Schur Algebras in Type B

by

Dinushi Munasinghe

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Department of Mathematics
University of Toronto

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Abstract

We compare two type B generalizations of the q-Schur algebra: the cyclotomic q-Schur algebra of Dipper, James, and Mathas, and an algebra constructed to maintain the type B Schur duality of Bao, Wang, and Watanabe, introduced by Lai and Luo. By writing the latter algebra as an idempotent truncation of the former, we leverage its properties to establish cellularity and study the crystal graph structure of the simples of the endomorphism algebra, investigating parameter values at which these algebras are Morita equivalent and quasi-hereditary. We then investigate its blocks, also by comparison with those of the cyclotomic q-Schur algebra and type B Hecke algebra.

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Chapter 1

Introduction

A fundamental object in representation theory is the Schur algebra, which arises as the algebra of $\mathfrak{S}_n$-invariant endomorphisms of $V^{\otimes n}$ via Schur–Weyl duality. In 1986, Jimbo generalized this duality to the quantum case [Jim86], considering the commutant of the type A Hecke algebra $\mathcal{H}_n^A(q)$. This naturally suggests the definition of a q-Schur algebra; it was introduced by Dipper and James in 1989 [DJ89], and shown to satisfy a commutant relation with the Hecke algebra in [Du92, (2.d)]. For any base field $\Bbbk$ and parameter $q \in \Bbbk^{\times}$, this algebra is a cellular, quasi-hereditary cover of $\mathcal{H}_n^A(q)$, mirroring the relationship of the classical Schur algebra to the group algebra of the symmetric group in characteristic p, and can be constructed equivalently as $\mathcal{H}_n^A(q)$-invariant endomorphisms on the tensor space or as homomorphisms over permutation modules.

$$\begin{tikzcd}
GL(V) \arrow[r, "\rho"] & \mathrm{End}(V^{\otimes n}) & \Bbbk\mathfrak{S}_n \arrow[l, "\sigma"'] \\
& S(m,n) \cong \rho(GL(V)) \cong \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes n}) \arrow[u, hook] &
\end{tikzcd}$$

Figure 1.1: The classical case

The core of this setup is the classical context where the type A Weyl group $\mathfrak{S}_n$ and the general linear group $GL(V)$ act on a tensor space, satisfying a double centralizer property. The Schur algebra arises as the image of $GL(V)$ under the representation map. Introducing the quantum deformation sees the Hecke algebra of the type A Weyl group take the place of $\mathfrak{S}_n$, the quantum group $U_q(\mathfrak{g})$ that of $GL(V)$, and the q-Schur algebra that of the classical Schur algebra. Key structures of this algebra have proven useful in enabling the study of the representation theories of all objects involved in the type A context - in particular, quasi-hereditarity.

Naturally, one might consider other types: this thesis focuses on type B. Changing the Weyl group at the core of this setup to type B, we see its Hecke algebra, $\mathcal{H}_n^B(Q,q)$ act on $V^{\otimes n}$ - the question is then what other algebras might step into the other positions.

The approach of Hecke-invariant endomorphisms aims to preserve a version of the double-centralizer property and Schur–Weyl duality seen in the type A case. This allows for the preservation of compatibility with induction from parabolic subgroups, the categorification by singular Soergel bimodules as in the work of Williamson [Wil10], and the investigation of other related structures

$$
\begin{tikzcd}
\text{Categorified level:} & \text{singular Soergel bimodules} & \text{Soergel bimodules} \\
& \text{Schur alg.} = \rho(\text{coideal subalg.}) \cong \mathrm{End}_{\mathcal{H}}(V^{\otimes n}) \arrow[u, dashed] \arrow[d, hook] \arrow[dr, "\text{Schur functor}"] & \\
\text{algebraic group} \arrow[r, "\rho"] & \mathrm{End}(V^{\otimes n}) & \text{Hecke algebra } \mathcal{H} \arrow[l, "\sigma"'] \arrow[uu, dashed] \\
\text{coideal subalg.} \arrow[u, hook, "\text{quantum symmetric pair}"'] & &
\end{tikzcd}
$$

Figure 1.2: The Type A case

— for example, coideal subalgebras and quantum symmetric pairs. From this perspective, one constructs a type B Schur algebra which we will call $\mathcal{L}^n(m)$ as the commutant of an action of the type B Hecke algebra $\mathcal{H}_n^B(Q,q)$ on $V^{\otimes n}$. Defined by Lai and Luo for the unequal parameters case ($Q \neq q$), Bao, Wang, and Watanabe showed that this commutant can also be written as a quotient of a coideal subalgebra of $U_q(\mathfrak{sl}_m)$ [LL21; BWW17].

We will see that the dependence of the Morita equivalence class of $\mathcal{L}^n(m)$ on the parameter $m = \dim V$ is limited — the number of simple representations depends on m for small values of M, but as soon as $m \geq n$ it stabilizes and only depends on the parity of m. For this reason, we will say that m is **large** if $m \geq 2n$, and **large odd** or **large even** depending on its parity.

To preserve the relationship between the Hecke algebra and the representation theory of its quasi-hereditary cover in order to maintain the resulting benefits necessitates the construction of the type B Hecke algebra as a cyclotomic quotient of the affine type A Hecke algebra. By work of Dipper–James–Mathas, and Du–Scott [DJM98; DS00] the permutation modules can be generalized to construct a highest weight cover of the module category $\mathcal{H}_n^B(Q,q)$-mod. The resulting algebra is the **cyclotomic q-Schur algebra** of Dipper, James, and Mathas which we will denote $\mathcal{S}^n(\Lambda)$. This algebra depends on the choice of a saturated set Λ of bicompositions. However, up to Morita equivalence, only the set of bipartitions contained in Λ matters. We will most frequently choose $\Lambda = \Lambda_{n,m}$, the set of weak bicompositions (allowing zeroes) where the first component has n parts and the second has $\lfloor \frac{m}{2} \rfloor$, and so will denote this by $\mathcal{S}^n(m) := \mathcal{S}^n(\Lambda_{n,m})$. For this to be a highest weight cover, we require Π_n, the full set of bipartitions of n, to lie in $\Lambda_{n,m}$. This holds if m is large — beyond this, up to Morita equivalence, the algebra $\mathcal{S}^n(m)$ does not depend on m. For both $\mathcal{L}^n(m)$ and $\mathcal{S}^n(m)$, unless otherwise stated, m large can be assumed.

Remark 1.0.1. *We will work with the unequal parameters case: all type B algebras will depend on two parameters, q and Q, where $\{q,-1\}$ and $\{Q,-1\}$ are the roots of the quadratic relation for reflection in long and short roots, respectively. As we will not change these parameters often, we will suppress this notation and write $\mathcal{H}_n$ for the type B Hecke algebra $\mathcal{H}_n^B(Q,q)$ and similarly suppress (Q,q) in the notation of $\mathcal{S}^n(m)$ and $\mathcal{L}^n(m)$.*

While in type A, the algebra of $\mathcal{H}_n^A$-invariant endomorphisms of $V^{\otimes n}$ and the quasi-hereditary cover of $\mathcal{H}_n^A$ built via permutation modules are isomorphic, these two approaches do not coincide in the case of type B. We will see that $\mathcal{L}^n(m)$ can be written as an idempotent truncation of $\mathcal{S}^n(m)$. This allows us to study the algebra $\mathcal{L}^n$, which generalizes Figure 1 to Type B, by leveraging the well-established structure and representation theory of $\mathcal{S}^n(m)$.

Given two such non-isomorphic generalizations of an algebra, it is natural to ask when they are Morita equivalent — this will also mean that $\mathcal{L}^n$ inherits the quasi-hereditary structure of $\mathcal{S}^n(m)$. The category of modules over $\mathcal{S}^n(m)$ has been studied in [DJM98] and [DS00], and that of $\mathcal{L}^n(m)$ in [LNX20]. It has been established that at generic values of the parameters Q, q both algebras are Morita equivalent to $\mathcal{H}_n$, and in fact to the group algebra of the type B Weyl group. However, just as the Morita equivalence of $\mathcal{S}^n(m)$ to $\mathcal{H}_n$ has been shown to fail at a closed set of parameters (see [Ari94]), the Morita equivalence of $\mathcal{L}^n(m)$ and $\mathcal{S}^n(m)$ will also fail at special values — we prove a modified conjecture of Lai, Nakano and Xiang [LNX20], establishing a necessary and sufficient condition for this Morita equivalence to hold.

The key connection between the representation theory of Hecke and Schur algebras is the Schur functor Ω_n. This can be realized for m large as

$$\begin{aligned}\Omega_n : \mathcal{S}^n(m)\text{-mod} &\to \mathcal{H}_n\text{-mod}\\ M &\mapsto Me_{\mathcal{H}_n}\end{aligned}$$

for an idempotent $e_{\mathcal{H}_n}$ such that $e_{\mathcal{H}_n}\mathcal{S}^n(m)e_{\mathcal{H}_n} = \mathcal{H}_n$. That is, the Hecke algebra is realized as an idempotent truncation of the cyclotomic q-Schur algebra, and $\mathcal{H}$-mod is realized as a quotient of $\mathcal{S}^n$-mod by the subcategory of modules sent to zero by $e_{\mathcal{H}_n}$. We will show that this quotient functor factors through the category $\mathcal{L}^n(m)$-mod for large m. That is:

Theorem 1.0.2. *There is an explicitly constructible idempotent e_n such that $\mathcal{L}^n(m) = e_n\mathcal{S}^n(m)e_n$. Thus we have a quotient functor $\mathcal{S}^n(m)$-mod $\to \mathcal{L}^n(m)$-mod. For large m, the Schur functor Ω_n factors through this quotient:*

$$\mathcal{S}^n(m)\text{-mod} \to \mathcal{L}^n(m)\text{-mod} \to \mathcal{H}_n\text{-mod}$$

At all parameter values Q, q, the algebra $\mathcal{S}^n(m)$ will be a quasi-hereditary cover of $\mathcal{L}^n(m)$, which will itself be a cover of $\mathcal{H}_n$. However, $\mathcal{L}^n(m)$ will only be a quasi-hereditary cover of $\mathcal{H}_n$ when it is Morita equivalent to $\mathcal{S}^n(m)$.

We will study the parameter values when the quasi-hereditarity of $\mathcal{L}^n(m)$ fails, beginning with an overview of the relevant algebraic structures and categorical machinery required.

Chapter 2

Background: Structures

2.1 Cellular Algebras

2.1.1 Construction

Let R be a commutative ring with identity. In [GL96] Graham and Lehrer defined a class of associative algebras over R, referred to as **cellular algebras** as follows:

Definition 2.1.1 ([GL96, Definition 1.1])**.** *A **cellular algebra** $\mathcal{A}$ over R is an associative unital algebra with cell datum $(\Lambda, \mathcal{T}, C, *)$ such that:*

(C1) Λ is a poset, and for each $\lambda \in \Lambda$, $\mathcal{T}(\lambda)$ is a finite set such that $C : \bigsqcup_{\lambda\in\Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to \mathcal{A}$ gives an R-basis $\{C^{\lambda}_{ST}\}$ indexed by (ordered) pairs of elements of $\mathcal{T}(\lambda)$ (tableaux of shape λ) as λ runs over Λ.

(C2) $*$ is an R-linear anti-involution of $\mathcal{A}$ such that $(C^{\lambda}_{ST})^* = C^{\lambda}_{TS}$.

(C3) if $\lambda \in \Lambda$ and $S, T \in \mathcal{T}(\lambda)$, then for any $a \in \mathcal{A}$:

$$aC^{\lambda}_{ST} \equiv \sum_{U\in\mathcal{T}(\Lambda)} r_a(U, S)C^{\lambda}_{UT} \mod \mathcal{A}_{<\lambda}$$

where $\mathcal{A}_{<\lambda}$ is the R-submodule of $\mathcal{A}$ generated by "lower-order-terms":

$$\mathcal{A}_{<\lambda} = R\{C^{\mu}_{VW} : \mu < \lambda, V, W \in \mathcal{T}(\mu))\}$$

A cellular algebra is therefore one that has a special basis indexed by pairs of "tableaux" on a given "shape". These shapes are elements of a poset, and we will say that two basis elements belong to the same "cell" if they are indexed by tableaux of the same shape. Intuitively, the multiplication condition states that multiplying a two elements of the algebra together can never "raise" the cell (it is not possible to create a basis element corresponding to a shape higher in the poset by multiplying two lower elements together).

König and Xi have since given an equivalent basis-free definition of cellular algebras:

Definition 2.1.2 ([KX98, Definition 3.2])**.** *Let $\mathcal{A}$ be an R-algebra with anti-involution i. A two-sided ideal $J \subset \mathcal{A}$ is a* ***cell ideal*** *if and only if $i(J) = J$, and there is a left ideal $\Delta \subset J$, free and finitely generated over R, such that $J \cong \Delta \otimes_R i(\Delta)$ as $\mathcal{A}$-bimodules making the following diagram commute:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

An R-algebra $\mathcal{A}$ with anti-involution i is ***cellular*** *if it has an R-module decomposition*

$$\mathcal{A} = J'_1 \oplus \cdots \oplus J'_n$$

such that for for all j the following hold:

- $i(J'_j) = J'_j$
- $J_j := \oplus_{k=1}^{j} J'_k$ *is a two-sided ideal of $\mathcal{A}$*
- $J'_j = J_j/J_{j-1}$ *is a cell ideal of the quotient A/J_{j-1}.*

The chain of i-stable ideals $0 \subset J_1 \subset \cdots \subset J_n = \mathcal{A}$ is then called a ***cellular chain****.*

The two definitions Definition 2.1.1 and Definition 2.1.2 are equivalent [KX98, §3].

2.1.2 Module Structures

Cell Modules and the Bilinear Form

Let us look closer at the requirement (C3) from Definition 2.1.1: If we have two cellular basis vectors C^λ_{ST} and C^λ_{SR}, then it states that:

$$aC^\lambda_{ST} \equiv \sum_{U \in \mathcal{T}(\Lambda)} r_a(U,S) C^\lambda_{UT} \mod \mathcal{A}_{<\lambda}$$

$$aC^\lambda_{SR} \equiv \sum_{U \in \mathcal{T}(\Lambda)} r_a(U,S) C^\lambda_{UR} \mod \mathcal{A}_{<\lambda}$$

In particular, in moving from C^λ_{ST} to C^λ_{SR} the coefficients in the expansion haven't changed, and $r_a(U,S)$ is the coefficient in front of C^λ_{UT} in the first expansion if and only if it is the coefficient in front of C^λ_{UR} in the second. These coefficients can therefore be used to construct a well-defined $\mathcal{A}$-action on the R-span of $\{C^\lambda_S : S \in \mathcal{T}(\lambda)\}$:

Definition 2.1.3. *The cell module $\Delta^\lambda_{\mathcal{A}}$ corresponding to λ is the formal R-span of the symbols $\{C^\lambda_S : S \in \mathcal{T}(\lambda)\}$, with action given by*

$$aC^\lambda_S \equiv \sum_{U \in \mathcal{T}(\lambda)} r_a(U,S) C^\lambda_U \mod \mathcal{A}_{<\lambda}$$

for $a \in \mathcal{A}$.

Now, given any cell module $\Delta^\lambda_{\mathcal{A}}$, we can define a bilinear form $\langle\ ,\ \rangle$ on the cell module by requiring that:

$$C^\lambda_{ST}C^\lambda_{UV} \equiv \langle C^\lambda_T, C^\lambda_U\rangle C^\lambda_{SV} \bmod \mathcal{A}_{<\lambda}$$

$\mathcal{A}$ has simple modules given by quotients of cell modules by the radical of their bilinear forms. Letting

$$\mathcal{D}^\lambda_{\mathcal{A}} \cong \Delta^\lambda_{\mathcal{A}}/\langle,\rangle$$

this gives us a parameterization of the simple modules of a cellular algebra: they are indexed by the cells for which the cellular inner product is nonzero:

Theorem 2.1.4 ([GL96, Theorem 3.4]). *Given a cellular algebra with cell datum $(\Lambda, \mathcal{T}, C, *)$, the set $\{\lambda \in \Lambda \mid \langle a, b\rangle \neq 0$ for some $a, b \in \Delta^\lambda_{\mathcal{A}}\}$ gives a complete parameterization of its simple modules as the unique simple quotients of the cell modules $\Delta^\lambda_{\mathcal{A}}$.*

Blocks of Cellular Algebras

Given an algebra $\mathcal{A}$, its blocks are defined to be its indecomposable direct summand two-sided ideals, giving us a decomposition $\mathcal{A} = B_1 \oplus \cdots \oplus B_k$. Each block comes with a primitive central **block idempotent** e_i corresponding to projection onto this summand.

Given a module $M \in \mathcal{A}$-mod, we say M is in a block B_i if all of its composition factors are. This is equivalent to $M = Me_i$.

Definition 2.1.5 ([GL96, p. 3.9.8], [Mat99]). *Given a cellular algebra $\mathcal{A}$ with cell datum $(\Lambda, \mathcal{T}, C, *)$, $\lambda, \mu \in \Lambda$ are **cell linked** if there is a sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$ of elements in Λ such that for each i, the cell modules $\Delta^{\lambda_{i-1}}_{\mathcal{A}}$ and $\Delta^{\lambda_i}_{\mathcal{A}}$ share a composition factor.*

For cellular algebras over a field, with cells indexed by a finite set Λ, these notions coincide:

Lemma 2.1.6 ([Mat99, Corollary 2.22]). *The poset elements λ and μ are cell linked if and only if the corresponding cell modules belong to the same block.*

2.2 Quasi-Hereditary Algebras

2.2.1 Highest Weight Categories

Definition 2.2.1 ([CPS88, Definition 3.1]). *A locally artinian category $\mathcal{C}$ over a field $\Bbbk$ is **highest weight** if there is an interval finite poset Λ such that*

- *Λ indexes a complete collection of non-isomorphic simple objects $\{S(\lambda)\}_{\lambda\in\Lambda}\}$ of $\mathcal{C}$.*
- *For each $\lambda \in \Lambda$, there is an object $A(\lambda)$ and an embedding $S(\lambda) \hookrightarrow A(\lambda)$ such that all simple composition factors $S(\mu)$ of the quotient $A(\lambda)/S(\lambda)$ satisfy $\mu < \lambda$. Furthermore, for any $\lambda, \mu \in \Lambda$, $\dim_\Bbbk \mathrm{Hom}_{\mathcal{C}}(A(\lambda), A(\mu)))$ and the multiplicities $[A(\lambda) : S(\mu)]$ are finite.*
- *Each simple $S(\lambda)$ has an injective envelope $I(\lambda) \in \mathrm{Obj}(\mathcal{C})$ with a filtration with subquotients of the form $A(\mu)$, beginning with $A(\lambda)$.*

The elements of Λ are referred to as the **weights** of the category, and this structure generalizes that seen with Verma modules in the BGG Category $\mathcal{O}$ with the representation theory of complex semisimple Lie algebras. The modules $A(\lambda)$ are the **(dual) standard objects** of the category [CPS88, Example 3.3].

2.2.2 Quasi-hereditary Algebras

Quasi-hereditary algebras over a field $\Bbbk$ were introduced by Cline, Parshall and Scott [CPS88]. Let $\mathcal{A}$ be a $\Bbbk$-algebra. We begin with the definition of a heredity ideal:

Definition 2.2.2. *An ideal $J \subset \mathcal{A}$ is a* ***heredity ideal*** *if it is idempotent, $J(\operatorname{rad}\mathcal{A})J = 0$, and J is projective.*

A **heredity chain** for an algebra $\mathcal{A}$ (as defined in [KX99], also referred to as a **defining system of ideals** in [CPS88]), if it exists, is a finite chain of ideals $0 = J_0 \subset J_1 \subset \cdots \subset J_n = \mathcal{A}$ such that for each i, J_i/J_{i-1} is a heredity ideal in $\mathcal{A}/J_{i-1}$.

