Symplectic Theta Functions and Theta Lifts of Modular Forms to Split Orthogonal Groups

by

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Abstract

We define a notion of modularity for a function on the symmetric space $\mathcal{D}_m = G/K$ for $G = O(m,m)$. We define symplectic theta functions which are functions on $\mathcal{D}_m \times \mathcal{H}_n$, where $\mathcal{H}_n$ is the Siegel upper half space of genus n, which are modular in both variables. We pair these symplectic theta functions with modular forms on $\mathcal{H}_n$ to obtain modular forms on $\mathcal{D}_m$, and we compute their Fourier coefficients. We do this for cusp forms and weakly holomorphic modular forms in the case when $n = 1$, and for cusp forms when $n > 1$. In the case where $n = 2$ we obtain complete explicit formulas, and in the case where $n > 2$ we reduce the computation of the Fourier coefficients to the calculation of a single integral.

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To Oscar and Riley

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Chapter 1

Introduction

This thesis concerns automorphic forms for the split algebraic group $O_{m,m}$ defined over $\mathbb{Q}$, from a classical point of view. Denote by G the real points of $O_{m,m}$:

$$G = O(m,m) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : Q[g] = Q \right\}$$

where $Q = \begin{pmatrix} & 1_m \\ 1_m & \end{pmatrix}$, and for matrices A and B, we write $A[B] = {}^tBAB$, where ${}^tB$ denotes the transpose of B. The corresponding Riemannian symmetric space has a model given by:

$$\mathcal{D} = \{\xi = X + Y \in M_m(\mathbb{R}) : \ {}^tX = -X, \ {}^tY = Y, \ Y > 0\}.$$

and the action of G on $\mathcal{D}$ is given by

$$g \cdot \xi = (a\xi + b)(c\xi + d)^{-1}.$$

and this action has a pair of factors of automorphy,

$$j^{\pm}(g, \xi) = c\xi^{\pm} + d$$

where $\xi^{\pm} = X \pm Y$. By a modular form for G we will mean a function $\Phi : \mathcal{D} \to \mathcal{V}_\rho$, where $\mathcal{V}_\rho$ is a representation of $GL_m(\mathbb{R})$, that transforms under translations by $\gamma \in \Gamma = O_{m,m}(\mathbb{Z})$ as:

$$\Phi(\gamma\xi) = \rho(j^-(\gamma, \xi))\Phi(\xi) \tag{1.1}$$

(More generally we could have a representation ρ of $GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$ and take modular to mean $\Phi(\gamma, \xi) = \rho\big(j^+(\gamma, \xi), j^-(\gamma, \xi)\big)\Phi(\xi)$, but in this work we will consider ones that only depend on j^-). We compare this to the case of classical and Siegel modular forms. In that case we have the group $G' = Sp_n(\mathbb{R})$, given by:

$$G' = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : J[g] = J \right\}$$

where $J = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$. This group has a Hermitian symmetric space $\mathcal{H}$:

$$\mathcal{H} = \{\tau = x + iy \in M_n(\mathbb{C}) : \ {}^t x = x, \ {}^t y = y, \ y > 0\}$$

with the action of G' on $\mathcal{H}$ given by

$$g \cdot \tau = (a\tau + b)(c\tau + d)^{-1}.$$

This action has the $GL_n(\mathbb{C})$ valued factor of automorphy $j(g, \tau) = c\tau + d$. A modular form for $Sp_n(\mathbb{R})$ is then a holomorphic function $f : \mathcal{H} \to \mathcal{V}_\kappa$, where κ is a holomorphic representation of $GL_n(\mathbb{C})$, that satisfies:

$$f(\gamma\tau) = \kappa(j(\gamma, \tau))f(\tau) \tag{1.2}$$

for $\gamma \in \Gamma' = Sp_n(\mathbb{Z})$, together with a finiteness condition at the cusp when $n = 1$. The most commonly studied are the scalar valued modular forms that satisfy $f(\gamma\tau) = (c\tau + d)^\kappa f(\tau)$ or $f(\gamma\tau) = \det(c\tau + d)^\kappa f(\tau)$ when $n = 1$ or $n > 1$, respectively. We seek to provide classical formulas for modular forms for $O_{m,m}(\mathbb{R})$ in analogy to the theory for $Sp_n(\mathbb{R})$. More specifically, we will construct modular forms on $\mathcal{D}$ by integrating modular forms on $\mathcal{H}$ against a theta kernel.

In Chapter 2 we will fix some notation about G and G', and also discuss this space $\mathcal{D}$ in more detail, as well as some natural functions associated to this space. There is another notion of factors of automorphy on $\mathcal{D}$ valued in $O(m) \times O(m)$ that essentially come from the Cartan decomposition of G, which we will denote by $k^\pm(g, \xi)$, and we will describe their relation to $j^\pm(g, \xi)$. We will describe a set of generators of Γ. We will also recall a low dimensional exceptional isomorphism $SO_0(2, 2) \cong SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ that leads to an identification $\mathcal{D}_2 \cong \mathcal{H}_1 \times \mathcal{H}_1$.

In Chapter 3 we will discuss modular forms on $\mathcal{H}$ of full level $\Gamma' = Sp_n(\mathbb{Z})$, and in particular the theory of Hecke operators. Most of this will be review of the standard theory, however we will also look at a less common class of Hecke operators that change the level of a modular form. We will call these operators $T(D)$ (Definition 3.2.7) indexed by diagonal matrices D whose entries are positive integers that satisfy a certain divisibility condition.

Next we will define the notion of a modular form for $O(m, m)$ (1.1). Just as with Siegel modular forms these functions have a Fourier expansion, except in this case it is with respect to *skew symmetric* matrices:

$$\Phi(\xi) = \sum_{S \in \mathrm{Skew}_m(\mathbb{Z})} e^{\pi i \,\mathrm{tr}\, SX} a_S(Y)$$

where $a_S : \mathrm{Sym}_m^+(\mathbb{R}) \to \mathcal{V}_\rho$ are some functions of a positive definite matrix variable. We will also describe how to convert between the notion of modularity in (1.1) and a notion of modularity with respect to the factors of automorphy $k^\pm$.

We will describe how the two notions of modularity on $\mathcal{D}_2 \cong \mathcal{H}_1 \times \mathcal{H}_1$ in the senses of (1.2) and (1.1) are related (in this case it is more convenient to use the notion of modularity on $O(2,2)$ with respect to $k^\pm$). In particular for a modular form $\widetilde{\Phi}$ on $\mathcal{D}_2$ that satisfies $\widetilde{\Phi}(\gamma\xi) = \rho\big(k^+(\gamma,\xi), k^-(\gamma,\xi)\big)\widetilde{\Phi}(\xi)$, we will describe how to obtain a function $F(\tau_1,\tau_2)$ on $\mathcal{H}_1 \times \mathcal{H}_1$ that satisfies $F(\gamma_1\tau_1, \gamma_2\tau_2) = j(\gamma_1,\tau_1)^{\kappa_1} j(\gamma_2,\tau_2)^{\kappa_2} F(\tau_1,\tau_2)$, where (κ_1,κ_2) are two integers depending on the representation ρ.

In the last section of chapter 3, we will define a class of functions on $\mathcal{D} \times \mathcal{H}$ which we will call symplectic theta functions:

Definition 1.0.1 (Symplectic Theta Functions - Definition 3.4.1)**.** *For $\xi = X + Y \in \mathcal{D}$, $\tau = x + iy \in \mathcal{H}$, and $p \in \mathbb{C}[M_{m,n}(\mathbb{C})]$ a polynomial of an $m \times n$ matrix variable, define:*

$$\Theta(\xi,\tau;p) = \sum_{w \in M_{m,2n}(\mathbb{Z})} p(\eta_\tau(w)) e^{\pi i \,\mathrm{tr}\, X\langle w,w\rangle} e^{-\pi \,\mathrm{tr}\, Y(w,w)_\tau} \tag{1.3}$$

where $\langle w,w\rangle = wJ\,{}^t w$ is an integral skew symmetric matrix, and $(w,w)_\tau$ is an $m \times m$ symmetric matrix depending on τ and w such that $\mathrm{tr}(w,w)_\tau$ is a positive definite symmetric bilinear form on $M_{m,2n}(\mathbb{R})$. This defines a function $\Theta : \mathcal{D} \times \mathcal{H} \to \mathbb{C}[M_{m,n}(\mathbb{C})]^*$.

Theorem 1.0.2 (Theorem 3.4.2)**.** *These functions are modular on both $\mathcal{D}$ and $\mathcal{H}$, in the sense that:*

$$\Theta(\gamma\xi, \gamma'\tau; p) = |\det j^-(\gamma,\xi)|^n \Theta\Big(\xi,\tau;\ \sigma\big(j^-(\gamma,\xi), j(\gamma',\tau)\big)^{-1} p\Big) \tag{1.4}$$

where σ is the representation of $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ given by $\sigma(\alpha,a)p(\eta) = p(\alpha^{-1}\eta a)$, and $\eta_\tau(w) = w_1\tau + w_2$.

The proof is similar to the proof of modularity for the classical theta function and is proved by an application of Poisson summation. $\Theta(\xi,\tau)$ takes values in an infinite dimensional vector space $\mathbb{C}[M_{m,n}(\mathbb{C})]^*$. We have that $\mathbb{C}[M_{m,n}(\mathbb{C})] \cong \bigoplus_\kappa (\mathcal{V}_\kappa^{(m)})^* \otimes \mathcal{V}_\kappa^{(n)}$ where $\mathcal{V}_\kappa^{(n)}$ is an irreducible representation of $GL_n(\mathbb{C})$, and $\mathcal{V}_\kappa^{(m)}$ is the corresponding representation of $GL_m(\mathbb{C})$. We can then take projections in $\mathbb{C}[M_{m,n}(\mathbb{C})]^*$ to obtain functions $\Theta^\kappa : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_\kappa^{(m)} \otimes (\mathcal{V}_\kappa^{(n)})^*$. These functions satisfy:

$$\Theta^\kappa(\gamma\xi, \gamma'\tau) = |\det j^-(\gamma,\xi)|^n \big(\kappa^{(m)}(j^-(\gamma,\xi))\big) \otimes \big((\kappa^{(n)})^*(j(\gamma,\tau))\big)\Theta^\kappa(\xi,\tau)$$

In chapters 4 and 5 we will pair $\Theta^\kappa(\xi,\tau)$ with a modular form $f(\tau)$ of weight κ, to obtain a function invariant under Γ' in τ which we will denote by $(f,\Theta)(\xi,\tau)$, given by the natural pairing between $\mathcal{V}_\kappa$ and $\mathcal{V}_\kappa^*$. We will then define the *theta lift* of f to be:

$$\Phi_f(\xi) = \int_{\Gamma'\backslash\mathcal{H}} (f,\Theta)(\xi,\tau)\frac{dxdy}{\det y^{n+1}} \tag{1.5}$$

(see Definitions 4.1.1, 4.2.3, and 5.1.1 for the precise definitions in context). The function $\Phi_f(\xi)$ is an example of a modular form on $\mathcal{D}$ of weight $\rho_\kappa = |\det|^n \otimes \kappa^{(m)}$ and we will endeavor to find explicit formulas for it, in particular for its Fourier coefficients. We identify the underlying space $\mathcal{V}_{\rho_\kappa}$ that Φ_f takes values in with $\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})])^*$, and will often express formulas for Φ_f in terms of its evaluation on a $P : \mathcal{V}_\kappa \to \mathbb{C}[M_{m,n}(\mathbb{C})]$. Given a $P \in \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})])$ we have $P(f\tau)) \in \mathbb{C}[M_{m,n}(\mathbb{C})]$, and we have:

$$\Phi_f(\xi;P) = \int_{\Gamma'\backslash\mathcal{H}} \sum_{w\in M_{m,2n}(\mathbb{Z})} P(f(\tau))(\eta_\tau(w))e^{\pi i \operatorname{tr} X\langle w,w\rangle}e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-1-n} dxdy$$

When $n=1$ or κ is a scalar representation we have that $\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})]) \cong \mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa$, the κ isotypic component of $\mathbb{C}[M_{m,n}(\mathbb{C})]$, and we will write p for a polynomial in place of P. In this case we have simply that $P(f(\tau))(\eta) = f(\tau)p(\eta)$

In Chapter 4 we will consider theta lifts of modular forms from $SL_2(\mathbb{R})$ $(n=1)$. We will do this for both cusp forms and for weakly holomorphic modular forms, where we allow for a pole at the cusp. The latter will require a regularized integral in equation 1.5, and we follow [1] for this (see Definition 4.2.3). We will define functions $\tau_1(\xi,w)$, $\tau_2(\xi,w) \in \mathcal{H}_1$ that depend on a $\xi \in \mathcal{D}$ and a $w \in M_{m,2}(\mathbb{R})$ of rank 2. For such $w=(w_1,w_2)$ we will also define $\eta(\xi,w) = \tau_2(\xi,w)w_1 + w_2$.

Theorem 1.0.3 (Theorem 4.1.16)**.** *Suppose that f is a Hecke cusp form of weight κ. Then for $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$ we have:*

$$\Phi_f(\xi;p) = 2\sum_{\substack{S_0\in\mathcal{S}^1\\ \mu>0}} e^{\pi i \operatorname{tr}\mu S_0 X} a_\mu p\big(\eta(Y,w_0)\big) y_1(Y,w_0)^{-1} e^{-2\pi\mu y_1(Y,w_0)} f\big(\tau_2(Y,w_0)\big) \tag{1.6}$$

where $\mathcal{S}^1$ is the set of primitive rank 2 matrices in $\mathrm{Skew}_m(\mathbb{Z})$, and for $S_0 \in \mathcal{S}^1$, $w_0 \in M_{m,2}(\mathbb{Z})$ is such that $\langle w_0, w_0\rangle = S_0$, and we have written $\tau_2(Y,w_0)$, etc. in place of $\tau_2(\xi,w_0)$, etc to emphasize that those terms depend only on the Y variable of ξ, and $y_1(Y,w_0)$ for the imaginary part of $\tau_1(Y,w_0)$.

The expression in the theorem above hints at a relationship between the lift to $O(m,m)$ and the lift to $O(2,2)$. We can express this more precisely with a general theorem that will hold for lifts of cusp forms from any $Sp_n(\mathbb{R})$, which states roughly that the lift to $O(m,m)$ for $m > 2n$ is “comes from” by the lift to $O(2n,2n)$. More precisely:

Theorem 1.0.4 (Theorems 4.1.17 and 5.2.1)**.** *Suppose that f is a cusp form on $\mathcal{H}_n$ of weight κ, and write $\Phi_f^{(m)}$ for the lift to $O(m,m)$ (1.5). If $m < 2n$, then $\Phi_f^{(m)}$ is identically 0. For $m > 2n$, denote by $\overline{P}_{2n}(\mathbb{Z}) \subset GL_m(\mathbb{Z})$ the subgroup $\overline{P}_{2n}(\mathbb{Z}) = \begin{pmatrix} GL_{2n}(\mathbb{Z}) & \\ * & GL_{m-2n}(\mathbb{Z}) \end{pmatrix}$. For $\xi \in \mathcal{D}_m$, denote by $[\xi]_{m,2n} \in \mathcal{D}_{2n}$ the matrix obtained by taking the top left $2n \times 2n$ minor of ξ. We will write $\mathcal{V}_{\rho_\kappa}^{(m)}$ for the domain of $\Phi_f^{(m)}$. We will also define a certain map $[-]_{m,2n} : \mathcal{V}_{\rho_\kappa}^{(2n)} \to \mathcal{V}_{\rho_\kappa}^{(m)}$ (this map is essentially the dual to the map $\mathbb{C}[M_{m,n}(\mathbb{C})] \twoheadrightarrow \mathbb{C}[M_{2n,n}(\mathbb{C})]$ obtained by setting the bottom $m-2n$ rows to 0). Then we have*

$$\Phi_f^{(m)}(\xi) = \sum_{A \in \overline{P}_{2n}(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \rho_\kappa({}^tA)[\Phi_f^{(2n)}([A\xi\,{}^tA]_{m,2n})]_{m,2n} \tag{1.7}$$

We will then describe what $\Phi_f(\xi)$ is on $\mathcal{D}_2$ in terms of the exceptional isomorphism $\mathcal{D}_2 \cong \mathcal{H}_1 \times \mathcal{H}_1$. As we described earlier we have a way of going from Φ_f, a modular form on $\mathcal{D}_2$, to F_f, a modular form on $\mathcal{H}_1 \times \mathcal{H}_1$. This function F_f is given by:

Theorem 1.0.5 (Theorem 4.1.18)**.** *When $f(\tau)$ is a Hecke eigenform, we that the Theta lift of f to $\mathcal{H}_1 \times \mathcal{H}_1$ is*

$$F_f(\tau_1, \tau_2) = f(\tau_1) f(\tau_2)$$

The main computations involve a two step unfolding process. From its definition (1.5), Φ_f is immediately expressible as a sum over $w \in M_{m,2}(\mathbb{Z})$ of integrals over a fundamental region. The first step is to group these w according to right $SL_2(\mathbb{Z})$ orbits, which has the effect of enlarging the domain of integration (Lemma 4.1.3). Due to f being a cusp form all terms except those corresponding to w of rank 2 integrate to 0. We then obtain that each Fourier coefficient is expressible as a finite sum of integrals over $\mathcal{H}$. This finite sum can then be grouped together for the effect of acting on f by a Hecke operator (Lemma 4.1.13). Ultimately the result is to condense many integrals into a single integral which is evaluated. This process is mirrored in chapter 5 where we examine lifts of cusp forms from $Sp_n(\mathbb{R})$ to $O(m,m)$.

In the second part of Chapter 4 we handle the lifts of weakly holomorphic modular forms. These are modular functions on $\mathcal{H}_1$ that are allowed to have a pole at the cusp, while otherwise being holomorphic on the interior of $\mathcal{H}_1$. In this case the integral defining Φ_f does not converge, so we follow the regularization procedure used in [1] to define the theta lift Φ_f (Definition 4.2.3). These give modular forms on $\mathcal{D}_m$ with singularities along sub-manifolds $\mathcal{D}_\lambda$, corresponding to positive length vectors $\lambda \in \mathbb{Z}^{m,m}$, such that $a_{-(\lambda,\lambda)/2} \neq 0$ in the Fourier expansion of f, (Theorem 4.2.5). In this case the unfolding method gives two classes of orbits corresponding to $w \in M_{m,2}(\mathbb{Z})$ of rank 1 and 2, and we will separate their sums to define $\Phi_{f,1}$ and $\Phi_{f,2}$ with $\Phi_f = \Phi_{f,1} + \Phi_{f,2}$. The function $\Phi_{f,1}$ is the constant term of the Fourier expansion of Φ_f. For $z \in \mathbb{C}$ with $\mathrm{Re}(z)$ large and $Y \in \mathrm{Sym}_m^+(\mathbb{R})$, we define Epstein zeta functions (Definition 4.2.13):

$$\zeta(z, Y; p) = \sum_{u \in \mathbb{Z}^m \setminus \{0\}} \frac{p(u)}{({}^t u Y u)^z} \tag{1.8}$$

In Lemma 4.2.14 we follow the same steps as [2] (Theorem 3 of §1.5) to prove a meromorphic continuation and functional equation for these functions without the assumption that p is Y-spherical. This was likely known but was not written down due to being unwieldy, but it is necessary for us to input arbitrary polynomials into these Zeta functions. These zeta functions are used to express $\Phi_{f,1}$.

We then examine the rank 2 pieces, $\Phi_{f,2}$. We show that on a region of $\mathcal{D}$ where the minimal eigenvalue of Y is bounded below by some number depending on f that $\Phi_{f,2}$ defines a real analytic function of ξ. On that region we have:

Theorem 1.0.6 (Theorem 4.2.26)**.** *Suppose that the weakly holomorphic modular form f has the Fourier expansion $f(\tau) = \sum_{n \geq -n_0} a_n e^{2\pi i n \tau}$. The on a region of the form*

$$\mathcal{R} = \{\xi \in \mathcal{D} :\ {}^t u Y u > C_f \text{ for all } u \in \mathbb{Z}\setminus\{0\}\}$$

(where C_f is some positive constant depending on f), the regularized lift $\Phi_f : \mathcal{D} \to \mathcal{V}_{\rho_\kappa}$ (4.2.3) has the Fourier expansion:

$$\Phi_f(\xi; p) = a_0 \frac{1}{\pi} \zeta(1, Y; p) + 2 \sum_{\substack{S_0 \in \mathcal{S}^1 \\ \mu > 0}} e^{\pi i \mu \operatorname{tr} X S_0} p(\eta(Y, w_0)) y_1(Y, w_0) e^{-2\pi \mu y_1(Y, w_0)} \sum_{n=-\infty}^{\infty} c_{\mu,n} e^{2\pi i n \tau_2(\xi, S_0)}$$

where $\zeta(1, Y; p)$ is the Epstein zeta function (where we take the constant term at $z = 1$ in case $m = 2$ and $\zeta(z, Y; p)$ has a pole at $z = 1$), and $c_{\mu,n}$ are some coefficients given as finite sums of the coefficients of f.

Under the correspondence for modular forms on $\mathcal{D}_2$ and $\mathcal{H}_1 \times \mathcal{H}_1$ these together give what Borcherds calls the Singular Shimura Correspondence in Chapter 14 of [1].

Theorem 1.0.7 (Theorem 4.2.27)**.** *When f is a weakly holomorphic modular form, the Singular Shimura Correspondence to $O(2, 2)$ is:*

$$F_f(\tau_1, \tau_2) = a_0 \frac{(\kappa - 1)!\zeta(\kappa)}{i^\kappa \pi^\kappa} E_\kappa(\tau_2) + 2^{\kappa+1} \sum_{\substack{m > 0 \\ n \geq -n_0}} c_{m,n} e^{2\pi i n \tau_1} e^{2\pi i m \tau_2} \tag{1.9}$$

where $c_{m,n}$ are some coefficients defined in terms of the coefficients of f, and E_κ is the weight κ holomorphic Eisenstein series. This is a modular form of weight (κ, κ) on $\mathcal{H}_1 \times \mathcal{H}_1$, with singularities that are poles of weight κ along the divisors $\tau_1 = \gamma \tau_2$ for $\gamma \in M_2(\mathbb{Z})$ with $\det \gamma = n > 0$ and the $-n$ Fourier coefficient of f non-zero.

This theorem exists implicitly in [1], however it is not given explicitly for the correspondence to $O(2,2)$, and we come about this formula in a different way.

In Chapter 5 we will pair the theta functions with Siegel cusp forms, where we will be able to lift arbitrary holomorphic Siegel cusp forms, including the more exotic vector valued ones where $\mathcal{V}_\kappa$ is not a one dimensional representation of $GL_n(\mathbb{C})$. To calculate the Fourier coefficients $a_S(Y)$ of Φ_f, we follow a similar two step unfolding procedure as in the previous chapter. First we show that all orbits except for those with rank $w = 2n$ integrate to 0 in Lemmas 5.1.7 and 5.1.9. This is crucial in the proof of Theorem 1.0.4 and in the calculation of the Fourier coefficients for the lift to $O(2n, 2n)$, which are then obtained by the sum of a finite number of integrals over $\mathcal{H}_n$. The modularity of Φ_f implies relationships between the Fourier coefficients, and so it is sufficient to calculate $a_D(Y) := a_{J(D)}(Y)$, where $J(D) = \begin{pmatrix} & D \\ -D & \end{pmatrix}$ for D diagonal and integral. For $D_0 = \operatorname{diag}(d_1, \dots, d_{n-1}, 1)$, with $d_{n-1}| \dots |d_1$, we define:

$$\phi_{f,D_0}(\xi) = \sum_{d=1}^{\infty} e^{\pi i \operatorname{tr} dJ(D_0)X} a_{dD_0}(Y), \tag{1.10}$$

and then we have another expression for Φ_f:

$$\Phi_f(\xi) = \sum_{D_0} \sum_{A \in GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)} \rho_\kappa(A) \phi_{f,D_0}(\xi[A])$$

Where D_0 ranges over matrices as given above, and

$$Sp_n(\mathbb{Z}; D_0) = \{\gamma \in GL_{2n}(\mathbb{Z}) : \ J(D_0)[{}^t\gamma] = J(D_0)\}$$

The second step of the unfolding procedure involves grouping all of the integrals giving $a_D(Y)$ for the effect of acting on f by a Hecke operator. We define a particular integral $I(Y; f; P)$

$$I(Y; f; P) = \int_{\mathcal{H}} P(f(\tau))(\eta_\tau(1)) e^{-\pi \operatorname{tr} Y(1,1)_\tau} \det y^{-1-n} dx dy$$

We have that $a_1(Y; P) = 2I(Y; f; P)$, and more generally:

Lemma 1.0.8 (Lemma 5.3.4)**.**

$$a_D(Y; P) = 2\mu(D)^{n(n+1)/2} I\Big(T(D)f;\ \mu(D)Y_{D'};\ P_{D'}\Big) \tag{1.11}$$

where $T(D)$ is the Hecke operator associated to the matrix D (Definition 3.2.7), and Y_{D_0} and P_{D_0} are some slightly modified versions of Y and P depending on D_0. When $D = dI_n$ is a scalar matrix these are simply the usual degree d Hecke operators, and when f is a Hecke eigenform the Hecke eigenvalues of f will appear in this way.

When $n = 2$ we are able to completely evaluate $I(Y; f; P)$ to obtain explicit formulas for $a_D(Y)$. The key integration is performed in Lemma 5.3.9. Despite being significantly more complicated than the main integration in chapter 4 in Lemma 4.1.12, it follows a similar path. In this case matrix argument K-Bessel functions come up, and again there is a useful special form for $K^{(2)}_{1/2}(z)$ due to Herz [3]. It is also a key feature here that the N that appears as the summation index for the modular form $f(\tau) = \sum_N a_N e^{2\pi i \operatorname{tr} N\tau}$ appears linearly inside an exponential in the final formula, allowing for us to deal with polynomial factor cleanly 5.3.12. The integration involved in the lift from $n = 2$ is similar in many key aspects to the lift from $n = 1$, which leads to hope that the pattern can be extended to $n > 2$. Unfortunately we are not yet able to work out the integral when $n > 2$, stopping further progress in calculating the Fourier coefficients for $n > 2$.

Theorem 1.0.9 (Theorem 5.3.13)**.** *Suppose that f scalar is a Hecke eigenform of genus 2 with Hecke eigenvalues $T(d)f = \lambda(d)f$. Then:*

$$a_{dI_2}(Y; p) = \lambda(d)\frac{e^{-2\pi d \operatorname{tr}|M(Y)|}}{2\det Y^{1/2}\operatorname{tr}|M(Y)|}\mathcal{E}(Y; p)f\big(\tau(Y)\big)$$

where $\tau(Y)$ and $M(Y)$ are certain $\mathcal{H}_2$ and $M_2(\mathbb{R})$ valued functions of Y (Definition 5.3.7), and $\mathcal{E}(Y; p)$ is a function given by evaluating the polynomial p at a matrix $\eta(Y)$ depending on Y (Definition 5.3.11).

The functions $\phi_{f,D_0}(\xi)$ have a relationship to the Spin L-function of f.

Theorem 1.0.10 (Theorem 5.3.19)**.** *For Y and p fixed, consider $t^3\phi_{I_2}(tY; p)$ as a function of $t \in \mathbb{R}_{>0}$. Its Mellin transform with respect to t is*

$$\mathcal{M}_t(t^3\phi_{I_2}(tY; p))(s) = \frac{f(\tau(Y))\mathcal{E}(Y; p)}{2\det Y^{1/2}\operatorname{tr}|M(Y)|} \times \frac{\Gamma(s)L(s; f; \mathrm{spin})}{(2\pi \operatorname{tr}|M(Y)|)^s\zeta(2s - 2\kappa + 4)} \tag{1.12}$$

Where $L(s, f; Spin)$ is the Spin L-function of f.

We hope that deeper investigation of the Hecke operators $T(D)$ for D_0 not equal to I_2 can reveal similar formulas for ϕ_{f,D_0}, possibly involving other L-functions of f.

This work takes place within the wider context of the Theta correspondence where extensive work has been done for a few decades at least. This thesis distinguishes itself from the existing work by its focus on obtaining explicit formulas for the functions obtained via the theta lift. In the literature most work is framed in terms of the representation theoretic aspect of the correspondence. Given a holomorphic cuspidal Hecke eigenform, f, on $\mathcal{H}_n$, this determines in a standard way a cuspidal automorphic representation of $Sp_n(\mathbb{A})$, π_f. This representation factors as $\pi_f = \bigotimes_{p\le\infty}\pi_{f,p}$. The Archimedean place, $\pi_{f,\infty}$ is a discrete series representation determined by the weight of f, and the p-adic factors are spherical representations whose Satake parameters are related to the Hecke eigenvalues of f. The corresponding representation, $\theta(\pi_f)$ of $O_{m,m}(\mathbb{A})$ under the Theta correspondence is then determined as the product of local correspondences: $\theta(\pi_f) = \bigotimes_{p\leq\infty}\theta(\pi_{f,p})$. It long understood how these correspondences go, see for example [4] for $p = \infty$ and [5] for $p < \infty$. This does not shed light on what sort of functions actually live inside of $\theta(\pi_f)$, however. This thesis hopes to shed light on an answer at least when $n = 1$ and $n = 2$.

It should also be noted that the vanishing of the lift to $O(m, m)$ for $m < 2n$ (part of theorem 1.0.4) is known due to the vanishing of the lift of the holomorphic discrete series from $Sp_n(\mathbb{R})$ to $O(m, m)$ with $m < 2n$.

Chapter 2

Symplectic and Orthogonal Groups

2.1 Orthogonal Groups

Denote by $Q = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$, and V the vector space $\mathbb{R}^{2m}$ (written as column vectors) with symmetric bilinear form given by Q which we will denote by $(v, v') = {}^t v Q v$. We will usually write elements $v \in V$ as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, so that $(v, v') = {}^t v_1 v_2' + {}^t v_2 v_1'$. We will also write $\mathbb{R}^{m,m}$ to denote the same space. Define the group G by:

$$G = O(m, m) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2m}(\mathbb{R}) : \ Q[g] = Q \right\} \tag{2.1}$$

the group of orthogonal transformations preserving the bilinear form on V. We will think of G as acting on vectors in V on the left. We mention this in contrast to later when we will define $Sp_n(\mathbb{R})$ that will be thought of as acting on row vectors on the right. We will define P to be the maximal parabolic subgroup stabilizing the maximal isotropic subspace spanned by the columns of $\begin{pmatrix} 1_m \\ 1 \end{pmatrix}$. P has a Levi decomposition $P = NM$ with:

$$\begin{aligned} M &= \left\{ m(\alpha) = \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix} : \ \alpha \in GL_m(\mathbb{R}) \right\}, \\ N &= \left\{ n(\beta) = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} : \ \beta \in \mathrm{Skew}_m(\mathbb{R}) \right\} \end{aligned} \tag{2.2}$$

Where $\mathrm{Skew}_m(\mathbb{R})$ denotes the set of skew symmetric matrices with entries in $\mathbb{R}$. We also define the maximal compact subgroup:

$$K = \left\{ (k_+, k_-) = \frac{1}{2} \begin{pmatrix} k_+ + k_- & k_+ - k_- \\ k_+ - k_- & k_+ + k_- \end{pmatrix} : \ k_+, k_- \in O(m) \right\}. \tag{2.3}$$

We have $K = K^+ \times K^-$ where $K^\pm$ is the subgroup with $k_\mp = 1$, with both factors isomorphic to $O(m)$. We define $K^\Delta \subset K$ as the subgroup with $k^+ = k^-$, and we have $K^\Delta = K \cap M$.

Definition 2.1.1 (The Symmetric Space $\mathcal{D}_m$)**.** *Define $\mathcal{D}_m$ to be:*

$$\mathcal{D}_m = \{\xi = X + Y \in M_m(\mathbb{R}) : \; X \in \mathrm{Skew}_m(\mathbb{R}), \; Y \in \mathrm{Sym}_m^+(\mathbb{R})\} \tag{2.4}$$

Where $\mathrm{Sym}_m^+(\mathbb{R})$ denotes the set of positive definite symmetric matrices. For $\xi \in \mathcal{D}_m$, write:

$$\xi^\pm = X \pm Y$$

We will write $\mathcal{D}$ without the subscript if we are not worried about ambiguity.

Lemma 2.1.2. *Points $\xi \in \mathcal{D}$ correspond to pairs (V_ξ^+, V_ξ^-) where $V_\xi^\pm$ are maximal $\pm$-definite subspaces of V, are mutually perpendicular, and we have that $V = V_\xi^+ \oplus V_\xi^-$ is an orthogonal decomposition of V. The space $V_\xi^\pm$ is given by:*

$$V_\xi^\pm = \mathrm{colspan}\begin{pmatrix} \xi^\pm \\ 1 \end{pmatrix} \tag{2.5}$$

Proof. Suppose first that we have a maximal positive definite subspace of V. Choose some basis of that space and write it as the columns of the matrix $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Consider now the linear (but not orthogonal) projection $V \to \mathbb{R}^m$ given by $\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} \mapsto v_2'$. Its kernel is spanned by the columns of $\begin{pmatrix} 1_m \\ 0 \end{pmatrix}$, which is isotropic. Thus the kernel of this map and our positive definite subspace must intersect only trivially, so that we conclude that the projection is surjective, and so v_2 is invertible. Then the columns of $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} v_2^{-1} = \begin{pmatrix} v_1 v_2^{-1} \\ 1 \end{pmatrix}$ also span the same maximal positive definite subspace, as they are an invertible linear combination of the original columns. Thus we have seen that any maximal positive definite subspace can be given as the column span of a matrix of the form $\begin{pmatrix} \xi \\ 1 \end{pmatrix}$ for some $\xi \in M_m(\mathbb{R})$.

Now we will consider which such $\xi = X+Y$ with $X \in \mathrm{Skew}_m(\mathbb{R})$ and $Y \in \mathrm{Sym}_m(\mathbb{R})$ may appear. As the space is maximal positive definite, the matrix of bilinear products between the basis vectors must be a positive definite matrix. This product is $\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix}, \begin{pmatrix} \xi \\ 1 \end{pmatrix}\right) = \xi + {}^t\xi = 2Y$. Thus the condition is that Y must be positive definite, i.e. $Y \in \mathrm{Sym}_m^+(\mathbb{R})$. Likewise the columns span of any matrix of the form $\begin{pmatrix} \xi \\ 1 \end{pmatrix}$ will be a maximal positive definite subspace so long as $Y > 0$, and this subspace will be uniquely specified by ξ.

Thus suppose we have some maximal positive definite subspace V_ξ^+ specified. Denote by V_ξ^- its perpendicular, which will be a maximal negative definite subspace. We can see that $\left(\begin{pmatrix}\xi^+\\1\end{pmatrix}, \begin{pmatrix}\xi^-\\1\end{pmatrix}\right) = {}^t\xi^+ + \xi^- = 0$, so that V_ξ^- is spanned by the columns of $\begin{pmatrix}\xi^-\\1\end{pmatrix}$. □

From the interpretation of $\mathcal{D}$ given in the previous lemma, $\mathcal{D}$ inherits an action of G via the action of G on subspaces of V.

Lemma 2.1.3. *The action of G on $\mathcal{D}$ is given by:*

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\cdot\xi = (a\xi+b)(c\xi+d)^{-1} \tag{2.6}$$

Proof. From the previous lemma we write $V_\xi^+ = \text{colspan}\begin{pmatrix}\xi\\1\end{pmatrix}$. Then we have:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\left(\text{colspan}\begin{pmatrix}\xi\\1\end{pmatrix}\right) = \text{colspan}\begin{pmatrix}a\xi+b\\c\xi+d\end{pmatrix} = \text{colspan}\begin{pmatrix}(a\xi+b)(c\xi+d)^{-1}\\1\end{pmatrix}$$

□

We will record the action of some particular elements of G on $\mathcal{D}$:

$$\begin{aligned} m(\alpha)\cdot\xi &= \alpha\xi\,{}^t\alpha = \xi[{}^t\alpha] \\ n(\beta)\cdot\xi &= \xi+\beta \\ Q\cdot\xi &= \xi^{-1} \end{aligned} \tag{2.7}$$

and also

$$K = \text{stab}\,_G(1) \tag{2.8}$$

Define a subgroup $B_{0,0}\subset GL_m(\mathbb{R})$ as:

$$B_{0,0} = \{\alpha\in GL_m(\mathbb{R}):\ \alpha \text{ is lower triangular with positive diagonal entries}\} \tag{2.9}$$

And define a subgroup $B_0\subset P$:

$$B_0 = \{n(\beta)m(\alpha):\ \beta\in\text{Skew}_m(\mathbb{R}),\ \alpha\in B_{0,0}\} \tag{2.10}$$

We have that $B_0\cap K = \{1\}$, and $G = B_0K$, so that every $g\in G$ can be uniquely written as $g = bk$ with $b\in B_0, k\in K$. We also obtain an identification of B_0 with $\mathcal{D}$:

Definition 2.1.4 (g_ξ)**.** *For $\xi = X+Y\in\mathcal{D}$, let α be the unique $\alpha\in B_{0,0}$ such that $\alpha\,{}^t\alpha = Y$.*

Then define:

$$g_\xi = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix} \tag{2.11}$$

so that we have $g_\xi \cdot 1 = \xi$.

The group G has 4 connected components, corresponding to the connected components of K, with $\pi_0(G) \cong \{\pm 1\}^2$. The map sending $g \in G$ to the element of $\pi_0(G)$ corresponding to its connected component is given by $\pi_0(g) = (\det k_+, \det k_-)$ for $g = bk$ with $b \in B_0$ and $k \in K$. We will write $G_0 = SO_0(m,m)$ for the connected component of the identity. For $m(\alpha) \in M$, we have $\pi_0(m(\alpha)) = (\operatorname{sgn}\det\alpha, \operatorname{sgn}\det\alpha)$, and we have $\pi_0(Q) = (1, (-1)^m)$.

We define $\Gamma \subset G$ to be the subgroup of integral matrices:

$$\Gamma = GL_{2m}(\mathbb{Z}) \cap O(m,m). \tag{2.12}$$

Equivalently, Γ is the stabilizer of the lattice $\mathbb{Z}^{2m} \subset \mathbb{R}^{2m}$. Define subgroups of Γ: $P_\Gamma = \Gamma \cap P$, $M_\Gamma = \Gamma \cap M$, and $N_\Gamma = \Gamma \cap N$, so that:

$$\begin{aligned} M_\Gamma &= \left\{ \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix} : \alpha \in GL_m(\mathbb{Z}) \right\} \\ N_\Gamma &= \left\{ \begin{pmatrix} 1 & S \\ & 1 \end{pmatrix} : S \in \mathrm{Skew}_m(\mathbb{Z}) \right\} \end{aligned} \tag{2.13}$$

and $P_\Gamma = N_\Gamma M_\Gamma$. We will also define some elements related to the Weyl group of G. For $I \subseteq \{1, \dots, m\}$, define 1_I to be the $m \times m$ matrix:

$$(1_I)_{ij} = \begin{cases} 1 & i = j, \text{ and } i \in I \\ 0 & \text{otherwise} \end{cases}$$

This is an $m \times m$ identity matrix with diagonal entries whose indices are in I replaced by 0. Let $\overline{I} = \{1, \dots, m\} - I$. Define $Q_I \in O(m,m)$ by:

$$Q_I = \begin{pmatrix} 1_{\overline{I}} & 1_I \\ 1_I & 1_{\overline{I}} \end{pmatrix} \tag{2.14}$$

When $I = \emptyset$ have $Q_I = 1$, and when $I = \{1, \dots, m\}$ we have $Q_I = Q$. We have $\pi_0(Q) = (1, (-1)^{|I|})$.

Lemma 2.1.5. Γ *is generated by the subgroups* M_Γ *and* N_Γ*, and the elements* Q_I*. More specifically* Γ *is generated by the subgroups* M_Γ*,* N_Γ*, and the element* Q_I *with* $I = \{1\}$*.*

Proof. Suppose that we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We may multiply on the left and right by elements of M_Γ so that w.l.o.g. we may assume that c is in Smith normal form, so that the matrix is:

$$\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ D & 0 & d_{11} & d_{12} \\ 0 & 0 & d_{21} & d_{22} \end{pmatrix}$$

with $D = \mathrm{diag}(c_1, \dots, c_r)$ with $c_r | \dots | c_1$, each positive and non-zero. As this matrix is in $O(m, m)$, we may infer that $a_{11} = a_{12} = b_{12} = 0$, and $Db_{11} = 1_r$. Thus we have that $D = 1_r$ as b_{11} and D are integral. Thus the matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ a_{12} & a_{22} & b_{12} & b_{22} \\ 1 & 0 & d_{11} & d_{12} \\ 0 & 0 & d_{21} & d_{22} \end{pmatrix}$$

after which left multiplication by Q_I with $I = \{1, \dots, r\}$ puts it in M_Γ. Finally to show that just $Q_{\{1\}}$ is sufficient, if σ is a permutation of $\{1, \dots, m\}$ we will identify it with its $m \times m$ permutation matrix. Then we have that conjugating Q_I by $m(\sigma)$ gives $Q_{\sigma I}$. Thus from $Q_{\{1\}}$ and these permutation matrices we can obtain all other Q_I with I a singleton. Finally we have that $Q_I Q_{I'} = Q_{I \triangle I'}$ where here we use $I \triangle I'$ for the symmetric difference of I and I', so that we can obtain any Q_I. $\square$

2.1.1 Factors of Automorphy and Majorized Inner Products

Definition 2.1.6 (Factors of Automorphy for the action of G on $\mathcal{D}$ valued in $GL_m(\mathbb{R})$)**.** *For* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ *and* $\xi \in \mathcal{D}$*, define:*

$$j^{\pm}(g, \xi) = c\xi^{\pm} + d \tag{2.15}$$

(recall $\xi^{\pm} = X \pm Y$*).*

Remark 2.1.7. *These functions* $j^{\pm}(g, \xi)$ *are analogous to the standard factor of automorphy (and its complex conjugate) for the action of* $Sp_n(\mathbb{R})$ *on* $\mathcal{H}_n$*, the Siegel upper half space, which we will review in the next section.*

Lemma 2.1.8. $j^{\pm}(g, \xi)$ *are factors of automorphy for the action of* G *on* $\mathcal{D}$*. That is we have:*

$$j^{\pm}(gg', \xi) = j^{\pm}(g, g'\xi) j^{\pm}(g', \xi) \tag{2.16}$$

Proof. Consider the action of a $g \in G$ on the matrix $\begin{pmatrix} \xi^{\pm} \\ 1 \end{pmatrix}$. We have that

$$g \begin{pmatrix} \xi^{\pm} \\ 1 \end{pmatrix} = \begin{pmatrix} a\xi^{\pm} + b \\ c\xi^{\pm} + d \end{pmatrix} = \begin{pmatrix} (g\xi)^{\pm} \\ 1 \end{pmatrix} j^{\pm}(g, \xi).$$

It follows that we have:

$$\begin{pmatrix} ((gg')\xi)^{\pm} \\ 1 \end{pmatrix} j^{\pm}(gg',\xi) = \begin{pmatrix} (g(g'\xi))^{\pm} \\ 1 \end{pmatrix} j^{\pm}(g,g'\xi) j^{\pm}(g',\xi)$$

and so (2.16) follows as $\begin{pmatrix} (gg'\xi)^{\pm} \\ 1 \end{pmatrix}$ is full rank. □

We will record some special values for $j^{\pm}(g,\xi)$ below:

Lemma 2.1.9. *Suppose that $k = (k_+, k_-) \in K$, $m(\alpha) \in M$ for $\alpha \in GL_m(\mathbb{R})$, and $n(\beta) \in N$ for $\beta \in \mathrm{Skew}_m(\mathbb{R})$. Then:*

$$\begin{aligned} j^{\pm}(k,1) &= k_{\pm} \\ j^{\pm}(m(\alpha),\xi) &= {}^t\alpha^{-1} \\ j^{\pm}(n(\beta),\xi) &= 1 \\ j^{\pm}(Q,\xi) &= (\xi^{\pm})^{-1} \end{aligned} \tag{2.17}$$

Remark 2.1.10. *We can ask as well what component of $GL_m(\mathbb{R})$ these automorphy factors live in. Recall that $\pi_0(GL_m(\mathbb{R})) = \{\pm 1\}$, with the map given by $\pi_0(\alpha) = \operatorname{sgn}\det\alpha$. The maps $j^{\pm} : G \times \mathcal{D}$ are continuous, so that we have that $\pi_0 \circ j^{\pm}$ is constant on the connected components of G. From the top line of (2.17) we have that $(\det j^+(g,\xi), \det j^-(g,\xi)) = \pi_0(g)$. We also have a formula for evaluating $\pi(j^{\pm}(g,\xi))$ directly by evaluating at $\xi = 1$, where we obtain that:*

$$\operatorname{sgn}\det j^{\pm}(g,\xi) = \operatorname{sgn}\det(d \pm c) \tag{2.18}$$

There are a number of natural functions associated to $\xi \in \mathcal{D}$ that have transformation properties with respect to these $j^{\pm}$.

Lemma 2.1.11. *The function $\xi \mapsto Y$ on $\mathcal{D}$ transforms as:*

$$Y(g\xi) = Y(\xi)[j^{\pm}(g,\xi)^{-1}] \tag{2.19}$$

where $\pm$ should be chosen consistently in this formula. One consequence of this formula is:

$$|\det j^+(g,\xi)| = |\det j^-(g,\xi)|. \tag{2.20}$$

Proof.

$$\left(\begin{pmatrix} \xi^{\pm} \\ 1 \end{pmatrix}, \begin{pmatrix} \xi^{\pm} \\ 1 \end{pmatrix} \right) = \pm 2Y(\xi).$$

Thus we have

$$
\begin{aligned}
\pm 2Y(g\cdot\xi) &= \left(\begin{pmatrix}(g\xi)^{\pm}\\ 1\end{pmatrix}, \begin{pmatrix}(g\xi)^{\pm}\\ 1\end{pmatrix}\right)\\
&= {}^{t}j^{\pm}(g,\xi)^{-1}\left(\begin{pmatrix}\xi^{\pm}\\ 1\end{pmatrix}, \begin{pmatrix}\xi^{\pm}\\ 1\end{pmatrix}\right) j^{\pm}(g,\xi)^{-1}\\
&= \pm 2Y(\xi)[j^{\pm}(g,\xi)^{-1}]
\end{aligned}
$$

using that $(gv, gv') = (v, v')$. □

We now define some functions from V to $\mathbb{R}^m$ that depend on ξ.

Definition 2.1.12 ($\nu_\xi^{\pm}$)**.** *For $\xi \in \mathcal{D}$, define functions $\nu_\xi^{\pm} : V \to \mathbb{R}^m$ by:*

$$\nu_\xi^{\pm}(v) := \left(\begin{pmatrix}\xi^{\pm}\\ 1\end{pmatrix}, v\right) = v_1 + {}^{t}\xi^{\pm} v_2 = v_1 + (\pm Y - X)v_2 \tag{2.21}$$

These functions are the products v with the basis of $V_\xi^{\pm}$ we have defined (2.5). They satisfy a transformation property with respect to G:

$$\nu_\xi^{\pm}(g^{-1}v) = {}^{t}j^{\pm}(g,\xi)\nu_{g\xi}^{\pm}(v) \tag{2.22}$$

Denote by $\mathrm{pr}_\xi^{\pm}$ the orthogonal projection to $V_\xi^{\pm}$. We can express this in terms of $\nu_\xi^{\pm}$:

$$\mathrm{pr}_\xi^{\pm}(v) = \pm\frac{1}{2}\begin{pmatrix}\xi^{\pm}\\ 1\end{pmatrix} Y^{-1}\nu_\xi^{\pm}(v) \tag{2.23}$$

Definition 2.1.13 (Majorized Bilinear Form on V)**.** *For $\xi \in \mathcal{D}$, define a positive definite symmetric bilinear form by:*

$$(v, v')_\xi = (v, v')_\xi^{+} + (v, v')_\xi^{-} \tag{2.24}$$

where $(v, v')_\xi^{\pm}$ are the positive semi-definite bilinear forms defined by:

$$(v, v')_\xi^{\pm} = \pm(\mathrm{pr}_\xi^{\pm}(v), \mathrm{pr}_\xi^{\pm}(v')) \tag{2.25}$$

We have that $(v, v') = (v, v')_\xi^{+} - (v, v')_\xi^{-}$, so that another way to write $(v, v')_\xi$ is

$$(v, v')_\xi = (v, v') + 2(v, v')_\xi^{-}$$

These can be expressed in terms of $\nu_\xi^{\pm}$ as well:

$$(v, v')_\xi^{\pm} = \frac{1}{2}\,{}^{t}\nu_\xi^{\pm}(v) Y^{-1}\nu_\xi^{\pm}(v') \tag{2.26}$$

and

$$\begin{aligned}(v,v)_\xi &= \frac{1}{2}\left({}^t\nu_\xi^+(v)Y^{-1}\nu_\xi^+(v) + {}^t\nu_\xi^-(v)Y^{-1}\nu_\xi^-(v)\right)\\ &= {}^t(v_1 - Xv_2)Y^{-1}(v_1 - Xv_2) + {}^tv_2Yv_2\end{aligned} \tag{2.27}$$

These bilinear forms satisfy a transformation property with respect to the action of G:

$$\begin{aligned}(g^{-1}v, g^{-1}v')_\xi &= (v,v')_{g\xi},\\ (g^{-1}v, g^{-1}v')_\xi^\pm &= (v,v')_{g\xi}^\pm\end{aligned} \tag{2.28}$$

We also extend $\nu_\xi^\pm$ and all the products associated to V to elements in $M_{2m,n}(\mathbb{R}) \cong V^n$, thought of as row vectors of elements of V. Then for $v, v' \in V^n$, (v,v'), $(v,v')_\xi$, etc. will be $n \times n$ matrices, and $\nu_\xi^\pm(v)$ will be an $m \times n$ matrix. When we do this we have that (2.22), (2.27), and (2.28) all continue to hold.

We will also define some factors of automorphy for the action of G on $\mathcal{D}$ that are valued in K. These are the standard K valued factors of automorphy for $\mathcal{D} \cong G/K$ associated to the Cartan decomposition.

Before considering G and $\mathcal{D}$, first consider the (transitive) action of $GL_m(\mathbb{R})$ on $\mathrm{Sym}_m^+(\mathbb{R})$ given by $A\cdot Y = Y[{}^tA]$. For each Y there is a unique $\alpha(Y) \in B_{0,0}$ such that $\alpha(Y)\,{}^t\alpha(Y) = Y$. We have that $B_{0,0} \cap O(m) = \{1\}$ and $GL_m(\mathbb{R}) = B_{0,0}O(m)$ so that any $A \in GL_m(\mathbb{R})$ can be uniquely written as $A = \alpha K$ with $\alpha \in B_{0,0}$ and $k \in O(m)$. Thus we define:

Definition 2.1.14 (Factor of Automorphy for the action of $GL_m(\mathbb{R})$ on $\mathrm{Sym}_m^+(\mathbb{R})$)**.** *For $A \in GL_m(\mathbb{R})$ and $Y \in \mathrm{Sym}_m^+(\mathbb{R})$, define $k(A,Y) \in O(m)$ by:*

$$A\alpha(Y) = \alpha(A\cdot Y)k(A,Y) \tag{2.29}$$

It is straightforward to check that this is a factor of automorphy, i.e. that $k(AA',Y) = k(A, A'\cdot Y)k(A',Y)$.

In the same way we define a K valued factor of automorphy for the action of G on $\mathcal{D}$:

Definition 2.1.15 (Factor of Automorphy for the action of G on $\mathcal{D}$)**.** *For $g \in G$ and $\xi \in \mathcal{D}$, define $k(g,\xi) \in K$ by:*

$$gg_\xi = g_{g\cdot\xi}k(g,\xi) \tag{2.30}$$

and define $k^\pm(g,\xi) \in K^\pm$ by

$$k(g,\xi) = (k^+(g,\xi), k^-(g,\xi)) \tag{2.31}$$

We will record some special values for these factors of automorphy:

Lemma 2.1.16. *Suppose that* $k = (k_+, k_-) \in K$*,* $m(\alpha) \in M$*,* $n(\beta) \in N$*, and* $\xi \in \mathcal{D}$*. Then:*

$$\begin{aligned} k(k, 1) &= k \\ k^{\pm}(k, 1) &= k_{\pm} \\ k(m(\alpha), \xi)) &= (k(\alpha, Y), k(\alpha, Y)) \\ k(n(\beta), \xi) &= 1 \end{aligned}$$

We also have a way to relate the two pairs of factors of automorphy that we have defined for the action of G on $\mathcal{D}$.

