum_{\gamma \subseteq \delta} f_G(\delta)$$

for all $\delta \in \Delta_{\mathrm{rs}}(G)$.

We give $\Delta(G)$ the quotient topology from $G(F)$. Then $\Delta_{\mathrm{rs}}(G)$ and $\Delta_{\mathrm{sr}}(G)$ are open dense locally compact Hausdorff subspaces of $\Delta(G)$, and $\Gamma_{\mathrm{sr}}(G)$ is naturally an $F$-analytic manifold. For $f \in C_c(G(F), \zeta)$ or $f \in \mathcal{C}(G, \zeta)$, the function $\delta \mapsto SO_\delta(f)$ is a continuous on $\Delta_{\mathrm{rs}}(G)$.

Two maximal tori $T, T'$ of $G$ are said to be stably conjugate if the exists $g \in G(\overline{F})$ such that $\mathrm{Int}(g) : T \to T'$ is defined over $F$. We have a stable version of the Weyl integration formula:

$$\int_{G(F)} f(x)\, dx = \sum_T |W(G,T)(F)|^{-1} \int_{T(F)} |D^G(t)| SO_t(f)\, dt,$$

whenever one side and hence the other converges, where the sum is over a set of representatives of stable conjugacy classes of maximal tori of $G$. As in the invariant case, one can define a Radon measure $d\delta$ on $\Delta_{\mathrm{rs}}(G)$ by

$$\int_{\Delta_{\mathrm{rs}}(G)} \varphi(\delta)\, d\delta = \sum_T |W(G,T)(F)|^{-1} \int_{T(F)} \varphi(t)\, dt,$$

for all $\varphi \in C_c(\Gamma_{\mathrm{rs}}(G))$, where the sum runs over a set of representatives of the stable conjugacy classes of maximal tori of $G$. The stable Weyl integration formula then becomes

$$\int_{G(F)} f(x)\, dx = \int_{\Delta_{\mathrm{rs}}(G)} |D^G(\delta)| SO_\delta(f)\, d\delta.$$

Note that $\Delta_{\mathrm{sr}}(G)$ has comeasure zero in $\Delta_{\mathrm{rs}}(G)$. This restricts to a Radon measure measure on the set $\Delta_{\mathrm{rs,ell}}(G)$ of elliptic regular semisimple stable conjugacy classes of $G$. One can rewrite the stable Weyl integration formula in terms of conjugacy classes semistandard Levi subgroups a similar way as the Weyl integration formula was rewritten.

We define the subspace of unstable functions in $\mathcal{C}_{(c)}(G, \zeta)$ by

$$\begin{aligned} \mathcal{C}^{\mathrm{unst}}_{(c)}(G, \zeta) &= \mathrm{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{SO_\delta : \delta \in \Delta_{\mathrm{sr}}(G)\}) \\ &= \{f \in \mathcal{C}_{(c)}(G, \zeta) : f^G(\delta) = 0,\ \forall \delta \in \Delta_{\mathrm{sr}}(G)\} \end{aligned}$$

and its image in $\mathcal{I}_{(c)}(G,\zeta)$,

$$\mathcal{I}_{(c)}^{\mathrm{unst}}(G,\zeta) = \mathrm{Ann}_{\mathcal{I}_{(c)}(G,\zeta)}(\{SO_\delta : \delta \in \Delta_{\mathrm{sr}}(G)\}).$$

We define the space of stable orbital integrals

$$\mathcal{S}_{(c)}(G,\zeta) = \{f^G : f \in \mathcal{C}_{(c)}(G,\zeta)\}.$$

We have

$$\mathcal{S}_{(c)}(G,\zeta) = \mathcal{C}_{(c)}(G,\zeta)/\mathcal{C}_{(c)}^{\mathrm{unst}}(G,\zeta) = \mathcal{I}_{(c)}(G,\zeta)/\mathcal{I}_{(c)}^{\mathrm{unst}}(G,\zeta)$$

and we give it the natural quotient topology. We also define the spaces $\mathcal{C}_c^{\mathrm{unst}}(G,\zeta,K)$, $\mathcal{I}_f^{\mathrm{unst}}(G,\zeta) = \mathcal{I}_c^{\mathrm{unst}}(G,\zeta,K)$, and $\mathcal{S}_f(G,\zeta) = \mathcal{S}_c(G,\zeta,K)$ in a similar way.

The space of $\zeta$-equivariant stable distributions is defined to be $\mathcal{S}_c(G,\zeta)'$ and the space of tempered $\zeta$-equivariant stable distributions is defined to be $\mathcal{S}(G,\zeta)'$. Note that we have a continuous linear injection $\mathcal{S}_c(G,\zeta) \to \mathcal{S}(G,\zeta)$ with dense image. Its transpose is a continuous linear injection $\mathcal{S}(G,\zeta)' \to \mathcal{S}_c(G,\zeta)'$, which enables us to identify each tempered $\zeta$-equivariant stable distribution with a $\zeta$-equivariant stable distribution.

We may identify $\mathcal{S}_{(c)}(G,\zeta)'$ with a subspace of $\mathcal{I}_{(c)}(G,\zeta)'$ via the transpose of the quotient map $\mathcal{I}_{(c)}(G,\zeta) \to \mathcal{S}_{(c)}(G,\zeta)$. Thus, we have a further identification of $\mathcal{S}_{(c)}(G,\zeta)'$ as a subspace of $\mathcal{C}_{(c)}(G,\zeta)'$. As vector spaces, we have

$$\begin{aligned}\mathcal{S}_{(c)}(G,\zeta)' &= \mathrm{Ann}_{\mathcal{C}_{(c)}(G,\zeta)'}(\mathrm{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{SO_\delta : \delta \in \Delta_{\mathrm{sr}}(G)\}))\\ &= \mathrm{cl}_{\mathcal{C}_{(c)}(G,\zeta)',\text{weak-}*}(\{SO_\delta : \delta \in \Delta_{\mathrm{sr}}(G)\}).\end{aligned}$$

That is, an distribution in $\mathcal{C}_{(c)}(G,\zeta)'$ belongs to $\mathcal{S}_{(c)}(G,\zeta)'$ if and only if it lies in the weak-$*$ closure in $\mathcal{C}_{(c)}(G,\zeta)'$ of the linear span of the set of strongly regular stable orbital integrals of $G$. A locally integrable function $\Theta$ on $G(F)$ that is continuous $G_{\mathrm{sr}}(F)$ defines a stable distribution of $G$ if and only if $\Theta$ is stably invariant on $G_{\mathrm{sr}}(F)$.

We say that a virtual representation is stable if its character, which we identify it with, is stable. We define $D_{\mathrm{spec}}^{\mathrm{st}}(G,\zeta)$ (resp. $D_{\mathrm{temp}}^{\mathrm{st}}(G,\zeta)$, $D_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta)$) to be the subspace of stable elements in $D_{\mathrm{spec}}(G,\zeta)$ (resp. $D_{\mathrm{temp}}(G,\zeta)$, $D_{\mathrm{ell}}(G,\zeta)$).

The parabolic descent map $\mathcal{I}_{(c)}(G,\zeta) \to \mathcal{I}_{(c)}(M,\zeta)^{W^G(M)}$ descends to a continuous map $\mathcal{S}_{(c)}(G,\zeta) \to \mathcal{S}_{(c)}(M,\zeta)^{W^G(M)}$, which we write as $f^G \mapsto f^M$ and also call parabolic descent. Thus, the parabolic induction map $I_M^G : \mathcal{I}_{(c)}(M,\zeta)'/W^G(M) \to \mathcal{I}_{(c)}(G,\zeta)'$ restricts to a continuous map $I_M^G : \mathcal{S}_{(c)}(M,\zeta)'/W^G(M) \to \mathcal{S}_{(c)}(G,\zeta)'$, namely the transpose of the parabolic descent map $\mathcal{S}_{(c)}(G,\zeta) \to \mathcal{S}_{(c)}(M,\zeta)^{W^G(M)}$. Consequently, parabolic induction $I_M^G$ preserves stability of distributions and in particular of virtual characters. Let $\Delta_{G\text{-rs}}(M)$ denote the set of $G$-regular semisimple stable conjugacy classes in $M(F)$. For all $\delta \in \Delta_{G\text{-rs}}(M)$ we have $f^M(\delta) = f^G(\delta)$.

## 4.2 $L$-groups

In order to define the spectral objects in stable harmonic analysis on $G$, we need the notion of the $L$-group of $G$. We give a review of $L$-groups emphasising the functoriality of various constructions. A similar exposition is given in [Ngô20]. Other useful references on $L$-groups are [Bor79; Kot84; MW16a; SZ18; Var]. A reader familiar with the subject can safely skip to Section 4.2.3, where we introduce some notation and terminology that is not standard.

### 4.2.1 Based root data

Let $k$ be a field and let $G$ be a connected reductive group over $k$. We begin by focusing on the case when $G$ is split. For each Borel pair $(B,T)$ of $G$, we have a (reduced) based root datum

$$\Psi(G,B,T) = (X^*(T), \Phi(G,T), \Delta(B,T), X_*(T), \Phi^\vee(G,T), \Delta^\vee(B,T)).$$

We will usually write $\Phi(G,B,T) = (X^*(T), \Delta(B,T), X_*(T), \Delta^\vee(B,T))$ since the right hand side determines $\Phi(G,B,T)$.

Any two Borel pairs $(B_1,T_1),(B_2,T_2)$ of $G$ are $G(k)$-conjugate. Moreover, up to multiplication on the right by an element of $T_1(k)$ there is a unique $g \in G(k)$ such that $(B_2,T_2) = (gB_1g^{-1}, gT_1g^{-1})$. Thus, we obtain a canonical isomorphism

$$\mathrm{Int}(g) : T_1 \longrightarrow T_2.$$

The adjoint isomorphisms $\mathrm{Int}(g)^* : X^*(T_2) \xrightarrow{\sim} X^*(T_1)$ and $\mathrm{Int}(g)_* : X_*(T_1) \xrightarrow{\sim} X_*(T_2)$ form a canonical isomorphism $\Psi(G,B_1,T_1) \xrightarrow{\sim} \Psi(G,B_2,T_2)$ of based root data. The canonical based root datum $\Psi(G)$ of $G$ is defined to be the limit $\Psi(G) = \lim_{(B,T)} \Psi(G,B,T)$ in the category whose objects are of based root data and whose morphisms are isomorphisms of based root data. By definition, for each Borel pair $(B,T)$ of $G$ there is a canonical isomorphism $\Psi(G) \cong \Psi(G,B,T)$. We write $\Psi(G) = (X_G, \Phi_G, \Delta_G, X_G^\vee, \Phi_G^\vee, \Delta_G^\vee)$, or more briefly, $\Psi(G) = (X_G, \Delta_G, X_G^\vee, \Delta_G^\vee)$. We also write

$X(G) = X_G$, etc. if it is preferable. We omit "$G$" from this notation if it is clear from context. We recall that the Existence Theorem states that for any based root datum $\Psi$ there exists a split connected reductive group $G$ over $k$ such that $\Psi(G) \cong \Psi$.

We will now discuss the functoriality of $G \mapsto \Psi(G)$. Suppose that $(G, B, T)$, $(G', B', T')$ are two split connected reductive groups with Borel pairs over $k$. Every homomorphism $\eta : (G, B, T) \to (G', B', T')$ gives rise to a pair of homomorphisms $\eta^* : X^*(T') \to X^*(T)$ and $\eta_* : X_*(T) \to X_*(T')$. The homomorphisms $\eta^*$ and $\eta_*$ are adjoint with respect to the canonical perfect pairings between character and cocharacter groups of split tori: $\langle \eta^*(\alpha'), \beta^\vee \rangle = \langle \alpha', \eta_*(\beta) \rangle$ for all $\alpha' \in X^*(T')$ and $\beta \in X_*(T)$. Thus, $\eta^*$ determines $\eta_*$ and vice versa.

If $\eta$ is an isomorphism, then $\eta^*$ (or the pair $(\eta^*, \eta_*)$) is an isomorphism $\Psi(G, B, T) \to \Psi(G', B', T')$ of based root data. Therefore, every isomorphism $\eta : (G, B, T) \to (G', B', T')$ gives rise to an isomorphism $\Psi(\eta) : \Psi(G, B, T) \to \Psi(G', B', T')$ and thus an isomorphism $\Psi(\eta) : \Psi(G) \to \Psi(G')$. For each $t' \in (T'/Z_{G'})(k)$, we have an inner automorphism $\mathrm{Int}(t') : (G', B', T') \to (G', B', T')$, and $\Psi(\mathrm{Int}(t'))$ is the identity automorphism of $\Psi(G, B, T)$.

The Isomorphism Theorem asserts that $\eta \mapsto \Psi(\eta)$ is a bijection

$$\Psi : (\mathrm{Isom}((G, B, T), (G', B', T'))/ \sim) \longrightarrow \mathrm{Isom}(\Psi(G), \Psi(G')),$$

where in the domain we identify isomorphisms that differ by an inner automorphism $\mathrm{Int}(t')$ with $t' \in (T'/Z_{G'})(k)$. It follows that we obtain a bijection

$$\Psi : (\mathrm{Isom}(G, G')/ \sim) \longrightarrow \mathrm{Isom}(\Psi(G), \Psi(G')),$$

where in the domain we identify automorphisms that differ by an inner automorphism of $G'$, that is an automorphism of the form $\mathrm{Int}(g')$ with $g' \in G'_{\mathrm{ad}}(k) = (G'/Z_{G'})(k)$. A corollary is that we have the following short exact sequence

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\Psi(G)) \longrightarrow 1,$$

where $\mathrm{Int}(G) = \mathrm{Int}(G_{\mathrm{ad}}(k))$. Thus, $\mathrm{Out}(G) = \mathrm{Aut}(\Psi(G))$.

Recall that a pinning $\{X_\alpha\}_{\alpha \in \Delta(B,T)}$ of $(G, B, T)$ is a choice of a nonzero element $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta(B, T)$. For $\alpha \in \Phi(G, T)$, the isomorphisms $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$, where $U_\alpha$ is the root subgroup of $G$ attached to $\alpha$, are in bijective correspondence with non-zero elements $X_\alpha \in \mathfrak{g}_\alpha$ via $dx_\alpha(1) = X_\alpha$. Thus, one may define a pinning using isomorphisms $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ instead. We also call $(B, T, \{X_\alpha\}_\alpha)$ a pinning of $G$

if $\{X_\alpha\}_\alpha$ is a pinning of $(G, B, T)$. We call $(G, B, T, \{X_\alpha\}_\alpha)$ a pinned split connected reductive group over $k$. Fix pinnings $\{X_\alpha\}_\alpha$ and $\{X'_{\alpha'}\}_{\alpha'}$ for $(G, B, T)$ and $(G', B', T')$ respectively. Then the map $\eta \mapsto \Psi(\eta)$ restricts to a bijection

$$\mathrm{Isom}((G, B, T, \{X_\alpha\}_\alpha), (G', B', T', \{X'_{\alpha'}\}_{\alpha'})) \longrightarrow \mathrm{Isom}(\Psi(G), \Psi(G')).$$

In particular, a choice of pinning $(B, T, \{X_\alpha\}_\alpha)$ of $G$ determines a splitting

$$\mathrm{Aut}(\Psi(G)) \longrightarrow \mathrm{Aut}(G)$$

of the above short exact sequence, and thus a semidirect product decomposition $\mathrm{Aut}(G) = \mathrm{Int}(G) \rtimes \mathrm{Aut}(\Psi(G))$.

There is a generalisation of the Isomorphism Theorem called the Isogeny Theorem, and yet a further generalisation involving homomorphisms with normal image. Let $\eta : G \to G'$ be a homomorphism. We call $\eta$ a normal homomorphism if $\eta(G)$ normal is in $G'$. We say that $\eta$ is separable if $\eta : G \to \eta(G)$ is separable. Recall the homomorphism $\eta^*$ and its adjoint $\eta_*$ defined above. Write $\Psi(G) = (X, \Delta, X^\vee, \Delta^\vee)$ and $\Psi(G') = (X', \Delta', (X')^\vee, (\Delta')^\vee)$. Let $\Delta_1 = \Delta \cap \mathrm{im}\, \eta^*$ and $\Delta_2 = \Delta \setminus \Delta_1$. Let $\Delta'_2 = \Delta' \cap \ker \eta^*$ and $\Delta'_1 = \Delta' \setminus \Delta'_2$. Let $p$ be the characteristic exponent of $k$. As stated in [Spr79, §2.11], the homomorphism $\eta^*$ satisfies the following properties.

1. The decompositions $\Delta = \Delta_1 \coprod \Delta_2$ and $\Delta' = \Delta'_1 \coprod \Delta'_2$ are orthogonal (that is $\langle \Delta_1, \Delta_2^\vee \rangle = \langle \Delta_2, \Delta_1^\vee \rangle = 0$ and $\langle \Delta'_1, (\Delta'_2)^\vee \rangle = \langle \Delta_2, (\Delta'_1)^\vee \rangle = 0$).

2. There is a bijection $\Delta_1 \to \Delta'_1, \alpha \mapsto \alpha'$ and a function $q : \Delta_1 \to p^{\mathbb{Z}_{\geq 0}}$ such that for all $\alpha \in \Delta_1$, $\eta^*(\alpha') = q(\alpha)\alpha$ and $\eta_*(\alpha^\vee) = q(\alpha)(\alpha')^\vee$.

3. $\eta^*(\Delta'_2) = 0$ and $\eta_*(\Delta_2) = 0$.

These properties define what we call a $p$-morphism

$$\eta^* : (X, \Delta, X^\vee, \Delta^\vee) \longrightarrow (X', \Delta', (X')^\vee, (\Delta')^\vee)$$

of based root data. We remark that often $p$-morphism has a more restricted meaning. We will denote by $\Psi(\eta) : \Psi(G) \to \Psi(G')$ the $p$-morphism $\eta^*$ determined by a normal homomorphism $\eta : G \to G'$. We say that a $p$-morphism is separable if $q(\alpha) = 1$ for all $\alpha \in \Delta_1$. Note that the notion of separable $p$-morphisms does not depend on $p$, so we will also call them simply morphisms of based root data. We have that a normal homomorphism is separable if and only if $\Psi(\eta)$ is separable. We say that a

$p$-morphism $\eta^*$ is surjective if $\Delta_1' = \Delta'$, and $\eta^*$ is injective, and we call $\eta^*$ a $p$-isogeny if furthermore $\Delta_1 = \Delta$ and $\operatorname{im} \eta^*$ has finite index in $X$. In [Spr79], $p$-isogenies are called $p$-morphisms and the notion of $p$-morphism given here is not named.

By [Spr79, §2.9, §2.10(ii)], the map $\eta \mapsto \Psi(\eta)$ is a bijection from the set of $G'_{\mathrm{ad}}(k)$-conjugacy classes of isogenies $\eta : G \to G'$ to the set of $p$-isogenies $\Psi(G) \to \Psi(G')$, and the separable $p$-isogenies correspond to $G'_{\mathrm{ad}}(k)$-conjugacy classes of separable (or equivalently central) isogenies. More generally, Steinberg proved that the map $\eta \mapsto \Psi(\eta)$ is a bijection from the set of $G'_{\mathrm{ad}}(k)$-conjugacy classes of surjective homomorphisms $\eta : G \to G'$ to the set of surjective $p$-morphisms $\eta^* : \Psi(G) \to \Psi(G')$ [Ste99, §5]. By using almost direct product decompositions, it follows that the map $\eta \mapsto \Psi(\eta)$ is a bijection from the set of $G'_{\mathrm{ad}}(k)$-conjugacy classes—or as we shall simply, equivalence classes—of normal homomorphisms $\eta : G \to G'$ to the set of $p$-morphisms $\Psi(G) \to \Psi(G')$. We obtain a functor $\Psi : \mathsf{Spl}_k \to \mathsf{BRD}_p$ from the category $\mathsf{Spl}_k$ of split connected reductive groups over $k$ and normal homomorphisms to the category $\mathsf{BRD}_p$ of based root data and $p$-morphisms. It restricts to a functor $\Psi : \mathsf{Spl}_k^{\mathrm{sep}} \to \mathsf{BRD}$, where in the domain we restrict to separable normal homomorphisms and in the codomain we restrict to (separable) morphisms of based root data. These functors are essentially surjective by the Existence Theorem. We use a subscript "Out" to indicate that we identify equivalent morphisms. For example, we write $\mathsf{Spl}_{k,\mathrm{Out}}$ and $\mathsf{Spl}_{k,\mathrm{Out}}^{\mathrm{sep}}$ to indicate the categories obtained by identifying equivalence classes of normal homomorphisms. We have an equivalence of categories $\Psi : \mathsf{Spl}_{k,\mathrm{Out}} \to \mathsf{BRD}_p$, which restricts to an equivalence of categories $\Psi : \mathsf{Spl}_{k,\mathrm{Out}}^{\mathrm{sep}} \to \mathsf{BRD}_p$.

### The general case

We now treat arbitrary connected reductive groups over $k$. Let $k_s$ be a separable closure of $k$ and let $\Gamma_k = \Gamma_{k_s/k}$. If $G$ is any connected reductive group over $k$, we can apply the above to the split connected reductive group $G_{k_s}$. Let $(B, T)$ be a Borel pair of $G_{k_s}$. For each $\sigma \in \Gamma_k$, we have an isomorphism $\sigma : X^*(T) \to X^*(\sigma_G(T))$ defined by $\chi^\sigma = \sigma_{\mathbb{G}_m} \circ \chi \circ \sigma_G^{-1}$. This defines an isomorphism

$$\sigma : \Psi(G_{k_s}, B, T) \longrightarrow \Psi(G_{k_s}, \sigma_G(B), \sigma_G(T)),$$

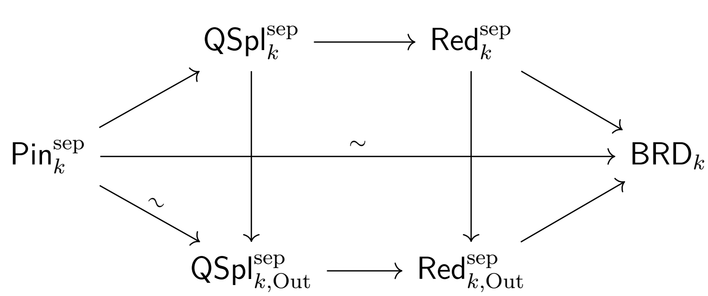
and thus an automorphism $\sigma \in \operatorname{Aut}(\Psi(G_{k_s}))$. This defines a continuous right action of $\Gamma_k$ on $\Psi(G_{k_s})$, that is a homomorphism $\Gamma_k \to \operatorname{Aut}(\Psi(G_{k_s}))^{\mathrm{op}}$, which is continuous with respect to the discrete topology on $\operatorname{Aut}(\Psi(G_{k_s}))^{\mathrm{op}}$. (If $k'/k$ is a finite Galois subextension of $k_s/k$ that splits $G$, then the open subgroup $\Gamma_{k_s/k'}$ of $\Gamma_k$ acts trivially on $\Psi(G_{k_s})$.) We prefer to work with the corresponding continuous left action $\Gamma_k \to \mathrm{Aut}(\Psi(G_{k_s}))$. We write $\Psi(G)$ as a shorthand for $\Psi(G_{k_s})$ together with the continuous left action of $\Gamma_k$. If $G'$ is another connected reductive group over $k$, we have $\Psi(G) \cong \Psi(G')$ if and only if $G$ and $G'$ are inner forms of each other.

If $\eta : G \to G'$ is a normal homomorphism, then $\Psi(\eta) : \Psi(G_{k_s}) \to \Psi(G'_{k_s})$ intertwines the actions of $\Gamma_k$. Thus, $\Psi$ is a functor from the category $\mathsf{Red}_k$ (resp. $\mathsf{Red}_k^{\mathrm{sep}}$) of connected reductive groups over $k$ and normal homomorphisms (resp. separable normal homomorphisms) to the category $\mathsf{BRD}_{k,p}$ (resp. $\mathsf{BRD}_k$) of based root data with continuous left actions of $\Gamma_k$ and $\Gamma_k$-equivariant $p$-morphisms (resp. morphisms of based root data). The functor $\Psi$ identifies equivalent normal homomorphisms and is exact. We thus have exact functors $\Psi : \mathsf{Red}_{k,\mathrm{Out}} \to \mathsf{BRD}_{k,p}$ and $\Psi : \mathsf{Red}^{\mathrm{sep}}_{k,\mathrm{Out}} \to \mathsf{BRD}_k$, however these are typically not equivalences. We will show that these functors admit natural sections.

Let $\mathsf{QSpl}_k$ (resp. $\mathsf{QSpl}_k^{\mathrm{sep}}$) denote the category of quasisplit connected reductive groups over $k$ and (resp. separable) normal homomorphisms. Let $\mathsf{QSpl}_{k,\mathrm{Out}}$ (resp. $\mathsf{QSpl}^{\mathrm{sep}}_{k,\mathrm{Out}}$) denote the categories obtained by identifying equivalent morphisms. Let $\mathsf{Pin}_k$ (resp. $\mathsf{Pin}_k^{\mathrm{sep}}$) denote the category of pinned connected reductive groups $(G, B, T, \{X_\alpha\}_\alpha)$ over $k$ and (resp. separable) normal homomorphisms

$$(G, B, T, \{X_\alpha\}) \to (G', B', T', \{X'_{\alpha'}\}_{\alpha'}).$$

The forgetful functors $\mathsf{Pin}_k \to \mathsf{QSpl}_{k,\mathrm{Out}}$ (resp. $\mathsf{Pin}_k^{\mathrm{sep}} \to \mathsf{QSpl}^{\mathrm{sep}}_{k,\mathrm{Out}}$) are equivalences and so are their compositions with $\mathsf{QSpl}_{k,\mathrm{Out}} \to \mathsf{BRD}_{k,p}$ (resp. $\mathsf{QSpl}^{\mathrm{sep}}_{k,\mathrm{Out}} \to \mathsf{BRD}_k$). In summary, we have following commutative diagram




and this commutative diagram extends to the commutative diagram obtained by removing all occurrences of "sep" and replacing $\mathsf{BRD}_k$ with $\mathsf{BRD}_{k,p}$. We call the inverse functor of $\Psi : \mathsf{QSpl}^{\mathrm{sep}}_{k,\mathrm{Out}} \to \mathsf{BRD}_k$, which is unique up to isomorphism, the realisation functor $Re : \mathsf{BRD}_k \to \mathsf{QSpl}^{\mathrm{sep}}_{k,\mathrm{Out}}$.

**The relative root datum, and parabolic and Levi subgroups**

Let $G$ be a connected reductive group over $k$. We will recall the relative root datum $\Psi_k(G)$ of $G$ and some basic facts about parabolic subgroups and Levi subgroups in relation to the $\Psi_k(G)$ and $\Psi(G)$. Since we assumed that based root data are reduced, we will use the term "relative based root datum" to mean a not necessarily reduced based root datum.

Let $(P_0, M_0)$ be a minimal parabolic pair of $G$. Let $A_0$ be the split component of the centre of $M_0$, which is a maximal split torus of $G$. Then $M_0 = C_G(A_0)$. Let $N_0$ denote the unipotent radical of $P_0$. We denote the relative based root datum of $(G, P_0, M_0)$ by

$$\Psi_k(G, P_0, M_0) = (X^*(A_0), \Phi_k(G, A_0), \Delta_k(P_0, A_0), X_*(A_0), \Phi_k^\vee(G, A_0), \Delta_k^\vee(P_0, A_0)).$$

The set of positive roots determined by $P_0$ is the set of relative roots of $(P_0, A_0)$ or equivalently $(N_0, A_0)$, which we denote by $\Phi_k(P_0, A_0) = \Phi_k(N_0, A_0)$.

Let $(P_0', M_0')$ be another minimal parabolic pair and define $A_0'$ and $N_0'$ as above. There exists $g \in G(k)$ such that $g \cdot (P_0, M_0) = (P_0', M_0')$ and $M_0'(k)g$ is unique. It follows that we have a canonical isomorphism $\mathrm{Int}(g) : A_0 \to A_0'$, and the canonical isomorphism $\mathrm{Int}(g)^* : X^*(A_0') \to X^*(A_0)$ is an isomorphism of relative based root data $\Psi_k(G, P_0, M_0) \to \Psi_k(G, P_0', M_0')$. We define the relative based root datum of $G$ to be the limit $\Psi_k(G) = \lim_{(P_0, M_0)} \Psi_k(G, P_0, M_0)$ with respect to the above canonical isomorphisms. We write

$$\Psi_k(G) = (X_{k,G}, \Phi_{k,G}, \Delta_{k,G}, X_{k,G}^\vee, \Phi_{k,G}^\vee, \Delta_{k,G}^\vee)$$

and omit "$G$" if it is clear from context. We also write $X_k(G) = X_{k,G}$, if it is preferable.

Note that when $G$ is split, we have $\Psi(G) = \Psi_k(G)$. In general, the relative based root datum of $G$ is related to the based root datum $\Psi(G)$ of $G$ by a canonical surjective homomorphism $X \to X_k$. Fix a minimal parabolic pair $(P_0, M_0)$ of $G$ and a Borel pair $(B, T)$ of $G_{k_s}$ with $P_{0,k)s} \supseteq B$ and $M_{0,k_s} \supseteq T$, which we abbreviate as $(P_0, M_0) \supseteq (B, T)$. Then $T \supseteq A_{0,k_s}$, restriction of characters defines a surjective homomorphism $X^*(T) \to X^*(A_0)$, and this determines a surjective homohomorphsim $X \to X_k$. To see that this homomorphism is canonical, suppose that we replace $(P_0, M_0) \supseteq (B, T)$ by a different choice $(P_0', M_0') \supseteq (B', T')$. There exists $g \in G(k_s)$ such that $g \cdot (P_{0,k_s}, M_{0,k_s}, B, T) = (P_{0,k_s}', M_{0,k_s}', B', T')$, and $T(k_s)g$ is unique. Thus the isomorphism $\mathrm{Int}(g)^* : X^*(T') \to X^*(T)$ is one of the canonical isomorphisms used to define $\Psi(G)$. Let $g_0 \in G(k)$ such that $g_0 \cdot (P_0, M_0) = (P_0', M_0')$. Then $g_0 \cdot (P_{0,k_s}, M_{0,k_s}) = (P'_{0,k_s}, M'_{0,k_s})$. It follows that $M_0(k_s)g = M_0(k_s)g_0$, so the isomorphism $\mathrm{Int}(g)^* : X^*(A_0') \to X^*(A_0)$ is one of the canonical isomorphisms used to define $\Psi_k(G)$. Since the diagram

$$\begin{array}{ccc} X^*(T') & \xrightarrow{\mathrm{Int}(g)^*} & X^*(T) \\ \downarrow & & \downarrow \\ X^*(A_0') & \xrightarrow{\mathrm{Int}(g)^*} & X^*(A_0) \end{array}$$

it follows that the homomorphism $X \to X_k$ is canonical.