Definition 2.2.3. *If an algebra $\mathcal{A}$ has a heredity chain, it is* ***quasi-hereditary****.*

Theorem 2.2.4 ([CPS88, Theorem 3.6])**.** *Given $\mathcal{A}$ a finite-dimensional algebra over a field $\Bbbk$ $\mathcal{A}$ is a quasi-hereditary algebra if and only if $\mathcal{A}$-mod is a highest weight category.*

2.2.3 Quasi-hereditarity and Cellularity

For this section, let $\mathcal{A}$ be a finite-dimensional algebra over a field $\Bbbk$ with cell datum $(\Lambda, \mathcal{T}, C, *)$, and cell chain

$$0 = J_0 \subset J_1 \subset \dots J_n = \mathcal{A}$$

Let $n_s(\mathcal{A})$ be the number of isomorphism classes of simple $\mathcal{A}$-modules, and let $n_c(\mathcal{A}) = n$ be the number of cells.

The number of simples $n_s(\mathcal{A})$ is equal to the number of cell modules with non-zero bilinear form and, in particular, $n_s(\mathcal{A}) \leq n_c(\mathcal{A})$. The quasi-hereditarity of a cellular algebra is determined by the relationship between these numbers.

Theorem 2.2.5 ([KX98, Theorem 3.1])**.** *The cellular algebra $\mathcal{A}$ is quasi-hereditary if and only if $n_c(\mathcal{A}) = n_s(\mathcal{A})$.*

This occurs precisely when any cell chain of $\mathcal{A}$ is a heredity chain [KX99, Theorem 3.1].

2.2.4 Idempotent Truncations

We will be looking at multiple instances of idempotent truncations of algebras — that is, given an idempotent e in an algebra $\mathcal{A}$ we will be looking at the algebra $e\mathcal{A}e \subseteq \mathcal{A}$, and comparing their structure and module categories.

Lemma 2.2.6 ([Mor58], [Lam99, Section 18.30])**.** *If the map $\alpha : \mathcal{A}e \otimes_{e\mathcal{A}e} e\mathcal{A} \to \mathcal{A}$ is surjective, we have the following mutually inverse equivalences of categories:*

$$\begin{aligned} F : \mathcal{A}\text{-mod} &\to e\mathcal{A}e\text{-mod} & \qquad G : e\mathcal{A}e\text{-mod} &\to \mathcal{A}\text{-mod} \\ M &\mapsto M \otimes_{\mathcal{A}} \mathcal{A}e \cong \operatorname{Hom}_{\mathcal{A}}(e\mathcal{A}, M) & M &\mapsto M \otimes_{e\mathcal{A}e} e\mathcal{A} \cong \operatorname{Hom}_{e\mathcal{A}e}(\mathcal{A}e, M) \end{aligned}$$

The surjectivity of α, which is equivalent to e generating $\mathcal{A}$ as a two-sided ideal, will be dependent on the parameters (in our case Q, q) involved in the structure coefficients of the algebras, and will fail on a closed set of these parameters.

Proposition 2.2.7 ([KX98, Proposition 4.3]). *Let $\mathcal{A}$ be a cellular algebra with cellular datum $(\Lambda, \mathcal{T}, C, i)$. If an idempotent $e \in \mathcal{A}$ is fixed by the involution, i.e. $i(e) = e$, then the idempotent truncation $e\mathcal{A}e$ is cellular with cellular datum $(\Lambda^{(e)}, \mathcal{T}^{(e)}, C, i)$, where*

$$\Lambda^{(e)} = \{\lambda \in \Lambda \mid e\Delta^\lambda_{\mathcal{A}} \neq 0\}$$

$$\mathcal{T}^{(e)}(\lambda) = \{S \in \mathcal{T}(\lambda) \mid eC_S \neq 0\}$$

So, the algebra $e\mathcal{A}e$ is cellular, inheriting the cellular structure obtained by restricting that of $\mathcal{A}$. Given a cell ideal $J \subset \mathcal{A}$ such that $J \cong \Delta^\lambda_{\mathcal{A}} \otimes_{\Bbbk} i(\Delta^\lambda_{\mathcal{A}})$, $eJe \subset e\mathcal{A}e$ is a cell ideal, and

$$eJe \cong \Delta^\lambda_{e\mathcal{A}e} \otimes_{\Bbbk} i(\Delta^\lambda_{e\mathcal{A}e})$$

Assuming that we begin with a quasi-hereditary cellular algebra $\mathcal{A}$, and then consider a corner algebra $e\mathcal{A}e$, how can we determine if $e\mathcal{A}e$ is quasi-hereditary?

By Proposition 2.2.7, we have that $n_c(e\mathcal{A}e) \leq n_c(\mathcal{A})$, with equality if and only if $e\Delta^\lambda_{\mathcal{A}} \neq 0$ for all λ, that is, if and only if $\Lambda = \Lambda^{(e)}$.

Lemma 2.2.8. *The algebras $\mathcal{A}$ and $e\mathcal{A}e$ are Morita equivalent if and only if $n_s(\mathcal{A}) = n_s(e\mathcal{A}e)$.*

Proof. ($\Rightarrow$) If $\mathcal{A}$ and $e\mathcal{A}e$ are Morita equivalent then this equivalence gives a bijection between their simples, and so $n_s(\mathcal{A}) = n_s(e\mathcal{A}e)$.
($\Leftarrow$) Assume that $n_s(\mathcal{A}) = n_s(e\mathcal{A}e)$. First, note that for any simple L over $\mathcal{A}$, the image Le is a simple $e\mathcal{A}e$-module or 0. Furthermore, if L is a composition factor of $\mathcal{A}/\mathcal{A}e\mathcal{A}$, then we have $Le = 0$, so if $\mathcal{A}$ and $e\mathcal{A}e$ are not Morita equivalent, the pigeonhole principle shows that there must be a simple $e\mathcal{A}e$-module S which cannot be written as $S = Le$. But the module $S' = S \otimes_{e\mathcal{A}e} e\mathcal{A}$ satisfies $S'e = S$, so some composition factor L of S' satisfies $Le = S$, a contradiction. Thus, $\mathcal{A}$ and $e\mathcal{A}e$ must be Morita equivalent. □

Combining these observations, we find that:

Corollary 2.2.9. *If $n_c(e\mathcal{A}e) = n_c(\mathcal{A})$ and $\mathcal{A}$ is quasi-hereditary, then the following are equivalent:*

1. *The algebra $e\mathcal{A}e$ is quasi-hereditary.*
2. *We have an equality $n_s(e\mathcal{A}e) = n_s(\mathcal{A})$.*
3. *The algebras $\mathcal{A}$ and $e\mathcal{A}e$ are Morita equivalent.*
4. *The bilinear form is non-zero on $e\Delta^\lambda_{\mathcal{A}}$ for all cell modules $\Delta^\lambda_{\mathcal{A}}$ of $\mathcal{A}$.*

Therefore $e\mathcal{A}e$ is Morita equivalent to the algebra $\mathcal{A}$ and, as a consequence, quasi-hereditary, if and only if no simple modules are killed by the corresponding quotient of module categories:

$$\mathcal{A} - \text{mod} \to e\mathcal{A}e - \text{mod}$$
$$S \mapsto Se$$

2.3 Crystals and Combinatorics

We now look at **Kashiwara crystals** as combinatorial objects, following the exposition of [BS17, Chapter 2] and [BSW20a].

Definition 2.3.1. *Given a root system Φ indexed by U with weight lattice X, a* ***crystal*** *of type Φ is a set $\mathcal{B} \neq \emptyset$ as well as maps indexed by $u \in U$*

$$\tilde{e}_u, \tilde{f}_u : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \qquad \textbf{\textit{Kashiwara operators}}$$
$$\varepsilon_u, \varphi_u : \mathcal{B} \to \mathbb{Z} \qquad \textbf{\textit{string lengths}}$$

as well as a ***weight function*** $\mathrm{wt} : \mathcal{B} \to X$ *such that the following hold:*

- *For $x, y \in \mathcal{B}$, $\tilde{e}_u(x) = y$ if and only if $\tilde{f}_u(y) = x$. In this case,*

$$\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_u, \quad \varepsilon_u(y) = \varepsilon_u(x) - 1, \quad \varphi_u(y) = \varphi_u(x) + 1$$

- $\langle \mathrm{wt}(x), \alpha_u^\vee \rangle = \varphi_u(x) - \varepsilon_u(x)$

This can be visualized as a directed (acyclic) graph with nodes indexed by $x \in B$ and edges labelled with elements from $u \in U$. The crystal operators $\tilde{e}_u(x) = y$ and $\tilde{f}_u(y) = x$ then correspond to edges of the form $x \xrightarrow{u} y$.

We will be working with **seminormal** crystals, so $\varphi_u(x) = \max\{r \in \mathbb{N} \mid \tilde{f}_u^r(x) \neq 0\}$ and $\varepsilon_u(x) = \max\{r \in \mathbb{N} \mid \tilde{e}_u^r(x) \neq 0\}$.

Starting at a node x, for a fixed $u \in U$, the operators $\tilde{e}_u$ and $\tilde{f}_u$ move up and down the **root string** containing x. This will be of the form

$$x_1 \xrightarrow{u} x_2 \xrightarrow{u} \dots \xrightarrow{u} x_m$$

where $\tilde{e}_u(x_m) = 0$. The weights of consecutive nodes will differ by α_u, so if $\mathrm{wt}(x_1) = \mu$, then $\mathrm{wt}(x_r) = \mu + r\alpha_u$.

So, for a node x in a root string, $\varepsilon_u(x)$ counts the number of times we can still move in the $\tilde{e}_u$ direction from the node x (with the arrow orientation we have chosen, this is arrows in the forward direction along the string), while $\varphi_u(x)$ counts the number of $\tilde{f}_u$ steps remaining (i.e. backwards). The difference between these two values is required to be the inner product between the weight of the node and the simple coroot corresponding to u.

Definition 2.3.2 ([BS17, Definition 2.35])**.** *Given $x \in \mathcal{B}$ and $k = \langle \mathrm{wt}(x), \alpha_u^\vee \rangle$, set*

$$\sigma_u(x) = \begin{cases} e_u^{-k}(x) & k < 0 \\ x & k = 0 \\ f_u^k(u) & k > 0 \end{cases}$$

This operation reflects the root string about its centre by way of iterated Kashiwara operators.

Proposition 2.3.3 ([BS17, Proposition 2.36])**.** *The map σ_u satisfies $\mathrm{wt}(\sigma_u(x)) = s_u(\mathrm{wt}(x))$, where $s_u(\mu) = \mu - \langle \mu, \alpha_u^\vee \rangle \alpha_u$ is the simple reflection of the weight lattice through the hyperplane orthogonal to the simple root α_u.*

This can be used to define an action of the Weyl group on the crystal that is compatible with the normal action on the weight lattice X.

Theorem 2.3.4 ([BS17, Theorem 11.14])**.** *Given a crystal $\mathcal{B}$ for a root system Φ with Weyl group W, the Weyl group acts on $x \in \mathbb{B}$ via $s_u \cdot x = \sigma_u(x)$.*

In our case, the nodes of the crystal will be indexed by bipartitions, and the weights will depend on fundamental weights determined by the parameter Q, and on the number of boxes of a given residue in the Young diagram. The reflection $s_u \in W$ then has the effect of adding all addable boxes and removing all removable boxes of a given residue u. This has the same effect on the weight as the operator σ_u, but unlike σ_u does not preserve the crystal graph.

2.4 Categorical Actions

2.4.1 The quantum Heisenberg category

In this section we will go through the definitions of some categories whose actions underlie the crystal structure used in Lemma 5.3.9.

Definition 2.4.1 (The Category $\mathcal{E}nd(\mathcal{C})$)**.** *Given a category $\mathcal{C}$, $\mathcal{E}nd(\mathcal{C})$ is the strict monoidal category with objects endofunctors on $\mathcal{C}$ and morphisms given by natural transformations.*

Definition 2.4.2 (The Category $\mathcal{AH}(z)$ [BSW20a])**.** *We define $\mathcal{AH}(z)$ to be the strict $\Bbbk$-linear monoidal category with generating object $\uparrow$ and generating morphisms $x :\uparrow\to\uparrow$ and $\tau :\uparrow \otimes \uparrow\to\uparrow \otimes \uparrow$, subject to the following relations:*

1. $\tau \circ (1_\uparrow \otimes x) \circ \tau = x \otimes 1_\uparrow$
2. $\tau \circ \tau = z\tau + 1_{\uparrow\otimes\uparrow}$
3. $(\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\tau) = (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau)$

This has a diagrammatic presentation in which the rank n affine Hecke algebra AH_n can be identified with $\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$ via its usual generators: x_i given by a dot on the i-th string, and τ_j given by the positive crossing of the jth and $j+1$st strings. For details of this presentation, we [BSW20a, §3.2].

Definition 2.4.3 ([BSW20a])**.** *The **quantum Heisenberg category** $\mathcal{H}eis_k(z,t)$ is formed from $\mathcal{AH}(z)$, by adjoining a new generating object $\downarrow$ which is right dual to the generating object $\uparrow$, as well as the generating morphisms "cup" and "cap".*

Definition 2.4.4 ([BSW20a])**.** *Given a category $\mathcal{C}$, a **categorical Heisenberg action** is the data of a strict monoidal functor $\mathcal{H}eis_k(z,t) \to \mathcal{E}nd(\mathcal{C})$.*

The structure of the Heisenberg category necessitates endofunctors $E, F : \mathcal{C} \to \mathcal{C}$ and natural transformations such that:

1. E and F are adjoint
2. AH_d acts on E^d for any $d \geq 0$

3. There is an explicit isomorphism of functors lifting the relation $[e, f] = k$ in the Heisenberg algebra of central charge k (for details, see [BSW20a, Section 3.30-3.32])

The generating objects E and F of the category formalize the notion of induction and restriction-type functors carrying the action of finite Hecke algebras and Jucys–Murphy elements, and Item 3 generalizes the satisfaction of a Mackey theorem [BSW20a].

2.4.2 The Kac–Moody 2-Category

Kac–Moody 2-categories were introduced independently by Khovanov and Lauda in [KL09] and Rouquier [Rou08], and proven to be equivalent by Brundan [Bru15].

Given a Dynkin diagram with connected components of type A_∞ or $A^{(1)}_{p-1}$ with generalized Cartan matrix $A = (a_{ij})_{i,j\in U}$ let $\mathfrak{g}$ be the Kac–Moody Lie algebra $\mathfrak{sl}_U$ generated by the elements $\{e_u, f_u, \alpha_u^\vee\}$, subject to the Serre relations given by the matrix A. This will be the direct sum of Kac–Moody algebras of the form $\mathfrak{sl}_\infty$ and $\widehat{\mathfrak{sl}}_p$, each corresponding to a connected component of the Dynkin diagram. Let $\mathfrak{h}$ be its Cartan subalgebra, and let $X \subset \mathfrak{h}^*$ be its weight lattice. This is generated by the **fundamental weights** Λ_i, where each Λ_i is defined by $\langle \Lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$. The **dominant weights** are $X^+ := \bigoplus_{u\in U} \mathbb{Z}_{\geq 0}\Lambda_u = \{\lambda \in X \mid \forall u \in U,\ \langle\lambda, \alpha_u^\vee\rangle \geq 0,\ \sum_{u\in U}\langle\lambda, \alpha_u^\vee\rangle < \infty\}$. The **simple roots** are then given by $\alpha_u = \sum_{j\in U} a_{uj}\Lambda_j$. Note that these are not necessarily linearly independent.

Definition 2.4.5. *[Bru15, Definition 1.1]*

*Given a symmetrizable generalized Cartan matrix A indexed by a set I, and the associated Kac–Moody algebra $\mathfrak{g} := \mathfrak{g}(A)$ with Cartan subalgebra $\mathfrak{h}$ and weight lattice X, the **Kac–Moody 2-category** $\mathcal{U}(\mathfrak{g})$ is the strict additive $\Bbbk$-linear 2-category with:*

- $\mathrm{Obj}\,(\mathcal{U}(\mathfrak{g})) = X$
- *generating 1-morphisms*

$$E_u 1_\lambda : \lambda \to \lambda + \alpha_u \qquad F_u 1_\lambda : \lambda \to \lambda - \alpha_u$$

- *and generating 2-morphisms*

$$x : E_u 1_\lambda \to E_u 1_\lambda \qquad \tau : E_u E_w 1_\lambda \to E_w E_u 1_\lambda \qquad \eta : 1_\lambda \to F_u E_w 1_\lambda \qquad \epsilon : E_u F_u 1_\lambda \to 1_\lambda$$

for $u \in U$ and $\lambda \in X$.

This 2-category also has a diagrammatic presentation, and its relations can be given locally. See [BSW20a, §3.3] for the complete diagrammatic presentation.

Given a family of categories $\{\mathcal{C}_\lambda\}_{\lambda\in X}$ indexed by weights of $\mathfrak{g}$, the 2-category $\mathcal{U}(\mathfrak{g})$ acts via $\lambda \mapsto \mathcal{C}_\lambda$, with functors $E_u : \mathcal{C}_\lambda \to \mathcal{C}_{\lambda+\alpha_u}$ and $F_u : \mathcal{C}_{\lambda+\alpha_u} \to \mathcal{C}_\lambda$ corresponding to the Chevalley generators.

For a fixed $d \geq 0$ the quiver Hecke algebra (KLR algebra) with Cartan type matching that of $\mathfrak{g}$ acts on E^d, where $E := \oplus_{u\in U} E_u$. This 2-functor from $\mathcal{U}(\mathfrak{g})$ to the 2 category of categories, functors, and natural transformations formalizes the notion of decomposing the category into the weight spaces of the $\mathfrak{g}$-action.

The Chevalley functor action reflects the root-string structure discussed in Section 2.3 (Kashiwara crystals were constructed based on Lie theory). Most importantly, the action we have is nilpotent, giving us root strings of finite length. As a result:

Theorem 2.4.6 ([BD16, Theorem 4.31]). *Given a nilpotent categorical action on a Schurian category $\mathcal{C}$ with irreducible objects $\{L(b) : b \in B\}$. There is a unique crystal structure on $B = \bigsqcup_{\lambda \in P} B_\lambda$ such that the following hold:*

- $\tilde{e}_u(b) \neq 0$ *if and only if* $E_u L(b) \neq 0$
- $\tilde{f}_u(b) \neq 0$ *if and only if* $F_u L(b) \neq 0$.

The simple module corresponding to the node $\tilde{e}_u(b)$ is then isomorphic to the unique simple head and the unique simple submodule of $E_u L(b)$, and that of the node $\tilde{f}_u(b)$ corresponds to that of $F_u L(b)$.

2.4.3 From a Quantum Heisenberg to a Kac–Moody Categorical Action

Given a quantum Heisenberg action, we can decompose E and F into eigenfunctors E_u and F_u by projecting onto the generalized eigenspaces for each eigenvalue u. In [BSW20a], it is proved that these operators satisfy the definition of a Kac–Moody categorical action, playing the role of the E_u and F_u (respectively) in Definition 2.4.5. This Kac–Moody categorical action will be constructed starting with its Dynkin diagram: let $U = Qq^{\mathbb{Z}} \cup -q^{\mathbb{Z}} \subseteq \Bbbk$ be the set of eigenvalues of the Heisenberg operators E and F. These will label the vertices of the quiver, and will include arrows $u \to q^{-1}u$ for $u \in U$.