Proposition 2.1.17. *The factors of automorphy* $j^{\pm}(g, \xi)$ *and* $k^{\pm}(g, \xi)$ *have the following relationship with each other:*

$$ {}^t\alpha(g\xi) j^{\pm}(g, \xi) = k^{\pm}(g, \xi)\, {}^t\alpha(\xi) \tag{2.32}$$

Proof. We have that:

$$\begin{aligned} j^{\pm}(g, \xi) &= j^{\pm}(g, g_\xi \cdot 1) \\ &= j^{\pm}(g g_\xi, 1) j^{\pm}(g_\xi, 1)^{-1} \\ &= j^{\pm}(g_{g\xi}, 1) j^{\pm}(k(g, \xi), 1) j^{\pm}(g_\xi, 1)^{-1} \end{aligned}$$

From (2.17) we have that this is ${}^t\alpha(g\xi)^{-1} k^{\pm}(g, \xi)\, {}^t\alpha(\xi)$, which gives the result. □

2.2 Symplectic Groups

Denote by $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and by W the vector space isomorphic to $\mathbb{R}^{2n}$, written as row vectors, with the alternating form $\langle w, w' \rangle = wJ\, {}^t w'$. We will write elements of W as $w = (w_1, w_2)$, and then $\langle w, w' \rangle = w_1\, {}^t w'_2 - w_2\, {}^t w'_1$. Write G' to be the group of symplectic linear transformations of this space:

$$G' = Sp_n(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(\mathbb{R}) : \; J[g] = J \right\}$$

which acts on W on the right. Define P' to be the Siegel parabolic subgroup, the maximal parabolic subgroup consisting of elements of the form $\begin{pmatrix} * & * \\ & * \end{pmatrix}$. We will define M' and N'

to be its subgroups:

$$M' = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} : \ a \in GL_n(\mathbb{R}) \right\}$$
$$N' = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : \ b \in \mathrm{Sym}_n(\mathbb{R}) \right\}$$

so that $P' = N'M'$. Let K' be the maximal compact subgroup given by elements of the form:

$$K' = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in G' \right\}$$

Let $\mathcal{H}_n$ be the genus n Siegel upper half space:

$$\mathcal{H}_n = \{\tau = x + iy : \ x \in \mathrm{Sym}_n(\mathbb{R}), \ y \in \mathrm{Sym}_n^+(\mathbb{R})\}$$

We will drop the subscript n unless we feel it necessary to avoid ambiguity. The group G' acts on $\mathcal{H}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1} \tag{2.33}$$

The subgroup K' is the stabilizer of $iI_n \in \mathcal{H}$. The action has a factor of automorphy valued in $GL_n(\mathbb{C})$ given by:

$$j(g, \tau) = c\tau + d \tag{2.34}$$

The factor of automorphy gives an isomorphism $K' \cong U(n)$, the group of unitary matrices in $GL_n(\mathbb{C})$, via $j(-, iI_n)$:

$$j\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, iI_n \right) = a - ib$$

Definition 2.2.1 (g_τ)**.** *For $\tau = x + iy \in \mathcal{H}$, let a be the unique element of $B_{0,0}$ such that $a\,{}^t a = y$. Then define:*

$$g_\tau = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \tag{2.35}$$

Define B_0' to be the subgroup of P' consisting of elements $n(b)m(a)$ for $a \in B_{0,0}$. The elements g_τ defined above give an identification between $\mathcal{H}$ and B_0'. We note that this allows to a way to define a factor of automorphy for the action of G' on $\mathcal{H}$ valued in K' by $gg_\tau = g_{g\tau}k(g, \tau)$.

Using $\tau \in \mathcal{H}_n$ we can define an identification between W and $\mathbb{C}^n$ as follows:

Definition 2.2.2 ($\eta_\tau(w)$ and $\overline{\eta}_\tau(w)$)**.** *For $w = (w_1, w_2) \in W$ and $\tau \in \mathcal{H}$, define:*

$$\eta_\tau(w) = w_1\tau + w_2, \qquad \overline{\eta}_\tau(w) = -w_1\overline{\tau} + w_2 \tag{2.36}$$

These functions satisfy transformation properties with respect to G':

$$\eta_\tau(wg) = \eta_{g\tau}(w)j(g,\tau), \qquad \overline{\eta}_\tau(wg) = \overline{\eta}_{g\tau}(w)\overline{j(g,\tau)} \tag{2.37}$$

For $\tau \in \mathcal{H}$, we also define an element in $J_\tau \in G'$:

$$J_\tau = g_\tau J g_\tau^{-1} \tag{2.38}$$

We have $gJ_\tau g^{-1} = J_{g\tau}$. The element J_τ gives a complex structure on W as $J_\tau^2 = -1$, and is compatible with the above identifications of W with $\mathbb{C}^m$:

$$\eta_\tau(wJ_\tau) = (-i)\eta_\tau(w), \qquad \overline{\eta}_\tau(wJ_\tau) = i\overline{\eta}_\tau(w) \tag{2.39}$$

They can also be used to form a positive definite inner product on W.

Definition 2.2.3 (Positive Definite Product on W)**.** *For $\tau \in \mathcal{H}$, define the positive definite product:*

$$(w, w')_\tau := \langle w, w'J_\tau\rangle = wg_\tau\, {}^t g_\tau\, {}^t w' \tag{2.40}$$

If $w = (w_1, w_2)$, then we have:

$$(w, w)_\tau = w_1 y\, {}^t w_1 + (w_1 x + w_2)y^{-1}\, {}^t(w_1 x + w_2)$$

This product has a transformation property given by:

$$(wg, w'g)_\tau = (w, w')_{g\tau} \tag{2.41}$$

We will also use η_τ and the products $\langle w, w'\rangle$ and $(w, w')_\tau$ to denote similar things on $W^m \cong M_{m,2n}(\mathbb{R})$, thought of as column vectors of m elements in W, so that $\eta_\tau(w)$ will be an $m \times n$ complex matrix, and $\langle w, w'\rangle$ and $(w, w')_\tau$ are $m \times m$ matrices. Then we have:

$$\eta_\tau(w)y^{-1}\, {}^t\overline{\eta}_\tau(w) = \langle w, w\rangle + i(w, w)_\tau$$

where $\langle w, w\rangle$ will be a skew symmetric matrix, and $(w, w)_\tau$ a positive semi-definite matrix.

We will also make use of the group $GSp_n(\mathbb{R})$:

$$GSp_n(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(\mathbb{R}) : \; J[g] = \mu(g)J \right\}$$

for a homomorphism $\mu : GSp_n(\mathbb{R}) \to \mathbb{R}^\times$ called the similitude of g. We will write $GSp_n^+(\mathbb{R})$ for the subgroup with $\mu(g) > 0$. $GSp_n^+(\mathbb{R})$ contains $G' = Sp_n(\mathbb{R})$ as the subgroup of elements with $\mu(g) = 1$. $GSp_n^+(\mathbb{R})$ acts on $\mathcal{H}$ via the same formula as (2.33), and has the same factor of automorphy as in (2.34). Formulas (2.37) and (2.38) also hold, and we have:

$$\langle wg, w'g\rangle = \mu(g)\langle w, w'\rangle, \qquad (wg, w'g)_\tau = \mu(g)(w, w')_{g\tau}$$

2.3 Exceptional Isomorphism $SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R}) \cong SO_0(2,2)$

Define two embeddings $\iota_1, \iota_2 : SL_2(\mathbb{R}) \hookrightarrow O(2,2)$ by:

$$\iota_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & & & -b \\ & a & b & \\ & c & d & \\ -c & & & d \end{pmatrix}$$
$$\iota_2(g) = \begin{pmatrix} {}^t g^{-1} & \\ & g \end{pmatrix}$$

These two embeddings commute, and so define a map $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ to $O(2,2)$. The kernel of this map is $\{(1,1), (-1,-1)\}$. Define $SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ to be the quotient of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ by this subgroup, so that $\iota = \iota_1 \times \iota_2$ defines an isomorphism of $SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ onto its image, which is $SO_0(2,2)$. The maximal compact subgroup $SO(2) \times_{\pm 1} SO(2)$ is carried into K, so that $\iota_1 \times \iota_2$ descends to a map $\mathcal{H}_1 \times \mathcal{H}_1 \cong \mathcal{D}_2$. The map can be described explicitly:

$$\begin{aligned} \xi(\tau_1, \tau_2) &= X(\tau_1) + Y(\tau_1, \tau_2) = -x_1 J + y_1 \, {}^t g_{\tau_2}^{-1} g_{\tau_2}^{-1} \\ &= \begin{pmatrix} & -x_1 \\ x_1 & \end{pmatrix} + \frac{y_1}{y_2}\begin{pmatrix} 1 & -x_2 \\ -x_2 & x_2^2 + y_2^2 \end{pmatrix}, \qquad (2.42) \\ (\tau_1(\xi), \tau_2(\xi)) &= \left(-X_{12} + i\sqrt{\det Y}, -\frac{Y_{12}}{Y_{11}} + i\frac{\sqrt{\det Y}}{Y_{11}}\right) \end{aligned}$$

Furthermore, we have:

$$\iota_1(g_{\tau_1})\iota_2(g_{\tau_2}) = g_{\xi(\tau_1,\tau_2)} \qquad (2.43)$$

it will also be worthwhile to record $\alpha(\tau_1, \tau_2)$, the unique lower triangular matrix with positive diagonal entries so that $\alpha(\tau_1, \tau_2)\, {}^t\alpha(\tau_1, \tau_2) = Y(\tau_1, \tau_2)$:

$$\alpha(\tau_1, \tau_2) = \sqrt{y_1}\, {}^t g_{\tau_2}^{-1} = \sqrt{\frac{y_1}{y_2}}\begin{pmatrix} 1 & \\ -x_2 & y_2 \end{pmatrix} \qquad (2.44)$$

The isomorphism can be described in another way. Consider the vector space $M_2(\mathbb{R})$ with the bilinear form defined by polarizing $(v, v) = 2\det v$. Then if we write $v =$

$\begin{pmatrix} v_{11} & v_{12} \\ -v_{22} & v_{21} \end{pmatrix}$, we have $\det v = v_{11}v_{21} + v_{12}v_{22}$, giving an isometry with $V = \mathbb{R}^{2,2}$. $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ acts on $M_2(\mathbb{R})$ by $(g_1, g_2)v = g_1 v g_2^{-1}$, which is identified with the embeddings ι_1, ι_2 under the isometry. From this it is clear that the isomorphism takes $SL_2(\mathbb{Z}) \times_{\pm 1} SL_2(\mathbb{Z})$ into $SO_0(2,2) \cap \Gamma$.

As mentioned the image of $SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ is only the connected component of the identity of $O(2,2)$, and it it worth recording how the other connected components of $O(2,2)$ act on $\mathcal{H}_1 \times \mathcal{H}_1$. Write $\epsilon \in O(2)$ for the element:

$$\epsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

We have the elements $(\epsilon, \epsilon), (\epsilon, 1) \in K$ in the $(-1,-1)$ and $(-1,1)$ components of G, respectively, and their action in terms of τ is given by:

$$\begin{aligned} (\epsilon, \epsilon) \cdot \xi(\tau_1, \tau_2) &= \xi(-\overline{\tau}_1, -\overline{\tau}_2) \\ (\epsilon, 1) \cdot \xi(\tau_1, \tau_2) &= \xi(\tau_2, \tau_1) \end{aligned} \tag{2.45}$$

As mentioned above, the maximal compact subgroup $SO(2) \times_{\pm 1} SO(2) \subset SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ is mapped isomorphically onto $K_0 \cong SO(2) \times SO(2) \subset SO_0(2,2)$. Writing an element of $SO(2)$ as:

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

we have that:

$$\iota(k_{\theta_1}, k_{\theta_2}) = (k_{\theta_2 - \theta_1}, k_{\theta_1 + \theta_2}) \tag{2.46}$$

There are potentially 3 different choices of factors of automorphy that could apply. First there is the standard factor of automorphy on $\mathcal{H}_1 \times \mathcal{H}_1$ for the action of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ given by $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$ in each factor. Then there is also the factor of automorphy valued in $K \cong O(2) \times O(2)$ for the action of $O(2,2)$ on $\mathcal{D}_2$, and the factors $j^{\pm}$ valued in $GL_2(\mathbb{R})$.

To begin with, given the standard factor of automorphy on $\mathcal{H}_1$ for the action of $SL_2(\mathbb{R})$, we can construct a $U(1) \cong SO(2)$ valued factor by:

$$\chi(g, \tau) = \frac{j(g, \tau)}{|j(g, \tau)|}$$

We will fix an isomorphism between $U(1)$ and $SO(2)$ such that $e^{i\theta}$ is identified with k_θ. We will write $k(e^{i\theta}) = k_\theta$ for one direction and $\chi(k_\theta) = e^{i\theta}$ for the other. Recall that we have

$gg_\tau = g_{g\tau}k(g,\tau)$ for some $k(g,\tau) \in SO(2)$. This is given specifically by:

$$gg_\tau = g_{g\tau}k\left(\frac{j(g,\tau)}{|j(g,\tau)|}\right)^{-1},$$

and matrix $k\left(\frac{j(g,\tau)}{|j(g,\tau)|}\right)$ is equal to:

$$k\left(\frac{j(g,\tau)}{|j(g,\tau)|}\right) = \frac{1}{|j(g,\tau)|}\begin{pmatrix} cx+d & cy \\ -cy & cx+d \end{pmatrix}$$

We can thus describe the relationship between k on $\mathcal{D}_2$ and (j_1, j_2) on $\mathcal{H}_1 \times \mathcal{H}_1$, in the following:

Lemma 2.3.1. *for $(g_1, g_2) \in SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ and $(\tau_1, \tau_2) \in \mathcal{H}_1 \times \mathcal{H}_1$, we have:*

$$\begin{aligned} &k\big(\iota(g_1,g_2), \xi(\tau_1,\tau_2)\big) \\ &\quad = \left(k\left(\frac{j(g_1,\tau_1)}{|j(g_1,\tau_1)|}\right)k\left(\frac{j(g_2,\tau_2)}{|j(g_2,\tau_2)|}\right)^{-1},\ k\left(\frac{j(g_1,\tau_1)}{|j(g_1,\tau_1)|}\right)^{-1}\left(\frac{j(g_2,\tau_2)}{|j(g_2,\tau_2)|}\right)^{-1}\right) \end{aligned}$$

Proof. As we have ι restricts to an isomorphism between the maximal compact subgroups and we have equation 2.43, we have:

$$\iota(g_1,g_2)g_{\xi(\tau_1,\tau_2)} = g_{\xi(g_1\tau_1, g_2\tau_2)}k(\iota(g_1,g_2),\xi(\tau_1,\tau_2))$$

on one hand, and

$$\iota(g_1,g_2)g_{\xi(\tau_1,\tau_2)} = \iota(g_1g_{\tau_1}, g_2g_{\tau_2}) = \iota(g_{g_1\tau_1}, g_{g_2\tau_2})\iota\left(k\left(\frac{j(g_1,\tau_1)}{|j(g_1,\tau_1)|}\right)^{-1}, k\left(\frac{j(g_2,\tau_2)}{|j(g_2,\tau_2)|}\right)^{-1}\right)$$

on the other. Thus we have

$$k(\iota(g_1,g_2),\xi(\tau_1,\tau_2)) = \iota\left(k\left(\frac{j(g_1,\tau_1)}{|j(g_1,\tau_1)|}\right)^{-1}, k\left(\frac{j(g_2,\tau_2)}{|j(g_2,\tau_2)|}\right)^{-1}\right)$$

and the result follows from equation 2.46. □

We can also express the factors of automorphy $j^\pm$, and we have:

$$j^\pm\big(\iota(g_1,g_2),\xi(\tau_1,\tau_2)\big) = \big(c_1\left(x_1 \pm y_1 J_{g_2\tau_2}\right) + d_1\big)g_2$$

If we examine the first part $\left(c_1\left(x_1 \pm y_1 J_{g_2\tau_2}\right) + d_1\right)$, this is an element of $GL_2(\mathbb{R})$, and if we identify $\mathbb{R}^2$ with $\mathbb{C}$ via the complex structure given by $J_{g_2\tau_2}$, this is multiplication by $c_1\tau_1 + d_1$ or $c_1\overline{\tau}_1 + d_1$, depending on whether $\pm = +$ or $-$, respectively. Thus we can think of the factors of automorphy $j^\pm$ as the usual factor of automorphy for the first $\mathcal{H}_1$ (or it's complex conjugate), with the complex structure modified by the second $\mathcal{H}_1$, up to this factor g_2 on the right.

Chapter 3

Modular Forms

3.1 Modular Forms on $SL_2(\mathbb{R})$

Definition 3.1.1 (Modular Forms on $\mathcal{H}_1$). *We will say that a function $f : \mathcal{H} \to \mathbb{C}$ is modular of weight $\kappa \in \mathbb{Z}$ if it satisfies:*

$$f(\gamma\tau) = j(\gamma, \tau)^\kappa f(\tau). \tag{3.1}$$

for all $\gamma \in \Gamma' = SL_2(\mathbb{Z})$. If f is holomorphic on $\mathcal{H}_1$, then it has a Fourier expansion of the form:

$$f(\tau) = \sum_n a_n e^{2\pi i n\tau}$$

We will say that f is a holomorphic modular form if we have $a_n = 0$ for all $n < 0$, a cusp form if further $a_n = 0$, and a weakly holomorphic modular form if there is an $n_0 > 0$ so that $a_{-n_0} \neq 0$ and $a_n = 0$ for all $n < -n_0$. In this case we will say that the weakly holomorphic modular form has a pole of order n_0 at ∞.

If f is holomorphic on $\mathcal{H}$, We will denote by $S_\kappa(\Gamma')$ the space of cusp forms of weight κ for Γ', and $M^!_\kappa(\Gamma')$ the space of weakly holomorphic modular forms for Γ'. Note that $S_\kappa(\Gamma')$ is always finite dimensional, while $M^!_\kappa(\Gamma')$ is infinite dimensional, but has a filtration by finite dimensional subspaces given by bounding the order of the pole at the cusp. Denote by $M_2^\mu(\mathbb{Z}) = \{\gamma \in M_2(\mathbb{Z}) : \det\gamma = \mu\}$, and $M_2^+(\mathbb{Z}) = M_2(\mathbb{Z}) \cap GL_2^+(\mathbb{R})\{\gamma \in M_2(\mathbb{Z}) : \det\gamma > 0\}$. For $g \in GL_2^+(\mathbb{R})$, define the slash operator:

$$f|^\kappa_g(\tau) = (\det g)^{\kappa-1} j(g, \tau)^{-\kappa} f(g\tau) \tag{3.2}$$

We have that $f|^\kappa_\gamma = f$ for all $\gamma \in \Gamma'$. For $\mu \in \mathbb{Z}_{>0}$, define the Hecke operators on $\mathcal{H}_1$ to be:

$$T(\mu) = \sum_{\gamma \in \Gamma' \backslash M_2^\mu(\mathbb{Z})} f|^\kappa_\gamma(\tau) \tag{3.3}$$

These operators are endomorphisms of $S_\kappa(\Gamma')$, and set $\{T(\mu):\ \mu \in \mathbb{Z}_{>0}\}$ is a commuting set of operators. $S_\kappa(\Gamma')$ has a simultaneous eigenbasis for this action, and we will call an simultaneous eigenvector for all of the operators $T(\mu)$ an eigenform. We will assume that an eigenform is normalized so that $a_1 = 1$. For such a form we have that

$$T(\mu)f = a_\mu f \tag{3.4}$$

so that the coefficients of f coincide with the eigenvalues of the Hecke operators.

3.2 Siegel Modular Forms

Definition 3.2.1 (Siegel Modular Form)**.** *Suppose that $(\mathcal{V}_\kappa, \kappa)$ is a holomorphic representation of $GL_n(\mathbb{C})$. We will say that a function $f : \mathcal{H}_n \to V_\kappa$ is modular of weight κ and level $\Gamma' = Sp_n(\mathbb{Z})$ if we have:*

$$f(\gamma\tau) = \kappa(j(\gamma,\tau))f(\tau) \tag{3.5}$$

for all $\gamma \in \Gamma'$ and $\tau \in \mathcal{H}_n$. We say that it is a Siegel modular form if moreover it is holomorphic. The special case where $V_\kappa = \mathbb{C}$ and κ is a power of the determinant will be called a classical Siegel modular form.

For a classical Siegel modular form we will slightly abuse notation and also write κ for the power of the determinant, so that a classical Siegel modular form satisfies the more familiar equation:

$$f(\gamma\tau) = \det(c\tau + d)^\kappa f(\tau)$$

A Siegel modular form has a Fourier series:

$$f(\tau) = \sum_{N \in \mathrm{Sym}_n(\mathbb{Z})^*} a_N e^{2\pi i \operatorname{tr} N\tau} \tag{3.6}$$

where $\mathrm{Sym}_n(\mathbb{Z})^*$ is the dual to $\mathrm{Sym}_n(\mathbb{Z})$ under the trace form, consisting of symmetric matrices whose diagonals are integral and whose non-diagonal entries are in $\frac{1}{2}\mathbb{Z}$. The coefficients $a_N \in \mathcal{V}_\kappa$ are constant with respect to τ due to holomorphicity. The Koecher principle implies that $a_N = 0$ unless $N \geq 0$, and we say that f is a cusp form if $a_N = 0$ unless $N > 0$. We will denote by $S_\kappa(\Gamma')$ the space of cusp forms of weight κ. The action of matrices of the form $\begin{pmatrix} {}^t\alpha^{-1} & \\ & \alpha \end{pmatrix}$ for $\alpha \in GL_n(\mathbb{Z})$ implies relations between the coefficients:

$$\kappa(\alpha)a_N = a_{\alpha N\, {}^t\alpha}$$

The irreducible representations κ of $GL_n(\mathbb{C})$ are parameterized by non-increasing sequences of integers $\kappa_1 \geq \ldots \geq \kappa_n$. The representation $\det^\kappa$ corresponds to the sequence $(\kappa, \ldots, \kappa)$.

Define the *total weight* of κ to be:

$$d(\kappa) = \sum_i \kappa_i \tag{3.7}$$

We have that for scalar matrices $zI_n \in GL_n(\mathbb{C})$ that $\kappa(zI_n) = z^{d(\kappa)}$.

Remark 3.2.2. *There is a positivity condition on the weights κ that can occur for a non-zero cusp form. Namely we have $S_\kappa(\Gamma') = \{0\}$ unless $\kappa_n > 0$ (see for example proposition 1 on page 192 of [6]).*

3.2.1 Hecke Operators

Definition 3.2.3 (Slash Operator)**.** *For $A \in GSp_n(\mathbb{R})$, and $f : \mathcal{H}_n \to \mathcal{V}_\kappa$, define the slash operator of weight κ associated to A to be:*

$$f|_A^\kappa(\tau) = \mu(A)^{d(\kappa)-n(n+1)/2}\kappa(j(A,\tau)^{-1})f(A\tau) \tag{3.8}$$

We will omit κ when it is clear from context.

Remark 3.2.4. *For a classical Siegel modular form of weight $\det^\kappa$ we have that (3.8) is*

$$f|_A^\kappa(\tau) = \mu(A)^{n\kappa-n(n+1)/2}\det j(A,\tau)^{-\kappa}f(A\tau)$$

The slash operator gives a right action of $GSp_n^+(\mathbb{R})$ on the space of functions on $\mathcal{H}_n$, i.e. $(f|_{g_1}^\kappa)|_{g_2}^\kappa = f|_{g_1g_2}^\kappa$ for all $g_1, g_2 \in GSp_n^+(\mathbb{R})$. This action preserves holomorphicity of f. Suppose that f is modular of weight κ with respect to a subgroup $\Gamma'' \subset Sp_n(\mathbb{R})$, (i.e. that it satisfies 3.5 for $\gamma \in \Gamma''$). We have first of all that $f|_\gamma = f$, for all $\gamma \in \Gamma''$, and if $A \in GSp_n^+(\mathbb{R})$, we have that $f|_A$ is modular of weight κ for $A\Gamma''A^{-1}$.

When $A \in GSp_n(\mathbb{Q})$ has the form $A = \begin{pmatrix} \mu m & \\ & {}^t m^{-1} \end{pmatrix}$ for a $m \in GL_n(\mathbb{Q})$, and f is a Siegel modular form whose Fourier expansion is given in (3.6), then $f|_A$ also has a Fourier expansion:

$$f|_A(\tau) = \mu^{d(\kappa)-n(n+1)/2} \sum_{N \in \mu\, {}^t m \mathrm{Sym}_n(\mathbb{Z})^* m} \kappa({}^t m) a_{\mu^{-1}\, {}^t m^{-1} N m^{-1}} e^{2\pi i \operatorname{tr} N\tau}$$

Another special case is:

$$f|_{rI_{2n}}^\kappa(\tau) = r^{d(\kappa)-n(n+1)}f(\tau)$$

Define $GSp_n(\mathbb{Z}) = GSp_n(\mathbb{Q}) \cap GL_{2n}(\mathbb{Z})$ and for $\mu \in \mathbb{Z}$,

$$\begin{aligned} GSp_n^{\mu}(\mathbb{Z}) &= \{g \in GSp_n(\mathbb{Z}) : \ \mu(g) = \mu\} \\ GSp_n^{+}(\mathbb{Z}) &= \{g \in GSp_n(\mathbb{Z}) : \ \mu(g) > 0\} \end{aligned} \tag{3.9}$$

$GSp_n(\mathbb{Z})$ is a monoid under multiplication, and we have $GSp_n^{\mu_1}(\mathbb{Z}){\cdot}GSp_n^{\mu_2}(\mathbb{Z}) = GSp_n^{\mu_1\mu_2}(\mathbb{Z})$. $\Gamma' = GSp_n^1(\mathbb{Z})$ is the set of invertible elements of $GSp_n^+(\mathbb{Z})$. It is a standard fact that for a given μ we have:

$$\Gamma'\backslash GSp_n^{\mu}(\mathbb{Z})/\Gamma' = \left\{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_n \end{pmatrix} : \ a_i, d_i > 0, \ a_i d_i = \mu, \ a_1 | \ldots | a_n | d_1 | \ldots | d_n \right\}. \tag{3.10}$$

In particular $\Gamma'\backslash GSp_n^{\mu}(\mathbb{Z})/\Gamma'$ has finitely many elements. We can consider these double cosets to be the orbits of Γ' acting on $\Gamma'\backslash GSp_n^+(\mathbb{Z})$ on the right. it is another standard fact that each of these orbits are finite, namely:

$$\Gamma' g \Gamma' = \bigsqcup_i \Gamma' g_i \tag{3.11}$$

where i ranges over some finite index set and g_i are representatives for the action of Γ' on $\Gamma'\backslash GSp_n^+(\mathbb{Z})$. We have as well that the representatives of $\Gamma'\backslash GSp_n^+(\mathbb{Z})$ may be taken to be elements of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \tag{3.12}$$

with a upper triangular, and b is unique modulo $Sym_n(\mathbb{Z})d$. Define $\mathscr{H}_n$ to be the set of formal $\mathbb{Z}$-linear sums of double cosets:

$$\mathscr{H}_n = \mathbb{Z}\{\Gamma'\backslash GSp_n^+(\mathbb{Z})/\Gamma'\} \tag{3.13}$$

Definition 3.2.5 (Hecke Operators)**.** *For $g \in GSp_n^+(\mathbb{Z})$, define $T(g)$ to be the element of $\mathscr{H}_n$ given by the double coset $\Gamma' g \Gamma'$. For $\mu \in \mathbb{Z}_{>0}$, we define:*

$$T(\mu) = \sum_{g \in \Gamma'\backslash GSp_n(\mathbb{Z})/\Gamma'} T(g) \tag{3.14}$$

We will call $T(\mu)$ the total Hecke operator of degree μ.

It is well known that $\mathscr{H}_n$ may be given a commutative algebra structure, and it acts on the space $S_\kappa(\Gamma')$ by:

$$T(g)f(\tau) = \sum_i f|_{g_i}(\tau) \tag{3.15}$$

where the g_i are the representatives in (3.11). In particular, the total Hecke operators act as:

$$T(\mu)f(\tau) = \sum_{g \in \Gamma' \backslash GSp_n^\mu(\mathbb{Z})} f_g(\tau) \tag{3.16}$$

The action of $\mathscr{H}_n$ on $S_\kappa(\Gamma')$ has a common eigenbasis, and we call a simultaneous eigenvector for the action of $\mathscr{H}_n$ a Hecke eigenform. Unlike when $n = 1$, the Hecke theory is more complicated and there is not a straightforward relationship between the coefficients of f and the eigenvalues of Hecke operators. Given a Hecke eigenform, f, we will define it's *μ-total Hecke eigenvalue* to be the $\lambda(\mu)$ such that:

$$T(\mu)f = \lambda(\mu)f \tag{3.17}$$

We will be led to consider some more uncommon spaces of Hecke operators that change the level of a modular form. Suppose now that Γ'' is a finite index subgroup of Γ'. Denote by $\mathscr{H}_n(\Gamma', \Gamma'')$ the module of $\mathbb{Z}$ linear combinations of double cosets:

$$\mathscr{H}_n(\Gamma', \Gamma'') = \mathbb{Z}\{\Gamma' \backslash GSp_n^+ / \Gamma''\} \tag{3.18}$$

As Γ'' is finite index, we have as well that:

$$\Gamma' \cdot g \cdot \Gamma'' = \bigsqcup_i \Gamma' \cdot g_i \tag{3.19}$$

for i ranging over some finite index subset and g_i representatives of $\Gamma' \backslash GSp_n^+(\mathbb{Z})$.

Lemma 3.2.6 (Level Changing Hecke Operators)**.** *Suppose that $g \in GSp_n^+(\mathbb{Z})$. Define $T(g; \Gamma', \Gamma'') \in \mathscr{H}_n(\Gamma', \Gamma'')$ to be $\Gamma' g \Gamma''$. $\mathscr{H}_n(\mathbb{Z})$ defines elements of $\mathrm{Hom}(S_\kappa(\Gamma'), S_\kappa(\Gamma''))$ by:*

$$T(g; \Gamma', \Gamma'')f = \sum_i f|_{g_i} \tag{3.20}$$

Proof. It is quick to verify that $T(g; \Gamma', \Gamma'')f$ is modular of weight κ for Γ''. For $\gamma \in \Gamma''$, we have that γ permutes the cosets $\Gamma' g_i$, so that:

$$(T(g; \Gamma', \Gamma'')f)|_\gamma = \sum_i f|_{g_i \gamma} = \sum_i f|_{\gamma_i g_i} = \sum_i f|_{g_i} = T(g; \Gamma', \Gamma'')f$$

where γ_i are some elements in Γ'. □

We will be particularly interested in such Hecke operators for certain paramodular subgroups we will denote by $\Gamma'(D')$. Given a diagonal matrix D' with positive integral entries $D' = \operatorname{diag}(d_1', \ldots, d_n')$ with $d_i | d_{i+1}$, we define $\Gamma'(D')$ to be:

$$\Gamma'(D') = \Gamma' \cap \begin{pmatrix} D' & \\ & 1 \end{pmatrix}^{-1} GL_{2n}(\mathbb{Z}) \begin{pmatrix} D' & \\ & 1 \end{pmatrix} \tag{3.21}$$

These groups $\Gamma'(D')$ are finite index subgroups of Γ', and we can give a concrete description of them:

$$\Gamma'(D') = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' : \ c(D')^{-1} \in M_n(\mathbb{Z}), D'a(D')^{-1} \in M_n(\mathbb{Z}) \right\} \tag{3.22}$$

Definition 3.2.7 (D-total Hecke operator)**.** *Suppose that D is a positive integral diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_{i+1}|d_i$.* [1] *For such a D define $\mu(D) = d_1$. Define the D-total Hecke operator to be:*

$$T(D)f = \sum_{\substack{g \in \Gamma' \backslash GSp_n^{\mu(D)}(\mathbb{Z}) \\ \mu(D)^{-1} g_{11} D \in M_n(\mathbb{Z})}} f|_g \tag{3.23}$$

where $g = \begin{pmatrix} g_{11} & g_{12} \\ & g_{22} \end{pmatrix}$ is a representative for $\Gamma' \backslash GSp_n^{\mu(D)}(\mathbb{Z})$ of the form (3.12).

For a diagonal matrix D as defined above, we will define $D' = \mu(D)D^{-1}$. Then $D' = \operatorname{diag}(1, d_2', \ldots, d_n')$ with $d_i | d_{i+1}$. Each D is given uniquely as $D = dD_0$ with $d = d_n$ and $D_0 = \operatorname{diag}(d_1/d_n, \ldots, 1)$. We have that $D' = D_0'$, so this assignment $D \mapsto D'$ depends only on D_0.

Lemma 3.2.8. *The total Hecke operator $T(D)$ is an element of $\mathscr{H}_n\big(\Gamma', \Gamma'(D')\big)$.*

Proof. We have $(D')^{-1} = \mu(D)^{-1} D$. Thus we need to show that the set

$$\left\{ \begin{pmatrix} g_{11} & g_{12} \\ & g_{22} \end{pmatrix} : \ g_{11}(D')^{-1} \in M_n(\mathbb{Z}) \right\} \tag{3.24}$$

is right $\Gamma'(D')$ stable. Suppose that $g = \begin{pmatrix} g_{11} & g_{12} \\ & g_{22} \end{pmatrix}$, and $\gamma \in \Gamma'(D')$, so that

$$\gamma = \begin{pmatrix} D' & \\ & 1 \end{pmatrix}^{-1} \gamma' \begin{pmatrix} D' & \\ & 1 \end{pmatrix}$$

[1] Note the opposite order to the definition of D' above. In general we will use the apostrophe (D') when the condition is increasing divisibility down the diagonal and no apostrophe (D) when the condition is decreasing divisibility

for some $\gamma' \in GL_{2n}(\mathbb{Z})$. Let $g' = \begin{pmatrix} g'_{11} & g'_{12} \\ & g'_{22} \end{pmatrix}$ be the representative of the class of $g\gamma$, so that $g' = \gamma'' g\gamma$ for some $\gamma'' \in \Gamma'$. We have that $g_{11} = g''_{11}D'$ for an integral matrix g''_{11} by supposition, so that we have:

$$\begin{pmatrix} g'_{11} & g'_{12} \\ & g'_{22} \end{pmatrix} = \gamma'' \begin{pmatrix} g''_{11}D' & g_{12} \\ & g_{22} \end{pmatrix} \begin{pmatrix} D' & \\ & 1 \end{pmatrix}^{-1} \gamma' \begin{pmatrix} D' & \\ & 1 \end{pmatrix} = \gamma'' \begin{pmatrix} g''_{11} & g_{12} \\ & g_{22} \end{pmatrix} \gamma' \begin{pmatrix} D' & \\ & 1 \end{pmatrix}$$

We have that $\gamma'' \begin{pmatrix} g''_{11} & g_{12} \\ & g_{22} \end{pmatrix} \gamma'$ is integral as each matrix in the product is, so that it follows that $g'_{11}(D')^{-1}$ is integral as well. □

Remark 3.2.9. *We note that when $D = \operatorname{diag}(d, \dots, d)$ is a scalar matrix we have $\Gamma'(D') = \Gamma'$ so that $T(D) = T(d) \in \mathscr{H}_n$ is the degree d total Hecke operator (3.14).*

We have for $f \in S_\kappa(\Gamma')$, that $T(D)f \in S_\kappa(\Gamma'(D'))$. As we mentioned D' depends only on D_0, so for $D_0 = \operatorname{diag}(d_1, \dots, d_{n-1}, 1)$ any positive integral diagonal matrix with $d_{i+1}|d_i$, varying $d \in \mathbb{Z}_{>0}$ gives a sequence of operators $T(dD_0) \in \operatorname{Hom}\big(S_\kappa(\Gamma'), S_\kappa(\Gamma'(D'_0))\big)$.

We have that $\mathscr{H}_n(\Gamma', \Gamma'(D'))$ is a right module for $\mathscr{H}_n(\Gamma')$, with the action given by precomposition. We will not at this time explore this in detail, however we note that when $\gcd(\mu(D), \mu) = 1$ that we have $T(D)T(\mu) = T(\mu D)$.

3.3 Modular Forms on $O(m, m)$

Definition 3.3.1 (Modular forms on $\mathcal{D}$)**.** *Suppose that $(\mathcal{V}_\rho, \rho)$ is an algebraic representation of $GL_m(\mathbb{R})$. Then we say that a function $f : \mathcal{D} \to \mathcal{V}_\rho$ is modular of weight ρ if:*

$$\Phi(\gamma\xi) = \rho(j^-(\gamma, \xi))\Phi(\xi) \tag{3.25}$$

for all $\gamma \in \Gamma = O_{m,m}(\mathbb{Z})$. We we will say moreover that Φ is a modular form if it is also smooth.

Example 3.3.2. *The function $\Phi(\xi) = \det Y$ on $\mathcal{D}$ is a modular form of weight $|\det|^{-2}$, i.e. we have $\Phi(\gamma\xi) = |\det j^-(\gamma, \xi)|^{-2}\Phi(\xi)$ for all $\gamma \in \Gamma$, $\xi \in \mathcal{D}$.*

Remark 3.3.3. *Earlier we defined two factors of automorphy, $j^\pm(g, \xi)$ on $\mathcal{D}$ that are valued in $GL_m(\mathbb{R})$. We can use both of these to define a wider class of functions as modular forms, namely if ρ is a representation of $GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$ we would instead require that a function transform as:*

$$\Phi(\gamma\xi) = \rho\big(j^+(\gamma, \xi), j^-(\gamma, \xi)\big)\Phi(\xi) \tag{3.26}$$

We have that $|\det j^+(\gamma,\xi)| = |\det j^-(\gamma,\xi)|$ *so that if* Φ *is modular of weight* ρ*, is it also automatically modular of weight* $\rho\otimes(|\det|^s\otimes|\det|^{-s})$*, where* $|\det|^s\otimes|\det|^{-s}$ *is understood to be the* $GL_m(\mathbb{R})\times GL_m(\mathbb{R})$ *representation give by* $(\alpha_1,\alpha_2)\mapsto|\det\alpha_1|^s|\det\alpha_2|^{-s}$*.*

We have that $j^\pm(n(S),\xi)=1$ for all $n(S)=\begin{pmatrix}1 & S\\ & 1\end{pmatrix}$, so that if Φ is modular of weight ρ, then we have that $\Phi(\xi)=\Phi(\xi+S)$ for all $S\in\mathrm{Skew}_m(\mathbb{Z})$. Thus Φ has a Fourier expansion in the X variable. For $\xi=X+Y$, we have:

$$\Phi(\xi)=\sum_{S\in\mathrm{Skew}_m(\mathbb{Z})} e^{\pi i\,\mathrm{tr}\,XS}a_S(Y) \tag{3.27}$$

where $a_S:\mathrm{Sym}_m^+(\mathbb{R})\to\mathcal{V}_\rho$ are the Fourier coefficients. Note that the dual of $\mathrm{Skew}_m(\mathbb{Z})$ under the trace pairing is $\frac{1}{2}\mathrm{Skew}_m(\mathbb{Z})$, which is why there is no factor of 2 in the exponent. The modularity of Φ implies some relations between certain Fourier coefficients. We have that $j^\pm\big(m({}^tA^{-1},\xi)\big)=A$ for $m(A)=\begin{pmatrix}{}^tA^{-1} & \\ & A\end{pmatrix}$ so that applying the modularity condition with elements of the form $m({}^tA^{-1})$ for $A\in GL_m(\mathbb{Z})$ implies that $\Phi(\xi[A^{-1}])=\rho(A)\Phi(\xi)$.[2] Then we have:

$$a_{S[{}^tA]}(Y)=\rho(A)a_S(Y[A]) \tag{3.28}$$

for all $A\in GL_m(\mathbb{Z})$. For a diagonal matrix D we will define $J(D)$ to be the skew-symmetric matrix:

$$J(D)=\begin{pmatrix} & D & \\ -D & & \\ & & 0\end{pmatrix} \tag{3.29}$$

Definition 3.3.4 (Skew Normal Form)**.** *We say that* $S\in\mathrm{Skew}_m(\mathbb{Z})$ *is in skew normal form if* $S=J(D)$ *for a matrix* $D=\mathrm{diag}(d_1,\dots,d_r)$ *with* $2r=\mathrm{rank}\ S$*, and* $d_1,\dots,d_r$ *positive integers with* $d_n|\dots|d_1$*.*

We have that $GL_m(\mathbb{Z})$ acts on $\mathrm{Skew}_m(\mathbb{Z})$ by $A\cdot S=S[{}^tA]$, and we have (for example from [7], page 57) that each $GL_m(\mathbb{Z})$ orbit on $\mathrm{Skew}_m(\mathbb{Z})$ contains exactly one element in skew normal form.

Definition 3.3.5 (Symplectic Divisors)**.** *For* $S\in\mathrm{Skew}_m(\mathbb{Z})$*, we say that* S *has symplectic divisors* D *if* $S=J(D)[{}^tA]$ *for some* $A\in GL_m(\mathbb{Z})$*. In this case we write* $\mathrm{sd}(S)=D$*.*

Because of the relations between the Fourier coefficients (3.28), all of the Fourier coefficients are determined by the coefficients for S in skew normal form

[2] If we were to use the notion of modularity in (3.26) then we understand A to be in the diagonally embedded $GL_m(\mathbb{R})$

Definition 3.3.6 (Representative Fourier Coefficients)**.** *For $D = \operatorname{diag}(d_1, \ldots, d_r)$ with $2r \leq m$, and $d_1, \ldots, d_r$ positive integers with $d_r | \ldots | d_1$, we will define the representative Fourier coefficient $a_D(Y)$ to be defined by:*

$$a_D(Y) = a_{J(D)}(Y) \tag{3.30}$$

Remark 3.3.7 (Modular forms with respect to the factors of automorphy $k^{\pm}$)**.** *We will also note that it is possible to alternatively define the notion of a modular form on $\mathcal{D}$ in terms of the factors of automorphy $k^{\pm}(g, \xi)$ defined in definition 2.1.15. Let $(\mathcal{V}_{\widetilde{\rho}}, \widetilde{\rho})$ be a representation of $O(m) \times O(m)$. We will say that a smooth function $\widetilde{\Phi} : \mathcal{D} \to \mathcal{V}_{\widetilde{\rho}}$ is modular with respect to the factors of automorphy $k^{\pm}$ of weight $\widetilde{\rho}$ if it satisfies*

$$\widetilde{\Phi}(\gamma\xi) = \widetilde{\rho}\big(k^+(\gamma, \xi), k^-(\gamma, \xi)\big)\widetilde{\Phi}(\xi) \tag{3.31}$$

for all $\gamma \in \Gamma$ and $\xi \in \mathcal{D}$. This is related to the notion of modular with respect to the factors of automorphy $j^{\pm}$ in (3.26). Suppose that ρ is a representation of $GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$, and define $\widetilde{\rho}$ to be the representation of $O(m) \times O(m)$ obtained by restricting ρ to the subgroup $O(m) \times O(m) \subset GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$. Suppose that $\Phi : \mathcal{D} \to \mathcal{V}_\rho$ is a modular form of weight ρ. Then define a function $\widetilde{\Phi} : \mathcal{D} \to \mathcal{V}_\rho$ by:

$$\widetilde{\Phi}(\xi) = \rho({}^t\alpha(\xi))\Phi(\xi) \tag{3.32}$$

where ${}^t\alpha(\xi)$ is the element of $B_{0,0}$ such that $\alpha(\xi)\,{}^t\alpha(\xi) = Y$, and is embedded into $GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$ diagonally. Then we have as a consequence of (2.32) that $\widetilde{\Phi}$ is a modular form of weight $\widetilde{\rho}$ in the sense of (3.31). As we have $k^{\pm}(n(S), \xi) = 1$ we have a Fourier expansion for $\widetilde{\Phi}$:

$$\widetilde{\Phi}(\xi) = \sum_{S \in \operatorname{Skew}_m(\mathbb{Z})} e^{\pi i \operatorname{tr} SX} \widetilde{a}_S(Y) \tag{3.33}$$

for Fourier coefficients $\widetilde{a}_S : \operatorname{Sym}_m^+(\mathbb{R}) \to \mathcal{V}_{\widetilde{\rho}}$. If Φ and $\widetilde{\Phi}$ are related via (3.32), then we have $\widetilde{a}_S(Y) = \rho({}^t\alpha(Y))a_S(Y)$. We also have relations between the Fourier coefficients similar to (3.28):

$$\widetilde{a}_S(Y[{}^tA]) = \widetilde{\rho}(k(A, Y))\widetilde{a}_{S[A]}(Y) \tag{3.34}$$

where $k(A, Y) \in O(m)$ is embedded diagonally in to $O(m) \times O(m)$.

We will make two notes about this idea of obtaining this $\widetilde{\Phi}$ from Φ. First of all, even if a $GL_m(\mathbb{R})$ representation is irreducible, it will usually decompose into many different $O(m)$ representations, including ones of different irreducible types. Thus one can obtain many different $\widetilde{\Phi}$ from taking projections onto different $O(m) \times O(m)$ sub-representations, all of which will be modular forms. Functions obtained this way need not have any relations between them; it could be the case that some of them are identically 0 while others are not.

Second, it is possible to do this process in reverse. Namely if $\widetilde{\Phi}$ is a modular form *of weight $\widetilde{\rho}$ with respect to the factors of automorphy $k^{\pm}$, and ρ is a representation of $GL_m(\mathbb{R}) \times GL_m(\mathbb{R})$ that restricts to $\widetilde{\rho}$, then we can define $\Phi(\xi) = \rho({}^t\alpha(\xi)^{-1})\widetilde{\Phi}$, which will be modular of weight ρ with respect to the factors of automorphy $j^{\pm}$. This process is not as straightforward, however, as many $GL_m(\mathbb{R})$ representations can restrict to a given $O(m)$ representation, so that there is no notion of uniqueness in terms of which $GL_m(\mathbb{R})$ we should lift to.*

3.3.1 Modular Forms on $O(2,2)$

We will now look more closely at modular forms for $O(2,2)$. In this section we will use the notion of modular on $\mathcal{D}_2$ as in (3.31). We have that $K \cong O(2) \times O(2)$, and as $O(2)$ is closely related to the abelian $SO(2)$, the representation theory of $O(2)$ is significantly different from that of $O(m)$ for $m > 2$. The irreducible representations of $SO(2)$ are given by $\mathcal{V}_n = \mathbb{C}$ for $n \in \mathbb{Z}$, with $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acting on $\mathcal{V}_n$ by $e^{in\theta}$. We will also write χ_n for the character $\chi_n(k_\theta) = e^{in\theta}$ of $SO(2)$. We have that $O(2) \cong SO(2) \rtimes \langle \epsilon \rangle$ where

$$\epsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

and $\epsilon k_\theta \epsilon = k_{-\theta}$. We will fix a basis vector $\mathbf{v}_n$ of $\mathcal{V}_n$, so that $k_\theta \cdot \mathbf{v}_n = \chi_n(k_\theta)\mathbf{v}_n = e^{in\theta}\mathbf{v}_n$. The irreducible representations of $O(2)$ are classified by non-negative integers, n, and we will write them as $\mathcal{V}_{|n|}$. When $n = 0$ the representation $\mathcal{V}_{|0|}$ is the trivial representation, and for $n \geq 1$, $\mathcal{V}_{|n|}$ decomposes as $\mathcal{V}_{-n} \oplus \mathcal{V}_n$ under the restriction to $SO(2)$, with the element ϵ interchanging the factors: $\epsilon \cdot \mathbf{v}_{\pm n} = \mathbf{v}_{\mp n}$.

Now suppose that $\mathcal{V}_\rho$ is an irreducible representation of $O(2) \times O(2)$ where $\mathcal{V}_\rho \cong \mathcal{V}_{|\kappa_+|} \otimes \mathcal{V}_{|\kappa_-|}$ for κ_+, κ_- non-negative integers, and $\widetilde{\Phi} : \mathcal{D}_2 \to \mathcal{V}_\rho$ is a modular form of weight ρ on $\mathcal{D}_2$ in the sense of (3.31). We will denote by $\widetilde{\Phi}_{\pm_1,\pm_2}$ the coefficients of Φ when written in the basis we have fixed:

$$\widetilde{\Phi} = \widetilde{\Phi}_{+,+}(\mathbf{v}_{\kappa_+} \otimes \mathbf{v}_{\kappa_-}) + \widetilde{\Phi}_{+,-}(\mathbf{v}_{\kappa_+} \otimes \mathbf{v}_{-\kappa_-}) + \widetilde{\Phi}_{-,+}(\mathbf{v}_{-\kappa_+} \otimes \mathbf{v}_{\kappa_-}) + \widetilde{\Phi}_{-,-}(\mathbf{v}_{-\kappa_+} \otimes \mathbf{v}_{-\kappa_-}) \quad (3.35)$$

where we miss out the certain subscripts and terms if κ_+ or κ_- are 0. When $\gamma \in \Gamma_0$ we have that $k(\gamma, \xi) \in SO(2) \times SO(2)$ and we have:

$$\widetilde{\Phi}_{\pm_1,\pm_2}(\gamma\xi) = \chi_{\pm_1\kappa_+}(k^+(\gamma,\xi))\chi_{\pm_2\kappa_-}(k^-(\gamma,\xi))\widetilde{\Phi}_{\pm_1,\pm_2}(\xi). \quad (3.36)$$

We will describe now how the notion of modular on $\mathcal{D}_2$ translates to the notion of modular on $\mathcal{H}_1 \times \mathcal{H}_1$.

Lemma 3.3.8 (Correspondence between modular forms on $\mathcal{D}_2$ and $\mathcal{H}_1 \times \mathcal{H}_1$)**.** *Suppose that $\widetilde{\Phi}(\xi)$ is a modular form in the sense of definition 3.31 on $\mathcal{D}_2$ of weight $\rho = \mathcal{V}_{|\kappa_+|} \otimes \mathcal{V}_{|\kappa_-|}$.*

Define $\kappa_1 = \kappa_- - \kappa_+$ and $\kappa_2 = \kappa_+ + \kappa_-$. Then the function $F(\tau_1, \tau_2)$ defined by:

$$F(\tau_1, \tau_2) = y_1^{-\kappa_1/2} y_2^{-\kappa_2/2} \widetilde{\Phi}_{-,-}(\xi(\tau_1, \tau_2)) \tag{3.37}$$

is modular in the sense of (3.1) with weight (κ_1, κ_2), i.e.:

$$F(\gamma_1\tau_1, \gamma_2\tau_2) = j(\gamma_1, \tau_1)^{\kappa_1} j(\gamma_2, \tau_2)^{\kappa_2} F(\tau_1, \tau_2)$$

for all $(\tau_1, \tau_2) \in \mathcal{H}_1 \times \mathcal{H}_1$ and $(\gamma_1, \gamma_2) \in SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$.

Proof. Suppose that $(\gamma_1, \gamma_2) \in \Gamma' \times \Gamma'$. Then we have

$$F(\gamma_1\tau_1, \gamma_2\tau_2) = |j(\gamma_1, \tau_1)|^{\kappa_1} |j(\gamma_2, \tau_2)|^{\kappa_2} y_1^{-\kappa_1/2} y_2^{-\kappa_2/2} \widetilde{\Phi}_{-,-}(\xi(\gamma_1\tau_1, \gamma_2\tau_2))$$

We have that $\iota(\gamma_1, \gamma_2) \in \Gamma_0$, so that by lemma 2.3.1 and (3.36), we have

$$\widetilde{\Phi}_{-,-}(\xi(\gamma_1\tau_1, \gamma_2\tau_2)) = \left(\frac{j(\gamma_1, \tau_1)}{|j(\gamma_1, \tau_1)|}\right)^{\kappa_1} \left(\frac{j(\gamma_2, \tau_2)}{|j(\gamma_2, \tau_2)|}\right)^{\kappa_2} \widetilde{\Phi}_{-,-}(\xi(\tau_1, \tau_2))$$

showing the lemma. □

Remark 3.3.9. *In particular when ρ is trivial with respect to the K_+ factor, the F in the lemma above will be modular of weight κ_- in both variables.*

Remark 3.3.10. *We can also do lemma 3.3.8 in reverse to explain how to obtain a modular form for $\mathcal{D}_2$ from a modular form on $\mathcal{H}_1 \times \mathcal{H}_1$. Suppose that κ_1 and κ_2 are integers with $\kappa_2 \geq \max(\kappa_1, 0)$ and $\kappa_1 + \kappa_2$ even. Then define $\kappa_\pm = (\kappa_2 \mp \kappa_1)/2$, which will both be non-negative integers by the hypothesis. Then define $\widetilde{\Phi}_{-,-}(\xi) = y_1(\xi)^{\kappa_1/2} y_2(\xi)^{\kappa_2/2} F(\tau_1(\xi), \tau_2(\xi))$, and obtain the other components in (3.35) via (2.45).*

3.4 Symplectic Theta Functions

There is a class of functions on $\mathcal{D} \times \mathcal{H}$ called *Siegel theta functions* that are simultaneously modular in both variables. The simplest example of such a function is:

$$\sum_{v \in M_{2m,n}(\mathbb{Z})} \det y^{m/2} e^{\pi i \operatorname{tr}(v,v)x} e^{-\pi \operatorname{tr}(v,v)_\xi y} = \sum_{v \in M_{2m,n}(\mathbb{Z})} e^{\pi i \operatorname{tr}((v,v)_{\xi^+} \tau - (v,v)_{\xi^-} \overline{\tau})} \tag{3.38}$$

for $(\xi, \tau) \in \mathcal{D} \times \mathcal{H}$, where (v, v) is the split bilinear form on V from Section 2.1, and $(v, v)_\xi$ is the majorized inner product define in Definition 2.1.13. It can be modified by adding some appropriate choice of polynomial factors to obtain functions that transform with respect to other weights. We will take a different perspective on these sorts of functions. In particular we note that in this function τ appears $\mathbb{R}$-linearly in the exponent, so that the Fourier expansion with respect to the x variable of τ is straightforward. In comparison, the formula for $(v,v)_\xi$ in terms of ξ is fairly complicated, and as a consequence it is not readily apparent what the Fourier expansion is with respect to the X variable of ξ.