Define $\Delta_0 = \Delta \cap \ker(X \to X_k)$ and $\Phi_0 = \Phi \cap \ker(X \to X_k)$. The sets $\Delta_0$ and $\Phi_0$ are $\Gamma_k$ stable, and thus so are $\Delta \setminus \Delta_0$ and $\Phi \setminus \Phi_0$. The homomorphism $X \to X_k$ restricts to a surjective maps $\Delta \setminus \Delta_0 \to \Delta_k$ and $\Phi \setminus \Phi_0 \to \Phi_k$, and the fibres of these maps are the $\Gamma_k$-orbits.

We recall that a subset $S$ of a root system $\Phi$ is said to be closed if $\alpha + \beta \in S$ for all $\alpha, \beta \in S$ with $\alpha + \beta \in \Phi$; symmetric if $S = -S$; a subsystem if $S$ is closed and symmetric; parabolic if $S$ is closed and $S \cup (-S) = \Phi$. If $S \subseteq \Phi$ is a parabolic subset, then $S \cap (-S)$ is a subsystem and we call it its associated Levi subsystem. For a subset $S \subseteq \Phi$ we will write $\Phi_S^+ = \Phi \cap \mathbb{Z}_{\geq 0} S$, $\Phi_S^- = \Phi \cap \mathbb{Z}_{\leq 0} S$, and $\Phi_S = \Phi \cap \mathbb{Z} S$. By Proposition 20 and Corollary 6 of [Bou02, Ch. VI, §1, no. 7], a subset $S \subseteq \Phi$ is parabolic if and only if there exists a base $\Delta \subseteq \Phi$ and a subset $I \subseteq \Delta$ such that $S = \Phi_\Delta^+ \cup \Phi_I^-$, and in this case $I$ is a a base of the Levi subsystem $S \cap (-S) = \Phi_I = \Phi_I^+ \cup \Phi_I^-$. Let $\Delta \subseteq \Phi$ be a fixed base. For each $I \subseteq \Delta$, we have a parabolic subset $\Phi_\Delta^+ \cup \Phi_I^-$ and its associated Levi subsystem is $\Phi_I = \Phi_I^+ \cup \Phi_I^-$ and has $I$ as a base. We call these parabolic subsets and Levi subsystems standard (with respect to $\Delta$).

Fix a minimal Levi subgroup $M_0$ of $G$. For each semistandard parabolic (resp. Levi) subgroup of $G$ its set of relative roots with respect to $A_0$ is a parabolic subset (resp. Levi subsystem) of $\Phi_k(G, A_0)$. Moreover, the map $P \mapsto \Phi_k(P, A_0)$ (resp. $M \mapsto \Phi_k(M, A_0)$) from the set of semistandard parabolic (resp. Levi) subgroups of $G$ to the set of parabolic subsets (resp. Levi subsystems) of $\Phi(G, A_0)$ is a bijection. A semistandard parabolic subgroup $P$ (resp. Levi subgroup $M$) is recovered as the subgroup of $G$ generated by $M_0$ and the relative root subgroups $U_{k,\alpha}$ for $\alpha \in \Phi(P, A_0)$ (resp. $\alpha \in \Phi(M, A_0)$). If $N$ is the unipotent radical of a semistandard parabolic subgroup $P$, then $\Phi(N, A_0) = \Phi(P, A_0) \setminus \Phi(M, A_0)$ and $N$ is the subgroup of $G$ generated by $U_{k,\alpha}$ for $\alpha \in \Phi(N, A_0)$. The split component $A_M$ of the centre of a semistandard Levi subgroup $M$ is the reduced identity component of $\bigcap_{\alpha \in \Phi(M,A_0)} \ker \alpha$. If we fix a minimal parabolic pair $(P_0, M_0)$, then the standard parabolic (resp. Levi) subgroups of $G$ correspond to the standard parabolic subsets (resp. Levi subsystems) of $\Phi_k(G, A_0)$ with respect to the base $\Delta_k(P_0, A_0)$.

Let $(P, M)$ be a parabolic pair of $G$. Choose a minimal parabolic pair $(P_0, M_0) \subseteq (P, M)$ of $G$. Then $(M \cap P_0, M_0)$ is a minimal parabolic pair of $M$. We have isomorphisms $\Psi_k(G) \cong \Psi_k(G, P_0, M_0)$ and $\Psi_k(M) \cong \Psi_k(M, M \cap P_0, M_0)$. Moreover, the (co)roots and simple (co)roots of $\Psi_k(M, M \cap P_0, M_0)$ are subsets of those of $\Psi_k(G, P_0, M_0)$. Let

$$\Psi_k(G)_{(P,M)} = (X_k(G), \Phi_k(G)_{(P,M)}, \Delta_k(G)_{(P,M)}, X_k^\vee(G), \Phi_k^\vee(G)_{(P,M)}, \Delta_k^\vee(G)_{(P,M)})$$

be the relative based root datum obtained from $\Psi_k(M, M \cap P_0, M_0)$ and the isomorphism $\Psi_k(G) \cong \Psi_k(G, P_0, M_0)$ by transport of structure. By construction, we have isomorphisms

$$\Psi_k(M) \cong \Psi_k(M, M \cap P_0, M_0) \cong \Psi_k(G)_{(P,M)}.$$

Moreover, $\Psi_k(G, M)_{(P,M)}$ and the composite isomorphism $\Psi_k(M) \cong \Psi_k(G)_{(P,M)}$ does not depend on the choice of $(P_0, M_0) \subseteq (P, M)$. Indeed, suppose that $(P_0', M_0') \subseteq (P, M)$ is another choice of minimal parabolic pair. It suffices to show that there exists $m \in M(k)$ with $m \cdot (P_0, M_0) = (P_0', M_0')$. There exists $p \in P(k)$ such that $p \cdot (P_0, M_0) = (P_0', M_0')$. Then $M$ and $p \cdot M$ are both Levi factors of $P$ containing the minimal Levi $M_0'$. Since there is a unique such Levi factor we have $p \cdot M = M$, and thus $p \in M(k)$. The isomorphism $\Psi(M) \to \Psi(G)_{(P,M)}$ determines an isomorphism of their relative Weyl groups $W_k(M) \to W_k(G)_{(P,M)} \subseteq W(G)$. The isomorphism is compatible with the bijection $\Phi_k(M) \to \Phi_k(G)_{(P,M)}$ and $W_k(G)_{(P,M)}$ is the subgroup of $W(G)$ generated by the reflections corresponding to roots in $\Phi_k(G)_{(P,M)}$. Suppose that $(P, M)$ and $(P', M')$ are conjugate parabolic pairs of $G$ and let $g \in G(k)$ such that $g \cdot (P, M) = (P', M')$. Then $M'(k)g$ is uniquely determined, and thus the isomorphism $\mathrm{Int}(g)^* : \Psi_k(M) \to \Psi_k(M')$ is canonical. It relates the isomorphisms $\Psi_k(M) \to \Psi_k(G)_{(P,M)}$ and $\Psi_k(M') \to \Psi(G)_{(P',M')}$.

For each $I \subseteq \Delta_k(G)$, we have a corresponding standard Levi subsystem $\Phi_k(G)_I$ of $\Phi_k(G)$, and a relative based root datum

$$\Psi_k(G)_I = (X_k(G), \Phi_k(G)_I, I, X_k^\vee(G), \Phi_k^\vee(G)_{I^\vee}, I^\vee).$$

Fix a minimal parabolic pair $(P_0, M_0)$ of $G$. For each standard parabolic $P$ of $G$ with standard Levi factor $M_P$, write $\Psi_k(G)_P = \Psi_k(G)_{(P,M_P)}$ and let $\Delta_k(G)_P$ be its set of simple roots. The map $P \mapsto \Delta_k(G)_P$ from the set of standard parabolic subgroups of $G$ (with respect to $P_0$) to the set of subsets of $\Delta_k(G)$ is a bijection.

We can apply the above considerations to $G_{k_s}$ and $\Psi(G) = \Psi_{k_s}(G_{k_s})$. Let $(P, M)$ be a parabolic pair of $G$. We will write

$$\Psi(G)_{(P,M)} = (X(G), \Delta(G)_{(P,M)}, X^\vee(G), \Delta^\vee(G)_{(P,M)})$$

for $\Psi_{k_s}(G_{k_s})_{P_{k_s},M_{k_s}}$. The isomorphisms $\Psi(M) \to \Psi(G)_{(P,M)}$ and $\Psi_k(M) \to \Psi_k(G)_{(P,M)}$ are compatible with respect to $X(G) \to X_k(G)$. It follows that $\Psi(G)_{(P,M)}$ inherits a continuous action of $\Gamma_k$ from $\Psi(G)$ and that the isomorphism $\Psi_k(M) \to \Psi_k(G)_{(P,M)}$ is $\Gamma_k$-equivariant. The Levi subsystems of $\Phi(G)$ that arise from Levi subgroups $M$ of $G$ are those that are $\Gamma_k$-stable and contain $\Phi_0(G)$. It follows that $G$ is quasisplit if and only if $\Phi_0(G) = \varnothing$, or equivalently $\Delta_0(G) = \varnothing$.

Let $(P, M)$ be a parabolic pair of $G$. The isomorphism $\Psi(M) \to \Psi(G)_{(P,M)}$ gives $\Gamma_k$-equivariant isomorphism of their Weyl groups $W(M) \to W(G)_{(P,M)} \subseteq W(G)$. We have $W(M)^{\Gamma_k} \cong W_k(M)$ and $W(G)^{\Gamma_k}_{(P,M)} \cong W_k(G)_{(P,M)}$, and the restricted isomorphism $W(M)^{\Gamma_k} \to W(G)^{\Gamma_k}_{(P,M)}$ coincides with the isomorphism $W_k(M) \to W_k(G)_{(P,M)}$ above.

### 4.2.2 Dual groups

Let $C$ be a field. Let $G$ be a split connected reductive group over $C$. We define an $L$-action to be a continuous left action $\Gamma_k \to \mathrm{Aut}(G)$ that preserves a pinning. Let $\mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k}$ denote the category whose objects are split connected reductive groups $G$ over $C$ with a continuous actions $\Gamma_k \to \mathrm{Aut}(G)$ and whose morphisms are $\Gamma_k$-equivariant separable normal homomorphisms. Let $\mathsf{Spl}^{\mathrm{sep}}_{C,L}$ denote the full subcategory of objects whose continuous action of $\Gamma_k$ is an $L$-action. We have a functor $\Psi : \mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k} \to \mathsf{BRD}_k$. (If the characteristic of $C$ is equal to the characteristic of $k$, this extends to a functor from the category $\mathsf{Spl}_{C,\Gamma_k}$ obtained from $\mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k}$ by dropping the separability requirement on morphisms to the cateogry $\mathsf{BRD}_{k,p}$, and there are evident generalisations of what follows.) We define the equivalence class of a $\Gamma_k$-equivariant homomorphism $G \to G'$ to be its $G'_{\mathrm{ad}}(C)^{\Gamma_k}$-conjugacy class. The above functor identifies equivalent $\Gamma_k$-equivariant separable normal homomorphisms. Thus, we have a functor $\Psi : \mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k,\mathrm{Out}} \to \mathsf{BRD}_k$, where $\mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k,\mathrm{Out}}$ is obtained from $\mathsf{Spl}^{\mathrm{sep}}_{C,\Gamma_k}$ by identifying equivalent morphisms.

We remark that if $C$ is algebraically closed and $G$ is a connected reductive group over $C$ with an $L$-action of $\Gamma_k$, then the canonical homomorphism $G^{\Gamma_k} \to G_{\mathrm{ad}}^{\Gamma_k}$ is surjective and the pinnings of $G$ that are $\Gamma_k$-stable are all $G^{\Gamma_k}$-conjugate by [Kot84, §1.6, §1.7].

The restricted functor $\Psi : \mathsf{Spl}_{C,L}^{\mathrm{sep}} \to \mathsf{BRD}_k$ is an equivalence. Indeed, let $\mathsf{PinSpl}_{C,\Gamma_k}^{\mathrm{sep}}$ whose objects are pinned split connected reductive groups $(G, B, T, \{X_\alpha\}_\alpha)$ over $C$ with a continuous action $\Gamma_k \to \mathrm{Aut}(G, B, T, \{X_\alpha\}_\alpha)$ and whose morphisms are $\Gamma_k$-equivariant separable normal homomorphisms. The forgetful functor $\mathsf{PinSpl}_{C,\Gamma_k}^{\mathrm{sep}} \to \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}}$ is an equivalence and so is its composition with $\Psi : \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}} \to \mathsf{BRD}_k$. We have the following commutative diagram of functors

$$\begin{array}{ccccccc}
 & & \mathsf{Spl}_{C,L}^{\mathrm{sep}} & \longrightarrow & \mathsf{Spl}_{C,\Gamma_k}^{\mathrm{sep}} & & \\
 & \nearrow & \downarrow & & \downarrow & \searrow & \\
\mathsf{PinSpl}_{C,\Gamma_k}^{\mathrm{sep}} & & & \xrightarrow{\sim} & & & \mathsf{BRD}_k \\
 & \searrow^{\sim} & \downarrow & & \downarrow & \nearrow & \\
 & & \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}} & \longrightarrow & \mathsf{Spl}_{C,\Gamma_k,\mathrm{Out}}^{\mathrm{sep}} & &
\end{array}$$

We call the inverse functor of $\Psi : \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}} \to \mathsf{BRD}_k$, which is unique up to isomorphism, the realisation functor $Re : \mathsf{BRD}_k \to \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}}$.

There is a natural duality operation on based root data: $(X, \Delta, X^\vee, \Delta^\vee) \mapsto (X^\vee, \Delta^\vee, X, \Delta)$. It defines a contravariant equivalence from $\mathsf{BRD}_p$ (resp. $\mathsf{BRD}$) to itself. The duality functor further determines contravariant equivalences from $\mathsf{BRD}_{k,p}$ (resp. $\mathsf{BRD}_k$) to itself. Applying the duality functor to $\Psi(G)$, we obtain $\Psi(G)^\vee$ and a continuous homomorphism $\Gamma_k \to \mathrm{Aut}(\Psi(G_{k_s})^\vee)$. We write $\Psi(G)^\vee$ as a shorthand for $\Psi(G_{k_s})^\vee$ together with the continuous left action of $\Gamma_k$. The composite

$$\mathsf{Red}_{k,\mathrm{Out}}^{\mathrm{sep}} \xrightarrow{\Psi} \mathsf{BRD}_k \xrightarrow{(\cdot)^\vee} \mathsf{BRD}_k \xrightarrow{Re} \mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}}$$

is an exact contravariant functor called the Langlands dual group functor. We denote it on objects by $G \mapsto G^\vee$ and on morphisms by $\eta \mapsto \eta^\vee$. Note that $G^\vee$ comes with an $L$-action $\rho_G : \Gamma_k \to \mathrm{Aut}(G^\vee)$ and a $\Gamma_k$-equivariant isomorphism $\eta_G : \Psi(G)^\vee \to \Psi(G^\vee)$. The $L$-action $\rho_G$ is trivial if $G$ is split and $G^\vee = (G^\vee, \rho_G, \eta_G)$ only depends on the inner class of $G$. Note that the dual group functor has a natural section

$$\mathsf{Spl}_{C,L,\mathrm{Out}}^{\mathrm{sep}} \xrightarrow{\Psi} \mathsf{BRD}_k \xrightarrow{(\cdot)^\vee} \mathsf{BRD}_k \xrightarrow{Re} \mathsf{QSpl}_{k,\mathrm{Out}}^{\mathrm{sep}}$$

There is a canonical $\Gamma_k$-equivariant isomorphism $X^*(G_{k_s}) \to X_*(Z(G^\vee)^\circ)$ defined as follows. The isomorphism $\eta_G : \Psi(G)^\vee \to \Psi(G^\vee)$ gives us a $\Gamma_k$-equivariant isomorphism $X(G) \cong X^\vee(G^\vee)$ and a compatible isomorphism $W(G) \cong W(G^\vee)$. Fixing Borel pairs $(B, T)$ and $(\mathcal{B}^0, \mathcal{T}^0)$ for $G$ and $G^\vee$, respectively, we obtain a $\Gamma_k$-equivariant isomorphism $X^*(T)^{W(G_{k_s},T)} \cong X_*(\mathcal{T}^0)^{W(G^\vee,\mathcal{T}^0)}$, where the action of $\Gamma_k$ on the left is inherited from the action of $\Gamma_k$ on $X(G)$. We have $Z(G^\vee)^\circ = ((\mathcal{T}^0)^{W(G^\vee,\mathcal{T}^0)})^\circ$, and thus $X_*(\mathcal{T}^0)^{W(G^\vee,\mathcal{T}^0)} = X_*(Z(G^\vee)^\circ)$. Moreover, restriction gives us an isomorphism $X^*(G_{k_s}) \to X^*(T)^{W(G_{k_s},T)}$. (See [SZ18, §4.1.1] or [KS13, Lemma 13].) Thus, we obtain an isomorphism $X^*(G_{k_s}) \cong X_*(Z(G^\vee)^\circ)$ and this is seen to not depend on the choices of Borel pairs $(B, T)$ and $(\mathcal{B}^0, \mathcal{T}^0)$. The isomorphism $X^*(G_{k_s}) \cong X_*(Z(G^\vee)^\circ)$ is $\Gamma_k$-equivariant for the action of $\Gamma_k$ on $X^*(G_{k_s})$ obtained by transporting the action of $\Gamma_k$ on $X(G)$, and this is the same as the usual $\Gamma_k$-action on $X^*(G_{k_s})$. By passing to $\Gamma_k$-invariants and using $X_*(Z(G^\vee)^\circ)^{\Gamma_k} = X_*(Z(G^\vee)^{\Gamma_k,\circ})$, we obtain an isomorphism

$$X^*(G) \longrightarrow X_*(Z(G^\vee)^{\Gamma_k,\circ}),$$

which we write as $\theta \mapsto \theta^\vee$.

### Duality and Levi subgroups

Suppose that $(\mathcal{P}^0, \mathcal{M}^0)$ is a $\Gamma_k$-stable parabolic pair of $G^\vee$. Then $\Psi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$ inherits a continuous $\Gamma_k$-action from $\Psi(G^\vee)$, and $\Psi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$ is the relative based root datum associated to the $\Gamma_k$-stable subset $\Delta_k(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$ of $\Delta_k(G^\vee)$. Conversely, suppose that $I \subseteq \Delta_k(G^\vee)$ is $\Gamma_k$ stable, then the relative based root datum $\Psi(G^\vee)_I$ inherits a continuous $\Gamma_k$-action from $\Psi(G^\vee)$. Moreover, there exists a $\Gamma_k$-stable parabolic pair $(\mathcal{P}^0, \mathcal{M}^0)$ of $G^\vee$ such that $\Psi(G^\vee)_I = \Psi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$. For example, let $(\mathcal{B}^0, \mathcal{T}^0)$ be a $\Gamma_k$-stable Borel pair of $G^\vee$ and let $(\mathcal{P}^0, \mathcal{M}^0)$ be the standard parabolic pair of $G^\vee$ coresponding to $I$. Since $\mathcal{P}^0$ (resp. $\mathcal{M}^0$) is generated by $\mathcal{T}^0$ and the root subgroups of $G^\vee$ attached to the roots in $\Phi(G^\vee)^+ \cup I$ (resp. $I$), and since this is $\Gamma_k$-stable, it follows that $\mathcal{P}^0$ (resp. $\mathcal{M}^0$) is $\Gamma_k$-stable.

Now, let $(P, M)$ be a parabolic pair of $G$. The isomorphism $\eta_G : \Psi(G)^\vee \to \Psi(G^\vee)$ and the based root datum $\Psi(G)_{(P,M)}$ determines a based root datum $\Psi(G^\vee)_{(P,M)}$ that inherits a continuous $\Gamma_k$-action from $\Psi(G^\vee)$ and a $\Gamma_k$-equivariant isomorphism $\eta_G : \Psi(G)^\vee_{(P,M)} \to \Psi(G^\vee)_{(P,M)}$. Let $(\mathcal{P}^0, \mathcal{M}^0)$ be a $\Gamma_k$-stable parabolic pair of $G^\vee$ such that $\Psi(G^\vee)_{(P,M)} = \Psi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$. We say that $\mathcal{M}^0$ is a $\Gamma_k$-stable $G$-relevant Levi subgroup of $G^\vee$. The $\Gamma_k$-stable $G$-relevant Levi subgroups of $G^\vee$ are precisely the $\Gamma_k$-stable Levi subgroups $\mathcal{M}^0$ of $G^\vee$ such that for some, and hence any, $\Gamma_k$-stable parabolic subgroup $\mathcal{P}^0$ with Levi factor $\mathcal{M}^0$ the set $\Phi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)}$ contains the set of roots corresponding to $\Phi(G)_0^\vee$ under $\eta_G$. More generally, we say that a Levi subgroup of $G^\vee$ is $G$-relevant if it is conjugate to a $\Gamma_k$-stable $G$-relevant Leci subgroup. Note that if $\mathcal{M}^0$ is a $G$-relevant Levi subgroup of $G^\vee$ and $\mathcal{L}^0$ is a Levi subgroup of $G^\vee$ with containing $\mathcal{M}^0$, then $\mathcal{L}$ is $G$-relevant. We have a $\Gamma_k$-equivariant isomorphism $f_{P,\mathcal{P}^0} : \Psi(M^\vee) \to \Psi(\mathcal{M}^0)$ defined to be the composite of

$$\Psi(M^\vee) \xrightarrow{\eta_M^{-1}} \Psi(M)^\vee \longrightarrow \Psi(G)^\vee_{(P,M)} \xrightarrow{\eta_G} \Psi(G^\vee)_{(\mathcal{P}^0,\mathcal{M}^0)} \longrightarrow \Psi(\mathcal{M}^0).$$

Let $f_{P,\mathcal{P}^0} : M^\vee \to \mathcal{M}^0$ denote the corresponding equivalence class of $\Gamma_k$-equivariant isomorphisms.

Suppose that $(P_i, M_i)$, $(\mathcal{P}_i^0, \mathcal{M}_i^0)$ for $i = 1, 2$ are parabolic pairs as above. Let

$$W^G(L_1, L_2) = \{g \in G(k) : g \cdot L_1 = L_2\}/L_1(k)$$

and

$$W^G(\mathcal{M}_1^0, \mathcal{M}_2^0) = \{g \in G^\vee(C) : g \cdot \mathcal{M}_1^0 = \mathcal{M}_2^0\}/\mathcal{M}_1^0(C).$$

There is a unique bijection $W^G(M_1, M_2) \to W^G(\mathcal{M}_1^0, \mathcal{M}_2^0)$ compatible with the isomorphisms $f_{P_i,\mathcal{P}_i^0}$. A proof of this is given in [Var, Lemma 2.4.8]. It follows from the proof, which makes use of the argument in the proof of [Bor79, §6.2], that the elements of $W(\mathcal{M}_1^0, \mathcal{M}_2^0)$ have $\Gamma_k$-fixed representatives.

It follows from this and the bijection $W^G(M_1, M_2) \to W^G(\mathcal{M}_1^0, \mathcal{M}_2^0)$ that $M_1$ and $M_2$ are $G(k)$-conjugate if and only if $\mathcal{M}_1^0$ and $\mathcal{M}_2^0$ are $G^\vee(C)^{\Gamma_k}$-conjugate.

Consequently, for each equivalence class of $\Gamma_k$-equivariant isomorphisms $f_{P,\mathcal{P}^0}$ : $M^\vee \to \mathcal{M}^0$, composing with the $\Gamma_k$-equivariant embedding $\mathcal{M}^0 \to G^\vee$ gives an equivalence class $\iota_M^G : M^\vee \to G^\vee$ of $\Gamma_k$-equivariant embeddings that does not depend on $P$ or $\mathcal{P}^0$, and identifies $M^\vee$ with a $\Gamma_k$-stable $G$-relevant Levi subgroup of $G^\vee$. The map $M \mapsto \iota_M^G(M)$ defines a bijection between the set of $G(F)$-conjugacy classes of Levi subgroups of $G$ and $G^\vee(C)^{\Gamma_k}$-conjugacy classes of $\Gamma_k$-stable $G$-relevant Levi subgroups of $G^\vee$. If $L \subseteq M$ are Levi subgroups of $G$, then $\iota_M^G \circ \iota_L^M = \iota_L^G$. (See [Var, Proposition 2.4.15].)

### 4.2.3 $L$-groups and $\lambda$-groups

Now, we assume that $k = F$ is a local or global field of arbitrary characteristic. From now on, we take $C = \mathbb{C}$ in any discussion of Langlands dual groups. We define the $L$-group of $G$ to be ${}^LG = G^\vee \rtimes W_F$, where $W_F$ acts by the composition of $W_F \to \Gamma_F$

and $\rho_G : \Gamma_F \to \mathrm{Aut}(G^\vee)$. The $L$-group ${}^LG$ is a locally compact group. Note that ${}^L1 = W_F$.

Given a surjective homomorphism $\mathcal{G} \to W_F$, we will write $\mathcal{G}^0 = \ker(\mathcal{G} \to W_F)$. Note that we have used a superscript "0" instead of a superscript "$\circ$", which we use to denote identity components.

Let $\mathcal{G}$ be a second countable locally compact group together with a continuous surjective homomorphism $\mathcal{G} \to W_F$. By the open mapping theorem, the continuous surjective homomorphism $\mathcal{G} \to W_F$ is open. Suppose that the kernel $\mathcal{G}^0 = \ker(\mathcal{G} \to W_F)$ is the group of $\mathbb{C}$-points of a connected reductive group $\mathcal{G}^0$ over $\mathbb{C}$ and that for all $g \in \mathcal{G}$ the automorphism $\mathrm{Int}(g)|_{\mathcal{G}^0} : \mathcal{G}^0 \to \mathcal{G}^0$ is algebraic. The resulting homomorphism $\mathcal{G} \to \mathrm{Aut}(\Psi(\mathcal{G}^0))$ then factors through $W_F$. Suppose that it further factors through $W_F/W_K = \Gamma_{K/F}$ for some finite Galois subextension $K/F$ of $F_s/F$, and thus extends to a continuous homomorphism $\Gamma_F \to \mathrm{Aut}(\Psi(\mathcal{G}^0))$ along $W_F \to \Gamma_F$. We then say that $\mathcal{G} \to W_F$, or just $\mathcal{G}$, is a $\lambda$-group. We say that a $\lambda$-group is global or local according to whether $F$ is global or local. We denote the preimage of $W_F^1 = \ker(W_F \to \Gamma_F)$ in $\mathcal{G}$ by $\mathcal{G}^1$.

An element $g \in \mathcal{G}$ is said to be semisimple if $\mathrm{Int}(g)|_{\mathcal{G}^0}$ is a semisimple automorphism of $\mathcal{G}^0$ (after embedding $\mathcal{G}^0$ in a general linear group it can be realised as conjugation by a semisimple element). If $g \in \mathcal{G}^0$, then $g$ is semisimple as an element of $\mathcal{G}$ if and only if it is semisimple as an element of $\mathcal{G}^0$. If $\mathcal{G}$ and $\mathcal{G}'$ are $\lambda$ groups, an isomorphism of topological groups $\mathcal{G} \to \mathcal{G}'$ over $W_F$ maps semisimple elements to semisimple elements.

Note that an $L$-group ${}^LG = G^\vee \rtimes W_F$ of a connected reductive group $G$ over $F$ together with the projection ${}^LG \to W_F$ is a $\lambda$-group. The semisimple elements of ${}^LG$ are those of the form $(g, w)$, where $g$ is a semisimple element of $G^\vee$.

Suppose that $\mathcal{G}$ is a global $\lambda$-group. For each place $v$ of $F$, choose an embedding $\overline{F}_v \to \overline{F}$ over $F$. This determines a continuous injective homomorphism $\Gamma_{F_v} \to \Gamma_F$ whose $\Gamma_F$-conjugacy class does not depend on the choice of $\overline{F}_v \to \overline{F}$. There exists a continuous injective homomorphism $W_{F_v} \to W_F$ such that the following diagram commutes

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & \Gamma_{F_v} \\ \downarrow & & \downarrow \\ W_F & \longrightarrow & \Gamma_F \end{array}$$

and the $W_F^1$-conjugacy class of $W_{F_v} \to W_F$ does not depend on the choice of $\overline{F}_v \to \overline{F}$. (See [Tat79] for more details.) Let $\mathcal{G}_v \to W_{F_v}$ be the pullback of $\mathcal{G} \to W_F$ along

$W_{F_v} \to W_F$. It provides us with a commutative diagram of second countable locally compact groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{G}_v^0 & \longrightarrow & \mathcal{G}_v & \longrightarrow & W_{F_v} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & \mathcal{G} & \longrightarrow & W_F & \longrightarrow & 1 \end{array}$$

where $\mathcal{G}_v^0 \to \mathcal{G}^0$ is an isomorphism of topological groups and $\mathcal{G}_v \to \mathcal{G}$ is a continuous injection. We have that $\mathcal{G}_v$ is a $\lambda$-group and the $\mathcal{G}^1$-conjugacy class of $\mathcal{G}_v \to \mathcal{G}$ does not depend on the choice of $\overline{F}_v \to \overline{F}$.

If $\mathcal{H}, \mathcal{G}$ are global $\lambda$-groups, a continuous homomorphism of $\xi : \mathcal{H} \to \mathcal{G}$ over $W_F$ pulls back to a continuous homomorphism $\xi_v : \mathcal{H}_v \to \mathcal{G}_v$ over $W_{F_v}$, that is, making the diagram

$$\begin{array}{ccc} \mathcal{H}_v & \xrightarrow{\xi_v} & \mathcal{G}_v \\ \downarrow & & \downarrow \\ \mathcal{H} & \xrightarrow{\xi} & \mathcal{G} \end{array}$$

commute, and the $\mathcal{G}_v^0$-conjugacy class of $\xi_v$ only depends on the $\mathcal{G}^0$-conjugacy class of $\xi$.

