THE NULL GLUING PROBLEM AND CONSERVATION LAWS FOR THE MAXWELL EQUATIONS

by

saeyon mylvaganam

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University of Toronto

ABSTRACT

The Null Gluing Problem and Conservation Laws for the Maxwell Equations

Saeyon Mylvaganam

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We study the null gluing problem for Maxwell's equations along null hypersurfaces. By studying a weaker formulation of the gluing problem, which we call the kth-order gluing problem, we classify all possible conservation laws by proving that they are the only obstructions to gluing. We derive sets of conserved charges for the zeroth-order gluing problem along general null hypersurfaces and the first-order gluing problem along extremal horizons. We derive an elliptic structure related to a foliation with 2-spheres of a null hypersurface, using a similar method introduced in [9] by Aretakis. We also show the non-existence of zeroth-order conservation laws along extremal horizons and the non-existence of kth-order conservation laws for spherically symmetric extremal horizons by using a hierarchy of v-weighted integrals of the Maxwell equations. Finally, we determine how the space of these conserved charges changes under a change of foliation by understanding the gauge covariance of the elliptic structure.

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CONTENTS

1 INTRODUCTION

The *gluing problem* asks given two disjoint regions with prescribed data to extend the data in the connecting region to satisfy given constraint equations. Obstructions to gluing have provided deeper insights into the underlying constraint equations. For example, for the Einstein equations, they have provided insights into the rigidity properties of the geometry and for the wave equation, they have been used to understand the decay and asymptotics of solutions, as shown by Aretakis in [3] and [4].

In this thesis, we address the gluing problem for Maxwell's equations along null hypersurfaces and show that the obstructions take the form of *conservation laws*. The Maxwell equations are given by

$$\mathbf{Div}F = 0 \quad dF = 0$$

where the Maxwell tensor F is a 2-form defined on a Lorentzian manifold. We can show that these equations are equivalent to the following null decomposition

$$\frac{1}{\phi}\nabla\!\!\!/_4(\phi\Omega\underline{\alpha}) = +\frac{\Omega}{2}\left(d\!\!\!/\rho + 2\underline{\eta}\rho\right) - \frac{\Omega}{2}\left(d\!\!\!/\sigma + 2\underline{\eta}\sigma\right)^* + \underline{\hat{\chi}}^\sharp\cdot(\Omega\alpha) \tag{1.1}$$

$$\frac{1}{\phi}\nabla\!\!\!/_3(\phi\Omega\alpha) = -\frac{\Omega}{2}\left(d\!\!\!/\rho - 2\underline{\eta}\rho\right) - \frac{\Omega}{2}\left(d\!\!\!/\sigma - 2\underline{\eta}\sigma\right)^* + \hat{\chi}^\sharp\cdot(\Omega\underline{\alpha}) \tag{1.2}$$

$$\mathrm{d\!\!\!/iv}(\Omega\underline{\alpha}) = +\frac{1}{\phi^2}\underline{L}(\phi^2\rho) \tag{1.3}$$

$$\mathrm{cu\!\!\!/rl}(\Omega\underline{\alpha}) = +\frac{1}{\phi^2}\underline{L}(\phi^2\sigma) \tag{1.4}$$

$$\mathrm{d\!\!\!/iv}(\Omega\alpha) = -\frac{1}{\phi^2}L(\phi^2\rho) \tag{1.5}$$

$$\mathrm{cu\!\!\!/rl}(\Omega\alpha) = +\frac{1}{\phi^2}L(\phi^2\sigma) \tag{1.6}$$

on a general four-dimensional Lorentzian manifold $(\mathcal{M}, g)$ where $(\alpha, \underline{\alpha}, \rho, \sigma)$, defined in Definition 2.5, are the components of the F tensor. Here, the 4 and 3 indices refer to the outgoing and ingoing normalized null vector fields e_3 and e_4 and operators $\mathrm{d\!\!\!/iv}, \mathrm{cu\!\!\!/rl}$ are defined with respect to a family of Riemannian metrics $g\!\!\!/$, which are the restrictions of the metric g to the spherical sections S_v of the outgoing null hypersurface. The simplest example of a conserved charge is given by

$$Q_0(v) = \int_{S_v} \rho d\mu_{\not{g}} \quad Q_1(v) = \int_{S_v} \sigma d\mu_{\not{g}} \tag{1.7}$$

which for all solutions of (1.5) and (1.6) satisfy the conservation law

$$\partial_v Q_0(v) = 0 \quad \partial_v Q_1(v) = 0$$

In Section 1.1, we introduce the concept of conserved charges for Maxwell's equations, which includes the above example. The remaining charges will require a characterization of null hypersurfaces that possess such charges in terms of the kernel of an elliptic operator, a novel idea introduced by Aretakis in [9]. We show that these charges are all of the conserved quantities along the null hypersurface by showing that they are the only obstructions to the null gluing problem for Maxwell's equations. We study "gluing constructions for the characteristic initial value problem" (as outlined in Section 1.2), and show that the existence of conserved charges (in our sense) on a null hypersurface $\mathcal{H}$ is the primary obstruction to gluing.