We will describe a class of functions that we will call *symplectic theta functions*. At the end of this section we will show how to translate between them and Siegel theta functions.

Definition 3.4.1 (Symplectic Theta Functions)**.** *Let $p(\eta) \in \mathbb{C}[M_{m,n}(\mathbb{C})]$ be any polynomial, and $\xi = X+Y \in \mathcal{D}$, $\tau = x+iy \in \mathcal{H}$. For $w = (w_1, w_2) \in M_{m,2n}(\mathbb{R})$, define $\eta_\tau(w) = w_1\tau + w_2$ as in Definition 2.2.2. We define the symplectic theta function:*

$$\begin{aligned} \Theta(\xi,\tau;p) &= \sum_{w\in M_{m,2n}(\mathbb{Z})} p(\eta_\tau(w)) e^{\pi i \operatorname{tr} \xi \eta_\tau(w) y^{-1}\, {}^t\overline{\eta}_\tau(w)} \\ &= \sum_{w\in M_{m,2n}(\mathbb{Z})} p(\eta_\tau(w)) e^{\operatorname{tr} X\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(w,w)_\tau} \end{aligned} \tag{3.39}$$

where $(w,w)_\tau$ is the majorized symmetric bilinear form on W described in definition 2.2.3. This defines a function $\Theta : \mathcal{D}_m \times \mathcal{H}_n \to \mathbb{C}[M_{m,n}(\mathbb{C})]^*$ given by $\Theta(\xi,\tau)(p) = \Theta(\xi,\tau;p)$.

When $p = 1$, we can compare this to the function 3.38. It appears that the variables ξ and τ have switched places, with ξ now appearing linearly in the exponent, and τ appearing in a complicated formula giving a majorized symmetric bilinear form. First of all we note that as $\langle w,w\rangle \in \mathrm{Skew}_n(\mathbb{Z})$, we have that $\Theta(\xi+S,\tau;p) = \Theta(\xi,\tau;p)$ for all $S \in \mathrm{Skew}_n(\mathbb{Z})$. One of these advantages of these symplectic theta functions is that the Fourier expansion with respect to the X variable of ξ is apparent:

$$\Theta(\xi,\tau;p) = \sum_{S\in \mathrm{Skew}_m(\mathbb{Z})} e^{\pi i \operatorname{tr} XS} \left(\sum_{\substack{w\in M_{m,2n}(\mathbb{Z}) \\ \langle w,w\rangle = S}} p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \right) \tag{3.40}$$

We will describe how these symplectic theta functions change under the action of $\Gamma \times \Gamma'$ on $\mathcal{D}_m \times \mathcal{H}_n$. To do so, first we note that the vector space $\mathbb{C}[M_{m,n}(\mathbb{C})]$ carries a representation of $G_m(\mathbb{C}) \times GL_n(\mathbb{C})$ by left and right translation, respectively, which we will denote by σ:

$$\big(\sigma(\alpha,a)p\big)(\eta) = p(\alpha^{-1}\eta a) \tag{3.41}$$

This induces a representation of $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathbb{C}[M_{m,n}(\mathbb{C})]^*$ given by

$$\big(\sigma^*(\alpha,a)\Theta\big)(p) = \Theta(\sigma(\alpha,a)^{-1}p) \tag{3.42}$$

for $\Theta \in \mathbb{C}[M_{m,n}(\mathbb{C})]^*$.

Theorem 3.4.2 (Modularity of Symplectic Theta Functions)**.** *Recall the standard factor of automorphy on $\mathcal{H}$ given by $j(g',\tau) = c'\tau + d' \in GL_n(\mathbb{C})$, and we defined a factor of* *automorphy on $\mathcal{D}$ valued in $GL_m(\mathbb{C})$ given by $j^-(g,\xi) = c\xi^- + d$, described in Definition 2.1.6. Then for all $(\gamma,\gamma') \in \Gamma \times \Gamma'$ and $(\xi,\tau) \in \mathcal{D} \times \mathcal{H}$, we have:*

$$\Theta(\gamma\xi, \gamma'\tau; p) = |\det j^-(\gamma,\xi)|^n \Theta\Big(\xi, \tau; \sigma\big(j^-(\gamma,\xi), j(\gamma',\tau)\big)^{-1} p\Big) \tag{3.43}$$

or in other words:

$$\Theta(\gamma\xi, \gamma'\tau) = |\det j^-(\gamma,\xi)|^n \sigma^*\big(j^-(\gamma,\xi), j(\gamma',\tau)\big)\Theta(\xi,\tau) \tag{3.44}$$

Remark 3.4.3. *The appearance of $\det j^-(\gamma,\xi)$ inside of absolute values may seem somewhat strange. In light of remark 2.1.10, if we restrict to $\gamma \in \Gamma_0 = \Gamma \cap G_0$, we can do away with the absolute values as $\det j^-(\gamma,\xi) > 0$ there. Alternately we may define $\epsilon(g) = \operatorname{sgn}\det(d-c)$ (2.18), and then we have $|\det j^-(\gamma,\xi)|^n = \epsilon(\gamma)^n \det j^-(\gamma,\xi)^n$. The character $\epsilon(\gamma)$ of Γ depends only on the second component of the connected component homomorphism $\pi_0 : G \to \pi_0(G)$.*

Remark 3.4.4. *The Fourier coefficients in (3.40) themselves also transform modularly as functions on $\mathcal{H}$ with respect to Γ', as can be inferred from a slight modification in the following proof and the fact that $\langle w\gamma', w\gamma'\rangle = \langle w, w\rangle$. They are not in any sense modular with respect to Γ.*

Proof. First we show the transformation property for τ. From (2.37) and (2.41) we have for $\gamma' \in \Gamma'$ and $\tau \in \mathcal{H}$:

$$\begin{aligned} \eta_{\gamma'\tau}(w) &= \eta_\tau(w\gamma') j(\gamma',\tau)^{-1} \\ \eta_{\gamma'\tau}(w)(\operatorname{Im}\gamma'\tau)^{-1}\, {}^t\overline{\eta}_{\gamma'\tau}(w) &= \eta_\tau(w\gamma') y^{-1}\, {}^t\overline{\eta}_\tau(w\gamma') \end{aligned}$$

so that we have:

$$\begin{aligned} \Theta(\xi, \gamma'\tau; p) &= \sum_{w \in M_{m,2n}(\mathbb{Z})} p\big(\eta_\tau(w\gamma') j(\gamma',\tau)^{-1}\big) e^{\pi i \operatorname{tr} \xi \eta_\tau(w\gamma') y^{-1}\, {}^t\overline{\eta}_\tau(w\gamma')} \\ &= \Theta\big(\xi, \tau; \sigma(j(\gamma',\tau))^{-1} p\big) \end{aligned}$$

as $w \mapsto w\gamma'$ simply permutes the terms of the sum. Now for the action of Γ on $\mathcal{D}$. First of all we note that it suffices to show it for elements $m(A)$ for $A \in GL_m(\mathbb{Z})$, $n(S)$ for $S \in \operatorname{Skew}_m(\mathbb{Z})$, and $Q_1 := Q_{\{1\}}$ (2.14) as by lemma 2.1.5 these elements generate Γ. As we already observed we have that $\Theta(\xi + S, \tau) = \Theta(\xi,\tau)$, proving it for elements $n(S)$. Next, we have

$$\begin{aligned}\Theta(\xi[{}^tA],\tau;p) &= \sum_{w\in M_{m,2n}(\mathbb{Z})} p(\eta_\tau(w))e^{\pi i\,\mathrm{tr}\,A\xi\,{}^tA\eta_\tau(w)y^{-1}\,{}^t\overline{\eta}_\tau(w)}\\ &= \sum_{w\in M_{m,2n}(\mathbb{Z})} p\big({}^tA^{-1}\eta_\tau({}^tAw)\big)e^{\pi i\,\mathrm{tr}\,\xi\eta_\tau({}^tAw)y^{-1}\,{}^t\overline{\eta}_\tau({}^tAw)}\\ &= \Theta\big(\xi,\tau;\sigma({}^tA)p\big)\end{aligned}$$

Where the second equality comes from $w\mapsto {}^tAw$ simply permutes the terms in the sum on w. Thus we have $\Theta\big(m(A)\cdot\xi,\tau\big)=\sigma^*({}^tA^{-1})\Theta(\xi,\tau)=\sigma^*\big(j^-(m(A),\xi)\big)\Theta(\xi,\tau)$, showing the claim for elements $m(A)$. Finally we seek to show it for the element Q_1. It is interesting to note that the proof is almost identical to the proof of showing modularity of the standard theta function. Namely we will perform a Fourier transform and apply Poisson summation. We will write ξ in block form:

$$\xi=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}=\begin{pmatrix}0 & X_{12}\\ -{}^tX_{12} & X_{22}\end{pmatrix}+\begin{pmatrix}Y_{11} & Y_{12}\\ {}^tY_{12} & Y_{22}\end{pmatrix}$$

with ξ_{11} a 1×1 matrix and ξ_{22} an $(m-1)\times(m-1)$ matrix (similarly for the X's and Y's). We have that:

$$\begin{aligned}Q_1\cdot\xi &= \begin{pmatrix}1 & \\ \xi_{21} & \xi_{22}\end{pmatrix}\begin{pmatrix}\xi_{11} & \xi_{12}\\ & 1\end{pmatrix}^{-1}\\ X(Q_1\cdot\xi) &= \begin{pmatrix}0 & -Y_{11}^{-1}Y_{12}\\ Y_{11}^{-1}\,{}^tY_{12} & X_{22}+Y_{11}^{-1}({}^tX_{12}Y_{12}-{}^tY_{12}X_{12})\end{pmatrix},\\ Y(Q_1\cdot\xi) &= \begin{pmatrix}Y_{11}^{-1} & -Y_{11}^{-1}X_{12}\\ -Y_{11}\,{}^tX_{12} & Y_{22}+Y_{11}^{-1}({}^tX_{12}X_{12}-{}^tY_{12}Y_{12})\end{pmatrix}\end{aligned}$$

and

$$j^-(Q_1,\xi)=\begin{pmatrix}-Y_{11} & X_{12}-Y_{12}\\ & 1_m\end{pmatrix},\quad j^-(Q_1,\xi)^{-1}=\begin{pmatrix}-Y_{11}^{-1} & Y_{11}^{-1}(X_{12}-Y_{12})\\ & 1_m\end{pmatrix}$$

Along with this decomposition of ξ, will write elements of $W\cong M_{m,2n}(\mathbb{R})$ as $\begin{pmatrix}w\\ \mathbf{w}\end{pmatrix}=\begin{pmatrix}w_1 & w_2\\ \mathbf{w}_1 & \mathbf{w}_2\end{pmatrix}$ with $w=(w_1,w_2)\in\mathbb{R}^{2n}$ and $\mathbf{w}=(\mathbf{w}_1,\mathbf{w}_2)\in M_{m-1,2n}(\mathbb{R})$. For $p\in\mathbb{C}[M_{m,n}(\mathbb{C})]$ some polynomial, define the function:

$$h\left(\begin{pmatrix} w \\ \mathbf{w} \end{pmatrix}\right) = \int_{\mathbb{R}^{2n}} e^{2\pi i\langle w', w\rangle} p\left(\eta_\tau \begin{pmatrix} w' \\ \mathbf{w} \end{pmatrix}\right) \exp\left[\pi i \operatorname{tr} X \left\langle \begin{pmatrix} w' \\ \mathbf{w} \end{pmatrix}, \begin{pmatrix} w' \\ \mathbf{w} \end{pmatrix} \right\rangle\right] \times \\ \times \exp\left[-\pi \operatorname{tr} Y \begin{pmatrix} w' \\ \mathbf{w} \end{pmatrix} g_\tau \, {}^t g_\tau \, {}^t \begin{pmatrix} w' \\ \mathbf{w} \end{pmatrix}\right] \, dw' \tag{3.45}$$

Next, we will write $u = wg_\tau$, $\mathbf{u} = \mathbf{w}g_\tau$, and perform the change of variables $w' \mapsto w' g_\tau^{-1}$, and then (3.45) is:

$$\int_{\mathbb{R}^{2n}} e^{2\pi i\langle w', u\rangle} p\left(\begin{pmatrix} iw_1' + w_2' \\ i\mathbf{u}_1 + \mathbf{u}_2 \end{pmatrix} {}^t a^{-1}\right) \\ \times \exp\left[\pi i \operatorname{tr} X \left\langle \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} \right\rangle - \pi \operatorname{tr} Y \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} {}^t \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix}\right] \, dw'$$

where $a \in GL_n(\mathbb{R})$ is such that $a\,{}^t a = y$. We have that:

$$\operatorname{tr} X \left\langle \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} \right\rangle = 2w_2' \, {}^t\mathbf{u}_1 \, {}^t X_{12} - 2w_1' \, {}^t\mathbf{u}_2 \, {}^t X_{12} + \operatorname{tr} X \langle \mathbf{u}, \mathbf{u} \rangle$$

and

$$\operatorname{tr} Y \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} {}^t \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} = Y_{11}(w_1' \, {}^t w_1' + w_2' \, {}^t w_2') + 2(w_1' \, {}^t\mathbf{u}_1 \, {}^t Y_{12} + w_2' \, {}^t\mathbf{u}_2 \, {}^t Y_{12}) + \operatorname{tr} Y_{22} \mathbf{u} \, {}^t\mathbf{u}$$

so that we have:

$$\operatorname{tr} Y \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} {}^t \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} - i \operatorname{tr} X \left\langle \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} w' \\ \mathbf{u} \end{pmatrix} \right\rangle \\ = \operatorname{tr} Y_{22} \mathbf{u} \, {}^t\mathbf{u} - i \operatorname{tr} X_{22} \langle \mathbf{u}, \mathbf{u} \rangle \\ + Y_{11}\big(w_1' + Y_{11}^{-1}(Y_{12}\mathbf{u}_1 + iX_{12}\mathbf{u}_2)\big) \, {}^t\big(w_1' + Y_{11}^{-1}(Y_{12}\mathbf{u}_1 + iX_{12}\mathbf{u}_2)\big) \\ + Y_{11}\big(w_2' + Y_{11}^{-1}(Y_{12}\mathbf{u}_2 - iX_{12}\mathbf{u}_1)\big) \, {}^t\big(w_2' + Y_{11}^{-1}(Y_{12}\mathbf{u}_2 - iX_{12}\mathbf{u}_1)\big) \\ - Y_{11}^{-1}(Y_{12}\mathbf{u}_1 + iX_{12}\mathbf{u}_2) \, {}^t(Y_{12}\mathbf{u}_1 + iX_{12}\mathbf{u}_2) - Y_{11}^{-1}(Y_{12}\mathbf{u}_2 - iX_{12}\mathbf{u}_1) \, {}^t(Y_{12}\mathbf{u}_2 - iX_{12}\mathbf{u}_1)$$

and then performing the change of variables $w_1' \mapsto w_1' - Y_{11}^{-1}(Y_{12}\mathbf{u}_1 + iX_{12}\mathbf{u}_2)\big)$, $w_2' \mapsto w_2' - Y_{11}^{-1}(Y_{12}\mathbf{u}_2 - iX_{12}\mathbf{u}_1)\big)$, and collecting out terms that do not depend on w', the (3.45) is:

$$\exp\left[\pi i \operatorname{tr} X(Q_1\cdot\xi)\left\langle \begin{pmatrix} u \\ \mathbf{u}\end{pmatrix}, \begin{pmatrix} u \\ \mathbf{u}\end{pmatrix}\right\rangle - \pi \operatorname{tr} Y(Q_1\cdot\xi)\begin{pmatrix} u \\ \mathbf{u}\end{pmatrix}{}^t\begin{pmatrix} u \\ \mathbf{u}\end{pmatrix} + \pi \operatorname{tr} Y_{11}^{-1} u\,{}^t u\right]$$
$$\times \int_{\mathbb{R}^n + iY_{11}^{-1}X_{12}\mathbf{u}_2}\int_{\mathbb{R}^n - iY_{11}^{-1}X_{12}\mathbf{u}_1} e^{2\pi i (w_1'\,{}^t u_2 - w_2'\,{}^t u_1)}$$
$$\times\, p\left(\begin{pmatrix} iw_1' + w_2' + Y_{11}^{-1}(X_{12}-Y_{12})(i\mathbf{u}_1+\mathbf{u}_2) \\ i\mathbf{u}_1 + \mathbf{u}_2\end{pmatrix}{}^t a^{-1}\right) e^{-\pi Y_{11}(w_1'\,{}^t w_1' + w_2'\,{}^t w_2')}\, dw_1' dw_2' \tag{3.46}$$

We note that by the contour integration trick for Gaussians the domain of integration can be converted back into an integration over $\mathbb{R}^{2n}$. We also use another trick related to Fourier transforms and polynomials. If $p'(w_1', w_2')$ is some polynomial in w_1', w_2', then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\langle w', u\rangle} p'(w_1', w_2')\varphi(w_1', w_2') dw_1' dw_2'$$
$$= p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial u_2}, -\frac{1}{2\pi i}\frac{\partial}{\partial u_1}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\langle w', u\rangle}\varphi(w_1', w_2') dw_1' dw_2'$$

where $p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial u_2}, -\frac{1}{2\pi i}\frac{\partial}{\partial u_1}\right)$ is the differential operator obtained by replacing the w' variables by the prescribed differentiation with respect to u. It is characterized by:

$$p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial u_2}, -\frac{1}{2\pi i}\frac{\partial}{\partial u_1}\right) e^{2\pi i\langle w', u\rangle} = p'(w_1', w_2') e^{2\pi i\langle w', u\rangle}$$

We thus obtain that the integral in (3.46) is:

$$Y_{11}^{-n} p\left(\begin{pmatrix} \frac{1}{2\pi}\left(-i\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right) + Y_{11}^{-1}(X_{12}-Y_{12})(i\mathbf{u}_1+\mathbf{u}_2) \\ i\mathbf{u}_1 + \mathbf{u}_2\end{pmatrix}{}^t a^{-1}\right) e^{-\pi Y_{11}^{-1} u\,{}^t u}$$

Next, we have that for $p'' \in \mathbb{C}[\mathbb{C}^n]$, that

$$p''\left(\frac{1}{2\pi}\left(-i\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)\right) e^{-\pi Y_{11}^{-1} u\,{}^t u} = p''\big(-Y_{11}^{-1}(iu_1 + u_2)\big)$$

since we have

$$\frac{1}{2}\left(-i\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right) e^{u\,{}^t u} = (iu_1 + u_2) e^{u\,{}^t u}$$

and further that

$$\frac{1}{2}\left(-i\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)(iu_1 + u_2) = 0.$$

Then, after putting the u and $\mathbf{u}$ terms back in terms of w and $\mathbf{w}$, we obtain:

$$h\left(\begin{pmatrix} w \\ \mathbf{w} \end{pmatrix}\right) = Y_{11}^{-n} p\big(j^-(Q_1,\xi)^{-1}\eta_\tau(w)\big) e^{-\pi \operatorname{tr} X(Q_1\cdot\xi)\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(Q_1\cdot\xi)(w,w)_\tau}$$

Thus by Poisson summation applied to (3.45) we have that:

$$\sum_{w\in M_{m,2n}(\mathbb{Z})} p(\eta_\tau(w)) e^{\pi i \operatorname{tr} X(\xi)\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(\xi)(w,w)_\tau}$$
$$= |\det j^-(Q_1,\xi)|^n \sum_{w\in M_{m,2n}(\mathbb{Z})} p\big(j^-(Q_1,\xi)^{-1}\eta_\tau(w)\big) e^{\pi i \operatorname{tr} X(Q_1\cdot\xi)\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(Q_1\cdot\xi)(w,w)_\tau}$$

so that $|\det j^-(Q_1,\xi)|^n \Theta\big(Q_1\cdot\xi,\tau;\sigma(j^-(Q_1,\xi))p\big) = \Theta(\xi,\tau;p)$, which finishes the proof upon replacing p with $\sigma(j^-(Q_1,\xi))^{-1}p$ and dividing by $|\det j^-(Q_1,\xi)|^n$. □

The function $\Theta(\xi,\tau)$ so far constructed takes values in an infinite dimensional vector space. This is undesirable and we will now describe how we can use it to make some functions that take values in finite dimensional vector spaces.

There is a sesquilinear form on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ given by $(p(\eta),q(\eta)) = \overline{q}\left(\frac{\partial}{\partial\eta}\right)p\Big|_{\eta=0}$ where $\overline{q}$ denotes the polynomial obtained by conjugating the coefficients of q. If η^I, $\eta^{I'}$ are monomials, then we have $(\eta^I,\eta^{I'}) = |I|\delta_{I,I'}$, so that the monomials provide an orthogonal basis with respect to this Hermitian form. This gives a (sesquilinear) isomorphism between $\mathbb{C}[M_{m,n}(\mathbb{C})]$ and $\mathbb{C}[M_{m,n}(\mathbb{C})]^*$, and under this isomorphism we get $\sigma^*(\alpha,a) = \sigma({}^t\overline{\alpha}^{-1},{}^t\overline{a}^{-1})$. Now, consider the representation σ' of $GL_n(\mathbb{C})\times GL_m(\mathbb{C})$ on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ given by:

$$\sigma'(\alpha,a)p(\eta) = p({}^t\alpha\eta a) \tag{3.47}$$

From the preceding discussion we have that $\sigma'|_{GL_m(\mathbb{R})} = (\sigma|_{GL_m(\mathbb{R})})^*$. For $\kappa = (\kappa_1,\ldots,)$ we will say that $\kappa\geq 0$ if $\kappa_i \geq 0$ for all i, and if $\kappa\geq 0$, we define $\operatorname{len}(\kappa)$ to be the greatest i so that $\kappa_i\neq 0$. If $\kappa\geq 0$ and $\operatorname{len}(\kappa)\leq\min(m,n)$ we can interpret κ as a weight for both $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$ by having 0's in the appropriate number of places, and we will denote by $\mathcal{V}_\kappa^{(m)}$ and $\mathcal{V}_\kappa^{(n)}$ the corresponding irreducible representations of $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$, respectively. It is a standard result that we have:

$$(\mathbb{C}[M_{m,n}(\mathbb{C})],\sigma') \cong \bigoplus_{\substack{\kappa\geq 0 \\ \operatorname{len}(\kappa)\leq\min(m,n)}} \mathcal{V}_\kappa^{(m)}\otimes\mathcal{V}_\kappa^{(n)} \tag{3.48}$$

and so we have:

$$(\mathbb{C}[M_{m,n}(\mathbb{C})],\sigma|_{GL_m(\mathbb{R})\times GL_n(\mathbb{C})}) \cong \bigoplus_{\substack{\kappa\geq 0 \\ \operatorname{len}(\kappa)\leq\min(m,n)}} (\mathcal{V}_\kappa^{(m)})^*\otimes\mathcal{V}_\kappa^{(n)} \tag{3.49}$$

where we interchangeably use $\mathcal{V}_\kappa^{(m)}$ for the representation of $GL_m(\mathbb{C})$ and its restriction to $GL_m(\mathbb{R})$. We will denote by:

$$\mathbb{C}[M_{m,n}(\mathbb{C})]^{\kappa}$$

The κ isotypic component (as a representation of $GL_n(\mathbb{C})$). We have that $\mathbb{C}[M_{m,n}(\mathbb{C})]^{\kappa} \cong (\mathcal{V}_{\kappa}^{(m)})^* \otimes \mathcal{V}_{\kappa}^{(n)}$ is spanned by the $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ translates of the function:

$$\Delta^{\kappa}(\eta) = \prod_{j=1}^{\min(m,n)} \det \begin{pmatrix} \eta_{11} & \dots & \eta_{1j} \\ \vdots & \ddots & \vdots \\ \eta_{j1} & \dots & \eta_{jj} \end{pmatrix}^{\kappa_j - \kappa_{j+1}} \tag{3.50}$$

We obtain $GL_m(\mathbb{R}) \times GL_n(\mathbb{C})$ linear projections $\mathbb{C}[M_{m,n}(\mathbb{C})]^* \twoheadrightarrow \mathcal{V}_{\kappa}^{(m)} \otimes (\mathcal{V}_{\kappa}^{(n)})^*$ dual to the inclusions $(\mathcal{V}_{\kappa}^{(m)})^* \otimes \mathcal{V}_{\kappa}^{(n)} \hookrightarrow \mathbb{C}[M_{m,n}(\mathbb{C})]$.

Definition 3.4.5 (The symplectic theta functions Θ^{κ})**.** *For each $\kappa \geq 0$ with $\operatorname{len}(\kappa) \leq \min(m,n)$, define $\Theta^{\kappa} : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\kappa}^{(m)} \otimes (\mathcal{V}_{\kappa}^{(n)})^*$ to be the composition of Θ with the projection map $\mathbb{C}[M_{m,n}(\mathbb{C})]^* \twoheadrightarrow \mathcal{V}_{\kappa}^{(m)} \otimes (\mathcal{V}_{\kappa}^{(n)})^*$.*

Remark 3.4.6. *The function Θ defined above is essentially the direct sum off all of the functions Θ^{κ} as κ ranges over the κ with $\kappa \geq 0$ and $\operatorname{len}(\kappa) \leq \min(m,n)$.*

Lemma 3.4.7. *The functions Θ^{κ} defined above are modular of weight $|\det|^n \otimes \kappa^{(m)}$ on $\mathcal{D}$ and $(\kappa^{(n)})^*$ on $\mathcal{H}$, i.e. for $(\gamma, \gamma') \in \Gamma \times \Gamma'$ we have:*

$$\Theta^{\kappa}(\gamma\xi, \gamma'\tau) = |\det j^-(\gamma,\xi)|^n \Big(\big(\kappa^{(m)}(j^-(\gamma,\xi))\big) \otimes \big((\kappa^{(n)})^*(j(\gamma',\tau))\big) \Big) \Theta^{\kappa}(\xi,\tau)$$

The proof of this is immediate from theorem 3.4.2 and the above discussion. For brevity we will give a special name to the representation $|\det|^n \otimes \mathcal{V}_{\kappa}^{(m)}$

Definition 3.4.8 (The representation $(\mathcal{V}_{\rho_\kappa}, \rho_\kappa)$ of $GL_m(\mathbb{R})$)**.** *For κ a representation of $GL_n(\mathbb{C})$, with $\kappa \geq 0$ and $\operatorname{len}(\kappa) \leq \min(m,n)$, define the representation $(\mathcal{V}_{\rho_\kappa}, \rho_\kappa)$ of $GL_m(\mathbb{R})$ by:*

$$(\mathcal{V}_{\rho_\kappa}, \rho_\kappa) = \big(\mathcal{V}_{\kappa}^{(m)}, |\det|^n \otimes (\kappa^{(m)}|_{GL_m(\mathbb{R})})\big) \tag{3.51}$$

Remark 3.4.9. *Thus we can rephrase lemma 3.4.7 as:*

$$\Theta^{\kappa}(\gamma\xi, \gamma'\tau) = \big(\rho_\kappa(j^-(\gamma,\xi)) \otimes (\kappa^{(n)})^*(j(\gamma',\tau))\big)\Theta^{\kappa}(\xi,\tau) \tag{3.52}$$

These functions Θ^{κ} can be paired with modular forms of the appropriate weight.

Definition 3.4.10 (The pairing (f, Θ^{κ}))**.** *Suppose that $f : \mathcal{H} \to \mathcal{V}_{\kappa}^{(n)}$ is a function that is modular of weight κ. There is a natural pairing $\mathcal{V}_{\kappa}^{(n)} \times (\mathcal{V}_{\kappa}^{(n)})^*$ that is invariant under the action of $GL_n(\mathbb{C})$, so we can pair Θ^{κ} and f. we will write:*

$$(f, \Theta^{\kappa}) : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\rho_\kappa} \tag{3.53}$$

to be the function obtained by this pairing.

We have immediately that

Lemma 3.4.11. *The function* (f,Θ^κ) *defined in definition 3.4.10 transform as:*

$$(f,\Theta^\kappa)(\gamma\xi,\gamma'\tau) = \rho_\kappa(j^-(\gamma,\xi))(f,\Theta^\kappa)(\xi,\tau) \tag{3.54}$$

for $(\gamma,\gamma') \in \Gamma\times\Gamma'$*. In particular we note that this invariant under* Γ'*.*

We will now describe how to have a more concrete handle on this function (f,Θ^κ). We identified $(\mathcal{V}_\kappa^{(m)})^*\otimes\mathcal{V}_\kappa^{(n)}$ with $\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa$, the κ-isotypic component of $\mathbb{C}[M_{m,n}(\mathbb{C})]$. We have an identification:

$$\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa \cong \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa,\mathbb{C}[M_{m,n}(\mathbb{C})])\otimes\mathcal{V}_\kappa^{(n)}$$

and so as a vector space:

$$\mathcal{V}_{\rho_\kappa} \cong \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa,\mathbb{C}[M_{m,n}(\mathbb{C})])^* \tag{3.55}$$

Thus we can describe the pairing $(f,\Theta^\kappa)(\xi,\tau)$ by its values on $GL_n(\mathbb{C})$ linear maps $P:\mathcal{V}_\kappa\to\mathbb{C}[M_{m,n}(\mathbb{C})]$. Generally we will use the capital letter P when we are referring to these maps. In the case where $\dim\mathcal{V}_\kappa^{(n)}=1$ these maps are identified with individual polynomials and we will use a lowercase p. We have:

$$(f,\Theta^\kappa)(\xi,\tau)(P) = \sum_{w\in M_{m,2n}(\mathbb{Z})} P(f(\tau))(\eta_\tau(w))e^{\pi i\,\mathrm{tr}\,X\langle w,w\rangle}e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau} \tag{3.56}$$

where $P(f(\tau))$ will be a polynomial in $\mathbb{C}[M_{m,n}(\mathbb{C})]$, and can be evaluated at elements in $M_{m,n}(\mathbb{C})$. For scalar $\mathcal{V}_\kappa$ this is simply:

$$(f,\Theta^\kappa)(\xi,\tau)(p) = \sum_{w\in M_{m,2n}(\mathbb{Z})} f(\tau)p(\eta_\tau(w))e^{\pi i\,\mathrm{tr}\,X\langle w,w\rangle}e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau} \tag{3.57}$$

We note that the expression $P(f(\tau))$ transforms as:

$$P(f(\gamma\tau)) = \sigma(j(\gamma,\tau))P(f(\tau))$$

for all $\gamma\in\Gamma'$.

We will now discuss how these functions are related to the more usual Siegel theta functions like the function in (3.38). This will be useful to us for locating the singularities of the regularized lift of weakly holomorphic modular forms from $SL_2(\mathbb{R})$ which we will look at in latter half of the next chapter. We will derive the following equality directly via Poisson summation here.

Lemma 3.4.12. *For $Y \in \mathrm{Sym}_m^+(\mathbb{R})$, define $\triangle_{ij}^Y$ to be the differential operator on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ defined by:*

$$\triangle_{ij}^Y = \sum_{s,t=1}^{m} (Y^{-1})_{s,t} \frac{\partial^2}{\partial \eta_{s,i} \partial \eta_{t,j}} \tag{3.58}$$

and write $\triangle^Y$ for the $n \times n$ matrix of differential operators with entries $\triangle_{ij}^Y$. Then we have:

$$\Theta(\xi, \tau; p) = \frac{\det y^{m/2}}{\det Y^{n/2}} \sum_{v \in M_{2m,n}(\mathbb{Z})} \left(\exp\left(\frac{1}{4\pi} \operatorname{tr} \triangle^Y y \right) p \right) \left(-iY^{-1} \nu_\xi^-(v) y \right) e^{\pi i \operatorname{tr}(v,v)x} e^{-\pi \operatorname{tr}(v,v)_\xi y} \tag{3.59}$$

where ν_ξ^- is as in (2.21).

Proof. Again we will use Poisson summation, this time with the Fourier transform:

$$\int_{M_{m,n}(\mathbb{R})} e^{-2\pi i \operatorname{tr} w_2 {}^t v_1} p(v_2\tau + w_2) e^{\pi i \operatorname{tr} X \left\langle (v_2,w_2),(v_2,w_2) \right\rangle} e^{-\pi \operatorname{tr} Y \left((v_2,w_2),(v_2,w_2) \right)_\tau} dw_2 \tag{3.60}$$

we have $\left((v_2, w_2), (v_2, w_2) \right)_\tau = v_2 y {}^t v_2 + (v_2 x + w_2) y^{-1} {}^t (v_2 x + w_2)$. We can perform the translation $w_2 \mapsto w_2 - v_2 x$ and we obtain that (3.60) is:

$$e^{2\pi i \operatorname{tr} {}^t v_1 v_2 x} e^{-\pi \operatorname{tr} Y v_2 y {}^t v_2} \times \int_{M_{m,n}(\mathbb{R})} e^{-2\pi i \operatorname{tr} w_2 {}^t v_1} p(i v_2 y + w_2) e^{2\pi i \operatorname{tr} w_2 {}^t (X v_2)} e^{-\pi \operatorname{tr} Y w_2 y^{-1} {}^t w_2} dw_2$$

as we have

$$\left\langle (v_2, w_2 - v_2 x), (v_2, w_2 - v_2 x) \right\rangle = \left\langle (v_2, w_2), (v_2, w_2) \right\rangle$$

and

$$\operatorname{tr} X \left\langle (v_2, w_2), (v_2, w_2) \right\rangle = 2 \operatorname{tr} w_2 {}^t (X v_2)$$

We can complete the square in the exponent:

$$\begin{aligned} & e^{-2\pi i \operatorname{tr} w_2 {}^t v_1 + 2\pi i \operatorname{tr} w_2 {}^t (X v_2) - \pi \operatorname{tr} Y w_2 y^{-1} {}^t w_2} \\ &= e^{-\pi \operatorname{tr} Y^{-1} (v_1 - X v_2) y {}^t (v_1 - X v_2)} e^{-\pi \operatorname{tr} Y \left(w_2 + i Y^{-1} (v_1 - X v_2) y \right) y^{-1} {}^t \left(w_2 + i Y^{-1} (v_1 - X v_2) y \right)} \end{aligned}$$

then translate $w_2 \mapsto w_2 - iY^{-1}(v_1 - X v_2) y$, and perform the same contour integration trick as in the proof of theorem 3.4.2 to obtain that (3.60) is:

$$e^{2\pi i \operatorname{tr} {}^t v_1 v_2 x} e^{-\pi \operatorname{tr} Y v_2 y {}^t v_2 - \pi \operatorname{tr} Y^{-1} (X v_2 - v_1) y {}^t (X v_2 - v_1)} \times \int_{M_{m,n}(\mathbb{R})} p\left(i v_2 y - i Y^{-1} (v_1 - X v_2) y + w_2 \right) e^{-\pi \operatorname{tr} Y w_2 y^{-1} {}^t w_2} dw_2$$

We have that $Y^{-1}(v_1 - Xv_2) - v_2 = Y^{-1}\nu_\xi^-(v)$, and from (2.27):

$$e^{2\pi i \operatorname{tr} {}^t v_1 v_2 x} e^{-\pi \operatorname{tr} Y v_2 y {}^t v_2 - \pi \operatorname{tr} Y^{-1}(Xv_2 - v_1) y {}^t(Xv_2 - v_1)} = e^{\pi i \operatorname{tr}(v,v) x} e^{-\pi \operatorname{tr}(v,v)_\xi y}$$

and finally we have that:

$$\int_{M_{m,n}(\mathbb{R})} p'(w_2) e^{-\pi \operatorname{tr} Y w_2 y^{-1} {}^t w_2} dw_2 = \det Y^{-n/2} \det y^{m/2} \exp\left(\frac{1}{4\pi} \operatorname{tr} \triangle^Y y\right) p' \Big|_{\eta=0}$$

Thus we get that (3.60) is equal to:

$$\left(\exp\left(\frac{1}{4\pi} \operatorname{tr} \triangle^Y y\right) p\right)(-i\alpha^{-1}\nu_\xi^-(v) y) e^{\pi i \operatorname{tr}(v,v) x} e^{-\pi \operatorname{tr}(v,v)_\xi y}$$

and by applying Poisson summation we have the result. □

We will now describe how we can interpret these symplectic theta functions as modular functions in the sense of (3.31). For $p \in \mathbb{C}[M_{m,n}(\mathbb{C})]$ and $(\xi, \tau) \in \mathcal{D} \times \mathcal{H}$, define:

$$\widetilde{\Theta}(\xi, \tau; p) = \det Y^{n/2} \sum_{w \in M_{m,2n}(\mathbb{Z})} p({}^t\alpha \eta_\tau(w)) e^{\pi i \operatorname{tr} X \langle w,w \rangle} e^{-\pi \operatorname{tr} Y (w,w)_\tau} \tag{3.61}$$

where $\alpha \in B_{0,0}$ is such that $\alpha\, {}^t\alpha = Y$. This is obtained from Θ by the process outlined in remark 3.3.7, and the function satisfies

$$\widetilde{\Theta}(\gamma\xi, \gamma'\tau; p) = \widetilde{\Theta}\big(\xi, \tau; \sigma^{-1}(k^-(\gamma, \xi), j(\gamma', \tau))p\big) \tag{3.62}$$

For $\mathcal{V}_{\widetilde{\rho}}$ a subrepresentation of $\mathcal{V}_\kappa^{(m)}|_{O(m)}$, we can define functions $\widetilde{\Theta}^{\widetilde{\rho},\kappa} : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\widetilde{\rho}} \otimes (\mathcal{V}_\kappa^{(n)})^*$. We will also describe some of the structure of the $GL_m(\mathbb{R})$ representations we have in terms of how they restrict to $O(m)$ representations. To begin with, for $1 \leq i, j \leq n$, we have the differential operators $\triangle_{ij}$ on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ given by:[3]

$$\triangle_{ij} = \sum_{s=1}^m \frac{\partial^2}{\partial \eta_{s,i} \partial \eta_{s,j}} \tag{3.63}$$

These differential operators commute with the action of $O(m)$.

Remark 3.4.13. *We also have a version of lemma 3.4.12 for $\widetilde{\Theta}^{\widetilde{\rho},\kappa}$. For $p \in \mathbb{C}[M_{m,n}(\mathbb{C})]$, we have:*

$$\widetilde{\Theta}(\xi, \tau; p) = \det y^{m/2} \sum_{v \in M_{2m,n}(\mathbb{Z})} \left(\exp\left(\frac{1}{4\pi} \operatorname{tr} \triangle y\right) p\right)(-i\alpha^{-1}\nu_\xi^-(v) y) e^{\pi i \operatorname{tr}(v,v) x} e^{-\pi \operatorname{tr}(v,v)_\xi y} \tag{3.64}$$

We will call a $p \in \mathbb{C}[\eta]$ *pluriharmonic* if $\triangle_{ij} p = 0$ for all $1 \leq i, j \leq n$, and we define:

[3]this is the same as $\triangle_{ij}^1$ defined in lemma 3.4.12

$$\mathscr{H}[M_{m,n}(\mathbb{C})] = \{p \in \mathbb{C}[M_{m,n}(\mathbb{C})] : \triangle_{ij} p = 0 \text{ for } 1 \leq i, j \leq n\} \tag{3.65}$$

to be the space of pluriharmonic polynomials. The space $\mathscr{H}[M_{m,n}(\mathbb{C})]$ is $O(m) \times GL_n(\mathbb{C})$ stable under the restriction of σ to $O(m) \times GL_n(\mathbb{C})$ on $\mathbb{C}[M_{m,n}(\mathbb{C})]$. Note that it is not $GL_m(\mathbb{C})$ stable. Define $\mathscr{I}[M_{m,n}(\mathbb{C})]$ to be the subspace of $\mathbb{C}[M_{m,n}(\mathbb{C})]$ generated by the coefficients of ${}^t\eta\eta$. Equivalently $\mathscr{I}[M_{m,n}(\mathbb{C})]$ is the space of $O(m)$ invariants. We have $\mathscr{H} \cap \mathscr{I} = \mathbb{C} \cdot 1$, and $\mathscr{H} \cdot \mathscr{I} = \mathbb{C}[M_{m,n}(\mathbb{C})]$. For κ a $GL_n(\mathbb{C})$ representation, denote by $\mathscr{H}[M_{m,n}(\mathbb{C})]^\kappa$ the κ isotypic component of $\mathscr{H}[M_{m,n}(\mathbb{C})]$. We have a description of the space of pluriharmonic polynomials:

Lemma 3.4.14 (From [8]). *Suppose that $m \geq n$, and $\kappa \geq 0$. Then there is a unique representation $\widetilde{\rho}(\kappa)$ of $O(m)$ so that:*

$$\mathscr{H}[M_{m,n}(\mathbb{C})]^\kappa \cong \mathcal{V}_{\widetilde{\rho}(\kappa)} \otimes \mathcal{V}_\rho \tag{3.66}$$

We have further than $\mathrm{Hom}_{O(m)\times GL_n(\mathbb{C})}(\mathcal{V}_{\widetilde{\rho}(\kappa)} \otimes \mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})]) = 1$. *When $m \geq 2n$, we can describe this space as follows. Define the function $\Delta^\kappa \in \mathbb{C}[M_{n,n}(\mathbb{C})]$ by:*

$$\Delta^\kappa(x) = \prod_{i=1}^{n} \det \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix}^{\kappa_i - \kappa_{i+1}} \tag{3.67}$$

Then the space $\mathscr{H}^\kappa$ is spanned by the $O(m) \times GL_n(\mathbb{C})$ translates of

$$p_0^\kappa(\eta) = \Delta^\kappa\left(\begin{pmatrix} 1_n & i1_n & 0 \end{pmatrix} \eta\right)$$

Example 3.4.15 ($n = 1$)**.** *When $n = 1$, representations of $GL_1(\mathbb{C})$ correspond to integers $\kappa \in \mathbb{Z}$. The representation κ occurs in $\mathbb{C}[M_{m,1}(\mathbb{C})] \cong \mathbb{C}[\mathbb{C}^m]$ iff $\kappa \geq 0$, in which case $\mathbb{C}[\mathbb{C}^m]^\kappa$ consists of the homogeneous polynomials of degree κ. We will say $\mathbf{u} \in \mathbb{C}^m$ is isotropic if ${}^t\mathbf{u}\mathbf{u} = 0$. This is equivalent to $\mathbf{u} = u_1 + iu_2$ with $u_1, u_2 \in \mathbb{R}^m$ and ${}^tu_1u_2 = 0$, ${}^tu_1u_1 = {}^tu_2u_2$. The space $\mathscr{H}[\mathbb{C}^m]^\kappa$ is spanned by the polynomials $p_{\mathbf{u}}^\kappa(\eta) = ({}^t\mathbf{u}\eta)^\kappa$ with $\mathbf{u}$ isotropic. The action of $O(m) \times GL_1(\mathbb{C})$ for these polynomials is simple to describe:*

$$\sigma(k, a) p_{\mathbf{u}}^\kappa = a^\kappa p_{k\mathbf{u}}^\kappa \tag{3.68}$$

The subspace $\mathscr{I}$ is given by the powers of ${}^t\eta\eta$, and we have:

$$\mathbb{C}[\mathbb{C}^m]^\kappa = \bigoplus_{k=0}^{\lfloor \kappa/2 \rfloor} ({}^t\eta\eta)^k \mathscr{H}[\mathbb{C}^m]^{\kappa - 2k} \tag{3.69}$$

When $m = 2$, then up to a scalar there are only two isotropic vectors: $\mathbf{u}_\pm = \begin{pmatrix} \mp i \\ 1 \end{pmatrix}$. We *define corresponding polynomials:*

$$p^\kappa_\pm(\eta) = p^\kappa_{\mathbf{u}_\pm}(\eta) = \left({}^t\eta \begin{pmatrix} \mp i \\ 1 \end{pmatrix} \right)^\kappa \tag{3.70}$$

Recall that $O(2)$ is generated by elements of the form:

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \epsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

with the k_θ forming a subgroup isomorphic to $SO(2)$. We have:

$$\sigma(k_\theta)p^\kappa_\pm = e^{\pm i\kappa\theta} p^\kappa_\pm$$
$$\sigma(\epsilon)p^\kappa_\pm = p^\kappa_\mp$$

As a $O(2)$ representation, the space $\mathscr{H}[\mathbb{C}^2]^\kappa$ is isomorphic to $\mathcal{V}_{|\kappa|}$ defined in section 3.3.1. The dual space $(\mathscr{H}[\mathbb{C}^2]^\kappa)^*$ is also isomorphic to $\mathcal{V}_{|\kappa|}$, with the basis $\{p^\kappa_-, p^\kappa_+\}$ having dual basis $\{\mathbf{v}^\kappa_+, \mathbf{v}^\kappa_-\}$ (note the order).

Example 3.4.16 (Scalar $GL_n(\mathbb{C})$ representations)**.** *We have non-trivial scalar valued representations of $GL_n(\mathbb{C})$ inside of $\mathbb{C}[M_{m,n}(\mathbb{C})]$ iff $m \geq n$. The scalar representations of $GL_n(\mathbb{C})$ are given by integers $\kappa \in \mathbb{Z}$, corresponding to powers of the determinant, with $\det^\kappa$ occurring in $\mathbb{C}[M_{m,n}(\mathbb{C})]$ iff $\kappa \geq 0$. In this case $\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa$ will occur within the subspace of homogeneous polynomials of degree κn, but it will not be all of these polynomials. Given any matrix $\mathbf{u} \in M_{m,n}(\mathbb{C})$, define the polynomial:*

$$p^\kappa_{\mathbf{u}}(\eta) = \det({}^t\eta\mathbf{u})^\kappa \tag{3.71}$$

As $\mathbf{u}$ varies over the elements of $M_{m,n}(\mathbb{C})$, these polynomials span $\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa$. They transform as

$$\sigma(\alpha, a)p^\kappa_{\mathbf{u}} = \det a^\kappa p^\kappa_{{}^t\alpha^{-1}\mathbf{u}}$$

We now will suppose as well that $m \geq 2n$, and we will describe the space $\mathscr{H}[M_{m,n}(\mathbb{C})]^\kappa$. We will call $\mathbf{u} = u_1 + iu_2 \in M_{m,n}(\mathbb{C})$ isotropic if ${}^tu_1u_2 = 0$ and ${}^tu_1u_1 = {}^tu_2u_2$. This implies that ${}^t\mathbf{u}\mathbf{u} = 0$, however the converse is not true as ${}^t\mathbf{u}\mathbf{u}$ only implies that ${}^tu_1u_2$ is skew-symmetric, not necessarily 0. As $\mathbf{u}$ varies over the isotropic matrices, the polynomials $p^\kappa_{\mathbf{u}}$ span the space $\mathscr{H}[M_{m,n}(\mathbb{C})]^\kappa$. They transform under $O(m) \times GL_n(\mathbb{C})$ as:

$$\sigma(k, a)p^\kappa_{\mathbf{u}} = \det a^\kappa p^\kappa_{k\mathbf{u}}$$

Example 3.4.17 (Non-scalar $GL_n(\mathbb{C})$ representations)**.** *Suppose now that κ is not necessarily a scalar representation of $GL_n(\mathbb{C})$. We have that $\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa$ is spanned by polynomials* *of the form:*

$$p_{\mathbf{u}}^{\kappa} = \Delta^{\kappa}({}^{t}\eta\mathbf{u}) \tag{3.72}$$

as $\mathbf{u}$ *varies over elements of* $M_{m,n}(\mathbb{C})$*. This reduces to (3.71) in the case that* κ *is scalar. They transform as:*

$$\sigma(\alpha, a)p_{\mathbf{u}}^{\kappa} = \det a^{\kappa} p_{{}^{t}\alpha^{-1}\mathbf{u}}^{\kappa}$$

If we suppose that $m \geq 2n$*, then again we have that* $\mathscr{H}[M_{m,n}(\mathbb{C})]^{\kappa}$ *is spanned by polynomials of the form* $p_{\mathbf{u}}^{\kappa}$ *where* $\mathbf{u}$ *varies over isotropic matrices. For such polynomials we have* $\sigma(k)p_{\mathbf{u}}^{\kappa} = p_{k\mathbf{u}}^{\kappa}$ *for* $k \in O(m)$*.*

Chapter 4

Theta Lifts from $SL_2(\mathbb{R})$

4.1 Lifts of Cusp Forms

In this chapter we will be exploring theta lifts of modular forms from $\mathcal{H} = \mathcal{H}_1$ to $\mathcal{D} = \mathcal{D}_m$. First we will examine the lifts of cusp forms. Let $f : \mathcal{H} \to \mathbb{C}$ be a cusp form for $SL_2(\mathbb{Z}) = \Gamma'$ of weight κ:

$$f(\gamma\tau) = j(\gamma, \tau)^{\kappa} f(\tau)$$

for all $\gamma \in \Gamma'$. Suppose as well that f has the Fourier expansion:

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$$

Recall in the last chapter we defined functions $\Theta^{\kappa} : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\rho_\kappa}$ where $\mathcal{V}_{\rho_\kappa}$ (3.51) is the representation of $GL_m(\mathbb{R})$ given by $\rho_\kappa = |\det| \otimes \kappa^{(m)}$, where $\kappa^{(m)}$ is the irreducible representation of $GL_m(\mathbb{R})$ with highest weight $(\kappa, 0, \ldots, 0)$. We have:

$$\Theta^{\kappa}(\gamma\xi, \gamma'\tau) = j(\gamma', \tau)^{-\kappa} \rho_\kappa\big(j^{-}(\gamma, \xi)\big)\Theta(\xi, \tau)$$

for $(\gamma, \gamma') \in \Gamma \times \Gamma'$. We can pair this with $f(\tau)$ to obtain a function that is invariant under $\tau \mapsto \gamma'\tau$ (3.53).

Definition 4.1.1 (Theta lift for $SL_2(\mathbb{Z})$ cusp forms)**.** *With f a cusp form, define:*

$$\Phi_f(\xi) = \int_{\mathcal{F}} (f, \Theta^{\kappa})(\xi, \tau) \frac{dxdy}{y^2} \tag{4.1}$$

where $\mathcal{F}$ is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$.

The integral will converge due to f being a cusp form and the moderate growth of Θ^{κ}.

Proposition 4.1.2 (Modularity of Φ_f)**.** *The function Φ_f is modular of weight ρ_κ with* *respect to* Γ. *In other words, we have* $\Phi_f : \mathcal{D}_m \to \mathcal{V}_{\rho_\kappa}$, *and for all* $\gamma \in \Gamma$, *we have:*

$$\Phi_f(\gamma\xi) = \rho_\kappa(j^-(\gamma, \xi))\Phi_f(\xi) \tag{4.2}$$

Proof. This proposition follows immediately from the same transformation property for (f, Θ^κ) in (3.54). □

Recall we have an identification $\mathcal{V}_{\rho_\kappa} \cong \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[\mathbb{C}^m])^* \cong (\mathbb{C}[\mathbb{C}^m]^\kappa)^*$ (where $\mathbb{C}[\mathbb{C}^m]^\kappa$ is the subspace of homogeneous polynomials of degree κ). Thus we can describe Φ_f by its evaluation on polynomials $p(\eta) \in \mathbb{C}[\mathbb{C}^m]^\kappa$. This is given by:

$$\Phi_f(\xi; p) = \int_{\mathcal{F}} f(\tau) \sum_{w \in M_{m,2}(\mathbb{Z})} e^{\pi i \operatorname{tr} X\langle w,w\rangle} p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^2} \tag{4.3}$$

where $\langle w, w\rangle$ and $(w, w)_\tau$ are as in 2.2.3. It is our goal to calculate explicit formulas for these theta lifts. The introduction of the symplectic theta functions serves two purposes. First, as mentioned earlier, the lifts Φ_f have a readily apparently Fourier expansion with respect to the X variable of ξ due to the same for the symplectic theta function Θ^κ. Second, they also allow for a process we will call unfolding that allows us to transform the integrals over $\mathcal{F}$ into integrals over simpler regions.