This can take four possible forms:

1. If q is a primitive eth root of unity and $Q \in -q^{\mathbb{Z}}$, then the quiver is a single e-cycle and $\mathfrak{g}_U \cong \hat{\mathfrak{sl}}_e$
2. If q has infinite multiplicative order and $Q \in -q^{\mathbb{Z}}$, then the quiver is a single copy of the A_∞ Dynkin diagram and $\mathfrak{g}_U \cong \mathfrak{sl}_\infty$.
3. If q is a primitive eth root of unity and $Q \notin -q^{\mathbb{Z}}$, then the quiver is disjoint union of two e-cycles and $\mathfrak{g}_U \cong \hat{\mathfrak{sl}}_e \oplus \hat{\mathfrak{sl}}_e$
4. If q has infinite multiplicative order and $Q \notin -q^{\mathbb{Z}}$, then the quiver is a single copy of the A_∞ Dynkin diagram and $\mathfrak{g}_U \cong \mathfrak{sl}_\infty \oplus \mathfrak{sl}_\infty$.

In each case, we obtain the generalized Cartan matrix as a direct sum of the corresponding type A or affine type A Cartan matrices, and can build the corresponding Kac–Moody algebra and categorical action:

Theorem 2.4.7 ([BSW20a, Theorem A]). *Given a Schurian $\Bbbk$-linear category $\mathcal{C}$ equipped with a quantum Heisenberg categorical action, there is a canonical Kac–Moody categorical action on the decomposition of $\mathcal{C} = \bigsqcup_{\lambda \in X} \mathcal{C}_\lambda$ into Serre subcategories corresponding to weight spaces. The Kac–Moody action is given by the eigenfunctors E_u (respectively, F_u) of E (respectively, F) discussed above.*

Chapter 3

Hecke Algebras, Schur Algebras, and Dualities

We begin with a classical result:

Theorem 3.0.1 (Double Centralizer Theorem [Eti+11])**.** *Let E be a finite dimensional vector space, and let $A, B \subseteq \mathrm{End}(E)$ be subalgebras. Assume that A is semisimple and that $B = \mathrm{End}_A(E)$. Then the following hold:*

- $A = \mathrm{End}_B(E)$
- *B is semisimple*
- *There is a natural bijection between irreducible representations of A and B given by the decomposition*
$$E = \bigoplus_{i \in I} V_i \otimes W_i$$
of E as an $A \otimes B$ representation.

Let $\Bbbk$ be a field, and let $\mathfrak{S}_n$ denote the symmetric group on n letters with generating transpositions $s_i = (i\; i+1), i = 1, \ldots, n-1$. Let V be an m-dimensional $\Bbbk$ vector space, and let $GL(V)$ denote the group of invertible linear transformations on V. The natural action of $GL(V)$ commutes with that of the symmetric group given by permuting the tensor factors:

$$GL(V) \circlearrowright V^{\otimes n} \circlearrowleft \mathfrak{S}_n$$

Applying the double centralizer theorem to the images of $A = \Bbbk\mathfrak{S}_n$ and $B = \mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(E)$ with $E = V^{\otimes n}, \dim V = m$ gives the statement known as **Classical Schur–Weyl Duality**. The **Classical Schur Algebra** then arises as:

$$S(m,n) = \mathrm{End}_{\mathfrak{S}_n}(V^{\otimes n})$$

See [Eti+11, Chapter 5] for a more detailed discussion.

3.1 Type A

The (finite) type A Hecke algebra $\mathcal{H}_n^A(q)$ of rank n is obtained by deforming the relations of the group algebra of the symmetric group $\mathfrak{S}_n$. It has a presentation given by generators $T_1, \ldots, T_{n-1}$ subject to the relations:

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \text{ for } 1 \leq i \leq n-1 \\ T_iT_j &= T_jT_i \text{ for } |i-j| > 1 \\ (T_i - q)(T_i + 1) &= 0 \text{ for } 1 \leq i \leq n-1 \end{aligned}$$

Remark 3.1.1. *Many different conventions for the roots of the minimal polynomials of the T_i appear in the literature, but these are all equivalent up to algebra automorphism and parameter changes. We have chosen to follow the conventions of Dipper, James, and Mathas [DJM98].*

Schur–Weyl duality was generalized to the quantum case by Jimbo in [Jim86], establishing commuting actions by the quantum group $U_q(\mathfrak{g})$ and the type A Hecke algebra $\mathcal{H}_n^A(q)$, with the Hecke action given as follows:

$$v_{\vec{d}}T_t = \begin{cases} v_{\vec{d}\cdot s_t} & d_t < d_{t+1} \\ qv_{\vec{d}\cdot s_t} & d_t = d_{t+1} \\ v_{\vec{d}\cdot s_t} + (q-1)v_{\vec{d}} & d_t > d_{t+1} \end{cases}$$

$v_{\vec{d}} = v_{d_1} \otimes \cdots \otimes v_{d_n} \in V^{\otimes n}$ is a vector in the tensor product, and the symmetric group permutes indices: $d_i \mapsto d_{s_t(i)}$.

This suggests a generalization of the classical Schur algebra to the quantum case. The type A q-Schur algebra $\mathcal{S}_n^A(q)$ was introduced by Dipper and James in 1989:

Definition 3.1.2 ([DJ89, Definition 2.9])**.**

$$\mathcal{S}_n^A(q) := \mathrm{End}_{\mathcal{H}_n^A(q)}\Big(\bigoplus_{\lambda \vdash n} x_\lambda \mathcal{H}_n^A(q)\Big)$$

where, given a partition $\lambda \vdash n$, $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$ is the Young symmetrizer.

This algebra was shown to satisfy a double centralizer relation relation with the type A Hecke algebra in [Du92, Lemma 2.1]. Thus, we can equivalently construct this algebra as

$$\mathcal{S}_n^A(q) \cong \mathrm{End}_{\mathcal{H}_n^A(q)}(V^{\otimes n}).$$

Thus, in type A, the q-Schur algebra can be constructed equivalently as $\mathcal{H}_n^A(q)$-invariant module endomorphisms on the tensor space or as homomorphisms over permutation modules.

For any base field $\Bbbk$ and parameter $q \in \Bbbk^\times$, this algebra is a cellular, quasi-hereditary cover of $\mathcal{H}_n^A(q)$, mirroring the relationship of the classical Schur algebra to the group algebra of the symmetric group.

3.2 Type B

Let $\Bbbk$ be a field of any characteristic. Let $W_n = W_{B_n}$ denote the type B Weyl group of rank n with generators $s_0, s_1, \ldots, s_{n-1}$, then its Hecke algebra is $\mathcal{H}_n^B(Q,q)$, the algebra generated by the elements $T_0, T_1, \ldots, T_{n-1}$ subject to the relations

$$\begin{aligned}
T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \text{ for } 1 \leq i \leq n-1 \\
T_iT_j &= T_jT_i \text{ for } |i-j| > 1 \\
(T_0 - Q)(T_0 + 1) &= 0 \\
(T_i - q)(T_i + 1) &= 0 \text{ for } 1 \leq i \leq n-1
\end{aligned}$$

where $Q, q \in \Bbbk^\times$ are scalars. We will also define $L_1 := T_0$ and inductively $L_{i+1} := q^{-1}T_iL_iT_i$. These are the Jucys–Murphy elements as defined in [DJM98, §2]. Recall that as a consequence of these defining relations, any symmetric polynomial in these Jucys–Murphy elements L_i is central in $\mathcal{H}_n^B(Q,q)$. In particular, note that

$$T_iL_iL_{i+1} = L_iL_{i+1}T_i.$$

In 2017, Bao, Wang, and Watanabe established a version of quantum Schur duality for the unequal parameters case of type B [BWW17].

Remark 3.2.1. *In [BWW17] as in [LNX20], the roots of the quadratic relations of the Hecke algebra are $q_i^{-1}, -q_i$. As noted in Remark 3.1.1, we have followed the conventions for Hecke algebras from [DJM98]: the quadratic relations are $(T_0 - Q)(T_0 + 1) = 0$ and $(T_i - q)(T_i + 1) = 0$ for $i \neq 0$. Where necessary, we have adjusted all theorems and formulas to match those of [DJM98]. In the paper [LNX20], d refers to the rank of the Hecke algebra. To follow the conventions of Dipper, James, and Mathas [DJM98], we will call this n. Lai, Nakano, and Xiang use n to refer to the dimension of the vector space that is being acted on by the Hecke algebra, which we have called m. Note that a choice of isomorphism between the Hecke algebra in our conventions and those of [LNX20] requires a choice of square roots $\sqrt{Q}, \sqrt{q}$ in our base field or in an extension. We will further simplify the type B Hecke algebra as $\mathcal{H}_n^B(Q,q) = \mathcal{H}_n$.*

Let $r = \lfloor \frac{m}{2} \rfloor$. We set $I(m) = \{-r, \ldots -1, 0, 1, \ldots, r\}$ if m is odd and $\{-r, \ldots -1, 1, \ldots, r\}$ if m is even. We will be working with a vector space V of dimension m, with basis vectors indexed by $I(m)$ (of appropriate parity). The set $\overline{I(m)} := (I(m) \pm \frac{1}{2}) \setminus \{-r - \frac{1}{2}, r + \frac{1}{2}\}$ indexes the roots of the corresponding system of type A_{m-1}. Let $\overline{I(m)}_{>0}$ denote the strictly positive indices in this set.

Let $U_q(\mathfrak{sl}_m)$ be the usual quantum group of type A_{m-1} over $\mathbb{Q}(q)$ with standard generators $E_i, F_i, K_i^{\pm 1}, i \in \overline{I(m)}$, and let $U = U_q(\mathfrak{sl}_m) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(Q,q)$ be the quantum group they generate over $\mathbb{Q}(Q,q)$.

For $s_t \in W_n$ and any tuple $\vec{d} = (d_1, \ldots, d_n) \in I(m)^n$ let

$$(d_1, \ldots, d_n) \cdot s_t = \begin{cases} (-d_1, d_2, \ldots, d_n) & t = 0 \\ (d_1, \ldots, d_{t-1}, d_{t+1}, d_t, d_{t+2}, \ldots, d_n) & t \neq 0 \end{cases}$$

For $T_t \in \mathcal{H}_n$, the action is given by

$$v_{\vec{d}}T_0 = \begin{cases} v_{\vec{d}\cdot s_0} & 0 < d_1 \\ Qv_{\vec{d}\cdot s_0} & 0 = d_1 \\ v_{\vec{d}\cdot s_0} + (Q-1)v_{\vec{d}} & 0 > d_1 \end{cases} \qquad v_{\vec{d}}T_t = \begin{cases} v_{\vec{d}\cdot s_t} & d_t < d_{t+1} \\ qv_{\vec{d}\cdot s_t} & d_t = d_{t+1} \\ v_{\vec{d}\cdot s_t} + (q-1)v_{\vec{d}} & d_t > d_{t+1} \end{cases} \tag{3.1}$$

for all $t \in \{1, \ldots, n-1\}$.

3.3 Quantum Symmetric Pairs

We follow [BWW17, §2 & 4] for the case of m even and odd, respectively.

Assume that $m = \dim V = 2r+1$ is odd. Let $U^j \subset U$ denote the $\mathbb{Q}(Q,q)$-subalgebra generated by the following elements, indexed by $\overline{I(m)}_{>0}$:

$$k_i := K_i K_{-i}^{-1}, \qquad e_i := E_i + F_{-i}K_i^{-1} \left(i \neq \frac{1}{2}\right), \qquad f_i := E_{-i} + K_{-i}^{-1}F_i \left(i \neq \frac{1}{2}\right) \tag{3.2}$$

$$e_{\frac{1}{2}} := E_{\frac{1}{2}} + QF_{-\frac{1}{2}}K_{\frac{1}{2}}^{-1}, \; f_{-\frac{1}{2}} = E_{-\frac{1}{2}} + K_{-\frac{1}{2}}^{-1}F_{\frac{1}{2}} \tag{3.3}$$

Taking the natural representation V of U with action given as in Equation (3.1), $V^{\otimes n}$ becomes a U^j-module by restriction.

For $m = 2r = \dim V$ even, let $U^i \subset U$ denote the $\mathbb{Q}(Q,q)$-subalgebra generated by the following elements, indexed by $\overline{I(m)}_{>0}$:

$$k_i := K_i K_{-i}^{-1}, \qquad t = E_0 + F_0 K_0^{-1} + \frac{1-Q}{1-q}K_0^{-1}, \tag{3.4}$$

$$e_i := E_i + F_{-i}K_i^{-1}, \qquad f_i := E_{-i} + K_{-i}^{-1}F_i \tag{3.5}$$

This is a right coideal subalgebra of U, with a presentation over $\mathbb{Q}(Q,q)$ in the generators k_i, e_i, f_i, t that does not depend on the parameter Q (see [BWW17, §2.3]).

Theorem 3.3.1 ([BWW17, Theorem 2.6, 4.4])**.** *Let $U' = U^j$ if $m = 2r+1$ is odd, and U^i if $m = 2r$ is even. The algebras (U, U') form a quantum symmetric pair. The actions Φ of U' and Ψ of $\mathcal{H}_n$ on $V^{\otimes n}$ commute, forming double centralizers. Thus:*

$$\Phi(U') = \mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})$$

3.3.1 The Algebra $\mathcal{L}^n(m)$

This multiparameter Schur–Weyl duality led to Lai and Luo's construction of a double centralizer version of the Schur algebra in type B:

Definition 3.3.2 ([LL21, §3])**.** $\mathcal{L}^n(m) := \mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})$.

Theorem 3.3.1 gives this algebra as a quotient of the coideal subalgebra U^j (resp. U^i) which itself can be written in terms of the generators of the quantum group U. Simple modules of $\mathcal{L}^n(m)$ can therefore be inflated to those of U^j (resp. U^i).

3.4 Quasi-hereditary Covers

In [Rou07] Rouquier developed the relationship between the q-Schur algebra and the type A Hecke algebra in terms of the notions of highest weight categories of Cline, Parshall, and Scott [CPS88]. To formalize the situation of double centralizers, Rouquier defines:

Definition 3.4.1 ([Rou07, Definition 4.34]). *Let $\Bbbk$ be a field, $\mathcal{A}$ a $\Bbbk$-algebra, $\mathcal{C} = \mathcal{A}\text{-mod}$, and $P \in \mathcal{A}\text{-mod}$ a finitely generated projective. Let $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(P)$; then $(\mathcal{A}, P)$ is a **cover** of $\mathcal{B}$ if the restriction of the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ to $\mathcal{A}\text{-proj}$ is fully faithful. In this case we also say that $(\mathcal{C}, F)$ is a **cover** of $\mathcal{B}\text{-mod}$, where $F = \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, -) : \mathcal{A}\text{-mod} \to \mathcal{B}\text{-mod}$.*

If, further, the category $\mathcal{C} = \mathcal{A}\text{-mod}$ has a highest weight structure, we say it is a **highest weight cover** of $\mathcal{B}\text{-mod}$ [Rou07, §4.2]. As in the work of [CPS88], the algebra $\mathcal{A}$ is then called quasi-hereditary.

One example is the classical Schur algebra: if $\mathrm{char}(\Bbbk) = 0$ or $p < r$, the Schur algebra $S(n, r) \cong \mathrm{End}_{\mathfrak{S}_n}(\Bbbk^r)^{\otimes n}$ is semisimple, and hence quasi-hereditary, and forms a highest weight cover of $\Bbbk\mathfrak{S}_n$ [Xi92]. Another example is category $\mathcal{O}$ for a semisimple Lie algebra: Soergel's functor $\mathbb{V}$ makes the principal block of category $\mathcal{O}$ for a semisimple Lie algebra into a highest weight cover over the module category of the coinvariant algebra [Soe90].

3.4.1 The Algebra $\mathcal{S}^n(m)$

In this section, we define a quasi-hereditary cover of $\mathcal{H}_n$. This is the level 2 case of the cyclotomic q-Schur algebra of Dipper, James, and Mathas, constructed in [DJM98].

We define a **weak bicomposition** of a number n to be a pair $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of tuples of non-negative integers $\lambda^{(i)} = (\lambda^{(i)}_1, \dots, \lambda^{(i)}_{\ell_i})$ that sum to n. A **(strong) bicomposition** is a weak bicomposition where $\lambda^{(i)}_k > 0$ for all i and $1 \leq k \leq \ell_i$.

Let us now recall some more notation from [DJM98, §3]: for such a weak bicomposition, let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup of the symmetric group $\mathfrak{S}_n$. That is,

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda^{(1)}_1} \times \cdots \times \mathfrak{S}_{\lambda^{(1)}_{\ell_1}} \times \mathfrak{S}_{\lambda^{(2)}_1} \times \dots \mathfrak{S}_{\lambda^{(2)}_{\ell_2}}.$$

We define

$$u_\lambda := \prod_{i=1}^{|\lambda^{(1)}|} (L_i + 1) \qquad x_\lambda := \sum_{\omega \in \mathfrak{S}_\lambda} T_\omega \qquad m_\lambda := u_\lambda x_\lambda. \tag{3.6}$$

Here, we depart slightly from the notation of [DJM98]; the element u_λ would be denoted there by $u^+_{(0,|\lambda^{(1)}|)}$. Note also that $x_\lambda \in \mathcal{H}^A_n$ whereas $u_\lambda \notin \mathcal{H}^A_n$ unless $\lambda^{(1)} = \emptyset$.

Let $\Lambda_{n,m}$ denote the set of weak bicompositions of n such that the first component has n parts and the second component has $r = \lfloor m/2 \rfloor$ parts. If we have a strong bicomposition (i.e., one whose parts are all positive) such that the second component has at most r parts, we can think of this as an element of $\Lambda_{n,m}$ by appending parts that are 0 until we reach the desired number; let $\Lambda^{>0}_{n,m}$ be the subset of elements of $\Lambda_{n,m}$ which are obtained from strong bicompositions this way. Let $\Lambda^+_{n,m} \subset \Lambda^{>0}_{n,m}$ denote the subset where the bicomposition is a bipartition. Of course, if m is large, then $\Lambda^{>0}_{n,m}$ is the set of bicompositions of n and $\Lambda^+_{n,m} = \Pi_n$ is the set of all bipartitions of n. This set inherits a partial order called **dominance order**: we write $\nu \rhd \mu$ if for all $i \in \mathbb{Z}_{\geq 0}$, we have

$$\sum_{j=1}^{i} \nu_j^{(1)} \geq \sum_{j=1}^{i} \mu_j^{(1)} \qquad |\nu^{(1)}| + \sum_{j=1}^{i} \nu_j^{(2)} \geq |\mu^{(1)}| + \sum_{j=1}^{i} \mu_j^{(2)}.$$

When ν and μ have different numbers of parts, we can still compare them in dominance order by extending both to weak bicompositions by appending parts which are zero. For ease of notation, we will often not write 0's that come at the end of components.