Let $\mathcal{H}, \mathcal{G}$ be two $\lambda$-groups. A continuous homomorphism $\xi : \mathcal{H} \to \mathcal{G}$ over $W_F$ is said to be an $L$-homomorphism if its restriction $\xi^0 : \mathcal{H}^0 \to \mathcal{G}^0$ is algebraic and it is locally semisimple, which means the following: if $F$ is local, then $\xi$ maps semisimple elements to semisimple elements, and if $F$ is global, then all localisations $\xi_v$ of $\xi$ map semisimple elements to semisimple elements. Two $L$-homomorphisms $\xi, \xi' : \mathcal{H} \to \mathcal{G}$ are said to be equivalent if there exists $g \in \mathcal{G}^0$ such that $\xi' = \mathrm{Int}(g) \circ \xi$. We define a morphism of $\lambda$-groups to be an equivalence class of $L$-homomorphisms and denote the category of $\lambda$-groups by $\lambda\mathsf{Gp}$. We call an $L$-homomorphism $\xi : \mathcal{H} \to \mathcal{G}$ an $L$-embedding if it is a topological embedding, in which case $\xi(\mathcal{H})$ is a closed subgroup of $\mathcal{G}$ and $\xi^0 : \mathcal{H}^0 \to \mathcal{G}^0$ is a closed embedding of algebraic groups. We will often switch between thinking of $\xi$ as an equivalence class and a single $L$-homomorphism

For a separable normal homomorphism $\eta : G \to G'$, we have an equivalence class of $L$-homomorphisms ${}^L\eta = \eta^\vee \rtimes \mathrm{id}_{W_F} : {}^LG' \to {}^LG$ and this gives an exact contravariant functor $\mathsf{Red}_{F,\mathrm{Out}}^{\mathrm{sep}} \to \lambda\mathsf{Gp}$ from the category $\mathsf{Red}_{F,\mathrm{Out}}^{\mathrm{sep}}$ whose objects are connected reductive groups over $F$ and whose morphisms are equivalence classes (codomain conjugacy classes) separable normal homomorphisms. We denote its essential image by $L\mathsf{Gp}$ and call it the category of $L$-groups. A $\lambda$-group $\mathcal{G}$ lies in $L\mathsf{Gp}$ if and only if there is a splitting $c : W_F \to \mathcal{G}$ (i.e. continuous homomorphic section of $\mathcal{G} \to W_F$) such that the action $\rho_c : W_F \to \mathrm{Aut}(\mathcal{G}^0)$ defined by $\rho_c(w) = \mathrm{Int}(c(w))|_{\mathcal{G}^0}$ preserves a pinning, in which case $\rho_c$ factors through $\Gamma_{K/F}$ if $W_F \to \mathrm{Aut}(\Psi(\mathcal{G}^0))$ does.

Suppose that $\mathcal{H}$ is a closed subgroup of ${}^L G$ that surjects onto $W_F$ and $\mathcal{H}^0 = \ker(\mathcal{H} \to W_F)$ is connected reductive group. Then the homomorphism $W_F \to \mathrm{Aut}(H^\vee)$ factors through $W_F/W_K = \Gamma_{K/F}$ for some finite Galois subextension $K/F$ of $F_s/F$, and thus $\mathcal{H}$ is a $\lambda$-group and the inclusion $\mathcal{H} \to {}^L G$ is an $L$-embedding. (See p. 24 of [KS99] where a similar fact is proved for the subgroups $\mathcal{H}$ that appear in endoscopy. The key idea is the introduction of a subgroup $\mathcal{U} \subseteq \mathcal{H}$ as in the proof of [KS99, Lemma 2.2.A].)

Every $L$-homomorphism $\xi : {}^L H \to {}^L G$ can be written in the form $\xi(h, w) = (\xi^0(h)a_\xi(w), w)$, where $\xi^0 : H^\vee \to G^\vee$ is a morphism of algebraic groups and $a_\xi \in Z^1_c(W_F, G^\vee)$. The cohomology class $a_\xi \in H^1_c(W_F, G^\vee)$ only depends on the equivalence class of $\xi$. One says that $\xi$ is tempered (or bounded, or of unitary type) if the image of $a_\xi$ in $G^\vee$ is bounded, that is, has compact closure. We will use the notation $a_\xi$ and $\xi^0$ without comment. Sometimes we write $\xi_0$ for $\xi^0$.

### Parabolic and Levi subgroups of $L$-groups

A parabolic subgroup ${}^L G$ is defined to be a closed subgroup $\mathcal{P}$ that maps onto $W_F$ such that $\mathcal{P}^0$ is a parabolic subgroup of $G^\vee$. (Recall that $\mathcal{P}^0 = \ker(\mathcal{P} \to W_F)$ as defined above.) We have $\mathcal{P} = N_{{}^L G}(\mathcal{P}^0)$. The unipotent radical $\mathcal{N}_\mathcal{P}$ of $\mathcal{P}$ is defined to be the unipotent radical $N_{\mathcal{P}^0}$ of $\mathcal{P}^0$. It is a normal subgroup of $\mathcal{P}$. A Levi factor of $\mathcal{P}$ is defined to be a closed subgroup $\mathcal{M}$ of ${}^L G$ such that $\mathcal{M}$ maps onto $W_F$ and $\mathcal{M}^0$ is a Levi factor of $\mathcal{P}^0$. We have $\mathcal{M} = N_\mathcal{P}(\mathcal{M}^0)$ and $\mathcal{P} = \mathcal{M} \ltimes \mathcal{N}_\mathcal{P}$. A Levi subgroup of ${}^L G$ is a defined to be a Levi factor of a parabolic subgroup of ${}^L G$, that is, a closed subgroup $\mathcal{M}$ of ${}^L G$ such that $\mathcal{M}$ maps onto $W_F$ and $\mathcal{M}^0$ is a Levi subgroup of $G^\vee$. We say that a Levi subgroup $\mathcal{M}$ of ${}^L G$ is $G$-relevant if $\mathcal{M}^0$ is $G$-relevant. Note that if $\mathcal{M}$ is a $G$-relevant Levi subgroup of ${}^L G$ and if $\mathcal{L}$ is a Levi subgroup of ${}^L G$ containing $\mathcal{M}$, then $\mathcal{L}$ is $G$-relevant.

If $\mathcal{S}^0 \subseteq G^\vee$ is a torus such that $C_{{}^L G}(\mathcal{S}^0)$ maps onto $W_F$, then $C_{{}^L G}(\mathcal{S}^0)$ is a Levi subgroup of ${}^L G$. Moreover, each Levi subgroup $\mathcal{M}$ of ${}^L G$ is obtained in this way since $\mathcal{M} = C_{{}^L G}(Z(\mathcal{M}^0)^{\Gamma_F, \circ})$. (See [SZ18, §5.4].)

If $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ and $\mathcal{P}^0$ is $\Gamma_F$-stable, then $\mathcal{P} = \mathcal{P}^0 \rtimes W_F$. Conversely, if $\mathcal{P}^0$ is a $\Gamma_F$-stable parabolic subgroup of $G^\vee$, then $\mathcal{P}^0 \rtimes W_F$ is a parabolic subgroup of ${}^L G$. Suppose that $\mathcal{P}^0$ is $\Gamma_F$-stable parabolic subgroup of $G^\vee$, and let $\mathcal{P} = \mathcal{P}^0 \rtimes W_F$. If $\mathcal{M}$ is a Levi factor of $\mathcal{P}$ and $\mathcal{M}^0$ is $\Gamma_F$-stable, then $\mathcal{M} = \mathcal{M}^0 \rtimes W_F$.

Conversely, if $\mathcal{M}^0$ is a $\Gamma_F$-stable Levi factor of $\mathcal{P}^0$, then $\mathcal{M} \rtimes W_F$ is a Levi factor of $\mathcal{P}$.

Fix a $\Gamma_F$-stable Borel pair $(\mathcal{B}^0, \mathcal{T}^0)$ of $G^\vee$. We refer to parabolic and Levi subgroups of ${}^LG$ containing $\mathcal{T}^0 \rtimes W_F$ as semistandard, and we refer to a parabolic subgroups of ${}^LG$ containing $\mathcal{B}^0 \rtimes W_F$ as standard. The semistandard parabolic (resp. Levi) subgroups of ${}^LG$ are those of the form $\mathcal{P}^0 \rtimes W_F$ (resp. $\mathcal{M}^0 \rtimes W_F$), where $\mathcal{P}^0$ (resp. $\mathcal{M}^0$) is a $\Gamma_F$-stable semistandard parabolic (resp. Levi) subgroup of $G^\vee$. Every semistandard parabolic subgroup $\mathcal{P}^0 \rtimes W_F$ of ${}^LG$ has a unique semistandard Levi factor, namely $\mathcal{M}^0 \rtimes W_F$, where $\mathcal{M}^0$ is the unique semistandard Levi factor of $\mathcal{P}^0$. We refer to the semistandard Levi factor of a standard parabolic subgroup of ${}^LG$ as standard. Every parabolic subgroup of ${}^LG$ is $G^\vee$-conjugate to a unique standard parabolic subgroup of ${}^LG$, and every Levi subgroup of ${}^LG$ is $G^\vee$-conjugate to a standard Levi subgroup of ${}^LG$.

Recall that for each Levi subgroup $M$ of $G$, we have a canonical equivalence class of $\Gamma_F$-equivariant embeddings $\iota_M^G : M^\vee \to G^\vee$, identifying $M^\vee$ with a $\Gamma_F$-stable Levi subgroup $\mathcal{M}^0$ of $G^\vee$. Consequently, we have a canonical equivalence class of $L$-embeddings $\iota_M^G : {}^LM \to {}^LG$, identifying ${}^LM$ with a $G$-relevant Levi subgroup of ${}^LG$. The map $M \mapsto \iota_M^G({}^LM)$ defines a bijection between the set of $G(F)$-conjugacy classes of Levi subgroups of $G$ and the $G^\vee$-conjugacy classes of relevant Levi subgroups of ${}^LG$. If $L \subseteq M$ are Levi subgroups of $G$, then $\iota_M^G \circ \iota_L^M = \iota_L^G$.

Let $\mathcal{H}$ be a subgroup of ${}^LG$ that maps onto $W_F$. By [Bor79, Proposition 3.6] the Levi subgroups $\mathcal{M}_\mathcal{H}$ of ${}^LG$ that contain $\mathcal{H}$ minimally are all conjugate by $C_{G^\vee}(\mathcal{H})$. We say that $\mathcal{H}$ is $G$-relevant if $\mathcal{H}$ is only contained in $G$-relevant Levi subgroups of ${}^LG$, or equivalently if $\mathcal{M}_\mathcal{H}$ is $G$-relevant. We say that $\mathcal{H}$ is elliptic if it is not contained in a proper Levi subgroup of ${}^LG$. If $\mathcal{H}$ is a $\lambda$-group, an $L$-homomorphism $\xi : \mathcal{H} \to {}^LG$ is said to be $G$-relevant (resp. elliptic) if its image is $G$-relevant (resp. elliptic), and we write $\mathcal{M}_\xi = \mathcal{M}_{\xi(\mathcal{H})}$.

## 4.3 $L$-parameters

We return to assuming that $F$ is a local field of characteristic zero. Let $L_F$ denote the local Langlands group, which is defined by

$$L_F = \begin{cases} {}^L1 = W_F & \text{if } F \text{ is archimedean,} \\ {}^L\mathrm{PGL}_2 = \mathrm{SL}_2(\mathbb{C}) \times W_F & \text{if } F \text{ is non-archimedean.} \end{cases}$$

We write the homomorphism $L_F \to W_F$ as $l \mapsto w(l)$. For an $L$-homomorphism $\phi : L_F \to {}^L(G)$, we write $\phi(l) = (a_\phi(l), w(l))$. We denote by $\widetilde{\Phi}(G)$ the set of $G$-relevant $L$-homomorphisms $\phi : L_F \to {}^L G$. An $L$-parameter of $G$ is a $G^\vee$-conjugacy class in $\widetilde{\Phi}(G)$, and we denote the set of $L$-parameters of $G$ by $\Phi(G)$. We denote the set of tempered or bounded elements of $\widetilde{\Phi}(G)$ by $\widetilde{\Phi}_{\text{temp}}(G)$, and we denote its image in $\Phi(G)$ by $\Phi_{\text{temp}}(G)$. We call the elements of $\Phi_{\text{temp}}(G)$ tempered $L$-parameters. Furthermore, we define $\widetilde{\Phi}_2(G)$ to be the subset of all elliptic elements of $\widetilde{\Phi}_{\text{temp}}(G)$, and we denote its image in $\Phi_{\text{temp}}(G)$ by $\Phi_2(G)$. We call the elements of $\Phi_2(G)$ discrete $L$-parameters.

### 4.3.1 Central and cocentral characters, and the Langlands pairing

In order to formulate various properties of the local Langlands correspondence we need two constructions originally due to Langlands [Lan89]. A more intrinsic approach is given by Kaletha in [Kal15] using the cohomology of crossed modules. The first is the central character map

$$H^1_c(W_F, G^\vee) \longrightarrow \Pi(Z_G(F)).$$

The classes $H^1_{c,\text{bdd}}(W_F, G^\vee)$ represented by bounded 1-cocycles map into the group $\Pi_u(Z_G(F))$ of unitary central characters. We have a map $\Phi(G) \to H^1_c(W_F, G^\vee)$ defined by $\phi \mapsto a_\phi$, and this restricts to a map $\Phi(G) \to H^1_{c,\text{bdd}}(W_F, G^\vee)$. Thus, we have a central character map $\Phi_{(\text{temp})}(G) \to \Pi_{(u)}(Z_G(F))$, which we write as $\phi \mapsto \zeta_\phi$. For a central datum $(\mathcal{Z}, \zeta)$ of $G(F)$, we can thus define $\Phi(G, \zeta) = \{\phi \in \Phi(G) : \chi_\phi|_{\mathcal{Z}} = \zeta\}$. We define $\Phi_{\text{temp}}(G, \zeta) = \Phi_{\text{temp}}(G) \cap \Phi(G, \zeta)$ and $\Phi_2(G, \zeta) = \Phi_2(G) \cap \Phi(G, \zeta)$, which are empty unless $\zeta$ is unitary.

The second construction is a the cocentral character homomorphism

$$H^1_c(W_F, Z(G^\vee)) \longrightarrow \text{Hom}_c(G(F), \mathbb{C}^\times)$$
$$a \longmapsto \chi_a$$

The corresponding pairing $H^1_c(W_F, Z(G^\vee)) \times G(F) \to \mathbb{C}^\times$ is called the Langlands pairing. (See also [KS99, §5.1], Appendix A by Labesse and Lapid in [LM15].) Let $Z(G^\vee)^1$ denote the maximal compact subgroup of $Z(G^\vee)$. The above homomorphism restricts to a homomorphism

$$H^1_c(W_F, Z(G^\vee)^1) \longrightarrow \mathrm{Hom}_c(G(F), \mathbb{C}^1)$$

and thus we have the corresponding pairing $H^1_c(W_F, Z(G^\vee)^1) \times G(F) \to \mathbb{C}^1$. Following [KP23], we write $G(F)^\natural = \mathrm{im}(G_{\mathrm{sc}}(F) \to G(F))$, where $G_{\mathrm{sc}} \to G_{\mathrm{der}}$ is the simply connected cover of the derived group of $G$. The construction in [Kal15] makes it clear that the map $H^1_c(W_F, Z(G^\vee)) \to \mathrm{Hom}_c(G(F), \mathbb{C}^\times)$ factors the group $\mathrm{Hom}_c(G(F)/G(F)^\natural, \mathbb{C}^\times)$ of cocentral characters of $G(F)$. This explains the terminology.

The cocentral character homomorphism and its unitary restriction are isomorphisms if $G$ is quasisplit, in particular if $G$ is a torus. The homomorphism is injective if $F$ is non-archimedean. It is surjective if $G_{\mathrm{sc}}(F)$ is perfect, or equivalently if $F$ is archimedean or $G_{\mathrm{sc}}$ does not contain a simple factor of the form $\mathrm{Res}_{E/F}(\mathrm{SL}_1(D))$ for a finite-dimensional non-commutative division algebra $D$ over a finite separable (this works over positive characteristic) extension $E$ of $F$. (See Appendix A by Labesse and Lapid in [LM15].)

### 4.3.2 The local Langlands correspondence for tori

Suppose that $G = T$ is a torus. Then we have a bijection $\Phi(T) \cong H^1_c(W_F, T^\vee)$ defined by $\phi \mapsto a_\phi$, and we transport the group structure from $H^1_c(W_F, T^\vee)$ to $\Phi(T)$ so that this bijection becomes a group isomorphism. It restricts to a isomorphism $\Phi_{\mathrm{temp}}(T) = H^1_c(W_F, (T^\vee)^1)$, where $(T^\vee)^1$ is the maximal compact subgroup of $T^\vee$. Furthermore, we have $\Pi(T) = \mathrm{Hom}_c(T(F), \mathbb{C}^\times)$ and $\Pi_{\mathrm{temp}}(T) = \mathrm{Hom}_c(T(F), \mathbb{C}^1)$. The cocentral character homomorphism in this case thus gives us an isomorphism $\mathrm{rec}_T : \Phi(T) \to \Pi(T)$, which restricts to an isomorphism $\mathrm{rec}_T : \Phi_{\mathrm{temp}}(T) \to \Pi_{\mathrm{temp}}(T)$. It is called the local Langlands correspondence or local reciprocity map for tori and was first constructed in [Lan97]. (See [Bor79, §9] for an overview and [Lab85] for a slightly different approach.)

### 4.3.3 Unramified characters

Consider the subgroup

$$H^1_c(W_F, Z(G^\vee)^{\Gamma_F,\circ}) = \mathrm{Hom}_c(W_F, Z(G^\vee)^{\Gamma_F,\circ})$$

of $H^1_c(W_F, Z(G^\vee))$. The unramified elements of this group are by definition those that are trivial on $W_F^1 = \ker(|\cdot| : W_F \to \mathbb{R}_{>0})$, that is, the elements of

$$H^1_c(W_F/W_F^1, Z(G^\vee)^{\Gamma_F,\circ}) = \mathrm{Hom}_c(W_F/W_F^1, Z(G^\vee)^{\Gamma_F,\circ}).$$

We call the homomorphism

$$H^1_c(W_F/W_F^1, Z(G^\vee)^{\Gamma_F,\circ}) \longrightarrow H^1_c(W_F, Z(G^\vee)) \longrightarrow \mathrm{Hom}_c(G(F), \mathbb{C}^\times)$$

the unramified cocentral character homomorphism. We will show that its image is the group $X^{\mathrm{nr}}(G)$ of unramified characters of $G$ by relating it to the isomorphism $\mathfrak{a}^*_{G,\mathbb{C}}/\mathfrak{a}^\vee_{G,F} \to X^{\mathrm{nr}}(G), \lambda \mapsto |\cdot|^\lambda_G$.

Recall that there is a surjective homomorphism $a^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G), \lambda \mapsto |\cdot|^\lambda_G$, where $|g|^\lambda_G = e^{\langle \lambda, H_G(g)\rangle}$. Further, recall the canonical isomorphism

$$\begin{aligned} X^*(G) &\longrightarrow X_*(Z(G^\vee)^{\Gamma_F,\circ}) \\ \theta &\longmapsto \theta^\vee \end{aligned}$$

which we defined above. This induces an isomorphism

$$\mathfrak{a}^*_{G,\mathbb{C}} = X^*(G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\ \sim\ } X_*(Z(G^\vee)^{\Gamma_F,\circ}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Lie}(Z(G^\vee)^{\Gamma_F,\circ}),$$

which we write as $\lambda \mapsto \lambda^\vee$. Here, we have used that for any complex torus $T$ we have identifications

$$\begin{array}{ccc} X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times & \xrightarrow{\ \sim\ } & T \\ {\scriptstyle \mathrm{id}\otimes\exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ X_*(T) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\ \sim\ } & \mathrm{Lie}(T) \end{array}$$

For $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, we define $\|\cdot\|^{\lambda^\vee} \in H^1_c(W_F/W_F^1, Z(G^\vee)^{\Gamma_F,\circ})$ by

$$\|w\|^{\lambda^\vee} = \exp((\log \|w\|)\lambda^\vee)$$

for all $w \in W_F$. This determines a homomorphism

$$\mathfrak{a}^*_{G,\mathbb{C}} \longrightarrow H^1_c(W_F/W_F^1, Z(G^\vee)^{\Gamma_F,\circ}).$$

For $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, let $a_\lambda$ be the image of $\|\cdot\|^{\lambda^\vee}$ in $H^1_c(W_F, Z(G^\vee))$. We have

$$\chi_{a_\lambda} = |\cdot|^\lambda_G.$$

(See the discussion below 4.7 in [SZ18].) It follows that the unramified cocentral character homomorphism maps into $X^{\mathrm{nr}}(G)$ and the following diagram commutes

$$\begin{array}{ccc} & \mathfrak{a}^*_{G,\mathbb{C}} & \\ \swarrow & & \searrow \\ H^1_c(W_F/W^1_F, Z(G^\vee)^{\Gamma_F,\circ}) & \longrightarrow & X^{\mathrm{nr}}(G) \end{array}$$

As a consequence, one obtains that the bottom homomorphism is surjective.

Suppose that $F$ is archimedean. We have $W_F/W^1_F = \mathbb{R}_{>0}$, and therefore the homomorphism $\lambda \mapsto \|\cdot\|^{\lambda^\vee}$ can be viewed as an isomorphism $\mathfrak{a}^*_{G,\mathbb{C}} \to \mathrm{Lie}(Z(G^\vee)^{\Gamma_F,\circ})$. The homomorphism $\mathfrak{a}^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G)$ is also an isomorphism. Consequently the unramified cocentral character homomorphism

$$\mathrm{Lie}(Z(G^\vee)^{\Gamma_F,\circ}) = H^1_c(W_F/W^1_F, Z(G^\vee)^{\Gamma_F,\circ}) \longrightarrow X^{\mathrm{nr}}(G)$$

is an isomorphism. Moreover, the homomorphism

$$H^1_c(W_F/W^1_F, Z(G^\vee)^{\Gamma_F,\circ}) \to H^1_c(W_F/W^1_F, Z(G^\vee))$$

is easily seen to be injective, so the homomorphism $\lambda \mapsto a_\lambda$ is injective.

Suppose that $F$ is non-archimedean. We have $W^1_F = I_F$ and $W_F/W^1_F = \langle \mathrm{Fr} \rangle$ and the norm $\|\cdot\| : W_F/W^1_F \to q_F^{\mathbb{Z}}$ is defined by $\|\mathrm{Fr}\| = q_F$. The homomorphism $\lambda \mapsto \|\cdot\|^{\lambda^\vee}$ can be viewed as a surjective homomorphism $\mathfrak{a}^*_{G,\mathbb{C}} \to Z(G^\vee)^{\Gamma_F,\circ}$, which descends to an isomorphism $\mathfrak{a}^*_{G,\mathbb{C}}/\frac{2\pi i}{\log q_F} X^*(G) \to Z(G^\vee)^{\Gamma_F,\circ}$. The homomorphism $\mathfrak{a}^*_{G,\mathbb{C}} \to X^{\mathrm{nr}}(G)$ has kernel $\mathfrak{a}^\vee_{G,F}$, and the kernel of the unramified cocentral character homomorphism

$$Z(G^\vee)^{\Gamma_F,\circ} = H^1_c(W_F/I_F, Z(G^\vee)^{\Gamma_F,\circ}) \longrightarrow X^{\mathrm{nr}}(G)$$

is the finite subgroup of $H^1_c(W_F/I_F, Z(G^\vee)^{\Gamma_F,\circ})$ isomorphic to $\mathfrak{a}^\vee_{G,F}/\frac{2\pi i}{\log q_F} X^*(G)$. As in [Hai14, §3.3], we can describe the image of $\lambda \mapsto a_\lambda$ using Kottwitz homomorphism for $G$.

The Kottwitz homomorphism was introduced in [Kot97]. See [KP23, Ch. 11] for a detailed exposition. The Kottwitz homomorphism for $G$ is a surjective homomorphism

$$\kappa_G : G(F) \twoheadrightarrow (X^*(Z(G^\vee))_{I_F})^{\mathrm{Fr}} = (X^*((Z(G^\vee)^{I_F})_{\mathrm{Fr}}).$$

We define $G(F)_1 = \ker \kappa_G \subseteq G(F)^1$. The open subgroup $G(F)_1$ of $G(F)$ is the subgroup of $G(F)$ generated by its parahoric subgroups. A (continuous) character of $G(F)$ is said to be weakly unramified if it is trivial on $G(F)_1$. We denote the group of weakly unramified characters of $G(F)$ by $X^{\mathrm{wnr}}(G) = \mathrm{Hom}(G(F)/G(F)_1, \mathbb{C}^\times)$. Since Kottwitz homomorphism is functorial and trivial on simply connected groups [KP23, Proof of Prop. 11.5.4], it follows that $G(F)^\natural \subseteq G(F)_1$. Therefore weakly unramified characters are cocentral. By definition we have an isomorphism

$$\kappa_G : G(F)/G(F)_1 \xrightarrow{\sim} (X^*((Z(G^\vee)^{I_F})_{\mathrm{Fr}}).$$

Consequently, we obtain an isomorphism

$$(Z(G^\vee)^{I_F})_{\mathrm{Fr}} \xrightarrow{\sim} \mathrm{Hom}((X^*((Z(G^\vee)^{I_F})_{\mathrm{Fr}}), \mathbb{C}^\times) \xrightarrow{\kappa_G^*} X^{\mathrm{wnr}}(G).$$

We also have the following homomorphism obtained from inflation homomorphism and the cocentral character homomorphism

$$(Z(G^\vee)^{I_F})_{\mathrm{Fr}} = H^1_c(W_F/I_F, Z(G^\vee)^{I_F}) \hookrightarrow H^1_c(W_F, Z(G^\vee)) \longrightarrow \mathrm{Hom}_c(G(F), \mathbb{C}^\times).$$

Thus, we have two methods parametrising cocentral characters of $G(F)$ by $(Z(G^\vee)^{I_F})_{\mathrm{Fr}}$, one from the Kottwitz homomorphism and one from Langlands's cocentral character homomorphism. As explained in [Hai14, §3.3.1], it follows from [Kal15, Prop. 4.5.2] these two methods are the same.

The Kottwitz homomorphism induces an isomorphism

$$\kappa_G : G(F)/G(F)^1 \xrightarrow{\sim} X^*((Z(G^\vee)^{I_F})_{\mathrm{Fr}})/\mathrm{Tor} = X^*((Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}),$$

where Tor denotes the torsion subgroup and $(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} := ((Z(G^\vee)^{I_F})_{\mathrm{Fr}})^\circ$. Therefore we obtain an isomorphism

$$(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} \xrightarrow{\sim} \mathrm{Hom}(X^*((Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}), \mathbb{C}^\times) \xrightarrow{\kappa_G^*} X^{\mathrm{nr}}(G).$$

giving a description of $X^{\mathrm{nr}}(G)$. By the compatibility of the cocentral character homomorphism with the Kottwitz homomorphism above, we obtain that this isomorphism coincides with the homomorphism

$$(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} = H^1_c(W_F/I_F, (Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}) \hookrightarrow H^1_c(W_F, Z(G^\vee)) \longrightarrow \mathrm{Hom}_c(G(F), \mathbb{C}^\times)$$

obtained from inflation and the cocentral character homomorphism. This is also proved [Var, Lemma 2.5.8]. Consequently, we have the following factorisation of the unramified cocentral character homomorphism

$$\begin{array}{ccc} & \mathfrak{a}^*_{G,\mathbb{C}} & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ Z(G^\vee)^{\Gamma_F,\circ} = H^1_c(W_F/I_F, Z(G^\vee)^{\Gamma,\circ}) & \longrightarrow\!\!\!\!\rightarrow & X^{\mathrm{nr}}(G) \\ \downarrow\!\!\downarrow & & \uparrow \kappa_G^* \\ (Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} = H^1_c(W_F/I_F, (Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}) & \xrightarrow{\sim} & \mathrm{Hom}(X^*(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}), \mathbb{C}^\times) \end{array}$$

Composing the arrows on the left with the natural injection

$$(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} = H^1_c(W_F/I_F, (Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}}) \hookrightarrow (Z(G^\vee)^{I_F})_{\mathrm{Fr}} = H^1_c(W_F/I_F, Z(G^\vee)^{I_F})$$

and the inflation homomorphism

$$(Z(G^\vee)^{I_F})_{\mathrm{Fr}} = H^1_c(W_F/I_F, Z(G^\vee)^{I_F}) \hookrightarrow H^1_c(W_F, Z(G^\vee)).$$

gives the homomorphism $\mathfrak{a}_{G,\mathbb{C}^*} \to H^1_c(W_F, Z(G^\vee)), \lambda \mapsto a_\lambda$. As a consequence, we obtain that the homomorphism $\lambda \mapsto a_\lambda$ has image $(Z(G^\vee)^{I_F})^\circ_{\mathrm{Fr}} \hookrightarrow H^1_c(W_F, Z(G^\vee))$ and kernel $\mathfrak{a}^\vee_{G,F}$, which will be important in what follows.

### 4.3.4 Twists

Let $a \in Z^1_c(W_F, Z(G^\vee))$ and $\phi \in \widetilde{\Phi}(G)$. Recall that we write $\phi(l) = (a_\phi(l), w(l))$. We define $a \cdot \phi \in \widetilde{\Phi}(G)$ by $(a \cdot \phi)(l) = (a(w(l))a_\phi(l), w(l))$. This gives an action of the group $Z^1_c(W_F, Z(G^\vee))$ on the pointed set $\widetilde{\Phi}(G)$, and the action of $Z^1_c(W_F, Z(G^\vee)^1)$ preserves $\widetilde{\Phi}_2(G)$ and $\widetilde{\Phi}_{\mathrm{temp}}(G)$. This action descends to a well-defined action of the group $H^1_c(W_F, Z(G^\vee))$ on the pointed set $\Phi(G)$, and the action of $H^1_c(W_F, Z(G^\vee)^1)$ preserves $\Phi_2(G)$ and $\Phi_{\mathrm{temp}}(G)$.

Pulling back along homomorphism $\mathfrak{a}^*_{G,\mathbb{C}}/\mathfrak{a}^\vee_{G,F} \to H^1_c(W_F, Z(G^\vee)), \lambda \mapsto a_\lambda$, we obtain an action of $\mathfrak{a}^*_{G,\mathbb{C}}/\mathfrak{a}^\vee_{G,F}$ on $\Phi(G)$, and the action of $i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,F}$ preserves $\Phi_2(G)$

and $\Phi_{\text{temp}}(G)$. We define

$$\Phi_{\text{temp}}(G)_{\mathbb{C}} = \mathfrak{a}^*_{G,\mathbb{C}} \cdot \Phi_{\text{temp}}(G) \quad , \quad \Phi_2(G)_{\mathbb{C}} = \mathfrak{a}^*_{G,\mathbb{C}} \cdot \Phi_2(G).$$

The set $\Phi_2(G)_{\mathbb{C}}$ is precisely the set of elements of $\Phi(G)$ that are elliptic but not necessarily tempered. We call the elements of $\Phi_2(G)_{\mathbb{C}}$ essentially discrete $L$-parameters and the elements of $\Phi_{\text{temp}}(G)_{\mathbb{C}}$ essentially tempered $L$-parameters. We caution the reader that sometimes in the literature $\Phi_2(G)_{\mathbb{C}}$ and $\Phi_2(G)$ are denoted by $\Phi_{2,\text{temp}}(G)$ and $\Phi_{2,\text{temp}}(G)$, respectively.