We note that due f being a cusp form we are able to interchange the order of integration and summation, and so we can write out the Fourier expansion of $\Phi_f(\xi)$:

$$\Phi_f(\xi; p) = a_0(Y; p) + \sum_{\substack{S \in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S=2}} e^{\pi i \operatorname{tr} SX} a_S(Y; p) \tag{4.4}$$

where $a_S : \mathrm{Sym}_m^+(\mathbb{R}) \to (\mathbb{C}[\mathbb{C}^m]^\kappa)^*$ are the Fourier coefficients, given by:

$$a_S(Y; p) = \sum_{\substack{w \in M_{m,2}(\mathbb{Z}) \\ \langle w,w\rangle = S}} \int_{\mathcal{F}} f(\tau) p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^2} \tag{4.5}$$

Note as well that the sum is only over skew symmetric matrices of rank at most 2 as that is the maximum rank of $\langle w, w\rangle$ for $w \in M_{m,2}(\mathbb{Z})$. We can further refine these sums into more tractable integrals to consider:

Lemma 4.1.3 (Unfolding for $SL_2(\mathbb{Z})$)**.** *For* $w \in M_{m,2}(\mathbb{Z})$, *and* $w \neq 0$. *Write* Γ'_w *for the stabilizer of* w *inside of* Γ', *and write* $\mathcal{F}_w$ *for a fundamental domain for the action of* Γ'_w *on* $\mathcal{H}$. *Then we have:*

$$\sum_{w' \in w\cdot\Gamma'} \int_{\mathcal{F}} f(\tau) p(\eta_\tau(w')) e^{-\pi \operatorname{tr} Y(w',w')_\tau} \frac{dxdy}{y^2}$$
$$= 2 \int_{\mathcal{F}_w} f(\tau) p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^2}$$

Proof. Let $\{\gamma_i'\}$ denote a set of representatives for $\Gamma_w'\backslash\Gamma'$. Then we have the left hand side is:

$$\sum_i \int_{\mathcal{F}} f(\tau)p(\eta_\tau(w\gamma_i))e^{-\pi \operatorname{tr} Y(w\gamma_i, w\gamma_i)_\tau}\frac{dxdy}{y^2}$$

This is:

$$\sum_i \int_{\mathcal{F}} f(\tau)p\big(\eta_{\gamma_i\tau}(w)j(\gamma_i,\tau)\big)e^{-\pi \operatorname{tr} Y(w,w)_{\gamma_i\tau}}\frac{dxdy}{y^2}$$

and we have that $f(\tau)p\big(\eta_{\gamma_i\tau}(w)j(\gamma_i,\tau)\big) = j(\gamma_i,\tau)^\kappa f(\tau)p(\eta_{\gamma_i\tau}(w))$ so this is:

$$\sum_i \int_{\gamma_i\mathcal{F}} f(\tau)p(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau}\frac{dxdy}{y^2}$$

Then as γ_i ranges over the set of representatives we will double cover $\mathcal{F}_w$ as $-1 \notin \Gamma_w'$. When $w = 0$ we have that $-I$ is in $\Gamma_w' = \Gamma'$, however we have that $p(\eta_\tau(0)) = 0$ as p is non-constant homogeneous. $\square$

Thus we are able to group together terms corresponding to w in the same Γ' orbit in 4.5 into terms with easier integrals. We will first classify the orbits of $M_{m,2}(\mathbb{Z})$ under Γ'.

Lemma 4.1.4 ($SL_2(\mathbb{Z})$ orbits of $M_{m,2}(\mathbb{Z})$)**.** *Suppose that* $[w] \in M_{m,2}(\mathbb{Z})/SL_2(\mathbb{Z})$*. Then either:*

1. $w = 0$*, in which case* $\mathcal{F}_w = \mathcal{F}$ *and* $\Gamma_w' = \Gamma'$*.*
2. $\operatorname{rank} w = 1$*, in which case we may take the representative* w *to be* $(0, u)$*, where* u *ranges over* $\mathbb{Z}^m/\{0\}$*,* u *unique up to multiplication by* ± 1*. In this case* $\Gamma_w' = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$*, and* $\mathcal{F}_w = \{\tau \in \mathcal{H}_1 : \ 0 \le x \le 1\}$*. We have* $\langle w, w\rangle = 0$*.*
3. $\operatorname{rank} w = 2$*, in which case* $\Gamma_w' = \{1\}$ *and* $\mathcal{F}_w = \mathcal{H}$*. In this case* w *can be taken to be:*

$$w = Aw_0^{\pm}\gamma_{a,b,d}$$

where $A \in GL_m(\mathbb{Z})/P_2(\mathbb{Z})$*, where:*

$$P_2(\mathbb{Z}) = \begin{pmatrix} GL_2(\mathbb{Z}) & * \\ & GL_{m-2}(\mathbb{Z}) \end{pmatrix}$$

$w_0^{\pm}$ *is the matrix:*

$$w_0^{\pm} = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \end{pmatrix}$$

and $\gamma_{a,b,d} = \begin{pmatrix} a & b \\ & d \end{pmatrix}$ *with* $a, b, d \in \mathbb{Z}$, $a, d \geq 0$, *and* $0 \leq b < a$. *In this case*

$$\langle w, w \rangle = \pm A \begin{pmatrix} 0 & ad & \dots & 0 \\ -ad & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} {}^tA$$

We will offer another description of the orbits in the rank 2 case that provides a somewhat more intrinsic characterization. First of all, we will denote by $\mathcal{O}^S$:

$$\mathcal{O}^S = \{w \in M_{m,2}(\mathbb{Z}) : \ \langle w, w \rangle = S\}/SL_2(\mathbb{Z}) \tag{4.6}$$

From the above considerations when S is a rank 2 matrix in $\mathrm{Skew}_m(\mathbb{Z})$ we have that $\mathcal{O}^S$ is a finite set. We will define $M^+_{m,2}(\mathbb{Z})$ to be the subset of rank 2 elements of $M_{m,2}(\mathbb{Z})$, and $M^1_{m,2}(\mathbb{Z})$ to be the subset of $w \in M^+_{m,2}(\mathbb{Z})$ that can be completed to an element of $GL_m(\mathbb{Z})$. We will also call such w *primitive*. Similarly we will denote by $\mathrm{Skew}^1_m(\mathbb{Z})$ the subset of $S \in \mathrm{Skew}_m(\mathbb{Z})$ such that $\frac{1}{n}S \in \mathrm{Skew}_m(\mathbb{Z})$ iff $n \in \{1, -1\}$. This is equivalent to $\mathrm{sd}(S) = 1$ (Definition 3.3.5). We will call $S \in \mathrm{Skew}^1_m(\mathbb{Z})$ *primitive* as well. For a primitive $S_0 \in \mathrm{Skew}^1_m(\mathbb{Z})$, there is a primitive $w_0 \in M^1_{m,2}(\mathbb{Z})$ so that $S_0 = \langle w_0, w_0 \rangle$. From the lemma this w_0 is unique up to action of $SL_2(\mathbb{Z})$ on the right. We have $\mathcal{O}^{S_0} = \{w_0\}$.

Given any $S \in \mathrm{Skew}_m(\mathbb{Z})$ of rank 2, there is a unique $S_0 \in \mathrm{Skew}^1_m(\mathbb{Z})$ and $\mu \in \mathbb{Z}_{>0}$ so that $S = \mu S_0$, and we have

$$\mathcal{O}^S = \{w_0 \gamma_{a,b,d} : \ a, b, d \in \mathbb{Z}_{\geq 0}, \ ad = \mu, \ 0 \leq b < a\} \tag{4.7}$$

Corollary 4.1.5. *The constant Fourier coefficient is given by:*

$$a_0(Y; p) = \int_{\mathcal{F}} p(0) f(\tau) \frac{dxdy}{y^2} + \sideset{}{'}\sum_{u \in \mathbb{Z}^m} \int_0^\infty \int_0^1 p(0, u) f(\tau) e^{-\pi \, {}^tuYuy} \frac{dxdy}{y^2} \tag{4.8}$$

where the apostrophe (′) on the sum denotes excluding $u = 0$. The non-constant coefficients are given by:

$$a_S(Y; p) = 2 \sum_{w \in \mathcal{O}^S} \int_{\mathcal{H}} p(\eta_\tau(w)) f(\tau) e^{-\pi \operatorname{tr} Y \langle w, w \rangle_\tau} \frac{dxdy}{y^2} \tag{4.9}$$

Note, in the sum of the rank 1 terms we dropped the coefficient of 2 in lemma 4.1.3 by summing both u and $-u$, which are included in the same Γ' orbit. When f is a cusp form we have:

Lemma 4.1.6. *Suppose that f is a cusp form. Then the constant term of Φ_f is 0, i.e., $a_0(Y; p) = 0$ identically for all $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$.*

Proof. As we noted before, we have that as p is a homogeneous non-constant polynomial we have $p(0) = 0$, so that the rank 0 term contributes 0. Next, we have that the only dependence in the integrand on x of:

$$\int_0^\infty \int_0^1 p(0,u) f(\tau) e^{-\pi\, {}^t u Y u y} \frac{dxdy}{y^2}$$

is in $f(\tau)$, and as f is a cusp form the integral $\int_0^1 f(\tau)dx = 0$, so that the rank 1 terms contribute 0 as well. □

We will now define some quantities that will come up in the computation of the rank 2 Fourier coefficients of Φ_f.

Definition 4.1.7. *Write $M^+_{m,2}(\mathbb{R})$ for the subset of $M_{m,2}(\mathbb{R})$ of matrices of rank 2. For $w = (w_1, w_2) \in M^+_{m,2}(\mathbb{R})$ and $\xi \in \mathcal{D}_m$, define $\tau_1(\xi, w), \tau_2(\xi, w) \in \mathcal{H}_1$ to be:*

$$\begin{aligned} \tau_1(\xi, w) &= \frac{1}{2} \operatorname{tr} X \langle w, w \rangle + i \sqrt{\det {}^t w Y w} \\ \tau_1(\xi, w) &= -\frac{{}^t w_1 Y w_2}{{}^t w_1 Y w_1} + i \frac{\sqrt{\det {}^t w Y w}}{{}^t w_1 Y w_1} \end{aligned} \tag{4.10}$$

and define $\eta(\xi, w)$ to be:

$$\eta(\xi, w) = \tau_2(\xi, w) w_1 + w_2 \tag{4.11}$$

Remark 4.1.8. *We will write $y_1(\xi, w)$ and $y_2(\xi, w)$ for the imaginary parts of $\tau_1(\xi, w)$ and $\tau_2(\xi, w)$, respectively. We will also use $y_1(Y, w)$, $\tau_2(Y, w)$, and $\eta(Y, w)$ with Y in the place of ξ since these quantities are independent of X.*

We will describe how these quantities change under the right action of $GL_2(\mathbb{R})$ on $M_{m,2}(\mathbb{R})$. First of all, for $g \in GL_2^+(\mathbb{R})$, and $\epsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ we have:

$$\begin{aligned} \tau_1(\xi, wg) &= (\det g) \tau_1(\xi, w) & \tau_1(\xi, w\epsilon) &= -\overline{\tau}_1(\xi, w) \\ \tau_2(\xi, wg) &= g^{-1} \cdot \tau_2(\xi, w) & \tau_2(\xi, w\epsilon) &= -\overline{\tau}_2(\xi, w) \end{aligned} \tag{4.12}$$

and we have:

$$\eta(\xi, wg) = j\big(g^{-1}, \tau_2(\xi, w)\big)^{-1} \eta(\xi, w), \qquad \eta(\xi, w\epsilon) = \overline{\eta(\xi, w)} \tag{4.13}$$

Remark 4.1.9. *When $m = 2$, and $w = 1_2$, these reduce to the identification $(\tau_1, \tau_2) : \mathcal{D}_2 \cong \mathcal{H}_1 \times \mathcal{H}_1$, and $\eta = \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}$. We will examine this in more detail when we discuss specializing the lift to $m = 2$ later in this section.*

We will provide an interpretation of what these quantities are. For $w = (w_1, w_2) \in M_{m,2}(\mathbb{Z})$, w being primitive is equivalent to $\{w_1, w_2\}$ being a $\mathbb{Z}$ basis of $U_{\mathbb{Z}}(w)$. Otherwise and otherwise the columns span a full rank sub-lattice of $U_{\mathbb{Z}}(w)$.

Next, $\xi \in \mathcal{D}_m$ provides a positive definite bilinear form on $\mathbb{R}^m$, which we will denote by $(u, u')_Y = {}^t u Y u'$, and a (possibly degenerate) alternating form which we will denote by $\langle u, u' \rangle_X = {}^t u X u'$. We can restrict both of these forms to the lattice $U_{\mathbb{Z}}(w)$ to obtain a positive definite bilinear form and an alternating form on $U_{\mathbb{Z}}(w)$, which we will denote by the same. We have

$$\det {}^t w Y w = \det \begin{pmatrix} (w_1, w_1)_Y & (w_1, w_2)_Y \\ (w_2, w_1)_Y & (w_2, w_2)_Y \end{pmatrix}$$

which is the determinant of the lattice spanned $\{w_1, w_2\}$ with respect to $(-,-)_Y$. The vector $\eta(\xi, w)$ is an element of $U_{\mathbb{Z}}(w) \otimes \mathbb{C} = \mathbb{C}\{w_1, w_2\}$, and is isotropic with respect to $(-,-)_Y$. Its real and imaginary parts have length $\sqrt{y_1(\xi, w) y_2(\xi, w)}$ with respect to $(-,-)_Y$. The imaginary part of $\eta(\xi, w)$ is in the w_1 direction, and the real part is in the direction of the projection with respect to $(-,-)_Y$ of w_2 onto the perpendicular of w_1.

As we mentioned the space $U_{\mathbb{Z}}(w)$ is a $\mathbb{Z}^2$ lattice equipped with a positive definite symmetric bilinear form $(-,-)_Y$, a (possibly degenerate) alternating form $\langle -,- \rangle_X$, and an orientation determined by the ordering w_1, w_2. A different choice of oriented basis $\{w_1, w_2\}$ of $U_{\mathbb{Z}}(w)$ amounts to acting on w by a $\gamma \in SL_2(\mathbb{Z})$ on the right which we saw (4.12) leaves τ_1 invariant and acts on τ_2 by γ^{-1}. Thus the τ_1 and τ_2 variables give an invariant of this triple of data that is valued in $\mathcal{H} \times (\mathcal{H}/\Gamma')$.

Lemma 4.1.10. *Suppose that $w \in M^+_{m,2}(\mathbb{R})$, and $\xi \in \mathcal{D}$, and $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$. Then we have:*

$$\begin{aligned} e^{\pi i \operatorname{tr} X \langle w, w \rangle} & \int_{\mathcal{H}} f(\tau) p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w) - \tau} y^{-2} dx dy \\ &= y_1(\xi, w)^{-1} p\big(\eta(\xi, w)\big) e^{2\pi i \tau_1(\xi, w)} f(\tau_2(\xi, w)) \end{aligned} \tag{4.14}$$

Remark 4.1.11. *We could ask the question what happens to the formula in (4.14) when w is replaced by $w\gamma$ for $\gamma \in SL_2(\mathbb{Z})$. From (4.12) we have that for $\gamma \in SL_2(\mathbb{Z})$ we have that $\tau_1(\xi, w\gamma^{-1}) = \tau_1(\xi, w)$, $\tau_2(\xi, w\gamma^{-1}) = \gamma \cdot \tau_2(\xi, w)$ and $\eta(\xi, w\gamma^{-1}) = j(\gamma, \tau_2(\xi, w))^{-1} \eta(\xi, w)$. Then when we consider (4.14) under $w \mapsto w\gamma^{-1}$, and using that p is homogeneous of degree κ, we have:*

$$\begin{aligned} & y_1(\xi, w\gamma^{-1})^{-1} p\big(\eta(\xi, w\gamma^{-1})\big) e^{2\pi i \tau_1(\xi, w\gamma^{-1})} f(\tau_2(\xi, w\gamma^{-1})) \\ &\quad = y_1(\xi, w)^{-1} j(\gamma, \tau_2(\xi, w))^{-\kappa} p\big(\eta(\xi, w)\big) e^{2\pi i \tau_1(\xi, w)} j(\gamma, \tau_2(\xi, w))^{\kappa} f(\tau_2(\xi, w)) \\ &\quad = y_1(\xi, w)^{-1} p\big(\eta(\xi, w)\big) e^{2\pi i \tau_1(\xi, w)} f(\tau_2(\xi, w)) \end{aligned}$$

for all $\gamma \in SL_2(\mathbb{Z})$, so that (4.14) is independent of the choice of representative for the

$SL_2(\mathbb{Z})$ action on $M_{m,2}(\mathbb{R})$.

Before we prove lemma 4.1.10 we will prove the following lemma about the integral at the heart of the matter.

Lemma 4.1.12. *Suppose that $p(\tau)$ is a polynomial in τ, and that $A, B, C \in \mathbb{R}$ are such that $Ax^2 + 2Bx + C > 0$ for all $x \in \mathbb{R}$, and $n > 0$. Then*

$$\begin{aligned}&\int_{\mathcal{H}} p(\tau)e^{2\pi i n\tau}e^{-\pi Ay-\pi(Ax^2+2Bx+C)y^{-1}}y^{-2}dxdy\\ &=\frac{1}{\sqrt{AC-B^2}}p\left(-\frac{B}{A}+i\frac{\sqrt{AC-B^2}}{A}\right)e^{-2\pi i nBA^{-1}}e^{-2\pi A^{-1}(n+A)\sqrt{AC-B^2}}\end{aligned}$$

Proof. First we note that this integral may be obtained from a simpler integral via differentiation with respect to n:

$$p\left(\frac{1}{2\pi i}\frac{d}{dn}\right)\int_{\mathcal{H}} e^{2\pi i n\tau}e^{-\pi Ay-\pi(Ax^2+2Bx+C)y^{-1}}y^{-2}dxdy$$

Looking at just the integral now, we complete the square in x to obtain:

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty e^{2\pi i n\tau}e^{-\pi Ay-\pi\left(A(x+B)^2+(C-B^2/A)\right)y^{-1}}y^{-2}dxdy\\ =&A^{-1/2}e^{-2\pi i nBA^{-1}}\int_0^\infty y^{-3/2}e^{-\pi A^{-1}(n+A)^2y-\pi(C-B^2/A)y^{-1}}dy\end{aligned}$$

as the integral over x amounts to a Fourier transform of a Gaussian. For the integral on y, we use the following two formulas (K_ν is the modified Bessel function of the second kind of order ν):

$$\int_0^\infty e^{-\pi E^2y-\pi F^2y^{-1}}y^{-\nu-1}dy = 2\left(\frac{E}{F}\right)^\nu K_\nu(2\pi EF)$$
$$K_{1/2}(2\pi r) = \frac{1}{2\sqrt{r}}e^{-2\pi r}$$

so that we get:

$$\frac{1}{\sqrt{AC-B^2}}e^{-2\pi i nBA^{-1}}e^{-2\pi A^{-1}(n+A)\sqrt{AC-B^2}}$$

and then differentiation gives the formula. □

of lemma 4.1.10. First we have that $\mathrm{tr}(w,w)_\tau = {}^tw_1Yw_1 + {}^t(xw_1+w_2)Y(xw_1+w_2)$, and we can apply lemma 4.1.12 with $A = {}^tw_1Yw_1$, $B = {}^tw_1Yw_2$ and $C = {}^tw_2Yw_2$ and $p'(\tau) = p(\eta_\tau(w))$. We then plug these values to obtain that:

$$\begin{aligned}
&\int_{\mathcal{H}} p(\eta_\tau(w))e^{2\pi i n\tau}e^{-\pi \operatorname{tr} Y(w,w)_\tau}y^{-2}dxdy \\
&\qquad = \frac{1}{y_2(\xi,w)}p\left(i\frac{\sqrt{\det {}^t wYw}}{(w_1,w_1)_Y}w_1 + w_2 - \frac{(w_1,w_2)_Y}{(w_1,w_1)_Y}w_1\right)e^{-2\pi y_1(\xi,w)}e^{2\pi i n\tau_2(\xi,w)} \\
&\qquad = \frac{1}{y_2(\xi,w)}p\big(\eta(\xi,w)\big)e^{-2\pi y_1(\xi,w)}e^{2\pi i n\tau_2(\xi,w)}
\end{aligned}$$

We obtain the lemma by summing over n with $f(\tau) = \sum_n a_n e^{2\pi i n\tau}$. □

By this point we have described each of the terms in the sum giving $a_S(Y;p)$ in (4.9). We will now discuss how we can simplify these further. Let $S \in \operatorname{Skew}_m(\mathbb{Z})$ be rank 2, and let S_0 and w_0 be as in the discussion after lemma 4.1.4, with $S = \mu S_0$. Then we have:

$$a_{\mu S_0}(Y;p) = 2\sum_{\substack{a,b,d \\ ad=\mu \\ 0\le b<a}} \int_{\mathcal{H}_1} p(\eta_\tau(w_0\gamma_{a,b,d}))f(\tau)e^{-\pi \operatorname{tr} Y(w_0\gamma_{a,b,d},w_0\gamma_{a,b,d})_\tau}\frac{dxdy}{y^2} \tag{4.15}$$

We can handle these sums tidily with the following lemma.

Lemma 4.1.13. *Recall we denote by $T(\mu)$ the degree μ Hecke operator (3.14). Then if $w \in M^+_{m,2}(\mathbb{R})$ we have:*

$$\begin{aligned}
&\sum_{\substack{a,b,d\in\mathbb{Z} \\ a,d>0 \\ ad=\mu \\ 0\le b<a}} \int_{\mathcal{H}} p\big(\eta_\tau(w\gamma_{a,b,d})\big)f(\tau)e^{-\pi \operatorname{tr} Y(w\gamma_{a,b,d},w\gamma_{a,b,d})_\tau}y^{-2}dxdy \\
&\qquad = \mu\int_{\mathcal{H}} p\big(\eta_\tau(w)\big)\big(T(\mu)f\big)(\tau)e^{-\pi\mu \operatorname{tr} Y(w,w)_\tau}y^{-2}dxdy
\end{aligned} \tag{4.16}$$

Proof. Write $M_2^\mu(\mathbb{Z})$ for the set of integral 2×2 matrices with determinant μ. Recall:

$$T(\mu)f(\tau) = \mu^{\kappa-1}\sum_{\gamma\in SL_2(\mathbb{Z})\backslash M_2^\mu(\mathbb{Z})} j(\gamma,\tau)^{-\kappa}f(\gamma\tau)$$

The expression on the left hand side of (4.16) is equal to:

$$\sum_{\gamma\in M_2^\mu(\mathbb{Z})/SL_2(\mathbb{Z})} \int_{\mathcal{H}} p\big(\eta_\tau(w\gamma)\big)f(\tau)e^{\pi i \operatorname{tr} \xi\eta_\tau(w\gamma)(\operatorname{Im}\tau)^{-1}\overline{\eta}_\tau(w\gamma)}y^{-2}dxdy$$

We have $\eta_\tau(w\gamma)(\operatorname{Im}\tau)^{-1}\overline{\eta}_\tau(w\gamma) = \mu\eta_{\gamma\tau}(w)(\operatorname{Im}\gamma\tau)^{-1}\overline{\eta}_{\gamma\tau}(w)$ so that after performing the change of variables $\tau \mapsto \gamma^{-1}\tau$, it is:

$$\sum_{\gamma \in M_2^\mu(\mathbb{Z})/SL_2(\mathbb{Z})} \int_{\mathcal{H}} p\big(\eta_\tau(w) j(\gamma^{-1},\tau)^{-1}\big) f(\gamma^{-1}\tau) e^{\mu\pi i \operatorname{tr} \xi \eta_\tau(w) (\operatorname{Im}\tau)^{-1} \overline{\eta}_\tau(w)} y^{-2} dx dy$$

We note that as γ ranges over $M_2^\mu(\mathbb{Z})/SL_2(\mathbb{Z})$, $\mu\gamma^{-1}$ ranges over $SL_2(\mathbb{Z})\backslash M_2^\mu(\mathbb{Z})$, so the above is:

$$\sum_{\gamma \in SL_2(\mathbb{Z})\backslash M_2^\mu(\mathbb{Z})} \int_{\mathcal{H}} p\big(\eta_\tau(w) j(\mu^{-1}\gamma,\tau)^{-1}\big) f(\mu^{-1}\gamma\tau) e^{\mu\pi i \operatorname{tr} \xi \eta_\tau(w) (\operatorname{Im}\tau)^{-1} \overline{\eta}_\tau(w)} y^{-2} dx dy$$

We have $j(\mu^{-1}\gamma,\tau)^{-1}) = \mu j(\gamma,\tau)^{-1}$ and $\mu^{-1}\gamma\tau = \gamma\tau$, and p is a homogeneous degree κ polynomial, so that

$$p(\eta_\tau(w) j(\mu^{-1}\gamma,\tau)^{-1}) f(\mu^{-1}\gamma\tau) = \mu p\big({}^t\alpha\eta_\tau(w)\big)\big(\mu^{\kappa-1} j(\gamma,\tau)^{-\kappa} f(\gamma\tau)\big)$$

and summing over γ we obtain $\mu T(\mu) f$. □

Corollary 4.1.14. *Suppose that w_0, S_0 are as above, and f is a cusp form. We have:*

$$a_{\mu S_0}(Y;p) = 2y_1(\xi,w_0)^{-1} p\big(\eta(\xi,w_0)\big) e^{2\pi i \mu \tau_1(\xi,w_0)} (T(\mu)f)(\tau_2(\xi,w_0)) \tag{4.17}$$

Proof. This is just an application of (4.15) and lemma 4.1.13, and we use the fact that $\tau_2(\mu\xi,w_0) = \tau_2(\xi,w_0)$ and $\tau_1(\mu\xi,w_0) = \mu\tau_1(\xi,w_0)$. □

Remark 4.1.15. *Instead of (4.17) we could write:*

$$a_{\mu S_0}(Y;p) = 2y_1(\xi,S_0)^{-1} p\big(\eta(\xi,S_0)\big) e^{2\pi i \mu \tau_1(\xi,S_0)} (T(\mu)f)(\tau_2(\xi,S_0)) \tag{4.18}$$

as in light of remark 4.1.11, we have that this formula is invariant of substitutions $w_0 \mapsto w_0\gamma$ for $\gamma \in SL_2(\mathbb{Z})$, so it depends only on the class $S_0 \in \mathrm{Skew}_m^1(\mathbb{Z})$.

Putting this all together we obtain:

Theorem 4.1.16. *Suppose that $f = \sum_n a_n e^{2\pi i n\tau}$ is a Hecke cusp eigenform, i.e. that $T(\mu)f = a_\mu f$ for all $\mu > 0$. Write $\mathrm{Skew}_m^1(\mathbb{Z})$ for the set of primitive elements rank 2 of $\mathrm{Skew}_m(\mathbb{Z})$, and for $S_0 \in \mathrm{Skew}_m^1(\mathbb{Z})$ we will write $\tau_1(\xi,S_0)$, etc. for $\tau_1(\xi,w_0)$ where $w_0 \in M_{m,2}^1(\mathbb{Z})$ is such that $\langle w_0,w_0\rangle = S_0$. Then we have that*

$$\Phi_f(\xi;p) = 2 \sum_{S_0 \in \mathrm{Skew}_m^1(\mathbb{Z})} p\big(\eta(\xi,S_0)\big) y_1(\xi,S_0)^{-1} f\big(\tau_1(\xi,S_0)\big) f\big(\tau_2(\xi,S_0)\big) \tag{4.19}$$

note that the Fourier expansion of Φ_f is contained in that of $f(\tau_2(\xi,S_0))$. If we write out *the Fourier expansion explicitly it is:*

$$\Phi_f(\xi; p) = 2 \sum_{\substack{S_0 \in \mathrm{Skew}^1_m(\mathbb{Z}) \\ \mu > 0}} e^{\pi i \operatorname{tr} \mu S_0 X} a_\mu p\big(\eta(Y, S_0)\big) y_1(Y, S_0)^{-1} e^{-2\pi\mu y_1(Y,S_0)} f\big(\tau_2(Y, S_0)\big) \quad (4.20)$$

where we have written $\tau_2(Y, S_0)$, etc. in place of $\tau_2(\xi, S_0)$ to emphasize that those terms depend only on the Y variable of ξ, so that the $a_{\mu S_0}$ Fourier coefficient is:

$$a_{\mu S_0}(Y; p) = 2 a_\mu p\big(\eta(Y, S_0)\big) y_1(Y, S_0)^{-1} e^{-2\pi\mu y_1(Y,S_0)} f\big(\tau_2(Y, S_0)\big) \quad (4.21)$$

It is also of interest to consider how these lifts are related when we vary m. For $m \geq 2$, denote by $\Phi_f^{(m)}$ the theta lift of f to $\mathcal{D}_m$. We define a function $[\xi]_{m,2}$ from $\mathcal{D}_m$ to $\mathcal{D}_2$ obtained by taking the top left 2×2 minor of ξ. In terms of the variables X and Y is also simply amounts to taking the top 2×2 minor. Each minor of a skew symmetric matrix will be skew symmetric, and each minor of a positive definite matrix is positive definite, verifying that $[\xi]_{m,2}$ is in $\mathcal{D}_2$. Next, for a polynomial $p \in \mathbb{C}[\mathbb{C}^m]$, define $[p]_{m,2} \in \mathbb{C}[\mathbb{C}^2]$ to be the polynomial obtained by setting all variables except the first two to 0, i.e. the map dual to the inclusion $\mathbb{C}^2 \hookrightarrow \mathbb{C}^m$. If $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$, then $[p]_{m,2} \in \mathbb{C}[\mathbb{C}^2]^\kappa$. We then obtain a map in the reverse direction between dual spaces which we will also denote by $[-]_{m,2} : \mathbb{C}[\mathbb{C}^2]^* \to \mathbb{C}[\mathbb{C}^m]^*$. This map sends $(\mathbb{C}[\mathbb{C}^2]^\kappa)^*$ into $(\mathbb{C}[\mathbb{C}^m]^\kappa)^*$, and is given by $[\Phi]_{m,2}(p) = \Phi([p]_{m,2})$ for $\Phi \in \mathbb{C}[\mathbb{C}^2]^*$ and $p \in \mathbb{C}[\mathbb{C}^m]$. Note that we are using the notation $[-]_{m,2}$ for 3 different maps, but we hope from context it is clear what is meant.

We will also define $\overline{P}_2(\mathbb{Z}) \subset GL_m(\mathbb{Z})$ to be the subgroup:

$$\overline{P}_2(\mathbb{Z}) = \begin{pmatrix} GL_2(\mathbb{Z}) & \\ * & GL_{m-2}(\mathbb{Z}) \end{pmatrix} \quad (4.22)$$

With this setup we have the following theorem relating the lifts $\Phi_f^{(m)}$ to $\Phi_f^{(2)}$:

Theorem 4.1.17.

$$\Phi_f^{(m)}(\xi) = \sum_{A \in \overline{P}_2(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \rho_\kappa({}^t A) \big[\Phi_f^{(2)}([A \xi\, {}^t A]_{m,2})\big]_{m,2}$$

or, evaluated on a polynomial:

$$\Phi_f^{(m)}(\xi; p) = \sum_{A \in \overline{P}_2(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \Phi_f^{(2)}\big([A \xi\, {}^t A]_{m,2}; [\sigma({}^t A^{-1}) p]_{m,2}\big)$$

Proof. For this proof only we will sometimes write $w_{0,m}^\pm$ and $w_{0,2}^\pm$ for the matrix $w_0^\pm$ defined in lemma 4.1.4 for $M_{m,2}(\mathbb{Z})$ and $M_{2,2}(\mathbb{Z})$, respectively. From that same lemma we have:

$$\Phi_f^{(m)}(\xi;p) = \sum_{A\in GL_m(\mathbb{Z})/P_2(\mathbb{Z})} \sum_{\substack{a,b,d\in\mathbb{Z}\\ a,d>0\\ 0\le b<a\\ \pm=+,-}} \int_{\mathcal{H}} p(\eta_\tau(Aw_0^\pm\gamma_{a,b,d}))f(\tau)e^{\pi i\,\mathrm{tr}\,\xi\eta_\tau(Aw_0^\pm\gamma_{a,b,d})(\mathrm{Im}\,\tau)^{-1}\overline{\eta}_\tau(Aw_0^\pm\gamma_{a,b,d})}y^{-2}dxdy \tag{4.23}$$

Looking at at single one of these terms, we can move A around and get:

$$\int_{\mathcal{H}}(\sigma(A^{-1})p)(\eta_\tau(w_0^\pm\gamma_{a,b,d}))f(\tau)e^{\pi i\,\mathrm{tr}\,{}^tA\xi A\eta_\tau(w_0^\pm\gamma_{a,b,d})(\mathrm{Im}\,\tau)^{-1}\overline{\eta}_\tau(w_0^\pm\gamma_{a,b,d})}y^{-2}dxdy$$

Then we have $w_0^\pm\gamma_{a,b,d}$ is zero in all except for the top two rows, so that

$$\sigma(A^{-1})p(w_{0,m}^\pm\gamma_{a,b,d}) = [\sigma(A^{-1})p]_{m,2}(\eta_\tau(w_{0,2}^\pm\gamma_{a,b,d})$$

Then similarly we have that $\eta_\tau(w_{0,m}^\pm\gamma_{a,b,d})y^{-1}\,{}^t\overline{\eta}_\tau(w_{0,m}\gamma_{a,b,d}))$ is 0 except in the top 2×2 block, so that

$$\mathrm{tr}\,{}^tA\xi A\eta_\tau(w_{0,m}^\pm\gamma_{a,b,d})y^{-1}\,{}^t\overline{\eta}_\tau(w_{0,m}^\pm\gamma_{a,b,d})) = \mathrm{tr}[{}^tA\xi A]_{m,2}\eta_\tau(w_{0,2}^\pm\gamma_{a,b,d})y^{-1}\,{}^t\overline{\eta}_\tau(w_{0,2}^\pm\gamma_{a,b,d}))$$

thus the inside sum of equation (4.23) is

$$\Phi_f^{(2)}([{}^tA\xi A]_{m,2})([\sigma(A)^{-1}p).$$

Then as A ranges over $GL_m(\mathbb{Z})/P_2(\mathbb{Z})$, ${}^tA$ ranges over $\overline{P}_2(\mathbb{Z})\backslash GL_m(\mathbb{Z})$. □

The previous theorem implies that the essential case for the lift to $O(m,m)$ from $SL_2(\mathbb{R})$ is at $m=2$. To finish off this section we will examine the $m=2$ case more closely here in the context of the identification $\mathcal{D}_2 \cong \mathcal{H}_1\times\mathcal{H}_1$ and the correspondence of modular forms for $O(2,2)$ and $SL_2(\mathbb{R})\times_{\pm 1}SL_2(\mathbb{R})$ described in section 3.3.1. This is achieved by first converting Φ_f into $\widetilde{\Phi}_f$, which is modular with respect to the $O(m)$ valued factors of automorphy $k^\pm$ on $\mathcal{D}_2$, and then converting that into $F_f(\tau_1,\tau_2)$ as in Section 3.3.1.

Given $\Phi_f(\xi)$, we obtain $\widetilde{\Phi}_f(\xi) = \rho_\kappa({}^t\alpha(\xi))\Phi_f(\xi)$ as in 3.32. Combined with (4.19) when f is a Hecke eigenform, we have that evaluating $\widetilde{\Phi}_f$ on a polynomial is:

$$\widetilde{\Phi}_f(\xi;p) = \det Y^{1/2}\sum_{S_0\in\mathrm{Skew}_m^1(\mathbb{Z})} p\big({}^t\alpha(\xi)\eta(\xi,S_0)\big)y_1(\xi,S_0)^{-1}f\big(\tau_1(\xi,S_0)\big)f\big(\tau_2(\xi,S_0)\big)$$

When $m=2$, there are only $S_0=\pm J$ in (4.19). We have $w_0^\pm = \begin{pmatrix}\pm1 & \\ & 1\end{pmatrix}$, with

$\langle w_0^{\pm}, w_0^{\pm}\rangle = \pm J$. As we noted before, under the identification $\mathcal{D}_2 \cong \mathcal{H}_1 \times \mathcal{H}_1$, we have:

$$\begin{aligned} \tau_1(\xi(\tau_1,\tau_2), w_0^+) &= \tau_1 \qquad & \tau_1(\xi(\tau_1,\tau_2), w_0^-) &= -\overline{\tau}_1 \\ \tau_2(\xi(\tau_1,\tau_2), w_0^+) &= \tau_2 \qquad & \tau_2(\xi(\tau_1,\tau_2), w_0^-) &= -\overline{\tau}_2 \end{aligned} \tag{4.24}$$

We have

$$\eta(\xi(\tau_1,\tau_2), w_0^{\pm}) = \begin{pmatrix} \tau_2^{\pm} \\ 1 \end{pmatrix} \tag{4.25}$$

(where $\tau_2^{\pm}$ means τ_2 for $\pm = +$ and $-\overline{\tau}_2$ when $\pm = -$. Combined with (2.44), we have

$${}^t\alpha(\xi(\tau_1,\tau_2))\eta(\xi(\tau_1,\tau_2)) = y_1^{1/2} y_2^{1/2} \begin{pmatrix} \pm i \\ 1 \end{pmatrix}$$

We also have $\det Y(\xi(\tau_1,\tau_2))^{1/2} = y_1$, so that:

$$\widetilde{\Phi}_f(\xi(\tau_1,\tau_2); p) = 2 y_1^{\kappa/2} y_2^{\kappa/2} \left(f(\tau_1) f(\tau_2) p \begin{pmatrix} i \\ 1 \end{pmatrix} + f(-\overline{\tau}_1) f(-\overline{\tau}_2) p \begin{pmatrix} -i \\ 1 \end{pmatrix} \right) \tag{4.26}$$

Next, recall the decomposition of $\mathbb{C}[\mathbb{C}^2]^{\kappa}$ into $O(m)$ irreducible representations. We have polynomials $p_{\pm}^{\kappa}(\eta) = \left({}^t\eta \begin{pmatrix} \mp i \\ 1 \end{pmatrix} \right)^{\kappa}$ of weight $\pm\kappa$ with respect to $SO_2(\mathbb{R})$. They span the $O(2)$ irreducible subspace $\mathscr{H}[\mathbb{C}^2]^{\kappa}$ of harmonic polynomials of degree κ. We have $\mathbb{C}[\mathbb{C}^2]^{\kappa} = \bigoplus_{k=0}^{\lfloor \kappa/2 \rfloor} ({}^t\eta\eta)^{\kappa} \mathscr{H}[\mathbb{C}^2]^{\kappa-2k}$. If we evaluate any polynomial that is divisible by ${}^t\eta\eta$ at $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$ we obtain 0, due to those vectors being isotropic. Thus the lift (4.26) is identically 0 except for polynomials $p \in \mathscr{H}[\mathbb{C}^2]^{\kappa}$, so we will simply assume that $\widetilde{\Phi}_f$ takes values in $(\mathscr{H}[\mathbb{C}^2]^{\kappa})^*$. If we denote by $\{\mathbf{v}_-^{\kappa}, \mathbf{v}_+^{\kappa}\}$ the dual basis to $\{p_+^{\kappa}, p_-^{\kappa}\}$ (note the sign flip), we have that $\mathbf{v}_{\pm}^{\kappa}$ are weight $\pm\kappa$ with respect to $SO(2)$, and also that

$$p_{\pm_1}^{\kappa} \begin{pmatrix} \pm_2 i \\ 1 \end{pmatrix} = \begin{cases} 2^{\kappa} & \pm_1 = \pm_2 \\ 0 & \pm_1 = \mp_2 \end{cases}$$

and so we can write (4.26) as:

$$\widetilde{\Phi}_f(\xi(\tau_1,\tau_2)) = 2^{\kappa+1} y_1^{\kappa/2} y_2^{\kappa/2} \big(f(\tau_1) f(\tau_2) \mathbf{v}_-^{\kappa} + f(-\overline{\tau}_1) f(-\overline{\tau}_2) \mathbf{v}_+^{\kappa} \big) \tag{4.27}$$

Putting this all together we have:

Theorem 4.1.18. *From the correspondence between modular forms on $\mathcal{D}_2$ and $\mathcal{H}_1 \times \mathcal{H}_1$ outlined in section 3.3.1, when f is a Hecke cusp form for $SL_2(\mathbb{Z})$, the lift $\Phi_f(\xi)$ on $\mathcal{D}_2$*

corresponds to:

$$F_f(\tau_1, \tau_2) = 2^{\kappa+1} f(\tau_1) f(\tau_2) \tag{4.28}$$

on $\mathcal{H}_1 \times \mathcal{H}_1$*.*

Remark 4.1.19. *When f is not a normalized Hecke cusp form this formula does not apply (indeed the lift is linear and this formula is clearly not). Hecke cusp forms provide a basis for $S_\kappa(\Gamma')$ so that the above formula extends linearly to give the lift for cusp forms that are not Hecke eigenforms. Alternately a slight modification of the above argument and using (4.17) we obtain that the lift is:*

$$F_f(\tau_1, \tau_2) = 2^{\kappa+1} \sum_{\mu>0} e^{2\pi i \mu \tau_1} (T(\mu) f)(\tau_2)$$

which reduces to (4.28) when f is a normalized Hecke cusp form.

4.2 Theta Lifts of Weakly Holomorphic Modular Forms

In [1], the theta lift is extended to modular forms that are allowed to have singularities at the cusp. This process requires a regularization procedure as the defining integral for the lift no longer converges due to the singularity at $i\infty$. After this regularization process, the resulting function on $\mathcal{D}$ obtains the transformation properties from the theta function as before. In general the lifts will have singularities along submanifolds of $\mathcal{D}$, whose locations correspond to vectors in the lattice Λ with certain positive lengths that are controlled by the order of the pole at the cusp of the modular form input. Borcherds considers functions on $\mathcal{D}_{m_+,m_-} \times \mathcal{H}$ (where $\mathcal{D}_{m_+,m_-}$ is the Grassmannian of maximal negative definite subspaces in $\mathbb{R}^{m_+,m_-}$) of the form:

$$\theta(\xi, \tau; \Lambda, \lambda, p) = \sum_{v \in \lambda + \Lambda} \left(\exp\left(-\frac{\triangle}{8\pi y} \right) p \right) (\nu_\xi(v)) e^{\pi i (v,v) x} e^{-\pi (v,v)_\xi y} \tag{4.29}$$

where Λ is an even integral lattice in $\mathbb{R}^{m_+,m_-}$ and Λ^* is its dual, $\lambda \in \Lambda^*$, p is a polynomial on $\mathbb{R}^{m_+ + m_-}$, and ν_ξ is an identification between $\mathbb{R}^{m_+,m_-}$ and $\mathbb{R}^{m_+ + m_-}$ that depends on ξ. These functions are modular in τ and have transformation properties with respect to analogues of the K valued factors of automorphy we have considered. By varying the lattice Λ the level can be changed, and by varying p the weight may be changed.

These functions are essentially the right hand side of (3.64) when $m_+ = m_- = m$ and $\Lambda = \mathbb{Z}^{m,m}$.

Borcherds develops a general theory for pairing (4.29) with functions on $\mathcal{H}$ with singularities at the cusp, and develops the theory for a fairly wide class of functions having Fourier expansions of the form

$$f(\tau) = \sum_{\substack{n \gg 0, \\ 0 \le k \le K}} a_{n,k} y^{-k} e^{2\pi i n \tau}$$

and computes the Fourier expansion of the regularized lift with respect to a unipotent group stabilizing an isotropic line. One point of departure we have here is that we are instead calculating the expansion with respect to the stabilizer of a maximal isotropic plane. One of the main results of [1] is that when $p = 1$ in 4.29, $m_+ = 2$, and f is weakly holomorphic with integral coefficients, the non-constant terms of the Fourier expansion of the lift can be interpreted as the logarithm of the modulus of a modular form on $O(2, m_-)$, which carries a Hermitian structure. This provides a product expansion for a large class of modular forms on $O(2, m_-)$ that have prescribed singularities, and whose weight is determined by the constant term of f.

Borcherds also defines a correspondence he calls the *Singular Shimura correspondence*, which is a linear map from weakly holomorphic modular forms on $\mathcal{H}$ to meromorphic modular functions on $O(2, m_-)$, by taking p to be certain specifically chosen harmonic polynomials in (4.29). These polynomials are essentially the p_+^κ in example 3.4.15. In example 14.4 of [1] the correspondence is worked out for $O(2, 1)$, which is related to $SL_2(\mathbb{R})$ by a double cover $SL_2(\mathbb{R}) \to SO_0(2, 1)$, obtaining a Fourier expansion for the image under the correspondence. Implicit in the formulas that Borcherds develops is the lift to $O(2, m_-)$ for other m_-, however it is not written explicitly.

Using our expansion along a parabolic stabilizing a maximal isotropic plane we will work out this correspondence for $O(2, 2)$, related to $SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$ by (2.42), and obtain the Fourier expansion for it. We will also work out what the image of the regularized theta lift is to $O(m, m)$. As in the case of lifting cusp forms there is a phenomenon where the $O(m, m)$ case with $m > 2$ is obtained from lifts to groups with smaller m. Unlike for the cusp forms the orbits of rank 1 will not simply contribute 0.

Definition 4.2.1 (Weakly Holomorphic Modular Form)**.** *A weakly holomorphic modular form of weight κ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that:*

1. *$f(\gamma\tau) = (c\tau + d)^\kappa f(\tau)$ for $\gamma \in SL_2(\mathbb{Z})$,*

2. *f has a Fourier expansion:*

$$f(\tau) = \sum_{n \ge -n_0} a_n e^{2\pi i \,\mathrm{tr}\, n\tau} \tag{4.30}$$

   for some $n_0 \ge 0$.

We will generally assume that the n_0 in the above Fourier expansion is the least such possible n_0, so that $a_{-n_0} \ne 0$ but $a_n = 0$ for $n < -n_0$.

We then want to form theta lifts using Θ^κ similar to the previous chapter:

$$\Phi_f(\xi) = \int_{\mathcal{F}} f(\tau)\Theta^\kappa(\xi,\tau)\frac{dxdy}{y^2}$$

This runs into the issue that this integral is not convergent, due to the singularity of f. Following [1] we use the following regularization procedure.

Definition 4.2.2 (Regularized Integral)**.** *For $T \geq 1$, define the truncated fundamental domain: $\mathcal{F}_T = \{\tau \in \mathcal{F} : \operatorname{Im}(\tau) \leq T\}$. Next, suppose that $\phi(\tau)$ is an $SL_2(\mathbb{Z})$ invariant function on $\mathcal{H}$, and that for $\operatorname{Re}(s) \geq s_0$, that:*

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} \phi(\tau)\frac{dxdy}{y^{2+s}}$$

exists and defines and entire function of s on that region. If we can meromorphically extend this in s to $s = 0$, then we define the regularized integral to be the constant term of the Laurent expansion at $s = 0$:

$$\int_{\mathcal{F}}^{reg} \phi(\tau)\frac{dxdy}{y^2} := CT_{s=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T} \phi(\tau)\frac{dxdy}{y^{2+s}}\right] \tag{4.31}$$

Definition 4.2.3 (Regularized Theta Lift)**.** *Suppose that f is a weakly holomorphic modular form of weight κ. Define the regularized theta lift of f to be:*

$$\Phi_f(\xi) := \int_{\mathcal{F}}^{reg} f(\tau)\Theta^\kappa(\xi,\tau)\frac{dxdy}{y^2} \tag{4.32}$$

for all ξ where the regularized integral exists.

Similar to the previous section this will define a function $\Phi_f : \mathcal{D} \to \mathcal{V}_{\rho_\kappa} \cong (\mathbb{C}[\mathbb{C}^m]^\kappa)^*$ that transforms as

$$\Phi_f(\gamma\xi) = \rho_\kappa(j^-(\gamma,\xi))\Phi_f(\xi) \tag{4.33}$$

for all $\gamma \in \Gamma$. It is evaluated on a $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$ by:

$$\Phi_f(\xi;p) = \int_{\mathcal{F}}^{reg} f(\tau)\Theta^\kappa(\xi,\tau;p)\frac{dxdy}{y^2} \tag{4.34}$$

For the values of ξ where this integral exists, and the transformation property is:

$$\Phi_f(\gamma\xi;p) = |\det j^-(\gamma,\xi)|\Phi_f\big(\xi;\sigma(j^-(\gamma,\xi)^{-1})p\big) \tag{4.35}$$

The regularized integral will have singularities occurring along sub-manifolds of $\mathcal{D}_m$ given by lower dimensional Grassmannians. For $\lambda \in \mathbb{Z}^{m,m}$ a vector of with $(\lambda,\lambda) > 0$, define $\mathcal{D}_\lambda$ to be the submanifold consisting of points ξ that correspond to negative definite subspaces perpendicular to λ:

$$\mathcal{D}_\lambda := \{\xi \in \mathcal{D} : \ \xi \perp \lambda\}. \tag{4.36}$$

We can identify $\mathcal{D}_\lambda$ in a non-canonical way with $\mathcal{D}_{m-1,m}$, the Grassmannian of negative definite m-planes in $\mathbb{R}^{m-1,m}$. It is a (real) co-dimension m submanifold. Given a weakly holomorphic modular form f, define: $\mathscr{I}(f)$ to be the (negative of) the indices in the Fourier expansion corresponding to poles of f:

$$\mathscr{I}(f) := \{n \in \mathbb{N} : \ a_{-n} \neq 0\}$$

This is a finite set whose maximal element is n_0. Define $\mathscr{D}_f$ to be:

$$\mathscr{D}_f := \bigcup_{\substack{n \in \mathscr{I}(f) \\ \lambda \in \mathbb{Z}^{m,m} \\ (\lambda,\lambda)=2n}} \mathcal{D}_\lambda$$

and define a $\xi \in \mathcal{D}$ to be *regular with respect to* f if is not contained in $\mathscr{D}_f$. Note that if $\xi \perp \lambda$, then $\xi \perp c\lambda$ for all scalars non-zero c, so $\mathcal{D}_\lambda = \mathcal{D}_{c\lambda}$, so that the same $\mathcal{D}_\lambda$ may appear multiple times on the right. It is not hard to show that any compact set will intersect only finitely many of the $\mathcal{D}_\lambda$ appearing in $\mathscr{D}_f$. For $\xi \in \mathcal{D}$, define $\Lambda(\xi)$ to be the set:

$$\Lambda(\xi) = \{\lambda \in \mathbb{Z}^{m,m} : \ \xi \perp \lambda\}$$

We have that $\Lambda(\xi) =$ iff ξ is regular, and from the previous observation for any $\xi \in \mathscr{D}_f$ the set $\Lambda(\xi)$ has only finitely many elements modulo scaling. If we define:

$$\Lambda_f(\xi) = \{\lambda \in \mathbb{Z}^{m,m} : \ \xi \perp \lambda, \ a_{-(\lambda,\lambda)/2} \neq 0\}$$

Then $\Lambda_f(\xi)$ is actually finite.

Definition 4.2.4 (Singularities)**.** *If Φ is a function on $\mathcal{D}$ and $\xi' \in \mathcal{D}$, then we will say that Φ has a singularity of type Ψ if $\Phi - \Psi$ is the restriction (to the intersection of the domains of Φ and Ψ) of a real analytic function defined in a neighborhood of ξ'.*

Proposition 4.2.5 (Singularities of Φ_f, from [1])**.** *Suppose that $p \in \mathbb{C}[\mathbb{C}^m]^\kappa$, and f is a weakly holomorphic modular form of weight κ. Then Φ_f defines a real analytic function for ξ that are regular with respect to f. For $\xi' \in \mathscr{D}_f$, we have that $\Phi_f(\xi; p)$ has a singularity of type:*

$$\det Y^{-1/2} \sum_{\lambda \in \Lambda_f(\xi)} \sum_{k=0}^{\lfloor \kappa/2 \rfloor} a_{-(\lambda,\lambda)/2} \frac{1}{k!} \left(\left(\frac{\triangle^Y}{4\pi} \right)^k p \right) (-iY^{-1}\nu_\xi^-(\lambda)) \frac{\Gamma(m/2 - 1 + \kappa - k)}{(2\pi(\lambda,\lambda)_\xi^-)^{m/2-1+\kappa-k}} \tag{4.37}$$

for ξ near ξ'.