Lemma 3.4.2. *The set $\Lambda_{n,m}^{>0}$ is* **saturated***, that is, if $\mu \in \Lambda_{n,m}^{>0}$ and ν is an arbitrary bipartition of n such that $\nu \rhd \mu$, then $\nu \in \Lambda_{n,m}^{+}$.*

Proof. Since $\nu \rhd \mu$, we must have $|\nu^{(1)}| + \sum_{i=1}^{r} \nu_i^{(2)} \geq n$. On the other hand, ν only has n boxes, so this means that $\nu^{(2)}$ cannot have more than r boxes. □

These are exactly the conditions on a set of bicompositions required at the start of [DJM98, §6]. Let

$$M^{\lambda} := m_{\lambda}\mathcal{H}_n \qquad\qquad M^{\Lambda_{n,m}} := \bigoplus_{\lambda \in \Lambda_{n,m}} M^{\lambda}.$$

Based on the definition of the cyclotomic q-Schur algebra [DJM98, Def 6.1]:

Definition 3.4.3. *We define the* **Dipper–James–Mathas Type B q-Schur algebra** *of rank n to be:*

$$\mathcal{S}^n(m) = \mathcal{S}^n(\Lambda_{n,m}) := \operatorname{End}_{\mathcal{H}}\left(M^{\Lambda_{n,m}}\right) \cong \bigoplus_{\mu,\nu \in \Lambda_{n,m}} \operatorname{Hom}_{\mathcal{H}_n}(M^{\mu}, M^{\nu}).$$

Definition 3.4.4 ([DJM98, Definition 4.2])**.** *Given a bipartition λ, a bicomposition μ, and a standard tableau $\mathfrak{t} \in \mathcal{T}(\lambda)$, let $\mu(\mathfrak{t})$ be tableau with filling obtained by replacing an entry m with i_k if m occurs in row i of the k-th component of $\mathfrak{t}^{\mu}$.*

For example, if $\lambda = ((3,2),(2))$ and $\mu = ((2),(5))$,

$$\mu(\mathfrak{t}^{\lambda}) = \left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 1_2 \\ \hline 1_2 & 1_2 & \\ \cline{1-2}\end{array}, \begin{array}{|c|c|}\hline 1_2 & 1_2 \\ \hline\end{array} \right).$$

Definition 3.4.5. *Given any bipartition λ, bicomposition μ, and semi-standard tableau $T \in \mathcal{T}_0(\lambda,\mu)$ of shape λ and type μ let*

$$\mathcal{A}(\lambda, T) = \{\mathfrak{t} \in \mathcal{T}(\lambda) \mid \mu(\mathfrak{t}) = T\}. \tag{3.7}$$

Definition 3.4.6 ([DJM98, Definition 6.2, 6.4])**.** *Let λ be a bipartition and μ, ν be weak bicompositions of n. Let $S \in \mathcal{T}_0(\lambda,\mu)$ and $T \in \mathcal{T}_0(\lambda,\nu)$. Consider the elements*

$$m_{ST} = \sum_{\mathfrak{s} \in \mathcal{A}(\lambda,S), \mathfrak{t} \in \mathcal{A}(\lambda,T)} T^{*}_{d(\mathfrak{s})} m_{\lambda} T_{d(\mathfrak{t})}.$$

Let $\varphi_{ST} \in \operatorname{Hom}_{\mathcal{H}_n}(M^{\nu}, M^{\mu})$ be the unique module homomorphism such that $\varphi_{ST}(m_{\nu}h) = m_{ST}h$ for any $h \in \mathcal{H}_n$.

Theorem 3.4.7 ([DJM98, Theorem 6.12]). *The algebra $\mathcal{S}^n(m)$ has cellular basis*

$$\{\varphi_{ST} : S \in \mathcal{T}_0(\lambda,\mu), T \in \mathcal{T}_0(\lambda,\nu) \text{ for some } \mu,\nu \in \Lambda_{n,m}, \lambda \in \Lambda^+_{n,m}\}$$

Theorem 3.4.8 ([Rou07, Theorem 6.6]). *The algebra $\mathcal{S}^n(m)$ is a quasi-hereditary cover of the Hecke algebra $\mathcal{H}_n$.*

Given $\lambda \in \Lambda_{n,m}$, we let $\varphi_\lambda \in \mathcal{S}^n(m)$ denote the identity map on the module M^λ. Note that this is an element of the cellular basis: $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$, where T^λ is the unique semi-standard tableau of shape and type λ.

The cell modules of this cellular structure are the Weyl modules defined in [DJM98, Def. 6.13]. We can write these as a quotient:

$$C^\lambda_{\mathcal{S}} := \mathcal{S}^n(m)\cdot(\varphi_\lambda + \overline{N}^\lambda_{\mathcal{S}})/\overline{N}^\lambda_{\mathcal{S}} \subset \mathcal{S}^n(m)/\overline{N}^\lambda_{\mathcal{S}}$$

of a left ideal by the two-sided ideal

$$\overline{N}^\lambda_{\mathcal{S}} := \{\varphi_{UV} \mid U \in \mathcal{T}_0(\alpha,\mu), V \in \mathcal{T}_0(\alpha,\nu) \text{ for some } \mu,\nu \in \Lambda_{n,m}, \alpha \in \Lambda^+_{n,m}, \alpha \rhd \lambda\}.$$

For a semi-standard λ-tableau S, let $\varphi_S = \varphi_{ST^\lambda} + \overline{N}^\lambda_{\mathcal{S}}$ denote the coset of φ_{ST^λ} in $C^\lambda_{\mathcal{S}}$. There is a unique inner product defined on these Weyl modules by the formula

$$\varphi_{T^\lambda S}\varphi_{TT^\lambda} \equiv \langle \varphi_S, \varphi_T\rangle \varphi_\lambda \mod \overline{N}^\lambda_{\mathcal{S}}.$$

This is always non-zero since $\langle \varphi_{T^\lambda}, \varphi_{T^\lambda}\rangle = 1$ for all λ. As we'll see below, this is a manifestation of the fact that $\mathcal{S}^n(m)$ is quasi-hereditary for all m and all choices of parameters $q, Q \in \Bbbk^\times$.

Chapter 4

Idempotent Truncation

4.1 Constructing the Idempotent

Consider the set

$$B_n(m) = \{\alpha = (\alpha_i)_{i \in I(m)} \mid \alpha_0 \in 2\mathbb{Z}+1, \alpha_{-i} = \alpha_i, \sum_i \alpha_i = 2n+1\}.$$

Note that if m is even, then $0 \notin I(m)$, so the condition on α_0 is vacuous.

Definition 4.1.1. *For $\alpha \in B_n(m)$, we define the bicomposition $\lambda(\alpha)$ to be $((\lfloor \frac{\alpha_0}{2} \rfloor), (\alpha_1, \ldots \alpha_r))$.*

Let

$$\Lambda^B_{n,m} := \{\lambda(\alpha) \mid \alpha \in B_n(m)\} \subseteq \Lambda_{n,m}.$$

Note that if m is odd, then $\Lambda^B_{n,m}$ is the set of all weak bicompositions where the first component has one part and the second has r parts. If m is even, then it is all bicompositions where the first component is trivial and the second has r parts.

Given $\alpha \in B_n(m)$ we define the set of transpositions

$$G_\alpha := \{s_0, \ldots, s_{n-1}\} - \{s_{\lfloor \frac{\alpha_0}{2} \rfloor}, s_{\lfloor \frac{\alpha_0}{2} \rfloor + \alpha_1}, \ldots, s_{\lfloor \frac{\alpha_0}{2} \rfloor + \alpha_1 + \cdots + \alpha_r}\}.$$

The corresponding Weyl group $W_\alpha := \langle G_\alpha \rangle$ is the subgroup generated by these transpositions. Note that this is the product of the type B Weyl group generated by $\{s_0, \ldots s_{\lfloor \frac{\alpha_0}{2} \rfloor - 1}\}$ with the type A Weyl group $\mathfrak{S}_\mu$ for $\mu = (\alpha_1, \ldots, \alpha_r)$. We will denote by $\mathcal{H}_\alpha$ the subalgebra of $\mathcal{H}_n$ generated by $\{T_i : s_i \in G_\alpha\}$.

Consider the element

$$x^B_\alpha := \sum_{\omega \in W_\alpha} T_\omega = \sum_{T_\omega \in \mathcal{H}_\alpha} T_\omega.$$

By [LNX20, (2.5)] we have an isomorphism

$$\mathcal{L}^n(m) \cong \bigoplus_{\alpha, \beta \in B_n(m)} \mathrm{Hom}_{\mathcal{H}_n}(x^B_\alpha \mathcal{H}_n, x^B_\beta \mathcal{H}_n).$$

Note that the set G_α and thus the module $x^B_\alpha \mathcal{H}_n$ is unchanged if we swap a part of α with one which is 0, or add additional 0's to the partition. We can include $\Lambda_{m,n}$ into $\Lambda_{m',n}$ by allowing extra parts as long as $m \leq m'$ and either both m, m' are even or at least m' is odd. Let $\mathsf{e}(m', m)$ denote the projection to weight spaces obtained by the inclusion $\Lambda_{m,n} \hookrightarrow \Lambda_{m',n}$. Therefore:

Lemma 4.1.2. *If $m' \geq m \geq 2n$ and $m \equiv m' \pmod 2$, the idempotent $\mathsf{e}(m', m)$ induces a Morita equivalence of $\mathcal{L}^n(m)$ to $\mathcal{L}^n(m')$.*

Proof. The composition α can only have n non-zero parts, so if we remove all parts which are 0, we will arrive at $\alpha' \in B_n(m')$ such that $G_\alpha = G_{\alpha'}$, and so $x^B_\alpha \mathcal{H}_n \cong x^B_{\alpha'}\mathcal{H}_n$. This shows that the image of $\mathsf{e}(m', m)$ contains a copy of every indecomposable summand of $V^{\otimes n}$; the desired Morita equivalence follows immediately. □

Lemma 4.1.3. *For a fixed $\alpha \in B_n(m)$, there is a unique line $\ell \subset \mathcal{H}_\alpha$ such that $(T_i - q_i) \cdot \ell = 0$ for i such that $s_i \in G_\alpha$, and it is spanned by x^B_α.*

Proof. Consider an element $a = \sum_\omega a_\omega T_\omega \in \mathcal{H}_\alpha$. Using the decomposition of each group element ω into cosets of $s_i \in G_\alpha$ we obtain:

$$a = \sum_{\omega \in W_\alpha} (a_{s_i\omega} T_i T_\omega + a_\omega T_\omega)$$

so that

$$\begin{aligned} T_i a &= \sum_{\omega \in W_\alpha} (a_{s_i\omega} T_i^2 T_\omega + a_\omega T_i T_\omega) \\ &= \sum_{\omega \in W_\alpha} (a_{s_i\omega}(q_i - 1) T_i T_\omega + q_i a_{s_i\omega} T_\omega + a_\omega T_i T_\omega) \\ &= \sum_{\omega \in W_\alpha} ((a_{s_i\omega}(q_i - 1) + a_\omega) T_i T_\omega + q_i a_{s_i\omega} T_\omega). \end{aligned}$$

Therefore, if we are to have $T_i a = q_i a$ we must have $a_{s_i\omega} = a_\omega$ for all $s_i \in W_\alpha$. □

Lemma 4.1.4. *Given $\alpha \in B_n(m)$, we have equality of the modules*

$$x^B_\alpha \mathcal{H}_n = m_{\lambda(\alpha)} \mathcal{H}_n$$

Proof. We can write $m_\lambda = x_\lambda u_\lambda = u_\lambda x_\lambda$ since u_λ is a polynomial in the L_i's which is symmetric under $\mathfrak{S}_\lambda$. Note also that if $\lambda^{(1)} \neq \emptyset$, then u_λ is divisible by $T_0 + 1$, and so $(T_0 - Q)u_\lambda = 0$. On the other hand, for any composition λ, we have that $x_\lambda = x^B_{\alpha'}$ for

$$\alpha'_i = \begin{cases} 1 & i = 0 \\ \lambda_i & i > 0. \end{cases}$$

Applying Lemma 4.1.3 in this case, we find that if $s_i \in \mathfrak{S}_\lambda$, then $(T_i - q)x_\lambda = 0$. Now, fix α, and consider $m_{\lambda(\alpha)}$. If $\alpha_0 > 1$, we have $s_0 \in G_\alpha$. Since $\lambda^{(1)} = (\lfloor \frac{\alpha_0}{2} \rfloor)$, we have that

$$(T_0 - Q)m_{\lambda(\alpha)} = (T_0 - Q)u_{\lambda(\alpha)}x_{\lambda(\alpha)} = 0.$$

On the other hand, if $s_i \in G_\alpha$ for $i > 0$, then $s_i \in \mathfrak{S}_{\lambda(\alpha)}$ and so

$$(T_i - q)m_{\lambda(\alpha)} = (T_i - q)x_{\lambda(\alpha)}u_{\lambda(\alpha)} = 0. \qquad \square$$

Thus, each direct summand of $V^{\otimes n}$ appears in $M^{\Lambda_{n,m}}$. However, not all summands appear, since $\Lambda^B_{n,m}$ is a proper subset of $\Lambda_{n,m}$ if $n > 1$. Let

$$e_n = \sum_{\lambda \in \Lambda^B_{n,m}} \varphi_\lambda. \tag{4.1}$$

Lemma 4.1.4 shows that as modules over the Hecke algebra:

$$e_n M^{\Lambda_{n,m}} = V^{\otimes n}. \tag{4.2}$$

Thus, we have that:

Theorem 4.1.5. *There is an isomorphism $\mathcal{L}^n(m) \cong e_n \mathcal{S}^n(m) e_n$.*

Proof. We note that φ_λ is zero on all M^μ for $\mu \neq \lambda$. Thus,

$$e_n M^{\Lambda_{n,m}} \cong \bigoplus_{\lambda \in \Lambda^B_{n,m}} M^\lambda. \tag{4.3}$$

Using the bijection of $B_n(m)$ with $\Lambda^B_{n,m} \subset \Lambda_{n,m}$ and the isomorphism of Lemma 4.1.4, we can conclude $e_n M^{\Lambda_{n,m}} = V^{\otimes n}$. Thus,

$$e_n \mathcal{S}^n(m) e_n = \mathrm{End}_{\mathcal{H}_n}(e_n M^{\Lambda_{n,m}}) = \mathrm{End}_{\mathcal{H}_n}(V^{\otimes n}) = \mathcal{L}^n(m). \qquad \square$$

This tells us that $\mathcal{L}^n(m)$ and $\mathcal{S}^n(m)$ are Morita equivalent if and only if no simple $\mathcal{S}^n(m)$ modules are killed by e_n — since simples occur as quotients of cell modules by the radicals of their inner products, this will happen when the inner products do not become identically zero when truncating to $\mathcal{L}^n(m)$.

4.2 Compatibility with the Schur Functor

Assume that $m \geq 2n$ (i.e. the case of large m). In this case, we can consider the idempotent

$$e_{\mathcal{H}_n} = \varphi_{(\emptyset,(1^n))}. \tag{4.4}$$

Since $e_{\mathcal{H}_n} V^{\otimes n}$ is a free module of rank 1 over the Hecke algebra, we have $e_{\mathcal{H}_n} \mathcal{S}^n(m) e_{\mathcal{H}_n} = \mathcal{H}_n$. The basis vectors of $e_{\mathcal{H}_n} C^\lambda_{\mathcal{L}}$ correspond to standard tableaux with the entries $\{1, \ldots, n\}$. The module $C^\lambda_{\mathcal{H}} = e_{\mathcal{H}_n} C^\lambda_{\mathcal{L}} = e_{\mathcal{H}_n} e_n C^\lambda_{\mathcal{S}}$ is non-zero for all bipartitions λ.

The idempotent from Theorem 4.1.5 satisfies $e_n \cdot e_{\mathcal{H}_n} = e_{\mathcal{H}_n}$, where $e_{\mathcal{H}_n}$ is the idempotent giving the type B Hecke algebra $\mathcal{H}_n$ as an idempotent truncation of the cyclotomic q-Schur algebra $\mathcal{S}^n$ (referred to as $\varphi_{T^\omega T^\omega}$ in [JM00]). So,

$$\mathcal{H}_n = e_{\mathcal{H}_n} \mathcal{L}^n e_{\mathcal{H}_n} = e_{\mathcal{H}_n}(e_n \mathcal{S}^n e_n) e_{\mathcal{H}_n} = e_{\mathcal{H}_n} \mathcal{S}^n e_{\mathcal{H}_n}.$$

That is to say, the Schur functor factors through the corresponding quotient of module categories.

$$\mathcal{S}^n\text{-mod} \xrightarrow{e_n} \mathcal{L}^n\text{-mod} \xrightarrow{e_{\mathcal{H}_n}} \mathcal{H}_n\text{-mod}$$

As a result, for m large, $n_c(\mathcal{S}^n(m)) = n_c(\mathcal{L}^n(m)) = n_c(\mathcal{H}_n)$

This gives us the opportunity to apply Theorem 2.2.5: if we have used the Schur functor to truncate down to the Hecke algebra and have not lost any simple modules, then we cannot have lost any simples along the way. That is,

$$n_s(\mathcal{H}_n) \leq n_s(\mathcal{L}^n) \leq n_s(\mathcal{S}^n)$$

with equality if and only if all three algebras are Morita equivalent, and therefore all quasi-hereditary, and all semisimple.

It is known when this semisimplicity holds:

Theorem 4.2.1 ([Ari94] Main Theorem). *The Hecke algebra $\mathcal{H}_n$ is semisimple and quasi-hereditary if and only if*

$$P_n(Q,q) = \prod_{m=2}^{n-1} \frac{q^m - 1}{q - 1} \cdot \prod_{i=1-n}^{n-1} (Q + q^i) \neq 0.$$

4.3 Kleshchev and LNX Bipartitions

Ariki's polynomial $P_n(Q,q)$ (see Theorem 4.2.1) tells us at which parameter values the Hecke algebra $\mathcal{H}_n$ is Morita equivalent to $\mathcal{S}^n$ - these are the values at which the simples of $\mathcal{H}_n$ are indexed by the full set of bipartitions. In general, bipartitions λ such that the simple module $D^\lambda_{\mathcal{H}} \neq 0$ are known as **Kleshchev** (see [AM00]), so $P_n(Q,q) \neq 0$ if and only if all bipartitions are Kleshchev. These can be constructed iteratively from the empty bipartition by the addition of "good nodes" via the Kashiwara operators which we will encounter in Lemma 5.3.9.

Similarly, we will say that a bipartition λ is LNX_m if the simple module $D^\lambda_{\mathcal{L}}$ is nonzero in $\mathcal{L}^n(m)$-mod. Since we will be mainly dealing with $m \geq 2n$, when behaviour stabilizes based on parity, we will say λ is LNX_o if it is LNX_m for $m \geq 2n$ odd, and LNX_e if it is LNX_m for $m \geq 2n$ even.

If a bipartition is Kleshchev at a given choice of parameters, the corresponding simple has survived the Schur functor, and so it cannot have been sent to zero by the idempotent e_n - therefore, it must also be LNX_*. However, as we will see, there are LNX_* bipartitions which are not Kleshchev - in terms of the crystal graph constructed in Lemma 5.3.9, Kleshchev bipartitions are those that lie in the connected component of the empty bipartition, but LNX_* bipartitions can lie in other components, as long as the component in question is not deleted by the idempotent e_n.

Bipartitions which are LNX_* but are not necessarily Kleshchev can be found using the following:

Lemma 4.3.1. *Every bipartition in $\Lambda^B_{n,m}$ is LNX_m for all parameters (Q,q).*

Proof. If $\lambda \in \Lambda^B_{n,m}$, then $\lambda = (\lambda^{(1)}, \lambda^{(2)}) = ((\lambda^{(1)}_1), \lambda^{(2)})$, that is, the first partition has only one part. Let T^λ be the unique tableau in $\mathcal{T}_0(\lambda,\lambda)$. As noted above, $\varphi_{T^\lambda T^\lambda}$ is the identity map on M^λ, and so

$$\langle \varphi_{T^\lambda}, \varphi_{T^\lambda} \rangle = 1$$

regardless of parameters. Therefore $D^{\lambda}_{\mathcal{L}} \neq 0$ and so λ is LNX_m. $\square$

As an example, see Figure 5.1. The bipartition $((1), \emptyset)$ is not Kleshchev, as it is not in the crystal component of $(\emptyset, \emptyset)$, but it is LNX_o by Lemma 4.3.1.