### 4.3.5 Infinitesimal characters and $L$-parameters of real groups

Assume that $G$ is a real group. Let $\phi \in \Phi(G)$. As explained in [Lan89], one can attach an infinitesimal character $\mu_\phi \in \operatorname{Hom}_{\mathbb{C}-\text{alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C})$ of $G$ to $\phi$ as follows. Choose a Borel pair $(B, T)$ of $G_{\mathbb{C}}$ and a $\Gamma_{\mathbb{R}}$-stable Borel pair $(\mathcal{B}^0, \mathcal{T}^0)$ of $G^\vee$. We choose a representative of $\phi$ such that $\phi(\mathbb{C}^\times) \subseteq \mathcal{T}^0$. There exists $\mu, \nu \in X_*(\mathcal{T}^0) \otimes_{\mathbb{Z}} \mathbb{C} = \operatorname{Lie}(\mathcal{T}^0)$ with $\mu - \nu \in X_*(\mathcal{T}^0)$ such that $\phi(z) = z^\mu \overline{z}^\nu$ for all $z \in \mathbb{C}^\times$. The $W(G^\vee, \mathcal{T}^0)$-orbit of $\mu$ does not depend on the choice of representative of $\phi$. The Borel pairs and the isomorphism $\eta_G : \Psi(G)^\vee \to \Psi(G^\vee)$ give us an isomorphism $X_*(\mathcal{T}^0) \cong X^*(T)$ and a compatible isomorphism $W(G^\vee, \mathcal{T}^0) \cong W(G_{\mathbb{C}}, T)$. Using the identification $\mathfrak{t}^* = \operatorname{Lie}(T)^*$ with $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$, we obtain an isomorphism $\operatorname{Lie}(\mathcal{T}^0)/W(G^\vee, \mathcal{T}^0) \cong \mathfrak{t}^*/W(G_{\mathbb{C}}, T)$. Thus, we obtain an element $\mu_\phi \in \mathfrak{t}^*/W(G_{\mathbb{C}}, T)$. Recall that the Harish-Chandra isomorphism $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \cong \operatorname{Sym}(\mathfrak{t})^{W(G_{\mathbb{C}}, T)}$ gives an isomorphism $\operatorname{Hom}_{\mathbb{C}-\text{alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C}) \cong \mathfrak{t}^*/W(G_{\mathbb{C}}, T)$. Thus, we have an infinitesimal character $\mu_\phi$ of $G$. It does not depend on any of the choices made.

### 4.3.6 Classification of $L$-parameters

For each $M \in \mathcal{L}^G(M_0)$, the conjugacy class of $L$-homomorphisms ${}^LM \to {}^LG$ gives rise to a map

$$\Phi_2(M)_{\mathbb{C}} \to \Phi(G),$$

which restricts to a map

$$\Phi_2(M) \to \Phi_{\text{temp}}(G).$$

These are both quotients for the natural actions of $W^G(M)$. We have a decomposition

$$\Phi(G) = \coprod_{M \in \mathcal{L}^G(M_0)/W_0^G} \Phi_2(M)_{\mathbb{C}}/W^G(M)$$

(cf. [Var, Lemma 2.10.6, Corollary 2.10.7]) and this restricts to a decomposition

$$\Phi_{\text{temp}}(G) = \coprod_{M \in \mathcal{L}^G(M_0)/W_0^G} \Phi_2(M)/W^G(M).$$

(cf. the proof of [Var, Theorem 2.10.10]). This latter decomposition is an analogue for tempered $L$-parameters of the Harish-Chandra classification of irreducible tempered representations. The first decomposition is analogous to the classification of irreducible admissible representations obtained by combining the Langlands classification with the Harish-Chandra classification.

Implicit in [Lan89] is a classification of $\Phi(G)$ that is analogous to the Langlands classification of $\Pi(G)$. This was elaborated on and given an explicit formulation and detailed proof in [SZ18]. We define a Langlands datum for $L$-parameters to be a triple $((P, M), \phi, \lambda)$, where $(P, M)$ is a parabolic pair, $\sigma \in \Pi_{\text{temp}}(M)$, and $\lambda \in (\mathfrak{a}_M^*)^{P,+}$. To each Langlands datum $((P, M), \phi, \lambda)$, we may assign an $L$-parameter $\iota_M^G(\phi_\lambda) \in \Phi(G)$ and the Langlands classification for $L$-parameters asserts that this gives a bijection between the set of $G(F)$-conjugates of Langlands data and $\Phi(G)$. See [SZ18] for more details and a description of the inverse of this bijection.

## 4.4 The local Langlands correspondence

The local Langlands correspondence for $G$, which is still hypothetical for $p$-adic groups in general, is a natural surjective "local reciprocity" map

$$\text{rec}_G : \Pi(G) \longrightarrow \Phi(G)$$

with finite fibres. The fibre of $\text{rec}_G$ above an $L$-parameter $\phi \in \Phi(G)$ is called the $L$-packet of $\phi$ and denoted by $\Pi_\phi$. The local reciprocity map $\text{rec}_G$ is natural in the sense that it satisfies several desiderata, which say that $\text{rec}_G$ is compatible with various structures. Among these, is the assertion that $\text{rec}_G$ is a a natural transformation of functors from the underlying groupoid of $\mathsf{Red}_{k,\text{Out}}^{\text{sep}}$ to $\mathsf{Set}$. The following is a partial list of desiderata.

**Desiderata.**

1. Compatibility with central characters: for each $\pi \in \Pi(G)$, we have $\zeta_\pi = \zeta_{\phi_\pi}$.

2. Compatibility with by cocentral twists: for each $\pi \in \Pi(G)$ and $a \in H^1(W_F, Z(G^\vee))$,

we have $\phi_{\pi\otimes\chi_a} = a\cdot\phi_\pi$. In particular, we have compatibility with unramified twists: for each $\pi\in\Pi(G)$ and $\lambda\in\mathfrak{a}^*_{G,\mathbb{C}}$, we have $\phi_{\pi_\lambda} = (\phi_\pi)_\lambda$.

3. Compatibility with temperedness and discreteness: for $\pi\in\Pi(G)$, we have $\pi\in\Pi_{\mathrm{temp}}(G)$ (resp. $\pi\in\Pi_2(G)$) if and only if $\phi_\pi\in\Phi_{\mathrm{temp}}(G)$ (resp. $\phi_\pi\in\Phi_2(G)$).

4. Naturality: if $\eta : G\to H$ is a separable normal homomorphism with abelian kernel and cokernel and if $\pi\in\Pi(H)$, then the irreducible constituents of the restriction $\pi\circ\eta$, which is a direct sum of finitely many irreducible admissible representations of $G(F)$, all belong to $\Pi_{^L\eta\circ\phi}$. In particular, if $\eta : G\to H$ is an isomorphism, then the following diagram commutes:

$$\begin{array}{ccc} \Pi(H) & \xrightarrow{\Pi(\eta)} & \Pi(G) \\ {\scriptstyle \mathrm{rec}_H}\downarrow & & \downarrow{\scriptstyle \mathrm{rec}_G} \\ \Phi(H) & \xrightarrow{\Phi(\eta)} & \Phi(G) \end{array}$$

where $\Pi(\eta) = \eta^*$ and $\Phi(\eta) = ({}^L\eta)_*$.

5. Compatibility with Weil restriction: If $K/F$ is a finite subextension of $F_s/F$, $G'$ is a connected reductive group over $K$, and $G = \mathrm{Res}_{K/F} G'$, then we have a commutative diagram

$$\begin{array}{ccc} \Pi(G') & \xrightarrow{\sim} & \Pi(G) \\ \downarrow{\scriptstyle \mathrm{rec}_{G'}} & & \downarrow{\scriptstyle \mathrm{rec}_G} \\ \Phi(G') & \xrightarrow{\sim} & \Phi(G) \end{array}$$

where the upper horizontal arrow is the isomorphism coming from the identification $G(F) = G'(K)$ and the lower horizontal arrow is a canonical bijection coming from Shapiro's lemma. See [Bor79, §4, §5, §8.4] for more details.

6. Compatibility with the Langlands classification: if $((P,M),\phi,\lambda)$ is a Langlands triple, then $\Pi_{\iota^G_M(\phi)}$ is the set of Langlands quotients $J((P,M),\sigma,\lambda)\in\Pi(G)$, for $\sigma\in\Pi_\phi$.

7. Compatibility with the Harish-Chandra classification: if $M$ is a Levi subgroup of $G$ and $\phi\in\Phi_2(M)$, then $\Pi_{\iota^G_M(\phi)}$ is the set of irreducible subrepresentations of the various parabolically induced representations $I^G_M(\sigma)$ for $\sigma\in\Pi_\phi$.

8. Stable tempered characters: for each $\phi\in\Phi_{\mathrm{temp}}(G)$, there exists a stable virtual

tempered representation

$$\pi_\phi = \sum_{\pi \in \Pi_\phi} c_{\phi,\pi}\pi$$

with $c_{\phi,\pi} \in \mathbb{Z}_{>0}$ for all $\pi \in \Pi_\phi$, such that the following properties hold:

(a) Compatibility with parabolic induction: If $M$ is a Levi subgroup of $G$ and $\phi \in \Phi_{\mathrm{temp}}(M)$, then $I_M^G(\pi_\phi) = \pi_{\iota_M^G(\phi)}$

(b) The set $\{\pi_\phi : \phi \in \Phi_{\mathrm{temp}}(G)\}$ is a basis of the space $D^{\mathrm{st}}_{\mathrm{temp}}(G)$ of stable tempered virtual representations of $G$. Consequently, the set $\{\pi_\phi : \phi \in \Phi_2(G)\}$ is a basis of the space $D^{\mathrm{st}}_{\mathrm{ell}}(G)$.

The character $\Theta_\phi = \Theta_{\pi_\phi}$ of $\pi_\phi$ is a stable tempered distribution called the stable tempered character of $\phi$, and called a stable discrete series character if $\phi \in \Phi_2(G)$. Note that the stable tempered virtual representations are linearly independent. We will often identify $\phi = \pi_\phi = \Theta_\phi$.

9. Compatibility with infinitesimal characters in the case $F = \mathbb{R}$: for each $\pi \in \Pi(G)$, we have $\mu_\pi = \mu_{\phi_\pi}$.

It follows from desideratum 5 that the local Langlands correspondence for groups over arbitrary fields is determined by and can be constructed from the local Langlands correspondence for groups over $F = \mathbb{R}$ or $F = \mathbb{Q}_p$ for a prime $p$. It follows from the Langlands classification of admissible representations and the parallel Langlands classification of $L$-parameters, that a local Langlands correspondence $\mathrm{rec}_G : \Pi(G) \to \Phi(G)$ satisfying the above desiderata is determined by its restricted tempered local Langlands correspondence $\mathrm{rec}_G : \Pi_{\mathrm{temp}}(G) \to \Phi_{\mathrm{temp}}(G)$. Furthermore, it follows from what we have called the Harish-Chandra classification of tempered representations and the parallel Harish-Chandra classification of tempered $L$-parameters that a local Langlands correspondence satisfying the above desiderata is determined by its restricted discrete local Langlands correspondence $\mathrm{rec}_G : \Pi_2(G) \to \Phi_2(G)$. Conversely, given a discrete local reciprocity map $\mathrm{rec}_G : \Pi_2(G) \to \Phi_2(G)$ satisfying the above desiderata in the forms that make sense, one can construct a local reciprocity map $\mathrm{rec}_G : \Pi(G) \to \Phi(G)$ satisfying the above desiderata. (See [Bor79, §11.7], [KT, §6.1.2], [Var, p. 2.10] for more on this.)

Note that $\Pi_\phi$ can be recovered from $\Theta_\phi$ since the characters of irreducible admissible representations are linearly dependent. Thus the local Langlands correspondence is determined by the stable discrete series characters and can be constructed by defining stable discrete series characters in such a way that the associated discrete reciprocity map satisfies the above desiderata.

For real groups, and thus for complex groups by restriction of scalars, Langlands constructed the local Langlands correspondence in this way. Suppose that $G$ is a connected reductive group over $\mathbb{R}$ and $\phi \in \Phi_2(G)$. Langlands showed that $G$ has an elliptic maximal torus $T$. Moreover, a choice of a Borel subgroup $B$ of $G_{\mathbb{C}}$ containing $T$ determines an $L$-embedding ${}^LT \to {}^LG$ through which $\phi$ factors as a tempered $L$-parameter of $\phi_T \in \Phi_{\mathrm{temp}}(T)$. The unitary character of $\theta_\phi$ of $T(\mathbb{R})$ corresponding to $\phi_T$ under the local Langlands correspondence for tori is dominant with respect to $B$ and its orbit under the absolute Weyl group $W(G,T)$ does not depend on $B$. In Harish-Chandra's papers on discrete series, he defined a stable tempered distribution $\Theta_{\theta_\phi}$ that is uniquely determined by its formula on the set $T(\mathbb{R})'$ of regular elements in $T(\mathbb{R})$:

$$\Theta_{\theta_\phi}(t) = (-1)^{q(G)} \sum_{w \in W(G,T)} \frac{\theta_\phi(w^{-1} \cdot t)}{\prod_{\alpha >_B 0}(1 - \alpha(w^{-1} \cdot t)^{-1})},$$

where $q(G) = 1/2 \dim(G(\mathbb{R})/K)$ for any maximal compact subgroup $K$ of $G(\mathbb{R})$ and $\alpha >_B 0$ indicates that $\alpha$ is a positive root with respect to the positive system determined by $B$. Harish-Chandra showed that $\Theta_{\theta_\phi}$ is the sum of $\Theta_\pi$ for all $\pi \in \Pi_2(G)$ with $\zeta_\pi = \theta_\phi|_{Z_G(\mathbb{R})}$ and $\mu_\pi = d\theta_\phi$. Note that we have $\zeta_\phi = \theta_\phi|_{Z_G(\mathbb{R})}$ and $\mu_\phi = d\theta_\phi$. Langlands constructed the local Langlands correspondence for $G$ by requiring that $\Theta_\phi = \Theta_{\theta_\phi}$. Consequently, we have that $\Pi_\phi$ consists of all $\pi \in \Pi_2(G)$ with infinitesimal character $\mu_\pi = \mu_\phi$ and central character $\zeta_\pi = \zeta_\phi$, and we have $\Theta_\phi = \sum_{\pi \in \Pi_\phi} \Theta_\pi$. In fact, in [AV16] Adams and Vogan prove that $\Pi_\phi$ consists of all $\pi \in \Pi_2(G)$ with infinitesimal character $\mu_\pi = \mu_\phi$ and what they call "split radical character" $\zeta_\pi|_{A_G(\mathbb{R})} = \zeta_\phi|_{A_G(\mathbb{R})}$. For $\phi \in \Phi_2(G)$, the cardinality of $\Pi_\phi$ is bounded by $|W(G,T)/W_{\mathbb{R}}(G,T)|$, where $T$ is an elliptic torus of $G$. For $\phi \in \Phi_{\mathrm{temp}}(G)$, we also have $\pi_\phi = \sum_{\pi \in \Pi_\phi} \pi$. This follows as in the proof of [She79, Lemma 3.1] using [Kna76, Theorem] and [SV80, Theorem 2.9]. Desideratum 8.(b) is a corollary of [ABV92, Lemma 18.11], which follows from the proof of [She79, Lemma 5.2].

For non-archimedean groups, it is expected that stable discrete series characters $\Theta_\phi$ can be constructed directly in terms of $\phi$, as was the case for real groups. This would then give rise to characterisation of the local Langlands correspondence and a characterisation of it. (See [Kal; Kal23] for further discussion.)

We will make use of the following hypothesis on the local Langlands correspondence for non-archimedean groups.

**Hypothesis 4.4.1.** *Let $G$ be a connected reductive group over a non-archimedean local* *field $F$. The group $G$ and its Levi subgroups have local reciprocity maps satisfying the following subset of the above desiderata: 1, 2 in the special case of unramified twists, 3, 4 in the special case when $\eta$ is of the form $\mathrm{Int}(g) : M \to g \cdot M$ for $g \in G(F)$ and $M$ a Levi subgroup of $G$, 6, 7, and 8.*

We remark that by [Var, §2.7.7], we have that for non-archimedean groups the first sentence of Desideratum 8 follows from the second sentence of Desideratum 8 in Hypothesis 4.4.1.

Hypothesis 4.4.1 is known for tori. The local reciprocity map is the cocentral character isomorphism given by the Langlands pairing and the stable tempered character of a tempered $L$-parameter can be taken to simply be the corresponding character.

In the case $G = \mathrm{GL}_n$, Hypothesis 4.4.1 was proved by Harris and Taylor, Henniart, and Scholze [HT01; Hen00; Sch13]. The $L$-packets are singletons and the stable tempered characters are simply the irreducible tempered characters.

If $G$ is an inner form of a quasisplit symplectic, odd special orthogonal, unitary, or odd general spin group, then Hypothesis 4.4.1 is satisfied. (See [Var, Theorem 7.3.3].) This follows from the works of Arthur, Mok, and Moeglin [Art13; Mok15; Mœg14], which use the theory of twisted endoscopy and the local Langlands correspondence for $\mathrm{GL}_n$.

From now on, if $F$ is non-archimedean we assume we assume Hypothesis 4.4.1 for all connected reductive groups over $F$.

### 4.4.1 Infinitesimal characters for $p$-adic groups

Suppose that $F$ is archimedean. An infinitesimal character (also called an infinitesimal parameter) of $G$ is defined to be a $G^\vee$-conjugacy class of $L$-homomorphisms $\mu : W_F \to {}^LG$. This notion originates in [Vog93]. Another useful reference is [Cun+22]. Every $L$-parameter $\phi : L_F = \mathrm{SL}_2(\mathbb{C}) \times W_F \to {}^LG$ of $G$ has an associated infinitesimal character $\mu_\phi : W_F \to {}^LG$ defined by $\mu_\phi(w) = \phi(d_w, w)$, where $d_w = \mathrm{diag}(\|w\|^{1/2}, \|w\|^{-1/2})$. Each $\phi \in \Phi_{\mathrm{temp}}(G)$ can be recovered from its infinitesimal character $\mu_\phi$ [Var, Lemma 2.9.5]. In general, there are at most finitely many elements of $\Phi(G)$ with a given infinitesimal character [Vog93, Corollary 4.6].

For each $\pi \in \Pi(G)$, we define its infinitesimal character $\mu_\pi$ by $\mu_\pi = \mu_{\phi_\pi}$. It follows from the finiteness of $L$-packets that for each infinitesimal character $\mu$ of $G$, there are finitely many $\pi \in \Pi(G)$ with $\mu_\pi = \mu$.

Let $M$ be a Levi subgroup of $G$. If $\mu$ is an infinitesimal characters of $M$, then $\iota^G_M \circ \mu$ is an infinitesimal characters of $G$. Moreover, if $\mu_1$ and $\mu_2$ are infinitesimal characters of $M$, then $\iota_M^G \circ \mu_1 = \iota_M^G \circ \mu_2$ if and only if $\mu_1$ and $\mu_2$ are in the same $W^G(M)$-orbit.

For each $\tau = (M, \sigma, \widetilde{r}) \in T_{\mathrm{temp}}(G)$, we define its infinitesimal character to be$\mu_\tau = \iota_M^G \circ \mu_\sigma$. Note that $\mu_\tau$ only depends on the image of $\tau$ in $T_{\mathrm{temp}}(G)/\mathbb{C}^1$. There are at most finitely many elements in $T_{\mathrm{temp}}(G)/\mathbb{C}^1$ with a given infinitesimal character.

We have an action of $\mathfrak{a}_{G,\mathbb{C}}^*/\mathfrak{a}_{G,F}^\vee$ on the set of infinitesimal characters by $(\lambda, \mu) \mapsto a_\lambda \cdot \mu$. The assignment of infinitesimal characters to elements of $\Pi(G)$ (resp. $T_{\mathrm{temp}}(G)$) is equivariant with respect to the actions of $i\mathfrak{a}_G^*$ (resp. $\mathfrak{a}_{G,\mathbb{C}}^*$).

## 4.5 The stable Fourier transform

For $f \in \mathcal{C}(G, \zeta)$, we define its stable Fourier transform to be the function $f^G : \Phi_{\mathrm{temp}}(G, \zeta) \to \mathbb{C}$ defined by $f^G(\phi) = \Theta_\phi(f)$.

We define the space of stable Fourier transforms

$$\widehat{\mathcal{S}_{(c)}}(G, \zeta) = \{f^G : f \in \mathcal{C}_{(c)}(G, \zeta)\}.$$

We have

$$\widehat{\mathcal{S}_{(c)}}(G, \zeta) = \mathcal{C}_{(c)}(G, \zeta) / \operatorname{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{\Theta_\phi : \phi \in \Phi_{\mathrm{temp}}(G, \zeta)\})$$

and we give it the natural quotient topology. The stable Fourier transform is a continuous surjective linear map $\mathcal{F}^{\mathrm{st}} : \mathcal{C}_{(c)}(G, \zeta) \to \widehat{\mathcal{S}_{(c)}}(G, \zeta)$.

The following lemma is an immediate consequence of the Harish-Chandra regularity theorem and the stable Weyl integration formula

**Lemma 4.5.1.** *If $f \in \mathcal{C}_c^{\mathrm{unst}}(G, \zeta)$, then $f^G(\phi) = 0$ for all $\phi \in \Phi(G, \zeta)$. Consequently, $\Theta_\phi \in \mathcal{S}_c(G, \zeta)'$ for all $\phi \in \Phi(G, \zeta)$. (Here $\zeta$ does not need to be unitary.)*

*If $f \in \mathcal{C}^{\mathrm{unst}}(G, \zeta)$, then $f^G(\phi) = 0$ for all $\phi \in \Phi_{\mathrm{temp}}(G, \zeta)$. Consequently, $\Theta_\phi \in \mathcal{S}(G, \zeta)'$ for all $\phi \in \Phi_{\mathrm{temp}}(G, \zeta)$.*

It follows that the stable Fourier transform $\mathcal{F}^{\mathrm{st}} : \mathcal{C}_{(c)}(G, \zeta) \to \widehat{\mathcal{S}_{(c)}}(G, \zeta)$ descends to a continuous surjective linear map

$$\mathcal{F}^{\mathrm{st}} : \mathcal{S}_{(c)}(G, \zeta) \longrightarrow \widehat{\mathcal{S}_{(c)}}(G, \zeta).$$

We call the property of injectivity of this map stable spectral density for $\mathcal{S}_{(c)}(G, \zeta)$.

Stable spectral density for $\mathcal{S}_{(c)}(G,\zeta)$ is equivalent to

$$\mathcal{C}_{(c)}^{\mathrm{unst}}(G,\zeta) = \mathrm{Ann}_{\mathcal{C}_{(c)}(G,\zeta)}(\{\Theta_\phi : \phi \in \Phi_{\mathrm{temp}}(G,\zeta)\}).$$

It follows from stable spectral density for $\mathcal{C}_{(c)}(G,\zeta)$ that a distribution in $\mathcal{C}_{(c)}(G,\zeta)'$ is stable if and only if it lies in the weak-$*$ closure the linear span of $\{\Theta_\phi\}_{\phi\in\Phi_{\mathrm{temp}}(G,\zeta)}$.

For archimedean $F$, stable spectral density for $\mathcal{C}(G,\zeta)$ was proved by Shelstad in [She79, Lemma 5.3] for $\mathcal{Z}=1$ and in [She08, Theorem 4.1] for $\mathcal{Z}=Z(F)$, where $Z$ is a central torus of $G$. For non-archimedean $F$, stable spectral density for $\mathcal{C}_c(G,\zeta)$ was proved by Arthur in [Art96] when $\mathcal{Z}=Z(F)$ for $Z$ a central induced torus of $G$. The proof uses a global argument and is generalised in [MW16b, p. XI.5.2] in the case $\mathcal{Z}=1$. In particular, we have stable spectral density for $\mathcal{C}_c(G)$ and $C^\infty_{c,\mathrm{cusp}}(G,\zeta,K)$ and we will use these below to prove the stable Paley–Wiener theorems below. Stable spectral density for $\mathcal{C}(G)$ in the non-archimedean case does not appear to be in the literature. It is established below (Theorem 4.6.16) in the course of the proof of the stable Paley–Wiener theorem for $\mathcal{C}(G)$.

## 4.6 Stable Paley–Wiener theorems

In this section, we will give various stable Paley–Wiener theorems, the main ones being for the stable Fourier transforms on the spaces $\mathcal{S}_{(c)}(G)$ and $\mathcal{S}_f(G)$.

### 4.6.1 The space of $L$-parameters

Recall that we have an action of $\mathfrak{a}^*_{G,\mathbb{C}}$ on $\Phi(G)$, and that the action of $i\mathfrak{a}^*_G$ preserves $\Phi_{\mathrm{temp}}(G)$ and $\Phi_2(G)$. We denote the isotropy subgroup of $\phi\in\Phi(G,\zeta)$ in $\mathfrak{a}^*_{G,\mathbb{C}}$ by $\mathfrak{a}^\vee_{G,\tau}$. Just as was the case for $\mathfrak{a}^\vee_{G,\tau}$, we have

$$\mathfrak{a}^\vee_{G,F} \subseteq \mathfrak{a}^\vee_{G,\phi} \subseteq \widetilde{\mathfrak{a}}^\vee_{G,F}$$

Thus, if $F$ is archimedean we have $\mathfrak{a}^\vee_{G,\phi}=0$, and if $F$ is non-archimedean we have that $\mathfrak{a}^\vee_{G,\phi}$ is a full lattice in $i\mathfrak{a}^*_G$. Consequently, $\Phi_2(G)$ is naturally a smooth manifold with countably many components $i\mathfrak{a}^*_G\cdot\phi = i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,\phi}$, which are Euclidean spaces if $F$ is archimedean and compact tori if $F$ is non-archimedean. Moreover, $\Phi_2(G)_{\mathbb{C}}$ is the complexification of $\Phi_2(G)$ with connected components $\mathfrak{a}^*_{G,\mathbb{C}} \cdot \phi = \mathfrak{a}^*_{G,\mathbb{C}}/\mathfrak{a}^\vee_{G,\phi}$. Since

$$\Phi_{\mathrm{temp}}(G) = \coprod_{L \in \mathcal{L}^G(M_0)/W_0^G} \Phi_2(L)/W^G(L),$$

we have that $\Phi_{\mathrm{temp}}(G)$, and similarly $\Phi(G)$ is naturally a topological space. We say that a function on $\Phi_{\mathrm{temp}}(G)$ is smooth if pulls back to a smooth function on each $\Phi_2(L)$.

### 4.6.2 The stable Paley–Wiener theorems

We define $\|\phi\| = \|\mu_\phi\|$ in the archimedean case. Note that for any countable set $E \subseteq \Phi_2(L)$, we have Paley–Wiener and Schwartz spaces defined on the space $\Lambda = \bigcup_{\phi \in E} \Lambda_\phi$ with $\Lambda_\phi = i\mathfrak{a}^*_L/\mathfrak{a}^\vee_{L,\phi}$.

We define $\mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(L)$ to be the space of smooth functions $\varphi : \Phi_2(L)$ such that for some (and hence any) set of representatives $B^{\mathrm{st}}_{\mathrm{ell}}(L) \subseteq \Phi_2(L)$ for the connected components of $\Phi_2(L)$, we have

$$\varphi \in \mathscr{S}\left( \coprod_{\phi \in B^{\mathrm{st}}_{\mathrm{ell}}(G)} i\mathfrak{a}^*_L/\mathfrak{a}^\vee_{L,\phi} \right).$$

We define $PW^{\mathrm{st}}_{\mathrm{ell}}(L)$ (resp. $PW^{\mathrm{st}}_{\mathrm{ell},f}(L)$) in the same way as $\mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(L)$, except that we replace $\mathscr{S}(\cdot)$ by $PW(\cdot)$ (resp. $PW_f(\cdot)$).

We define

$$\mathscr{S}^{\mathrm{st}}(G) = \left( \bigoplus_{L \in \mathcal{L}^G(M_0)} \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(L) \right)^{W_0^G} = \bigoplus_{L \in \mathcal{L}^G(M_0)/W_0^G} \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(L)^{W^G(L)}$$

and similarly we define $PW^{\mathrm{st}}(G)$ and $PW^{\mathrm{st}}_f(G)$. These are naturally spaces of smooth functions on $\Phi_{\mathrm{temp}}(G)$. The aim of this section is to prove the following stable Paley–Wiener theorems.

**Theorem 4.6.1.** *The stable Fourier transform is an isomorphism of topological vector spaces*

$$\mathcal{S}(G) \longrightarrow \mathscr{S}^{\mathrm{st}}(G)$$

*and restricts to isomorphisms of topological vector spaces*

$$\mathcal{S}_c(G) \longrightarrow PW^{\mathrm{st}}(G)$$

*and*

$$\mathcal{S}_f(G) \longrightarrow PW_f^{\mathrm{st}}(G).$$

**Example 4.6.2.** We describe the above spaces explicitly in the case when $G = \mathrm{SL}_2$ and $F = \mathbb{R}$. Let $T$ be the maximal anisotropic torus of $\mathrm{SL}_2$ with $T(\mathbb{R})$ consisting of elements of the form

$$t(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $S$ be the subgroup of diagonal elements of $\mathrm{SL}_2$, a split maximal torus. For $a \in \mathbb{R}^\times$, we write $s(x) = \mathrm{diag}(x, x^{-1}) \in S(\mathbb{R})$. The set $\{S, \mathrm{SL}_2\}$ is a set of representatives for the conjugacy classes of Levi subgroups of $G$. The Weyl group $W_0^G = W^G(S)$ is of order two, with the non-trivial element acting by $s(x^{-1}) \mapsto s(x^{-1})$.

The space $\Phi_2(G) = \{\phi_n\}_{n=1}^\infty$ is countable and discrete, with $\phi_n$ denoting the $L$-parameter of the whose $L$-packet consists of the discrete series representations $\Theta_{\pm n}$, whose characters are given on the regular elements of $T(\mathbb{R})$ by the formula

$$\Theta_{\pm n}(t(\theta)) = -\frac{\pm e^{\pm in\theta}}{e^{i\theta} - e^{-i\theta}}.$$

The stable discrete series character of $\phi_n$ is

$$\Theta_{\phi_n}(t(\theta)) = -\frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}.$$

We identify $\Phi_2(G) = \mathbb{Z}_{\geq 1}$ via $\phi_n \mapsto n$. We have

$$\mathscr{S}_{\mathrm{ell}}^{\mathrm{st}}(G) = PW_{\mathrm{ell}}^{\mathrm{st}}(G) = \mathscr{S}(\mathbb{Z}_{\geq 1}),$$

the space of rapidly decreasing functions $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{C}$. Furthermore,

$$PW_{\mathrm{ell},f}^{\mathrm{st}}(G) = C_c(\mathbb{Z}_{\geq 1}),$$

the space of finitely supported functions $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{C}$.