Remark 4.2.6. *Note that as $\xi \to \lambda^\perp$, we have $(\lambda,\lambda)^-_\xi \to 0$, leading to a singularity given by negative powers of $(\lambda,\lambda)^-_\xi$, but also $\nu^-_\xi(\lambda) \to 0$, so that the polynomial term is going to 0 as well, leading to some cancellation in the singularity. We will explore this in more detail in example 4.2.7.*

We will outline the proof of this, missing out some details as they are covered in [1]. Some notes about comparison to [1]: we interpret $\mathcal{D}_m$ as the space of maximal negative definite subspaces, while Borcherds interprets it as the space of maximal positive definite subspaces, leading to the differences between the above theorem and Theorem 6.2 in [1]. We also have a slightly different theta function (compare (3.59) in ours to (4.29)).

Proof. From (3.59), we have that $\Phi_f(\xi;p)$ is the constant term at $s=0$ of the meromorphic continuation of:

$$\det Y^{-1/2} \lim_{T\to\infty} \int_{\mathcal{F}_T} \sum_{n,v} a_n \left(\exp\left(\frac{1}{4\pi}\triangle^Y y\right) p\right)(-iyY^{-1}\nu^-_\xi(v)) \\ \times e^{2\pi i(n+\frac{1}{2}(v,v))x} e^{-2\pi(n+\frac{1}{2}(v,v)_\xi)y} \frac{dxdy}{y^{2+s-m/2}}$$

The factor of $\det Y^{-1/2}$ will not effect anything so we will drop it throughout the proof and remember to put it back on at the end. If we write $\mathcal{F}_T = \mathcal{F}_1 \cup \mathcal{F}'_T$ where $\mathcal{F}'_T = \{x+iy : |x| \le 1/2, 1 \le y \le T\}$, then we have $\int_{\mathcal{F}_T} = \int_{\mathcal{F}_1} + \int_{\mathcal{F}'_T}$, with the latter integral being an integration over a rectangular region. We note that it is only this latter integral that may contribute to singularities, as the integral over $\mathcal{F}_1$ is an integral over a compact region that does not change as T increases. Thus the singularities will all come from (the constant term at $s=0$) of:

$$\lim_{T\to\infty} \int_1^T \int_0^1 \sum_{n,v} a_n \left(\exp\left(\frac{1}{4\pi}\triangle^Y y\right) p\right)(-iyY^{-1}\nu^-_\xi(v)) \\ \times e^{2\pi i(n+\frac{1}{2}(v,v))x} e^{-2\pi(n+\frac{1}{2}(v,v)_\xi)y} \frac{dxdy}{y^{2+s-m/2}}$$

The integration over x kills off all terms except where $(v,v) = -2n$, so this is:

$$\lim_{T\to\infty} \int_1^T \sum_{\frac{1}{2}(v,v)=-n} a_n \left(\exp\left(\frac{1}{4\pi}\triangle^Y y\right) p\right)(-iyY^{-1}\nu^-_\xi(v)) e^{-2\pi(n+\frac{1}{2}(v,v)_\xi)y} \frac{dy}{y^{2+s-m/2}}$$

The possibility of divergence will ultimately come from whether $(n+\frac{1}{2}(v,v)_\xi) > 0$ or not. For the terms with $n > 0$ this will never happen, and when $n=0$ we have $(v,v)_\xi = 0$ iff $v=0$, but this term is killed off by the polynomial factor, as the polynomial is homogeneous of positive degree. Thus there can only be divergences from the $n<0$ terms. Then due to the restriction of summation that $\frac{1}{2}(v,v) = -n$, we have $n + \frac{1}{2}(v,v)_\xi = -(v,v)_{\xi^-}$, so that we will only get divergences when this is 0, or equivalently that $\xi \perp v$, or in other words,

$v = \lambda \in \Lambda(\xi)$. These singular terms are then:

$$\sum_{\lambda\in\Lambda(\xi)} \int_1^\infty a_{-(\lambda,\lambda)/2} \left(\exp\left(\frac{1}{4\pi}\triangle^Y y\right) p\right) (-iyY^{-1}\nu_\xi^-(\lambda)) e^{-2\pi(\lambda,\lambda)_\xi^-)y} y^{m/2-2-s} dy$$

Note that if p is homogeneous of degree κ, $(\triangle^Y)^k p$ is homogeneous of degree $\kappa - 2k$. Thus we can extract the y from inside the polynomial and it appears as $y^{\kappa-2k}$ on the outside. Then when we take the constant term at $s = 0$, this is:

$$\sum_{\lambda\in\Lambda(\xi)} \sum_k a_{-(\lambda,\lambda)/2} \frac{1}{k!} \left(\left(\frac{\triangle^Y}{4\pi}\right)^k p\right) (-iY^{-1}\nu_\xi^-(\lambda)) CT_{s=0} \left[\int_1^\infty e^{-2\pi(\lambda,\lambda)_\xi^- y} y^{m/2-2-s+\kappa-k} dy\right]$$

The singularity is coming from $(\lambda,\lambda)_\xi^- \to 0$, and Lemma 6.1 of [1]calculates that the integral is a singularity of type:

$$(2\pi(\lambda,\lambda)_\xi^-)^{s+1-m/2-\kappa+k}\Gamma(m/2-1+\kappa-k-s)$$

provided $m/2-1+\kappa-k-s$ is not a non-positive integer. So long as $\kappa+m/2 > 1$, which will happen if $\kappa \geq 1$ or $m \geq 2$, and $k \leq \kappa/2$, we may simply take $s = 0$ in this formula and get that the singularities of Φ_f at the point in question are as claimed. □

Example 4.2.7 (Singularities for $\Phi_f(\xi; p_{\mathbf{u}}^\kappa)$)**.** *Suppose that $p(\eta)p_{\mathbf{u}}^\kappa = ({}^t\eta\mathbf{u})^\kappa$ for some $\mathbf{u} \in \mathbb{C}^m$. Such polynomials span the space $\mathbb{C}[\mathbb{C}^m]^\kappa$. We have that:*

$$\triangle^Y({}^t\eta\mathbf{u})^\kappa = \kappa(\kappa-1)({}^t\mathbf{u}Y^{-1}\mathbf{u})({}^t\eta\mathbf{u})^{\kappa-2}$$

Plugging this in to (4.37), we have that the singularities of $\Phi_f(\xi; p_{\mathbf{u}}^\kappa)$ are of the type:

$$\det Y^{-1/2} \sum_{\lambda\in\Lambda(\xi)} a_{-(\lambda,\lambda)/2} \sum_{k=0}^{\lfloor\kappa/2\rfloor} \frac{\kappa!\Gamma(m/2-1+\kappa-k)}{(4\pi)^k k!(\kappa-2k)!} \frac{({}^t\mathbf{u}Y^{-1}\mathbf{u})^k\big(-i\,{}^t\nu_\xi^-(\lambda)Y^{-1}\mathbf{u}\big)^{\kappa-2k}}{\big(\pi\,{}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi^-(\lambda)\big)^{m/2-1+\kappa-k}} \tag{4.38}$$

We have:

$$\frac{({}^t\mathbf{u}Y^{-1}\mathbf{u})^k\big(-i\,{}^t\nu_\xi^-(\lambda)Y^{-1}\mathbf{u}\big)^{\kappa-2k}}{\big(\pi\,{}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi^-(\lambda)\big)^{m/2-1+\kappa-k}} = \frac{({}^t\mathbf{u}Y^{-1}\mathbf{u})^k}{\big(\pi\,{}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi^-(\lambda)\big)^{m/2-1+\kappa/2}} \left(\frac{-i\,{}^t\nu_\xi^-(\lambda)Y^{-1}\mathbf{u}}{\big(\pi\,{}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi^-(\lambda)\big)^{1/2}}\right)^{\kappa/2-2k}$$

with the first factor blowing up at $\xi \perp \lambda$, and the second factor being bounded but discontinuous along $\mathcal{D}_\lambda$.

Example 4.2.8 (Singularities of $\widetilde{\Phi}_f(\xi)$). *Recall we defined* $\widetilde{\Phi}_f(\xi) = \rho_\kappa({}^t\alpha(\xi))\Phi_f(\xi)$ *in (3.32). We have then that* $\widetilde{\Phi}_f(\xi;p)$ *has singularities of type:*

$$\sum_{\lambda\in\Lambda(\xi)}\sum_k a_{-(\lambda,\lambda)/2}\frac{1}{k!}\left(\left(\frac{\triangle}{4\pi}\right)^k p\right)\big(-i\alpha^{-1}\nu_\xi^-(\lambda)\big)\frac{\Gamma(m/2-1+\kappa-k)}{(2\pi(\lambda,\lambda)_\xi^-)^{m/2-1+\kappa-k}} \tag{4.39}$$

Note in particular that there is no Y dependence for $\triangle$ for this formula. One advantage of this is that the operator $\triangle$ annihilates the subspace $\mathscr{H}^\kappa \subset \mathbb{C}[\mathbb{C}^m]^\kappa$, so that for $p\in\mathscr{H}^\kappa$ the singularities of $\widetilde{\Phi}_f(\xi,p)$ are simply:

$$\sum_{\lambda\in\Lambda(\xi)} a_{-(\lambda,\lambda)/2}p\big(-i\alpha^{-1}\nu_\xi^-(\lambda)\big)\frac{\Gamma(m/2-1+\kappa)}{(2\pi(\lambda,\lambda)_\xi^-)^{m/2-1+\kappa}} \tag{4.40}$$

Example 4.2.9 (Singularities with the identification $\mathcal{D}_2\cong\mathcal{H}_1\times\mathcal{H}_1$). *To begin with we will look more at the singularities of $\widetilde{\Phi}_f(\xi)$ given in (4.39) for $m=2$. Recall we defined the polynomials* $p_\pm^\kappa(\eta) = \left({}^t\eta\begin{pmatrix}\mp i\\ 1\end{pmatrix}\right)^\kappa = (\pm i\eta_1+\eta_2)^\kappa$ *as in example 3.4.15. We have that* $p_+(\eta)p_-(\eta) = {}^t\eta\eta$ *and so*

$$\mathbb{C}[\mathbb{C}^2]^\kappa = \bigoplus_{\kappa_1+\kappa_2=\kappa}\mathbb{C}\cdot p_+^{\kappa_1}(\eta)p_-^{\kappa_2}(\eta)$$

It is a straightforward verification that we have:

$$\triangle\big(({}^t\eta\eta)^{\kappa_1}p_\pm^{\kappa_2}(\eta)\big) = 2\kappa_1(\kappa_1+\kappa_2)({}^t\eta\eta)^{\kappa_1-1}p_\pm^{\kappa_2}(\eta).$$

We can adapt (4.39) to see that $\Phi_f\big(\xi;({}^t\eta\eta)^{\kappa_1}p_\pm^{\kappa_2}(\eta)\big)$, *with* $\kappa = 2\kappa_1+\kappa_2$, *has a singularity of the form:*

$$\sum_{\lambda\in\Lambda(\xi)} a_{-(\lambda,\lambda)/2}\sum_{k=0}^{\kappa_1}\frac{1}{(2\pi)^k}\frac{\kappa_1!(\kappa_1+\kappa_2-1)!}{(\kappa_1-k)!}\frac{p_\pm^{\kappa_2}\big(-i\alpha^{-1}\nu_\xi^-(\lambda)\big)}{\big(\pi\,{}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi(\lambda)\big)^{\kappa_1+\kappa_2}} \tag{4.41}$$

Now recall we have the identification $\mathcal{D}_2\cong\mathcal{H}_1\times\mathcal{H}_1$. We will phrase the above in terms of (τ_1,τ_2). We have that $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \lambda\perp\xi$ *when* $\lambda_1 = \xi\lambda_2$. *If we write* $\lambda_1 = \begin{pmatrix}a\\ b\end{pmatrix}, \lambda_2 = \begin{pmatrix}d\\ -c\end{pmatrix}$, *then* $\frac{1}{2}(\lambda,\lambda) = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \det\gamma$, *and using 2.42, we have that* $\lambda_1 = \xi(\tau_1,\tau_2)\lambda_2$ *iff* $\lambda_1 = (-x_2J + y_1\,{}^tg_{\tau_2}^{-1}g_{\tau_2}^{-1})\lambda_2$, *which is equivalent to* $y_2^{1/2}\,{}^tg_{\tau_2}\lambda_1 = \begin{pmatrix}y_1 & -x_1\\ x_1 & y_1\end{pmatrix}g_{\tau_2}^{-1}\lambda_2$. *Expanding out this out, the first and second components give the real and imaginary parts* *of the equality $a\tau_2 + b = \tau_1(c\tau_2 + d)$, so that we have:*

$$\tau_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau_2 = \gamma\tau_2$$

Thus we have that Φ_f has a singularity along the divisors $\tau_1 = \gamma\tau_2$ for $\gamma \in M_2(\mathbb{Z})$ with $\det\gamma = n$ for all the n with $a_{-n} \neq 0$. In other words $\xi(\tau_1, \tau_2)$ is not regular with respect to f if there is some $\gamma \in M_2^+(\mathbb{Z})$ with $\tau_1 = \gamma\tau_2$, and $a_{-\det\gamma} \neq 0$.

We have:

$$\alpha^{-1}\nu_\xi^-(\lambda) = (y_1y_2)^{-1/2} \begin{pmatrix} \mathrm{Im}\big((a\tau_2 + b) - \tau_1(c\tau_2 + d)\big) \\ \mathrm{Re}\big((a\tau_2 + b) - \tau_1(c\tau_2 + d)\big) \end{pmatrix}$$

so that with $p_\pm^\kappa$ as above, we have:

$$p_\pm^\kappa\big(- i\alpha^{-1}\nu_\xi^-(\lambda)\big) = \begin{cases} (y_1y_2)^{-\kappa/2}\big(- \overline{((a\tau_2 + b) - \tau_1(c\tau_2 + d))}\big)^\kappa & \pm = + \\ (y_1y_2)^{-\kappa/2}\big((a\tau_2 + b) - \tau_1(c\tau_2 + d)\big)^\kappa & \pm = - \end{cases}$$

and

$$2(\lambda, \lambda)_\xi^- = {}^t\nu_\xi^-(\lambda)Y^{-1}\nu_\xi^-(\lambda) = \frac{1}{y_1y_2}|(a\tau_2 + b) - \tau_1(c\tau_2 + d)|^2$$

Thus if $\xi(\tau_1', \tau_2')$ is not regular with respect to f, we have that $\widetilde{\Phi}_f\big(\xi(\tau_1, \tau_2), ({}^t\eta\eta)^{\kappa_1}p_\pm^{\kappa_2}(\eta)\big)$ has a singularity of type:

$$\begin{aligned} &\sum_{\substack{\gamma\in M_2^+(\mathbb{Z}) \\ \tau_1'=\gamma\tau_2'}} \frac{c_{\kappa_1,\kappa_2}a_{-\det\gamma}(y_1y_2)^{\kappa/2}}{(\gamma\tau_2 - \tau_1)^{\kappa_2}|\gamma\tau_2 - \tau_1|^{2\kappa_1}j(\gamma, \tau_2)^{\kappa_2}|j(\gamma, \tau_2)|^{2\kappa_1}} \ \text{if } \pm = + \\ &\sum_{\substack{\gamma\in M_2^+(\mathbb{Z}) \\ \tau_1'=\gamma\tau_2'}} \frac{c_{\kappa_1,\kappa_2}a_{-\det\gamma}(y_1y_2)^{\kappa/2}}{(\overline{\tau}_1 - \gamma\overline{\tau}_2)^{\kappa_2}|\overline{\tau}_1 - \gamma\overline{\tau}_2|^{2\kappa_1}j(\gamma, \overline{\tau}_2)^{\kappa_2}|j(\gamma, \overline{\tau}_2)|^{2\kappa_1}} \ \text{if } \pm = - \end{aligned} \tag{4.42}$$

near $\xi(\tau_1', \tau_2')$, where c_{κ_1,κ_2} is a constant depending only on κ_1 and κ_2. In particular when $\kappa_1 = 0$, we can form $F_f(\tau_1, \tau_2)$ as in section 3.3.1 and the end of the last section on cusp forms, and then $F_f(\tau_1, \tau_2)$ has a singularity of type

$$\sum_{\substack{\gamma\in M_2^+(\mathbb{Z}) \\ \tau_1'=\gamma\tau_2'}} a_{-\det\gamma}\frac{(\kappa - 1)!}{\pi^\kappa(\tau_1 - \gamma\tau_2)^\kappa j(\gamma, \tau_2)^\kappa} \tag{4.43}$$

near (τ_1', τ_2'). We note that these are poles of order κ on $\mathcal{H} \times \mathcal{H}$. We also note that it is possible for a point (τ_1, τ_2) to lie on multiple divisors of the form $\tau_1 = \gamma\tau_2$. If we have $\tau_1' = \gamma\tau_2'$ and $\tau_1' = \gamma'\tau_2'$, then we have $\gamma^{-1}\gamma' \in \mathrm{stab}\ (\tau_2')$, so this can potentially occur for any points τ_2 whose stabilizer intersects non-trivially (i.e. not just ± 1) with $GL_2^+(\mathbb{Q})$

Lemma 4.2.10. *Suppose that n_0 is the greatest positive integer n for which $a_{-n} \neq 0$. Then Φ_f has no singularities in the region*

$$\left\{\xi \in \mathcal{D}_m : \ {}^t u Y u > n_0 \text{ for all } u \in \mathbb{Z}^m/\{0\}\right\} \tag{4.44}$$

Remark 4.2.11. *A priori this is a region defined by an infinite amount of inequalities, however we can guarantee being in this region by bounding the minimal eigenvalue of Y from below.*

Proof. Suppose that $\lambda \perp \xi$ for some $\lambda \in \mathbb{Z}^{2m}$ with $\frac{1}{2}(\lambda, \lambda) = {}^t\lambda_1 \lambda_2 = n > 0$. As $\lambda \perp \xi$ we have $\lambda_1 = (X+Y)\lambda_2$, and plugging this into the previous equation we have ${}^t\lambda_2 Y \lambda_2 = n$. Thus if ${}^t u Y u > n_0$ for all $u \in \mathbb{Z}^m$, it would be impossible to find such a λ. □

Now we will move to calculation of the regularized lift. We again will break the summation over $w \in M_{m,2}(\mathbb{Z})$ that defines the symplectic theta function according to the rank of w.

Definition 4.2.12. *For $r = 0, 1, 2$, define:*

$$\Phi_{f,r}(\xi)(p) := \int_{\mathcal{F}}^{reg} \sum_{\substack{w \in M_{m,2}(\mathbb{Z}) \\ \text{rank } w = r}} f(\tau) p(\eta_\tau(w)) e^{\pi i \operatorname{tr} X \langle w,w \rangle} e^{-\pi \operatorname{tr} Y (w,w)_\tau} \frac{dxdy}{y^2} \tag{4.45}$$

Note the the integrand is Γ' invariant (Remark 3.4.4). We have clearly that $\Phi_f = \Phi_{f,0} + \Phi_{f,1} + \Phi_{f,2}$. We also have that $\Phi_{f,0}(\xi) = 0$ identically in what we are considering since the polynomials have no constant term. To calculate these pieces, we will do a similar unfolding procedure as in the previous section. Again we use the characterization of orbits under $SL_2(\mathbb{Z})$ in 4.1.4. We note as well that the constant term of Φ_f is equal to $\Phi_{f,1}$.

4.2.1 Rank 1 Terms

Definition 4.2.13 (Epstein Zeta Functions)**.** *Suppose that $Y \in \mathrm{Sym}_m^+(\mathbb{R})$, $v, w \in \mathbb{R}^m$, and $p \in \mathbb{C}[\mathbb{C}^m]$. Define the Epstein Zeta function for z with $\operatorname{Re} z$ large as:*

$$\zeta(Y, z; p, v, w) = \sum_{u \in \mathbb{Z}^m}' e^{2\pi i \, {}^t uv} \frac{p(u+w)}{\left(Y[u+w]\right)^z} \tag{4.46}$$

If v and w are 0 we leave them out, and if $p = 1$ we leave it out.

A priori this function converges absolutely uniformly in z for all z in any closed half plane to the right of $\mathrm{Re}(z) = \frac{m}{2} + \frac{\kappa}{2}$, where $\kappa = \deg p$. Note that this definition differs from the more common definition (see for example §1.5 in [2]) in that we allow for p to by any polynomial, instead of only homogeneous polynomials that are harmonic with respect to Y (i.e. $\triangle^Y p = 0$), and we have not made the common normalization (it is more common to write $(Y[u+w])^{z+\kappa/2}$ in the denominator instead). In [2] it is proven that these zeta functions (with the mentioned restrictions) can be extended to meromorphic functions that have a single pole at $m/2$ if p is constant and $v \in \mathbb{Z}^m$, and are entire otherwise, as well as satisfy a reflection formula in z. This is unsatisfactory for our purposes for two reasons. First of all we will desire to input in arbitrary polynomials p as these zeta functions will come up in the evaluation of $\Phi_{f,1}(\xi;p)$, and also we have in mind that we want to allow Y to vary, so that the set of polynomials that are harmonic with respect to Y will change. We will adapt the proof in [2] to obtain the following lemma:

Lemma 4.2.14. *Suppose that p is a polynomial of degree κ, and write $p(\eta) = \sum_j^\kappa p_j(\eta)$ with $p_j(\eta)$ homogeneous of degree j. Then $\zeta(z, Y; p, v, w)$ has a meromorphic continuation in z to all $\mathbb{C}$. If $v \notin \mathbb{Z}^m$ then it is entire, and otherwise it has a simple pole at $z = \frac{m}{2} + k$ for $k \in \{0, \dots, \lfloor \frac{\kappa}{2} \rfloor\}$ such that $\big((\triangle^Y)^k p_{2k}\big)(0) \neq 0$, where it has a residue of:*

$$\frac{\pi^{m/2}}{4^k k! \Gamma\left(\frac{m}{2}+k\right) \det Y^{1/2}} \big((\triangle^Y)^k p_{2k}\big)(0) \tag{4.47}$$

and ζ satisfies a functional equation of the form:

$$\begin{aligned}&\frac{\Gamma(z)}{\pi^z}\zeta(z, Y; p(\eta), v, w) = \\ &\frac{e^{-2\pi i\, {}^t v w}}{\det Y^{1/2}} \sum_{j=0}^{\kappa} \sum_{k=0}^{\lfloor j/2 \rfloor} \frac{(-i)^{j-2k}\Gamma\left(\frac{m}{2}+j-k-z\right)}{4^k k! \pi^{m/2+j-z}} \\ &\qquad\qquad \times \zeta\Big(\frac{m}{2}+j-k-z, Y^{-1}; \big((\triangle^Y)^k p_j\big)(Y^{-1}\eta), -w, v\Big)\end{aligned} \tag{4.48}$$

Remark 4.2.15. *Under the assumption that p is homogeneous of degree κ and is harmonic with respect to Y, this reduces to:*

$$\frac{\Gamma(z)}{\pi^z}\zeta(z, Y; p(\eta), v, w) = \frac{e^{-2\pi i\, {}^t v w}}{\det Y^{1/2}} \frac{\Gamma\left(\frac{m}{2}+\kappa-z\right)}{\pi^{m/2+\kappa-z}} \zeta\left(\frac{m}{2}+\kappa-z, Y^{-1}; p(Y^{-1}\eta), -w, v\right)$$

which is Theorem 3 in §1.5 of [2] after accounting for our difference of convention.

Proof. We follow the same structure as §1.5 in [2], making appropriate changes. Define:

$$g(Y, p; v, w) = \sum_{u \in \mathbb{Z}^m} e^{-\pi Y[u+w] + 2\pi i\, {}^t v u} p(u+w)$$

If we calculate the Fourier transform:

$$\begin{aligned}&\int_{\mathbb{R}^m} e^{2\pi i\, {}^t x y} e^{-\pi Y[x+w] + 2\pi i\, {}^t v x} p(x+w)\, dx \\ &= \det Y^{-1/2} e^{-2\pi i\, {}^t v w} e^{-\pi Y^{-1}[y+v] - 2\pi i\, {}^t w y} \left(\exp\left(\frac{1}{4\pi}\triangle^Y\right) p\right)(-iY^{-1}(y+v))\end{aligned}$$

so that by Poisson summation we have:

$$g(Y,p(\eta);v,w) = e^{-2\pi i\,{}^t vw} \det Y^{-1/2} g(Y^{-1}, (e^{\triangle^Y/4\pi}p)(-iY^{-1}\eta); -w, v)$$

We use the fact that:

$$\int_0^\infty t^z e^{-\pi A t}\frac{dt}{t} = \pi^{-z}\Gamma(z)A^{-z}$$

So that we have for z in a closed half plane to the right of $\mathrm{Re}(x) = \frac{m}{2} + \frac{\kappa}{2}$:

$$\begin{aligned} \frac{\Gamma(z)}{\pi^z}\zeta(z,Y;p,v,w) &= \sum_{u\in\mathbb{Z}^m}{}' e^{2\pi i\,{}^t vu}\int_0^\infty t^{z+\frac{\kappa}{2}} e^{-\pi t Y[u+w]}\frac{dt}{t} \\ &= \int_0^\infty t^{z+\frac{\kappa}{2}} \sum_{u\in\mathbb{Z}^m}{}' e^{-\pi t Y[u+w]+2\pi i\,{}^t vu}\frac{dt}{t} \end{aligned} \tag{4.49}$$

where we may interchange the sum and integral due to the absolute convergence of the sum in this right half plane. The expression inside the integral is almost equal to $g(Y,p;v,w)$ we defined above, except possibly the term where $u+w=0$. Define:

$$k(w) = \begin{cases} 1 & w \in \mathbb{Z}^m \\ 0 & w \notin \mathbb{Z}^m \end{cases}$$

then we have that:

$$\sum_{u\in\mathbb{Z}^m}{}' e^{-\pi t Y[u+w]+2\pi i\,{}^t vu} = g(Y,p;v,w) - e^{-2\pi i\,{}^t vw}p(0)k(w)$$

We break the integral over t from 0 to ∞ into one from 0 to 1 and another from 1 to ∞, and then (4.49) is:

$$\begin{aligned} &\int_1^\infty t^z \sum_{u\in\mathbb{Z}^m}{}' e^{-\pi t Y[u+w]+2\pi i\,{}^t vu}\frac{dt}{t} + \int_0^1 t^{z+\frac{\kappa}{2}}\Big(g(tY,p;v,w) - e^{-2\pi i\,{}^t vw}k(w)p(0)\Big)\frac{dt}{t} \\ &= \int_1^\infty t^z \sum_{u\in\mathbb{Z}^m}{}' e^{-\pi t Y[u+w]+2\pi i\,{}^t vu}\frac{dt}{t} + e^{-2\pi i\,{}^t vw}k(w)p(0)\int_0^1 t^z\frac{dt}{t} + \\ &\quad + \int_0^1 t^{z-\frac{m}{2}}\det Y^{-1/2} e^{-2\pi i\,{}^t vw}\Big(g\left(t^{-1}Y^{-1}, (e^{\triangle^{tY}/4\pi}p)(-it^{-1}Y^{-1}\eta); -w, v\right)\Big)\frac{dt}{t} \end{aligned}$$

The first integral defines an entire function of z, and the second is easily evaluated as:

$$e^{-2\pi i\,{}^t vw}k(w)p(0)\int_0^1 t^z\frac{dt}{t} = e^{-2\pi i\,{}^t vw}k(w)p(0)z^{-1}$$

For the third integral, consider that $\triangle^{tY} = t^{-1}\triangle^{Y}$, and we have:

$$(e^{\triangle^Y/4\pi t}p)(-it^{-1}Y^{-1}\eta) = \sum_{j=0}^{\kappa}\sum_{k=0}^{\lfloor j/2\rfloor}\frac{(-i)^{j-2k}}{(4\pi)^k k!}t^{k-j}\big((\triangle^Y)^k p_j\big)(Y^{-1}\eta)$$

The constant term of this is:

$$\sum_{k=0}^{\lfloor \kappa/2\rfloor}\frac{\big((\triangle^Y)^k p_{2k}\big)(0)}{(4\pi)^k k!}t^{-k}$$

so that we have that the third integral is equal to:

$$\int_0^1 t^{z-\frac{m}{2}}\det Y^{-1/2}e^{-2\pi i\,{}^t vw}\sideset{}{'}\sum_{u\in\mathbb{Z}^m} e^{-2\pi i\,{}^t uw-\pi t^{-1}Y^{-1}[u+v]} \times \sum_{j,k}\frac{(-i)^{j-2k}}{(4\pi)^k k!}t^{k-j}\big((\triangle^Y)^k p_j\big)\big(Y^{-1}(u+v)\big)\frac{dt}{t}$$
$$+\,k(u)\det Y^{-1/2}\sum_{k=0}^{\lfloor \kappa/2\rfloor}\frac{\big((\triangle^Y)^k p_{2k}\big)(0)}{(4\pi)^k k!}\int_0^1 t^{z-\frac{m}{2}-k}\frac{dt}{t}$$

Thus, we have:

$$\begin{aligned}
&\zeta(z,Y;p,v,w) = \\
&\frac{\pi^z}{\Gamma(z)}\Bigg(-e^{-2\pi i\,{}^t vw}k(w)p(0)z^{-1}+\frac{k(u)}{\det Y^{1/2}}\sum_{k=0}^{\lfloor \kappa/2\rfloor}\frac{\big((\triangle^Y)^k p_{2k}\big)(0)}{(4\pi)^k k!}\Big(z-\frac{m}{2}-k\Big)^{-1}+ \\
&+\int_1^\infty t^z\sideset{}{'}\sum_{u\in\mathbb{Z}^m}e^{-\pi tY[u+w]+2\pi i\,{}^t vu}\frac{dt}{t} \\
&+\frac{e^{-2\pi i\,{}^t vw}}{\det Y^{1/2}}\sum_{j,k}\frac{(-i)^{j-2k}}{(4\pi)^k k!}\int_1^\infty t^{\frac{m}{2}+k-j-z} \\
&\qquad\times\sideset{}{'}\sum_{u\in\mathbb{Z}^m}e^{-2\pi i\,{}^t uw-\pi tY^{-1}[u+v]}\big((\triangle^Y)^k p_j\big)(Y^{-1}(u+v))\frac{dt}{t}\Bigg)
\end{aligned}\tag{4.50}$$

We have that the pole from z^{-1} cancels with the 0 from $\frac{1}{\Gamma(z)}$, and we are left with all the poles coming from:

$$\frac{k(u)}{\det Y^{1/2}}\sum_{k=0}^{\lfloor \kappa/2\rfloor}\frac{\big((\triangle^Y)^k p_{2k}\big)(0)}{(4\pi)^k k!}\Big(z-\frac{m}{2}-k\Big)^{-1}$$

due to the two integrals defining an entire function of z. We are also able to deduce the functional equation from this line as well. If we start with the right hand side of (4.48) and perform the same steps we will arrive the right hand side of (4.50). □

Theorem 4.2.16. *The rank* 1 *piece is given by:*

$$\Phi_{f,1}(\xi;p) = a_0 CT_{z=1}\left[\frac{\Gamma(z)}{\pi^z}\zeta(z,Y;p)\right] \tag{4.51}$$

where ζ *is the Epstein zeta function defined above.*

We will give some examples where we can provide a more explicit formula for this after the proof.

Proof. We can group terms in (4.45) according to lemma 4.1.4, and we have the rank 1 term is the constant term at $s = 0$ of:

$$\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\substack{w=(0,u)\\ u\in\mathbb{Z}^m}}{}' \sum_{\gamma\in\Gamma'/\Gamma'_w}\int_{\mathcal{F}_T} f(\tau)p(\eta_\tau(w\gamma))e^{-\pi\,\mathrm{tr}\,Y(w\gamma,w\gamma)_\tau}\frac{dxdy}{y^{2+s}}$$

in the same way as in the proof of lemma 4.1.3, this is:

$$\lim_{T\to\infty}\sum_{\substack{w=(0,u)\\ u\in\mathbb{Z}^m}}{}' \sum_{\gamma\in\Gamma'/\Gamma'_w}\int_{\gamma\mathcal{F}_T} f(\tau)p(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}|j(\gamma^{-1},\tau)|^{2s}\frac{dxdy}{y^{2+s}}$$

For $w = (0,u)$, we have $(w,w)_\tau = uy^{-1}\,{}^tu$, and $\eta_\tau(w) = u$, so that this is:

$$\lim_{T\to\infty}\sum_{u\in\mathbb{Z}^m}{}' \sum_{\gamma\in\Gamma'/\Gamma'_w}\int_{\gamma\mathcal{F}_T} f(\tau)p(u)e^{-\pi\,{}^tuYuy^{-1}}|j(\gamma^{-1},\tau)|^{2s}\frac{dxdy}{y^{2+s}} \tag{4.52}$$

which differs from :

$$\sum_{u\in\mathbb{Z}^m}{}' \lim_{T\to\infty}\int_0^T\int_{-1/2}^{1/2} f(\tau)p(u)e^{-\pi\,\mathrm{tr}\,{}^tuYuy^{-1}}\frac{dxdy}{y^{2+s}} \tag{4.53}$$

by an entire function of s that goes to 0 at $s = 0$. We will take a minute to verify this claim. We consider that (4.52) and (4.53) are the same on the region with $y \geq 1$ (as this is contained in the region with $\gamma = 1$ in the first expression), so that their difference is:

$$\lim_{T\to\infty}\sum_{\substack{\gamma\in\Gamma'/\Gamma'_w\\ \gamma\neq 1}}\int_{\gamma\mathcal{F}_T} f(\tau)p(u)e^{-\pi\,{}^tuYuy^{-1}}(1-|j(\gamma^{-1},\tau)|^{2s})\frac{dxdy}{y^{2+s}} \tag{4.54}$$

We note that $|j(\gamma,\gamma^{-1}\tau)| = |j(\gamma^{-1},\tau)|$, and when $\tau\in\gamma\mathcal{F}$, we have $|j(\gamma^{-1},\tau)| < 1$, so that this integral is dominated by:

$$\int_0^1\int_0^1 |f(\tau)|e^{-\pi\,{}^tuYuy^{-1}}\frac{dxdy}{y^{2+s}} \tag{4.55}$$

We have that $f(-\tau^{-1}) = \tau^\kappa f(\tau)$, so that we have for y near 0, $f(x+iy) = (x+iy)^{-\kappa} f(\frac{-x+iy}{x^2+y^2})$. As $y \to \infty$ we have that $|f(\tau)| = O(e^{2\pi n_0 y})$, so that as $y \to 0$, and so we have $|f(\tau)| = O(y^{-\kappa} e^{2\pi n_0 y^{-1}})$. Thus so long as Y is in the region where ${}^t uYu > 2n_0$ for all $u \in \mathbb{Z}^m - \{0\}$, we have that (4.55) converges on this region. Then we have that (4.54) gives an entire function of s that goes to 0 at $s = 0$.

Thus we consider now (4.53). We can integrate in x which kills off all terms from $f(\tau)$ except for the constant term, and then taking the limit as $T \to \infty$, so that we have:

$$\Phi_{f,1}(\xi; p) = CT_{s=0} \left[a_0 \sum_{u \in \mathbb{Z}^m}{}' \int_0^\infty p(u) e^{-\pi \operatorname{tr} {}^t uYuy^{-1}} \frac{dy}{y^{2+s}} \right]$$

This integral over y is easily seen to be:

$$\int_0^\infty p(u) e^{-\pi \operatorname{tr} {}^t uYuy^{-1}} \frac{dy}{y^{2+s}} = \frac{\Gamma(1+s)}{\pi^{1+s}} \frac{p(u)}{({}^t uYu)^{1+s}} \Gamma(s+1)$$

and summing over u we obtain:

$$\sum_{u \in \mathbb{Z}^m}{}' \frac{\Gamma(1+s)}{\pi^{1+s}} \frac{p(u)}{({}^t uYu)^{1+s}} = \frac{\Gamma(1+s)}{\pi^{1+s}} \zeta(1+s, Y; p)$$

From the previous lemma this can be meromorphically continued to all of $\mathbb{C}$, and taking the constant term at $s = 0$ amounts to taking the constant term at $z = 1$. We note that the right hand side of (4.51) is real analytic for on all of $\operatorname{Sym}_m^+(\mathbb{R})$, so that since the two sides agree on the region described in the proof, they agree on all of $\operatorname{Sym}_m^+(\mathbb{R})$. $\square$

Example 4.2.17 ($m \geq 3$)**.** *When $m \geq 3$, the function $\zeta(z, Y; p)$ is regular at $z = 1$ due to all potential singularities being at $z = m/2$ and to the right of it, so that we may simply evaluate at $z = 1$ and we have:*

$$\Phi_{f,1}(\xi; p) = a_0 \frac{\zeta(1, Y; p)}{\pi} \tag{4.56}$$

where a_0 is the constant coefficient of f. In some cases we can do better than this and obtain a more explicit formula. The sum defining $\zeta(z, Y; p)$ for z large (4.46) does not converge at $z = 1$. Instead, we can use the reflection formula (4.48), to have:

$$\begin{aligned} &\frac{\zeta(1, Y; p(\eta))}{\pi} \\ &= \frac{(-1)^{\kappa/2} \det Y^{-1/2}}{\pi^{m/2+\kappa-1}} \sum_{k=0}^{\kappa/2} \frac{\Gamma\left(\frac{m}{2} + \kappa - k - 1\right)}{4^k k!} \zeta\left(\frac{m}{2} + \kappa - k - 1, Y^{-1}; \left((\triangle^Y)^k p\right)(Y^{-1}\eta)\right) \end{aligned} \tag{4.57}$$

the sums in the definition of the zeta function above are:

$$\sum_{u\in\mathbb{Z}^m}{}' \frac{\big((\triangle^Y)^k p\big)(Y^{-1}u)}{(Y^{-1}[u])^{\frac{m}{2}+\kappa-k-1}} \tag{4.58}$$

and the summand is on the order of $||u||^{-m-\kappa+2}$*, so that when* $\kappa \geq 3$*, the sum converges, giving:*

$$\Phi_{f,1}(\xi;p) = a_0 \frac{(-1)^{\kappa/2}}{\pi^{m/2+\kappa-1}\det Y^{1/2}} \sum_{k=0}^{\kappa/2} \frac{\Gamma\left(\frac{m}{2}+\kappa-k-1\right)}{4^k k!} \sum_{u\in\mathbb{Z}^m}{}' \frac{\big((\triangle^Y)^k)p\big)(Y^{-1}u)}{({}^tuY^{-1}u)^{m/2+\kappa-k-1}} \tag{4.59}$$

Example 4.2.18 $(\widetilde{\Phi}_{f,1})$**.** *We will work out some more explicit formulas for evaluating* $\widetilde{\Phi}_{f,1}(\xi;p)$*. We have first that:*

$$\widetilde{\Phi}_{f,1}(\xi;p) = \det Y^{1/2} CT_{z=1}\left[\frac{\Gamma(z)}{\pi^z}\zeta\big(z,Y;\sigma({}^t\alpha^{-1})p\big)\right] \tag{4.60}$$

The advantage to this form of the function is that the operator $\triangle^Y$ *behaves much more nicely in this case, due to the fact that* $\triangle^Y\sigma({}^t\alpha^{-1}p) = \sigma({}^t\alpha^{-1})\triangle p$*. Recall we have the decomposition:*

$$\mathbb{C}[\mathbb{C}^m]^\kappa = \bigoplus_{k=0}^{\lfloor\kappa/2\rfloor} ({}^t\eta\eta)^k \mathscr{H}[\mathbb{C}^m]^{\kappa-2k} \tag{4.61}$$

where $\mathscr{H}[\mathbb{C}^m]^\kappa$ *is the subspace of harmonic polynomials that are homogeneous of degree* κ*. The space* $\mathscr{H}[\mathbb{C}^m]^\kappa$ *is spanned by polynomials of the form*

$$p_{\mathbf{u}}^\kappa(\eta) = {}^t(\eta\mathbf{u})^\kappa$$

where $\mathbf{u}\in\mathbb{C}^m$ *is an isotropic vector. We have that*

$$\triangle\big(({}^t\eta\eta)^{\kappa_1}p_{\mathbf{u}}^{\kappa_2}(\eta)\big) = 2\kappa_1(\kappa_1+\kappa_2)({}^t\eta\eta)^{\kappa_1-1}p_{\mathbf{u}}^{\kappa_2}(\eta) \tag{4.62}$$

(the same holds for $p\in\mathscr{H}[\mathbb{C}^m]^{\kappa_2}$*). For* $p\in\mathscr{H}[\mathbb{C}^m]^{\kappa_2}$ *we have that*

$$\triangle^k\big(({}^t\eta\eta)^{\kappa_1}p\big)(0) = \begin{cases} 2^{\kappa_1}(\kappa_1!)^2 & \kappa_2 = 0 \text{ and } \kappa_1 = k \\ 0 & \text{otherwise}\end{cases} \tag{4.63}$$

Thus we have that $\zeta\big(z,Y,({}^t\eta\eta)^{\kappa_1}p\big)$ *is entire in* z *unless* $p=1$*, in which case it has a pole of order* 1 *at* $\frac{m}{2}+\kappa_1$ *of residue* $\frac{\pi^{m/2}\kappa_1!}{2^{\kappa_1}\Gamma\left(\frac{m}{2}+\kappa_1\right)}$*. Again we can use the reflection formula to obtain that if* $\kappa\geq 3$*, and* $p\in\mathbb{C}[\mathbb{C}^m]^\kappa$ *of the form* $p(\eta) = ({}^t\eta\eta)^{\kappa_1}p_0(\eta)$ *with* $p_0(\eta)\in\mathscr{H}[\mathbb{C}^m]^{\kappa_2}$ *and* $2\kappa_1+\kappa_2=\kappa$*, we have:*

$$\widetilde{\Phi}_{f,1}\big(\xi;({}^t\eta\eta)^{\kappa_1}p_0(\eta)\big) = a_0 C_{\kappa_1,\kappa_2} \sum_{u\in\mathbb{Z}^m}{}' \frac{p_0(\alpha^{-1}u)}{(Y^{-1}[u])^{m/2+\kappa_1+\kappa_2-1}} \tag{4.64}$$

where C_{κ_1,κ_2} is the constant:

$$C_{\kappa_1,\kappa_2} = \frac{(-1)^{\kappa_1}}{i^{\kappa_2}\pi^{m/2+\kappa-1}} \sum_{k=0}^{\kappa_1} \frac{(-1)^k\Gamma(m/2+\kappa-k-1)}{2^k}\binom{\kappa_1}{k}\frac{(\kappa_1+\kappa_2)!}{(\kappa_1+\kappa_2-k)!}$$

Example 4.2.19 (Identification $\mathcal{D}_2 \cong \mathcal{H}_1\times\mathcal{H}_1$)**.** *We have from (2.44) that:*

$$\alpha(\tau_1,\tau_2)^{-1}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \frac{1}{(y_1y_2)^{1/2}}\begin{pmatrix}y_2u_1\\x_2u_1+u_2\end{pmatrix}$$

so that

$$\big(({}^t\eta\eta)^{\kappa_1}p_\pm^{\kappa_2}\big)\left(\alpha(\tau_1,\tau_2)\begin{pmatrix}u_1\\u_2\end{pmatrix}\right) = (y_1y_2)^{-\kappa/2}|\tau_2u_1+u_2|^{2\kappa_1}\times\begin{cases}(\overline{\tau}_2u_1+u_2)^{\kappa_2} & \pm=+\\(\tau_2u_1+u_2)^{\kappa_2} & \pm=-\end{cases}$$

and we have $Y(\tau_1,\tau_2)^{-1}[u]=(y_1y_2)^{-1}|\tau_2u_1+u_2|^2$. Thus, for $\kappa\geq 3$ we have:

$$\widetilde{\Phi}_{f,1}\big(\xi(\tau_1,\tau_2);p_+^\kappa\big) = a_0\frac{(y_1y_2)^{\kappa/2}(\kappa-1)!}{i^\kappa\pi^\kappa}\sum_{u\in\mathbb{Z}^2}{}'\frac{1}{(\tau_2u_1+u_2)^\kappa} \tag{4.65}$$

If we write $E_\kappa(\tau)$ for the rank κ holomorphic Eisenstein series, normalized so that is has constant term 1, this is :

$$\widetilde{\Phi}_{f,1}\big(\xi(\tau_1,\tau_2);p_+^\kappa\big) = a_0 y_1^{\kappa/2}y_2^{\kappa/2}\frac{(\kappa-1)!\zeta(\kappa)}{i^\kappa\pi^\kappa}E_\kappa(\tau_2) \tag{4.66}$$

After we have calculated the rank 2 piece as well we will return to this to give an expression for $F_f(\tau_1,\tau_2)$.

4.2.2 Rank 2 Terms

We move on to the calculation of the rank 2 piece, $\Phi_{f,2}$. We will first need to do some preparation to deal with the regularization process and perform the unfolding. To begin with, recall we have:

$$\Phi_{f,2}(\xi;p) = CT_{s=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\substack{w\in M_{m,2}(\mathbb{Z})\\ \text{rank } w=2}} f(\tau)p(\eta_\tau(w))e^{\pi i\,\mathrm{tr}\,X\langle w,w\rangle}e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}\frac{dxdy}{y^{2+s}}\right] \tag{4.67}$$

Let $\mathcal{R}_A$ to be the region in $\mathrm{Sym}_m^+(\mathbb{R})$ defined by:

$$\mathcal{R}_A = \{Y \in \mathrm{Sym}_m^+(\mathbb{R}) : \ {}^t u Y u > A \text{ for all } u \in \mathbb{Z}^m \backslash \{0\}\} \tag{4.68}$$

We remark that this region is a cone in $\mathrm{Sym}_m^+(\mathbb{R})$ in the sense that if $Y \in \mathcal{R}_A$ then $tY \in \mathcal{R}_A$ for any $t \in \mathbb{R}_{\geq 1}$. Also note that this is the region described in Lemma 4.2.10 if we take $A = n_0$. For f a weakly holomorphic modular form, we have a positive constant c_f such that $|f(\tau)| \lesssim \max(e^{c_f y}, e^{c_f y^{-1}})$, where $f \lesssim g$ means that $f \leq cg$ for some constant c.

Lemma 4.2.20. *Let $c_f > 0$ be a positive constant such that $|f(\tau)| \lesssim \max(e^{c_f y}, e^{c_f y^{-1}})$. Then for $Y \in \mathcal{R}_{4c_f/3\pi}$, we have that $\Phi_{f,2}(\xi;p)$ is real analytic and is given by:*

$$\Phi_{f,2}(\xi;p) = \sum_{\substack{S \in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S=2}} e^{\pi i \operatorname{tr} SX} a_S(Y;p)$$

where $a_S(Y;p)$ is the constant term at $s = 0$ of:

$$a_S(Y;p;s) = 2 \sum_{\substack{a,b,d \in \mathbb{Z}_{\geq 0} \\ ad=\mu \\ 0 \leq b < a}} \int_{\mathcal{H}} f(\tau) p(\eta_\tau(w_0\gamma_{a,b,d})) e^{-\pi \operatorname{tr} Y(w_0\gamma_{a,b,d}, w_0\gamma_{a,b,d})_\tau} \frac{dxdy}{y^{2+s}}$$

where $S = \mu S_0$ with S_0 primitive, and $w_0 \in M^1_{m,2}(\mathbb{Z})$ is such that $\langle w_0, w_0 \rangle$ and $\operatorname{Im} \tau_2(Y, w_0) \geq \sqrt{3}/2$. This is an entire function in s.

Proof. Suppose that Y is in $\mathcal{R}_{4c_f/3\pi}$, and consider the expression

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \sum_{\substack{w \in M_{m,2}(\mathbb{Z}) \\ \mathrm{rank}\ w=2}} f(\tau) p(\eta_\tau(w)) e^{\pi i \operatorname{tr} X\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^{2+s}} \tag{4.69}$$

for s is in some right half plane $\mathrm{Re}(s) \geq s_0$. We have the estimate $\pi \operatorname{tr} Y(w,w)_\tau > \pi\, {}^t w_1 Y w_1 y > c_f y$, so that for $Y \in \mathcal{R}_{4c_f/3\pi}$, we have $|f(\tau)| e^{-\pi \operatorname{tr} Y(w,w)_\tau} \lesssim e^{-C_1 \pi \operatorname{tr} Y(w,w)_\tau}$ for some positive constant C_1. Thus we may interchange the sum and the limit in the above expression. Define $a'_S(Y;p;s)$ to be:

$$a'_S(Y;p;s) = \sum_{\substack{w \in M_{m,2}(\mathbb{Z}) \\ \langle w,w\rangle = S}} \int_{\mathcal{F}} f(\tau) p(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^{2+s}}$$

and so we have that (4.69) is equal to:

$$\sum_{\substack{S \in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S=2}} e^{\pi i \operatorname{tr} XS} a'_S(Y;p;s)$$

Now we consider these individual pieces a'_S. By the same unfolding process as in the proof of theorem 4.2.16, we have that

$$a'_S(Y;p;s) = \sum_{[w]\in\mathcal{O}^S} \sum_{\gamma\in SL_2(\mathbb{Z})} \int_{\gamma\mathcal{F}} f(\tau)p(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} |j(\gamma^{-1},\tau)|^{2s} \frac{dxdy}{y^{2+s}} \tag{4.70}$$

Note that $\sum_{\gamma\in SL_2(\mathbb{Z})}\int_{\gamma\mathcal{F}}$ is actually $\int_{\mathcal{H}}$, but we have written it like this because of the presence of $|j(\gamma^{-1},\tau)|^{2s}$. Also, we note that the sum $[w]\in\mathcal{O}^S$ is finite. For a given $S\in \operatorname{Skew}_m(\mathbb{Z})$ with rank $S=2$, there is a unique S_0 that is primitive and $S=\mu S_0$ for some $\mu\in\mathbb{Z}_{>0}$. There is also a $w_0\in M^+_{m,2}(\mathbb{Z})$ so that $\langle w_0,w_0\rangle = S_0$, and then the elements of $\mathcal{O}^{\mu S_0}$ have representatives of the form $w_0\gamma_{a,b,d}$ ($\gamma_{a,b,d} = \begin{pmatrix} a & b \\ & d\end{pmatrix}$), where $a,d>0$, $ad=\mu$, and $0\le b<a$. Define $a_S(Y;p;s)$ to be:

$$a_S(Y;p;s) = \sum_{[w]\in\mathcal{O}^S} \int_{\mathcal{H}} f(\tau)p(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^{2+s}} \tag{4.71}$$

and

$$a''_S(Y;p;s) = \sum_{[w]\in\mathcal{O}^S} \int_{\mathcal{H}} |f(\tau)p(\eta_\tau(w))|e^{-\pi \operatorname{tr} Y(w,w)_\tau} \frac{dxdy}{y^{2+s}}$$

We have that the integrand in the definition of a''_S is non-negative and dominates that of a_S and a'_S (this is obvious for a_S and for a'_S we use that $|j(\gamma^{-1},\tau)|<1$). We will show that for Y in the region described it converges absolutely uniformly on compact subsets of Y for all values of s, and so defines an entire function of s. After we have shown this we obtain the same for a'_S and a_S, and since their difference goes to 0 at $s=0$, we have that $a_S(Y;p;0)=a_S(Y;p;s)$. Now to show that the integral for a''_S converges for all s.

We will choose the representative $w_0=(w_{0,1},w_{0,2})$ for $\mathcal{O}^{S_0}$ such that

$$y_2(\xi,w_0) = \frac{\sqrt{\det {}^t w_0 Y w_0}}{{}^t w_{0,1} Y w_{0,1}} \ge \frac{\sqrt{3}}{2}$$

($y_2(\xi,w_0)$ is the imaginary part of $\tau_2(\xi,w_0)$). We may do this since a different choice of representative amounts to multiplying w_0 on the right by an element of $SL_2(\mathbb{Z})$, which amounts to acting on $\tau_2(\xi,w_0)$ by the inverse of that element. Thus by choice of representative we can fix it so that $\tau_2(\xi,w_0)$ is in the fundamental region for the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$, where $\operatorname{Im}\tau\ge\sqrt{3}/2$.