Chapter 5

The Structure of $\mathcal{L}^n(m) - \mathrm{mod}$

5.1 A Sufficient Condition for Quasi-Hereditarity

We have seen that Ariki's polynomial provides a sufficient condition for the quasi-hereditarity of $\mathcal{L}^n(m)$. While the quasi-hereditary behaviour of $\mathcal{L}^n(m)$ does not exactly mirror that of the underlying Hecke algebra, we can still apply Corollary 2.2.9: $\mathcal{L}^n(m)$ will remain quasi-hereditary if and only if the restriction of the bilinear form remains nonzero on $C_{\mathcal{L}}^{\lambda}$ for all $\lambda \in \Lambda_{n,m}$. That is:

Corollary 5.1.1. *For m large, the algebra $\mathcal{L}^n(m)$ is quasi-hereditary if and only if every bipartition of n is LNX_m.*

Remark 5.1.2. *Here we see immediately why $m < 2n$ will prove to be more difficult. In this case, we have a strict inequality $n_c(\mathcal{S}^n(m)) > n_c(\mathcal{L}^n(m))$ since we have already noted that $e_n C_{\mathcal{S}}^{((1^n),\emptyset)} = 0$. Thus, there can be bipartitions which are not LNX_m, but which do not cause a problem for quasi-hereditarity since their cell modules are trivial as well.*

Consider the factor $f_n(Q,q) = \prod_{i=1-n}^{n-1}(Q+q^i)$ of P_n. If q is not a root of unity, then $f_n(Q,q) = 0$ if and only if $P_n(Q,q) = 0$. If $f_n(Q,q) \neq 0$ then the quasi-hereditarity of $\mathcal{L}^n(m)$ is resolved:

Lemma 5.1.3 ([LNX20, Cor. 6.1.1])**.** *If $f_n(Q,q) \neq 0$, then $\mathcal{L}^n(m)$ is quasi-hereditary for all m and all bipartitions of n are LNX_m.*

In [LNX20, Conjecture 6.1.3], it was conjectured that the converse to this result holds. As we will establish, this is true for $m \geq 2n$ even, but as we will see in a small rank example, and then prove in larger generality, this polynomial overestimates parameter values at which quasi-hereditarity fails for $m \geq 2n$ odd (see Theorem 1.0.2).

5.1.1 Small Rank Examples and a Counterexample to Necessity

In this section, we will investigate $n = 2$ for small values of $\dim V$ to see both that the behaviour for small m is unpredictable, and that [LNX20, Conjecture 6.1.3] does not hold for $m \geq 2n$ odd.

In each case, we will have $\mathcal{S}^2(m) = \mathcal{S}^2(\Lambda_{2,m})$ where $\Lambda_{2,m}$ is the set of weak bicompositions of $n = 2$ where the first component has 2 parts and the second has $r = \lfloor \frac{m}{2} \rfloor$ parts, and $\mathcal{L}^2(m) = e_2 \mathcal{S}^2(m) e_2$, where $e_n = \sum_{\lambda \in \Lambda_{2,m}^B} \varphi_\lambda$, and $\Lambda_{2,m}^B$ is the set of weak bicompositions of $n = 2$ with 1 part if m is odd and 0 parts if m is even, and $r = \lfloor \frac{m}{2} \rfloor$ parts in the second component. The bipartitions in $\Lambda_{2,m}$ index the cell modules, and by Theorem 3.4.7, $\mathcal{S}^2(m)$ has cellular basis

$$\{\varphi_{ST} \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu) \text{ for some } \mu, \nu \in \Lambda_{2,m},\ \lambda \in \Lambda^+_{2,m}\}$$

whereas $\mathcal{L}^2(m)$ will have cellular basis

$$\{e_2\varphi_{ST}e_2 \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu) \text{ for some } \mu, \nu \in \Lambda^B_{2,m},\ \lambda \in \Lambda^+_{2,m}\}$$

For both algebras, cell and simple modules will be indexed by $\lambda \in \Lambda^+_{2,m}$. However, while for $\mathcal{S}^2(m)$ the cell module $C^\lambda_{\mathcal{S}}$ will have basis $\{\varphi_S \mid S \in \mathcal{T}_0(\lambda, \mu),\ \mu \in \Lambda_{2,m}\}$ while $C^\lambda_{\mathcal{L}}$ will have basis $\{\varphi_S \mid S \in \mathcal{T}_0(\lambda, \mu),\ \mu \in \Lambda^B_{2,m}\}$.

We consider the Hecke algebra $\mathcal{H}_2$ generated by T_0, T_1 subject to the relations

$$T_1^2 = (q-1)T_1 + q, \qquad T_0^2 = (Q-1)T_0 + Q, \qquad T_1T_0T_1T_0 = T_0T_1T_0T_1.$$

For the purposes of determining quasi-hereditarity, we need only consider bicompositions in $\Lambda_{2,m}$ with zero parts removed. In this particular case, these are precisely the bipartitions of 2:

$$\lambda = ((2), \emptyset) = \left(\ydiagram{2}, \emptyset\right) \qquad \lambda' = (\emptyset, (2)) = \left(\emptyset, \ydiagram{2}\right)$$

$$\mu = ((1^2), \emptyset) = \left(\ydiagram{1,1}, \emptyset\right) \qquad \mu' = (\emptyset, (1^2)) = \left(\emptyset, \ydiagram{1,1}\right)$$

$$\nu = ((1), (1)) = \left(\ydiagram{1}, \ydiagram{1}\right)$$

From the definition, we have:

$$\begin{aligned} m_\lambda &= (L_1+1)(L_2+1)(T_1+1) & m_{\lambda'} &= T_1 + 1 \\ m_\mu &= (L_1+1)(L_2+1) & m_{\mu'} &= 1 \\ m_\nu &= L_1 + 1 & & \end{aligned}$$

The Case $\mathbf{m = 1}$

In this case $\dim V = 1$, so $\mathcal{L}^2(1) \cong \mathrm{End}_{\mathcal{H}_2}(\Bbbk) \cong \Bbbk$ is semisimple, and therefore quasi-hereditary and Morita equivalent to $\mathcal{S}^2(1)$.

The Case $\mathbf{m = 2}$

$$\Lambda^+_{2,2} = \{\lambda, \mu, \lambda', \nu\} \qquad \Lambda_{2,2} = \{\lambda, \lambda', \mu, \nu\} \qquad \Lambda^B_{2,2} = \{\lambda'\}$$

Cell modules are indexed by $\Lambda^+_{2,2}$. The only filling permitted for bitableaux indexing basis vectors of cell modules of $\mathcal{L}^2(2)$ is $1, 1$, meaning that the cell module $C^\mu_{\mathcal{L}}$ is trivial. In this case, we have $n_s(\mathcal{S}^2(2)) = n_c(\mathcal{S}^2(2)) = 4$ whereas $n_s(\mathcal{L}^2(2)) \leq n_s(\mathcal{L}^2(2)) = 3$ so even at parameters where $\mathcal{L}^2(2)$ is quasi-hereditary (determined below), it is not Morita equivalent to $\mathcal{S}^2(2)$ even though both are quasi-hereditarity.

The bipartitions λ', ν are LNX_2, as we can write them as the images of elements from $B_n(m)$ (see Chapter 4), so we need only look at λ. The cell module $C_{\mathcal{L}}^{\lambda}$ has basis vector φ_S, where $S = \left(\boxed{1}\boxed{1}, \emptyset\right)$

This is a coset representative of the homomorphism $\varphi_{SS} \in \mathrm{Hom}_{\mathcal{H}_n}(M^{\lambda'}, M^{\lambda'})$, satisfying

$$\varphi_{SS}(m_{\lambda'}) = m_{\lambda} = (L_1 + 1)(L_2 + 1)m_{\lambda'}).$$

We now compute the bilinear form. By the argument given in the proof of [JM00, Prop. 3.7]:

$$\begin{aligned} \varphi_{SS}^2(m_{\lambda'}) &= (L_1 + 1)(L_2 + 1)\varphi_{SS}(m_{\lambda'}) \\ &\equiv (\mathrm{res}_{\lambda}(1) + 1)(\mathrm{res}_{\lambda}(2) + 1)\varphi_{SS}(m_{\lambda'}) \mod \overline{N}_{\lambda} \end{aligned}$$

implying that $\langle \varphi_S, \varphi_S \rangle = (Q+1)(Qq+1)$. Thus, this simple is lost and $\mathcal{L}^2(2)$ is not quasi-hereditary if and only if $Q = -1$ or $Q = -q^{-1}$.

The Case $\mathbf{m = 3}$

$$\Lambda_{2,3}^+ = \Lambda_{2,2}^+ \qquad \Lambda_{2,3} = \Lambda_{2,2} \qquad \Lambda_{2,3}^B = \{\lambda, \nu, \lambda'\}$$

With the parity change we are now allowed a single row in the first component of elements of $\Lambda_{2,3}^B$. These are all automatically LNX_3, so the only bipartition left to consider is μ.

Since we are now allowed bitableaux with the filling $0, 1$ (where 0 can only go in the first component) to index basis vectors, $C_{\mathcal{L}}^{\mu}$ is nonzero and spanned by φ_A, where $A = \left(\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}, \emptyset\right)$. $\varphi_{AA} \in \mathrm{Hom}_{\mathcal{H}_n}(M^{\nu}, M^{\nu})$, so again following [JM00, Prop. 3.7]:

$$\begin{aligned} \varphi_{AA}^2(m_{\nu}) &= \varphi_{AA}(m_{\mu}) \\ &= \varphi_{AA}((L_2 + 1)m_{\nu}) \\ &\equiv (\mathrm{res}_{\mu}(2) + 1)\varphi_{AA}(m_{\nu}) \mod \overline{N}_{\mu} \end{aligned}$$

implying that $\langle \varphi_A, \varphi_A \rangle = (Qq^{-1} + 1)$, so the simple is lost if and only if $Q = -q$. If $Q \neq -q$, then we have a Morita equivalence to $\mathcal{S}^2(2)$.

The Case $\mathbf{m = 4}$

$$\Lambda_{2,4}^+ = \{\lambda, \mu, \nu, \lambda', \mu'\} \qquad \Lambda_{2,4} = \Lambda_{2,4}^B = \{\lambda', \mu'\}$$

For $m = 4$ we are allowed two rows in the second component, meaning that $\mu' \in \Lambda_{2,m}^+$ now indexes a cell module. Since it can be written as the image of an element from $B_2(m)$, however, it is LNX_4. The other change this increase in m allows is that we can index basis vectors by bitableaux of type μ', i.e. with filling $\{1, 2\}$, as well as of types $((1), (0, 1))$, $(\emptyset, (0, 2))$, and $((0, 1), (0, 1))$. We also gain another possible filling with repeated letters via $(\emptyset, (0, 2)))$. That is, the dimension of the cell modules $C_{\mathcal{L}}^{\mu}$ depends on m.

The Case $\mathbf{m = 5}$

As we can have two rows in the second component, μ' now indexes a cell module. Since μ' can be written as the image of an element from $B_2(5)$, it is LNX_5, and the corresponding simple survives the idempotent truncation of Theorem 4.1.5. The only bipartition that is not of this form is μ, so we investigate further.

The tableaux of shape μ using the above described alphabet are

$$A = \left(\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline\end{array}, \emptyset\right) \qquad B = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \emptyset\right) \qquad C = \left(\begin{array}{|c|}\hline 0 \\ \hline 2 \\ \hline\end{array}, \emptyset\right)$$

As C is equivalent to A via the isomorphism $M^{((1),(1,0))} \cong M^{((1),(0,1))}$, we need only consider A and B. Thus, we just need to calculate the bilinear forms $\langle \varphi_A, \varphi_A \rangle$ and $\langle \varphi_B, \varphi_B \rangle$, and quasi-hereditarity will fail if and only if these are both 0.

We will see that we have the following equalities:

$$\langle \varphi_A, \varphi_A \rangle = q^{-1}Q + 1 \qquad \langle \varphi_B, \varphi_B \rangle = (Q+1)(q^{-1}Q+1).$$

We begin with the endomorphism

$$\varphi_{AA} \in End_{\mathcal{H}_n}(m_\nu) \qquad \varphi_{AA}(m_\nu) = m_\mu = (L_2+1)m_\nu$$

By the argument given in the proof of [JM00, Proposition 3.7]

$$\begin{aligned}\varphi_{AA}^2(m_\nu) &= (L_2+1)m_\mu \\ &\equiv (\mathrm{res}_\mu(2)+1)m_\mu \mod \overline{N}_\mu \\ &\equiv (q^{-1}Q+1)\varphi_{AA}(m_\nu) \mod \overline{N}_\mu\end{aligned}$$

which implies that $\langle \varphi_A, \varphi_A \rangle = q^{-1}Q + 1$. Now consider

$$\varphi_{BB} \in \mathrm{End}_{\mathcal{H}_n}(M^{\mu'}) \qquad \varphi_{BB}(m_{\mu'}) = m_\mu.$$

Following the same argument as above,

$$\begin{aligned}\varphi_{BB}^2(m_{\mu'}) &= m_\mu^2 \\ &= (L_1+1)(L_2+1)m_\mu \\ &\equiv (Q+1)(q^{-1}Q+1)\varphi_{BB}(m_{\mu'})\end{aligned}$$

implying that $\langle \varphi_B, \varphi_B \rangle = (Q+1)(q^{-1}Q+1)$. Therefore, this simple is lost if and only if $Q = -q$.

The case of m large

For $m \geq 2n$, the full set of bipartitions of n indexes the cell modules. Increasing m further amounts to adding zero rows to elements of $\Lambda_{n,m}$, which correspond to more possible fillings for tableaux which index the cellular basis vectors (and also basis vectors for cell modules). Thus, once we have established that each cell module has a basis vector that survives the idempotent truncation and quotient by the inner product, the addition of extra basis vectors cannot break quasi-hereditarity. As we saw in Chapter 4, increasing m of the same parity results in a Morita equivalent algebra.

These examples can be summarized in the following table:

Proposition 5.1.4. *The algebra $\mathcal{L}^2(m)$ demonstrates the following behaviour for parameters Q, q:*

	$Q = -q$	$Q = -1$	$Q = -q^{-1}$	$Q \notin \{-q, -1, -q^{-1}\}$
$m = 1$	*qh*	*qh*	*qh*	*qh*
$m = 2$	*qh*	*not qh*	*not qh*	*qh*
$m = 3$ *odd*	*not qh*	*qh*	*qh*	*qh*
$m \geq 4$ *even*	*not qh*	*not qh*	*not qh*	*qh*
$m \geq 5$ *odd*	*not qh*	*qh*	*qh*	*qh*

As we have seen, the behaviour for small values of m is erratic, but does stabilize for $m \geq 2n$ for a given parity. The examples also show that the converse to Lemma 5.1.3 is not true in a low rank case for large odd m: In the case where $n = 2$, if $Q \in \{-q^{-1}, 1\}$, then $f_n(Q,q) = \prod_{i=-1}^{1}(Q + q^i) = 0$, but the algebra $\mathcal{L}^n(m)$ is still quasi-hereditary for m large odd, for example, $m = 2n + 1 = 5$.

5.2 Cell Modules and Inner Products

5.2.1 Unstacking

The Odd Case

To generalize the example from Section 5.1.1, we consider the case of a bipartition λ of n such that $\lambda^{(2)} = \emptyset$. We will drop the superscripts on the parts for simplicity:

$$\lambda = ((\lambda_1, \ldots, \lambda_\ell), \emptyset).$$

Let $\mathcal{U}(\lambda)$ denote the bipartition given by unstacking the diagram for λ after the first row, and moving the second row and onward to the second partition, i.e.:

$$\mathcal{U}(\lambda) = ((\lambda_1), (\lambda_2, \ldots, \lambda_\ell)).$$

Note that

$$\mathfrak{S}_{\mathcal{U}(\lambda)} \cong \mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$$

giving us that $x_\lambda = x_{\mathcal{U}(\lambda)}$ and therefore that

$$m_\lambda = \left(\prod_{i=\lambda_1+1}^{n} (L_i + 1) \right) m_{\mathcal{U}(\lambda)} \tag{5.1}$$

Given any tableau A of shape λ with any filling, let $\mathcal{U}(A)$ denote the corresponding tableau of shape $\mathcal{U}(\lambda)$. This "unstacking" operation on Young diagrams has inverse given by "stacking" the two partitions: given a bicomposition $\rho = (\rho^{(1)}, \rho^{(2)})$, let $\mathcal{S}(\rho)$ denote the diagram with $\rho^{(1)}$ stacked on top of $\rho^{(2)}$ and $\emptyset$ in the second component:

$$\mathcal{S}(\rho) = ((\rho^{(1)}, \rho^{(2)}), \emptyset).$$

Note that stacking or unstacking a tableau does not change its type (i.e. filling). The only thing that may change is whether or not we maintain increasing entries down the columns. Recall Definition 3.4.5: the set $\mathcal{A}(\lambda, S)$ consists of standard tableaux s of shape λ such that the mapping on entries $m \mapsto i_k$ if m occurs in row i of component k of $\mathfrak{t}^\mu$, where μ is the type of the tableau S.

Lemma 5.2.1. *Fix a semi-standard tableau $S \in \mathcal{T}_0(\lambda, \mu)$ of shape λ and type μ. We have equalities of sets of standard tableaux:*

$$\{\mathcal{U}(s) | s \in \mathcal{A}(\lambda, S)\} = \mathcal{A}(\mathcal{U}(\lambda), \mathcal{U}(S))$$

$$\mathcal{A}(\lambda, S) = \{\mathcal{S}(s') | s' \in \mathcal{A}(\mathcal{U}(\lambda), \mathcal{U}(S)\}$$

Stacking and unstacking induce inverse bijections between these sets.

Proof. Given $s \in \mathcal{T}(\lambda)$ such that $\mu(s) = S$, we have $\mathcal{U}(s) \in \mathcal{T}(\mathcal{U}(\lambda))$ and $\mu(\mathcal{U}(S)) = \mathcal{U}(S)$. Consider $s' \in \mathcal{T}(\mathcal{U}(\lambda))$ such that $\mu(s') = \mathcal{U}(S)$. Since $\mu(s')$ sends an integer a to i_k whenever a is in the i^{th} row of the k^{th} component of t^μ, $\mu(\mathcal{S}(s')) = S$. What remains to be verified is that $\mathcal{S}(s') \in \mathcal{T}(\lambda)$.