By the local Langlands correspondence for $S$, we may identify $\Phi_2(S)$ with $\Pi_u(S)$, the set of unitary characters of $S(\mathbb{R})$. The unitary unramified characters of $S(\mathbb{R})$ are those of the form $s(x) \mapsto |x|^{i\lambda}$ for $\lambda \in \mathbb{R}$. A general unitary character of $S(\mathbb{R})$ can be written as

$$\chi_{m,\lambda}(s(x)) = \mathrm{sgn}(x)^m |x|^{i\lambda},$$

for unique $m \in \{0, 1\}$ and $\lambda \in \mathbb{R}$. The non-trivial element of $W^G(S)$ acts $\Pi_u(S)$ by

$\chi_{m,\lambda} \mapsto \chi_{m,-\lambda}$. We identify $\Phi_2(S) = \{0,1\} \times \mathbb{R} = \mathbb{R} \coprod \mathbb{R}$ via $\chi_{m,\lambda} \mapsto (m,\lambda)$. We have $\Phi_2(S)/W_0^G = \mathbb{R}/\{\pm 1\} \coprod \mathbb{R}/\{\pm 1\} = \mathbb{R}_{\geq 0} \coprod \mathbb{R}_{\geq 0}$. We have

$$\mathscr{S}^{\text{st}}_{\text{ell}}(S) = \mathscr{S}(\mathbb{R}) \oplus \mathscr{S}(\mathbb{R})$$

and

$$\mathscr{S}^{\text{st}}_{\text{ell}}(S)^{W^G(S)} = \mathscr{S}(\mathbb{R})^{\{\pm 1\}} \oplus \mathscr{S}(\mathbb{R})^{\{\pm 1\}}.$$

Similarly, we have

$$PW^{\text{st}}_{\text{ell}}(S) = PW^{\text{st}}_{\text{ell},f}(S) = PW(\mathbb{R}) \oplus PW(\mathbb{R})$$

and

$$PW^{\text{st}}_{\text{ell}}(S)^{W^G(S)} = PW^{\text{st}}_{\text{ell},f}(S)^{W^G(S)} = PW(\mathbb{R})^{\{\pm 1\}} \oplus PW(\mathbb{R})^{\{\pm 1\}}.$$

Finally, we have

$$\begin{aligned}
\mathscr{S}^{\text{st}}(G) &= \mathscr{S}(\mathbb{Z}_{\geq 1}) \oplus \mathscr{S}(\mathbb{R})^{\{\pm 1\}} \oplus \mathscr{S}(\mathbb{R})^{\{\pm 1\}}, \\
PW^{\text{st}}(G) &= \mathscr{S}(\mathbb{Z}_{\geq 1}) \oplus PW(\mathbb{R})^{\{\pm 1\}} \oplus PW(\mathbb{R})^{\{\pm 1\}}, \\
PW^{\text{st}}_f(G) &= C_c(\mathbb{Z}_{\geq 1}) \oplus PW(\mathbb{R})^{\{\pm 1\}} \oplus PW(\mathbb{R})^{\{\pm 1\}}.
\end{aligned}$$

Arthur proved the stable Paley–Wiener theorem for test functions on a quasisplit group as a consequence of his more general result [Art96, Theorem 6.1] on collective endoscopic transfer (see the discussion below the statement of Theorem 6.2 in [Art96]). Moeglin–Waldspurger proved the stable Paley–Wiener theorem for test functions on a quasisplit real group. In fact, they work in the more general setting of twisted spaces (also called twisted groups). Generalising Arthur's proof to Schwartz functions would require generalising several results that we do not need. Instead, we follow the argument of Moeglin–Waldspurger to prove the above theorem. Since we work with groups (and not general twisted groups), there are some simplifications. We also consider groups that are not quasisplit.

### 4.6.3 Stable and unstable elliptic tempered characters

Recall that the stable tempered characters are linearly independent and that we identify tempered $L$-parameters $\phi \in \Phi_{\text{temp}}(G,\zeta)$ with their associated virtual tempered representations $\pi_\phi = \sum_{\pi \in \Pi_\phi} c_\phi(\pi)\pi$ and with their stable tempered distribution characters $\Theta_\phi = \sum_{\pi \in \Pi_\phi} c_\phi(\pi)\Theta_\pi$.

By our hypothesis, we have $D^{\mathrm{st}}_{\mathrm{temp}}(G,\zeta) = \mathbb{C}\Phi_{\mathrm{temp}}(G,\zeta)$ and $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) = \mathbb{C}\Phi_2(G,\zeta)$. We define the space $D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$ to be the subspace of $D_{\mathrm{ell}}(G,\zeta)$ consisting of all elements orthogonal to $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$ with respect to the elliptic inner product.

**Lemma 4.6.3.** *We have* $D_{\mathrm{ell}}(G,\zeta) = D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) \oplus D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$.

Let us set some notation and terminology. Let $V$ be an inner product space (not necessarily complete) and let $W$ be a subspace of $V$. We say that $W$ has an orthogonal complement in $V$ if there exists a subspace $X$ of $V$ such that $X$ is orthogonal to $W$ and $V = W \oplus X$, in which case $X$ is

$$W^{\perp} = \mathrm{Ann}_V(W) = \{v \in V : (W,v) = 0\}$$

the largest subspace of $V$ orthogonal to $W$. If $V$ is complete and $W$ is closed (in particular, if $V$ is finite-dimensional), then $W$ has an orthogonal complement in $V$.

*Proof.* We have $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) = \mathbb{C}\Phi_2(G,\zeta) = \bigoplus_{\phi\in\Phi_2(G,\zeta)} \mathbb{C}\phi$. For each $\phi \in \Phi_2(G,\zeta)$, the space $\mathbb{C}\phi$ has an orthogonal complement in the finite-dimensional space $\mathbb{C}\Pi_\phi$. Since

$$\mathbb{C}\Pi_2(G,\zeta) = \bigoplus_{\phi\in\Phi_2(G,\zeta)} \mathbb{C}\Pi_\phi.$$

it follows that $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$ has an orthogonal complement in $\mathbb{C}\Pi_2(G,\zeta)$, namely

$$\mathrm{Ann}_{\mathbb{C}\Pi_2(G,\zeta)}(D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)) = \bigoplus_{\phi\in\Phi_2(G,\zeta)} \mathrm{Ann}_{\mathbb{C}\Pi_\phi}(\mathbb{C}\phi).$$

Since

$$D_{\mathrm{ell}}(G,\zeta) = \mathbb{C}\Pi_2(G,\zeta) \oplus \mathbb{C}(T_{\mathrm{ell}}(G,\zeta) \setminus \mathbb{C}^1\Pi_2(G,\zeta))$$

it follows that

$$D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta) = \mathrm{Ann}_{\mathbb{C}\Pi_2(G,\zeta)}(D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)) \oplus \mathbb{C}(T_{\mathrm{ell}}(G,\zeta) \setminus \mathbb{C}^1\Pi_2(G,\zeta))$$

and that $D_{\mathrm{ell}}(G,\zeta) = D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) \oplus D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$. □

### 4.6.4 Stable pseudocoefficients

Let $\zeta$ be a unitary character of $A_G(F)$. Recall the Paley–Wiener space $PW_{\mathrm{ell},f}(G,\zeta)$. Fix a set of representatives $E_{\mathrm{ell}}(G,\zeta) \subseteq T_{\mathrm{ell}}(G,\zeta)$ for $T_{\mathrm{ell}}(G,\zeta)/\mathbb{C}^1$ such that $\Pi_2(G,\zeta) \subseteq$

$E_{\mathrm{ell}}(G,\zeta)$. Then $E_{\mathrm{ell}}(G,\zeta)$ is a basis for $D_{\mathrm{ell}}(G,\zeta)$. As explained above, we have an identification $PW_{\mathrm{ell},f}(G,\zeta) = \bigoplus_{\tau\in E_{\mathrm{ell}}(G,\zeta)} \mathbb{C}$ defined by $\varphi \mapsto (\varphi(\tau))_{\tau\in E_{\mathrm{ell}}(G,\zeta)}$.

Let $B^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) = \Phi_2(G,\zeta)$, which is an orthogonal basis of $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$ (with respect to the elliptic inner product). Let $B^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$ be an orthogonal basis for $D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$ and define $B_{\mathrm{ell}}(G,\zeta) = B^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) \cup B^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$, which is an orthogonal basis of $D_{\mathrm{ell}}(G,\zeta)$. Define $PW^{?}_{\mathrm{ell},f}(G,\zeta) = \bigoplus_{b\in B^{?}_{\mathrm{ell}}(G,\zeta)} \mathbb{C}$, where $? = \mathrm{st}, \mathrm{unst}$.

We identify an element of $PW_{\mathrm{ell},f}(G,\zeta)$ (resp. $PW^{?}_{\mathrm{ell},f}(G,\zeta)$) with the linear functional on $D_{\mathrm{ell}}(G,\zeta)$ (resp. $D^{?}_{\mathrm{ell}}(G,\zeta)$) it defines by linear extension. Thus, for each $\varphi \in PW_{\mathrm{ell},f}(G,\zeta)$ the value $\varphi(b)$ is defined for all $b \in B_{\mathrm{ell}}(G,\zeta)$. Explicitly, for each $b \in B_{\mathrm{ell}}(G,\zeta)$, let us write $b = \sum_{\tau\in E_{\mathrm{ell}}(G,\zeta)} c_{b,\tau}\tau$ for $c_{b,\tau} \in \mathbb{C}$. Then $\varphi(b) = \sum_{\tau\in E_{\mathrm{ell}}(G,\zeta)} c_{b,\tau}\tau$. We have an isomorphism

$$PW_{\mathrm{ell},f}(G,\zeta) \xrightarrow{\sim} PW^{\mathrm{st}}_{\mathrm{ell},f}(G,\zeta) \oplus PW^{\mathrm{unst}}_{\mathrm{ell},f}(G,\zeta)$$

defined by $(\varphi(\tau))_{\tau\in E_{\mathrm{ell}}(G,\zeta)} \mapsto (\varphi(b))_{b\in B_{\mathrm{ell}}(G,\zeta)} = ((\varphi(b))_{b\in B^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)}, (\varphi(b))_{b\in B^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)})$, that is, by change-of-basis. We identify $PW_{\mathrm{ell},f}(G,\zeta) = PW^{\mathrm{st}}_{\mathrm{ell},f}(G,\zeta) \oplus PW^{\mathrm{unst}}_{\mathrm{ell},f}(G,\zeta)$ via the above isomorphism. Note that $PW^{\mathrm{unst}}_{\mathrm{ell},f}(G,\zeta)$ is the subspace of functions in $PW_{\mathrm{ell},f}(G,\zeta)$ that vanish on $D^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$ and thus does not depend on the choice of $B^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$.

Let $\mathcal{I}^{\mathrm{st}}_{f,\mathrm{cusp}}(G,\zeta)$ be closed subspace of $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$ consisting of all functions that are constant on strongly regular stable classes. Let $\mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G,\zeta)$ be the kernel of the quotient

$$\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta) \longrightarrow \mathcal{S}_{f,\mathrm{cusp}}(G,\zeta).$$

Note that $\mathcal{I}^{\mathrm{st}}_{f,\mathrm{cusp}}(G,\zeta) \cap \mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G,\zeta) = 0$.

**Proposition 4.6.4.** *The invariant Fourier transform*

$$\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta) \xrightarrow{\sim} PW_{\mathrm{ell},f}(G,\zeta)$$

*restricts to isomorphisms*

$$\mathcal{I}^{?}_{f,\mathrm{cusp}}(G,\zeta) \xrightarrow{\sim} PW^{?}_{\mathrm{ell},f}(G,\zeta)$$

*Consequently, we have*

$$\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta) = \mathcal{I}^{\mathrm{st}}_{f,\mathrm{cusp}}(G,\zeta) \oplus \mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G,\zeta).$$

*Proof.* It suffices to prove that the subspace of $\mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$ corresponding to $PW^{?}_{\mathrm{ell},f}(G,\zeta)$

lies in $\mathcal{I}^?_{f,\mathrm{cusp}}(G,\zeta)$.

Let $b \in B^?_{\mathrm{ell}}(G,\zeta)$. The inverse invariant Fourier transform of $\delta_b = (\delta_{b,b'})_{b' \in B_{\mathrm{ell}}(G,\zeta)} \in PW^?_{\mathrm{ell},f}(G,\zeta)$ is the normalised pseudocoefficient $\|b\|^{-1}_{\mathrm{ell}} f[b]_G \in \mathcal{I}_{f,\mathrm{cusp}}(G,\zeta)$ of $b$. Thus, it suffices to prove that the pseudocoefficient $f[b]_G$ lies in $\mathcal{I}^?_{f,\mathrm{cusp}}(G,\zeta)$

If $b \in B^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$, then $f[b]_G(\phi) = 0$ for all $\phi \in \Phi_2(G,\zeta) = B^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$, and thus $f[b]_G \in \mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G,\zeta)$ by stable spectral density for $C^\infty_{c,\mathrm{cusp}}(G,\zeta,K)$.

Suppose that $b \in B^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) = \Phi_2(G,\zeta)$ and let us write $b = \phi$ and $f[b]_G = f[\phi]_G$. Let $\gamma \in \Gamma_{\mathrm{sr}}(G)$. If $\gamma$ is non-elliptic, then $f[\phi]_G(\gamma) = 0$. If $\gamma$ is elliptic, then

$$f[\phi]_G(\gamma) = m(\gamma)^{-1}|D^G(\gamma)|^{1/2}\overline{\Theta_\phi(\gamma)}.$$

We have that $D^G$ and $\Theta_\phi$ are constant on stable strongly regular classes. Thus, it suffices to show that $m(\gamma)$ is constant on stable strongly regular classes. By definition, we have $m(\gamma) = \mathrm{vol}(G_\gamma(F)/A_G(F))$. Suppose that $\gamma'$ is stably conjugate to $\gamma$. Then there exists $g \in G(\overline{F})$ such that $\mathrm{Int}(g) : G_\gamma \to G_{\gamma'}$ is defined over $F$. The Haar measures on $G_\gamma(F)$ and $G_{\gamma'}(F)$ are normalised so that they correspond under $\mathrm{Int}(g)$, and thus $m(\gamma) = m(\gamma')$. □

We obtain the following stable Paley–Wiener theorem as a corollary

**Corollary 4.6.5.** *The stable Fourier transform gives an isomorphism*

$$\mathcal{S}_{f,\mathrm{cusp}}(G,\zeta) \xrightarrow{\ \sim\ } PW^{\mathrm{st}}_{\mathrm{ell},f}(G,\zeta).$$

*Proof.* The composition of the invariant Fourier transform

$$\mathcal{F} : \mathcal{I}_{f,\mathrm{cusp}}(G,\zeta) \longrightarrow PW_{\mathrm{ell},f}(G,\zeta)$$

with the natural projection $PW_{\mathrm{ell},f}(G,\zeta) \to PW^{\mathrm{st}}_{\mathrm{ell},f}(G,\zeta)$ is the stable Fourier transform. It is surjective and its kernel is $\mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G,\zeta)$ by Proposition 4.6.4, so it descends to an isomorphism $\mathcal{S}_{f,\mathrm{cusp}}(G,\zeta) \to PW^{\mathrm{st}}_{\mathrm{ell},f}(G,\zeta)$. □

### 4.6.5 The cuspidal case

Let $\zeta$ be a unitary character of $A_G(F)$ and let $\mu$ be an infinitesimal character of $G$. We denote by $T_{\mathrm{ell}}(G,\zeta,\mu)$ the set of $\tau \in T_{\mathrm{ell}}(G,\zeta)$ with $\mu_\tau = \mu$. Define $D_{\mathrm{ell}}(G,\zeta,\mu) = \mathbb{C}T_{\mathrm{ell}}(G,\zeta,\mu)$.

**Lemma 4.6.6.** *The quotient $T_{\mathrm{ell}}(G,\zeta,\mu)/\mathbb{C}^1$ is finite. If $F$ is archimedean, then its cardinality is bounded independently of $(\zeta,\mu)$.*

*Proof.* If $F$ is non-archimedean, there are finitely many elements of $T_{\mathrm{temp}}(G)/\mathbb{C}^1$ with a given infinitesimal character.

Suppose that $F$ is archimedean. We may assume that $F = \mathbb{R}$. Since $L^G(M_0)/W_0^G$ is finite, it suffices to show that for each $M \in \mathcal{L}^G(M_0)$, the number of possibilities for $\sigma \in \Pi_2(M)$ such that $(M, \sigma, \widetilde{r}) \in T_{\mathrm{ell}}(G, \zeta, \mu)$ is bounded independently of $(\zeta, \mu)$. There are at most $|W(G,T)/W(M,T)|$ possibilities for $\mu_\sigma$ since it maps to $\mu$. Consider the restriction $\zeta_\sigma|_{A_M(\mathbb{R})^\circ}$ of the central character of $\sigma$ to $A_M(\mathbb{R})^\circ$. Since $H_M : A_M(\mathbb{R})^\circ \to \mathfrak{a}_M$ is an isomorphism, we have $\zeta_\sigma|_{A_M(\mathbb{R})^\circ} = e^{\langle \lambda, H_M(\cdot)\rangle}$ for some $\lambda \in i\mathfrak{a}_M^*$. Every element of $W^G_{\mathrm{reg}}(\sigma)$ fixes $\zeta_\sigma|_{A_M(\mathbb{R})^\circ}$. Since $W^G_{\mathrm{reg}}(\sigma)$ is the subset of $w \in W^G_{\mathrm{reg}}(\sigma)$ such that $\mathfrak{a}_M^w = \mathfrak{a}_G$, we have $\lambda \in i\mathfrak{a}_G^*$. At the same time, $\zeta|_{A_G(\mathbb{R})^\circ} = \zeta_\sigma|_{A_G(\mathbb{R})^\circ} = e^{\langle \lambda, H_G(\cdot)\rangle}$. It follows that $\lambda$ and thus $\zeta_\sigma|_{A_M(\mathbb{R})^\circ}$ is determined by $\zeta$. There are $|A_M(\mathbb{R})/A_M(\mathbb{R})^\circ|$ ways that $\zeta_\sigma|_{A_M(\mathbb{R})}$ can extend $\zeta_\sigma|_{A_M(\mathbb{R})^\circ}$, so there are $|A_M(\mathbb{R})/A_M(\mathbb{R})^\circ|$ possibilities for $\zeta_\sigma|_{A_M(\mathbb{R})}$. As remarked above, in [AV16] Adams and Vogan prove that there are finitely many discrete series representations with a fixed infinitesimal character and split radical character and these representations form a discrete series $L$-packet. (The group $A_M(\mathbb{R})$ is the split radical of $M(\mathbb{R})$.) Since the cardinality of discrete series $L$-packets of $M$ is bounded, the lemma follows. □

Fix a set of representatives $\mathscr{X}^G$ for the set of $i\mathfrak{a}_G^*$-orbits of unitary characters of $A_G(F)$. We denote the $i\mathfrak{a}_G^*$-orbit of a pair $(\zeta, \mu)$ by $[\zeta, \mu]$. Fix a set of representatives $\mathscr{P}^G$ for the set of orbits $[\zeta, \mu]$, such that for each $(\zeta, \mu) \in \mathscr{P}^G$ we have $\zeta \in \mathscr{X}^G$.

Let $T_{\mathrm{ell}}(G, [\zeta, \mu])$ be the subspace of all $\tau \in T_{\mathrm{ell}}(G)$ such that $(\zeta_\tau|_{A_G(F)}, \mu_\tau) \in [\zeta, \mu]$. Let $\mathfrak{a}^\vee_{G,\mu}$ denote the isotropy subgroup of $\mu$ in $\mathfrak{a}^*_{G,\mathbb{C}}$. We have

$$\mathfrak{a}^\vee_{G,F} \subseteq \mathfrak{a}^\vee_{G,\mu} \subseteq \widetilde{\mathfrak{a}}^\vee_{G,F}.$$

Each connected component of $T_{\mathrm{ell}}(G, [\zeta, \mu])/\mathbb{C}^1$ has a representative in the quotient $T_{\mathrm{ell}}(G, \zeta, \mu)/\mathbb{C}^1$, and two elements of $T_{\mathrm{ell}}(G, \zeta, \mu)/\mathbb{C}^1$ lie in the same connected component of $T_{\mathrm{ell}}(G, [\zeta, \mu])/\mathbb{C}^1$ if and only if they are in the same orbit under the finite group $\mathfrak{a}^\vee_{G,\mu}/\mathfrak{a}^\vee_{G,F}$. Let $\underline{E}_{\mathrm{ell}}(G, \zeta, \mu) \subseteq T_{\mathrm{ell}}(G, \zeta, \mu)$ be a set of representatives for the connected components of $T_{\mathrm{ell}}(G, [\zeta, \mu])/\mathbb{C}^1$, or equivalently a set of representatives for the $\mathfrak{a}^\vee_{G,\mu}$-orbits in $T_{\mathrm{ell}}(G, \zeta, \mu)/\mathbb{C}^1$. Then the set of translates $\mathfrak{a}^\vee_{G,\mu} \cdot \underline{E}_{\mathrm{ell}}(G, \zeta, \mu) \subseteq T_{\mathrm{ell}}(G, \zeta, \mu)$ contains an orthogonal basis $E_{\mathrm{ell}}(G, \zeta, \mu)$ of $D_{\mathrm{ell}}(G, \zeta, \mu)$.

We define $\Phi_2(G, \zeta, \mu)$ to be the set of $\phi \in \Phi_2(G, \zeta)$ with $\mu_\phi = \mu$, and we define $\Phi_2(G, [\zeta, \mu])$ to be the set of all $\phi \in \Phi_2(G)$ with $(\zeta_\phi|_{A_G(F)}, \mu_\phi) \in [\zeta, \mu]$. Then $D^{\mathrm{st}}_{\mathrm{ell}}(G, \zeta, \mu) = \mathbb{C}\Phi_2(G, \zeta, \mu)$. Let $D^{\mathrm{unst}}_{\mathrm{ell}}(G, \zeta, \mu)$ be the orthogonal complement of $D^{\mathrm{st}}_{\mathrm{ell}}(G, \zeta, \mu)$ in $D_{\mathrm{ell}}(G, \zeta, \mu)$ with respect to the elliptic inner product. (Recall that

$D_{\mathrm{ell}}(G,\zeta,\mu)$ is finite-dimensional.) Let $\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu) \subseteq \Phi_2(G,\zeta,\mu)$ be a set of representatives for the $i\mathfrak{a}_G^*$-orbits in $\Phi_2(G,[\zeta,\mu])$, or equivalently a set of representatives for the $\mathfrak{a}_{G,\mu}^\vee$-orbits in $\Phi_2(G,\zeta,\mu)$. Then $\mathfrak{a}_{G,\mu}^\vee \cdot \underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu) = \Phi_2(G,\zeta,\mu)$. We write $B_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu) = \Phi_2(G,\zeta,\mu)$. The unitary representation of the finite group $\mathfrak{a}_{G,\mu}^\vee/\mathfrak{a}_{G,F}^\vee$ on $D_{\mathrm{ell}}(G)$ stabilises $D_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu)$. Let $B_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu) = \underline{B}_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu)$ be an orthonormal eigenbasis of $D_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu)$ for this representation. Define

$$\underline{B}_{\mathrm{ell}}(G,\zeta,\mu) = \underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu) \cup \underline{B}_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu)$$

and

$$B_{\mathrm{ell}}(G,\zeta,\mu) = B_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu) \cup B_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu).$$

Define $\underline{E}_{\mathrm{ell}}(G) = \coprod_{(\zeta,\mu)\in\mathscr{P}^G} \underline{E}_{\mathrm{ell}}(G,\zeta,\mu)$, and define $E_{\mathrm{ell}}(G)$, $\underline{B}_{\mathrm{ell}}^{?}(G)$, and $B_{\mathrm{ell}}^{?}(G)$ similarly. Let $\underline{B}_{\mathrm{ell}}(G) = \underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G) \cup \underline{B}_{\mathrm{ell}}^{\mathrm{unst}}(G)$ and $B_{\mathrm{ell}}(G) = B_{\mathrm{ell}}^{\mathrm{st}}(G) \cup B_{\mathrm{ell}}^{\mathrm{unst}}(G)$.

In order to be able to treat the case of test functions and Schwartz functions at the same time, we write $\mathscr{F} = \mathcal{C}_{(c)}$, $\mathcal{I}\mathscr{F} = \mathcal{I}_{(c)}$, and $\mathcal{S}\mathscr{F} = \mathcal{S}_{(c)}$. We also write $\widehat{\mathscr{F}} = \mathscr{S}$ (resp. $\widehat{\mathscr{F}} = PW$) if $\mathscr{F} = \mathcal{C}$ (resp. $\mathscr{F} = \mathcal{C}_c$).

By definition, we have identifications

$$\widehat{\mathscr{F}}_{\mathrm{ell}}(G) = \widehat{\mathscr{F}}\left(\coprod_{\tau\in\underline{E}_{\mathrm{ell}}(G)} i\mathfrak{a}_G^*\cdot\tau\right) \quad , \quad \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G) = \widehat{\mathscr{F}}\left(\coprod_{b\in\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G)} i\mathfrak{a}_G^*\cdot b\right).$$

Let $\widehat{\mathscr{F}}_{\mathrm{ell}}(G,[\zeta,\mu])$ (resp. $\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G,[\zeta,\mu])$) be the subspace of $\widehat{\mathscr{F}}_{\mathrm{ell}}(G)$ (resp. $\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G)$) consisting of functions supported on $T_{\mathrm{ell}}(G,[\zeta,\mu])$ (resp. $\Phi_2(G,[\zeta,\mu])$). The above identifications restrict to identifications

$$\widehat{\mathscr{F}}_{\mathrm{ell}}(G,[\zeta,\mu]) = \widehat{\mathscr{F}}\left(\coprod_{\tau\in\underline{E}_{\mathrm{ell}}(G,\zeta,\mu)} i\mathfrak{a}_G^*\cdot\tau\right) \quad , \quad \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G,[\zeta,\mu]) = \widehat{\mathscr{F}}\left(\coprod_{b\in\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu)} i\mathfrak{a}_G^*\cdot b\right).$$

We define

$$\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{unst}}(G) = \widehat{\mathscr{F}}\left(\coprod_{b\in\underline{B}_{\mathrm{ell}}^{\mathrm{unst}}(G)} i\mathfrak{a}_G^*\cdot b\right) \quad , \quad \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{unst}}(G,[\zeta,\mu]) = \widehat{\mathscr{F}}\left(\coprod_{b\in\underline{B}_{\mathrm{ell}}^{\mathrm{unst}}(G,\zeta,\mu)} i\mathfrak{a}_G^*\cdot b\right).$$

We have two orthogonal bases $E_{\mathrm{ell}}(G)$ and $B_{\mathrm{ell}}(G)$ of $\bigoplus_{(\zeta,\mu)\in\mathscr{P}^G} D_{\mathrm{ell}}(G,\zeta,\mu)$. We define change-of-basis matrices

$$b = \sum_{\tau\in E_{\mathrm{ell}}(G)} c_{b,\tau}\tau \quad , \quad \tau = \sum_{b\in B_{\mathrm{ell}}(G)} c_{\tau,b} b.$$

It follows from the decompositions

$$E_{\mathrm{ell}}(G) = \coprod_{(\zeta,\mu)\in\mathscr{P}^G} E_{\mathrm{ell}}(G,\zeta,\mu) \quad , \quad B_{\mathrm{ell}}(G) = \coprod_{(\zeta,\mu)\in\mathscr{P}^G} B_{\mathrm{ell}}(G,\zeta,\mu)$$

into orthogonal bases of $D_{\mathrm{ell}}(G,\zeta,\mu)$ that these change-of-basis matrices are block-diagonal. Let $\varphi = (\varphi_\tau)_{\tau\in\underline{E}_{\mathrm{ell}}(G)} \in \widehat{\mathscr{F}}_{\mathrm{ell}}(G)$. We identify $\varphi$ with its linear extension to $D_{\mathrm{ell}}(G)$. For $\tau \in E_{\mathrm{ell}}(G)$ and $b \in B_{\mathrm{ell}}(G)$, we write $\varphi_\tau(\lambda) = \varphi(\tau_\lambda)$ and $\varphi_b(\lambda) = \varphi(b_\lambda)$. Each element of $\tau \in E_{\mathrm{ell}}(G)$ (resp. $b \in B_{\mathrm{ell}}(G)$) is a translate of an element of $\underline{\tau} \in \underline{E}_{\mathrm{ell}}(G)$ (resp. $\underline{b} \in \underline{B}_{\mathrm{ell}}(G)$) by an element of $\lambda_0 \in \widetilde{\mathfrak{a}}^\vee_{G,F}$, and thus $\varphi_\tau(\lambda) = \varphi_{\underline{\tau}}(\lambda+\lambda_0)$ (resp. $\varphi_b(\lambda) = \varphi_{\underline{b}}(\lambda+\lambda_0)$). For $b \in \underline{B}_{\mathrm{ell}}(G)$ and $\lambda \in i\mathfrak{a}^*_G$, we have $b_\lambda = \sum_{\tau\in E_{\mathrm{ell}}(G)} c_{b,\tau}\tau_\lambda$, and therefore

$$\varphi_b(\lambda) = \sum_{\tau\in E_{\mathrm{ell}}(G)} c_{b,\tau}\varphi_\tau(\lambda) = \sum_{\underline{\tau},\lambda_0} c_{b,\underline{\tau}_{\lambda_0}}\varphi_{\underline{\tau}}(\lambda+\lambda_0).$$

Consider the linear map $\varphi \mapsto (\varphi_b)_{b\in\underline{B}_{\mathrm{ell}}(G)} = (\varphi^{\mathrm{st}},\varphi^{\mathrm{unst}})$, where $\varphi^? = (\varphi_b)_{b\in\underline{B}^?_{\mathrm{ell}}(G)}$. For each $(\zeta,\mu)\in\mathscr{P}^G$, this linear map restricts to an isomorphism of topological vector spaces

$$\widehat{\mathscr{F}}_{\mathrm{ell}}(G,\zeta,\mu) \longrightarrow \widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta,\mu) \oplus \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta,\mu).$$

with inverse

$$\varphi_\tau(\lambda) = \sum_{b\in B_{\mathrm{ell}}(G)} c_{\tau,b}\varphi_b(\lambda) = \sum_{\underline{b},\lambda_0} c_{\tau,\underline{b}_{\lambda_0}}\varphi_{\underline{b}}(\lambda+\lambda_0).$$

This follows easily since the spaces $\widehat{\mathscr{F}}_{\mathrm{ell}}(G,\zeta,\mu)$, $\widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta,\mu)$, and $\widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta,\mu)$ are each finite direct sums of classical $\widehat{\mathscr{F}}$-spaces.

**Lemma 4.6.7.** *We have an isomorphism of topological vector spaces*

$$\begin{aligned} \widehat{\mathscr{F}}_{\mathrm{ell}}(G) &\longrightarrow \widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(G) \oplus \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(G) \\ \varphi &\longmapsto (\varphi^{\mathrm{st}},\varphi^{\mathrm{unst}}) \end{aligned}$$

The argument is similar to that in [MW16a, Ch. IV, §1.5 and §2.2].

*Proof.* First consider the case where $F$ is non-archimedean. In this case, we have locally convex direct sum decompositions

$$\widehat{\mathscr{F}}_{\mathrm{ell}}(G) = \bigoplus_{(\zeta,\mu)\in\mathscr{P}^G} \widehat{\mathscr{F}}_{\mathrm{ell}}(G,\zeta,\mu) \quad , \quad \widehat{\mathscr{F}}^?_{\mathrm{ell}}(G) = \bigoplus_{(\zeta,\mu)\in\mathscr{P}^G} \widehat{\mathscr{F}}^?_{\mathrm{ell}}(G,\zeta,\mu).$$

Thus, it suffices to show that the columns of the change-of-basis matrices have finitely many non-zero entries. This is true since they are block-diagonal as remarked above and the dimension of $D_{\mathrm{ell}}(G, \zeta, \mu)$ is finite for each $(\zeta, \mu) \in \mathscr{P}^G$.