Now, we have that

$$\operatorname{tr} Y(w,w)_\tau = {}^t w_1 Y w_1 y + {}^t w_1 Y w_1 \left(x + \frac{{}^t w_1 Y w_2}{{}^t w_1 Y w_1}\right)^2 y^{-1} + \left({}^t w_2 Y w_2 - \frac{({}^t w_1 Y w_2)^2}{{}^t w_1 Y w_1}\right) y^{-1}$$

If we have that $w = w_0\gamma_{a,b,d}$ then we have that

$$ {}^t w_1 Y w_1 = a^2\, {}^t w_{0,1} Y w_{0,1}, $$
$$ \left({}^t w_2 Y w_2 - \frac{({}^t w_1 Y w_2)^2}{{}^t w_1 Y w_1}\right) = d^2\left({}^t w_{0,2} Y w_{0,2} - \frac{({}^t w_{0,1} Y w_{0,2})^2}{{}^t w_{0,1} Y w_{0,1}}\right) $$

Now, since Y is in $\mathcal{R}_{4c_f/3\pi}$, we have that $\pi w_{0,1} Y w_{0,1} > c_f$, and also that

$$ \pi\left({}^t w_{0,2} Y w_{0,2} - \frac{({}^t w_{0,1} Y w_{0,2})^2}{{}^t w_{0,1} Y w_{0,1}}\right) = \pi y_2(\xi, w_0)^2\, {}^t w_{0,1} Y w_{0,1} > c_f $$

Thus we can find some C_2 so that $|f(\tau) e^{-\pi \operatorname{tr} Y(w,w)_\tau}| \lesssim e^{-C_2 \operatorname{tr} Y(w,w)_\tau}$, and so we have

$$ |f(\tau) p(\eta_\tau(w)) e^{-C_2 \operatorname{tr} Y(w,w)_\tau} y^{-2-s}| \lesssim |y^{l-s} e^{-C_2 \operatorname{tr} Y(w,w)_\tau}| $$

for some integer l, depending on the degree of p, whose integral converges uniformly for all s in any compact region of $\mathbb{C}$ and Y in a compact subset of $\mathcal{R}_{4c_f/3\pi}$. □

Remark 4.2.21. *It is important that we can choose w_0 such that $\tau_2(\xi, w_0)$ is in the fundamental region for the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ to obtain the estimate we use. We are unable to do the estimate without this as there will be no way to guarantee $|f(\tau)| \lesssim e^{-\pi \operatorname{tr} Y(w,w)_\tau}$ for some fixed Y and arbitrary w. If we do not carefully choose w this way we will have the problem that $\left({}^t w_2 Y w_2 - \frac{({}^t w_1 Y w_2)^2}{{}^t w_1 Y w_1}\right)$ can be arbitrarily small.*

We have:

Theorem 4.2.22.
$$ \Phi_{f,2}(\xi; p) = \sum_{S_0 \in \operatorname{Skew}^1_m(\mathbb{Z})} \phi_{f,S_0}(\xi, ; p) $$

where
$$ \phi_{f,S_0}(\xi; p) = 2\frac{p\big(\eta(\xi, w_0)\big)}{y_1(\xi, w_0)} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} e^{2\pi i m \tau_1(\xi, w_0)} e^{2\pi i n \tau_2(\xi, w_0)} \tag{4.72} $$

where $c_{m,n}$ are coefficients given by:

$$ c_{m,n} = \sum_{d \mid \gcd(m,n)} d^{\kappa-1} a_{mn/d^2} \tag{4.73} $$

Proof. Fix some $S_0 \in \operatorname{Skew}^1_m(\mathbb{Z})$, and $w_0 \in M^+_{m,2}(\mathbb{Z})$ as in the previous lemma, so that $\langle w_0, w_0 \rangle = S_0$. Then we have that a set of representatives for $\mathcal{O}^{\mu S_0}$ is given by $\{w_0\gamma_{a,b,d} : a, d > 0, ad = \mu, 0 \le b < a\}$. From the same calculation that shows (4.14) we get

$$\begin{aligned} &CT_{s=0}\left[\int_{\mathcal{H}} f(\tau)p(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau}\frac{dxdy}{y^{2+s}}\right] \\ &= \frac{p\big(\eta(\xi,w)\big)}{y_1(\xi,w)}e^{2\pi i\tau_1(\xi,w)}f(\tau_2(\xi,w)) \end{aligned} \tag{4.74}$$

Next we consider what happens to the above expression when we replace w with $w\gamma_{a,b,d}$. From (4.12) and (4.13) We have that:

$$\tau_1(\xi,w\gamma_{a,b,d}) = ad\tau_1(\xi,w), \qquad \tau_2(\xi,w\gamma_{a,b,d}) = \frac{d\tau_2(\xi,w)-b}{a}, \qquad \eta(\xi,w\gamma_{a,b,d}) = d\eta(\xi,w)$$

so that replacing w with $w\gamma_{a,b,d}$ in (4.74) is:

$$\frac{d^{\kappa-1}}{a}\frac{p\big(\eta(\xi,w)\big)}{y_1(\xi,w)}e^{2\pi i ad\tau_1(\xi,w)}f\left(\frac{d\tau_2(\xi,w)-b}{a}\right)$$

We have thus that:

$$\phi_{S_0}(\xi,;p) = 2\frac{p\big(\eta(\xi,w_0)\big)}{y_1(\xi,w_0)}\sum_{\mu=1}^{\infty}\sum_{\substack{a,b,d\in\mathbb{Z}_{\geq 0}\\ ad=\mu\\ 0\leq b<a}}\frac{d^{\kappa-1}}{a}e^{2\pi i\mu\tau_1(\xi,w_0)}f\left(\frac{d\tau_2(\xi,w_0)-b}{a}\right) \tag{4.75}$$

Now consider the expression:

$$\sum_{\substack{a,b,d\in\mathbb{Z}_{\geq 0}\\ ad=\mu\\ 0\leq b<a}}\frac{d^{\kappa-1}}{a}f\left(\frac{d\tau_2(\xi,w_0)-b}{a}\right) = \sum_{\substack{a,b,d\in\mathbb{Z}_{\geq 0}\\ ad=\mu\\ 0\leq b<a}}\sum_{n\geq -n_0}\frac{d^{\kappa-1}}{a}a_n e^{2\pi i n\frac{d\tau_2(\xi,w_0)-b}{a}}$$

For fixed a,d,n, when we sum over b we get 0 unless $a|n$, in which case the sum over b contributes a. Thus the above is

$$\sum_{\substack{n,d\in\mathbb{Z}_{\geq 0}\\ ad=\mu\\ a|n}} d^{\kappa-1}a_n e^{2\pi i\frac{n}{a}d\tau_2(\xi,w_0)}$$

Thus we get that (4.75) is:

$$2\frac{p\big(\eta(\xi,w_0)\big)}{y_1(\xi,w_0)}\sum_{\substack{n\geq -n_0\\ a,d>0\\ ad\\ a|n}} a_n e^{2\pi i ad\tau_1(\xi,w_0)}e^{2\pi i\frac{n}{a}d\tau_1(\xi,w_0)}$$

and by relabeling ad as m, and $\frac{n}{a}$ as n, and collecting coefficients together, we obtain the result. □

Remark 4.2.23. *This route provides an alternate way to prove theorem 4.1.16. For a Hecke eigenform form* $f = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$ *and* $a_1 = 1$, *we have that* $c_{m,n} = a_n a_m$. *Alternately we also have the expression* $\phi_{f,S_0}(\xi; p) = 2\frac{p(\eta(\xi,S_0))}{y_1(\xi,S_0)} e^{2\pi i\mu\tau_1(\xi,S_0)}(T(\mu)f)(\tau_2(\xi, S_0))$ *but we eschew this for weakly holomorphic modular forms as the Hecke operators increase the order of the pole at infinity so that there are no Hecke eigenforms (see the summation from* $-\infty$ *to* ∞ *in* n *(4.72).*

We have a similar result as theorem 4.1.17, where the rank 2 piece for the lift to $O(m, m)$ is obtained from the rank 2 piece for the lift to $O(2, 2)$. The proof is identical.

Corollary 4.2.24. *Using the same notation as in theorem 4.1.17, we have:*

$$\Phi_{f,2}^{(m)}(\xi) = \sum_{A \in \overline{P}_2(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \rho_\kappa({}^tA)\left[\Phi_{f,2}^{(2)}([{}^tA\xi A]_{m,2})\right]_{m,2}$$

Remark 4.2.25. *Note that it is not all of* $\Phi_f^{(m)}$ *that is obtained from* $\Phi_f^{(2)}$, *as we do not get the constant term in this way, only the non-constant term. The rank 1 piece can be thought of as a lift from the degenerate* $O(1, 1)$. $\mathcal{D}_1 \cong \mathbb{R}_{>0}$, *parameterized by a single variable* $Y \in \mathbb{R}_{>0}$. *When we calculate the rank 1 piece we get* $\Phi_{f,1}^{(1)}(Y) = \frac{2\zeta(2)}{\pi Y}$. *Then summing over* $\overline{P}_1(\mathbb{Z})\backslash GL_m(\mathbb{Z})$ *is essentially breaking up the sum defining* $\zeta(z, Y; p)$ *into lines through the origin. We will not treat this idea in detail since it is a small aside.*

We combine Theorems 4.2.16 and 4.2.22

Theorem 4.2.26 (Expression for the Regularized Lift $\Phi_f(\xi)$)**.** *Suppose that* $f : \mathcal{H} \to \mathbb{C}$ *is a weakly holomorphic modular form of weight* κ, *with Fourier expansion* $f(\tau) = \sum_n a_n e^{2\pi i n\tau}$. *Then the regularized lift* $\Phi_f : \mathcal{D} \to \mathcal{V}_{\rho_\kappa}$ *(4.2.3) has the Fourier expansion:*

$$\Phi_f(\xi; p) = a_0 \frac{1}{\pi}\zeta(1, Y; p) + 2 \sum_{\substack{S_0 \in \mathrm{Skew}_m^1(\mathbb{Z}) \\ \mu > 0}} e^{\pi i\mu \operatorname{tr} XS_0} p(\eta(Y, S_0)) y_1(Y, S_0) e^{-2\pi\mu y_1(Y,S_0)} \sum_{n \geq = -\infty}^{\infty} c_{\mu,n} e^{2\pi i n\tau_2(Y,S_0)}$$

where $\zeta(1, Y; p)$ *is the Epstein zeta function (4.2.13), (where we take the constant term at* $z = 1$ *in case* $m = 2$ *and* $\zeta(z, Y; p)$ *has a pole there), and* $c_{\mu,n}$ *are the coefficients defined in (4.73). The constant term of the Fourier expansion is* $a_0 \frac{1}{\pi}\zeta(1, Y; p)$ *and the* μS_0*-th Fourier coefficient is:*

$$a_{\mu S_0}(Y; p) = 2e^{\pi i\mu \operatorname{tr} XS_0} p(\eta(Y, S_0)) y_1(Y, S_0) e^{-2\pi\mu y_1(Y,S_0)} \sum_{n \geq = -\infty}^{\infty} c_{\mu,n} e^{2\pi i n\tau_2(Y,S_0)}$$

Theorem 4.2.27 (Singular Shimura Correspondence)**.** *If $f(\tau) = \sum_{n\geq -n_0} a_n e^{2\pi i n\tau}$ is a weakly holomorphic modular form of weight κ, then the singular Shimura lift of f to $SO_0(2,2) \cong SL_2(\mathbb{R}) \times_{\pm 1} SL_2(\mathbb{R})$, as defined in [1] is given by:*

$$F_f(\tau_1, \tau_2) = a_0 \frac{(\kappa-1)!\zeta(\kappa)}{i^\kappa \pi^\kappa} E_\kappa(\tau_2) + 2^{\kappa+1} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} e^{2\pi i m\tau_1} e^{2\pi i n\tau_2} \tag{4.76}$$

where E_κ is the weight κ holomorphic Eisenstein series (normalized with constant coefficient 1), and $c_{m,n}$ are the constants defined in (4.73).

In particular this is a meromorphic modular function on $\mathcal{H}_1 \times \mathcal{H}_1$ with respect to $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ of weight (κ, κ), and it has singularities along the divisors $\tau_1 = \gamma\tau_2$ that are poles of order κ for all $\gamma \in M_2(\mathbb{Z})$ with $\det\gamma = n > 0$ and $a_{-n} \neq 0$. At a point (τ_1', τ_2') on such a divisor, the singularity is given as:

$$\sum_{\substack{\gamma \in M_2^+(\mathbb{Z}) \\ \tau_1' = \gamma\tau_2'}} a_{-\det\gamma} \frac{(\kappa-1)!}{\pi^\kappa (\tau_1 - \gamma\tau_2)^\kappa j(\gamma, \tau_2)^\kappa}$$

Proof. Most of the proof of this is contained in example 4.2.9 (for the singularities), and example 4.2.19 (for the rank 1 piece giving the term with the Eisenstein series). To obtain the sum in (4.76) we follow the same process as we obtained theorem 4.1.18. □

Chapter 5

Lifting Cusp Forms of Genus 2 and Higher

5.1 Fourier Coefficients of Lifts

In this section we will write $\mathcal{H}$ for $\mathcal{H}_n$, except when later on we will be specifically referring to $n = 2$. We will also write f for a Siegel cups form $f : \mathcal{H} \to \mathcal{V}_\kappa$ weight κ, with $(\mathcal{V}_\kappa, \kappa)$ an irreducible representation holomorphic of $GL_n(\mathbb{C})$. Recall in definition 3.4.5, we defined functions $\Theta^\kappa : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\rho_\kappa} \otimes \mathcal{V}_\kappa^*$, where $\mathcal{V}_{\rho_\kappa}$ (3.51) is the $GL_m(\mathbb{R})$ representation whose underlying space is $\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa)^*$, whose action is given by $(\rho_\kappa(\alpha)P)(v) = |\det \alpha|^n \sigma(\alpha)^{-1} P(v)$ for $v \in \mathcal{V}_\kappa$, $P \in \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})])$, where σ is the action on $\mathbb{C}[M_{m,n}(\mathbb{C})]$ given by $\sigma(\alpha)p(\eta) = p(\alpha^{-1}\eta)$.

The function Θ^κ is modular for $\Gamma \times \Gamma'$ on $\mathcal{D} \times \mathcal{H}$ of weight (ρ_κ, κ^*), i.e.:

$$\Theta^\kappa(\gamma\xi, \gamma'\tau) = \big(\rho_\kappa(j^-(\gamma, \xi)) \otimes \kappa^*(j(\gamma', \tau))\big)\Theta^\kappa(\xi, \tau)$$

and can pair with f to obtain a function $(f, \Theta^\kappa) : \mathcal{D} \times \mathcal{H} \to \mathcal{V}_{\rho_\kappa}$ that is modular of weight $(\rho_\kappa, 1)$. We recall that $P(f(\tau)) \in \mathbb{C}[M_{m,n}(\mathbb{C})]$, and (3.56) says:

$$(f, \Theta^\kappa)(\xi, \tau)(P) = \sum_{w \in M_{m,2n}(\mathbb{Z})} P(f(\tau))(\eta_\tau(w)) e^{\pi i \operatorname{tr} X\langle w,w\rangle} e^{-\pi \operatorname{tr} Y(w,w)_\tau}$$

Definition 5.1.1 (Theta Lift of f to $O(m, m)$)**.** *Define the Theta lift of f to $O(m, m)$ to be $\Phi_f : \mathcal{D} \to \mathcal{V}_{\rho_\kappa}$ to be the function:*

$$\Phi_f(\xi) = \int_{\mathcal{F}} (f, \Theta^\kappa)(\xi, \tau) \det y^{-n-1} dx dy$$

where $\mathcal{F}$ is a fundamental domain for the action of $\Gamma' = SP_n(\mathbb{Z})$ on $\mathcal{H}$.

This converges without issue as f is a cusp form.

Remark 5.1.2. *Note that a priori that the minimum m for which we can lift f to $O(m,m)$ is $m=n$, due to the weight of f being $\kappa=(\kappa_1,\dots,\kappa_n)$ with $\kappa_n\neq 0$. We will see that indeed for cusp forms the lifts are identically 0 until we reach $m=2n$.*

Proposition 5.1.3. *The function $\Phi_f:\mathcal{D}\to\mathcal{V}_{\rho_\kappa}$ defined in the previous definition is modular on $\mathcal{D}$ of weight ρ_κ:*

$$\Phi_f(\gamma\xi)=\rho_\kappa(j^-(\gamma,\xi))\Phi_f(\xi)$$

for all $\gamma\in\Gamma$, $\xi\in\mathcal{D}$. In terms of evaluation of $\Phi_f(\xi)$ on $P\in\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa,\mathbb{C}[\eta]^\kappa)$, we have:

$$\Phi_f(\gamma\xi;P)=|\det j^-(\gamma,\xi)|^n\Phi_f\big(\xi;\sigma(j^-(\gamma,\xi))^{-1}P\big)$$

From the modularity of Φ_f we have that it has a Fourier expansion:

$$\Phi_f(\xi)=\sum_{S\in\mathrm{Skew}_m(\mathbb{Z})}e^{\pi i\,\mathrm{tr}\,XS}a_S(Y) \tag{5.1}$$

for some $a_S(Y)$ taking values $\mathcal{V}_{\rho_\kappa}$. We will write $a_S(Y;P)$ for evaluating $a_S(Y)$ at a specific P. We obtain an expression for the Fourier coefficients immediately from the definition of Θ^κ:

$$a_S(Y;P)=\sum_{\substack{w\in M_{m,2n}(\mathbb{Z})\\ \langle w,w\rangle=S}}\int_{\mathcal{F}}P(f(\tau))(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-1-n}dxdy \tag{5.2}$$

We will now seek to obtain more explicit expressions for these Fourier coefficients. To begin the calculation of the Fourier coefficients we will record the following lemma. It is a simple calculation but it will be used frequently in this section.

Lemma 5.1.4. *Suppose that $f:\mathcal{H}\to\mathcal{V}_\kappa$ is modular of weight κ, $g\in GSP_n^+(\mathbb{R})$, $\mathcal{R}\subseteq\mathcal{H}$, $P\in\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa,\mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa)$, and $w\in M_{m,2n}(\mathbb{R})$. Then:*

$$\begin{aligned}&\int_{\mathcal{R}}P(f(\tau))(\eta_\tau(wg))e^{-\pi\,\mathrm{tr}\,Y(wg,wg)_\tau}\det y^{-n-1}dxdy\\ =&\mu(g)^{d(\kappa)-n(n+1)/2}\int_{g\mathcal{R}}P(f|_{g^{-1}}(\tau))(\eta_\tau(w))e^{-\pi\mu(g)\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-n-1}dxdy\end{aligned} \tag{5.3}$$

where $f|_g$ is the slash operator of weight κ for g applied to f (3.8).

Proof. First recall we have $\eta_\tau(wg)=\eta_{g\tau}(w)j(g,\tau)$ and $(wg,wg)_\tau=\mu(g)(w,w)_{g\tau}$. Then in the left hand side of (5.3) we change variables $\tau\mapsto g^{-1}\tau$, giving:

$$\int_{g\mathcal{R}}P(f(g^{-1}\tau))\big(\eta_\tau(w)j(g,g^{-1}\tau)\big)e^{-\pi\mu(g)\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-n-1}dxdy$$

We have $j(g, g^{-1}\tau) = j(g^{-1}, \tau)^{-1}$ so that

$$P(f(g^{-1}\tau))\big(\eta_\tau(w)j(g, g^{-1}\tau)\big) = P\big(\kappa(j(g^{-1}, \tau))^{-1}f\big)(\eta_\tau(w))$$

Then finally we use that $\kappa(j(g^{-1}, \tau))^{-1}f = \mu(g)^{d(\kappa)-n(n+1)/2}f|_{g^{-1}}$. □

We can now collect terms together in the sum on w in (5.2) according to their Γ' orbits. First we introduce some notation. For $S \in \mathrm{Skew}_m(\mathbb{Z})$, define:

$$\begin{aligned} M^{S,j}_{m,2n} &= \{w \in M_{m,2n}(\mathbb{Z}) : \ \langle w, w\rangle = S, \ \mathrm{rank}\ w = \mathrm{rank}\ S + j\} \\ \mathcal{O}^{S,j} &= M^{S,j}_{m,2n}/\Gamma' \end{aligned} \tag{5.4}$$

We will write elements of $\mathcal{O}^{S,j}$ as either w or $[w]$, depending on whether we want to value simpler notation, or to emphasize that they are orbits of $M_{m,2n}(\mathbb{Z})$. We can rephrase (5.2) in these terms:

$$a_S(Y; P) = \sum_j \sum_{[w]\in\mathcal{O}^{S,j}} \sum_{w'\in[w]} \int_{\mathcal{F}} P(f(\tau))(\eta_\tau(w'))e^{-\pi\,\mathrm{tr}\,Y(w',w')_\tau} \det y^{-1-n} dxdy \tag{5.5}$$

For D a diagonal matrix, we will also write $\mathcal{O}^D$, $\mathcal{O}^{D,j}$, $M^D_{m,2n}(\mathbb{Z})$ and $M^{D,j}_{m,2n}(\mathbb{Z})$ for $\mathcal{O}^{J(D)}$, $\mathcal{O}^{J(D),j}$, $M^{J(D)}_{m,2n}(\mathbb{Z})$ and $M^{J(D),j}_{m,2n}(\mathbb{Z})$, respectively, where $J(D) = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so that the representative Fourier coefficients (Definition 3.3.6) are given as:

$$a_D(Y; P) = \sum_j \sum_{[w]\in\mathcal{O}^{D,j}} \sum_{w'\in[w]} \int_{\mathcal{F}} P(f(\tau))(\eta_\tau(w'))e^{-\pi\,\mathrm{tr}\,Y(w',w')_\tau} \det y^{-1-n} dxdy$$

We will now prove a lemma that allows us to group together the inside sums in (5.5). Given a $w \in M_{m,2n}(\mathbb{Z})$, we will write

$$\Gamma'_w = \mathrm{stab}_{\ \Gamma'}(w) \tag{5.6}$$

and $\mathcal{F}_w$ for a fundamental domain inside of $\mathcal{H}_n$ for the action of Γ'_w.

Lemma 5.1.5 (Unfolding)**.** *Suppose that* $[w] \in M_{m,2n}(\mathbb{Z})/\Gamma'$*, we have:*

$$\begin{aligned} &\sum_{w'\in[w]} \int_{\mathcal{F}} P(f(\tau))(\eta_\tau(w'))e^{-\pi\,\mathrm{tr}\,Y(w',w')_\tau} \det y^{-n-1} dxdy \\ &= 2\int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau} \det y^{-n-1} dxdy \end{aligned} \tag{5.7}$$

Proof. The proof is essentially the same as Lemma 4.1.3, but we will write it out anyways. Let γ_i range over a set of representatives for $\Gamma'_w\backslash\Gamma'$. We have then that if h is some function of w':

$$\sum_{w'\in[w]} h(w') = \sum_i h(w\gamma_i)$$

By Lemma 5.1.4 we get that the left hand side of (5.7) is

$$\sum_i \int_{\gamma_i^{-1}\mathcal{F}} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1} dxdy$$

Their union of the regions $\gamma_i^{-1}\mathcal{F}$ gives $\mathcal{F}_w$. We have that the integral with $w = 0$ is identically 0 since the polynomial $P(f(\tau))$ has no constant term, so that all of the stabilizers do not contain -1. Thus both γ_i and $-\gamma_i$ appear in the sum over i, so that we are double counting the region $\mathcal{F}_w$, giving the coefficient of 2. □

In light of the lemma we can then refine (5.5):

$$a_S(Y;P) = 2\sum_j \sum_{w\in\mathcal{O}^{S,j}} \int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1} dxdy \qquad (5.8)$$

We have an action of $GL_m(\mathbb{Q})$ and $Sp_n(\mathbb{Q})$ on $M_{m,2n}(\mathbb{Q})$ on the left and the right, respectively. Given $w \in M_{m,2n}(\mathbb{Q})$ we can form $\operatorname{colspan}_{\mathbb{Q}}(w) \subseteq \mathbb{Q}^m$ and $\operatorname{rowspan}_{\mathbb{Q}}(w) \subseteq \mathbb{Q}^{2n}$. For $A \in GL_n(\mathbb{Q})$ and $g \in Sp_n(\mathbb{Q})$, we have $\operatorname{colspan}_{\mathbb{Q}}(Awg) = A(\operatorname{colspan}_{\mathbb{Q}}(w))$, and $\operatorname{rowspan}_{\mathbb{Q}}(Awg) = (\operatorname{rowspan}_{\mathbb{Q}}(w))g$.

Proposition 5.1.6. *Suppose that $w \in \mathcal{O}^{S,j}$, with $\operatorname{rank} S = 2r$ and $\operatorname{sd}(S) = D$. Then there is an $A \in GL_m(\mathbb{Z})$ and $g \in Sp_n(\mathbb{Q})$ so that $w = Aw_{D,j}g$ where:*

$$w_{D,j} = \begin{pmatrix} D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proof. We can find an $A_1 \in GL_m(\mathbb{Z})$ so that $A_1 w = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}$ has the bottom $m-2r-j$ rows zero, and the rows of w_1 linearly independent. Then the top $2r+j$ rows of A_1w. Then we can choose $A_2 \in GL_{2r+j}(\mathbb{Z})$ so that $A_2\langle w_1, w_1\rangle\, {}^tA_2$ is in skew normal form. Then writing $A^{-1} = \begin{pmatrix} A_2 & \\ & 1 \end{pmatrix} A_1$, we have that $A^{-1}w = \begin{pmatrix} w_2 \\ 0 \end{pmatrix}$ with the rows of w_2 linearly independent and $\langle w_2, w_2\rangle = J(D)$. If we consider the matrix $w_3 = \begin{pmatrix} D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_j & 0 \end{pmatrix}$, then we have $\langle w_3, w_3\rangle = \langle w_2, w_2\rangle$, and since both matrices have linearly independent rows, there is a $g \in Sp_n(\mathbb{Q})$ so that $w_3 g = w_2$. Then we have $w_{D,j} = \begin{pmatrix} w_3 \\ 0 \end{pmatrix}$ and $w = Aw_{D,j}g$. □

We will call a $w \in M_{m,2n}(\mathbb{Z})$ *degenerate* if rank $w >$ rank $\langle w, w\rangle$. This is equivalent to the rows of w spanning a degenerate symplectic subspace of $\mathbb{R}^{2n}$. It is immediate that one w in a Γ' orbit is degenerate iff all members of that orbit are degenerate, and we will call a Γ' orbit degenerate if its elements are degenerate.

Lemma 5.1.7 (Degenerate Orbits Contribute 0)**.** *Suppose that $w \in M_{m,2n}(\mathbb{Z})$ is degenerate. Then:*

$$\int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1} dxdy = 0 \tag{5.9}$$

Proof. Suppose that $w \in M_{m,2n}(\mathbb{Z})$ is degenerate. We have that $w \in \mathcal{O}^{S,j}$ for some S with rank $S = 2r$ and $j > 0$. From the lemma above there is an $A \in GL_m(\mathbb{Z})$, a $g \in SP_n(\mathbb{Q})$ such that $w = Aw_{D,j}g$. Then we have by Lemma 5.1.4 that the left hand side of (5.9) is:

$$\int_{\mathcal{F}'} P'\big(f|_{g^{-1}}(\tau)\big)(\eta_\tau(w_{D,j}))e^{-\pi \operatorname{tr} Y'(w_{D,j},w_{D,j})_\tau} \det y^{-n-1} dxdy \tag{5.10}$$

where $Y' = Y[A]$, $P' = \sigma(A)^{-1}P$, and $\mathcal{F}' = g\mathcal{F}_w$ is a fundamental domain for $g\Gamma'_w g^{-1}$. We have that $g(\Gamma')g^{-1} \cap \Gamma'$ is finite order in both $g(\Gamma')g^{-1}$ and Γ' so that there is a finite order subgroup $S \subseteq \operatorname{Sym}_n(\mathbb{Z})$ so that $g\Gamma' g^{-1} \cap \begin{pmatrix} 1 & \operatorname{Sym}_n(\mathbb{Z}) \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & S \\ & 1 \end{pmatrix}$. Let S^* be the dual to S under the trace pairing. We have that $f|_{g^{-1}}$ has a Fourier expansion with respect to S^*:

$$f|_{g^{-1}}(\tau) = \sum_{N \in S^*_+} a'_N e^{2\pi i \operatorname{tr} N\tau}$$

where the expansion will only be over positive definite elements of S^* as f is a cusp form. We have that $g(\Gamma'_w)g^{-1} \subset \operatorname{stab}_{SP_n(\mathbb{Q})}(w_{D,j})$, and that $\operatorname{stab}_{SP_n(\mathbb{Q})}(w_{D,j})$ is generated by the elements:

$$m(g') = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & a' & & & b' \\ & & & 1 & & \\ & & & & 1 & \\ & & c' & & & d' \end{pmatrix},$$

$$n(x,y) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & x_{22} & x_{23} \\ & & 1 & & {}^t x_{23} & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & 1 & y & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & -{}^t y & 1 \end{pmatrix}$$

for $g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SP_{n-r-j}(\mathbb{Q})$, $y \in M_{j,n-r-j}(\mathbb{Q})$, $x_{22} \in \mathrm{Sym}_j(\mathbb{Q})$ and $x_{23} \in M_{j,n-r-j}(\mathbb{Q})$. Let S_{22} be defined by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = S \cap \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have that S_{22} is a finite order subgroup of $\mathrm{Sym}_j(\mathbb{Z})$. We describe the action of $m(g')$ and $n(x,y)$ on $\mathcal{H}$:

$$m(g') \cdot \tau = \begin{pmatrix} \tau_{11} - \tau_{13} j(g',\tau_{33})^{-1} c' \tau_{31} & \tau_{12} - \tau_{13} j(g',\tau_{33})^{-1} c' \tau_{32} & \tau_{13} j(g',\tau_{33})^{-1} \\ \tau_{21} - \tau_{23} j(g',\tau_{33})^{-1} c' \tau_{31} & \tau_{22} - \tau_{23} j(g',\tau_{33})^{-1} c' \tau_{32} & \tau_{23} j(g',\tau_{33})^{-1} \\ a'\tau_{31} - (g' \cdot \tau_{33}) c' \tau_{31} & a'\tau_{32} - (g' \cdot \tau_{33}) c' \tau_{32} & g' \cdot \tau_{33} \end{pmatrix}$$

and

$$n(x,y) \cdot \tau = \begin{pmatrix} \tau_{11} & \tau_{12} + \tau_{13}\,{}^t y & \tau_{13} \\ \tau_{21} + y\tau_{31} & \tau_{22} + y\tau_{32} + \tau_{23}\,{}^t y + y\tau_{33}\,{}^t y + x_{22} & \tau_{23} + y\tau_{33} + x_{23} \\ \tau_{31} & \tau_{32} + {}^t y \tau_{33} + {}^t x_{23} & \tau_{33} \end{pmatrix}$$

Suppose now that $\gamma\tau = \tau + \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for some $s_{22} \in S_{22}$ and $\gamma \in g\Gamma'_w g^{-1}$. Then we have $\gamma = n(x,y)m(g')$ for some x, y, g'. Since $(\gamma\tau)_{33} = \tau_{33}$ we have that $g' = \pm 1$, and then $(\gamma\tau)_{31} = \tau_{31}$ so that (unless τ is in a set of measure 0), we have that $a' = 1$, so that $g' = 1$.

Then, looking at τ_{12} and τ_{23} we have that $y = 0$ and $x_{23} = 0$, (again unless τ is in a set of measure 0). Thus if h is some function on $\mathcal{F}'$, we can write:

$$\int_{\mathcal{F}'} h(\tau)d\tau = \int_{\mathcal{F}'/n(\mathrm{Sym}_j(\mathbb{R}))} \int_{\mathrm{Sym}_j(\mathbb{R})/S_{22}} h(n(x_{22})\tau')dx_{22}d\tau'$$

where $d\tau = dxdy$ is the usual volume element on $\mathcal{H}$, and $d\tau' = \prod_{ij\neq 22} dx_{ij} \prod_{ij} dy_{ij}$. We have that $\eta_{n(x_{22})\cdot\tau}(w_{D,j}) = \eta_\tau(w_{D,j})$ and $(w_{D,j}, w_{D,j})_{n(x_{22})\cdot\tau} = (w_{D,j}, w_{D,j})_\tau$, so that in the integration

$$\int_{\mathrm{Sym}_j(\mathbb{R})/S_{22}} P'\big(f|_{g^{-1}}(n(x_{22})\cdot\tau')\big)\big(\eta_{n(x_{22})\cdot\tau'}(w_{D,j})\big)e^{-\pi \operatorname{tr} Y'(w_{D,j},w_{D,j})_{n(x_{22})\cdot\tau'}} \det y^{-n-1}dx_{22}$$

all of the terms except $f|_{g^{-1}}(n(x_{22})\cdot\tau')$ are independent of x_{22}. We have that

$$P'\big(f|_{g^{-1}}(n(x_{22})\cdot\tau')\big) = \sum_{N\in S_+^*} P'(a'_N)e^{2\pi i \operatorname{tr} N\tau'}e^{2\pi i \operatorname{tr} N_{22}x_{22}}$$

Since N is positive definite we have that $N_{22} > 0$, so we have

$$\int_{\mathrm{Sym}_j(\mathbb{R})/S_{22}} e^{2\pi i \operatorname{tr} N_{22}x_{22}}dx_{22} = 0$$

so that (5.10) is 0. □

It follows from this lemma that

Corollary 5.1.8. *The contribution to the Fourier coefficients of Φ_f is entirely from the non-degenerate orbits:*

$$a_S(Y;P) = \sum_{w\in\mathcal{O}^S} \int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-1-n}dxdy$$

We will say that w is *full rank* if rank $w = 2n$ (when $m < 2n$ there are no possible full rank w). When w is full rank we have that $\Gamma'_w = 1$ and $\mathcal{F}_w = \mathcal{H}_n$, but when w is not full rank these stabilizers will be non-trivial and the fundamental domains $\mathcal{F}_w$ will be more complicated. We are lucky then that we have the following lemma.

Lemma 5.1.9 (Non Full Rank Fourier Coefficients are 0)**.** *Suppose that $w \in M_{m,2n}(\mathbb{Z})$ is non-degenerate and not full rank, and f is a cusp form. Then*

$$\int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1}dxdy = 0 \tag{5.11}$$

The proof will follow from a series of smaller lemmas related to the structure of $\mathcal{F}_w$ and the corresponding integral.

Lemma 5.1.10. *Suppose that* $w \in M_{m,2n}(\mathbb{Z})$ *is non-degenerate and* $\operatorname{rank} w = 2r < 2n$. *Then there is a* $M_w \in SP_n(\mathbb{Q})$, *such that* wM_w *is of the form:*

$$wM_w = \begin{pmatrix} * & 0 & * & 0 \end{pmatrix}$$

where the $*$ *blocks are* r *entries wide and the* 0 *blocks are* $n-r$ *entries wide. We have that:*

$$\mathcal{F}_w = M_w^{-1} \cdot \left\{ \begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & \tau_{22} \end{pmatrix} : \tau_{22} \in \mathcal{F}'_w, \tau_{12} \in M_{r,n-r}(\mathbb{C}), \tau_{11} \in \mathcal{H}_r, y_{11} > y_{22}^{-1}[{}^ty_{12}] \right\}$$

where $\mathcal{F}'_w$ *is a fundamental domain inside of* $\mathcal{H}_{n-r}$ *for a subgroup of* $SP_{n-r}(\mathbb{Q})$ *that is commensurate with* $SP_{n-r}(\mathbb{Z})$.

Proof. To begin, we may find a M_w in $SP_n(\mathbb{Q})$ so that

$$\operatorname{rowspan}_{\mathbb{Q}}(wM_w) = \operatorname{rowspan}_{\mathbb{Q}} \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 1_r & 0 \end{pmatrix}$$

as the former is a $2r$-dimensional non-degenerate subspace of $\mathbb{Q}^{2n}$. Write $\Gamma'_{n-r}(\mathbb{Q})$ for the subgroup:

$$\Gamma'_{n-r}(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SP_{n-r}(\mathbb{Q}) \right\}$$

Then $\Gamma'_{n-r}(\mathbb{Q}) = \operatorname{stab}_{SP_n(\mathbb{Q})} \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & 0 & 1_r & 0 \end{pmatrix}$ so that $\Gamma'_w \subset M_w^{-1}\Gamma'_{n-r}(\mathbb{Q})M_w$. Consider the subgroup of $SP_{n-r}(\mathbb{Q})$ identified with $M_w\Gamma'_wM_w^{-1} \cap \Gamma'_{n-r}(\mathbb{Q})$, which we will call $\Gamma'_{n-r,w}$. We claim that this subgroup is commensurate with $SP_{n-r}(\mathbb{Z})$. Suppose it were not. Let $\Gamma''_w = M_w^{-1}\Gamma'_wM_w \cap \Gamma'_{n-r}(\mathbb{Z})$. We have that $M_w^{-1}\Gamma'M_w \subset M_w^{-1}SP_n(\mathbb{Z})M_w$ and further that $\Gamma'_w = SP_n(\mathbb{Z}) \cap M_w\Gamma'_{n-r}(\mathbb{Z})M_w^{-1}$. Thus we have that $\Gamma'' = M_w^{-1}SP_n(\mathbb{Z})M_w \cap SP_n(\mathbb{Z}) \cap \Gamma'_{n-r}(\mathbb{Q})$. We have thus that if $[\Gamma'' : M_w^{-1}\Gamma'_wM_w] = \infty$ or $[\Gamma'' : \Gamma_{n-r}(\mathbb{Z})] = \infty$, then we would have that $M_w^{-1}SP_n(\mathbb{Z})M_w$ and $SP_n(\mathbb{Z})$ are not commensurable, which is a contradiction.

Via a similar argument as in lemma 5.1.7, we have that $M_w^{-1}\Gamma'_wM_w$ has a fundamental domain given of the form:

$$\left\{ \begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & \tau_{22} \end{pmatrix} : \tau_{22} \in \mathcal{F}'_w, \tau_{12} \in M_{r,n-r}(\mathbb{C}), \tau_{11} \in \mathcal{H}_r, y_{11} > y_{22}^{-1}[{}^ty_{12}] \right\}$$

and then translating this by M_w^{-1} gives a fundamental domain for Γ'_w. □

Lemma 5.1.11. *Suppose that* $0 < r < n$ *and* $w \in M_{2r,2n}(\mathbb{Q})$ *is of the form:*

$$w = \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & 0 & w_2 & 0 \end{pmatrix}$$

is non-degenerate and has rank $2r$*. Let* $\mathcal{R} \subset \mathcal{H}_n$ *be a region of the form:*

$$\mathcal{R} = \left\{ \begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & \tau_{22} \end{pmatrix} : \tau_{22} \in \mathcal{F}'_w, \tau_{12} \in M_{r,n-r}(\mathbb{C}), \tau_{11} \in \mathcal{H}_r, y_{11} > y_{22}^{-1}[{}^ty_{12}] \right\}$$

where $\mathcal{F}'_w$ *is a fundamental domain inside of* $\mathcal{H}_{n-r}$ *for a subgroup of* $SP_{n-r}(\mathbb{Q})$ *that is commensurate with* $SP_{n-r}(\mathbb{Z})$*. Suppose as well that* $Y \in \mathrm{Sym}^+_{2r}(\mathbb{R})$ *and* $N = \begin{pmatrix} N_{11} & N_{12} \\ {}^tN_{12} & N_{22} \end{pmatrix}$ *is symmetric positive definite of the same dimensions of* τ*. Then the integral:*

$$\int_{\mathcal{R}} e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1} dxdy \tag{5.12}$$

is independent of N_{12}*.*

Proof. To begin, write $W = \begin{pmatrix} w_1 & \\ & w_2 \end{pmatrix}$ and replace Y by $Y[W^{-1}]$ so that (5.12) is

$$\int_{\mathcal{R}} e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr} Y(w',w')_\tau} \det y^{-n-1} dxdy \tag{5.13}$$

with $w' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and we will prove the statement for (5.13). We will write $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ {}^tY_{12} & Y_{22} \end{pmatrix}$ with each block being $r \times r$. We have:

$$\begin{aligned} Y(w',w')_\tau = \operatorname{tr} Y_{11} y_{11} + \operatorname{tr} Y_{11} {}^t\left(\begin{pmatrix} {}^tx_{11} \\ {}^tx_{12} \end{pmatrix} + \begin{pmatrix} {}^tY_{12}Y_{11}^{-1} \\ 0 \end{pmatrix} \right) y^{-1} \left(\begin{pmatrix} {}^tx_{11} \\ {}^tx_{12} \end{pmatrix} + \begin{pmatrix} {}^tY_{12}Y_{11}^{-1} \\ 0 \end{pmatrix} \right) \\ + \operatorname{tr}(Y_{22} - Y_{11}^{-1}[\mathrm{Y}_{12}])(y_{11} - y_{22}^{-1}[{}^ty_{12}])^{-1} \end{aligned}$$

where we use:

$$y^{-1} = \begin{pmatrix} (y_{11} - y_{22}^{-1}[{}^ty_{12}])^{-1} & \\ & y_{22}^{-1} \end{pmatrix} \left[\begin{pmatrix} 1 & -y_{12}y_{22}^{-1} \\ & 1 \end{pmatrix} \right]$$

Now we move on the integration. We will write $\mathcal{S} = \mathrm{Sym}_r(\mathbb{R})$, $\mathcal{P} = \mathrm{Sym}^+_r(\mathbb{R})$, $\mathcal{M} = M_{r,r}(\mathbb{R})$, $\mathcal{M}' = M_{r,n-r}(\mathbb{R})$, and $\mathcal{A} = \mathrm{Skew}_r(\mathbb{R})$. The integration is:

$$\int\limits_{\mathcal{F}'_w}\int\limits_{\mathcal{M}'}\int\limits_{\mathcal{P}+y_{22}^{-1}[{}^ty_{12}]}\int\limits_{\mathcal{S}\times\mathcal{M}'} e^{2\pi i\,\mathrm{tr}\,N\tau}$$
$$\times \exp\left[-\pi\,\mathrm{tr}\,Y_{11}\,{}^t\left(\begin{pmatrix}{}^tx_{11}\\ {}^tx_{12}\end{pmatrix}+\begin{pmatrix}{}^tY_{12}Y_{11}^{-1}\\ 0\end{pmatrix}\right)y^{-1}\left(\begin{pmatrix}{}^tx_{11}\\ {}^tx_{12}\end{pmatrix}+\begin{pmatrix}{}^tY_{12}Y_{11}^{-1}\\ 0\end{pmatrix}\right)\right]$$
$$\times \exp\left[-\pi\,\mathrm{tr}\left(y_{11}Y_{11}+(Y_{22}-Y_{11}^{-1}[Y_{12}])(y_{11}-y_{22}^{-1}[{}^ty_{12}])^{-1}\right)\right]$$
$$\times \det y^{-n-1}d(x_{11},x_{12})dy_{11}dy_{12}d(x_{22},y_{22})$$

The variable x_{11} is a symmetric matrix, which complicates the integral. We will use a trick that will come up again later, where we interpret the x_{11} as variable in $\mathcal{M}$ whose $\mathcal{A}$ coordinate is 0. With this perspective we use the Fourier inversion theorem to introduce another variable s in $\mathcal{A}$, and the integral is:

$$=\int\limits_{\mathcal{F}'_w}\int\limits_{\mathcal{M}'}\int\limits_{\mathcal{P}+y_{22}^{-1}[{}^ty_{12}]}\int\limits_{\mathcal{A}}\int\limits_{\mathcal{M}\times\mathcal{M}'} e^{2\pi s\,{}^tx_{11}}e^{2\pi i\,\mathrm{tr}\,N\tau}$$
$$\times \exp\left[-\pi\,\mathrm{tr}\,Y_{11}\,{}^t\left(\begin{pmatrix}{}^tx_{11}\\ {}^tx_{12}\end{pmatrix}+\begin{pmatrix}{}^tY_{12}Y_{11}^{-1}\\ 0\end{pmatrix}\right)y^{-1}\left(\begin{pmatrix}{}^tx_{11}\\ {}^tx_{12}\end{pmatrix}+\begin{pmatrix}{}^tY_{12}Y_{11}^{-1}\\ 0\end{pmatrix}\right)\right]$$
$$\times \exp\left[-\pi\,\mathrm{tr}\left(y_{11}Y_{11}+(Y_{22}-Y_{11}^{-1}[Y_{12}])(y_{11}-y_{22}^{-1}[{}^ty_{12}])^{-1}\right)\right]$$
$$\times \det y^{-n-1}d(x_{11},x_{12})ds\,dy_{11}dy_{12}d(x_{22},y_{22})$$

We now integrate the x_{11} and x_{12} variables. We can interpret them together as a variable in $M_{r,n}(\mathbb{R})$, and then after translating $x_{11}\mapsto x_{11}-Y_{11}^{-1}Y_{12}$ this is simply the Fourier transform of the Gaussian $\exp\left(-\pi\,\mathrm{tr}\,Y_{11}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}y^{-1}\,{}^t\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}\right)$, and after integrating x_{11} and x_{12}, we get

$$=\det Y_{11}^{-n/2}\int\limits_{\mathcal{F}'_w}\int\limits_{\mathcal{M}'}\int\limits_{\mathcal{P}+y_{22}^{-1}[{}^ty_{12}]}\int\limits_{\mathcal{A}} e^{-2\pi i\,\mathrm{tr}(N_{11}+s)\,{}^tY_{12}Y_{11}^{-1}}e^{2\pi i\,\mathrm{tr}\,N_{22}\tau_{22}}$$
$$\times \exp\left[-\pi\,\mathrm{tr}\,Y_{11}^{-1}\,{}^t\begin{pmatrix}N_{11}+s\\ {}^tN_{12}\end{pmatrix}y\begin{pmatrix}N_{11}+s\\ {}^tN_{12}\end{pmatrix}-\pi\,\mathrm{tr}\,y_{11}Y_{11}-2\pi\,\mathrm{tr}(y_{11}N_{11}+y_{12}\,{}^tN_{12})\right]$$
$$\times \exp\left[-\pi\,\mathrm{tr}(Y_{22}-Y_{11}^{-1}[Y_{12}])(y_{11}-y_{22}^{-1}[{}^ty_{12}])^{-1}\right]\det y^{-n-1+r/2}ds\,dy_{11}dy_{12}d(x_{22},y_{22})$$

We move the integral over $\mathcal{A}$ further into the integration, and we translate y_{11} by $y_{22}^{-1}[{}^ty_{12}]$, and then we can shift the integration over y_{12} to happen first. We arrange terms to collect terms with y_{12} together, and obtain:

$$
\begin{aligned}
&= \det Y_{11}^{-n/2} \int_{\mathcal{F}'_w}\int_{\mathcal{A}}\int_{\mathcal{P}}\int_{\mathcal{M}} e^{-2\pi i \operatorname{tr}(N_{11}+s)\,{}^tY_{12}Y_{11}^{-1}} e^{2\pi i \operatorname{tr} N_{22}\tau_{22}} \\
&\quad \times \exp\Big[-\pi \operatorname{tr}\Big(Y_{11}^{-1}\,{}^t(N_{11}+s)y_{11}(N_{11}+s) + (Y_{11}+2N_{11})y_{11} + (Y_{22}-Y_{11}^{-1}[Y_{12}])y_{11}^{-1}\Big)\Big] \\
&\quad \times \exp\Big[-\pi \operatorname{tr}\Big(Y_{11}^{-1}\,{}^t(N_{11}+s)y_{12}y_{22}^{-1}\,{}^ty_{12}(N_{11}+s) + 2Y_{11}^{-1}\,{}^t(N_{11}+s)y_{12}\,{}^tN_{12}\Big)\Big] \\
&\quad \times \exp\Big[-\pi \operatorname{tr}\Big(Y_{11}^{-1}N_{12}y_{22}^{-1}\,{}^tN_{12} + 2y_{12}\,{}^tN_{12} + 2y_{12}y_{22}^{-1}\,{}^ty_{12}N_{11} + Y_{11}y_{12}y_{22}^{-1}\,{}^ty_{12}\Big)\Big] \\
&\quad \times \det y_{11}^{-n-1+r/2} y_{22}^{-n-1+r/2} dy_{12}\,ds\,dy_{11}\,d(x_{22},y_{22})
\end{aligned}
$$

We have that:

$$
\begin{aligned}
&\operatorname{tr}\Big(Y_{11}^{-1}\,{}^t(N_{11}+s)y_{12}y_{22}^{-1}\,{}^ty_{12}(N_{11}+s) + 2Y_{11}^{-1}\,{}^t(N_{11}+s)y_{12}\,{}^tN_{12} + Y_{11}^{-1}N_{12}y_{22}^{-1}\,{}^tN_{12}\Big) \\
&+ \operatorname{tr}\Big(2y_{12}\,{}^tN_{12} + 2y_{12}y_{22}^{-1}\,{}^ty_{12}N_{11} + Y_{11}y_{12}y_{22}^{-1}\,{}^ty_{12}\Big) \\
&= \operatorname{tr}\,{}^tCY_{11}^{-1}C\big(y_{12}+C^{-1}N_{12}y_{12}\big)y_{22}^{-1}\,{}^t\big(y_{12}+C^{-1}N_{12}y_{12}\big)
\end{aligned}
$$

where $C = Y_{11} + N_{11} + s$, which we note is invertible as is has positive definite symmetric part. Integrating over y_{12} amounts to integrating a Gaussian, and we get:

$$
\begin{aligned}
&= \det Y_{11}^{-r/2} \int_{\mathcal{F}'_w}\int_{\mathcal{A}}\int_{\mathcal{P}} e^{-2\pi i \operatorname{tr}(N_{11}+s)\,{}^tY_{12}Y_{11}^{-1}} e^{2\pi i \operatorname{tr} N_{22}\tau_{22}} \\
&\quad \times \exp\Big[-\pi \operatorname{tr}\Big(Y_{11}^{-1}\,{}^t(N_{11}+s)y_{11}(N_{11}+s) + Y_{11}y_{11} + 2N_{11}y_{11} + (Y_{22}-Y_{11}^{-1}[Y_{12}])y_{11}^{-1}\Big)\Big] \\
&\quad \times \det(Y_{11}+N_{11}+s)^{-n+r} \det y_{11}^{-n-1+r/2} y_{22}^{-n-1+r} ds\,dy_{11}\,d(x_{22},y_{22})
\end{aligned}
$$

This integral is manifestly independent of N_{12}. $\square$

Proof of lemma 5.1.9. Suppose that $w \in M_{m,2n}(\mathbb{Z})$ is non-degenerate and not full rank. We can find an $A \in GL_m(\mathbb{Z})$ and a $g \in SP_n(\mathbb{Q})$ so that $w = Aw'g$, with:

$$
w' = \begin{pmatrix} w'_1 & 0 & 0 & 0 \\ 0 & 0 & w'_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

for some full rank $w'_1, w'_2 \in M_r(\mathbb{Z})$. Write $Y' = {}^tAYA$, and $P' = \sigma(A)^{-1}P$. Then we have:

$$\begin{aligned}&\int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi\operatorname{tr}Y(w,w)_\tau}\det y^{-n-1}dxdy\\&=\int_{g^{-1}\mathcal{F}_w} P'\big(f|_{g^{-1}}(\tau)\big)(\eta_\tau(w'))e^{-\pi\operatorname{tr}Y'(w',w')_\tau}\det y^{-n-1}dxdy\end{aligned}\tag{5.14}$$

The function $f|_{g^{-1}}(\tau)$ has a Fourier decomposition:

$$f|_{g^{-1}}(\tau)=\sum_N a'_N e^{2\pi i\operatorname{tr}N\tau}$$

where a'_N are some coefficients in $\mathcal{V}_\kappa$ and N ranges over positive definite matrices in some lattice. Write p'_N for $P'(a'_N)$, so that (5.14) is:

$$\sum_N\int_{g^{-1}\mathcal{F}_w} p'_N(\eta_\tau(w'))e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}Y'(w',w')_\tau}\det y^{-n-1}dxdy\tag{5.15}$$

We have:

$$\eta_\tau(w')=\begin{pmatrix}w'_1\tau_{11} & w'_1\tau_{12}\\ w'_2 & 0\\ 0 & 0\end{pmatrix}\tag{5.16}$$

The polynomials p'_N are contained inside of the $GL_m(\mathbb{C})\times GL_n(\mathbb{C})$ translates of $\Delta^\kappa(\eta)$ (3.50). As f has weight $\kappa=(\kappa_1,\dots,\kappa_n)$ with $\kappa_n>0$ (this is true for all Siegel modular forms, see remark 3.2.2), we have that p'_N will be a sum of terms, each of which has some factor of $\Delta^{(1,\dots,1)}(A'\eta)$ for some $A'\in GL_m(\mathbb{C})$. When we evaluate $\Delta^{(1,\dots,1)}(A'\eta_\tau(w'))$ we see that as a polynomial of τ, it is contained in the ideal generated by the variables of τ_{12}. Thus so is $p'_N(\eta_\tau(w'))$, so that $p'_N(\eta_\tau(w'))$ is either 0 or has non-trivial dependence on some entry of τ_{12}. The integrals in (5.15) are obtained from:

$$\int_{g^{-1}\mathcal{F}_w} e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}Y'(w',w')_\tau}\det y^{-n-1}dxdy\tag{5.17}$$

by differentiation:

$$\begin{aligned}&\int_{g^{-1}\mathcal{F}_w} p'_N(\eta_\tau(w'))e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}Y'(w',w')_\tau}\det y^{-n-1}dxdy\\&=p''_N\left(\frac{\partial}{\partial N}\right)\int_{g^{-1}\mathcal{F}_w} e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}Y'(w',w')_\tau}\det y^{-n-1}dxdy\end{aligned}\tag{5.18}$$

with the differential operator $p''_N\left(\frac{\partial}{\partial N}\right)$ defined by:

$$p''_N\left(\frac{\partial}{\partial N}\right)e^{2\pi i\operatorname{tr}N\tau}=p'_N(\eta_\tau(w'))e^{2\pi i\operatorname{tr}N\tau}$$

As the previous lemma shows that (5.17) is independent of any of the variables occurring in N_{12}, however the differentiation in (5.18) involves differentiation with respect to some number of entries in N_{12} which yields 0. □

Corollary 5.1.12. *Suppose that f is a cusp form on $\mathcal{H}_n$, and Φ_f is the theta lift to $O(m,m)$.*

1. *If $m < 2n$, then $\Phi_f = 0$ identically.*
2. *If $m \geq 2n$, then:*
$$\Phi_f(\xi) = \sum_{\substack{S \in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S = 2n}} e^{\pi i \operatorname{tr} SX} a_S(Y)$$
and
$$a_S(Y;P) = \sum_{w \in \mathcal{O}^S} \int_{\mathcal{H}} P(f(\tau))(\eta_\tau(w)) e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-n-1} dx dy \tag{5.19}$$
(i.e.: all Fourier non-zero coefficients are full rank and are given entirely by the non-degenerate w).