Since entries are of weight $(\emptyset, (1^n))$ and increase along rows in s', they must also increase along rows in $\mathcal{S}(s')$. We now consider the i^{th} column of $\mathcal{S}(s')$. Let a_r denote the entry in row r of column i, and let x_r denote the entry in the same position in S. Since S is a semi-standard tableau, we must have $x_1 < x_2$. The map given by $\mu : a_r \mapsto x_r$ is weakly order preserving, so if we had $a_1 > a_2$ we would have had $x_1 \geq x_2$, a contradiction. Therefore $a_1 < a_2$, and so $\mathcal{S}(s') \in \mathcal{T}(\lambda)$. □

Given a standard $\mathcal{U}(\lambda)$ tableau t, recall that $d(t) \in \mathfrak{S}_n$ is the unique shortest permutation such that $t \cdot d(t) = t^{\mathcal{U}(\lambda)}$. For any $s \in \mathcal{T}(\lambda)$ and $s' \in \mathcal{U}(\lambda)$,

$$d(s) = d(\mathcal{U}(s)), \qquad d(s') = d(\mathcal{S}(s')).$$

Lemma 5.2.2. *Let $\lambda = ((\lambda_1, \ldots, \lambda_\ell), \emptyset)$ be a bipartition of n and μ a bicomposition of n. Fix $S \in \mathcal{T}_0(\lambda, \mu)$. Let $T \in \mathcal{T}_0(\lambda, \mathcal{U}(\lambda))$ be the semi-standard tableau with 0's in row 1 and r's in row $r+1$ for $1 \leq r \leq \ell - 1$:*

$$T = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \hline 1 & 1 & 1 & \ldots & 1 \\ \cline{1-5} \vdots & \ddots & \vdots & \\ \cline{1-4} \ell-1 & \ldots & \ell-1 \\ \cline{1-3} \end{array}, \emptyset \right)$$

We have an equality:

$$\varphi_{ST} = \varphi_{\mathcal{U}(S)\mathcal{U}(T)} \circ \varphi_{TT}$$

Proof. First, note that these maps can indeed be composed:

$$\varphi_{ST}, \varphi_{\mathcal{U}(S)\mathcal{U}(T)} \in \mathrm{Hom}_{\mathcal{H}_n}(M^{\mathcal{U}(\lambda)}, M^\mu), \qquad \varphi_{TT} \in \mathrm{Hom}_{\mathcal{H}_n}(M^{\mathcal{U}(\lambda)}, M^{\mathcal{U}(\lambda)})$$

If $\mathfrak{t} \in \mathcal{T}(\mathcal{U}(\lambda))$ is such that $\mathcal{U}(\lambda)(\mathfrak{t}) = \mathcal{U}(T)$, then $\mathfrak{t} = \mathfrak{t}^{\mathcal{U}(\lambda)}$ and $d(\mathfrak{t}) = 1$. Therefore,

$$m_{TT} = m_\lambda, \qquad m_{ST} = \sum_{s \in \mathcal{T}(\lambda), \mu(s) = S} T^*_{d(s)} m_\lambda, \qquad m_{\mathcal{U}(\lambda)} = \sum_{s' \in \mathcal{T}(\mathcal{U}(\lambda)), \mu(s') = \mathcal{U}(S)} T^*_{d(s')} m_{\mathcal{U}(\lambda)}$$

Combining this with Lemma 5.2.1, we get

$$\varphi_{TT}\left(m_{\mathcal{U}(\lambda)}\right) = m_{\mathcal{U}(\lambda)} \left(\prod_{i=\lambda_1+1}^{n} (L_i + 1) \right). \tag{5.2}$$

Thus, if we begin with the defining equation:

$$\varphi_{\mathcal{U}(S)\mathcal{U}(T)}(m_{\mathcal{U}(\lambda)}) = m_{\mathcal{U}(S)\mathcal{U}(T)} = \sum_{s' \in \mathcal{A}(\mathcal{U}(\lambda), \mathcal{U}(S))} T^*_{d(s')} m_{\mathcal{U}(\lambda)} \tag{5.3}$$

and multiply on the right by $\prod_{i=\lambda_1+1}^{n}(L_i + 1)$, and use Eq. (5.1) to reindex sums, we find that:

$$\begin{aligned}
\varphi_{\mathcal{U}(S)\mathcal{U}(T)} \circ \varphi_{TT}\left(m_{\mathcal{U}(\lambda)}\right) &\overset{(5.2)}{=} \varphi_{\mathcal{U}(S)\mathcal{U}(T)}\left(m_{\mathcal{U}(\lambda)} \prod_{i=\lambda_1+1}^{n} (L_i+1) \right) \\
&\overset{(5.3)}{=} \left(\sum_{s' \in \mathcal{A}(\mathcal{U}(\lambda), \mathcal{U}(S))} T^*_{d(s')} m_{\mathcal{U}(\lambda)} \right) \left(\prod_{i=\lambda_1+1}^{n} (L_i+1) \right) \\
&\overset{(5.2)}{=} \sum_{s' \in \mathcal{A}(\lambda, S)} T^*_{d(s')} m_\lambda
\end{aligned}$$

This shows that $\varphi_{\mathcal{U}(S)\mathcal{U}(T)} \circ \varphi_{TT}$ satisfies the defining property

$$\varphi_{ST}(m_{\mathcal{U}(\lambda)}) = \sum_{s \in \mathcal{A}(\lambda, S)} T^*_{d(s)} m_\lambda,$$

giving us the result. □

Since T is the only tableau of its type and shape, we have that $\langle \varphi_T, \varphi_S \rangle = 0$ for all $S \neq T$. Thus, φ_T lies in the radical of the cell module form if and only $\langle \varphi_T, \varphi_T \rangle = 0$. Therefore:

Corollary 5.2.3. *For m large odd, the vector φ_T generates $C^\lambda_{\mathcal{L}}$, and the inner product on this cell module will be zero if and only if $\langle \varphi_T, \varphi_T \rangle = 0$.*

Since $m_\lambda = m_{\mathfrak{t}\mathfrak{t}}$ where $\mathfrak{t} = \mathfrak{t}^{\mathcal{U}(\lambda)}$, by [JM00, Prop. 3.7], we have

$$m_\lambda L_i \equiv \mathrm{res}_\lambda(i) m_\lambda \mod \overline{N}_\lambda.$$

So,

$$\begin{aligned}\varphi_{TT}^2(m_{\mathcal{U}(\lambda)}) &= \prod_{i=\lambda_1+1}^{n}(L_i+1)\varphi_{TT}(m_{\mathcal{U}(\lambda)}) \\ &= \prod_{i=\lambda_1+1}^{n}(L_i+1)m_\lambda \\ &\equiv \prod_{i=\lambda_1+1}^{n}(\text{res}_\lambda(i)+1)m_\lambda \mod \overline{N}_\lambda\end{aligned}$$

which implies that

$$\langle\varphi_T,\varphi_T\rangle = \prod_{i=\lambda_1+1}^{n}(\text{res}_\lambda(i)+1).$$

Each row from $r=2$ onwards will contribute a factor of

$$\prod_{c=1}^{\lambda_r}(Qq^{c-r}+1).$$

Taking the product over all of these rows gives:

$$\langle\varphi_T,\varphi_T\rangle = \prod_{r=2}^{\ell}\left(\prod_{c=1}^{\lambda_r}(Qq^{c-r}+1)\right). \tag{5.4}$$

Corollary 5.2.4. *The bipartition $\lambda=((\lambda_1,\dots,\lambda_\ell),\emptyset)$ is LNX_o if and only if*

$$\prod_{r=2}^{\ell}\left(\prod_{c=1}^{\lambda_r}(Q+q^{r-c})\right)\neq 0.$$

In particular:

1. *The bipartition $\lambda=((1^n),\emptyset)$ is LNX_o if and only if $\prod_{i=1}^{n-1}(Q+q^i)\neq 0$.*
2. *The bipartition $\lambda=\left(\left(\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor\right),\emptyset\right)$ is LNX_o if and only if $\prod_{i=2-\lfloor\frac{n}{2}\rfloor}^{1}(Q+q^i)\neq 0$.*

This result can be interpreted using the residue function

$$\text{res}(r,c,j) = \begin{cases} Qq^{c-r} & j=1 \\ -q^{c-r} & j=2. \end{cases} \tag{5.5}$$

where (r,c,j) denotes the box in row r and column c of the jth component.

Visually, the residues of a bipartition are (generically/unsimplified) as follows:

We can rephrase the result above as saying that λ is LNX_o if and only if no box $(r,c,1)$ in the diagram of $\lambda^{(1)}$ with $r>1$ has residue -1.

Lemma 5.2.5. *If any bipartition of the form $\lambda=((\lambda_1,\dots\lambda_\ell),\emptyset)$ is not LNX_o, then $Q=-q^k$ for some $k\in[2-\lfloor\frac{n}{2}\rfloor,n-1]$.*

$$T = \left(\begin{array}{|c|c|c|c|c|} \hline Q & Qq & Qq^2 & \dots & Qq^{\mu_1^{(1)}-1} \\ \hline Qq^{-1} & Q & \dots & \multicolumn{2}{c}{} \\ \cline{1-3} \dots & \multicolumn{4}{c}{} \\ \cline{1-1} Qq^{1-\ell_1} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|c|c|} \hline -1 & -q & -q^2 & \dots & -q^{\mu_1^{(2)}-1} \\ \hline -q^{-1} & -1 & \dots & \multicolumn{2}{c}{} \\ \cline{1-3} \dots & \multicolumn{4}{c}{} \\ \cline{1-1} -q^{1-\ell_2} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \right)$$

Proof. By Corollary 5.2.4, we must have $Q = -q^{r-c}$, and there is a box (r, c) with $r > 1$ in the diagram of λ. We find the limits above by trying to maximize or minimize $r - c$.

Of course, $r - c < r \leq n$, which gives the desired upper bound. On the other hand, we must have

$$c - r \leq c - 2 \leq \lambda_2 - 2 \leq \frac{\lambda_1 + \lambda_2}{2} - 2 \leq \frac{n}{2} - 2.$$

This implies that $c - r \leq \lfloor \frac{n}{2} \rfloor - 2$, completing the proof. □

Note that if q is itself a root of unity of order e, then $Q = -q^k$ can be reduced to $Q = -q^{\overline{k}}$ with $0 \leq \overline{k} < e$, so that these bounds are extremal regardless of the quantum characteristic.

The Even Case

Let $m \geq 2n$ be even. Recall that in this case, elements of $\Lambda^B_{n,m}$ have empty first component, so cellular basis vectors of $\mathcal{L}^n(m)$ cannot be indexed by tableaux with 0 in their filling.

We again let $\lambda = (\lambda^{(1)}, \emptyset)$ be a bipartition of n, and μ a bicomposition of n, fixing $S \in \mathcal{T}_0(\lambda, \mu)$. In the place of T from the section above, we let $U \in \mathcal{T}_0(\lambda, (\emptyset, \lambda^{(1)}))$ be the semi-standard tableau with $i's$ in row i. The arguments from the odd case generalize readily to show that φ_U generates the cell module $C^\lambda_{\mathcal{L}}$ for m large even. As U is the only tableau of its type and shape, we have the even version of Corollary 5.2.3:

Corollary 5.2.6. *For m large even, the vector φ_U generates $C^\lambda_{\mathcal{L}}$, and the inner product on this cell module will be zero if and only if $\langle \varphi_U, \varphi_U \rangle = 0$.*

Similarly, again following [JM00, Proposition 3.7], we have

$$\begin{aligned} \varphi^2_{UU}(m_{(\emptyset,(\lambda^{(1)}))}) &= \prod_{i=1}^{n} (L_i + 1)\varphi_{UU}(m_{(\emptyset,(\lambda^{(1)}))}) \\ &\equiv \prod_{i=1}^{n} (\mathrm{res}_\lambda(i) + 1) m_\lambda \mod \overline{N}_\lambda \end{aligned}$$

so that

$$\langle \varphi_U, \varphi_U \rangle = \prod_{i=1}^{n} (\mathrm{res}_\lambda(i) + 1)$$

Note that in the even case this product runs over the residues for the entries in all boxes, not just those from the second row onwards. Taking the product over all rows gives us

$$\langle \varphi_U, \varphi_U \rangle = \prod_{r=1}^{\ell} \left(\prod_{c=1}^{\lambda_r} (Qq^{c-r} + 1) \right) \tag{5.6}$$

so that

Corollary 5.2.7. *The bipartition $\lambda = ((\lambda_1, \dots, \lambda_\ell), \emptyset)$ is LNX_e if and only if*

$$\prod_{r=1}^{\ell} \left(\prod_{c=1}^{\lambda_r} (Q + q^{r-c}) \right) \neq 0$$

In particular:

1. *The bipartition $\lambda = ((1^n), \emptyset)$ is LNX_e if and only if $\prod_{i=0}^{n-1}(Q + q^i) \neq 0$.*
2. *The bipartition $\lambda = ((n), \emptyset)$ is LNX_e if and only if $\prod_{i=1-n}^{1}(Q + q^i) \neq 0$.*

5.3 Crystal Graph Structure

5.3.1 Induction and Restriction Functors

Fix $n \in \mathbb{Z}_{>0}$. In this section, we fix $\delta \in \{0, 1\}$ and let $m = 2n + \delta$. Each time we induct we embed $V = \Bbbk^m$ into $\Bbbk^{m+2}$, so we will leave m out of the notation so that $\mathcal{L}^{n\pm1} = \mathcal{L}^{n\pm1}(2(n \pm 1) + \delta)$. Let ${}^{\mathcal{H}}\mathrm{Res}^n_{n-1} : \mathcal{H}_n\text{-mod} \to \mathcal{H}_{n-1}\text{-mod}$ and ${}^{\mathcal{H}}\mathrm{Ind}^{n+1}_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ denote the induction and restriction functors on the categories of right modules over the Hecke algebra.

We also recall the induction and restriction functors of [Wad14] for right $\mathcal{S}^n$-modules.

We have an inclusion $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$, and for any $\lambda \in \Lambda_{n,m}$, we have that $m_\lambda = m_{\lambda^+}$ where $\lambda^+ = (\lambda^{(1)}, (\lambda^{(2)}_1, \dots, \lambda^{(2)}_r, 1))$. Thus, we have a canonical isomorphism $M^\lambda \otimes_{\mathcal{H}_n} \mathcal{H}_{n+1} \cong M^{\lambda^+}$. The induced map $\mathrm{Hom}(M^\nu, M^\mu) \to \mathrm{Hom}(M^{\nu^+}, M^{\mu^+})$ thus induces a non-unital inclusion $\mathcal{S}^n(m) \to \mathcal{S}^{n+1}(m+2)$. Let ξ be the image of the identity under this map. This is an idempotent such that the ring map $\mathcal{S}^n \to \xi\mathcal{S}^{n+1}\xi$ is unital, and so we have a $\mathcal{S}^{n+1}$-$\mathcal{S}^n$ bimodule structure on $\mathcal{S}^{n+1}\xi$, and similarly with left and right reversed. We can then define

$$\mathrm{Ind}^{n+1}_n : \mathcal{S}^n\text{-mod} \to \mathcal{S}^{n+1}\text{-mod} \qquad \mathrm{Res}^n_{n-1} : \mathcal{S}^n\text{-mod} \to \mathcal{S}^{n-1}\text{-mod}$$

given by

$$\mathrm{Ind}^{n+1}_n(N) = \mathcal{S}^{n+1}\xi \otimes_{\mathcal{S}^n} N \qquad \mathrm{Res}^n_{n-1}(M) = \xi M \cong \mathrm{Hom}_{\mathcal{S}^n}(\mathcal{S}^n\xi, M).$$

Let $\Omega_n : \mathcal{S}^n\text{-mod} \to \mathcal{H}_n\text{-mod}$ be the Schur functor $\Omega_n(M) = Me_{\mathcal{H}_n}$ (where $e_{\mathcal{H}_n}$ is defined in Equation (4.4)).

Corollary 5.3.1 ([Wad14, Corollary 4.18])**.** *There is an isomorphism of functors*

$$\Omega_n \circ \mathrm{Res}^{n+1}_n \cong {}^{\mathcal{H}}\mathrm{Res}^{n+1}_n \circ \Omega_{n+1}$$

and

$$\Omega_{n+1} \circ \mathrm{Ind}^{n+1}_n \cong {}^{\mathcal{H}}\mathrm{Ind}^{n+1}_n \circ \Omega_n$$

Let $\Phi_n : \mathcal{S}^n\text{-mod} \to \mathcal{L}^n\text{-mod}$ be the truncation functor given by $\Phi_n(M) = Me_n$. Given any associative ring R, let $\Sigma_k : R\text{-mod} \to R\text{-mod}$ be the functor given by

$$\Sigma_k(M) = M^{\oplus k}$$

Recall that by Equation (4.3), we have an isomorphism:

$$e_n \mathcal{S}^n e_{\mathcal{H}_n} \cong V^{\otimes n}$$

where V is an m-dimensional vector space.

Lemma 5.3.2. *For any $M \in \mathcal{S}^n$-mod,*

$$\mathrm{Ind}_n^{n+1}(M)e_{n+1} \cong (Me_n)^{\oplus \dim V}$$

Proof. First, assume that P is a projective right $\mathcal{S}^n$ module. Then, we have

$$\mathrm{Ind}_n^{n+1}(P)e_{n+1} \cong \mathrm{Hom}_{\mathcal{S}^{n+1}}(e_{n+1}\mathcal{S}^{n+1}, \mathrm{Ind}_n^{n+1}(P)).$$

By the adjunction of [Wad14, Corollary 3.18(ii)]

$$\begin{aligned}\mathrm{Ind}_n^{n+1}(P)e_{n+1} &\cong \mathrm{Hom}_{\mathcal{S}^n}(\mathrm{Res}_n^{n+1}(e_{n+1}Sn+1, P)\\ &\cong \mathrm{Hom}_{\mathcal{H}^n}(\Omega_n({}^{\mathcal{H}}\mathrm{Res}_n^{n+1}(e_{n+1}Sn+1)), \Omega_n(P))).\end{aligned}$$

since the Schur functor is fully faithful on projectives [Wad14, p. 4.8]. Now, restricting the action of $\mathcal{H}_{n+1}$ to $\mathcal{H}_n$ gives us the isomorphism of $\mathcal{H}_n$ modules:

$$\begin{aligned}e_{n+1}\mathcal{S}^{n+1}e_{\mathcal{H}_n} &\cong V^{\otimes n+1}\\ &\cong (V^{\otimes n} \otimes V)\\ &\cong (e_n\mathcal{S}^n e_{\mathcal{H}_n})^{\oplus m}\end{aligned}$$

where $V = \Bbbk^{I(m)}$ is the vector space from the construction of $\mathcal{L}^n(m)$ in [LNX20], so that

$$\mathrm{Hom}_{\mathcal{H}_n}(\Omega_n({}^{\mathcal{H}}\mathrm{Res}_n^{n+1}(e_{n+1}\mathcal{S}^{n+1}), \Omega_n(P))) \cong \mathrm{Hom}_{\mathcal{H}_n}(\Omega_n((e_n\mathcal{S}^n)^{\oplus m}), \Omega_n(P)).$$

Since Ω_n is fully faithful on projectives, this is isomorphic to

$$\mathrm{Hom}_{\mathcal{S}^n}((e_n\mathcal{S}^n)^{\oplus m}, P) \cong (Pe_n)^{\oplus m}$$

giving us

$$\mathrm{Ind}_n^{n+1}(P)e_{n+1} \cong (Pe_n)^{\oplus m}.$$

Now, let M be an arbitrary right $\mathcal{S}^n$ module. We can write M as a cokernel of a map between projectives $P_1, P_2 \in \mathcal{S}^n$-proj. So, we have an exact sequence:

$$P_2 \xrightarrow{f} P_1 \to M \to 0$$

By [Wad14, Corollary 3.18(i)], induction is exact, as is multiplying with any idempotent, so we have the exact sequence:

$$\mathrm{Ind}_n^{n+1}(P_2)e_{n+1} \to \mathrm{Ind}_n^{n+1}(P_1)e_{n+1} \to \mathrm{Ind}_n^{n+1}(M)e_{n+1} \to 0 \tag{5.7}$$

Applying the exact functors $\Phi_{n+1} \circ \mathrm{Ind}_n^{n+1}$ and $\Sigma_k \circ \Omega_n$ to this sequence, joining the resulting sequences by the isomorphisms on projectives, and then applying the Five Lemma gives us the result. $\square$

Corollary 5.3.3. *For $M \in \mathcal{S}^n$ -mod,*

$$Me_n = 0 \iff \mathrm{Ind}_n^{n+1}(M)e_{n+1} = 0$$

Lemma 5.3.4. *For any $M \in \mathcal{S}^n$ -mod, we have an injective map of vector spaces*

$$\mathrm{Res}_{n-1}^n(M)e_{n-1} \hookrightarrow Me_n$$

Proof. First, assume $P \in \mathcal{S}^n$ -proj. By the adjunction of [Wad14, Corollary 3.18] and the Schur functor:

$$\begin{aligned}
\mathrm{Res}_{n-1}^n(P)e_{n-1} &\cong \mathrm{Hom}_{\mathcal{S}^{n-1}}(e_{n-1}\mathcal{S}^{n-1}, \mathrm{Res}_{n-1}^n(Pe_{n-1})) \\
&\cong \mathrm{Hom}_{\mathcal{S}^n}(\mathrm{Ind}_{n-1}^n(e_{n-1}\mathcal{S}^{n-1}), P) \\
&\cong \mathrm{Hom}_{\mathcal{H}_n}(\Omega_n(\mathrm{Ind}_{n-1}^n(e_{n-1}\mathcal{S}^{n-1})), \Omega_n(P)) \\
&\cong \mathrm{Hom}_{\mathcal{H}_n}(\Omega_n(\mathrm{Ind}_{n-1}^n(e_{n-1}\mathcal{S}^{n-1})), Pe_{\mathcal{H}_n})
\end{aligned}$$

By [Wad14, Corollary 4.18(iv)], this is then isomorphic to:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{H}_n}({}^{\mathcal{H}}\mathrm{Ind}_{n-1}^n(e_{n-1}\mathcal{S}^{n-1}e_{\mathcal{H}_{n-1}}), Pe_{\mathcal{H}_n}) &\cong \mathrm{Hom}_{\mathcal{H}_n}(({}^{\mathcal{H}}\mathrm{Ind}_{n-1}^n(V^{\otimes(n-1)}), Pe_{\mathcal{H}_n}) \\
&\in_{\oplus} \mathrm{Hom}_{\mathcal{H}_n}(e_n\mathcal{S}^n e_{\mathcal{H}_n}, Pe_{\mathcal{H}_n}) \\
&\cong \mathrm{Hom}_{\mathcal{S}^n}(e_n\mathcal{S}^n, P) \\
&\cong Pe_n
\end{aligned}$$

Therefore, there is a functor $\mathcal{Q}_{n-1}$ such that on projective modules

$$\left(\Phi_{n-1} \circ \mathrm{Res}_{n-1}^n\right) \oplus \mathcal{Q}_{n-1} = \Phi_n$$

Applying the Five Lemma to the exact sequences obtained by applying these functors to a projective resolution for an arbitrary $\mathcal{S}^n$ module

$$P_2 \to P_1 \to M \to 0$$

again gives us the result for general M. Therefore, $\mathrm{Res}_{n-1}^n(M)e_{n-1} \hookrightarrow Me_n$. $\square$

Corollary 5.3.5. *Given $M \in \mathcal{S}^n$ -mod,*

$$Me_n = 0 \implies \mathrm{Res}_{n-1}^n(M)e_{n-1} = 0$$

Theorem 5.3.6. *The induction and restriction functors of [Wad14] induce functors*

$$\mathrm{Ind}_n^{n+1} : \mathcal{L}^n\text{-mod} \to \mathcal{L}^{n+1}\text{-mod}$$

$$\mathrm{Res}_n^{n+1} : \mathcal{L}^n\text{-mod} \to \mathcal{L}^{n-1}\text{-mod}$$

compatible with the sequence of quotient functors in Theorem 1.0.2.