Now consider the archimedean case. As in [MW16a, Ch. IV, §1.5 and §2.2], it suffices to show that: (1) the number of nonzero entries in a column of the change-of-basis matrices is bounded; and (2) the entries of the change of basis matrices are bounded. Property (1) follows since the dimension of $D_{\mathrm{ell}}(G, \zeta, \mu)$ is bounded independently of $(\zeta, \mu)$ by Lemma 4.6.6. For property (2), recall that we have seen that $\{\|\tau\|_{\mathrm{ell}} : \tau \in T_{\mathrm{ell}}(G)\}$ is bounded. Moreover, by construction $\|b\|_{\mathrm{ell}} = 1$ for $b \in B^{\mathrm{unst}}_{\mathrm{ell}}(G)$. Suppose that $b \in B^{\mathrm{st}}_{\mathrm{ell}}(G) = \Phi_2(G)$ and write $b = \phi$. We have $\phi = \sum_{\pi \in \Pi_\phi} \pi$. Therefore $\|\phi\|_{\mathrm{ell}} = \sum_{\pi \in \Pi_\phi} \|\pi\|_{\mathrm{ell}}$. As mentioned above, for $\pi \in \Pi_2(G)$ we have $\|\pi\|_{\mathrm{ell}} = 1$. (See the discussion after the orthogonality relations Theorem 3.2.2.) Therefore $\|\pi\|_{\mathrm{ell}} = |\Pi_\phi|$, which is bounded. □

We will identify $\widehat{\mathscr{F}}_{\mathrm{ell}}(G) = \widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(G) \oplus \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(G)$ via the above isomorphism. Note that the subspace $\widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(G)$ of $\widehat{\mathscr{F}}_{\mathrm{ell}}(G)$ consists of those functions $\varphi$ whose linear extension to $D_{\mathrm{ell}}(G)$ vanishes on $D^{\mathrm{st}}_{\mathrm{ell}}(G)$. Thus, it does not depend on any of the choices made above.

Define $\mathcal{I}\mathscr{F}^{\mathrm{st}}_{\mathrm{cusp}}(G)$ to be the closed subspace of $\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G)$ consisting of normalised invariant orbital integrals that are constant on strongly regular stable classes. Define $\mathcal{I}\mathscr{F}^{\mathrm{unst}}_{\mathrm{cusp}}(G)$ to be the kernel of the quotient $\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G) \to \mathcal{S}\mathscr{F}_{\mathrm{cusp}}(G)$. Note that $\mathcal{I}\mathscr{F}^{\mathrm{st}}_{\mathrm{cusp}}(G) \cap \mathcal{I}\mathscr{F}^{\mathrm{unst}}_{\mathrm{cusp}}(G) = 0$.

**Proposition 4.6.8.** *The invariant Fourier transform*

$$\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G) \xrightarrow{\sim} \widehat{\mathscr{F}}_{\mathrm{ell}}(G)$$

*restricts to isomorphisms*

$$\mathcal{I}\mathscr{F}^{?}_{\mathrm{cusp}}(G) \xrightarrow{\sim} \widehat{\mathscr{F}}^{?}_{\mathrm{ell}}(G).$$

*Consequently, we have*

$$\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G) = \mathcal{I}\mathscr{F}^{\mathrm{st}}_{\mathrm{cusp}}(G) \oplus \mathcal{I}\mathscr{F}^{\mathrm{unst}}_{\mathrm{cusp}}(G).$$

**Corollary 4.6.9.** *The stable Fourier transform gives an isomorphism*

$$\mathcal{S}\mathscr{F}_{\mathrm{cusp}}(G) \xrightarrow{\sim} \widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(G).$$

We start with the following lemma which we will use to construct functions in $\mathscr{F}(G)$.

**Lemma 4.6.10.** *Let $f \in C_c^\infty(G,\zeta)$ and let $\phi \in C_c^\infty(\mathfrak{a}_{G,F})$ (resp. $\phi \in \mathscr{S}(\mathfrak{a}_{G,F})$). The function $f^\phi \coloneqq f \cdot (\phi \circ H_G)$ lies in $C_c^\infty(G)$ (resp. $\mathcal{C}(G)$).*

Note that if $f$ is $K$-finite, then so is $f^\phi$.

*Proof.* Let $f^\phi = f \cdot (\phi \circ H_G)$. Evidently, $f^\phi$ is smooth. Let $C \subseteq G(F)$ be a compact subspace such that $\operatorname{supp}(f) = CA_G(F)$.

First, we suppose that $\phi \in C_c^\infty(\mathfrak{a}_{G,F})$. We have

$$\operatorname{supp}(f^\phi) \subseteq \operatorname{supp}(\phi \circ H_G) \cap \operatorname{supp}(f) \subseteq H_G^{-1}(\operatorname{supp}(\phi)) \cap CA_G(F).$$

Thus, to show that $\operatorname{supp}(f^\phi)$ is compact, it suffices to show that $H_G^{-1}(\operatorname{supp}\phi) \cap CA_G(F)$ is compact. Let $x \in H_G^{-1}(\operatorname{supp}(\phi)) \cap CA_G(F)$ and write $x = ca$, where $c \in C$ and $a \in A_G(F)$. Then $H_G(a) = H_G(x) - H_G(c)$, which lies in the compact subspace $\operatorname{supp}(\phi) - H_G(C)$ of $\mathfrak{a}_{G,F}$. Let $\widetilde{C} = \widetilde{\mathfrak{a}}_{G,F} \cap (\operatorname{supp}(\phi) - H_G(C))$, a compact subspace of $\widetilde{\mathfrak{a}}_{G,F}$. Then $a \in H_G|_{A_G(F)}^{-1}(\widetilde{C})$, and $x \in CH_G|_{A_G(F)}^{-1}(\widetilde{C})$ Thus,

$$H_G^{-1}(\operatorname{supp}(\phi)) \cap CA_G(F) \subseteq CH_G|_{A_G(F)}^{-1}(\widetilde{C}).$$

Therefore it suffices to show that $H_G|_{A_G(F)}^{-1}(\widetilde{C})$ is compact. This follows since $H_G|_{A_G(F)} : A_G(F) \to \widetilde{\mathfrak{a}}_{G,F}$ is proper, as it is a continuous surjective homomorphism with compact kernel $A_G(F)^1$.

For the rest of the proof, we suppose that $\phi \in \mathscr{S}(\mathfrak{a}_{G,F})$. First we show that $f^\phi$ is rapidly decreasing. Let $r > 0$. We must show that $|f^\phi(g)| \ll \Xi(g)(1+\sigma(g))^{-r}$. Recall the decomposition $G(F) = KM_0(F)K$, that $\Xi$ is bi-$K$-invariant, and that $1 + \sigma(k_1 m_0 k_2) \asymp 1 + \sigma(m_0)$ and $1 + \sigma(m_0) \asymp 1 + \|H_0(m_0)\|$ for $k_1, k_2 \in K$ and $m_0 \in M_0(F)$. Therefore, it suffices to show $|f^\phi(k_1 m_0 k_2)| \ll \Xi(m_0)(1 + \|H_0(m_0)\|)^{-r}$.

Since $f \in \mathcal{C}(G,\zeta)$, we have that $|f(g)| \ll \Xi(g)(1+\sigma^{A_G}(g))^{-r}$. We have $1 + \sigma^{A_G}(k_1 m_0 k_2) \asymp 1 + \sigma^{A_G}(m_0)$, and therefore $|f(k_1 m_0 k_2)| \ll \Xi(m_0)(1+\sigma^{A_G}(m_0))^{-r}$. Since $\phi \in \mathscr{S}(\mathfrak{a}_{G,F})$, we have $|\phi(X)| \ll (1 + \|X\|)^{-r}$. Therefore

$$|f^\phi(k_1 m_0 k_2)| \ll \Xi(m_0)(1 + \|H_G(m_0)\|)^{-r}(1+\sigma^{A_G}(m_0))^{-r}.$$

Decompose $H_0(m_0) \in \mathfrak{a}_{M_0,F}$ as $H_0(m_0) = H_G(m_0) + H_0(m_0)^G$ according to the decomposition $\mathfrak{a}_{M_0,F} = \mathfrak{a}_{G,F} \oplus \mathfrak{a}^G_{M_0,F}$. Since $1 + \sigma(m_0) \asymp 1 + \|H_0(m_0)\|$, we have

$$1 + \sigma^{A_G}(m_0) \asymp 1 + \inf_{a \in A_G(F)} \|H_0(a) + H_0(m_0)\|.$$

Since $\mathfrak{a}_{M_0,F}/\widetilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F} \oplus \mathfrak{a}^G_{M_0,F}$ and $\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ is finite, we obtain

$$1 + \inf_{a \in A_G(F)} \|H_0(a) + H_0(m_0)\| \asymp 1 + \|H_0(m_0)^G\|.$$

Therefore

$$1 + \sigma^{A_G(F)}(m_0) \asymp 1 + \|H_0(m_0)^G\|.$$

Thus, we have

$$|f^\phi(k_1 m_0 k_2)|/\Xi(m_0) \ll (1 + \|H_G(m_0)\|)^{-r}(1 + \|H_0(m_0)^G\|)^{-r} \ll (1 + \|H_0(m_0)\|)^{-r}$$

as required.

Suppose that $F$ is non-archimedean. In this case, we must show that there exists a compact open subgroup $K_0$ of $G(F)$ such that $f^\phi$ is bi-$K_0$-invariant. Since $F$ is non-archimedean, supp$(f)$ is open and has compact open image in $G(F)/A_G(F)$. Thus, we may take $C$ to be compact open. Then there exists a compact open subgroup $K_0$ of $G(F)$ such that $K_0 C K_0 = C$ and $f$ is constant on the $K_0$ double cosets in $C$. Since $f$ is zero outside of $CA_G(F)$, it suffices to show that $f$ is bi-$K_0$-invariant on $CA_G(F)$. Write $C = \bigcup_{i=1}^M K_0 g_i K_0$ and define $c_i = f(K_0 g_i K_0)$ for $i = 1, \ldots, M$. Then $f(k_0 g_i k_0' a) = \zeta(a)^{-1} c_i$ for all $k_0, k_0' \in K_0$, $i = 0, \ldots, M$, and $a \in A_G(F)$. For any $k_0'', k_0'''$ we have

$$f(k_0' k_0 g_i k_0' a k_0'') = f(k_0' k_0 g_i k_0' k_0''' a) = \zeta(a)^{-1} c_i = f(k_0 g_i k_0' a).$$

Therefore $f$ is bi-$K_0$-invariant. Since $H_G$ is zero on all compact subgroups of $G(F)$, we have that $f^\phi$ is bi-$K_0$-invariant.

Now assume that $F$ is archimedean. We must show that for all $X, Y \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, the function $L(X)R(Y)f^\phi$ is rapidly decreasing. For $X \in \mathfrak{g}$, we have

$$(R(X)f^\phi)(g) = \big[D\phi(H_G(g)) \cdot (R(X)H_G)(g)\big] f(g) + (R(X)f)^\phi(g),$$

and therefore $R(X)f^\phi$ is a linear combination of functions of the form $f^\phi$. A similar formula hods for $L(X)$. It follows that for all $X, Y \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, the function $L(X)R(Y)f^\phi$ is a linear combination of functions of the form $f^\phi$, and is thus rapidly decreasing by what we have already shown. □

Define $\underline{B}^?_{\mathrm{ell}}(G,\zeta) = \bigcup_{(\zeta,\mu)\in\mathscr{P}^G} \underline{B}^?_{\mathrm{ell}}(G,\zeta,\mu)$, and define $\underline{B}_{\mathrm{ell}}(G,\zeta) = \underline{B}^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta) \cup \underline{B}^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$. Let $b \in \underline{B}_{\mathrm{ell}}(G,\zeta)$. Since the elements of $\underline{B}_{\mathrm{ell}}(G)$ are mutually orthogonal, the pseudocoefficient $f[b]_G$ satisfies

$$\int_{G(F)/A_G(F)} \Theta_{b'}(x) f[b](x)\, dx = \|b\|_{\mathrm{ell}}\delta_b(b'),$$

for all $b' \in \underline{B}_{\mathrm{ell}}(G,\zeta)$. The domain of integration $G(F)/A_G(F)$ fibres over the finite set $\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ with open fibres. We will need the following formula for the integral of $\Theta_{b'}(x)f[b](x)$ on a single fibre in the case when $b \in \underline{B}^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$.

**Proposition 4.6.11.** *Let $b \in \underline{B}^{\mathrm{st}}_{\mathrm{ell}}(G,\zeta)$. For $b' \in \underline{B}_{\mathrm{ell}}(G,\zeta)$ and $X \in \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$, we have*

$$\int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})/A_G(F)} \Theta_{b'}(x) f[b](x)\, dx = \|b\|_{\mathrm{ell}} |\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|^{-1} \delta_b(b') \mathbb{1}_{\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}}(X).$$

*Proof.* Let $b' \in B_{\mathrm{ell}}(G,\zeta)$. For $X \in \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$, define

$$f_{b,b'}(X) = \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})/A_G(F)} \Theta_{b'}(x) f[b](x)\, dx.$$

We will calculate the Fourier transform of $f_{b,b'}$. The Pontryagin dual of $\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ is $\widetilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,F}$. For $\lambda \in \widetilde{\mathfrak{a}}^\vee_{G,F}/\mathfrak{a}^\vee_{G,F}$ we have

$$\begin{aligned}
\widehat{f_{b,b'}}(\lambda) &= \sum_{X\in\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} e^{-\langle\lambda,X\rangle} f_{b,b'}(X) \\
&= \sum_{X\in\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} e^{-\langle\lambda,X\rangle} \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})/A_G(F)} \Theta_{b'}(x) f[b](x)\, dx \\
&= \sum_{X\in\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})/A_G(F)} e^{-\langle\lambda,H_G(x)\rangle} \Theta_{b'}(x) f[b](x)\, dx \\
&= \sum_{X\in\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})/A_G(F)} \Theta_{b'}(x) f[b_\lambda](x)\, dx \\
&= \int_{G(F)/A_G(F)} \Theta_{b'_{-\lambda}}(x) f[b](x)\, dx \\
&= \langle b, b'_{-\lambda}\rangle_{\mathrm{ell}}
\end{aligned}$$

If $b' \in \underline{B}^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$, then $b'_{-\lambda} \in D^{\mathrm{unst}}_{\mathrm{ell}}(G,\zeta)$ and thus $\langle b, b'_{-\lambda}\rangle_{\mathrm{ell}} = 0$. If that $b' \in$

$\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta) \subseteq \Phi_2(G,\zeta)$, then $b'_{-\lambda} \in \Phi_2(G,\zeta)$ and $\langle b, b'_{-\lambda}\rangle_{\mathrm{ell}} = \|b\|_{\mathrm{ell}}\delta_b(b'_{-\lambda})$. Thus, we have $\widehat{f_{b,b'}}(\lambda) = \|b\|_{\mathrm{ell}}\delta_b(b'_{-\lambda})$.

Suppose $b = b'_{-\lambda}$. Then $\mu_b = (-\lambda)\cdot\mu_{b'}$. It follows that $[\zeta,\mu_b] = [\zeta,\mu_{b'}]$. Since $(\zeta,\mu_b),(\zeta,\mu_{b'}) \in \mathscr{P}^G$, it follows that $\mu_b = \mu_{b'}$. Consequently, $\lambda$ lies in the the stabiliser $\mathfrak{a}_{G,b}^\vee$ of $b$ in $\widetilde{\mathfrak{a}}_{G,F}^\vee$. Moreover, since $b, b' \in \underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G,\zeta,\mu_b)$ and $b = b'_{-\lambda}$, we have $b = b'$. It follows that $\delta_b(b'_{-\lambda}) = \delta_b(b')\mathbb{1}_{\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee}(\lambda)$.

Thus, we have shown that

$$\widehat{f_{b,b'}} = \|b\|_{\mathrm{ell}}\delta_b(b')\mathbb{1}_{\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee}.$$

When $b' \neq b$, we obtain $f_{b,b'} = 0$ as claimed. It remains to be shown that

$$f_{b,b} = \|b\|_{\mathrm{ell}}|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|^{-1}\mathbb{1}_{\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}}.$$

We have

$$\widehat{f_{b,b}} = \|b\|_{\mathrm{ell}}\mathbb{1}_{\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee}.$$

Thus, we must show that the Fourier transform of $\mathbb{1}_{\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}}$ on $\mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ is $|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|\cdot \mathbb{1}_{\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee}$. This follows from the general formula for the Fourier transform of a characteristic function of a subgroup of a finite abelian group. Let $A$ be a finite abelian group and let $B$ be a subgroup of $A$. Then for $\chi \in \widehat{B}$, we have

$$\begin{aligned}\widehat{\mathbb{1}_B}(\chi) &= \sum_{a\in A}\chi(a)^{-1}\mathbb{1}_B(a)\\ &= \sum_{b\in B}\chi(b)^{-1}\\ &= |B|\cdot\mathbb{1}_{A^\perp}(\chi),\end{aligned}$$

where $A^\perp = \{\chi \in \widehat{A} : \chi|_B = 1\}$. If we take $A = \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}$ and $B = \mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}$, then $A^\perp = \mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee$, and we obtain that and we obtain the desired result. □

For $b \in \underline{B}_{\mathrm{ell}}(G)$, define $\widehat{\mathscr{F}}_{\mathrm{ell}}(G)_b$ to be the subspace of all $\varphi = (\varphi_{b'})_{b'\in\underline{B}_{\mathrm{ell}}(G)}$ in $\widehat{\mathscr{F}}_{\mathrm{ell}}(G)$ with $\varphi_{b'} = 0$ for $b' \neq b$.

**Proposition 4.6.12.** *Let $b \in \underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G)$ and let $\varphi = (\varphi_{b'})_{b'\in\underline{B}_{\mathrm{ell}}(G)} \in \widehat{\mathscr{F}}_{\mathrm{ell}}(G)_b$. Regarding $\varphi_b \in \widehat{\mathscr{F}}(i\mathfrak{a}_G^*/\mathfrak{a}_{G,b}^\vee)$ as an element of $\widehat{\mathscr{F}}(i\mathfrak{a}_G^*/\mathfrak{a}_{G,F}^\vee)$, let $\phi_b \in \mathscr{F}(\mathfrak{a}_{G,F})$ be its Fourier transform. The inverse invariant Fourier transform of $\varphi$ is*

$$\mathrm{vol}(A_G(F)^1)^{-1}\|b\|_{\mathrm{ell}}^{-1}\frac{|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|}{|\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee|}\phi_b(H_G(\gamma))f[b]_G(\gamma)$$

*and this lies in* $\mathcal{I}\mathscr{F}^{\mathrm{st}}_{\mathrm{cusp}}(G)$*.*

*Proof.* Let $\zeta = \zeta_b$ be the $A_G(F)$-character of $b$. Choose a function $f[b] \in C^\infty_{c,\mathrm{cusp}}(G,\zeta)$ representing the pseudocoefficient $f[b]_G \in \mathcal{I}C^\infty_{c,\mathrm{cusp}}(G,\zeta)$ of $b$. Define $f : G(F) \to \mathbb{C}$ by

$$f(x) = \phi_b(H_G(x)) f[b](x).$$

By Lemma 4.6.10, we have that $f \in \mathscr{F}(G)$. We have $f_G(\gamma) = \phi_b(H_G(\gamma)) f[b]_G(\gamma)$, which does not depend on the choice of $f[b]$ and lies in $\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G)$. By Proposition 4.6.4, we have $f[b]_G \in \mathcal{I}C^{\infty,\mathrm{st}}_{c,\mathrm{cusp}}(G,\zeta)$. That is, $f[b]_G(\gamma)$ is constant on strongly regular stable classes. Since $H_G$ is constant on elements of $G(F)$ that lie in the same $G(\overline{F})$-conjugacy class, we have that $f_G$ is constant on strongly regular stable classes, i.e. $f_G \in \mathcal{I}\mathscr{F}^{\mathrm{st}}_{\mathrm{cusp}}(G)$.

It remains for us to show that for all $b' \in \underline{B}_{\mathrm{ell}}(G)$ and $\lambda \in i\mathfrak{a}^*_G$ we have

$$f_G(b'_\lambda) = \mathrm{vol}(A_G(F)^1)\|b\|_{\mathrm{ell}} \frac{|\mathfrak{a}^\vee_{G,b}/\mathfrak{a}^\vee_{G,F}|}{|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|} \varphi_b(\lambda)\delta_b(b').$$

Let $\zeta' = \zeta_{b'}$ be the $A_G(F)$-character of $b'$. We have

$$\begin{aligned} f_G(b'_\lambda) &= \int_{G(F)} e^{\langle \lambda, H_G(x)\rangle} \Theta_{b'}(x)\phi_b(H_G(x)) f[b](x)\, dx \\ &= \int_{G(F)/A_G(F)} \int_{A_G(F)} e^{\langle \lambda, H_G(xa)\rangle} \Theta_{b'}(xa)\phi_b(H_G(xa)) f[b](xa)\, da\, dx \\ &= \int_{G(F)/A_G(F)} e^{\langle \lambda, H_G(x)\rangle} \Theta_{b'}(x) f[b](x) \\ &\quad \int_{A_G(F)} e^{\langle \lambda, H_G(a)\rangle} \zeta'(a)\zeta(a)^{-1}\phi_b(H_G(xa))\, da\, dx. \end{aligned}$$

The inner integral in the third equality is equal to

$$\begin{aligned} &\int_{A_G(F)/A_G(F)^1} \int_{A_G(F)^1} e^{\langle \lambda, H_G(aa^1)\rangle} \zeta'(aa^1)\zeta(aa^1)^{-1}\phi_b(H_G(xaa^1))\, da^1\, da \\ &= \int_{A_G(F)/A_G(F)^1} e^{\langle \lambda, H_G(a)\rangle} \zeta'(a)\zeta(a)^{-1}\phi_b(H_G(xa)) \int_{A_G(F)^1} \zeta'(a^1)\zeta(a^1)^{-1}\, da^1\, da. \end{aligned}$$

Now,

$$\int_{A_G(F)^1} \zeta'(a^1)\zeta(a^1)^{-1}\, da^1 = \mathrm{vol}(A_G(F)^1)\delta_{\zeta|_{A_G(F)^1}}(\zeta'|_{A_G(F)^1}),$$

which is equal to 1 if $\zeta, \zeta'$ lie in the same $i\mathfrak{a}^*_G$-orbit, and 0 otherwise. Since $\zeta, \zeta'$ lie in our set $\mathscr{X}^G$ of representatives of $i\mathfrak{a}_G^*$-orbits, we obtain

$$\int_{A_G(F)^1} \zeta'(a^1)\zeta(a^1)^{-1}\,da^1 = \mathrm{vol}(A_G(F)^1)\delta_\zeta(\zeta')$$

Therefore we have that $f_G(b'_\lambda) = 0$ if $\zeta' \neq \zeta$. Thus, we may assume that that $\zeta' = \zeta$. Then $\mathrm{vol}(A_G(F)^1)^{-1} f_G(b'_\lambda)$ is equal to

$$\begin{aligned}
&\int_{G(F)/A_G(F)} \Theta_{b'}(x) f[b](x) \\
&\int_{A_G(F)/A_G(F)^1} e^{\langle \lambda, H_G(x)+H_G(a)\rangle}\phi_b(H_G(x)+H_G(a))\,da\,dx \\
=&\int_{G(F)/A_G(F)} \Theta_{b'}(x) f[b](x) \int_{\widetilde{\mathfrak{a}}_{G,F}} e^{\langle \lambda, H_G(x)+\widetilde{X}\rangle}\phi_b(H_G(x)+\widetilde{X})\,d\widetilde{X}\,dx \\
=&\sum_{X\in \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})} \Theta_{b'}(x) f[b](x) \int_{\widetilde{\mathfrak{a}}_{G,F}} e^{\langle \lambda, H_G(x)+\widetilde{X}\rangle}\phi_b(H_G(x)+\widetilde{X})\,d\widetilde{X}\,dx \\
=&\sum_{X\in \mathfrak{a}_{G,F}/\widetilde{\mathfrak{a}}_{G,F}} \int_{H_G^{-1}(X+\widetilde{\mathfrak{a}}_{G,F})} \Theta_{b'}(x) f[b](x) \int_{\widetilde{\mathfrak{a}}_{G,F}} e^{\langle \lambda, X+\widetilde{X}\rangle}\phi_b(X+\widetilde{X})\,d\widetilde{X}\,dx
\end{aligned}$$

Consequently, by Proposition 4.6.11, we have

$$\begin{aligned}
f_G(b'_\lambda) =& \mathrm{vol}(A_G(F)^1)\|b\|_{\mathrm{ell}}|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|^{-1}\delta_b(b') \\
&\sum_{X\in \mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}} \int_{\widetilde{\mathfrak{a}}_{G,F}} e^{\langle \lambda, X+\widetilde{X}\rangle}\phi_b(X+\widetilde{X})\,d\widetilde{X} \\
&= \mathrm{vol}(A_G(F)^1)\|b\|_{\mathrm{ell}}|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|^{-1}\delta_b(b') \int_{\mathfrak{a}_{G,b}} e^{\langle \lambda, X\rangle}\phi_b(X)\,dX.
\end{aligned}$$

For $X \in \mathfrak{a}_{G,F}$, we have

$$\begin{aligned}
\phi_b(X) &= \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,F}^\vee} e^{-\langle \lambda, X\rangle}\varphi_b(\lambda)\,d\lambda \\
&= \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,b}^\vee} \sum_{\widetilde{\lambda}\in \mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee} e^{-\langle \lambda+\widetilde{\lambda}, X\rangle}\varphi_b(\lambda+\widetilde{\lambda})\,d\lambda \\
&= \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,b}^\vee} e^{-\langle \lambda, X\rangle}\varphi_b(\lambda) \sum_{\widetilde{\lambda}\in \mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee} e^{-\langle \widetilde{\lambda}, X\rangle}\,d\lambda \\
&= \int_{i\mathfrak{a}_G^*/\mathfrak{a}_{G,b}^\vee} e^{-\langle \lambda, X\rangle}\varphi_b(\lambda)|\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee|\mathbb{1}_{\mathfrak{a}_{G,b}}(X)\,d\lambda \\
&= |\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee|\mathbb{1}_{\mathfrak{a}_{G,b}}(X)\widehat{\varphi}_b(X).
\end{aligned}$$

Therefore

$$\int_{\mathfrak{a}_{G,b}} e^{\langle \lambda, X\rangle} \phi_b(X)\,dX = |\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee| \int_{\mathfrak{a}_{G,b}} e^{\langle \lambda, X\rangle} \widehat{\varphi}_b(X)\,dX = |\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee| \varphi_b(\lambda).$$

Thus, we have

$$f_G(b'_\lambda) = \mathrm{vol}(A_G(F)^1)\|b\|_{\mathrm{ell}} \frac{|\mathfrak{a}_{G,b}^\vee/\mathfrak{a}_{G,F}^\vee|}{|\mathfrak{a}_{G,b}/\widetilde{\mathfrak{a}}_{G,F}|} \varphi_b(\lambda)\delta_b(b')$$

as required. □

Recall that we denote the invariant Fourier transform by $\mathcal{F}$. Stable spectral density for $\mathscr{F}_{\mathrm{cusp}}(G)$ is equivalent to the assertion that $\mathcal{F}^{-1}(\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{unst}}(G)) \subseteq \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{unst}}(G)$. We have stable spectral density for $\mathcal{C}_{c,\mathrm{cusp}}(G)$.

**Lemma 4.6.13** (Stable spectral density for $\mathcal{C}_{\mathrm{cusp}}(G)$)**.** *We have $\mathcal{F}^{-1}(\mathscr{S}_{\mathrm{ell}}^{\mathrm{unst}}(G)) \subseteq \mathcal{I}_{\mathrm{cusp}}^{\mathrm{unst}}(G)$.*

*Proof.* By stable spectral density for $\mathcal{C}_c^\infty(G)$, we have $\mathcal{F}^{-1}(PW_{\mathrm{ell}}^{\mathrm{unst}}(G)) \subseteq \mathcal{I}_{c,\mathrm{cusp}}^{\mathrm{unst}}(G)$. Since $PW_{\mathrm{ell}}^{\mathrm{unst}}(G)$ is dense in $\mathscr{S}_{\mathrm{ell}}^{\mathrm{unst}}(G)$ and $\mathcal{I}_{\mathrm{cusp}}^{\mathrm{unst}}(G)$ is closed in $\mathcal{I}_{\mathrm{cusp}}(G)$, it follows that $\mathcal{F}^{-1}(\mathscr{S}_{\mathrm{ell}}^{\mathrm{unst}}(G)) \subseteq \mathcal{I}_{\mathrm{cusp}}^{\mathrm{unst}}(G)$. □

We now prove Proposition 4.6.8.

*Proof.* Let $\varphi \in \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G)$. Write $\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G)$ as a countable increasing union of finite sets $\underline{B}_{\mathrm{ell}}^{\mathrm{st}}(G) = \bigcup_{i=1}^\infty \underline{B}_{\mathrm{ell},i}^{\mathrm{st}}(G)$. For each $i$, define $\varphi_i = (\varphi_{i,b})_{b\in \underline{B}_{\mathrm{ell},i}(G)} \in \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G)$ by $\varphi_{i,b} = \varphi_b$ if $b \in \underline{B}_{\mathrm{ell},i}(G)$ and $\varphi_{i,b} = 0$ otherwise. Then $\lim_i \varphi_i = \varphi$. By Proposition 4.6.12, we have $\mathcal{F}^{-1}(\varphi_i) \in \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{st}}(G)$. Since $\mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{st}}(G)$ is closed in $\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G)$, it follows that $\mathcal{F}(\varphi) \in \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{st}}(G)$. Therefore $\mathcal{F}^{-1}(\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G)) \subseteq \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{st}}(G)$.

Since $\widehat{\mathscr{F}}_{\mathrm{ell}}(G) = \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G) \oplus \widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{unst}}(G)$, we have

$$\begin{aligned}\mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G) &= \mathcal{F}^{-1}(\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{st}}(G)) \oplus \mathcal{F}^{-1}(\widehat{\mathscr{F}}_{\mathrm{ell}}^{\mathrm{unst}}(G)) \\ &\subseteq \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{st}}(G) \oplus \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{\mathrm{unst}}(G) \\ &\subseteq \mathcal{I}\mathscr{F}_{\mathrm{cusp}}(G).\end{aligned}$$

Therefore $\mathcal{F}^{-1}(\widehat{\mathscr{F}}_{\mathrm{ell}}^{?}(G)) = \mathcal{I}\mathscr{F}_{\mathrm{cusp}}^{?}(G)$ as required. □

### $K$-finite functions

If we trace through the above replacing $\mathcal{I}\mathscr{F}$ with $\mathcal{I}_f$, $\mathcal{S}\mathscr{F}$ with $\mathcal{S}_f$, and $\widehat{\mathscr{F}}$ with $PW_f$, we obtain a proof of the following.