Remark 5.1.13. *The sums in (5.19) are finite; that is $|\mathcal{O}^S| < \infty$. This is not immediately obvious but it follows from the discussion in the proof of lemma 5.3.4 in the next section.*

5.2 Relations Between the Lift for Different m

We have a fact analogous to Theorem 4.1.17 that states that the lift to $O(m,m)$ for $m > 2n$ "comes from" the lift to $O(2n,2n)$. The setup and proof is much the same, but we will write it out in full here. Define subgroups of $GL_m(\mathbb{Z})$:

$$P_{2n}(\mathbb{Z}) = \begin{pmatrix} GL_{2n}(\mathbb{Z}) & * \\ & GL_{m-2n}(\mathbb{Z}) \end{pmatrix}, \qquad \overline{P}_{2n}(\mathbb{Z}) = {}^tP_{2n}(\mathbb{Z}) = \begin{pmatrix} GL_{2n}(\mathbb{Z}) & \\ * & GL_{m-2n}(\mathbb{Z}) \end{pmatrix}$$

Next, for $\xi \in \mathcal{D}_m$, let $[\xi]_{m,2n}$ be the top left $2n \times 2n$ minor of ξ, which is an element of $\mathcal{D}_{2n}$, and for $P \in \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,n}(\mathbb{C})]^\kappa)$, let $[P]_{m,2n} \in \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{2n,n}(\mathbb{C})]^\kappa)$ given by composition with the map $\mathbb{C}[M_{m,n}(\mathbb{C})] \to \mathbb{C}[M_{2n,n}(\mathbb{C})]$ given by setting the bottom $m-2n$ rows equal to 0. For $\Phi \in \mathbb{C}[M_{2n,n}(\mathbb{C})]^*$ denote by $[\Phi]_{m,2n} \in \mathbb{C}[M_{m,2n}(\mathbb{C})]^*$ the dual map, so that we have $[\Phi]_{m,2n}(P) = \Phi([P]_{m,2n})$. Write $\Phi_f^{(m)}$ for the lift of f to $O(m,m)$ and $\Phi_f^{(2n)}$ for the lift to $O(2n,2n)$.

Theorem 5.2.1. *If $m < 2n$ then the lift to $O(m,m)$ of a cusp form for $SP_n(\mathbb{R})$ is 0. If $m > 2n$, then we have:*
$$\Phi_f^{(m)}(\xi) = \sum_{A \in \overline{P}_{2n}(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \rho_\kappa({}^tA) \left[\Phi_f^{(2n)}([A\xi\,{}^tA]_{m,2n})\right]_{m,2n}$$

or, evaluated on a $P \in \mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{m,2n}(\mathbb{C})])$*:*

$$\Phi_f^{(m)}(\xi; P) = \sum_{A \in \overline{P}_{2n}(\mathbb{Z}) \backslash GL_m(\mathbb{Z})} \Phi_f^{(2n)}\big([A\xi\, {}^tA]_{m,2n}; [\sigma({}^tA^{-1})P]_{m,2n}\big)$$

Proof. That $\Phi_f^{(m)}$ is 0 identically when $m < 2n$ is the first statement of Corollary 5.1.12. For the rest of the statement, we have that:

$$\Phi_f^{(m)}(\xi; P) = \sum_{\substack{S \in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S = 2n}} a_S(Y; P) e^{\pi i \operatorname{tr} SX}$$

Let $Gr_{2n,m}(\mathbb{Q})$ be the space of $2n$ dimensional subspaces of $\mathbb{Q}^m$. We have a transitive action of $GL_m(\mathbb{Z})$ on $Gr_{2n,m}(\mathbb{Q})$, and if we fix the base point $W_0 = \mathrm{colspan}\begin{pmatrix} 1_{2n} \\ 0 \end{pmatrix}$ we have:

$$GL_m(\mathbb{Z})/P_{2n}(\mathbb{Z}) \cong Gr_{2n,m}(\mathbb{Q})$$

where $A \in GL_m(\mathbb{Z})$ is sent to AW_0. Given $W \in Gr_{2n,m}(\mathbb{Q})$, define:

$$\mathrm{Skew}_m^W(\mathbb{Z}) = \{S \in \mathrm{Skew}_m(\mathbb{Z}) :\ \mathrm{image}(S) = W\}$$

We have that $\mathrm{Skew}_m^{AW}(\mathbb{Z}) = A(\mathrm{Skew}_m^W(\mathbb{Z}))\, {}^tA$. For $S \in \mathrm{Skew}_{2n}(\mathbb{Z})$, write $S' = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Skew}_m(\mathbb{Z})$. If rank $S = 2n$ then we have that $\mathrm{image}(AS'\, {}^tA) = AW_0$. We can group Fourier coefficients of $\Phi_f^{(m)}$ according to image(S):

$$\Phi_f^{(m)}(\xi; P) = \sum_{A \in GL_m(\mathbb{Z})/P_{2n,m}(\mathbb{Z})} \sum_{\substack{S \in \mathrm{Skew}_{2n}(\mathbb{Z}) \\ \mathrm{rank}\ S = 2n}} a_{AS'\,{}^tA}(Y; P) e^{\pi i \operatorname{tr} AS'\, {}^tAX}$$

we have that:

$$a_{AS'\,{}^tA}(Y; P) = a_{S'}\big({}^tAYA; \sigma(A^{-1})P\big)$$

and further that $e^{\pi i \operatorname{tr} AS'\,{}^tAX} = e^{\pi i \operatorname{tr} S[{}^tAXA]_{m,2n}}$. We have that:

$$a_{S'}\big({}^tAYA; \sigma(A^{-1})P\big) = 2 \sum_{\substack{w \in M_{2n,2n}(\mathbb{Z})/SP_n(\mathbb{Z}) \\ \mathrm{rank}\ w = 2n \\ \langle w, w\rangle = S}} \int_{\mathcal{H}} (\sigma(A)^{-1}P)(f(\tau)) \left(\eta_\tau \begin{pmatrix} w \\ 0 \end{pmatrix}\right) \exp\left(-\pi \operatorname{tr} {}^tAYA\left(\begin{pmatrix} w \\ 0 \end{pmatrix}, \begin{pmatrix} w \\ 0 \end{pmatrix}\right)_\tau\right) \det y^{-n-1} dx dy \tag{5.20}$$

We have that

$$\operatorname{tr} {}^tAYA\left(\begin{pmatrix} w \\ 0 \end{pmatrix}, \begin{pmatrix} w \\ 0 \end{pmatrix}\right)_\tau = \operatorname{tr}[{}^tAYA]_{m,2n}(w,w)_\tau$$

and

$$(\sigma(A)^{-1}P)(f(\tau))\left(\eta_\tau \begin{pmatrix} w \\ 0 \end{pmatrix}\right) = \left[(\sigma(A)^{-1}P)(f(\tau))\right]_{m,2n}(\eta_\tau(w))$$

due to the presence of the 0's. Thus we have that

$$a_{AS'\,{}^tA}(Y;P) = a_S\big([{}^tAYA]_{m,2n}; [\sigma(A)^{-1}P]_{m,2n}\big)$$

Finally, we note that summing ${}^tA$ over $GL_m(\mathbb{Z})/P_{2n,m}(\mathbb{Z})$ is equivalent to instead summing A over $\overline{P}_{2n,m}(\mathbb{Z})\backslash GL_m(\mathbb{Z})$. □

Remark 5.2.2. *It is interesting to see how the choice of representatives of* $\overline{P}_{2n}(\mathbb{Z})\backslash GL_m(\mathbb{Z})$ *do not matter in the above formula. We have that if* $A = \begin{pmatrix} A_{11} & \\ A_{21} & A_{22} \end{pmatrix} \in \overline{P}_{2n}(\mathbb{Z})$*, we have* $[A\xi\,{}^tA]_{m,2n} = A_{11}[\xi]_{m,2n}\,{}^tA_{11}$*, and* $[\sigma(A)^{-1}P]_{m,2n} = \sigma(A_{11})^{-1}[P]_{m,2n}$*. This has the effect of permuting the* w *in the sum (5.20) by* A_{11} *which cancels with the action on* P*.*

Remark 5.2.3. *If is interesting to note that Lemmas 5.1.7 and 5.1.9 are essential for the above result. We could define pieces*

$$\Phi_{f,r,j}(\xi;P) = \sum_{\substack{S\in \mathrm{Skew}_m(\mathbb{Z}) \\ \mathrm{rank}\ S=2r}} e^{\pi i \operatorname{tr} XS} \sum_{[w]\in\mathcal{O}^{S,j}} 2\int_{\mathcal{F}_w} P(f(\tau))(\eta_\tau(w))e^{-\pi \operatorname{tr} Y(w,w)_\tau} \det y^{-1-n} dxdy$$

so that $\Phi_f = \sum_{r,j}\Phi_{f,r,j}$*. Each* $\Phi_{f,r,j}$ *is not modular for* $O(m,m)$*, but their sum is (they do transform modularly for* $\gamma \in N_\Gamma M_\Gamma$ *(2.13), but only their sum is guaranteed to be modular for all of* Γ*). Following the same proof we have above, we would have that* $\Phi_{f,r,j}^{(m)}$ *"comes from"* $\Phi_{f,r,j}^{(2r+j)}$ *in the same sense as the theorem. Thus it is due to the lemmas that* $\Phi_{f,r,j} = 0$ *identically unless* $j = 0$ *and* $r = n$ *that we get that the entirety of* $\Phi_f^{(m)}$ *"comes from"* $\Phi_f^{(n)}$*.*

5.3 Fourier Coefficients of the Lift to $O(2n, 2n)$

Now we will move on to calculation of the full rank Fourier coefficients. In light of Theorem 5.2.1, the key case to consider is the lift from $SP_n(\mathbb{R})$ to $O(2n,2n)$, and from Lemma 5.1.9 and Lemma 5.1.7, the only non-zero Fourier coefficients come from the full rank orbits, so that we have:

$$\Phi_f(\xi;P) = \sum_{\substack{S\in \mathrm{Skew}_{2n}(\mathbb{Z}) \\ \mathrm{rank}\ S=2n}} a_S(Y;P)e^{\pi i \operatorname{tr} XS}$$

with the Fourier coefficients give by:

$$a_S(Y;P) = 2\sum_{w\in\mathcal{O}^S}\int_{\mathcal{H}} P(f(\tau))(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-n-1}dxdy$$

From (3.28) there are relations between the Fourier coefficients. For each S we have that there is a unique representative Fourier coefficient (3.3.6), $a_D(Y)$, where $D = \mathrm{diag}(d_1,\ldots,d_n)$ is diagonal with positive integral entries along the diagonal and $d_n|\ldots|d_1$. For such D we defined $J(D) = \begin{pmatrix} & D \\ -D & \end{pmatrix}$, and the coefficient a_D is given by:

$$a_D = \sum_{w\in\mathcal{O}^D}\int_{\mathcal{H}} P(f(\tau))(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-n-1}dxdy$$

where $M_{2n}^D(\mathbb{Z}) = \{w \in M_{2n}(\mathbb{Z}) : \ \langle w,w\rangle = J(D)\}$, and $\mathcal{O}^D = M_{2n}^D(\mathbb{Z})/\Gamma'$. For any $S \in \mathrm{Skew}_{2n}(\mathbb{Z})$ of full rank we have that there is some $A \in GL_{2n}(\mathbb{Z})$ so that

$$a_S(Y;P) = a_D(Y[A];\sigma(A)^{-1}P)$$

where and $S = J(D)[{}^tA]$. The A is unique as a class in $GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D)$, where:

$$Sp_n(\mathbb{Z};D) = \{\gamma \in GL_{2n}(\mathbb{Z}) : \ J(D)[{}^t\gamma] = J(D)\} \tag{5.21}$$

Note that we have $Sp_n(\mathbb{Z};\mu D) = Sp_n(\mathbb{Z};D)$ for all positive integers μ, so that $Sp_n(\mathbb{Z};D)$ depends only on $D_0 = d_n^{-1}D$. We can arrange the representative Fourier coefficients according to which D_0 they correspond to.

Definition 5.3.1. *For $D_0 = \mathrm{diag}(d_1,\ldots,d_{n-1},1)$ with $d_1,\ldots,d_{n-1} \in \mathbb{Z}_{>0}$ with $d_{n-1}|\ldots|d_1$, define $\phi_{f,D_0}(\xi;P)$ by:*

$$\phi_{f,D_0}(\xi;P) = \sum_{d=1}^{\infty} e^{\pi i\,\mathrm{tr}\,dJ(D_0)X}a_{dD_0}(Y;P) \tag{5.22}$$

The following proposition is simply an arrangement of the terms in the Fourier series for Φ_f according to which D_0 they correspond to:

Proposition 5.3.2. *With D_0, $Sp_n(\mathbb{Z};D_0)$ and ϕ_{f,D_0} as above, we have:*

$$\Phi_f(\xi;P) = \sum_{D_0}\sum_{A\in GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)}\sum_{d=1}^{\infty} e^{\pi i\,\mathrm{tr}\,dJ(D_0)X[A]}a_{dD_0}(Y[A];\sigma(A)^{-1}P)$$

and grouping the inside sum together, we have:

$$\Phi_f(\xi;P) = \sum_{D_0}\sum_{A\in GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)} \phi_{f,D_0}(\xi[A];\sigma(A)^{-1}P)$$

where the outside sum ranges over $D_0 = \mathrm{diag}(d_1,\ldots,d_{n-1},1)$ as in the previous definition.

Remark 5.3.3. *Instead of summing over cosets of $GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)$, we could instead define an $S_0 \in \mathrm{Skew}_{2n}(\mathbb{Z})$ to be primitive if for all $\mu \in \mathbb{Z}$ we have $\mu S_0 \in \mathrm{Skew}_n(\mathbb{Z})$ iff $\mu \in \{-1,0,1\}$. Then for S_0 primitive we define:*

$$\phi_{f,S_0}(\xi) = \sum_{\mu=1}^{\infty} e^{\pi i \mu \operatorname{tr} XS_0} a_{\mu S_0}(Y)$$

and then we have

$$\Phi_f(\xi) = \sum_{\substack{S_0 \in \mathrm{Skew}_{2n}(\mathbb{Z}) \\ \mathrm{rank}\ S_0 = 2n \\ S_0 \text{ primitive}}} \phi_{f,S_0}(\xi)$$

For such an S_0 we have that $S_0 = AJ(D_0)\,{}^tA$ for some unique $A \in GL_{2n}(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)$, so that the space of primitive elements of $\mathrm{Skew}_m(\mathbb{Z})$ of rank $2n$ is identified with the disjoint union of all of the $GL_m(\mathbb{Z})/Sp_n(\mathbb{Z};D_0)$.

Now, we move on to evaluation of the representative Fourier coefficients a_D. We will define the following integral:

$$I(f;Y;P) = \int_{\mathcal{H}} P(f(\tau))(\eta_\tau(1)) e^{-\pi \operatorname{tr} Y g_\tau\, {}^t g_\tau} \det y^{-n-1} dx dy \tag{5.23}$$

We note first of all that we have:

$$a_1(Y;P) = 2I(f;Y;P) \tag{5.24}$$

as $\mathcal{O}^1 = \{1\}$. The key to our calculations of the Fourier coefficients is that we will be able to trade the sum over $\mathcal{O}^S$ in (5.3) for a Hecke operator acting on f, so that we will be able to relate all the coefficients to $I(f;Y;P)$ in a similar way to (5.24).

Lemma 5.3.4. *For $D = \mathrm{diag}(d_1,\dots,d_n)$ with $d_1|\dots|d_n$ positive integers, recall the D-total Hecke operator, $T(D)$, defined in definition 3.2.7, and define $\mu(D) = d_1$, $D' = \mu(D)D^{-1}$, and $\Delta_{D'} = \begin{pmatrix} 1 & \\ & D' \end{pmatrix}$. For convenience we will also define*

$$Y_{D'} := \Delta_{D'}^{-1} Y \Delta_{D'}^{-1} = Y[\Delta_{D'}^{-1}] \tag{5.25}$$

as it will appear frequently in the formulas we have later. Then the Fourier coefficient a_D is given by:

$$a_D(Y;P) = 2\mu(D)^{n(n+1)/2} I\Big(T(D)f;\ \mu(D)Y_{D'};\ \sigma(\Delta_{D'})P\Big) \tag{5.26}$$

Remark 5.3.5. *The D' defined here will be $D' = \mathrm{diag}(1, d_2', \dots, d_n')$, with $d_2'|\dots|d_n'$. Note that for D we have that the diagonal entries decrease in divisibility as we go down the diagonal, but for D' it is the opposite where they increase. Also we note that D' depends only on D_0, so that we have:*

$$\phi_{f,D_0}(\xi;P) = 2\sum_{d=1}^{\infty} d^{n(n+1)/2}\mu(D_0)^{n(n+1)/2}e^{\pi i\,\mathrm{tr}\,dXJ(D_0)}I\Big(T(dD_0)f;\ d\mu(D_0)Y_{D_0'};\ \sigma(\Delta_{D_0'})P\Big)$$

Proof. First, we will describe a set of representatives for $\mathcal{O}^D$, analogous to (3.11). More precisely, we claim that each class of $\mathcal{O}^D$ is represented by an element of the form $\begin{pmatrix} w_{11} & w_{12} \\ & w_{22}\end{pmatrix}$, where w_{11} is in (column) Hermite normal form such that $w_{22} = D\,{}^tw_{11}^{-1}$ is integral, and w_{12} is determined uniquely modulo $w_{22}\mathrm{Sym}_n(\mathbb{Z})$. Note, taking $D = \mu I_n$ gives a set of representatives for $GSP_n^{\mu}(\mathbb{Z})/\Gamma'$. This is essentially the transpose of (3.11).

To prove the claim, suppose that $w \in M_{2n}^D(\mathbb{Z})$. As $\langle w, w\rangle = \begin{pmatrix} 0 & D \\ -D & 0\end{pmatrix}$, the bottom n rows of w span an n-dimensional isotropic subspace, so that there is some $\gamma \in \Gamma'$ so that $w\gamma$ is of the form $\begin{pmatrix} w_{11} & w_{12} \\ & w_{22}\end{pmatrix}$. Next, acting by $\begin{pmatrix} a & \\ & {}^ta^{-1}\end{pmatrix} \in \Gamma'$ for some $a \in GL_n(\mathbb{Z})$ has the effect of sending w_{11} to $w_{11}a$. Thus we may put w_{11} in (column) Hermite normal form so that w_{11} is lower triangular. Then it follows from $w_{11}\,{}^tw_{22} = D$ that $w_{22} = D\,{}^tw_{11}^{-1}$ is uniquely determined by w_{11} and in upper triangular form. The Hermite normal form of w_{11} is unique, and acting by $\begin{pmatrix} 1 & * \\ & 1\end{pmatrix}$ makes w_{12} unique modulo $w_{22}\mathrm{Sym}_n(\mathbb{Z})$, proving the claim.

Now, for $w \in M_{2n}^D(\mathbb{Z})$, we have that $\Delta_{D'}w \in GSp_n^{\mu(D)}(\mathbb{Z})$, and the map $w \mapsto \Delta_{D'}w$ descends to an injective map $\mathcal{O}^D$ to $GSp_n^{\mu(D)}(\mathbb{Z})/\Gamma'$. The image of this map can be described in terms of representatives by:

$$\Delta_{D'}\mathcal{O}^D = \left\{\begin{pmatrix} g_{11} & g_{12} \\ & g_{22}\end{pmatrix} \in GSp_n^{\mu(D)}(\mathbb{Z})/\Gamma' :\ (D')^{-1}g_{22} \in M_n(\mathbb{Z})\right\} \tag{5.27}$$

Now, when we consider the integrals involved, we have:

$$\begin{aligned}
&\int_{\mathcal{H}} P(f(\tau))(\eta_\tau(w))e^{-\pi\,\mathrm{tr}\,Y(w,w)_\tau}\det y^{-n-1}dxdy \\
&= \int_{\mathcal{H}} P(f(\tau))\big(\Delta_{D'}^{-1}\eta_\tau(\Delta_{D'}w)\big)e^{-\pi Y_{D'}(\Delta_{D'}w,\Delta_{D'}w)_\tau}\det y^{-n-1}dxdy \\
&= \mu(D)^{d(\kappa)-n(n+1)/2}\int_{\mathcal{H}} P\big(f|^{\kappa}_{(\Delta_{D'}w)^{-1}}(\tau)\big)\big(\Delta_{D'}^{-1}\eta_\tau(1)\big)e^{-\pi\mu(D)Y_{D'}(1,1)_\tau}\det y^{-n-1}dxdy \\
&= \mu(D)^{n(n+1)/2}\int_{\mathcal{H}} P\big(f|^{\kappa}_{\mu(D)(\Delta_{D'}w)^{-1}}(\tau)\big)\big(\Delta_{D'}^{-1}\eta_\tau(1)\big)e^{-\pi\mu(D)\,\mathrm{tr}\,Y_{D'}g_\tau\,{}^tg_\tau}\det y^{-n-1}dxdy
\end{aligned}$$

Where the first equality is simply multiplying and dividing by Δ_D in certain positions, and the second is using lemma 5.1.4, and the third by the fact that for $r \in \mathbb{R}_{>0}$ and $g \in GSp_n^+(\mathbb{R})$, we have $f|^{\kappa}_{rg} = r^{d(\kappa)-n(n+1)}f|^{\kappa}_g$, and that $(1,1)_\tau = g_\tau\,{}^tg_\tau$.

Next, we claim that the map $g \mapsto \mu g^{-1}$ is an involution on $GSp_n^\mu(\mathbb{Z})$ that descends to a bijection between $GSp_n^\mu(\mathbb{Z})/\Gamma'$ and $\Gamma' \backslash GSp_n^\mu(\mathbb{Z})$. To show the first point, note that $gJ\,{}^tg = \mu J$ for $g \in GSp_n^\mu(\mathbb{Z})$, so that $\mu g^{-1} = J\,{}^tgJ^{-1}$, showing that μg^{-1} is integral, and then we note that $\mu(\mu g^{-1}) = \mu^2\mu(g)^{-1} = \mu$. Next, to show the second point, note that $\mu \times (g\gamma)^{-1} = \gamma^{-1}(\mu g^{-1})$ for $\gamma \in \Gamma'$, so that the map interchanges the right and left action of Γ' on $GSp_n^\mu(\mathbb{Z})$.

It will be helpful to note what the map $g \mapsto \mu(g)g^{-1}$ does to representatives of the form (5.27). We have:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ & g_{22} \end{pmatrix}, \quad \mu(g)g^{-1} = \begin{pmatrix} {}^tg_{22} & -{}^tg_{12} \\ & {}^tg_{11} \end{pmatrix}$$

so that:

$$\mu(D)(\Delta_D\mathcal{O}^D)^{-1} = \left\{ \begin{pmatrix} g_{11} & g_{12} \\ & g_{22} \end{pmatrix} \in \Gamma'\backslash GSp_n^{\mu(D)}(\mathbb{Z}) : \ g_{11}(D')^{-1} \in M_n(\mathbb{Z}) \right\}$$

We note that this is exactly the set described in (3.24). Then, we have:

$$\begin{aligned} a_D(Y;P) &= 2\int_{\mathcal{H}} P(f(\tau))(\eta_\tau(w))e^{-\pi\operatorname{tr}Y(w,w)_\tau}\det y^{-n-1}dxdy \\ &= 2\mu(D)^{n(n+1)/2}\sum_{w\in\mathcal{O}^D}\int_{\mathcal{H}} P\Big(f|^\kappa_{\mu(D)(\Delta_{D'}w)^{-1}}(\tau)\Big)\big(\Delta_{D'}^{-1}\eta_\tau(1)\big)e^{-\pi\mu(D)\operatorname{tr}Y_{D'}g_\tau\,{}^tg_\tau}\det y^{-n-1}dxdy \end{aligned}$$

and so the summation over $w \in \mathcal{O}^D$ gives the summation in the definition of $T(D)$ (3.23) so that

$$a_D(Y;P) = 2\mu(D)^{n(n+1)/2}\int_{\mathcal{H}} P\big(T(D)f(\tau)\big)\big(\Delta_D^{-1}\eta_\tau(1)\big)e^{-\pi\operatorname{tr}(\mu(D)Y_{D'})g_\tau\,{}^tg_\tau}\det y^{-n-1}dxdy$$

proving the proposition. □

Corollary 5.3.6. *Suppose that f is a Hecke eigenfunction, that is $T(d)f = \lambda(d)f$ for some scalars $\lambda(d)$. Then we have:*

$$a_{dI_n}(Y;P) = d^{n(n+1)/2}\lambda(d)a_{I_n}(dY;P) \tag{5.28}$$

(compare to 5.24). More generally, if $\gcd(d,\mu(D_0)) = 1$, then we have

$$a_{dD_0}(Y;P) = d^{n(n+1)/2}\lambda(d)a_{D_0}(dY;P)$$

as $T(dD_0) = T(D_0)T(d)$ in this case.

Next, we move on the calculation of these $I(f;Y;P)$. We have only been able to obtain a full calculation of these integrals for $n=2$. One path for the calculation of these integrals is a manipulation involving Fourier inversion similar to the proof of Lemma (5.1.11) to obtain the integral formula for a matrix argument K-Bessel function as in [3]. When $n=2$ there are identities available from [3] to evaluate the K-Bessel function in our formulas that allow for simplifications.

To express the result of the integral we will introduce some new coordinates on $\mathrm{Sym}^+_{2n}(\mathbb{R})$.

Definition 5.3.7. *For $Y=\alpha\,{}^t\alpha$ with $\alpha=\begin{pmatrix}\alpha_{11} & \\ \alpha_{21} & \alpha_{22}\end{pmatrix}$, define the following quantities:*

$$
\begin{aligned}
\beta(Y) &= {}^t\alpha_{11}\alpha_{21}\\
\delta(Y) &= ({}^t\alpha_{11}\alpha_{22}\,{}^t\alpha_{22}\alpha_{11})^{1/2}\\
\tau(Y) &= {}^t\alpha_{11}^{-1}\left(-\beta(Y)^+ + \frac{1}{(\mathrm{tr}\,\delta(Y))^2}[\beta(Y)^-,\delta(Y)^2] + i\frac{\mathrm{tr}\,|\delta(Y)+\beta(Y)^-|}{\mathrm{tr}\,\delta(Y)}\delta(Y)\right)\alpha_{11}^{-1}\\
M(Y) &= \delta(Y)+\beta(Y)^-
\end{aligned}
\tag{5.29}
$$

where for a square matrices C, we let $|C|:=({}^tCC)^{1/2}$, the symmetric positive semi-definite square root of ${}^tCC$, and $C^{\pm}$ denotes the symmetric and skew symmetric parts of C, respectively, and $[C,C']=CC'-C'C$ denotes the commutator of C and C'.

Remark 5.3.8. *As we mention these give a system of coordinates on $\mathrm{Sym}^+_{2n}(\mathbb{R})$ consisting of a $\tau\in\mathcal{H}_n$ and $M=\delta+\beta^-\in M_n(\mathbb{R})$ with δ positive definite symmetric and β^- skew symmetric. Given $\tau(Y)$, and $M(Y)$ we can work backwards to obtain the entries α, using that α_{11} and α_{22} are assumed lower triangular with positive diagonal entries to obtain Y, so that we have $\mathrm{Sym}^+_{2n}(\mathbb{R})\cong\mathcal{H}_n\times\mathcal{D}_n$.*

We will build up the calculation of $I(f;Y;P)$ in three steps. First we will calculate an analogous integral with $e^{2\pi i\,\mathrm{tr}\,N\tau}$ in place of $P(f(\tau))$. We will differentiate that with respect to N to obtain the integral with a polynomial $p'(\tau)$ in place of $P(f(\tau))$, and finally the full integral $I(f;Y;P)$ by summing over N.

Lemma 5.3.9. *Suppose that $Y=\alpha\,{}^t\alpha$ with $\alpha=\begin{pmatrix}\alpha_{11} & \\ \alpha_{21} & \alpha_{22}\end{pmatrix}$, and N is positive definite. Then:*

$$
\begin{aligned}
&\int_{\mathcal{H}_2} e^{2\pi i\,\mathrm{tr}\,N\tau}e^{-\pi\,\mathrm{tr}\,Yg_\tau\,{}^tg_\tau}\det y^{-3}dxdy\\
&\qquad = \frac{1}{\det Y^{1/2}\,\mathrm{tr}\,|M(Y)|}e^{2\pi i\,\mathrm{tr}\,\tau(Y)N}e^{-2\pi\,\mathrm{tr}\,|M(Y)|}
\end{aligned}
\tag{5.30}
$$

Remark 5.3.10. *It is crucial for us that N appears linearly in the exponent*

Proof. We can expand out the term in the second exponential of the integral as:

$$\operatorname{tr} Y g_\tau {}^t g_\tau = \operatorname{tr}\left(\alpha_{11}{}^t\alpha_{11} y + \alpha_{11}{}^t\alpha_{11}{}^t(x+\alpha_{21}\alpha_{11}^{-1})y^{-1}(x+\alpha_{21}\alpha_{11}^{-1}) + \alpha_{22}{}^t\alpha_{22}y^{-1}\right)$$

We can then transform the integral by the change of variables: $\tau \mapsto {}^t\alpha_{11}^{-1}\tau\alpha_{11}^{-1}$, to obtain that the integral we are seeking to calculate is:

$$\int_{\mathcal{H}_2} e^{2\pi i \operatorname{tr} \alpha_{11}^{-1} N {}^t\alpha_{11}^{-1}\tau} e^{-\pi \operatorname{tr}(y + {}^t(x+{}^t\alpha_{11}\alpha_{21})y^{-1}(x+{}^t\alpha_{11}\alpha_{21}) + ({}^t\alpha_{11}\alpha_{22}{}^t\alpha_{22}\alpha_{11})y^{-1})} \det y^{-3} dxdy$$

Writing ${}^t\alpha_{11}\alpha_{22} = \beta(Y) = \beta(Y)^+ + \beta(Y)^-$ we can perform the change of variables $x \mapsto x - \beta^+(Y)$, and write $\delta(Y)^2 = {}^t\alpha_{11}\alpha_{22}{}^t\alpha_{22}\alpha_{11}$ so that the above is:

$$e^{-2\pi \operatorname{tr} {}^t\alpha_{11}^{-1}\beta(Y)^+\alpha_{11}^{-1}N} \int\limits_{\mathcal{H}_2} e^{2\pi i \operatorname{tr} \alpha_{11}^{-1} N {}^t\alpha_{11}^{-1}\tau} e^{-\pi \operatorname{tr}(y + {}^t(x+\beta(Y)^-)y^{-1}(x+\beta(Y)^-) + \delta(Y)^2 y^{-1})} \det y^{-3} dxdy$$

Thus we will seek to calculate that:

$$\begin{aligned}&\int_{\mathcal{H}_2} e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr}(y + {}^t(x+A)y^{-1}(x+A) + B^2 y^{-1}} \det y^{-3} dxdy \\ &= \frac{1}{4 \det B \operatorname{tr}|B+A|} e^{-2\pi i (\operatorname{tr} B)^{-2} \operatorname{tr}[A,B^2]N} e^{-2\pi \frac{\operatorname{tr}|B+A|}{\operatorname{tr} B} \operatorname{tr} BN - 2\pi \operatorname{tr}|B+A|}\end{aligned} \tag{5.31}$$

For B a positive definite symmetric matrix and A a skew symmetric matrix. We obtain our result by replacing N, A, and B with $\alpha_{11}^{-1}N{}^t\alpha_{11}^{-1}$, $\beta(Y)^-$, and $\delta(Y)$, respectively.

To calculate (5.31), first we take advantage of the Fourier inversion theorem to change the integral over symmetric 2×2 matrices into an integral over all matrices. Write $\mathcal{P}_2$ for the space of 2×2 positive definite symmetric matrices, $\mathcal{S}_2$ for the space of symmetric matrices, and $\mathcal{M}_2$ for the space of all 2×2 matrices. Write $R = \begin{pmatrix} & r \\ -r & \end{pmatrix} = rJ$ and $S = \begin{pmatrix} & s/2 \\ -s/2 & \end{pmatrix} = \frac{s}{2}J$ (where $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$). We have the integral is:

$$\int\limits_{\mathcal{P}_2}\int\limits_{\mathcal{S}_2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-2\pi i \operatorname{tr} {}^tSR} e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr}\left(y + {}^t(x+R+A)y^{-1}(x+R+A) + B^2 y^{-1}\right)} \det y^{-3} drdsdxdy \tag{5.32}$$

As x spans over $\mathcal{S}$ and R spans over the space of skew matrices, $x+R$ spans over $\mathcal{M}_2$. We can absorb the A into R and then write $x+R=M$, and then $dxdr = 2^{-1}dM$, so that above is:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathcal{P}_2}\int_{\mathcal{M}_2} e^{2\pi i \operatorname{tr} {}^tSA} e^{2\pi i \operatorname{tr} {}^t(N+S)M} e^{-\pi \operatorname{tr}\left(y+2Ny+{}^tMy^{-1}M+B^2y^{-1}\right)} \det y^{-3} dM ds dy$$

Integrating over $\mathcal{M}_2$ is a Fourier transform of a Gaussian and gives:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathcal{P}_2} e^{2\pi i \operatorname{tr} {}^tSA} e^{-\pi \operatorname{tr}\left(y+2Ny\,{}^t(N+S)y(N+S)+B^2y^{-1}\right)} \det y^{-2} ds dy$$

We can collect: $\operatorname{tr}(y+2Ny+{}^t(N+S)y(N+S)) = \operatorname{tr}{}^t(1+N+S)y(1+N+S)$ due to the skew symmetry of S. Note that $(1+N+S)$ is invertible for all S as its symmetric part is positive definite. For brevity we will write N' for $N+1$. We note that the quantity $\det y^{-3/2}dy$ is invariant under transformations $y \mapsto ay\,{}^ta$. After such a change of variables in y, the above integral is:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{\mathcal{P}_2} e^{2\pi i \operatorname{tr} {}^tSA} \frac{\det(N'+S)^{1/2}}{\det B^{1/2}} e^{-\pi \operatorname{tr}(y+y^{-1})\left({}^t(N'+S)B^2(N'+S)\right)^{1/2}} \det y^{-2} dy ds \tag{5.33}$$

In [3] functions $K_\nu^{(n)}(Z)$ of a positive definite $n\times n$ matrix Z are defined:

$$K_\nu^{(n)}(Z) = 2^{-n}\int_{\mathcal{P}_n} e^{-\frac{1}{2}\operatorname{tr}(y+y^{-1})Z} \det y^{\nu} \frac{dy}{\det y^{(1+n)/2}} \tag{5.34}$$

Using this, we have that (5.33) is:

$$\int_{-\infty}^{\infty} e^{2\pi i \operatorname{tr} {}^tSA} \frac{\det(N'+S)^{1/2}}{\det B^{1/2}} K_{1/2}^{(2)}\left(2\pi({}^t(N'+S)B^2(N'+S))^{1/2}\right) ds \tag{5.35}$$

Also from [3] we have the formula:

$$K_{1/2}^{(2)}(2\pi Z) = \frac{1}{2}\det Z^{-1/2} K_0^{(1)}(2\pi \operatorname{tr} Z) \tag{5.36}$$

and so (5.35) is:

$$\frac{1}{2\det B}\int_{-\infty}^{\infty} e^{2\pi i \operatorname{tr} SA} K_0^{(1)}\left(2\pi \operatorname{tr}({}^t(N'+S)B^2(N'+S))^{1/2}\right) ds \tag{5.37}$$

For a 2×2 positive definite matrix C we have: $\operatorname{tr} C^{1/2} = \sqrt{\operatorname{tr} C + 2\det C^{1/2}}$. We have as well that $\det(N'+S) = \det N' + \det S$, which is true for 2×2 matrices with one symmetric and the other skew symmetric. We have:

$$\left(\operatorname{tr}({}^t(N'+S)B^2(N'+S))^{1/2}\right)^2$$
$$= \operatorname{tr}{}^tSB^2S + 2\det S\det B + \operatorname{tr} N'[S,B^2] + \operatorname{tr} N'B^2N' + 2\det B\det N'$$

rewriting $S = \frac{s}{2}J$ and completing the square this is:

$$\frac{(\operatorname{tr} B)^2}{4}\left(s + \frac{\operatorname{tr} N'[J,B^2]}{(\operatorname{tr} B)^2}\right)^2 + \operatorname{tr} N'B^2N' + 2\det B\det N' - \frac{1}{4}\frac{(\operatorname{tr} N'[J,B^2])^2}{(\operatorname{tr} B)^2}$$

Consider now the term:

$$\operatorname{tr} N'B^2N' + 2\det B\det N' - \frac{1}{4}\frac{(\operatorname{tr} N'[J,B^2])^2}{(\operatorname{tr} B)^2} \tag{5.38}$$

These formulas are invariant under any transformation $N', B \mapsto kN'k^{-1}, kBk^{-1}$ for $k \in O(2)$, so w.l.o.g. we may assume that $B = \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$ with $b_1, b_2 > 0$. If we then write $N' = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}$, we have:

$$\begin{aligned}
\operatorname{tr} N'B^2N' &= b_1^2n_{11}^2 + (b_1^2+b_2^2)n_{12}^2 + b_2^2n_{22}^2 \\
\det B\det N' &= b_1b_2(n_{11}n_{22} - n_{12}^2) \\
\operatorname{tr} N'[J,B^2] &= 2(b_1^2 - b_2^2)n_{12} \\
\operatorname{tr} B &= b_1 + b_2
\end{aligned}$$

and so after some simplifications (5.38) is $(b_1n_{11} + b_2n_{22})^2 = (\operatorname{tr} BN')^2$. It is somewhat miraculous that this gives the square of a linear function of N, and this is crucial for later steps. Writing $A = aJ$, we have that (5.37) is:

$$\frac{1}{2\det B}\int_{-\infty}^{\infty} e^{2\pi isa}K_0^{(1)}\left(2\pi\sqrt{\frac{(\operatorname{tr} B)^2}{4}\left(s + \frac{\operatorname{tr} N'[J,B^2]}{(\operatorname{tr} B)^2}\right)^2 + (\operatorname{tr} BN')^2}\right)ds \tag{5.39}$$

from [9] we have the formula:

$$\int_{-\infty}^{\infty} e^{2\pi ixy}K_0^{(1)}\left(2\pi p\sqrt{x^2+q^2}\right)dx = \frac{1}{4}(y^2+p^2)^{-1/2}e^{-2\pi q(y^2+p^2)^{1/2}} \tag{5.40}$$

so that (5.39) is:

$$\frac{1}{4\det B\sqrt{(\operatorname{tr} B)^2+4a^2}}e^{-2\pi i a\frac{\operatorname{tr} N'[J,B^2]}{(\operatorname{tr} B)^2}}\exp\left(-2\pi\frac{\sqrt{(\operatorname{tr} B)^2+4a^2}}{\operatorname{tr} B}\operatorname{tr} BN'\right)$$

Next, we note that $\operatorname{tr}{}^tAA = 2a^2$ and $\det A = a^2$, so that

$$\begin{aligned}(\operatorname{tr} B)^2+4a^2 &= \operatorname{tr} B^2 + 2\det B + \operatorname{tr}{}^tAA + 2\det A\\ &= \operatorname{tr}{}^t(B+A)(B+A) + 2\det(B+A)\\ &= \left(\operatorname{tr}({}^t(B+A)(B+A))^{1/2}\right)^2\end{aligned}$$

Where we need that A and B are 2×2 and alternating and symmetric, respectively. Thus we have

$$\sqrt{(\operatorname{tr} B)^2+4a^2} = \operatorname{tr}|B+A|$$

We have that $\operatorname{tr}[J,B^2]=0$ so that $\operatorname{tr} N'[J,B^2] = \operatorname{tr} N[J,B^2]$, so that $a\frac{\operatorname{tr} N'[J,B^2]}{(\operatorname{tr} B)^2} = \frac{\operatorname{tr} N[A,B^2]}{(\operatorname{tr} B)^2}$. Finally we have

$$\frac{\sqrt{(\operatorname{tr} B)^2+4a^2}}{\operatorname{tr} B}\operatorname{tr} BN' = \frac{\operatorname{tr}|B+A|}{\operatorname{tr} B}\operatorname{tr} BN + \operatorname{tr}|B+A|$$

This shows the equality in (5.31). □

Next, for convenience of expressing the final form of the Fourier coefficients we will define the following functions.

Definition 5.3.11. *Define a map* $\mathcal{E} : \operatorname{Sym}^+_{2n}(\mathbb{R}) \to \mathbb{C}[M_{2n,n}(\mathbb{C})]^*$ *by:*

$$\mathcal{E}(Y;p) = p\left(\begin{pmatrix}\tau(Y)\\ 1\end{pmatrix}\right) \tag{5.41}$$

where $\tau(Y)$ *is from (5.29), and we will also define* $\eta(Y)$ *to be:*

$$\eta(Y) = \begin{pmatrix}\tau(Y)\\ 1\end{pmatrix} \tag{5.42}$$

so that $\mathcal{E}(Y;p) = p(\eta(Y))$.

Lemma 5.3.12. *Suppose that* $f : \mathcal{H}_2 \to \mathcal{V}_\kappa$ *is a function that has a Fourier series* $f(\tau) = \sum_N a_N e^{2\pi i \operatorname{tr} N\tau}$ *that is uniformly convergent on any compact subset in* $\mathcal{H}$*. Then*

$$I(f;Y;P) = \frac{1}{4}\mathcal{E}\Big(Y;P\Big(f\big(\tau(Y)\big)\Big)\Big)\frac{e^{-2\pi\operatorname{tr}|M(Y)|}}{\det Y^{1/2}\operatorname{tr}|M(Y)|} \tag{5.43}$$

Proof. With $f(\tau) = \sum_N a_N e^{2\pi i\operatorname{tr} N\tau}$, write $p_N(\eta) \in \mathbb{C}[M_{4,2}(\mathbb{C})]$ to be $p_N = P(a_N)$. Then we have:

$$I(f;Y;P) = \sum_N \int_{\mathcal{H}_2} p_N(\eta_\tau(1)) e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr} Y g_\tau {}^t g_\tau} \det y^{-3} dx dy \tag{5.44}$$

We have that $\eta_\tau(1) = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$. For $p'(\tau)$ a polynomial in $\mathbb{C}[\mathrm{Sym}_2(\mathbb{C})]$, denote by $p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial N}\right)$ the differential operator defined by:

$$p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial N}\right) e^{2\pi i \operatorname{tr} N\tau} = p(\tau)$$

Writing $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$ and $N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix}$, this differential operator can be explicitly obtained by replacing each power of τ_{11}, τ_{12}, and τ_{22} with the same power of $\frac{1}{2\pi i}\frac{\partial}{\partial N_{11}}$, $\frac{1}{4\pi i}\frac{\partial}{\partial N_{12}}$, and $\frac{1}{2\pi i}\frac{\partial}{\partial N_{22}}$, respectively. We then have:

$$\begin{aligned}
&\int_{\mathcal{H}_2} p'(\tau) e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr} Y g_\tau {}^t g_\tau} \det y^{-3} dx dy \\
&= p'\left(\frac{1}{2\pi i}\frac{\partial}{\partial N}\right) \int_{\mathcal{H}_2} e^{2\pi i \operatorname{tr} N\tau} e^{-\pi \operatorname{tr} Y g_\tau {}^t g_\tau} \det y^{-3} dx dy \\
&= \frac{p'(\tau(Y))}{4 \det Y^{1/2} \operatorname{tr} |M(Y)|} e^{2\pi i \operatorname{tr} \tau(Y) N} e^{-2\pi \operatorname{tr} |M(Y)|}
\end{aligned}$$

with the last equality following from (5.30). Then we get (5.44) is

$$\frac{1}{4} \sum_N p_N\left(\begin{pmatrix} \tau(Y) \\ 1 \end{pmatrix}\right) e^{2\pi i \operatorname{tr} \tau(Y) N} \frac{e^{-2\pi \operatorname{tr} |\delta(Y) + \beta(Y)^-|}}{\det Y^{1/2} \operatorname{tr} |M(Y)|}$$

and the result is obtained by summing over N. □

From the above results we have calculated the representative Fourier coefficients explicitly in the case where $n = 2$. We summarize them in the following.

Theorem 5.3.13 (Fourier Coefficients for the Theta lift when $n = 2$)**.** *Suppose that $f : \mathcal{H}_2 \to \mathcal{V}_\kappa$ is a weight κ cusp form of full level and genus 2. For $Y \in \mathrm{Sym}_4^+(\mathbb{R})$, define $\tau(Y)$, $\delta(Y)$ and $\beta(Y)$ as in (5.29), and $Y_{D'}$ and $\Delta_{D'}$ as in Lemma 5.3.4, and $\mathcal{E}$ as in definition 5.3.11. Then the representative Fourier coefficients of Φ_f are given by:*

$$\begin{aligned}
a_D(Y;P) &= \frac{\det D'}{2 \det Y^{1/2}} \mathcal{E}\Big(Y_{D'};\ \big(\sigma(\Delta_{D'})P\big)\big(\big(T(D)f\big)\big(\tau(Y_{D'})\big)\big)\Big) \frac{e^{-2\pi\mu(D) \operatorname{tr} |M(Y_{D'})|}}{\operatorname{tr} |M(Y_{D'})|} \\
&= \frac{\det D'}{2 \det Y^{1/2}} \frac{e^{-2\pi\mu(D) \operatorname{tr} |M(Y_{D'})|}}{\operatorname{tr} |M(Y_{D'})|} \times P\Big(\big(T(D)f\big)\big(\tau(Y_{D'})\big)\Big) \begin{pmatrix} \tau(Y_{D'}) \\ (D')^{-1} \end{pmatrix}
\end{aligned} \tag{5.45}$$

Where $T(D)$ is the D-total Hecke operator defined in Definition 3.2.7.

Proof. The proof is essentially just combining Lemma 5.3.12 and Lemma 5.3.4. Combining them immediately gives:

$$a_D(Y;P) = \frac{\mu(D)^3}{2\det(\mu(D)Y_{D'})^{1/2}}\mathcal{E}\Big(\mu(D)Y_{D'};\ \big(\sigma(\Delta_{D'})P\big)\big(\big(T(D)f\big)\big(\tau(\mu(D)Y_{D'})\big)\big)\Big)\frac{e^{2\pi \operatorname{tr}|M(\mu(D)Y_{D'})|}}{\operatorname{tr}|M(\mu(D)Y_{D'})|}$$

And then we also use that for $r > 0$ we have that $M(rY) = rM(Y)$ and $\tau(rY) = \tau(Y)$, and $\det(\mu(D)Y_{D'}) = \mu(D)^4(\det D')^{-2}\det Y$. □

Now we will describe a parameterization of the subset of Y such that $\beta^-(Y) = 0$ that leads to a nice interpretation of the Fourier coefficients $a_D(Y;P)$ in the above theorem. Just as in the case when $n = 1$, we may define embeddings $\iota_1, \iota_2 : Sp_n(\mathbb{R}) \hookrightarrow O(2n, 2n)$ by:

$$\begin{aligned} \iota_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a & & & b \\ & a & -b & \\ & -c & d & \\ c & & & d \end{pmatrix} \\ \iota_2(g) &= \begin{pmatrix} {}^tg^{-1} & \\ & g \end{pmatrix} \end{aligned} \tag{5.46}$$

unlike when $n = 1$, the images of these embeddings do not commute, and so there is no sense in which we have some sort of identification of $Sp_n(\mathbb{R}) \times_{\pm 1} Sp_n(\mathbb{R})$ with a subgroup of $O(2n, 2n)$. We can however note that these embeddings carry the maximal compact subgroup of $Sp_n(\mathbb{R})$ into that of $O(2n, 2n)$, and induce embeddings $\mathcal{H}_n \hookrightarrow \mathcal{D}_{2n}$ which we will denote by the same symbols. We have:

$$\begin{aligned} \iota_1(\tau) &= \begin{pmatrix} & x \\ -x & \end{pmatrix} + \begin{pmatrix} y & \\ & y \end{pmatrix} \\ \iota_2(\tau) &= {}^tg_\tau^{-1} g_\tau^{-1} = \begin{pmatrix} y^{-1} & -y^{-1}x \\ -xy^{-1} y^{-1}x + y & \end{pmatrix} \end{aligned} \tag{5.47}$$

We note that ι_2 carries all of $Sp_n(\mathbb{R})$ into the Levi subgroup $M \subset O(2n, 2n)$, so that it embeds $\mathcal{H}_n$ entirely inside of the submanifold of $\mathcal{D}_{2n}$ where $X = 0$, which we identify with $\operatorname{Sym}^+_{2n}(\mathbb{R})$.

Definition 5.3.14 (Embedding $\operatorname{Sym}^+_n(\mathbb{R}) \times \mathcal{H}_n \hookrightarrow \operatorname{Sym}^+_{2n}(\mathbb{R})$)**.** *For $y_1 \in \operatorname{Sym}^+_n(\mathbb{R})$ and $\tau_2 = x_2 + iy_2 \in \mathcal{H}_n$, with $y_j = a_j\,{}^ta_j$, a_j lower triangular with positive diagonal entries,* *define:*

$$Y(y_1,\tau_2) = \iota_2(g_{\tau_2})\iota_1(g_{iy_1})\cdot 1 = \begin{pmatrix} {}^t a_2^{-1} a_1\, {}^t a_1 a_2^{-1} & \\ & a_2 a_1\, {}^t a_1\, {}^t a_2 \end{pmatrix}\left[\begin{pmatrix} 1 & -x_2 \\ & 1 \end{pmatrix}\right] \tag{5.48}$$

Remark 5.3.15. *We will write $\alpha(y_1,\tau_2)$ for the unique element of $B_{0,0}$ such that*

$$\alpha(y_1,\tau_2)\, {}^t\alpha(y_1,\tau_2) = Y(y_1,\tau_2)$$

Note that ${}^t a_2^{-1} a_1$ is not lower triangular. Let $k({}^t a_2^{-1} a_1) \in O(n)$ be the unique element such that ${}^t a_2^{-1} a_1 k({}^t a_2^{-1} a_1)$ is lower triangular with positive diagonal entries. Then we have:

$$\alpha(y_1,\tau_2) = \begin{pmatrix} 1 & \\ -x_2 & 1 \end{pmatrix}\begin{pmatrix} {}^t a_2^{-1} a_1 k({}^t a_2^{-1} a_1) & \\ & a_2 a_1 \end{pmatrix} \tag{5.49}$$

Formulas for the relevant quantities β^-, δ, τ in terms of y_1 and τ_2 can be readily found from the above expression. To express the all Fourier coefficients in theorem 5.3.13 we will also need to find these same quantities for $Y(y_1,\tau_2)_{D'}$ when D' is non-scalar as well. These are made difficult for values of τ_2 that do not commute with D' (in particular since β^- does not vanish), so we restrict attention to only such τ_2. The following is a simple verification using (5.49).