5.3.2 Crystal Graph Actions

Remark 5.3.7. *We let $m = 2n + \delta$ for some $\delta \in \{0, 1\}$. We choose square roots of q and Q to compare with the conventions of [Bru15].*

Lemma 5.3.8. *The categories $\bigoplus_n \mathcal{L}^n\text{-mod}$ and $\bigoplus_n \mathcal{S}^n\text{-mod}$ carry compatible quantum Heisenberg actions of level 2 by $\mathcal{H}eis_2(q^{-1/2} - q^{1/2}, \sqrt{-1/Q})$ for all choices of $q, Q \in \Bbbk$ with $E = \mathrm{Res}, F = \mathrm{Ind}$.*

Proof. The analogous result for modules over the Hecke algebra is proven in [BSW20b, Th. 6.3]; in the notation of that paper, we have $f(u) = u^2 + (1 - Q)u - Q$, so $t = \sqrt{-Q}$. First, note that the Schur functor is fully faithful on projectives and that the induction and restriction functors on Hecke modules commute with Ω_n. By standard arguments (see [Sha11, Th. 5.1], [BSW20b, Th. 7.1]), this allows us to uniquely define corresponding actions on $\bigoplus_n \mathcal{L}^n\text{-proj}$ and $\bigoplus_n \mathcal{S}^n\text{-proj}$.

Writing arbitrary modules in these categories as cokernels of maps between projectives, and then applying the relevant exact functors and the Five Lemma gives us the general result. □

The most important part of this action is the endomorphism $x\colon \mathrm{Res}_{n-1}^n \to \mathrm{Res}_{n-1}^n$ induced by multiplication by the Jucys–Murphy element L_n, and its mate $x^*\colon \mathrm{Ind}_n^{n-1} \to \mathrm{Ind}_n^{n-1}$. By [BSW20a, Theorem A], this allows us to construct compatible Kac–Moody categorical actions on $\bigoplus_n \mathcal{L}^n\text{-mod}$ and $\bigoplus_n \mathcal{S}^n\text{-mod}$.

Lemma 5.3.9. *The sets of simples in $\bigoplus_n \mathcal{L}^n\text{-mod}$ and $\bigoplus_n \mathcal{S}^n\text{-mod}$ form a crystal graph, with Kashiwara crystal operators $\tilde{e}_u, \tilde{f}_u$ defined as follows:*

- *If $E_u L \neq 0$, then $\tilde{e}_u(L)$ is isomorphic to the unique simple head (equivalently, simple submodule) of $E_u L$*
- *If $F_u L \neq 0$, then $\tilde{f}_u(L)$ is isomorphic to the unique simple head and submodule of $F_u L$*

The set of simples of of each category is identified with (possibly a subset of) the set of bipartitions of n. Considering all possible bipartitions, the crystal action can be seen as follows: let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition of n, and let $u \in U$.

- Let $\mathcal{AR}_u$ be the set of boxes of residue u in the Young diagram of λ which are either addable or removable.
- Order the boxes in $\mathcal{AR}_u$ via $(r, c, \ell) > (r', c', \ell')$ if $\ell < \ell'$ or if $\ell = \ell'$ and $c > c'$. Visually, we are moving south-west along the first component and then the second.
- List the boxes in $\mathcal{AR}_u$ in decreasing order, representing removable boxes with an R and addable ones with an A. Reduce the sequence by iteratively cancelling all occurrences of the

subsequence "RA". The Kashiwara operator $\tilde{f}_u$ acts by adding the lowest remaining addable box (which corresponds to the last A in the reduced sequence) and $\tilde{e}_u$ acts by removing the highest remaining removable box (which corresponds to first R in the reduced sequence).

The weight of a node λ is $\mathrm{wt}(\lambda) = \Lambda - \sum_{u \in U} b_\lambda(u)\alpha_u$, where $b_\lambda(u)$ is the number of boxes of residue u in the Young diagram of λ.

Since the Schur functor and the quotient induced by the idempotent e_n commute with the induction and restriction functors behind these categorical actions, these quotients will delete entire connected components of the crystal graph at a time. That is:

Corollary 5.3.10. *The subsets of LNX_o and LNX_e bipartitions (and their complements) are closed under the action of the Kashiwara operators. Any crystal graph component with non-empty intersection with $\Lambda^B_{n,m}$ for m odd consists of LNX_o bipartitions, and similarly for m even.*

As a result, we can determine the set of simples $\mathcal{L}^n(m)$ over all n by testing a single simple in each connected component of the graph - whether or not it is LNX_* will tell us the behaviour of all other simples in the component.

Example 5.3.11. *Let $q = Q = -1$. In this case $U = \{1, -1\}$ and the first few levels of the crystal graph are shown in Figure 5.1. For example, $(\emptyset, (2))$ has two addable boxes with residue -1: $(1,1,1) > (3,1,2)$. This gives the sequence AA. Nothing is cancelled, so $\tilde{e}_{-1}$ will add the lower ($(3,1,2)$) resulting in $(\emptyset, (3))$. A second application of $\tilde{e}_{-1}$ would add $(1,1,1)$ to give $((1),(3))$. On the other hand, $\mathcal{AR}_1 = \{(2,1,2) > (1,2,2)\}$, so the sequence is RA. This reduces to the empty sequence, so both $\tilde{e}_1$ and $\tilde{f}_1$ send this diagram to zero.*

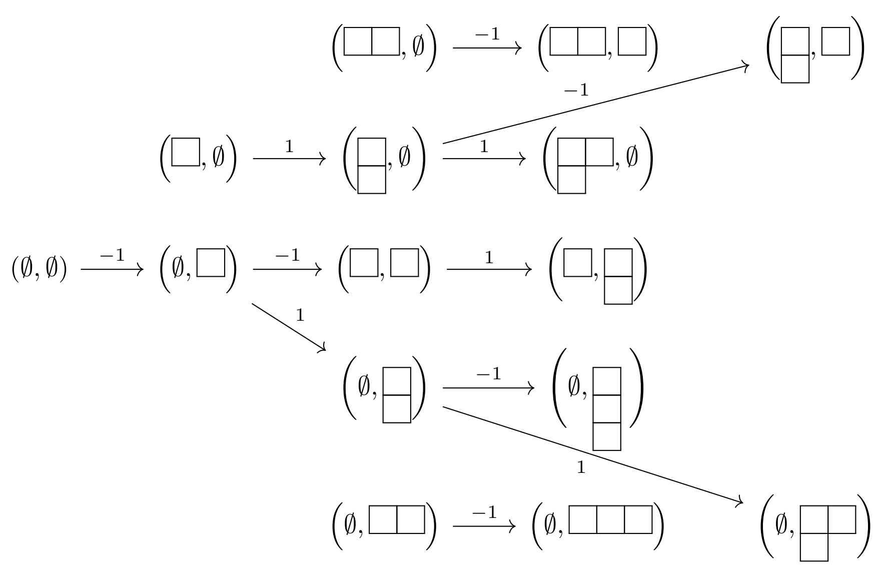

Figure 5.1: The crystal structure when $q = Q = -1$.

Example 5.3.12. *Now, consider the case $Q = -q$ for q generic. The first few levels of the graph are shown in Figure 5.2.*

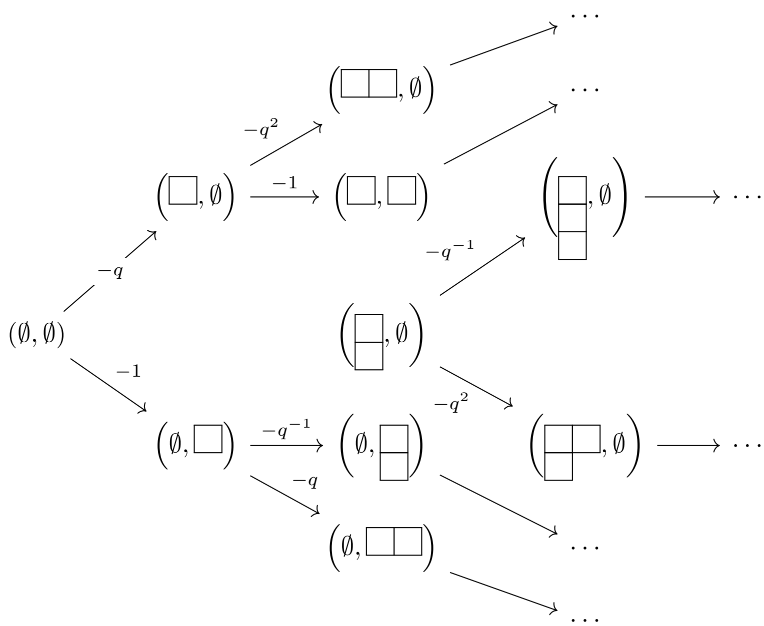

Figure 5.2: The crystal structure of Section 5.1.1 when $Q = -q$ for q generic.

Kleshchev bipartitions will be those in the component containing $(\emptyset, \emptyset)$. By the computations in Section 5.1.1, in Figure 5.1 all components are LNX_o, while the one containing $((1)^2, \emptyset)$ is not LNX_e, whereas in Figure 5.2, the component containing $((1^2), \emptyset)$ is neither LNX_o nor LNX_e.

5.4 The Main Theorem

We have seen already that the behaviour for small values of m is erratic - this section deals only with $m \geq 2n$, when the cells of $\mathcal{L}^n(m)$ are indexed by the full set of bipartitions of n. Recall Ariki's polynomial which determines the semisimplicity of the Hecke algebra:

$$P_n(Q, q) = \prod_{m=2}^{n-1} \frac{q^m - 1}{q - 1} \cdot \prod_{i=1-n}^{n-1} (Q + q^i) \neq 0$$

We call the second factor $f_n(Q, q)$, and expand it further as a product $f_n(Q, q) = b_n(Q, q) \cdot h_n(Q, q)$ where

$$b_n(Q, q) := \prod_{i=-(n-1)}^{1-\lfloor \frac{n}{2} \rfloor} (Q + q^i) \qquad h_n(Q, q) := \prod_{i=2-\lfloor \frac{n}{2} \rfloor}^{n-1} (Q + q^i)$$

The first factor says that if q is a root of unity of small order ($q^e = 1$ where $e < n$), the Hecke algebra will fail to be semisimple. While this does not guarantee that quasi-hereditarity of $\mathcal{L}^n(m)$ will fail, we will see that the second factor can take care of these cases as well.

We can summarize the cases covered thus far:

- If $P_n(Q,q) \neq 0$, the Hecke algebra is semisimple and $\mathcal{L}^n(m)$ is quasi-hereditary.
- If $f_n(Q,q) \neq 0$, [LNX20, Corollary 6.1.1] tells us that $\mathcal{L}^n(m)$ is quasi-hereditary.
- If $\prod_{i=1}^{n-1}(Q+q^i) = 0$, the bipartition $((1^n), \emptyset)$ is not LNX_m, so $\mathcal{L}^n(m)$ is not quasi-hereditary for m large of either parity.
- If $\prod_{i=2-\lfloor \frac{n}{2} \rfloor}^{1}(Q+q^i) = 0$, the bipartition $((\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor), \emptyset)$ is not LNX_o, so $\mathcal{L}^n(m)$ is not quasi-hereditary for m large odd.
- $\mathcal{L}^2(m)$ is quasi-hereditary if and only if $f_2(Q,q) \neq 0$ for m large even and if and only if $h_2(Q,q) \neq 0$ for m large odd.

Overall, we have established the following:

$Q = -q^i$	$i < -(n-1)$	$-(n-1), \ldots, 1 - \lfloor \frac{n}{2} \rfloor$	$2 - \lfloor \frac{n}{2} \rfloor, \ldots, 0$	$1, \ldots, n-1$	$n-1 < i$
$m \geq 2n$ odd	qh	?	not qh	not qh	qh
$m \geq 2n$ even	qh	?	?	not qh	qh

5.4.1 The Odd Case

Theorem 5.4.1. *Assume $m \geq 2n$ is odd. The algebra $\mathcal{L}^n(m)$ is quasi-hereditary if and only if $h_n(Q,q) \neq 0$.*

Proof. We have computed the case $n = 2$ (see Section 5.1.1) which will serve as a base case. Assume that the theorem holds for $\mathcal{L}^{n-1}$.
($\Leftarrow$) If $h_n(Q,q) = 0$ then $Q = -q^k$ for some $k \in \{2 - \lfloor \frac{n}{2} \rfloor, \ldots, n-1\}$. If $k > 0$, then the bipartition $((1^n), \emptyset)$ is not LNX_o, and if $k \leq 0$ then $((\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor, \emptyset)$ is not LNX_o by Corollary 5.2.4, so $\mathcal{L}^n(m)$ loses a simple and is not quasi-hereditary.

($\Rightarrow$) Assume that $h_n(Q,q) \neq 0$. If $f_n(Q,q)$ were also nonzero, quasi-hereditarity would be guaranteed by [LNX20, Cor 6.1.1]. If not, we must have $b_n(Q,q) = 0$.

First, note that under these assumptions q cannot be a small root of unity: if $q^e = 1$ for some $1 < e < n$ we would have $k \equiv k' \mod e$ for some $1 \leq k' \leq e \leq n-1$, meaning that $h_n(Q,q) = 0$, a contradiction.

Now, assume that we have a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of n that is not LNX_o. Applying the Kashiwara operator $\tilde{e}_u(\lambda)$ will either give us zero, or a bipartition of $n-1$ which is not LNX_o by Corollary 5.3.10. But then $\mathcal{L}^{n-1}(m)$ is not LNX_o, and $h_{n-1}(Q,q) = 0$ by the inductive hypothesis. Since $h_{n-1}(Q,q)$ divides $h_n(Q,q)$, this is a contradiction, and so we must have that $\tilde{e}_u(\lambda) = 0$ for all $u \in U$, i.e. that λ is highest weight.

By Lemma 5.2.5 we know that λ cannot be of the form $(\lambda^{(1)}, \emptyset)$. Furthermore, by Lemma 4.3.1, $\lambda^{(1)}$ must have at least two nonzero rows, otherwise it would be in $\Lambda^B_{n,m}$ and would therefore be LNX_m. So, consider the final box $(r, c, 2)$ in the last row of $\lambda^{(2)}$. It will be have residue $u = -q^{c-r}$. Since λ must be highest weight, this removable box must be cancelled in the sequence $\mathcal{AR}_u$, so we must have an addable box of the same residue that is lower in the order (otherwise we could apply the operator $\tilde{e}_u$ to remove it). The only addable or removable box lower in the order is $(r+1, 1, 2)$, which will have residue $-q^r$. If $(r, c, 2)$ and $(r+1, 1, 2)$ share a residue, then $-q^{c-r} = -q^r$ which implies that e divides c, meaning $1 \leq e \leq c < n$, a contradiction. Therefore, no such λ can exist. □

5.4.2 The Even Case

Theorem 5.4.2. *Assume $m \geq 2n$ is even. The algebra $\mathcal{L}^n(m)$ is quasi-hereditary if and only if $f_n(Q,q) \neq 0$*

Proof.
($\Leftarrow$) This is precisely [LNX20][Cor 6.1.1]: If $f_n(Q,q) \neq 0$ then $\mathcal{L}^n(m)$ is quasi-hereditary.
($\Rightarrow$) Assume that $f_n(Q,q) = 0$, so $Q = -q^k$ for some $k \in \{-(n-1), \ldots, n-1\}$. If $k \geq 0$ then $((1^n), \emptyset)$ is not LNX_m by Corollary 5.2.7; if $k < 0$ then $((n), \emptyset)$ is not LNX_m. In either case $\mathcal{L}^n(m)$ loses a simple and so is not quasi-hereditary and not Morita equivalent to $\mathcal{S}^n(m)$. □

To summarize, we can now complete our table:

$Q = -q^i$	$i < -(n-1)$	$-(n-1), \ldots, 1 - \lfloor \frac{n}{2} \rfloor$	$2 - \lfloor \frac{n}{2} \rfloor, \ldots, 0$	$1, \ldots, n-1$	$n - 1 < i$
$m \geq 2n$ odd	qh	qh	not qh	not qh	qh
$m \geq 2n$ even	qh	not qh	not qh	not qh	qh

Chapter 6

The Blocks of $\mathcal{L}^n$

6.1 Cell Linkage

We now look to the block structure of $\mathcal{L}^n$. We recall our series of idempotent truncations (equivalently, exact functors between module categories):

$$\mathcal{S}^n\text{-mod} \xrightarrow{e_n} \mathcal{L}^n\text{-mod} \xrightarrow{e_{\mathcal{H}_n}} \mathcal{H}_n\text{-mod}$$

These are all instances of a more general case that we have already seen: assume that $\mathcal{A}$ is a cellular algebra with cell datum $(\Lambda, \mathcal{T}, C, *)$, and that $e \in \mathcal{A}$ is an idempotent preserved by the cellular involution, so that $e\mathcal{A}e$ is also cellular (see Proposition 2.2.7). Since this idempotent preserves the cellular structure, cell modules are sent to cell modules indexed by the same poset element:

$$\Delta^\lambda_{\mathcal{A}} e = \Delta^\lambda_{e\mathcal{A}e}$$

allowing us to use the same cell linkage on both levels of the truncation. Recall that simples are obtained as quotients of cell modules $\mathcal{D}^\lambda_{\mathcal{A}} \cong \Delta^\lambda_{\mathcal{A}}/\langle\ ,\ \rangle$ and so are also indexed by poset elements (when they are nonzero).