**Proposition 4.6.14.** *The invariant Fourier transform*

$$\mathcal{I}_{f,\mathrm{cusp}}(G) \xrightarrow{\sim} PW_{\mathrm{ell},f}(G)$$

*restricts to isomorphisms*

$$\mathcal{I}^{?}_{f,\mathrm{cusp}}(G) \xrightarrow{\sim} PW^{?}_{\mathrm{ell},f}(G).$$

*Consequently, we have*

$$\mathcal{I}_{f,\mathrm{cusp}}(G) = \mathcal{I}^{\mathrm{st}}_{f,\mathrm{cusp}}(G) \oplus \mathcal{I}^{\mathrm{unst}}_{f,\mathrm{cusp}}(G).$$

**Corollary 4.6.15.** *The stable Fourier transform gives an isomorphism*

$$\mathcal{S}_{f,\mathrm{cusp}}(G) \xrightarrow{\sim} PW^{\mathrm{st}}_{\mathrm{ell},f}(G).$$

### 4.6.6 Proof of the main stable Paley–Wiener theorems

We apply the constructions and results of the preceding subsection to each semistandard Levi $L \in \mathcal{L}^G(M_0)$. We may therefore define

$$\widehat{\mathscr{F}}^{\mathrm{unst}}(G) = \Bigg(\bigoplus_{L\in\mathcal{L}^G(M_0)} \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(L)\Bigg)^{W_0^G} = \bigoplus_{L\in\mathcal{L}^G(M_0)/W_0^G} \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(L)^{W^G(L)}.$$

We have decompositions

$$\widehat{\mathscr{F}}(G) = \Bigg(\bigoplus_{L\in\mathcal{L}^G(M_0)} \widehat{\mathscr{F}}_{\mathrm{ell}}(G)\Bigg)^{W_0^G} = \bigoplus_{L\in\mathcal{L}^G(M_0)/W_0^G} \widehat{\mathscr{F}}_{\mathrm{ell}}(L)^{W^G(L)}$$

and $\widehat{\mathscr{F}}_{\mathrm{ell}}(L) = \widehat{\mathscr{F}}^{\mathrm{st}}_{\mathrm{ell}}(L) \oplus \widehat{\mathscr{F}}^{\mathrm{unst}}_{\mathrm{ell}}(L)$ for each $L \in \mathcal{L}^G(M_0)$. Thus, we obtain a decomposition $\widehat{\mathscr{F}}(G) = \widehat{\mathscr{F}}^{\mathrm{st}}(G) \oplus \widehat{\mathscr{F}}^{\mathrm{unst}}(G)$. By Corollary 3.3.2 the inclusion $PW^{\mathrm{unst}}(G) \to \mathscr{S}^{\mathrm{unst}}(G)$ is continuous with dense image.

We can now deduce the full stable spectral density theorem for $\mathcal{C}(G)$ from the stable spectral density theorem for $C_c^\infty(G)$.

**Theorem 4.6.16** (Stable spectral density)**.** *Let $f \in \mathcal{C}(G)$. If $f^G(\phi) = 0$ for all $\phi \in \Phi_{\mathrm{temp}}(G)$, then $f^G(\delta) = 0$ for all $\delta \in \Delta_{\mathrm{sr}}(G)$. Equivalently, $\mathcal{F}^{-1}(\mathscr{S}^{\mathrm{unst}}(G)) \subseteq \mathcal{IC}^{\mathrm{unst}}(G)$.*

*Proof.* By stable spectral density for $C_c^\infty(G)$, we have $\mathcal{F}^{-1}(PW^{\text{unst}}(G)) \subseteq \mathcal{I}C_c^{\infty,\text{unst}}(G)$. Since $PW^{\text{unst}}(G)$ is dense in $\mathscr{S}^{\text{unst}}(G)$ and $\mathcal{IC}^{\text{unst}}(G)$ is closed in $\mathcal{IC}(G)$, it follows that $\mathcal{F}^{-1}(\mathscr{S}^{\text{unst}}(G)) \subseteq \mathcal{IC}^{\text{unst}}(G)$. □

Note that $\mathcal{F}(\mathcal{I}\mathscr{F}^{\text{unst}}(G)) \subseteq \widehat{\mathscr{F}}^{\text{unst}}(G)$ follows immediately from the definitions. Thus, we have $\mathcal{F}^{-1}(\widehat{\mathscr{F}}^{\text{unst}}(G)) = \mathcal{I}\mathscr{F}^{\text{unst}}(G)$.

We now prove the main stable Paley–Wiener theorems.

*Proof.* The stable Fourier transform $\mathcal{F}^{\text{st}} : \mathcal{I}\mathscr{F}(G) \to \widehat{\mathscr{F}}^{\text{st}}(G)$ is the composition of the invariant Fourier transform $\mathcal{F}$ and the projection onto $\widehat{\mathscr{F}}^{\text{st}}(G)$. The kernel is $\mathcal{I}\mathscr{F}^{\text{unst}}(G)$, so it descends to a continuous bijection $\mathcal{F}^{\text{st}} : \mathcal{S}\mathscr{F}(G) \to \widehat{\mathscr{F}}^{\text{st}}(G)$. Since its inverse is the composition of continuous maps

$$\widehat{\mathscr{F}}^{\text{st}}(G) \hookrightarrow \widehat{\mathscr{F}}(G) \xrightarrow{\mathcal{F}^{-1}} \mathcal{I}\mathscr{F}(G) \twoheadrightarrow \mathcal{S}\mathscr{F}(G)$$

the stable Fourier transform $\mathcal{F}^{\text{st}} : \mathcal{S}\mathscr{F}(G) \to \widehat{\mathscr{F}}^{\text{st}}(G)$ is a topological isomorphism.

In the same way, one can prove that the stable Fourier transform restricts to an isomorphism of topological vector spaces $\mathcal{F}^{\text{st}} : \mathcal{S}_f(G) \to PW_f^{\text{st}}(G)$. □

Note that the stable Paley–Wiener theorems for $\mathcal{S}_{f,\text{cusp}}(G,\zeta)$ and $\mathcal{S}\mathscr{F}_{\text{cusp}}(G)$ given above can be proved in a similar way. The proof given above gives more information, namely that the subspace $\mathcal{I}^{\text{st}}_{f,\text{cusp}}(G,\zeta)$ (resp. $\mathcal{I}\mathscr{F}^{\text{st}}_{\text{cusp}}(G)$) corresponds to $\widehat{\mathscr{F}}^{\text{st}}_{\text{ell},f}(G,\zeta)$ (resp. $\widehat{\mathscr{F}}^{\text{st}}_{\text{ell}}(G)$) under the invariant Fourier transform.

Using the stable Paley–Wiener Theorem for $\mathcal{S}(G)$, a Fourier inversion formula for elements of $\mathcal{S}(G)$ can be obtained from Arthur's Fourier inversion formula for elements of $\mathcal{I}(G)$. The argument is the same as the one used by Arthur to prove [Art96, p. 6.3].

# 5 Stable Transfer

In this chapter we prove that stable transfer operators exist and preserve $K$-finiteness. We also compute some examples of stable transfer operators.

## 5.1 The theorem

Let $F$ be a local field of characteristic zero, let $H$ and $G$ be connected reductive groups over $F$, and let $\xi : {}^LH \to {}^LG$ be an equivalence class of $L$-homomorphism. Assume that $G$ is quasisplit so that $G$-relevance is automatic. Then we have a map

$$\xi_* : \Phi(H) \longrightarrow \Phi(G)$$

defined by $\xi_*(\phi) = \xi \circ \phi$. This restricts to a map

$$\xi_* : \Phi_{\text{temp}}(H) \longrightarrow \Phi_{\text{temp}}(G)$$

if and only if $\xi$ is tempered. Suppose that $\xi$ is injective and tempered. In [Lan13, Question A & B], Langlands asked whether for each $f^G \in \mathcal{S}_c(G)$ there exists $f^H \in \mathcal{S}_c(H)$ such that

$$f^H(\phi) = f^G(\xi_*(\phi))$$

for all $\phi \in \Phi_{\text{temp}}(H)$. Note that if $f^H$ exists it is uniquely determined by stable spectral density for test functions. More generally, one can ask whether for each $f^G \in \mathcal{S}(G)$ there exists $f^H \in \mathcal{S}(H)$ with the above property, which would also be uniquely determined by spectral density. The function $f^H$ is called the stable transfer of $f^G$ along $\xi$ and we denote it by $\mathcal{T}_\xi f^G$. It has also been called the functorial transfer of $f^G$, or to avoid with endoscopic transfer the stable-stable transfer of $f^G$.

We will prove the following.

**Theorem 5.1.1.** *Let $\xi : {}^LH \to {}^LG$ be an equivalence class of injective tempered $L$-homomorphisms. Then pullback along $\xi_*$ gives a well-defined continuous linear* *map*

$$\xi^* : \mathscr{S}^{\mathrm{st}}(G) \longrightarrow \mathscr{S}^{\mathrm{st}}(H).$$

*Furthermore, it restricts to continuous linear maps*

$$\xi^* : PW^{\mathrm{st}}(G) \longrightarrow PW^{\mathrm{st}}(H)$$

*and*

$$\xi^* : PW_f^{\mathrm{st}}(G) \longrightarrow PW_f^{\mathrm{st}}(H).$$

It follows that stable transfer along $\xi$ can be constructed by taking the stable Fourier transform, pulling back along $\xi_*$, and then taking the inverse stable Fourier transform:

$$\begin{array}{ccc} \mathcal{S}(G) & \xrightarrow{\mathcal{T}_\xi} & \mathcal{S}(H) \\ \big\downarrow{\scriptstyle \mathcal{F}^{\mathrm{st}}} & & \big\downarrow{\scriptstyle \mathcal{F}^{\mathrm{st}}} \\ \mathscr{S}(G) & \xrightarrow{\xi^*} & \mathscr{S}(H) \end{array}$$

As a corollary, we obtain the following, which answers [Lan13, Question A & B] in the affirmative.

**Corollary 5.1.2.** *Let $\xi : {}^LH \to {}^LG$ be an equivalence class of injective tempered L-homomorphisms. For each $f^G \in \mathscr{S}(G)$, the stable transfer $\mathcal{T}_\xi f^G \in \mathcal{S}(H)$ of $f^G$ along $\xi$ exists. Moreover,*

$$\mathcal{T}_\xi : \mathscr{S}(G) \longrightarrow \mathscr{S}(H)$$

*is a continuous linear map, which restricts to continuous linear maps*

$$\mathcal{T}_\xi : \mathscr{S}_c(G) \longrightarrow \mathscr{S}_c(H)$$

*and*

$$\mathcal{T}_\xi : \mathscr{S}_f(G) \longrightarrow \mathscr{S}_f(H).$$

## 5.2 Proof of the theorem

Let $\xi : {}^LH \to {}^LG$ be an equivalence class of injective tempered $L$-homomorphisms. To prove the first claim in Theorem 5.1.1 and thus that stable transfer along $\xi$ exists, we must show that pullback along $\xi_* : \Phi_{\mathrm{temp}}(H) \to \Phi_{\mathrm{temp}}(G)$ gives a well-defined continuous linear map

$$\xi^* : \mathscr{S}^{\mathrm{st}}(G) \longrightarrow \mathscr{S}^{\mathrm{st}}(H).$$

Recall that we have decompositions

$$\Phi_{\mathrm{temp}}(G) = \coprod_{M_G} \Phi_2(M_G)/W^G(M_G) \quad , \quad \Phi_{\mathrm{temp}}(H) = \coprod_{M_H} \Phi_2(M_H)/W^G(M_H)$$

and corresponding decompositions

$$\mathscr{S}^{\mathrm{st}}(G) = \bigoplus_{M_G} \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_G)^{W^G(M_G)} \quad , \quad \mathscr{S}^{\mathrm{st}}(H) = \bigoplus_{M_H} \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_H)^{W^H(M_H)}.$$

Thus, it suffices to show that for each $M_H$ and $M_G$, pullback along the partially defined map

$$\xi_*^{M_H,M_G} : \Phi_2(M_H)/W^H(M_H) \to \Phi_2(M_G)/W^G(M_G)$$

and extension by zero gives a well-defined continuous linear map

$$\xi^*_{M_H,M_G} : \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_G)^{W^G(M_G)} \longrightarrow \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_H)^{W^H(M_H)}.$$

In turn, it suffices to show that pullback along the partially defined map $\xi_*^{M_H,M_G}$ : $\Phi_2(M_H) \to \Phi_2(M_G)$ and extension by zero gives a well-defined continuous linear map

$$\xi^*_{M_H,M_G} : \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_G) \longrightarrow \mathscr{S}^{\mathrm{st}}_{\mathrm{ell}}(M_H).$$

We will do show by showing that we can apply Lemma 3.3.3, and this will establish Theorem 5.1.1 in full. We will now turn to an examination of the partially defined map $\xi_*^{M_H,M_G} : \Phi_2(M_H) \to \Phi_2(M_G)$ in more detail.

Let $\mathcal{G}$ be a $\lambda$-group. The outer action $\Gamma_F \to \mathrm{Out}(\mathcal{G}^0)$ determines an action $\Gamma_F \to \mathrm{Aut}(Z(\mathcal{G}^0))$. We write $A_{\mathcal{G}} = Z(\mathcal{G}^0)^{\Gamma_F,\circ}$.

**Lemma 5.2.1.** *Let $\xi : {}^LH \to {}^LG$ be an equivalence class of elliptic $L$-homomorphisms. Then $\xi$ restricts to a homomorphism*

$$\xi : A_{{}^LH} \longrightarrow A_{{}^LG}.$$

*Proof.* Let $\mathcal{M} = C_{{}^LG}(\xi(A_{{}^LH})) = C_{{}^LG}(\xi(Z(H^\vee)^{\Gamma_F,\circ}))$. Since $Z(H^\vee)^{\Gamma_F} = C_{H^\vee}({}^LH)$, we have $\xi({}^LH) \subseteq \mathcal{M}$. Therefore $\mathcal{M}$ maps onto $W_F$. Since $\xi(A_{{}^LH})$ is a torus in $G^\vee$, we have that $\mathcal{M}$ is Levi subgroup of ${}^LG$. Since $\xi$ is elliptic, we have $\mathcal{M} = {}^LG$, or in other words $\xi(A_{{}^LH}) \subseteq (C_{G^\vee}({}^LG)) = Z(G^\vee)^{\Gamma_F}$. The claim follows. □

To simplify notation, for a Levi subgroup $M$ of $G$ we write the canonical equivalence class of $L$-embeddings $\iota_M^G : {}^LM \to {}^LG$ as an inclusion ${}^LM \hookrightarrow {}^LG$. Fix

$\phi \in \Phi_2(M_H)$ and suppose that $\xi_*(\phi) \in \Phi_2(M_G)$. That is, $\phi$ is in the domain of the partially defined map $\xi_*^{M_H,M_G} : \Phi_2(M_H) \to \Phi_2(M_G)$.

Let $M_{H,\xi}$ be the Levi subgroup of $G$ determined up to $G(F)$-conjugacy by the property that ${}^L M_{H,\xi}$ belongs to the $G^\vee$-conjugacy class Levi subgroups of ${}^L G$ that contains $\xi({}^L M_H)$ minimally. Then $\xi$ factors through ${}^L M_{H,\xi} \hookrightarrow {}^L G$ as an equivalence class of injective elliptic $L$-homomorphisms $\xi : {}^L M_H \to {}^L M_{H,\xi}$. The above lemma gives us an injection $\xi : A_{{}^L M_H} \to A_{{}^L M_{H,\xi}}$.

Recall that we have an identification $\mathrm{Lie}(A_{{}^L M_H}) = X_*(A_{{}^L M_H}) \otimes_{\mathbb{Z}} \mathbb{C}$. Under this identification, we have that the injection $\mathrm{Lie}(\xi) : \mathrm{Lie}(A_{{}^L M_H}) \to \mathrm{Lie}(A_{{}^L M_{H,\xi}})$ is the map $\xi_* \otimes \mathrm{id}$, where $\xi_* : X_*(A_{{}^L M_H}) \to X_*(A_{{}^L M_{H,\xi}})$. We thus write $\xi_* = \mathrm{Lie}(\xi)$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Lie}(A_{{}^L M_H}) & \xrightarrow{\xi_*} & \mathrm{Lie}(A_{{}^L M_{H,\xi}}) \\ \uparrow & & \uparrow \\ X_*(A_{{}^L MH}) & \xrightarrow{\xi_*} & X_*(A_{{}^L M_{H,\xi}}) \end{array}$$

Recall that we have a canonical isomorphism $X^*(M_H) \cong X_*(A_{{}^L M_H})$ and that we write its extension $\mathfrak{a}^*_{M_H,\mathbb{C}} \cong \mathrm{Lie}(A_{{}^L M_H})$ as $\lambda \mapsto \lambda^\vee$. Thus, we may view the above commutative diagram as the commutative diagram

$$\begin{array}{ccc} \mathfrak{a}^*_{M_H,\mathbb{C}} & \xrightarrow{\xi_*} & \mathfrak{a}^*_{M_{H,\xi},\mathbb{C}} \\ \uparrow & & \uparrow \\ X^*(M_H) & \xrightarrow{\xi_*} & X^*(M_{H,\xi}) \end{array}$$

Note that the injection $\xi_* : \mathfrak{a}^*_{M_H,\mathbb{C}} \to \mathfrak{a}^*_{M_{H,\xi},\mathbb{C}}$ restricts to injections $\xi_* : \mathfrak{a}^*_{M_H} \to \mathfrak{a}^*_{M_{H,\xi}}$ and $\xi_* : i\mathfrak{a}^*_{M_H} \to i\mathfrak{a}^*_{M_{H,\xi}}$.

Since $\xi_*(\phi) \in \Phi_2(M_G)$ it follows that ${}^L M_G \hookrightarrow {}^L G$ factors through ${}^L M_{H,\xi} \hookrightarrow {}^L G$. Thus, we may assume that $M_G \subseteq M_{H,\xi}$, and therefore that $M_G$ is a Levi subgroup of $M_{H,\xi}$. Then $\mathfrak{a}^*_{M_{H,\xi}} \subseteq \mathfrak{a}^*_{M_G}$ and we have $\xi_* : i\mathfrak{a}^*_{M_H} \to i\mathfrak{a}^*_{M_G}$. Now, for all $\lambda \in \mathfrak{a}^*_{M_H,\mathbb{C}}$, we have

$$\xi_*(\phi_\lambda) = \xi_*(\phi)_{\xi_*(\lambda)}.$$

This explains how $\xi_*$ behaves on the component of $\phi$.

Suppose that $F$ is archimedean. We may assume that $F = \mathbb{R}$ by restriction of scalars. We must show that $\|\xi_*\phi\|_{M_G} \gg \|\phi\|_{M_H}$ for all $\phi$ in the domain of the partially defined map $\xi_*^{M_H,M_G} : \Phi_2(M_H) \to \Phi_2(M_G)$. Since $\|\cdot\|_{M_G} \asymp \|\cdot\|_G$ and $\|\cdot\|_{M_H} \asymp \|\cdot\|_H$, it suffices to show that $\|\xi_*\phi\|_G \gg \|\phi\|_H$ for $\phi \in \Phi_{\mathrm{temp}}(H)$.

Let $\mathcal{T}_H^0$ and $\mathcal{T}_G^0$ be $\Gamma_{\mathbb{R}}$-stable maximal tori of $H^\vee$ and $G^\vee$, respectively. Choose a representative $L$-homomorphism $\xi$ such that $\xi_0(\mathcal{T}_H^0) \subseteq \mathcal{T}_G^0$. Note that $a_\xi(\mathbb{C}^\times) \subseteq C_{G^\vee}(\xi_0(\mathcal{T}_H^0))$. Since $\xi_0(\mathcal{T}_H^0) \subseteq \mathcal{T}_G^0$, we have $\mathcal{T}_G^0 \subseteq C_{G^\vee}(\xi_0(\mathcal{T}_H^0))$. Therefore $\mathcal{T}_G^0$ is a maximal torus of $C_{G^\vee}(\xi_0(\mathcal{T}_H^0))$. By replacing $\xi$ by a $C_{G^\vee}(\xi_0(\mathcal{T}_H^0))$-conjugate, we may assume that both $\xi(\mathcal{T}_H^0)$ and $a_\xi(\mathbb{C}^\times)$ are contained in $\mathcal{T}_G^0$. Recall how the infinitesimal character attached to an $L$-parameter is defined. We may identify $\mu_\phi$ with an element of $\mathrm{Lie}(\mathcal{T}_H^0)/W(H^\vee, \mathcal{T}_H)$. Let $\mu_\xi = \mu_{\xi|_{W_{\mathbb{R}}}}$ denote the infinitesimal character attached to the $L$-parameter $\xi|_{W_{\mathbb{R}}}$. We may identify $\mu_{\xi_*\phi}$ and $\mu_\xi$ with elements of $\mathrm{Lie}(\mathcal{T}_G^0)/W(G^\vee, \mathcal{T}_G)$. We have $\mu_{\xi_*\phi} = \mathrm{Lie}(\xi)\mu_\phi + \mu_\xi$. Fix a $W(H^\vee, \mathcal{T}_H)$-invariant inner product on $\mathrm{Lie}(\mathcal{T}_H^0)$ and a $W(G^\vee, \mathcal{T}_G)$-invariant inner product on $\mathrm{Lie}(\mathcal{T}_G^0)$, and denote the associated norms by $\|\cdot\|_{H^\vee}$ and $\|\cdot\|_{G^\vee}$, respectively. We have $\|\phi\|_H \asymp \|\mu_\phi\|_{H^\vee}$ and $\|\xi_*\phi\|_G \asymp \|\mu_{\xi_*\phi}\|_{G^\vee}$, so it suffices to prove that $\|\mu_{\xi_*\phi}\|_{G^\vee} \gg \|\mu_\phi\|_{H^\vee}$. We have

$$\|\mu_{\xi_*\phi}\|_{G^\vee} = \|\mathrm{Lie}(\xi)\mu_\phi + \mu_\xi\|_{G^\vee} \asymp \|\mathrm{Lie}(\xi)\mu_\phi\|_{G^\vee}.$$

Since $\mathrm{Lie}(\xi) : \mathrm{Lie}(\mathcal{T}_H^0) \to \mathrm{Lie}(\mathcal{T}_G^0)$ is injective, we have that $\|\mathrm{Lie}(\xi)(\cdot)\|_{G^\vee}$ is a norm on $\mathrm{Lie}(\mathcal{T}_H^0)$, and is thus equivalent to $\|\cdot\|_{H^\vee}$. Thus,

$$\|\mu_{\xi_*\phi}\|_{G^\vee} \asymp \|\mu_\phi\|_{H^\vee}$$

as required.

In order to show that we can apply Lemma 3.3.3, and thus conclude the proof of Theorem 5.1.1, all that remains is for us to show that the partially defined map $\xi_*^{M_H,M)G} : \Phi_2(M_H) \to \Phi_2(M_G)$ maps at most finitely many connected components to a given connected component. The next section is devoted to this.

### 5.2.1 A finiteness property of functoriality

In this section we prove the following finiteness property of functoriality.

**Theorem 5.2.2.** *Let $H$ and $G$ be connected reductive groups over $F$ with $G$ quasisplit. If $\xi : {}^LH \to {}^LG$ is an injective $L$-homomorphism, then the preimage of a connected component of $\Phi(G)$ under $\xi_* : \Phi(H) \to \Phi(G)$ is a finite union of connected components of $\Phi(H)$, or in other words $\pi_0(\xi_*) : \pi_0(\Phi(H)) \to \pi_0(\Phi(G))$ has finite fibres.*

Since the restriction of $\xi_*$ to a connected component of $\Phi(H)$ is injective in the archimedean case and a quotient by a finite group action in the non-archimedean case, we obtain the following corollary.

**Corollary 5.2.3.** *The map $\xi_* : \Phi(H) \to \Phi(G)$ has finite fibres.*

We also obtain the following corollary in the tempered case, which gives what we need to conclude the proof of Theorem 5.1.1.

**Corollary 5.2.4.** *If $\xi : {}^LH \to {}^LG$ be a tempered injective $L$-homomorphism, then the preimage of a connected component of $\Phi_{\mathrm{temp}}(G)$ under $\xi_* : \Phi_{\mathrm{temp}}(H) \to \Phi_{\mathrm{temp}}(G)$ is a finite union of connected components of $\Phi_{\mathrm{temp}}(H)$, i.e. $\pi_0(\xi_*) : \pi_0(\Phi_{\mathrm{temp}}(H)) \to \pi_0(\Phi_{\mathrm{temp}}(G))$ has finite fibres.*

We now begin the proof of Theorem 5.2.2. We write write $L_F^1$ for the preimage of $W_F^1$ in $L_F$. If $F$ is non-archimedean, we have $L_F^1 = \mathrm{SL}_2 \times W_F^1 = \mathrm{SL}_2 \times I_F$. If $F$ archimedean, we have $L_F = W_F$, and thus $L_F^1 = W_F^1 = S^1 \cup S^1 j \subseteq \mathbb{H}^\times$. We also write ${}^LG^1 = G^\vee \rtimes W_F^1$.

For $\phi \in \Phi(G)$, we refer to $\phi|_{L_F^1}$ as its inertial $L$-parameter. (See [Lat] for a discussion of inertial $L$-parameters for $p$-adic groups.) For $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$, its associated cohomology class $a_\lambda \in H^1_c(W_F, Z(G^\vee))$ is trivial on $W_F^1$. Consequently, for $\phi \in \Phi(G)$ and $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$ we have $\phi_\lambda|_{L_F^1} = \phi|_{L_F^1}$, that is, $\phi_\lambda$ and $\phi$ have the same inertial $L$-parameter. Thus, the restriction map

$$\Phi(G) \longrightarrow \mathrm{Hom}_{c,W_F^1}(L_F^1, {}^LG^1)/G^\vee$$

is constant on connected components. Here, we have written $\mathrm{Hom}_{c,W_F^1}(L_F^1, {}^LG^1)$ to denote the set of continuous homomorphisms $L_F^1 \to {}^LG^1$ over $W_F^1$. We will use similar notation below.

**Proposition 5.2.5.** *Let $G$ be a connected reductive group over $F$. The fibres of the restriction map*

$$\Phi(G) \longrightarrow \mathrm{Hom}_{c,W_F^1}(L_F^1, {}^LG^1)/G^\vee$$

*are finite unions of connected components. When $F$ is archimedean, the fibres are the connected components of $\Phi(G)$.*

When $F$ is non-archimedean, the fibres can indeed be unions of more than one connected component. Suppose that $G$ is anisotropic modulo its centre. Then $G(F)^1$ is the unique maximal compact subgroup of $G(F)$ and $G(F)_1$ is the unique Iwahori subgroup of $G(F)$ [Hai14, §3.3.1]. Suppose further that $G$ is a torus. Note 1 at the end of [Rap05] gives a description of $G(F)_1$ and $G(F)^1$. The discussion there initially concerns a complete discretely valued field $L$ with algebraically closed residue field, however at the end of Note 1, Rapoport explains the necessary changes when $L$ is replaced by a discretely valued field $F$ with perfect residue field. The group $G$ has an lft Néron model $\mathcal{G}$ over $\operatorname{Spec}\mathcal{O}_F$. Let $\mathcal{G}^1$ be the maximal subgroup scheme of finite type over $\operatorname{Spec}\mathcal{O}_F$. Let $\mathcal{G}^{1,\circ}$ be the identity component of $\mathcal{G}^1$. Then $\mathcal{G}^1(\mathcal{O}_F) = G(F)^1$, $\mathcal{G}^{1,\circ}(\mathcal{O}_F) = G(F)_1$, and $G(F)_1$ has finite index in $G(F)^1$. Identify $\Phi(G)$ with $\Pi(G)$ using the local Langlands correspondence for tori. The fibre in the proposition containing a the trivial character $1 : G(F) \to \mathbb{C}^\times$ contains the group of weakly unramified characters $X^{\mathrm{wnr}}(G) = \operatorname{Hom}_c(G(F)/G(F)_1, \mathbb{C}^\times)$, but the connected component of 1 is the group of unramified characters $X^{\mathrm{nr}}(G) = \operatorname{Hom}_c(G(F)/G(F)^1, \mathbb{C}^\times)$. If $G(F)_1 \subsetneq G(F)^1$, then $X^{\mathrm{wnr}}(G) \supsetneq X^{\mathrm{nr}}(G)$.

The following proof was inspired by the proof of [Var, Lemma 4.2.3].

*Proof.* The fibres are unions of connected components. We must show that the fibres contain only finitely many connected components. Fix $\phi \in \Phi(G)$ and let $\phi_0 : L_F^1 \to {}^LG$ denote the restriction of $\phi$. Let $\phi' \in \Phi(G)$ be in the fibre containing $\phi$. By replacing $\phi'$ by a $G^\vee(\mathbb{C})$-conjugate if necessary, we may assume that $\phi'$ restricts to $\phi_0$.

Assume that $F$ is non-archimedean. Then $L_F^1 = \mathrm{SL}_2 \times I_F$. Let

$$C_{\phi_0} = C_{G^\vee}(\phi_0(\mathrm{SL}_2 \times I_F)) = C_{G^\vee}(\phi_0(I_F)) \cap C_{G^\vee}(\phi_0(\mathrm{SL}_2)).$$

Both $C_{G^\vee}(\phi_0(I_F))$ and $C_{G^\vee}(\phi_0(\mathrm{SL}_2))$ are reductive groups by Lemma 10.2.2 and the proof of Lemma 10.1.1 in [Kot84]. Therefore $C_{\phi_0}$ is reductive.

Both $\phi(\mathrm{Fr})$ and $\phi'(\mathrm{Fr})$ normalise $\phi_0(\mathrm{SL}_2 \times I_F)$, and therefore also normalise $C_{\phi_0}$. Let $\theta, \theta' \in \operatorname{Aut}(C_{\phi_0})$ denote the automorphisms defined by $\operatorname{Int}(\phi(\mathrm{Fr}))$ and $\operatorname{Int}(\phi'(\mathrm{Fr}))$. Since $\phi(\mathrm{Fr})$ and $\phi'(\mathrm{Fr})$ are semisimple elements of ${}^LG$, the automorphisms $\theta, \theta'$ are semisimple. Therefore they each preserve a Borel pair of $C_{\phi_0}$ by [Ste68, Theorem 7.5]. By replacing $\phi, \phi'$ by $C_{\phi_0}$-conjugates if necessary, we may assume that $\theta, \theta'$ preserve the same Borel pair $(\mathcal{B}, \mathcal{T})$ of $C_{\phi_0}^\circ$.