Lemma 5.3.16. *Suppose that τ_2 is such that $\tau_2 D' = D'\tau_2$. Then we have:*

$$\begin{aligned} \beta^-(Y(y_1,\tau_2)_{D'}) &= 0 \\ \delta(Y(y_1,\tau_2)_{D'}) &= k({}^t a_2^{-1} a_1)\, {}^t a_1 (D')^{-1} a_1 k({}^t a_2^{-1} a_1) \\ \tau(Y(y_1,\tau_2)_{D'}) &= \tau_2 (D')^{-1} \end{aligned}$$

In particular when D is scalar we have $D' = 1$ and simply:

$$\begin{aligned} \beta^-(Y(y_1,\tau_2)) &= 0 \\ \delta(Y(y_1,\tau_2)) &= k({}^t a_2^{-1} a_1)\, {}^t a_1 a_1 k({}^t a_2^{-1} a_1) \\ \tau(Y(y_1,\tau_2)) &= \tau_2 \end{aligned}$$

We can plug in these values with (5.45) to obtain:

Corollary 5.3.17. *Suppose that $\tau_2 D = D\tau_2$. Then the value of a_D on the subspace of $\mathrm{Sym}_4^+(\mathbb{R})$ consisting of elements of the form $Y(y_1,\tau_2)$ is given by:*

$$a_D(Y(y_1,\tau_2);P) = \frac{\mu(D)^2 \det D^{-1}}{2\operatorname{tr} y_1 \det y_1} e^{-2\pi \operatorname{tr} y_1 D} P\Big(\big(T(D)f\big)(\mu(D)^{-1}\tau_2 D)\Big)\left(\begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}\mu(D)^{-1} D\right)$$

Example 5.3.18 (Scalar Valued Cusp Forms on $\mathcal{H}_2$)**.** *The simplest case to consider is* *when the input form, f, is a scalar valued cusp form, $f(\gamma\tau) = \det j(\gamma,\tau)^\kappa f(\tau)$. As $\mathcal{V}_\kappa$ is one dimensional in this case, the space $\mathrm{Hom}_{GL_n(\mathbb{C})}(\mathcal{V}_\kappa, \mathbb{C}[M_{4,2}(\mathbb{C})])$ is identified with the space $\mathbb{C}[M_{4,2}(\mathbb{C})]^\kappa$, the space of polynomials that transform as $p(\eta a) = \det a^\kappa p(\eta)$ for $a \in GL_n(\mathbb{C})$. The space $\mathbb{C}[M_{4,2}(\mathbb{C})]^\kappa$ is spanned by polynomials of the form $p^\kappa_{\mathbf{u}}(\eta) = \det({}^t\eta\mathbf{u})^\kappa$ for $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in M_{4,2}(\mathbb{C})$. For such polynomials we have $\sigma(\alpha,a)p^\kappa_{\mathbf{u}} = \det a^\kappa p_{{}^t\alpha^{-1}\mathbf{u}}$. For these polynomials (5.45) is:*

$$a_D(Y; p^\kappa_{\mathbf{u}}) = \frac{\det D'}{2\det Y^{1/2}} \frac{e^{-2\pi\mu(D)\,\mathrm{tr}\,|M(Y_{D'})|}}{\mathrm{tr}\,|M(Y_{D'})|} \det\Big(\tau(Y_{D'})u_1 + (D')^{-1}u_2\Big)^\kappa \big(T(D)f\big)\big(\tau(Y_{D'})\big)$$

We can simplify further with some more assumptions. If $D\tau_2 = \tau_2 D$ then in the special coordinates described above we have that this is:

$$a_D(Y(y_1,\tau_2; p^\kappa_{\mathbf{u}}) = \frac{\mu(D)(\det D')^{1-\kappa}}{2\det y_1\,\mathrm{tr}\, y_1 D} e^{-2\pi\,\mathrm{tr}\, y_1 D} \det(\tau_2 u_1 + u_2)^\kappa (T(D)f)(\tau_2(D')^{-1})$$

and in particular if f is a Hecke eigenform with Hecke eigenvalues $T(d)f = \lambda(d)f$, then we have:

$$a_{dI_2}(Y(y_1,\tau_2); p^\kappa_{\mathbf{u}}) = \lambda(d) \frac{e^{-2\pi d\,\mathrm{tr}\, y_1}}{2\,\mathrm{tr}\, y_1 \det y_1} \det(\tau_2 u_1 + u_2)^\kappa f(\tau_2)$$

For a function $h : \mathbb{R}_{>0} \to \mathbb{C}$, whose argument we will denote by t, the *Mellin Transform* of h, denote by $(\mathcal{M}_t h)(s)$ is defined by:

$$(\mathcal{M}_t h(t))(s) = \int_0^\infty t^{s-1} h(t)\ dt \tag{5.50}$$

When f is a Hecke eigenform form on $\mathcal{H}_1$, normalized so that $a_1 = 1$, we have that

$$(\mathcal{M}_t f(it))(s) = \frac{\Gamma(s)}{(2\pi)^s} L(s; f) \tag{5.51}$$

where $L(s; f)$ is the L-function for f. For $n > 1$ we have the functions $\phi_{f,D_0}(\xi)$ defined in (5.22), given by:

$$\phi_{f,D_0}(\xi) = \sum_{\mu=1}^{\infty} e^{\pi i\mu\,\mathrm{tr}\, XJ(D_0)} a_{\mu D_0}(Y)$$

Scaling Y in (4.19) to tY and taking the Mellin transform would yield (up to a translation in s) yield terms that look like (5.51). We are able to say something similar for the lift to $\mathcal{H}_2$ for scalar cusp forms:

Theorem 5.3.19. *Suppose that f is a scalar Hecke cusp form of weight κ and genus 2. Recall we defined functions $\phi_{f,D_0}(\xi)$ in (5.22). Then the the Mellin transform of $t^3\phi_{f,I_2}(tY; p)$*

is:

$$\mathcal{M}_t(t^3\phi_{f,I_2}(tY;p))(s) = \frac{f(\tau(Y))\mathcal{E}(Y;p)}{2\det Y^{1/2}\operatorname{tr}|M(Y)|} \times \frac{\Gamma(s)L(s;f;\mathrm{spin})}{(2\pi\operatorname{tr}|M(Y)|)^s\zeta(2s-2\kappa+4)} \tag{5.52}$$

where $L(s;f;\mathrm{spin})$ is the spin L-function for f. (The factor of t^3 is chosen to eliminate a translation in s).

Proof. First note that we have

$$t^3\phi_{f,I_2}(tY;p) = \frac{\mathcal{E}(Y;p)}{2\det Y^{1/2}\operatorname{tr}|M(Y)|}\sum_{\mu=1}^{\infty}(T(\mu)f)(\tau(Y))e^{-2\pi t\mu\operatorname{tr}|M(Y)|}$$

For f a Hecke cusp form with Hecke eigenvalues $T(\mu)f = \lambda(\mu)$ this is:

$$\frac{f(\tau(Y))\mathcal{E}(Y;p)}{2\det Y^{1/2}\operatorname{tr}|M(Y)|}\sum_{\mu=1}^{\infty}\lambda(\mu)e^{-2\pi t\mu\operatorname{tr}|M(Y)|}$$

As only the part inside the sum depends on t this is the only part we need to calculate the Mellin transform of. We have:

$$\int_0^{\infty} t^s\sum_{\mu=1}^{\infty}\lambda(\mu)e^{-2\pi t\mu\operatorname{tr}|M(Y)|}\frac{dt}{t} = \sum_{\mu=1}^{\infty}\frac{\lambda(\mu)}{(2\pi\operatorname{tr}|M(Y)|)^s}\Gamma(s)$$

Then finally we have from [10] (exercise 3.10, page 21) that:

$$\sum_{\mu=1}^{\infty}\frac{\lambda(\mu)}{\mu^s} = \frac{L(s;f;\mathrm{spin})}{\zeta(2s-2\kappa+4)}$$

□

5.4 Future Work

There are some obvious further developments to do to achieve a more satisfactory result for the lifts from $Sp_n(\mathbb{R})$ to $O(2n,2n)$ of Siegel cusp forms.

1) We need to evaluate the integral $I(Y;f;P)$ (5.23) for $n>2$ as well. For that we have the following conjecture:

Conjecture 5.4.1. *Suppose that $Y \in \mathrm{Sym}^+_{2n}(\mathbb{R})$ and $N \in \mathrm{Sym}^+_n(\mathbb{R})$. Then*

$$\int_{\mathcal{H}_n} e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}Yg_\tau{}^tg_\tau}\det y^{-n-1}dxdy = h(Y)e^{2\pi i\operatorname{tr}\tau(Y)N}e^{-2\pi\operatorname{tr}|M(Y)|} \tag{5.53}$$

where $\tau:\mathrm{Sym}^+_{2n}(\mathbb{R})\to\mathcal{H}_n$, $M:\mathrm{Sym}^+_{2n}(\mathbb{R})\to\mathcal{D}_n$ and $h:\mathrm{Sym}^+_{2n}(\mathbb{R})\to\mathbb{R}$ are functions such that τ is homogeneous of degree 0, M is homogeneous of degree 1, and h is homogeneous of *degree* $-n(n+1)/2$.

If this conjecture is true we have a result similar to (5.45) to give the Fourier coefficients:

$$a_D(Y;P) = h(Y_{D'})\mathcal{E}\Big(Y_{D'};\ \big(\sigma(\Delta_{D'})P\big)\Big(\big(T(D)f\big)\big(\tau(Y_{D'})\big)\Big)\Big)e^{-2\pi\mu(D)\operatorname{tr}M(Y_{D'})}$$

When f is a Hecke eigenform with Hecke eigenvalues $\lambda(\mu)$, we would also have something analogous to (5.52):

$$\mathcal{M}_t\big(t^{n(n+1)/2}\phi_{f,D'}(tY;P)\big)(s) = h(Y)\mathcal{E}\Big(Y;\ P\Big(f\big(\tau(Y)\big)\Big)\Big)\frac{\Gamma(s)}{(2\pi\operatorname{tr}M(Y))^s}\sum_{\mu=1}^{\infty}\frac{\lambda(\mu)}{\mu^s}$$

Towards computing (5.53) we can follow the same steps as in Lemma 5.3.9, and we can get to:

$$\begin{aligned}&\int_{\mathcal{H}_n} e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}(y+{}^t(x+A)y^{-1}(x+A)+By^{-1}}\det y^{-n-1}dxdy\\ &=\int_{\mathcal{A}} e^{\pi i\operatorname{tr}SA}\frac{\det(N+1+S)^{1/2}}{\det B^{1/4}}K_{1/2}^{(n)}\left(2\pi({}^t(N+1+S)B(N+1+S))^{1/2}\right)dS\end{aligned}$$

One might hope that there is some sort of simplification for $K_{1/2}^{(n)}(Z)$ as there is for $K_{1/2}^{(2)}(Z) = \frac{1}{2}\det Z^{-1/2}K_0^{(1)}(2\pi\operatorname{tr}Z)$ like we used in the proof. The method used in [3] to obtain this formula does not seem to easily generalize to $n>2$, unfortunately. It is clear that $K_\nu^{(n)}(Z)$ will be some function of the elementary symmetric polynomials in the eigenvalues of Z, which suggests that even if they do exist for $n>2$, the formulas might be unwieldy if they involve symmetric functions of the eigenvalues other than $\operatorname{tr}Z$ and $\det Z$.

If we restrict attention to the subset of Y of the form $Y(y_1,\tau_2)$, it is sufficient to calculate the integral:

$$\int_{\mathcal{H}_n} e^{2\pi i\operatorname{tr}N\tau}e^{-\pi\operatorname{tr}(y+{}^txy^{-1}x+B^2y^{-1})}\det y^{-n-1}dxdy$$

for B and N positive symmetric definite, which is somewhat simpler than (5.53). (This is the n-dimensional version of (5.31) with $A=0$).

2) Some further investigation of the module structure of $\mathscr{H}(\Gamma',\Gamma'(D'))$ as a $\mathscr{H}_n$ module would shed more light on how to interpret the $T(D)f$ terms in (5.45) to obtain formulas like (5.52) for ϕ_{f,D_0} with $D_0\neq 1_2$, when f is a Hecke eigenform.

We have that $\mathscr{H}(\Gamma',\Gamma'(D'))\subset\operatorname{Hom}\big(S_\kappa(\Gamma'),S_\kappa(\Gamma'(D'))\big)$ is a $\mathscr{H}_n$ module via precomposition. When $\gcd(d,D_0)=1$ we have that $T(D_0)T(d)=T(dD_0)$, so that when f is a Hecke eigenform, we get that:

$$a_{dD_0}(Y) = \lambda(d)a_{D_0}(dY) \tag{5.54}$$

Conjecture 5.4.2. *Suppose that f is a Hecke eigenform. For a positive natural number, d, write $e_{max}(d)$ for the highest power of an exponent in the prime factorization of d. For* *each* $D_0 = \mathrm{diag}(d_1, \ldots, d_{n-1}, 1)$*, there is some* $k > 0$ *depending on* $e_{max}(d_1)$ *and* n *such that we have:*

$$T(dD_0) = \sum_{\substack{\mu | \gcd(d, d_1') \\ e_{max}(\mu) \le k}} \sum_{\mu(g)=\mu} c_{d/\mu}(g) T(\mu D_0) T(g)$$

for some constants $c_{d/\mu}(g)$*.*

If this conjecture is true then if f is a Hecke eigenform, we have:

$$a_{dD_0}(Y) = \sum_{\substack{\mu | \gcd(d, d_1') \\ e_{max}(\mu) \le k}} \sum_{\mu(g)=\mu} \left(\frac{d}{\mu}\right)^{n(n+1)/2} c_{d/\mu}(g) \lambda(g) a_{\mu D_0}(\frac{d}{\mu} Y)$$

Where $\lambda(g)$ is the Hecke eigenvalue of f with respect to $T(g)$. (5.54) is a special case of this with $c_{d/\mu}(g) = 0$ for $\mu > 1$ and $c_d(g) = 1$ for each g so that $\sum_g \lambda(g) = \lambda(d)$. We expect that formulas similar to (5.52) may be found that relate other L-functions related to f to the Mellin transforms of ϕ_{f,D_0}.

We expect that this can be worked out locally via $\mathscr{H}(\Gamma', \Gamma'(D')) \cong \bigotimes_p \mathscr{H}(\Gamma', \Gamma'(D'))_p$ and $\mathscr{H} \cong \bigotimes_p \mathscr{H}_p$ at each prime p.

Chapter 6

Appendix

6.1 Weil Representation

In this section we will describe the Weil representation of $G \times G' = O(m,m) \times Sp_n(\mathbb{R})$ which underlies the modular properties of the theta functions $\widetilde{\Theta}$ defined in (3.61). We will interpret $M_{2m,n}(\mathbb{R})$ as V^n, thought of as length n row vectors of elements of V, where V is $\mathbb{R}^{m,m}$, the quadratic space whose underlying vector space is $\mathbb{R}^{2m}$ together with quadratic the form given by the matrix $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Given $v, v' \in V$, we will denote by $(v, v') = {}^t v Q v'$, their symmetric bilinear product associated to Q. Writing $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (v, v') is given by ${}^t v_1 v_2' + {}^t v_2 v_1'$. We extend this product to V^n by taking the matrix of bilinear products between the different components of elements $v \in V^n$. For $v, v' \in V^n$ we still have the same formula for $(v, v') = {}^t v_1 v_2' + {}^t v_2 v_1'$, but this is an $n \times n$ matrix. This product is symmetric in the sense that ${}^t(v, v') = (v', v)$ and in particular we have (v, v) is a symmetric matrix for all $v \in M_{2m,n}(\mathbb{R})$. We have a left action of G and a right action of $GL_n(\mathbb{R})$ on $M_{2m,n}(\mathbb{R})$ given by matrix multiplication.

Definition 6.1.1 (Standard Model Weil Representation)**.** *Denote by $\mathcal{S} = \mathcal{S}(M_{2m,n}(\mathbb{R}))$ the space of Schwartz functions on $M_{2m,n}(\mathbb{R})$. We have an action of $G \times G'$ on $\mathcal{S}$, which we will denote by ω whose action is given for elements $g \in G$ by:*

$$\omega(g)\varphi(v) = \varphi(g^{-1}v) \tag{6.1}$$

and for elements in the Siegel parabolic subgroup of G' and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by:

$$\begin{aligned} \omega\left(\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}\right)\varphi(v) &= \det a^m \varphi(va) \\ \omega\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right)\varphi(v) &= e^{\pi i \operatorname{tr}(v,v)b}\varphi(v) \\ \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi(v) &= \int_{M_{2m,n}(\mathbb{R})} e^{2\pi i \operatorname{tr}(v,v')}\varphi(v')\, dv' \end{aligned} \tag{6.2}$$

A key feature of the Weil representation is the asymmetry between the action of the orthogonal group and the symplectic group, namely the former acts through a simple translation of the argument, while the latter acts in a significantly more complicated way. In particular the action of the maximal compact subgroup of G' acts in a much more obfuscated way. Because V is a split quadratic space there is an isomorphic representation that we can use that switches this dynamic. We will call this the *symplectic model Weil representation*, but this is terminology is not common as far as we are aware. We will now describe this representation.

We will denote by $W = \mathbb{R}^{2n}$, thought of as row vectors with the symplectic product associated to $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which we denote by $\langle w, w' \rangle = wJ\,{}^t w' = w_1\,{}^t w_2' - w_2\,{}^t w_1$. We will think of $M_{m,2n}(\mathbb{R})$ as W^m, the space of m-component column vectors of elements of w. We extend the symplectic product to elements of $M_{m,2n}(\mathbb{R})$ by taking the matrix of the symplectic products of the components of w, explicitly: $\langle w, w' \rangle = wJ\,{}^t w'$. This product is alternating in the sense that ${}^t\langle w, w' \rangle = -\langle w', w \rangle$, and in particular we have that $\langle w, w \rangle$ is a skew symmetric matrix for all $w \in W^m$.

Definition 6.1.2 (Symplectic Model Weil Representation)**.** *Define a representation ω' of $G \times G'$ on $\mathcal{S}' = \mathcal{S}(M_{m,2n}(\mathbb{R}))$ as follows. For $g \in G'$:*

$$\omega'(g)\varphi'(w) = \varphi'(wg) \tag{6.3}$$

and for elements of the Siegel parabolic of G *and* $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *by:*

$$\begin{aligned} \omega'\left(\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}\right) \varphi'(w) &= |\det a|^n \varphi'({}^t a w) \\ \omega'\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) \varphi'(w) &= e^{\pi i \operatorname{tr} b\langle w, w\rangle} \varphi'(w) \\ \omega'\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \varphi'(w) &= \int_{M_{m,2n}(\mathbb{R})} e^{2\pi i \operatorname{tr}\langle w', w\rangle} \varphi'(w')\, dw' \end{aligned} \tag{6.4}$$

In these representations we have that G acts by translations through ω, and G' acts by translations through ω', but the other group acts in a more complicated manner. We note that the elements $\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ do not generate all of G unlike how the analogous elements generate G'. We need in addition elements Q_I as in 2.14, and they have similar formulas in ω' as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where instead we leave out rows not in I from the integration. This will not be important for our purposes, but we point it out for thoroughness.

Definition 6.1.3 (Partial Fourier Transform)**.** *Define a map* $\mathscr{F} : \mathcal{S} \to \mathcal{S}'$ *by:*

$$\mathscr{F}\varphi(w_1, w_2) = \int_{M_{m,n}(\mathbb{R})} e^{2\pi i \operatorname{tr} {}^t w_2 v_1} \varphi\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} dv_1 \tag{6.5}$$

The inverse of $\mathscr{F}$ is given by:

$$\mathscr{F}^{-1}\varphi'\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \int_{M_{m,n}(\mathbb{R})} e^{-2\pi i \operatorname{tr} {}^t v_1 w_2} \varphi'(v_2, w_2)\, dw_2 \tag{6.6}$$

Lemma 6.1.4. *The map* $\mathscr{F}$ *is an intertwining isomorphism for the action of* $G \times G'$ *between the representations* $(\mathcal{S}, \omega)$ *and* $(\mathcal{S}', \omega')$

We will also record the action of the Lie algebras of G and G', which we will denote by $\mathfrak{g}_0 = \operatorname{Lie} G$ and $\mathfrak{g}_0' = \operatorname{Lie} G'$, respectively. We will generally use the subscript 0 for a real Lie algebra, and no subscript for its complexification. Also we will write matrices representing Lie algebra elements with square brackets, and elements in the groups with round brackets so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ while $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{g}_0$, for example. We will also write $\mathfrak{k}_0 = \operatorname{Lie} K$ and $\mathfrak{k}_0' = \operatorname{Lie} K'$. We obtain the action of the Lie algebras by differentiating the action described in Definitions 6.1.1 and 6.1.2. To simplify things we will introduce the following notation. Use l and r to denote the derivatives of left and right action on a function of a matrix variable:

$$
\begin{aligned}
l(a)\varphi(x) &= \frac{d}{dt}\varphi(e^{ta}x)\Big|_{t=0} = \operatorname{tr} a(x\,{}^t\nabla)\varphi(x) \\
r(a)\varphi(x) &= \frac{d}{dt}\varphi(xe^{ta})\Big|_{t=0} = \operatorname{tr} {}^ta({}^tx\nabla)\varphi(x)
\end{aligned}
\tag{6.7}
$$

where we use ∇ to represent the matrix of differential operators:

$$\nabla_{ij} = \left(\frac{\partial}{\partial x_{ij}}\right) \tag{6.8}$$

and we use the shorthand $\operatorname{tr} a(x\,{}^t\nabla)$ to express the sum $\sum_{i,j,s} a_{ij}x_{js}\frac{\partial\varphi}{\partial x_{is}}$. We also define some operators that are specific to $\mathcal{S}$ and $\mathcal{S}'$. On $\mathcal{S}$ we will define $\triangle^V$, an $n \times n$ matrix of second order differential operators whose ij coordinate is:

$$\triangle^V_{ij} = \sum_{s=1}^{m} \frac{\partial^2}{\partial(v_1)_{si}\partial(v_2)_{sj}} + \frac{\partial^2}{\partial(v_1)_{sj}\partial(v_2)_{si}} \tag{6.9}$$

This operator is essentially the Laplacian corresponding to the split symmetric bilinear form (v, v) on V, and is symmetric in the sense that $\triangle^V_{ij} = \triangle^V_{ji}$.

Lemma 6.1.5 (Lie Algebra Action on $\mathcal{S}$)**.** *For $a \in \mathfrak{g}_0$, we have:*

$$\omega(a)\varphi(v) = -l(a)\varphi(v) \tag{6.10}$$

Then and for $\begin{bmatrix} a & b \\ -c & -{}^ta \end{bmatrix} \in \mathfrak{g}_0'$, *we have:*

$$\omega\left(\begin{bmatrix} a & b \\ -c & -{}^ta \end{bmatrix}\right)\varphi(v) = \left(m \operatorname{tr} a + r(a) + \pi i \operatorname{tr}(v, v)b + \frac{1}{4\pi i}\operatorname{tr} c\triangle^V\right)\varphi(v) \tag{6.11}$$

On $\mathcal{S}'$, we define the differential operator $\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle$ as the $m \times m$ matrix of second order differential operators with entries given by:

$$\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle_{ij} = \sum_{s=1}^{n} \frac{\partial^2}{\partial(w_1)_{is}\partial(w_2)_{js}} - \frac{\partial^2}{\partial(w_1)_{js}\partial(w_2)_{is}} \tag{6.12}$$

This is something like a Laplacian, but instead of corresponding to a symmetric bilinear form, it corresponds to an alternating form. In particular the matrix $\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle$ is skew symmetric.

Lemma 6.1.6 (Lie Algebra Action on $\mathcal{S}'$)**.** *For $a \in \mathfrak{g}_0'$, the action on $\mathcal{S}'$ is given by:*

$$\omega'(a)\varphi'(w) = r(a)\varphi'(w) \tag{6.13}$$

For $\begin{bmatrix} a & b \\ c & -{}^t a \end{bmatrix} \in \mathfrak{g}_0$ *the action on* $\mathcal{S}'$ *is given by:*

$$\omega'\left(\begin{bmatrix} a & b \\ c & -{}^t a \end{bmatrix}\right)\varphi'(w) = \left(n \operatorname{tr} a + l({}^t a) + \pi i \operatorname{tr} b\langle w, w\rangle - \frac{1}{4\pi i}\operatorname{tr} c\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right\rangle\right)\varphi'(w) \tag{6.14}$$

Definition 6.1.7 (Majorized Gaussians and $\mathcal{S}_0$, $\mathcal{S}_0'$)**.** *Inside of* $\mathcal{S}$ *and* $\mathcal{S}'$ *we define a distinguished Gaussian function:*

$$\begin{aligned} \varphi_0(v) &= e^{-\pi \operatorname{tr} {}^t v v} = e^{-\pi \operatorname{tr}(v,v)_0} \in \mathcal{S} \\ \varphi_0'(w) &= e^{-\pi \operatorname{tr} {}^t w w} = e^{-\pi \operatorname{tr}(w,w)_0} \in \mathcal{S}' \end{aligned} \tag{6.15}$$

where $(v, v)_0$ *here is the same as* $(v, v)_1$ *in definition 2.1.13, and* $(w, w)_0$ *here is the same as* $(w, w)_i$ *in definition 2.2.3. We also define subspaces* $\mathcal{S}_0 \subset \mathcal{S}$ *and* $\mathcal{S}_0' \subset \mathcal{S}'$ *by:*

$$\begin{aligned} \mathcal{S}_0 &= \{p\varphi_0 : \ p \text{ is a polynomial on } M_{2m,n}(\mathbb{R})\} \\ \mathcal{S}_0' &= \{p\varphi_0' : \ p \text{ is a polynomial on } M_{m,2n}(\mathbb{R})\} \end{aligned} \tag{6.16}$$

We will write $\mathcal{P}$ *and* $\mathcal{P}'$ *for the spaces of polynomials on* $M_{2m,n}(\mathbb{R})$ *and* $M_{m,2n}(\mathbb{R})$*, respectively. We will interchangeably identify* $\mathcal{S}_0$ *and* $\mathcal{P}$ *as convenient.*

Lemma 6.1.8. *We have that* $\mathscr{F}\mathcal{S}_0 = \mathcal{S}_0'$*. For* $p \in \mathcal{P}$*, we will slightly abuse notation and write* $\mathscr{F}p \in \mathcal{P}'$ *for the polynomial so that* $\mathscr{F}(p\varphi_0) = (\mathscr{F}p)\varphi_0'$*. We can explicitly describe the map* $\mathscr{F} : \mathcal{P} \xrightarrow{\sim} \mathcal{P}'$*:*

$$(\mathscr{F}p)(w_1, w_2) = \exp\left(\frac{1}{4\pi}\triangle_1\right) p\begin{pmatrix} iw_2 \\ w_1 \end{pmatrix} \tag{6.17}$$

where $\triangle_1$ *is the differential operator:*

$$\triangle_1 \varphi \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sum_{i,j=1}^{m,n} \frac{\partial^2}{\partial (v_1)_{i,j}^2} \varphi \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6.18}$$

We note that as p as a polynomial the operator $\exp\left(\frac{1}{4\pi}\triangle_1\right)$ *reduces to a finite sum of orders of the operator* $\triangle_1$*. The inverse operator is given by:*

$$(\mathscr{F}^{-1}p')\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \exp\left(\frac{1}{4\pi}\triangle_2\right) p'(v_2, -iv_1) \tag{6.19}$$

where $\triangle_2$ *is the differential operator:*

$$\triangle_2 \varphi'(w_1, w_2) = \sum_{i,j=1}^{m,n} \frac{\partial^2}{\partial (w_2)_{i,j}^2} \varphi'(w_1, w_2) \tag{6.20}$$

The subspaces $\mathcal{S}_0$ and $\mathcal{S}_0'$ are closed under the actions of $\mathfrak{g}_0$ and $\mathfrak{g}_0'$. We will denote by $\omega_{\mathcal{P}}$ and $\omega_{\mathcal{P}}'$ the action induced on $\mathcal{P}$ and $\mathcal{P}'$ of the Lie algebras under the identifications between $\mathcal{S}_0, \mathcal{S}_0'$ and $\mathcal{P}, \mathcal{P}'$, respectively. The formulas for the action are given by:

Lemma 6.1.9 (Lie Algebra Actions on $\mathcal{S}_0$)**.** *For $x \in \mathfrak{g}_0$, the action on $\mathcal{P}$ is given by:*

$$\omega_{\mathcal{P}}(x)p(v) = -l(x)p(v) + 2\pi \operatorname{tr}(xv, v)_0 p(v) \tag{6.21}$$

For elements in $\mathfrak{g}_0'$ the actions is given by:

$$\begin{aligned}
\omega_{\mathcal{P}}\left(\begin{bmatrix} a & 0 \\ 0 & -{}^t a \end{bmatrix}\right) p(v) &= m \operatorname{tr} a p(v) + \operatorname{tr} {}^t a({}^t v \nabla) p(v) - 2\pi(\operatorname{tr}(v, v)_0 a) p(v) \\
\omega_{\mathcal{P}}\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) p(v) &= \pi i(\operatorname{tr}(v, v) b) p(v) \\
\omega_{\mathcal{P}}\left(\begin{bmatrix} 0 & 0 \\ -c & 0 \end{bmatrix}\right) p(v) &= \frac{1}{4\pi i} \operatorname{tr} c \triangle^V p(v) - \pi i(\operatorname{tr} c(v, v)) p(v) + i \operatorname{tr} c\, {}^t v Q \nabla p(v)
\end{aligned} \tag{6.22}$$

In particular for $x \in \mathfrak{k}_0$ we have the action simplifies to:

$$\omega_{\mathcal{P}}(x)p(v) = -l(x)p(v) \tag{6.23}$$

as $\operatorname{tr}(xv, v)_0 = \operatorname{tr} xv\, {}^t v = 0$ *due to* $v\, {}^t v$ *being symmetric and* x *skew symmetric. For* $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathfrak{k}_0'$*, we have that the action simplifies to:*

$$\omega_{\mathcal{P}}\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) p(v) = r({}^t a) p(v) + i \operatorname{tr} b\, {}^t v Q \nabla p(v) + \frac{1}{4\pi i} \operatorname{tr} b \triangle^V p(v) \tag{6.24}$$

Lemma 6.1.10 (Lie Algebra Actions on $\mathcal{S}_0'$)**.** *For $x \in \mathfrak{g}_0'$, we have:*

$$\omega_{\mathcal{P}}'(x)p'(w) = \left(r(x) + \operatorname{tr}((wx, w)_0)\right) p'(w) \tag{6.25}$$

and when $x \in \mathfrak{k}_0'$, the formula simplifies to:

$$\omega_{\mathcal{P}}'(x)p'(w) = r(x)p'(w) \tag{6.26}$$

For elements in $\mathfrak{g}_0$, the action is given by:

$$\begin{aligned}
\omega'_{\mathcal{P}}\left(\begin{bmatrix} a & \\ & -{}^t a \end{bmatrix}\right) p'(w) &= \left(n \operatorname{tr} a + l({}^t a) - 2\pi \operatorname{tr}(a\langle w, w\rangle_i)\right) p'(w) \\
\omega'_{\mathcal{P}}\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) p'(w) &= \pi i(\operatorname{tr} b\langle w, w\rangle) p'(w) \\
\omega'_{\mathcal{P}}\left(\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right) p'(w) &= \left(\frac{-1}{4\pi i} \operatorname{tr} c \left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle - \pi i \operatorname{tr} c\langle w, w\rangle + i \operatorname{tr} c \left\langle w, \frac{\partial}{\partial w} \right\rangle\right) p'(w)
\end{aligned} \tag{6.27}$$

When $\begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix} \in \mathfrak{k}_0$, *we the formula simplifies to:*

$$\omega'_{\mathcal{P}}\left(\begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix}\right) p'(w) = \left(-l(s_1) + \frac{-1}{4\pi i} \operatorname{tr} s_2 \left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle + i \operatorname{tr} s_2 \left\langle w, \frac{\partial}{\partial w} \right\rangle\right) p'(w) \tag{6.28}$$

We note that as a result of these formulas that the action of $\mathfrak{k}_0 \times \mathfrak{k}'_0$ does not increase the degree of a polynomial. As such we have that $\mathcal{S}_0$ (and $\mathcal{S}'_0$) has a filtration by finite dimensional subspaces (by degree of polynomials) that are invariant spaces for $\mathfrak{k}_0 \times \mathfrak{k}'_0$. As such the action can be exponentiated to obtain that $\mathcal{S}_0$ (resp. $\mathcal{S}'_0$) are $K \times K'$ finite vectors. Indeed, $\mathcal{S}_0$ (resp. $\mathcal{S}'_0$) is the underlying $(\mathfrak{g} \times \mathfrak{g}', K \times K')$ module of $\mathcal{S}$ (resp. $\mathcal{S}'$). We will use $\omega_{\mathcal{P}}$ and $\omega'_{\mathcal{P}}$ as well for the induced actions of $K \times K'$ on $\mathcal{P}$ and $\mathcal{P}'$, respectively. It is difficult to explicitly describe the action of $K \times K'$ on $\mathcal{P}$ or $\mathcal{P}'$. In either case one of the factors simply acts linearly, but the other factor has a much more complicated action. In particular we have:

$$\begin{aligned}
\omega_{\mathcal{P}}(k)p(v) &= p(k^{-1}v) \text{ for } k \in K, \\
\omega'_{\mathcal{P}}(k')p'(w) &= p'(wk') \text{ for } k' \in K'
\end{aligned} \tag{6.29}$$

We will want to introduce some alternate coordinates to express the polynomials in $\mathcal{P}$ and $\mathcal{P}'$ that makes the action of $K \cong O(m) \times O(m)$ and $K' \cong U(n)$ more clear.

Definition 6.1.11 (Polynomials Spaces $\mathbb{C}[\nu^+, \nu^-]$ and $\mathbb{C}[\eta^+, \eta^-]$)**.** *For elements* $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in M_{2m,n}(\mathbb{R})$, *denote by* $\nu^\pm$ *the matrices:*

$$\nu^\pm(v) = v_1 \pm v_2 \tag{6.30}$$

and define $\mathbb{C}[\nu^+, \nu^-]$ *to be the space of polynomials in the* $M_{m,n}(\mathbb{C})$ *variables* ν^+, ν^-. *We identify this space with* $\mathcal{P}$, *identifying* $p(\nu^+, \nu^-) \in \mathbb{C}[\nu^+, \nu^-]$ *with the polynomial* $q(v) \in \mathcal{P}$ *given by* $q(v) = p(v_1 + v_2, v_1 - v_2)$.

Similarly, for $w = (w_1, w_2) \in M_{m,2n}(\mathbb{R})$, denote by $\eta^\pm$ the matrices:

$$\eta^\pm(w) = \mp i w_1 + w_2 \tag{6.31}$$

and define $\mathbb{C}[\eta^+, \eta^-]$ to be the space of polynomials in the $M_{m,n}(\mathbb{C})$ variables η^+, η^-. We identify this space with $\mathcal{P}'$, identifying $p(\eta^+, \eta^-) \in \mathbb{C}[\eta^+, \eta^-]$ with $q(w) \in \mathcal{P}'$ given by $q(w) = p(-iw_1 + w_2, iw_1 + w_2)$.

The matrices $\nu^\pm$ may be thought of as something like the coordinates of V^n according to an orthogonal decomposition of $V = V^+ \oplus V^-$. These spaces are the ± 1 eigenspaces of the matrix $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As $\nu^\pm$ vary over $M_{m,n}(\mathbb{R})$, the vectors $\begin{pmatrix} \frac{1}{2}\nu^+ \\ \frac{1}{2}\nu^+ \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2}\nu^- \\ -\frac{1}{2}\nu^- \end{pmatrix}$ span a maximal positive definite and a perpendicular negative definite subspace, respectively. The coordinate ν^+ is 0 on the negative definite subspace and ν^- is 0 on the positive definite subspace. We denote by $\mathbb{C}[\nu^\pm]$ the subalgebra consisting of polynomials that only depend on the $\nu^\pm$ variable. In terms of functions on the underlying space, these only depend on $V^\pm$ component under the decomposition $V = V^+ \oplus V^-$ above. In particular $\mathbb{C}[\nu^\pm]$ is equal to the subspace of polynomials in $\mathbb{C}[\nu^+, \nu^-]$ that are invariant under precomposition with projection to $V^\pm$, given by the map $v \mapsto \frac{1}{2}(v \pm Qv)$.

There is a similar interpretation of $\eta^\pm$ giving coordinates on the $\pm i$ eigenspaces of $W \otimes \mathbb{C} = W^+_{\mathbb{C}} \oplus W^-_{\mathbb{C}}$ with respect to $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In particular, matrices of the form $\frac{1}{2}(\pm i\eta^\pm, \eta^\pm)$ span the $\pm i$ eigenspace of J on $M_{m,2n}(\mathbb{C})$, and under $w \mapsto \mp iw_1 + w_2$ these map to $\eta^\pm$, with the other coordinate $\eta^\mp = 0$. Similar to before we will denote by $\mathbb{C}[\eta^\pm]$ the subalgebra of polynomials that only depend on the $\eta^\pm$ variable, and these can be thought of as polynomials that depend only on the $W^\pm_{\mathbb{C}}$ variable in the decomposition $W \otimes \mathbb{C} = W^+_{\mathbb{C}} \oplus W^-_{\mathbb{C}}$. Likewise $\mathbb{C}[\eta^\pm]$ is equal to the subspace of polynomials that are invariant under precomposition with projection to $W^\pm_{\mathbb{C}}$, given by the map $w \mapsto \frac{1}{2}(w \mp iwJ)$.

The purpose of these coordinates is to simplify the action of the maximal compact subgroups K and K' in the two different models. Suppose that we have $k \in K$ identified with $(k_+, k_-) \in O(m) \times O(m)$, and $k' = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K'$. Then we have:

$$\begin{aligned} \nu^\pm(k^{-1}v) &= k_\pm^{-1}\nu^\pm(v) \\ \eta^\pm(wk) &= \eta^\pm(w)(a \pm ib) \end{aligned} \tag{6.32}$$

So that:

Lemma 6.1.12. *Consider the identification of $\mathcal{P} \cong \mathbb{C}[\nu^+, \nu^-]$ (resp. $\mathcal{P}' \cong \mathbb{C}[\eta^+, \eta^-]$) above, and the induced action of $K \times K'$ on these $\mathbb{C}[\nu^+, \nu^-]$ (resp. $\mathbb{C}[\eta^+, \eta^-]$) by the action $\omega_{\mathcal{P}}$ on*

$\mathcal{P}$ (resp. $\omega'_{\mathcal{P}}$ on $\mathcal{P}'$). The action of K (resp. K') is then given by:

$$\begin{aligned} &\omega_{\mathcal{P}}(k)p(\nu^+,\nu^-) = p(k_+^{-1}\nu^+, k_-^{-1}\nu_-) \text{ for } k \in K \\ &\omega'_{\mathcal{P}}(k')p(\eta^+,\eta^-) = p\big(\eta^+(a+ib), \eta_-(a-ib)\big) \text{ for } k' = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K' \end{aligned} \tag{6.33}$$

The action of K' on $\mathbb{C}[\nu^+,\nu^-]$ and K on $\mathbb{C}[\eta^+,\eta^-]$ is more difficult to explicitly describe but is obtained from the above via the partial Fourier transform $\mathscr{F} : \mathcal{P} \xrightarrow{\sim} \mathcal{P}'$. We will likewise use $\mathscr{F}$ to describe the induced isomorphism $\mathscr{F} : \mathbb{C}[\nu^+,\nu^-] \xrightarrow{\sim} \mathbb{C}[\eta^+,\eta^-]$. A formula analogous to that in Lemma 6.1.8 may be obtained by translating into the coordinates of $\nu^\pm$ and $\eta^\pm$:

Lemma 6.1.13 (Partial Fourier Transform $\mathscr{F} : \mathbb{C}[\nu^+,\nu^-] \xrightarrow{\sim} \mathbb{C}[\eta^+,\eta^-]$)**.** *The map $\mathscr{F} : \mathbb{C}[\nu^+,\nu^-] \xrightarrow{\sim} \mathbb{C}[\eta^+,\eta^-]$ is defined by the identifications $\mathbb{C}[\nu^+,\nu^-] \cong \mathcal{P}$ and $\mathbb{C}[\eta^+,\eta^-] \cong \mathcal{P}'$, and translating the formulas in terms of $\nu^\pm$ and $\eta^\pm$. We have the following:*

1. *The differential operator $\triangle_1$ on $\mathbb{C}[\nu^+,\nu^-]$ is given by:*

$$\sum_{s,i=1}^{m,n}\left(\frac{\partial}{\partial(\nu^+)_{s,i}} + \frac{\partial}{\partial(\nu^-)_{s,i}}\right)^2 = \sum_{s,i=1}^{m,n}\frac{\partial^2}{\partial(\nu^+)_{s,i}^2} + 2\frac{\partial^2}{\partial(\nu^+)_{s,i}\partial(\nu^-)_{s,i}} + \frac{\partial^2}{\partial(\nu^-)_{s,i}^2} \tag{6.34}$$

2. *The differential operator $\triangle_2$ on $\mathbb{C}[\eta^+,\eta^-]$ is given by:*

$$\sum_{s,i=1}^{m,n}\left(\frac{\partial}{\partial(\eta^+)_{s,i}} + \frac{\partial}{\partial(\eta^-)_{s,i}}\right)^2 = \sum_{s,i=1}^{m,n}\frac{\partial^2}{\partial(\eta^+)_{s,i}^2} + 2\frac{\partial^2}{\partial(\eta^+)_{s,i}\partial(\eta^-)_{s,i}} + \frac{\partial^2}{\partial(\eta^-)_{s,i}^2} \tag{6.35}$$

3. *With $\triangle_1$ and $\triangle_2$ as above, we have that for $p \in \mathbb{C}[\nu^+,\nu^-]$, $(\mathscr{F}p) \in \mathbb{C}[\eta^+,\eta^-]$ is given by:*

$$(\mathscr{F}p)(\eta^+,\eta^-) = \left(e^{\frac{1}{4\pi}\triangle_1}p\right)(i\eta^+, i\eta^-) \tag{6.36}$$

and for $p' \in \mathbb{C}[\eta^+,\eta^-]$, $(\mathscr{F}^{-1}p') \in \mathbb{C}[\nu^+,\nu^-]$ is given by:

$$(\mathscr{F}^{-1}p')(\nu^+,\nu^-) = \left(e^{\frac{1}{4\pi}\triangle_2}p\right)(-i\nu^+, -i\nu^-) \tag{6.37}$$

4. *We have that $\mathscr{F}$ restricts to an isomorphism between $\mathbb{C}[\nu^\pm]$ and $\mathbb{C}[\eta^\pm]$, i.e.: $\mathscr{F}(\mathbb{C}[\nu^\pm])$ $= \mathbb{C}[\eta^\pm]$. Writing $\triangle$ for the usual Laplacian operator on $\mathbb{C}[M_{m,n}(\mathbb{C})]$, the transformations are then:*

$$\begin{aligned} (\mathscr{F}p)(\eta^\pm) &= \left(e^{\frac{1}{4\pi}\triangle}p\right)(i\eta^\pm) \\ (\mathscr{F}^{-1}p')(\nu^\pm) &= \left(e^{\frac{1}{4\pi}\triangle}p'\right)(-i\nu^\pm) \end{aligned} \tag{6.38}$$

5. *When $p \in \mathbb{C}[M_{m,n}(\mathbb{C})]$ is harmonic the operator $\mathscr{F}$ has a particularly simple form:*

$$\begin{aligned}(\mathscr{F}p)(\eta^{\pm}) &= p(i\eta^{\pm}) \\ (\mathscr{F}^{-1}p)(\nu^{\pm}) &= p(-i\nu^{\pm})\end{aligned} \tag{6.39}$$

where we interpret p as a polynomial in $\mathbb{C}[\nu^{\pm}]$ in the first line and in $\mathbb{C}[\eta^{\pm}]$ in the second line.

There is a nice description for the action of $\mathfrak{g}'$ on $\mathbb{C}[\eta^+, \eta_-]$ that is more explicit than (6.25). We have the decomposition $\mathfrak{g}' = \mathfrak{p}'_- \oplus \mathfrak{k}' \oplus \mathfrak{p}'_+$, where

$$\mathfrak{p}'_{\pm} = \left\{ n_{\pm}(b) = \frac{1}{2}\begin{bmatrix} b & \pm ib \\ \pm ib & -b \end{bmatrix} : \ b \in \mathrm{Sym}_n(\mathbb{C}) \right\} \tag{6.40}$$

The action of elements in $\mathfrak{k}'$ on $\mathbb{C}[\eta^+, \eta^-]$ is obtained by differentiating (6.33). To describe the action of $\mathfrak{p}'_{\pm}$, first we note that:

$$\begin{aligned}r(n_{\pm}(b))\eta^{\pm}(w) &= 0 \\ r(n_{\pm}(b))\eta^{\mp}(w) &= -\eta^{\pm}(w)b\end{aligned} \tag{6.41}$$

Next, we can write $\varphi'_0(w) = e^{-\pi \operatorname{tr} \eta^+(w)\, {}^t\eta^-(w)}$. We write $\nabla^{\pm}$ for the gradient associated to $\eta^{\pm}$. Then we can calculate:

$$\omega_{\mathcal{P}}(n_{\pm}(b))p(\eta^+, \eta^-) = \big(-\operatorname{tr} \eta^{\pm} b\, {}^t\nabla^{\mp} + \pi \operatorname{tr} \eta^{\pm} b\, {}^t\eta^{\pm} \big) p(\eta^+, \eta^-) \tag{6.42}$$

by using (6.33) on $p\big(\eta^+(w), \eta^-(w)\big) e^{-\pi \operatorname{tr} \eta^+(w)\, {}^t\eta^-(w)}$.

6.2 Theta Distributions and Theta Functions

Definition 6.2.1 (Theta distribution on $\mathcal{S}$ and $\mathcal{S}'$)**.** *Denote by $\mathscr{D}(\mathcal{S})$ the space of tempered distributions on $\mathcal{S}$. Define $\theta \in \mathscr{D}(\mathcal{S})$ as:*

$$\theta(\varphi) = \sum_{v \in M_{2m,n}(\mathbb{Z})} \varphi(v) \tag{6.43}$$

and $\theta' \in \mathscr{D}(\mathcal{S}')$ as:

$$\theta'(\varphi') = \sum_{w \in M_{m,2n}(\mathbb{Z})} \varphi'(w) \tag{6.44}$$

We will also write $\mathscr{D}(\mathcal{S}_0)$ and $\mathscr{D}(\mathcal{S}'_0)$ for the dual spaces of $\mathcal{S}_0$ and $\mathcal{S}'_0$, respectively.

The partial Fourier transform $\mathscr{F} : \mathcal{S} \xrightarrow{\sim} \mathcal{S}'$ described in the previous section induces a isomorphism $\mathscr{F} : \mathscr{D}(\mathcal{S}') \xrightarrow{\sim} \mathscr{D}(\mathcal{S})$. Under this isomorphism θ and θ' are identified:

$$(\mathscr{F}\theta')(\varphi) = \theta'(\mathscr{F}\varphi) = \theta(\varphi) \tag{6.45}$$

The proof of this fact follows from an application of Poisson summation to (6.5). The groups G and G' act on $\mathscr{D}(\mathcal{S})$ through their action on $\mathcal{S}$:

$$(\omega^*(g,g')\theta)(\varphi) = \theta(\omega(g,g')^{-1}\varphi) \tag{6.46}$$

and similarly for an action on $\mathscr{D}(\mathcal{S}_0')$. As $\mathfrak{g}\times\mathfrak{g}'$ and $K\times K'$ stabilize $\mathcal{S}_0$, they act on $\mathscr{D}(\mathcal{S}_0)$ as well. For $(x,x')\in\mathfrak{g}\times\mathfrak{g}$ and $(k,k')\in K\times K'$ we have:

$$\begin{aligned}(\omega_{\mathcal{P}}^*(x,x')\theta)(p) &= -\theta(\omega_{\mathcal{P}}(x,x')p),\\ (\omega_{\mathcal{P}}^*(k,k')\theta)(p) &= \theta(\omega_{\mathcal{P}}(k,k')^{-1}p)\end{aligned} \tag{6.47}$$

and similarly for the action of $\mathscr{D}(\mathcal{S}_0)$. We will define some functions on $G\times G'$ that are valued in $\mathscr{D}(\mathcal{S}_0)$ (or $\mathscr{D}(\mathcal{S}_0'))$

Definition 6.2.2 (Theta Functions on $G\times G'$)**.** *We define a function $\Theta : G\times G'\to\mathscr{D}(\mathcal{S}_0)$ (and $\Theta' : G\times G'\to\mathscr{D}(\mathcal{S}_0')$) by:*

$$\Theta(g,g';p) = \theta(\omega(g,g')(p\varphi_0)), \quad \Theta'(g,g';p') = \theta'(\omega'(g,g')(p'\varphi_0')). \tag{6.48}$$

where we identify $\mathcal{S}_0$ and $\mathcal{S}_0'$ with $\mathcal{P}$ and $\mathcal{P}'$, respectively. In light of equation 6.45, we have that

$$\Theta(g,g';p) = \Theta'(g,g';\mathscr{F}p) \tag{6.49}$$

Recall we defined $\Gamma\subset G$ and $\Gamma'\subset G'$ as the subgroups $O_{m,m}(\mathbb{Z})$ and $Sp_n(\mathbb{Z})$, respectively. We have:

Lemma 6.2.3. *Suppose that $(\gamma,\gamma')\in\Gamma\times\Gamma'$, and $(k,k')\in K\times K$. Then for all $(g,g')\in G\times G'$, we have:*

$$\begin{aligned}\Theta(\gamma gk,\gamma' g'k') &= \omega_{\mathcal{P}}^*(k,k')^{-1}\Theta(g,g')\\ \Theta'(\gamma gk,\gamma' g'k') &= (\omega_{\mathcal{P}}')^*(k,k')^{-1}\Theta'(g,g')\end{aligned} \tag{6.50}$$

Proof. The invariance with respect to left translation from $(\gamma,\gamma')\in\Gamma\times\Gamma'$ follows from γ permuting the elements in the sum defining θ so that $\Theta(\gamma g,g') = \Theta(g,g')$, and γ' permuting the sum defining θ', so that $\Theta'(g,\gamma' g) = \Theta(g,g')$, and then the equality $\mathscr{F}(\Theta'(g,g')) = \Theta(g,g')$ gives invariance for the other argument. Next, if $(k,k')\in K\times K'$, then we have:

$$\begin{aligned}\Theta(gk,g'k')(p) &= \theta(\omega(g,g')\omega(k,k')(p\varphi_0))\\ &= \theta(\omega(g,g')((\omega_{\mathcal{P}}(k,k')p)\varphi_0))\\ &= \Theta(g,g')(\omega_{\mathcal{P}}(k,k')p)\end{aligned}$$

and similarly for Θ'. □

We will now describe how these are related to the function $\Theta(\xi, \tau)$ described in definition 3.4.1. First, we will denote by σ the action of $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathbb{C}[\eta^+, \eta^-]$ given by

$$\sigma(\alpha, a)p(\eta^+, \eta^-) = |\det \alpha|^n p(\alpha^{-1}\eta^+ \, {}^t a^{-1}, \alpha^{-1}\eta^- a) \tag{6.51}$$

When p depends only on η^- this reduces to the action σ in (3.41). Use σ^* to denote the dual action on $\mathbb{C}[\eta^+, \eta^-]^*$. Recall for $\xi = X + Y \in \mathcal{D}_m$ and $\tau = x + iy \in \mathcal{H}_n$, we have the elements $g_\xi \in G$, $g'_\tau \in G'$ given by:

$$g_\xi = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix}, \quad g'_\tau = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \tag{6.52}$$

where $Y = \alpha \, {}^t\alpha$, $y = a \, {}^t a$ with α, a both lower triangular with positive diagonal entries. Recall as well that we have factors of automorphy $k(g, \xi)$, and $k'(g', \tau)$ valued in K and K', respectively, defined by:

$$g g_\xi = g_{g\xi} k(g, \xi), \quad g' g'_\tau = g'_{g'\tau} k'(g', \tau) \tag{6.53}$$

We also have maps $\eta, \overline{\eta} : M_{m,2n}(\mathbb{R}) \mapsto M_{m,n}(\mathbb{C})$ by:

$$\eta_\tau(w) = w_1 \tau + w_2, \quad \overline{\eta}_\tau(w) = w_1 \overline{\tau} + w_2 \tag{6.54}$$

More generally than (3.39) we could consider functions of the form:

$$\Theta_{symp}(\xi, \tau; p) = \sum_{w \in M_{m,2n}(\mathbb{Z})} p(\overline{\eta}_\tau(w) y^{-1}, \eta_\tau(w)) e^{\pi i \operatorname{tr} \xi \eta_\tau(w) y^{-1} \, {}^t\overline{\eta}_\tau(w)} \tag{6.55}$$

for $p(\overline{\eta}, \eta) \in \mathbb{C}[M_{m,n}(\mathbb{C}) \times M_{m,n}(\mathbb{C})]$. Then we have:

$$\Theta_{symp}(\xi, \tau; p) = \sigma^*(\alpha, a)\Theta'(g_\xi, g'_\tau; p) \tag{6.56}$$

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