Proposition 6.1.1. *The cell modules $\Delta^\lambda_{\mathcal{A}}$ and $\Delta^\mu_{\mathcal{A}}$ lie in the same block of $\mathcal{A}$ if $\Delta^\lambda_{e\mathcal{A}e}$ and $\Delta^\mu_{e\mathcal{A}e}$ lie in the same block of $e\mathcal{A}e$.*

Proof. This is the first direction of the proof of [LM07, Proposition 2.3], which readily generalizes. Assume that $\Delta^\lambda_{e\mathcal{A}e}$ and $\Delta^\mu_{e\mathcal{A}e}$ lie in the same block of $e\mathcal{A}e$. By assumption, λ and μ are cell-linked in terms of $e\mathcal{A}e$ (see Definition 2.1.5). So, we need only consider the case where the modules $\Delta^\lambda_{e\mathcal{A}e}$ and $\Delta^\mu_{e\mathcal{A}e}$ have a common (nonzero) composition factor. By the transitivity of cell linkage and Lemma 2.1.6, we can further assume that this common composition factor is $\mathcal{D}^\mu_{e\mathcal{A}e}$, and so $[\Delta^\lambda_{e\mathcal{A}e} : \mathcal{D}^\mu_{e\mathcal{A}e}] \neq 0$. Since the idempotent e sends cell modules to cell modules and simples to simples (or zero, but we have assumed that this is not the case),

$$\begin{aligned}[\Delta^\lambda_{\mathcal{A}} : \mathcal{D}^\mu_{\mathcal{A}}] &= [\Delta^\lambda_{\mathcal{A}} e : \mathcal{D}^\mu_{\mathcal{A}} e] \\ &= [\Delta^\lambda_{e\mathcal{A}e} : \mathcal{D}^\mu_{e\mathcal{A}e}]\end{aligned}$$

This last quantity is nonzero, meaning that $\mathcal{D}^\mu_\mathcal{A}$ is a composition factor of $\Delta^\lambda_\mathcal{A}$, and so is a common composition factor for both $\Delta^\lambda_\mathcal{A}$ and $\Delta^\mu_\mathcal{A}$, showing that they lie in the same block of $\mathcal{A}$.

The general case follows by induction on the length of the cell linkage sequence. □

Applying this in the situations of $\mathcal{L}^n = e_n \mathcal{S}^n e_n$ and $\mathcal{H}_n = e_{\mathcal{H}_n} \mathcal{L}^n e_{\mathcal{H}_n} = e_{\mathcal{H}_n} \mathcal{S}^n e_{\mathcal{H}_n}$ tells us that $\mathcal{S}^n$ has at most as many blocks as $\mathcal{L}^n$, which itself has at most as many blocks as $\mathcal{H}_n$.

Proposition 6.1.2. *Let $M \in \mathcal{H}_n$-mod be such that $\mathcal{H}_n \Subset M$. Then $\mathrm{End}_{\mathcal{H}_n}(M)$ has at least as many blocks as $\mathcal{H}_n$.*

Proof. This is essentially the second direction of [LM07, Prop 2.3]. Let $\mathcal{H}_n = B_1 \oplus \cdots \oplus B_k$ be its block decomposition. Since $1 = \sum_i e_i$, where the block idempotents e_i are the identities in the subalgebras B_i, we obtain the decomposition $M\mathcal{H}_n = MB_1 \oplus \cdots \oplus MB_k$. Since this is a block decomposition, if $i \neq j$ then $\mathrm{Hom}_{\mathcal{H}_n}(B_i, B_j) = 0$, and so $\mathrm{Hom}_{\mathcal{H}_n}(MB_i, MB_j) = 0$. Therefore,

$$\begin{aligned}\mathrm{End}_{\mathcal{H}_n}(M) &= \mathrm{End}_{\mathcal{H}_n}(MB_1 \oplus \cdots \oplus MB_k) \\ &= \bigoplus_{i=1}^k \mathrm{End}_{\mathcal{H}_n}(MB_i)\end{aligned}$$

so that the endomorphism algebra has at least as many blocks as $\mathcal{H}_n$. □

Applying this to both $\mathcal{S}^n = \mathrm{End}_{\mathcal{H}_n}(M^{\Lambda_{n,m}})$ and $\mathcal{L}^n = \mathrm{End}_{\mathcal{H}_n}(M^{\Lambda^B_{n,m}})$, and combining it with the various applications of Proposition 6.1.1 aligns the blocks of all three algebras.

Corollary 6.1.3. *Let $\lambda, \mu \in \Lambda^+_{n,m}$. The following are equivalent:*

- *$C^\lambda_\mathcal{S}$ and $C^\mu_\mathcal{S}$ are in the same block as $\mathcal{S}^n$-modules*
- *$C^\lambda_\mathcal{L}$ and $C^\mu_\mathcal{L}$ are in the same block as $\mathcal{L}^n$-modules*
- *$C^\lambda_\mathcal{H}$ and $C^\mu_\mathcal{H}$ are in the same block as $\mathcal{H}_n$-modules*

6.2 Residue Equivalent Bipartitions and Blocks

Recall that $\mathrm{Res}(\Lambda^+_{n,m})$ is the set of all residues occurring in diagrams of elements of $\Lambda^+_{n,m}$. Given $u \in \mathrm{Res}(\Lambda^+_{n,m})$, let $b_\lambda(u)$ denote the number of boxes of this residue occurring in the diagram of λ.

Definition 6.2.1. *Given $\lambda, \mu \in \Lambda^+_{n,m}$, say that λ and μ are **residue equivalent** if $b_\lambda(u) = b_\mu(u)$ for all $u \in \mathrm{Res}(\Lambda^+_{n,m})$.*

It is shown in [LM07, Theorem 2.11] that the cell modules of the algebras $\mathcal{S}^n$ and $\mathcal{H}_n$ corresponding to the bipartitions λ, μ are in the same block if and only if λ and μ are residue equivalent. Combining this with Corollary 6.1.3, we find that:

Corollary 6.2.2. *The cell modules of the algebras $\mathcal{S}^n, \mathcal{L}^n$ and $\mathcal{H}_n$ corresponding to bipartitions $\lambda, \mu \in \Lambda^+_{n,m}$ lie in the same block if and only if λ and μ are residue equivalent.*

Since the residues of boxes correspond to eigenvalues of the Jucys–Murphy elements L_i, the block idempotent e_b is the projection to a simultaneous generalized eigenspace for the symmetric polynomials in the L_i's.

By Lemma 5.3.9, the simples, labelled by bipartitions, of $\bigoplus_n \mathcal{L}^n$ form a crystal. The block decomposition of the cyclotomic q-Schur algebra is given by equivalence classes of residue functions b (equivalently, by the weight function of the crystal): $\mathcal{S}^n \cong \bigoplus_b \mathcal{S}(b)$. Theorem 1.0.2 and Corollary 6.1.3 give the corresponding decomposition of $\mathcal{L}^n$:

$$\mathcal{L}^n \cong \bigoplus_b \mathcal{L}(b)$$

where $\mathcal{L}(b) = e_n \mathcal{S}(b) e_n$. Similarly, the blocks of $\mathcal{H}_n$ are given by $\mathcal{H}(b) = e_{\mathcal{H}_n} \mathcal{S}(b) e_{\mathcal{H}_n} = e_{\mathcal{H}_n} \mathcal{L}(b) e_{\mathcal{H}_n}$.

6.2.1 Highest Weights and Dominant Weights

By Lemma 5.3.9, the crystal graph encodes the structure of an integrable categorical module $\mathcal{C}$ over a Kac–Moody algebra $\mathfrak{g}$. The **support** of such a module is the set of weights χ such that $\mathcal{C}_\chi \neq 0$, i.e. such that the weight space is nonzero.

Each connected component of the graph corresponds to an irreducible highest weight module $V(\Lambda)$ over $\mathfrak{g}$ (see [Web23, §3.2]). The support of each such module will then be the weights which occur in that connected component.

A bipartition λ is highest weight if $\tilde{e}_u(\lambda) = 0$ for all $u \in U$. In terms of the crystal action defined in Lemma 5.3.9, a bipartition λ will be highest weight if for every residue u, the reduced sequence $\mathcal{AR}_u$ (i.e. after cancelling all RA-pairs) contains no Rs. This means that each removable u-box has an addable u-box lower than it in the bipartition which cancels it.

The empty bipartition has no removable boxes, and precisely two addable ones: these are of residue Q and -1. So, when constructing our crystal graph, $\Lambda = \mathrm{wt}((\emptyset, \emptyset)) = \Lambda_Q + \Lambda_{-1}$ as a sum of fundamental weights. Other weights in this crystal component are then obtained by applying Kashiwara operators $\tilde{e}_u$, and will be of the form $\Lambda_Q + \Lambda_{-1} - \sum_u b(u)\alpha_u$, where we have applied each Kashiwara operator $\tilde{e}_u$ $b(u)$ times, and added $b(u)$ boxes of this residue. Similarly, weights in other crystal components can be computed from the weight of the corresponding highest weight vectors.

6.3 Quasi-hereditary Blocks of $\mathcal{L}^n(m)$

As we have seen, the block decomposition of $\mathcal{L}^n$ partitions simples $D^\lambda_{\mathcal{L}}$ by residue equivalence classes, which are equivalent to weights. If a bipartition has residue function b, we will say that its weight is $\chi_b := \Lambda - \sum_{u \in U} b(u)\alpha_u$.

The block idempotents e_b are invariant under the cellular anti-involution, so the resulting block subalgebras are again cellular by Proposition 2.2.7. The blocks of a highest weight category are themselves highest weight, so each block algebra $\mathcal{S}(b)$ will be quasi-hereditary. As a result, we can again apply Corollary 2.2.9: the block algebra $\mathcal{L}(b)$ will be quasi-hereditary if and only if the idempotent e_n induces a Morita equivalence to $\mathcal{S}(b)$. This will fail when there is at least one bipartition λ of weight χ_b such that $D^\lambda_{\mathcal{L}} e_n = 0$.

Remark 6.3.1. *By Theorem 1.0.2, we only need to consider the cases when:*

- *For m large odd, $Q = -q^k$ for some $2 - \lfloor \frac{n}{2} \rfloor \leq k \leq n-1$.*
- *For m large even, $Q = -q^k$ for some $1 - n \leq k \leq n-1$.*

So, for the remainder of this thesis, we assume that $Q = -q^k$ for some $k \in \mathbb{Z}$.

Let λ be a bipartition in the crystal of weight $\mathrm{wt}(\lambda) = \chi_b$. Applying an iterated Kashiwara operator $\sigma_u(\lambda)$ corresponding to a Weyl group reflection (see Definition 2.3.2) puts us at a bipartition μ which we know will be LNX_* if and only if λ is LNX_*, by closure of the Kashiwara operators. By Theorem 2.3.4, the weight of the new bipartition will be $\mathrm{wt}(\mu) = s_u(\chi_b) = \chi_b - \langle \chi_b, \alpha_u^\vee \rangle \alpha_u$.

Given a series of residues $u_1, \ldots u_j$, let $\sigma_{\vec{u}}$ denote the series of operators $\sigma_{u_m} \cdot \cdots \cdot \sigma_{u_1}$, and similarly for a simple reflection $s_{\vec{u}}$. If $k = \langle \chi_b, \alpha_u^\vee \rangle < 0$, replace λ with $\sigma_u(\lambda) = \tilde{e}_u^{-k}(\lambda)$. This operation removes k boxes of residue u from the bipartition.

Repeating this process over all residues, we can reduce to the case where $\langle \chi_b, \alpha_u^\vee \rangle \geq 0$ for all $u \in U$ — i.e., where χ_b is a dominant weight. So, λ is LNX_* if and only if any element in its orbit under the Weyl group action is, and we need only check the dominant weight in a W-orbit.

Applying the same series of iterated Kashiwara operators (corresponding to Weyl group reflections) to all bipartitions μ of the same weight will result in a set of bipartitions $\sigma_{\vec{u}}(\mu)$ which are all cell-linked to each other, but which contain strictly fewer boxes than the original bipartition λ. Since this set was obtained by reflections, the operation is reversible. Therefore, the series of Kashiwara operators $\sigma_{\vec{u}}$ induces a bijection between the simples in $\mathcal{L}(\chi_b)$ and $\mathcal{L}(s_{\vec{u}}\chi_b)$. This preserves crystal graph components, and therefore preserves being LNX_*. Given a dominant weight $\chi_d \in X$, we will call the block algebra $\mathcal{L}(d)$ a **dominant block algebra**. Note that, by the bijection between simples, $\mathcal{L}(b)$ will be quasi-hereditary if and only if $\mathcal{L}(d)$ is.

Lemma 6.3.2. *A block algebra $\mathcal{L}(b)$ will be quasi-hereditary if and only if $\mathcal{L}(w \cdot b)$ is, where $w \in W$ is an element of the Weyl group. In particular, we can restrict to the case when b is a dominant weight.*

Let $K_+ = \{k \in \mathbb{Z}_{>0} \mid Q = -q^k\}$, and $K_- = \{k \in \mathbb{Z}_{\geq 0} \mid Q = -q^{-k}\}$. Given $k \in K_+$, by Corollary 5.2.4 the bipartition $\left(\left(1^{k+1}\right), \emptyset\right)$ is not LNX_o. For $k \in K_-$, the bipartition $\left(\left((k+2)^2\right), \emptyset\right)$ is not LNX_o. Let $K = K_+ \cup K_-$ and let

$$B_K^o = \left\{\left((1^{k+1}), \emptyset\right) \mid k \in K_+\right\} \cup \left\{\left(\left((k+2)^2\right), \emptyset\right) \mid k \in K_-\right\}$$

By Corollary 5.2.4 this is a set of "bad" (non-LNX_o) bipartitions based on K.

Note that any $\nu \in B_K^o$ is highest weight: if

$$\nu = \left(\begin{array}{|c|}\hline -q^k \\ \hline -q^{k-1} \\ \hline \dots \\ \hline -q \\ \hline -1 \\ \hline \end{array}, \emptyset \right)$$

then there are no removable boxes of residue $-q^r$ for $2 \leq r \leq k-1$, so $\tilde{e}_{-q^r}(\nu) = 0$ for such r. The only removable box is $(k+1, 1, 1)$ which is of residue -1, but $\mathcal{AR}_{-1} = R_{(k+1,1,1)}, A_{(1,1,2)}$ so $\tilde{e}_{-1}(\nu) = 0$. Similarly, if

$$\nu = \left(\begin{array}{|c|c|c|c|}\hline -q^{-k} & -q^{k+1} & \dots & -q \\ \hline -q^{-k-1} & -q^{-k} & \dots & -1 \\ \hline \end{array}, \emptyset \right)$$

then the only removable box is of residue -1, and $\mathcal{AR}_{-1} = R_{(2,k,1)} A_{(1,1,2)}$, telling us that ν is highest weight. For the even case, we take the same set K but instead take

$$B_K^e = \{((1^{k+1}), \emptyset) \mid k \in K_+\} \cup \{((k+1), \emptyset) \mid k \in K_-\}$$

$$\nu = \left(\begin{array}{|c|c|c|c|c|}\hline Q & Qq & \dots & Qq^{k-1} & Qq^k \\ \hline \end{array}, \emptyset \right)$$

to be our generating set of non-LNX_e highest weight bipartitions.

Theorem 6.3.3. *A dominant block algebra $\mathcal{L}(d)$ is quasi-hereditary if and only if $K \neq \emptyset$ and there exists a bipartition λ such that $\mathrm{wt}(\lambda) = \chi_d$ such that λ contains some $\nu \in B_K^*$ as a subdiagram.*

Proof. ($\Leftarrow$) Assume that there exists a λ such that $\mathrm{wt}(\lambda) = \chi_d$ which contains some $\nu \in B_K^*$ as a subdiagram. The bipartition ν is non-LNX_o by Corollary 5.2.4 and is highest weight by the discussion above. Let its weight be $\mathrm{wt}(\nu) = \chi_b$. By [Web23, Lemma 3.6], there is a bipartition μ in the crystal component of ν with $\mathrm{wt}(\mu) = \chi_d$ if and only if there is an element w in the Weyl group such that $w\chi_d$ is dominant and $w\chi_d \leq \chi_b$ in the weight lattice ordering. This holds by taking w to be the identity, since we are already considering a dominant weight, and χ_d is obtained from χ_b by subtracting simple roots. Therefore, there is a bipartition μ in the crystal component of ν — this will contain ν as a subdiagram, also be non-LNX_* and will be residue equivalent to λ, meaning that μ is the bipartition we set out to find.

($\Rightarrow$) Assume that $\mathcal{L}(d)$ is not quasi-hereditary. Then there is at least one non-LNX_* bipartition λ such that $\mathrm{wt}(\lambda) = \chi_d$. Let μ be the bipartition of highest weight in the crystal component of λ. If $\mu^{(2)} = \emptyset$ by Corollary 5.2.4 it must contain a subdiagram of the form $\nu \in B_K^*$ as desired.

So, assume $\mu^{(2)} \neq \emptyset$. Let ℓ_2 be the number of nonzero parts in $\mu^{(2)}$, so $(\ell_2, \mu^{(2)}_{\ell_2}, 2)$ is a removable box of residue $-q^{1-\ell_2+\mu^{(2)}_{\ell_2}}$. Since μ is of highest weight, we must have a lower addable box of the same residue — the only option is $(\ell_2+1, 1, 2)$, which will have residue $-q^{1-\ell_2}$. So, we must have some e such that $(-q)^e = 1$, giving us $1 - \ell_2 + \mu^{(2)}_{\ell_2} \equiv 1 - \ell_2 \mod e$, i.e. that this row is of length ae for some $a \geq 1$. Since $Q = -q^k$ for some $k \in \mathbb{Z}$, and since any residue is generically of the form Qq^k or $-q^k$, we can take e minimal such that $(-u)^e = 1$ for all $u \in U$. Visually, $\mu^{(2)}$ must contain an $a \times e$ rectangle as a subdiagram with the following residues:

-1	$-q$	$-q^2$	$\dots$	$-q^{ae-1}$
$-q^{-1}$	-1	$\dots$	$-q^{ae-3}$	$-q^{ae-2}$
$\dots$				
$-q^{1-\ell_2}$	$-q^{2-\ell_2}$	$\dots$	$-q^{ae-\ell_2-1}$	$q^{ae-\ell_2}$

Let k_+ be minimal in K_+. Note that we must have $k_+ \leq e$, since otherwise would contradict the minimality of k_+.

If $k < e$, the column $\nu = ((1^{k_++1}), \emptyset)$ will have distinct residues $-1, \dots, -q^{k_+}$, all of which are contained in the $a \times e$ box. Again applying [Web23, Lemma 3.6], we can build a bipartition ρ of dominant weight $\mathrm{wt}(\rho) = \mathrm{wt}(\mu) < \mathrm{wt}(\nu)$ in the crystal component of ν.

As ν, and hence ρ must also be non-LNX_*, and since $\mathrm{wt}(\lambda) < \mathrm{wt}(\mu)$ (in the weight lattice ordering), there is another bipartition λ' with $\mathrm{wt}(\lambda') = \mathrm{wt}(\lambda)$ in the crystal component of ν which contains ν as a subdiagram which is non-LNX_* by closure of the Kashiwara operators. This is the desired bipartition containing a subdiagram from B_K^*.

If $k = e$ then ν will have two boxes of residue -1. Since μ is by assumption not LNX_*, it must contain at least one box in $\mu^{(1)}$ — this will be of residue -1. With this box and the $a \times e$ rectangle in $\mu^{(2)}$ we are still guaranteed enough boxes of appropriate residue to cover those of ν. The rest of the argument follows. $\square$

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