Let $x = \phi'(\mathrm{Fr})^{-1}\phi(\mathrm{Fr}) \in G^\vee$. Since $\phi(\mathrm{Fr})$ and $\phi'(\mathrm{Fr})$ act on $\phi_0(\mathrm{SL}_2 \times I_F)$ in the same way, we have that $x$ acts trivially on $\phi_0(\mathrm{SL}_2 \times I_F)$, and thus $x \in C_{\phi_0}$. Moreover, $x \in N_{C_{\phi_0}}(\mathcal{B}, \mathcal{T})$. We have $N_{C_{\phi_0}}(\mathcal{B}, \mathcal{T})/\mathcal{T} \simeq C_{\phi_0}/C_{\phi_0}^\circ$ as explained in [DM94, §1]. Fix representatives $c_1, \dots, c_N$ of $N_{C_{\phi_0}}(\mathcal{B}, \mathcal{T})/\mathcal{T}$. For each $i = 1, \dots, N$, let $\phi_i : \mathrm{SL}_2 \times W_F \times \to {}^LG$ be the extension of $\phi_0$ that satisfies $\phi_i(\mathrm{Fr}) = c_i\phi(\mathrm{Fr})$. We will show that $\phi'$ lies in the component of $\phi_i$ for some $i = 1, \dots, N$.

Let $\mathcal{M} = C_{{}^LG}(\mathcal{T}^{\theta',\circ})$. Then $\mathcal{M}$ is a Levi subgroup of ${}^LG$ containing the image of $\phi'$. Let $\mathcal{M}' \subseteq \mathcal{M}$ be a minimal Levi of ${}^LG$ containing the image of $\phi'$. Then we have $\mathcal{T}^{\theta',\circ} \subseteq A_{\mathcal{M}'}$. Therefore, it suffices to show that there exists $t_0 \in \mathcal{T}^{\theta',\circ}$ and $t_1 \in \mathcal{T}$ such that $t_0 t_1 \phi'(\mathrm{Fr}) t_1^{-1} = c_i\phi(\mathrm{Fr})$ for some $i = 1, \dots, N$. Indeed, this says that the unramified twist of $\phi'$ by $t_1 \in A_{\mathcal{M}'} = H^1_c(W_F/I_F, A_{\mathcal{M}'})$ is equivalent to $\phi_i$, so that $\phi'$ lies in the connected component of $\phi_i$.

We may write $x = tc_i$ for some $t \in \mathcal{T}$ and some $i = 1, \dots, N$. Then $t^{-1}\phi'(\mathrm{Fr}) = c_i\phi(\mathrm{Fr})$. Let $(1-\theta')\mathcal{T} = \{t\theta'(t)^{-1} : t \in \mathcal{T}\}$. It follows from the Smith normal form that the homomorphism

$$\mathcal{T}^{\theta',\circ} \times (1-\theta')\mathcal{T} \longrightarrow \mathcal{T}$$

given by multiplication in $\mathcal{T}$ is surjective. Therefore, we may write $t^{-1} = t_0 t_1 \theta'(t_1)^{-1}$ for some $t_0 \in \mathcal{T}^{\theta',\circ}$ and $t_1 \in \mathcal{T}$. Then

$$\begin{aligned} c_i\phi(\mathrm{Fr}) &= t_0t_1\theta'(t_1)^{-1}\phi'(\mathrm{Fr}) \\ &= t_0t_1\phi'(\mathrm{Fr})t_1^{-1}\phi'(\mathrm{Fr})^{-1}\phi'(\mathrm{Fr}) \\ &= t_0t_1\phi'(\mathrm{Fr})t_1^{-1}. \end{aligned}$$

This concludes the proof in the non-archimedean case.

Now assume that $F$ is archimedean. By restriction of scalars, we may assume that $F = \mathbb{R}$. Recall that $L_\mathbb{R} = W_\mathbb{R} = W^1_\mathbb{R} \times \mathbb{R}_{>0}$. Since $\mathbb{R}_>0$ lies in the centre of $W_\mathbb{R}$ and acts trivially on $G^\vee$, we obtain that both $a_\phi(\mathbb{R}_{>0})$ and $a_{\phi'}(\mathbb{R}_{>0})$ centralise $\phi_0(W^1_\mathbb{R})$. Therefore $a_\phi(\mathbb{R}_{>0})$ and $a_{\phi'}(\mathbb{R}_{>0})$ are contained in $C_{\phi_0} \coloneqq C_{G^\vee}(\phi_0(W^1_\mathbb{R}))$. Since $a_\phi(\mathbb{R}_{>0})$ and $a_{\phi'}(\mathbb{R}_{>0})$ are connected abelian subgroups of $C_{\phi_0}$ consisting of semisimple elements, they are contained in maximal tori of $C_{\phi_0}$. Let $\mathcal{T}$ be a maximal torus of $C_{\phi_0}$. By replacing $\phi, \phi'$ by $C_{\phi_0}$-conjugates if necessary, we may assume that $a_\phi(\mathbb{R}_{>0})$ and $a_{\phi'}(\mathbb{R}_{>0})$ are both contained in $\mathcal{T}$. For each $w \in W_\mathbb{R}$, define $a(w) = \phi'(w)\phi(w)^{-1}$. For $w \in W^1_\mathbb{R}$ we have $a(w) = 1$, and for $r \in \mathbb{R}_{>0}$ we have $a(r) = a_{\phi'}(r)a_\phi(r)^{-1}$. Therefore $a \in \mathrm{Hom}_c(W_\mathbb{R}/W^1_\mathbb{R}, \mathcal{T})$.

Since $a_\phi(\mathbb{R}_{>0}) \subseteq \mathcal{T}$, it follows that that $\phi(\mathbb{R}_{>0})$ centralises $\mathcal{T}$. Since $\phi(W^1_\mathbb{R}) = \phi_0(W^1_\mathbb{R})$ also centralises $\mathcal{T}$, we have that $\phi(W_\mathbb{R})$ is contained in $\mathcal{M} \coloneqq C_{{}^LG}(\mathcal{T})$. Now, $\mathcal{M}$ is a Levi subgroup of ${}^LG$ containing the image of $\phi$. Let $\mathcal{M}_\phi \subseteq \mathcal{M}$ be a minimal Levi of ${}^LG$ containing the image of $\phi$. Then $\mathcal{T} \subseteq A_{\mathcal{M}_\phi}$. Therefore $a \in \mathrm{Hom}_c(W_\mathbb{R}/W^1_\mathbb{R}, A_{\mathcal{M}_\phi})$ and $\phi' = a \cdot \phi$, an unramified twist of $\phi$. Thus, $\phi'$ lies in the same connected component as $\phi$. □

We have the following commutative diagram

$$\begin{array}{ccc} \Phi(H) & \xrightarrow{\quad\xi_*\quad} & \Phi(G) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{c,W^1_F}(L^1_F, {}^LH)/H^\vee & \xrightarrow{\xi_*} & \mathrm{Hom}_{c,W^1_F}(L^1_F, {}^LG)/G^\vee \end{array}$$

The right vertical arrow maps each connected component to a point. The fibres of the left vertical arrow are finite unions of connected components. Therefore to prove Theorem 5.2.2, it suffices to show that the bottom map has finite fibres. To do so, we will make use of the following theorem of Vinberg [Vin96, Theorem 1].

**Theorem 5.2.6.** *Let $k$ be an algebraically closed field of characteristic zero. Let $\xi : \mathcal{H} \to \mathcal{G}$ be an embedding of (not necessarily connected) reductive groups over $k$. For each $n \in \mathbb{Z}_{\geq 0}$, the natural morphism*

$$\xi_* : \mathcal{H}^n//\mathcal{H} \longrightarrow \mathcal{G}^n//\mathcal{G}$$

*is finite. (The quotients are GIT quotients with respect to the conjugation actions.)*

**Proposition 5.2.7.** *Let $k$ be an algebraically closed field of characteristic zero. Let $\xi : \mathcal{H} \to \mathcal{G}$ be an embedding of (not necessarily connected) reductive groups over $k$. Equip $\mathcal{H}$ and $\mathcal{G}$ with a topology that is at least as fine as the Zariski topology. The pushforward map of conjugacy classes of continuous homomorphisms*

$$\xi_* : \mathrm{Hom}_c(\Gamma, \mathcal{H})/\mathcal{H} \longrightarrow \mathrm{Hom}_c(\Gamma, \mathcal{G})/\mathcal{G}$$

*has finite fibres.*

*Proof.* Let $\mathcal{H}_{\mathrm{Zar}}$ and $\mathcal{G}_{\mathrm{Zar}}$ denote $\mathcal{H}$ and $\mathcal{G}$ equipped with the Zariski topology. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_c(\Gamma, \mathcal{H})/\mathcal{H} & \xrightarrow{\xi_*} & \mathrm{Hom}_c(\Gamma, \mathcal{G})/\mathcal{G} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_c(\Gamma, \mathcal{H}_{\mathrm{Zar}})/\mathcal{H} & \xrightarrow{\xi_*} & \mathrm{Hom}_c(\Gamma, \mathcal{G})_{\mathrm{Zar}})/\mathcal{G} \end{array}$$

where the vertical arrows are inclusions. It follows that it suffices to show that the bottom arrow has finite fibres. We may thus assume that $\mathcal{H}$ and $\mathcal{G}$ are equipped with the Zariski topology.

Let $\gamma_1, \ldots, \gamma_n$ be a set of topological generators for $\Gamma$. This determines a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_c(\Gamma, \mathcal{H})/\mathcal{H} & \xrightarrow{\xi_*} & \mathrm{Hom}_c(\Gamma, \mathcal{G})/\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{H}^n//\mathcal{H} & \xrightarrow{\xi_*} & \mathcal{G}^n//\mathcal{G} \end{array}$$

The bottom arrow has finite fibres by Theorem 5.2.6, and therefore so does the top arrow, as claimed. □

**Proposition 5.2.8.** *Let $H$ and $G$ be connected reductive groups over $F$. Let $\xi : {}^L H \to {}^L G$ be a continuous injective homomorphism over $W_F$ such that the restriction $\xi_0 : H^\vee \to G^\vee$ is algebraic. The map*

$$\xi_* : \mathrm{Hom}_{c,W_F^1}(L_F^1, {}^L H^1)/H^\vee \longrightarrow \mathrm{Hom}_{c,W_F}(L_F^1, {}^L G^1)/G^\vee$$

*has finite fibres.*

Recall that we write $\xi(h,w) = (\xi_0(h)a_\xi(w), w)$ for all $w \in W_F$. Note that we can identify the map $\xi_*$ with the map

$$\xi_* : H_c^1(L_F^1, H^\vee) \longrightarrow H_c^1(L_F^1, G^\vee)$$

defined by $\xi_* a = \xi_{0,*} a \cdot a_\xi$, that is, $\xi_* a(l) = \xi_0 * (a(l)) a_\xi(w(l))$. We will identify $a_\xi$ with its inflation to $L_F$ and thus simply write $a_\xi(l) = a_\xi(w(l))$. It will be convenient to view $\xi_*$ in this way in order to save space and use language from group cohomology.

*Proof.* Let $a \in H_c^1(L_F^1, H^\vee)$. We will show that $\xi_*^{-1}(\xi_* a)$ is finite.

Assume that $F$ is non-archimedean. Then $W_F^1 = I_F$ and $L_F^1 = \mathrm{SL}_2 \times I_F$. There exists a compact open normal subgroup $J$ of $I_F$ such that $J$ acts trivially on $H^\vee$ and $G^\vee$, and $a(J) = a_\xi(J) = 1$. Since $a_\xi(J) = 1$, the homomorphism $\xi : H^\vee \rtimes W_F^1 \to G^\vee \rtimes W_F^1$ descends to a homomorphism

$$\xi : H^\vee \rtimes W_F^1/J \to G^\vee \rtimes W_F^1/J.$$

We have a commutative diagram

$$\begin{array}{ccc} H_c^1(L_F^1/J, H^\vee) & \xrightarrow{\xi_*} & H_c^1(L_F^1/J, H^\vee) \\ \big\downarrow & & \big\downarrow \\ H_c^1(L_F^1, H^\vee) & \xrightarrow{\xi_*} & H_c^1(L_F^1, H^\vee) \end{array}$$

where the vertical arrows are the injective inflation maps.

Since $\xi_* a = \xi_{0,*} a \cdot a_\xi$, we have $\xi_* a(J) = 1$. Let $a' \in \xi_*^{-1}(\xi_* a)$. Then there exists $g \in G^\vee$ such that

$$\xi_0(a'(l)) a_\xi(l) = \xi_* a'(l) = g \xi_* a(l) \, {}^l g^{-1},$$

for all $l \in L_F^1$. Since $\xi_*(J) = a_\xi(J) = 1$, we have $\xi(a'(J)) = 1$. Since $\xi$ is injective, we obtain $a'(J) = 1$. Thus, we have that $\xi_*^{-1}(\xi_* a)$ lies in $H_c^1(L_F^1/J, H^\vee) \hookrightarrow H_c^1(L_F^1, H^\vee)$. Consequently, we obtain the commutative diagram

$$\begin{array}{ccc} \xi_*^{-1}(\xi_* a) & \xrightarrow{\xi_*} & \{\xi_* a\} \\ \big\downarrow & & \big\downarrow \\ H_c^1(L_F^1/J, H^\vee) & \xrightarrow{\xi_*} & H_c^1(L_F^1/J, G^\vee) \end{array}$$

and it suffices to show that the bottom map has finite fibres. This map can be viewed as the map

$$\mathrm{Hom}_{c,I_F/J}(L_F^1/J, H^\vee \rtimes I_F/J)/H^\vee \xrightarrow{\xi_*} \mathrm{Hom}_{c,I_F/J}(L_F^1/J, G^\vee \rtimes I_F/J)/G^\vee$$

where we are writing $\mathrm{Hom}_{c,I_F/J}$ to indicate continuous homomorphisms over $I_F/J$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{c,I_F/J}(L_F^1/J, H^\vee \rtimes I_F/J)/H^\vee & \xrightarrow{\xi_*} & \mathrm{Hom}_{c,I_F/J}(L_F^1/J, G^\vee \rtimes I_F/J)/G^\vee \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{c,I_F/J}(L_F^1/J, H^\vee \rtimes I_F/J)/\mathrm{conj} & \xrightarrow{\xi_*} & \mathrm{Hom}_{c,I_F/J}(L_F^1/J, G^\vee \rtimes I_F/J)/\mathrm{conj} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_c(\mathrm{SL}_2 \times I_F/J, H^\vee \rtimes I_F/J)/\mathrm{conj} & \xrightarrow{\xi_*} & \mathrm{Hom}_c(\mathrm{SL}_2 \times I_F/J, G^\vee \rtimes I_F/J)/\mathrm{conj} \end{array}$$

where we have written "/conj" to indicate quotients by conjugacy of the codomain. The fibres of the upper vertical arrow on the left are finite as they are the sets of $H^\vee$-conjugacy classes that comprise a conjugacy class under $H^\vee \rtimes I_F/J$ and $I_F/J$ is finite. Since the bottom arrow has finite fibres by Proposition 5.2.7, it follows that the top arrow does and therefore $\xi_*^{-1}(\xi_* a)$ is finite.

Now, assume that $F$ is archimedean. By restriction of scalars, we may assume that $F = \mathbb{R}$. Recall that $L_\mathbb{R} = W_\mathbb{R} = \mathbb{C}^\times \cup \mathbb{C}^\times j$ and $L_\mathbb{R}^1 = W_\mathbb{R}^1 = S^1 \cup S^1 j$. We have $W_\mathbb{R} = W_\mathbb{R}^1 \times \mathbb{R}_{>0}$. We have an extension of topological groups

$$1 \longrightarrow \{\pm 1\} \longrightarrow S^1 \rtimes \langle j \rangle \longrightarrow W_\mathbb{R}^1 \longrightarrow 1$$

where the homomorphism $\{\pm 1\} \to S^1 \rtimes \langle j \rangle$ is the diagonal embedding and the homomorphism $S^1 \rtimes \langle j \rangle \to W_\mathbb{R}^1$ is given by multiplication in $W_\mathbb{R}^1$. We have a commutative diagram

$$\begin{array}{ccc} H^1_c(W^1_{\mathbb{R}}, H^\vee) & \xrightarrow{\xi_*} & H^1_c(W^1_{\mathbb{R}}, G^\vee) \\ \big\downarrow & & \big\downarrow \\ H^1_c(S^1 \rtimes \langle j \rangle, H^\vee) & \xrightarrow{\xi_*} & H^1_c(S^1 \rtimes \langle j \rangle, G^\vee) \end{array}$$

where the vertical arrows are the inflation maps. Thus, it suffices to show that the bottom map has finite fibres. This map can be viewed as the pushforward map

$$\xi_* : \mathrm{Hom}_{c,\langle j \rangle}(S^1 \rtimes \langle j \rangle, H^\vee \rtimes \langle j \rangle)/H^\vee \longrightarrow \mathrm{Hom}_{c,\langle j \rangle}(S^1, \rtimes \langle j \rangle, G^\vee \rtimes \langle j \rangle)/G^\vee$$

where $\mathrm{Hom}_{c,\langle j \rangle}$ indicates continuous homomorphisms over $\langle j \rangle$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{c,\langle j \rangle}(S^1 \rtimes \langle j \rangle, H^\vee \rtimes \langle j \rangle)/H^\vee & \xrightarrow{\xi_*} & \mathrm{Hom}_{c,\langle j \rangle}(S^1, \rtimes \langle j \rangle, G^\vee \rtimes \langle j \rangle)/G^\vee \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{c,\langle j \rangle}(S^1 \rtimes \langle j \rangle, H^\vee \rtimes \langle j \rangle)/\mathrm{conj} & \xrightarrow{\xi_*} & \mathrm{Hom}_{c,\langle j \rangle}(S^1 \rtimes \langle j \rangle, G^\vee \rtimes \langle j \rangle)/\mathrm{conj} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{c}(S^1 \rtimes \langle j \rangle, H^\vee \rtimes \langle j \rangle)/\mathrm{conj} & \xrightarrow{\xi_*} & \mathrm{Hom}_{c}(S^1 \rtimes \langle j \rangle, G^\vee \rtimes \langle j \rangle)/\mathrm{conj} \end{array}$$

As above, the left vertical map has finite fibres. By Proposition 5.2.7, the bottom map has finite fibres. Consequently, the top map has finite fibres. □

This completes the proof of Theorem 5.2.2 and thus Theorem 5.1.1 and its corollary Corollary 5.1.2, which in particular answers Questions A and B of [Lan13] in the affirmative.

## 5.3 Examples

We conclude by computing stable transfer in some cases. Let $H$ and $G$ be connected reductive groups over a local field $F$ of characteristic zero with $G$ quasisplit, and let $\xi : {}^LH \to {}^LG$ be an equivalence class of injective tempered $L$-parameters. First, we note some cases that go back to the work of Harish-Chandra.

Suppose that $H = 1$. We have ${}^LH = W_F$ and $\xi$ is just a tempered $L$-parameter of $G$. Let $f^G \in \mathcal{S}(G)$. Then the function $\mathcal{T}_\xi f^G$ on $H(F) = 1$ is just the constant

$f^G(\xi) = \Theta_\xi(f^G)$. That is, $\mathcal{T}_\xi$ is given by

$$\mathcal{T}_\xi f^G = \int_{\Delta_{\mathrm{sr}}(G)} |D^G(\delta)|^{1/2}\Theta_\xi(\delta) f^G(\delta)\, d\delta.$$

To obtain a formula for $\Theta_\xi(\delta)$, it suffices to treat the case when $\xi$ is discrete by the formula for parabolically induced characters [Sil79, Lemma 4.7.6]. Harish-Chandra obtained formulas for stable discrete series characters of real groups (see [GKM97]). For $p$-adic groups, formulas for stable discrete series characters are not known in general. See [Kal23] for the current state of the art.

Suppose that $H$ is a Levi subgroup $M$ of $G$ and $\xi : {}^LM \to {}^LG$ is the canonical equivalence class of $L$-embeddings. Then $\mathcal{T}_\xi : \mathcal{S}(G) \to \mathcal{S}(M)$ is parabolic descent. Thus, stable transfer coincides with suitably normalised endoscopic transfer in this case [AMV, §1]. If $f \in \mathcal{S}(G)$, then for $G$-regular semisimple elements $m \in M(F)$, we have $(\mathcal{T}_\xi f^G)(m) = f^G(m)$.

Suppose $G = H^*$ is the quasisplit inner form of $H$ and $\xi : {}^LH \to {}^LH^*$ is the identity. Endoscopic transfer $\mathcal{T}_{\mathrm{End}} : \mathcal{I}(H) \to \mathcal{S}(H^*)$ descends to a continuous linear map $\mathcal{T}_H^{H^*} : \mathcal{S}(H) \to \mathcal{S}(H^*)$. Furthermore, it can be normalised so that $(\mathcal{T}_H^{H^*} f^H)(\phi) = f^H(\phi)$ for all $\phi \in \Phi(H)$. This is proved for real groups in [She79] and is part of the refined local Langlands conjecture for $p$-adic groups [Kal16]. For $f^H \in \mathcal{S}(H)$ and $\phi \in \Phi(H)$ we have $(\mathcal{T}_\xi \mathcal{T}_H^{H^*} f^H)(\phi) = (\mathcal{T}_H^{H^*} f^H)(\phi) = f^H(\phi)$, and thus $\mathcal{T}_\xi \mathcal{T}_H^{H^*} f^H = f^H$ by spectral density. That is, endoscopic transfer gives a a section for stable transfer along $\xi$.

In general, stable transfer is not the same as endoscopic transfer. Indeed, the adjoint of stable transfer maps stable distributions to stable distributions, whereas the adjoint to endoscopic transfer does not do so in general.

### 5.3.1 Tori

Let $S$ and $T$ be tori over $F$ and let $\xi : {}^LS \to {}^LT$ be an injective tempered $L$-embedding. Since $T^\vee$ is abelian, the restriction $\xi_0 : S^\vee \to T^\vee$ of $\xi$ is $W_F$-equivariant, or equivalently $\Gamma_F$-equivariant. Let $\xi_0^* : T \to S$ be the homomorphism with $(\xi_0^*)^\vee = \xi_0$. Then $\xi_{0,*} : H^1_c(W_F, S^{\vee(1)}) \to H^1_c(W_F, T^{\vee(1)})$ and $\xi_0^* : T(F) \to S(F)$ are adjoint with respect to the Langlands pairings for $S$ and $T$.

Let $a \in H^1_c(W_F, S^{\vee 1}) = \Phi_{\mathrm{temp}}(S)$. We have $\xi_*(a) = a_\xi \cdot (\xi_{0,*}a)$, where $a_\xi \in Z^1_c(W_F, T^{\vee 1})$ is the 1-cocycle determined by $\xi$. For $t \in T(F)$, we have

$$\chi_{\xi_*(a)}(t) = \langle \xi_*(a), t\rangle = \langle a_\xi, t\rangle\langle(\xi_{0,*}a), t\rangle = \langle a_\xi, t\rangle\langle a, \xi_0^*(t)\rangle = \chi_{a_\xi}\chi_a(\xi_0^*(t)).$$

We will write $\chi_\xi = \chi_{a_\xi}$ a for brevity.

Since $\xi_0$ is injective, $\xi_0^* : T \to S$ is a quotient homomorphism. The Haar measure $dt$ on $T(F)$ disintegrates into measures $\mu_s$ on the fibres $(\xi_0^*)^{-1}(s)$ such that

$$\int_{T(F)} dt = \int_{S(F)} \int_{(\xi_0^*)^{-1}(s)} d\mu_s(t)\, ds.$$

We can describe the measures $\mu_s$ concretely as follows. Let $D = \ker(\xi_0^* : T(F) \to S(F))$. If $s = \xi_0^*(t)$ we have $(\xi_0^*)^{-1}(s) = tD$ and the measure $\mu_s$ on $(\xi_0^*)^{-1}(s)$ is obtained by transporting a suitably normalised Haar measure on $D$ along the map $D \to tD, d \mapsto td$.

Let $f \in \mathcal{S}(T) = \mathcal{C}(T)$. For all $a \in H_c^1(W_F, S^{\vee 1}) = \Phi_{\mathrm{temp}}(S)$, we have

$$\begin{aligned}
\int_{S(F)} \chi_a(s)(\mathcal{T}_\xi f)(s)\, ds &= (\mathcal{T}_\xi f)(a) \\
&= f(\xi_*(a)) \\
&= \int_{T(F)} \chi_{\xi_*(a)}(t) f(t)\, dt \\
&= \int_{T(F)} \chi_\xi(t)\chi_a(\xi_0^*(t))(t) f(t)\, dt \\
&= \int_{S(F)} \chi_a(s) \int_{(\xi_0^*)^{-1}(s)} \chi_\xi(t) f(t)\, d\mu_s(t)\, ds.
\end{aligned}$$

It follows that for all $s \in S(F)$ we have

$$\begin{aligned}
(\mathcal{T}_\xi f)(s) &= \int_{(\xi_0^*)^{-1}(s)} \chi_\xi(t) f(t)\, d\mu_s(t) \\
&= \int_{T(F)} \chi_\xi(t) f(t)\, d\mu_s(t)
\end{aligned}$$

where we have used $\mu_s$ to also denote the pushforward of $\mu_s$ to $T(F)$.

### 5.3.2 Complex groups

Let $H, G$ be connected reductive groups over $\mathbb{C}$ and let $\xi : {}^L H \to {}^L G$ be an equivalence class of injective tempered $L$-homomorphism. Let $S$ be a minimal Levi subgroup (maximal torus) of $H$ and let $T$ be a minimal Levi subgroup (maximal torus) of $G$.

We have a commutative diagram

$$\begin{array}{ccc} {}^L H & \xrightarrow{\xi} & {}^L G \\ \uparrow & & \uparrow \\ {}^L S & \xrightarrow{\xi} & {}^L T \end{array}$$

of equivalence classes of $L$-homomorphisms. We may apply the considerations of the preceding subsection to $\xi : {}^L S \to {}^L T$.

Define $s \in S$ to be $\xi$-regular if

$$\dim(T_{G\text{-sing}} \cap (\xi_0^*)^{-1}(s)) < \dim((\xi_0^*)^{-1}(s))$$

and $\xi$-singular otherwise. Let $S_{\xi\text{-reg}}$ (resp. $S_{\xi\text{-sing}}$) denote the set of $\xi$-regular (resp. $\xi$-singular) elements of $S$. If $s \in S_{\xi\text{-reg}}$, then $\mu_s(T_{G\text{-sing}} \cap (\xi_0^*)^{-1}(s)) = 0$. Let $D = \ker(\xi_0^*)$. We have a direct product decomposition $D = AD^\circ$ for a finite subgroup $A \subseteq D$.

**Lemma 5.3.1.** *We have* $S_{\xi\text{-sing}} = \bigcup_{a \in A} \bigcup_{\alpha \in \Phi(G,T), \alpha(D^\circ)=1} \xi_0^*(a \ker \alpha)$. *Furthermore,* $S_{\xi\text{-sing}}$ *is a closed subvariety of* $S$ *of positive codimension.*

*Proof.* Suppose that $s \in S_{\xi\text{-sing}}$, that is $s \in S$ with

$$\dim(T_{G\text{-sing}} \cap (\xi_0^*)^{-1}(s)) = \dim((\xi_0^*)^{-1}(s)).$$

Then $T_{G\text{-sing}}$ contains an irreducible component of $(\xi_0^*)^{-1}(s)$ of maximum dimension. Write $(\xi_0^*)^{-1}(s) = tD$. Then there exists $a \in A$ such that $taD^\circ \subseteq T_{G\text{-sing}}$. Let $\alpha \in \Phi(G,T)$ be a root such that $\alpha(taD^\circ) = 1$. Then $\alpha(ta) = 1$ and $\alpha(D^\circ) = 1$. It follows that $s \in \bigcup_{a \in A} \bigcup_{\alpha \in \Phi(G,T), \alpha(D^\circ)=1} \xi_0^*(a \ker \alpha)$.

Conversely, suppose that $s \in \bigcup_{a \in A} \bigcup_{\alpha \in \Phi(G,T), \alpha(D^\circ)=1} \xi_0^*(a \ker \alpha)$. There exist $a \in A$, $\alpha \in \Phi(G,T)$ with $\alpha(D^\circ) = 1$, and $c \in \ker \alpha$ such that $s = \xi_0^*(ak)$. Then $(\xi_0^*)^{-1}(s) = akD$. Now, $kD^\circ$ is an irreducible component of $(\xi_0^*)^{-1}(s)$ of maximum dimension that is contained in $\ker \alpha$, and thus $T_{G\text{-sing}}$. Therefore $s \in S_{\xi\text{-sing}}$.

We have shown that $S_{\xi\text{-sing}} = \bigcup_{a \in A} \bigcup_{\alpha \in \Phi(G,T), \alpha(D^\circ)=1} \xi_0^*(a \ker \alpha)$. Now, $\xi_0^*(a \ker \alpha) = \xi_0^*(a)\xi_0^*(\ker \alpha)$, and $\xi_0^*(\ker \alpha)$ is a closed subgroup of $S$. Therefore $S_{\xi\text{-sing}}$ is a closed subvariety of $S$. Moreover, we have

$$\begin{aligned}\dim \xi_0^*(a \ker \alpha) &= \dim \ker \alpha - \dim \ker \xi_0^* \\ &\leq \dim T - 1 - \dim \ker \xi_0^* \\ &= \dim S - 1.\end{aligned}$$

Thus, $S_{\xi\text{-sing}}$ has positive codimension in $S$. □

Let $f^G \in \mathcal{S}(G)$ and write $f^H$, $f^T$, $f^S$ for its associated stable transfers to $H$, $T$, and $S$, respectively. Let $s \in S_{H\text{-reg}}$. We have $f^H(s) = f^S(s)$ by parabolic descent. Using the notation of the preceding subsection, we obtain

$$f^H(s) = \int_T \chi_\xi(t) f^T(t)\, d\mu_s(t).$$

Suppose further that $s \in S_{\xi\text{-reg}}$. Then $\mu_s(T_{G\text{-sing}} \cap (\xi_0^*)^{-1}(s)) = 0$, and

$$f^H(s) = \int_{T_{G\text{-reg}}} \chi_\xi(t) f^T(t)\, d\mu_s(t).$$

Parabolic descent gives us

$$f^H(s) = \int_{T_{G\text{-reg}}} \chi_\xi(t) f^G(t)\, d\mu_s(t)$$

Let $c_G : T_{G\text{-reg}} \to T_{G\text{-reg}}/W(G,T) = \Delta_{\mathrm{rs}}(G)$ be the natural map. Then

$$f^H(s) = \int_{\Delta_{\mathrm{rs}}(G)} f^G(\delta)\, d(c_G)_*(\chi_\xi \mu_s)(\delta)$$

Since $S_{H\text{-sing}} \cup S_{\xi\text{-sing}}$ is a closed subvariety of $S$ of positive codimension, the set of $H$-regular and $\xi$-regular elements of $S$ is an open dense subset. Its image $\Delta_{\mathrm{rs},\xi\text{-reg}}(H)$ in $\Delta_{\mathrm{rs}}(H)$ is an open dense subset. Thus, the above formula determines $f^H$.

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