Conservation laws on degenerate horizons for the wave equation are particularly important due to their role in the instability properties of extremal black holes, leading to the so-called "horizon instability of extremal black holes," as discussed in [6, 11, 40, 39, 41, 42, 45]. Studying Maxwell's equations on general backgrounds may shed new light on the global evolution of hyperbolic equations on different backgrounds.

1.1 CONSERVATION LAWS FOR MAXWELL'S EQUATIONS

We will first present basic geometric definitions that will be important when we define the notion of conservation laws on null hypersurfaces. For more details about the geometric setting, see Section 2.1.

Null foliations

Suppose we have a regular null hypersurface, denoted as $\mathcal{H}$, within a four-dimensional Lorentzian manifold, represented as $(\mathcal{M}, g)$. A null hypersurface $\mathcal{H} \subset \mathcal{M}$ is a surface within the Lorentzian manifold if and only if every normal vector field L to $\mathcal{H}$ is null, i.e. $g(L, L) = 0$. Note that since L is orthogonal to itself, it is also tangent to $\mathcal{H}$. The integral curves of the vector field L can be shown to be null geodesics along $\mathcal{H}$. In fact, $\mathcal{H}$ will be generated by said null geodesics. A foliation of a null hypersurface is a collection of sections S_v that vary smoothly with a parameter v and $\cup_v S_v = \mathcal{H}$. We take each section S_v to be diffeomorphic to a 2-sphere $\mathbb{S}^2$ (although this approach can be extended to topologies with higher genus). This foliation can be uniquely defined by three components: the choice of an initial section S_0, a smooth function denoted as Ω on $\mathcal{H}$, and a null geodesic vector field L_{geod} that is tangent to the null generators of $\mathcal{H}$ and satisfies $\nabla_{L_{geod}} L_{geod} = 0$.

By introducing the vector field $L = \Omega^2 \cdot L_{geod}$ on $\mathcal{H}$ and considering an affine parameter v of L such that $Lv = 1$, with $v = 0$ on S_0, we can define the level sets S_v of v on $\mathcal{H}$, which correspond to the leaves of the foliation $\mathcal{S}$. This foliation can be represented as $\mathcal{S} = \left\langle S_0, L_{geod}, \Omega \right\rangle$.

The flow of the vector field L on $\mathcal{H}$ provides a diffeomorphism Φ_v between the sections S_v and the initial section S_0. Additionally, all sections can be equipped with the standard metric of the unit sphere, denoted as $\not{g}_{\mathbb{S}^2}$, via the diffeomorphism Φ. The volume form on S_v with respect to $\not{g}_{\mathbb{S}^2}$ is denoted as $d\mu_{\mathbb{S}^2}$.

Given any section S_v, there exists a unique metric $\hat{g}$ that is conformal to the induced metric $\not{g}$, ensuring that the volume forms $d\mu_{\hat{g}}$ and $d\mu_{\mathbb{S}^2}$ are equal. This conformal factor is represented by ϕ:

$$\not{g} = \phi^2 \cdot \hat{g}. \tag{1.8}$$

Conservation laws

We have already stated that the quantities $Q_0(v)$ and $Q_1(v)$ are conserved along the null hypersurface $\mathcal{H}$. Consider the linear space $\mathcal{V}_{\mathcal{H}}$ consisting of all smooth vector fields on $\mathcal{H}$ which have vanishing Lie derivative in the e_4 direction along $\mathcal{H}$, i.e.

$$\mathcal{V}_{\mathcal{H}} = \left\{ X \in \mathfrak{X}(\mathcal{H}) \,:\, \not{\mathcal{L}}_4 X = 0 \right\}. \tag{1.9}$$

The Lie derivative $\not{\mathcal{L}}_4$ is the restriction of the Lie derivative $\mathcal{L}_4$ to the spherical sections S_v. Let $\mathcal{S} = \left(S_v\right)_{v \in \mathbb{R}}$ be a foliation of $\mathcal{H}$ and let ϕ be the conformal factor defined above. We define the linear space $\mathcal{W}$ to be the subspace of $\mathcal{V}_{\mathcal{H}}$ such that for all $\Theta \in \mathcal{W}$ and for all solutions $(\alpha, \underline{\alpha}, \rho, \sigma)$ to the Maxwell equations (3.4), the integrals

$$H_0^{\Theta} = \int_{S_v} \left(\Psi^{-1}(v)(W(v) - Q_0 Z_1(v) - Q_1 Z_2(v)) \right) \Theta d\mu_{\mathbb{S}^2}$$

are conserved (i.e. independent of v). That is,

$$\mathcal{W} = \left\{ \Theta \in \mathfrak{X}(\mathcal{H}) \,:\, \not{\mathcal{L}}_4 \Theta = 0, \partial_v H_0^{\Theta} = 0 \right\} \tag{1.10}$$

where

$$W(v) = W(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma})(v) = \underline{\alpha} - \underline{\hat{\chi}}^{\sharp} \cdot \not{\mathcal{D}}^{-1}(-(\rho - \overline{\rho}), (\sigma - \overline{\sigma}))$$

and $\Psi(v)$ is a $(1,1)$ tensor that only depends on the background metric, defined in (3.6). Also, the vector fields $Z_i(v) = \Psi(v) \int_1^v \Psi^{-1}(s) \cdot Y_i(s) ds$ are given by

$$Y_1 = (-\underline{\eta} - \underline{\hat{\chi}}^{\sharp} \cdot \not{\mathcal{D}}^{-1}((\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}), 0))$$
$$Y_2 = (\underline{\eta} - \underline{\hat{\chi}}^{\sharp} \cdot \not{\mathcal{D}}^{-1}((\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}), 0))^*$$

and finally the operator $\not{\mathcal{D}}$ is the Hodge operator, defined in Section 2.9. We make the following definition:

Definition 1.1. (Conservation laws on $\mathcal{H}$): A null hypersurface $\mathcal{H}$ is said to admit *zeroth-order conservation laws* with respect to a foliation $\mathcal{S}$ of $\mathcal{H}$ if

$$\dim \mathcal{W} \geq 1. \tag{1.11}$$

If (1.11) holds then we will refer to the space $\mathcal{W}$ and the number $\dim \mathcal{W}$ as the kernel and the dimension of the conservation laws, respectively. We will call the integrals (1.1) conserved charges.

The initial definition, Definition 1.1, may appear to impose significant constraints on the concept of conservation laws but, as demonstrated in Theorem 1.4, the conservation laws defined in Definition 1.1 prove to be the sole form of 'zeroth order' conservation laws that a null hypersurface $\mathcal{H}$ can admit.

We can also establish higher order conservation laws by involving higher derivatives of variables such as $\underline{\alpha}$. However, we restrict our analysis to Extremal Horizons and to null cones of Minkowski.

1.2 THE NULL GLUING PROBLEM

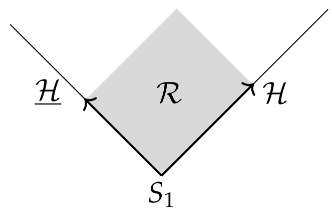
The null gluing problem for Maxwell's equations gives a formal way to represent the concept of conservation laws on null hypersurfaces accurately. We will introduce this problem in this section.

Consider a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$. Let S_1 be the 2-sphere that is the intersection of two regular null hypersurfaces $\mathcal{H}$ and

$\underline{\mathcal{H}}$. We define u, v to be optical functions such that $\mathcal{H} = \{u = 1\}$ and $\underline{\mathcal{H}} = \{v = 1\}$. Following the description and well-posedness of the null initial value problem of hyperbolic equations in [48], we prescribe characteristic initial data for the Maxwell equations (1.1)-(1.6) corresponding to the restriction of $(\alpha, \underline{\alpha}, \rho, \sigma)$ on the union $\mathcal{H} \cup \underline{\mathcal{H}}$. Let us define the following regions

$$\mathcal{A} = \mathcal{H} \cap \{1 \leq v \leq v_f\}$$
$$\underline{\mathcal{A}} = \underline{\mathcal{H}} \cap \{1 \leq u \leq u_f\}$$

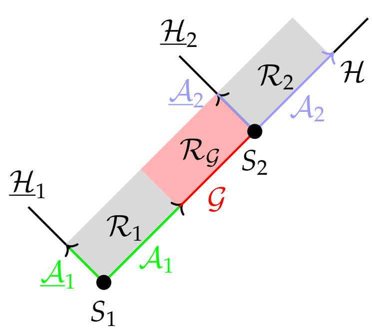
for some $v_f > 1, u_f > 1$. Let us further prescribe initial data $(\alpha, \underline{\alpha}, \rho, \sigma)$ on these regions. By [48], there exists a smooth unique solution to Maxwell's equations in the domain of dependence $\mathcal{R}$, shown below.



We can now state the gluing construction problem. Consider our original null hypersurface $\mathcal{H}$ and two conjugate null hypersurfaces $\underline{\mathcal{H}}_1$ and $\underline{\mathcal{H}}_2$ intersecting $\mathcal{H}$ at the two-dimensional spheres $S_1 = \{v = 1\}$ and $S_2 = \{v = v_1\}$, for some $v_1 > 1$. We prescribe initial data for the Maxwell's equations (1.1)-(1.6) on the hypersurfaces

$$\begin{aligned} \mathcal{A}_1 =& \mathcal{H} \cap \{1 \leq v \leq v_0\} \\ \underline{\mathcal{A}}_1 =& \underline{\mathcal{H}}_0 \cap \{1 \leq u \leq u_f\} \\ \mathcal{A}_2 =& \mathcal{H} \cap \{v_1 \leq v \leq v_f\} \\ \underline{\mathcal{A}}_2 =& \underline{\mathcal{H}}_1 \cap \{1 \leq u \leq u_f\} \end{aligned}$$

In Figure 1.2, we see the prescribed data on their respective regions and the hyperbolic development in regions $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively.



We would like to extend the data on the truncated hypersurface $\mathcal{G} = \mathcal{H} \cap \{v_0 \leq v \leq v_1\}$ such that there exists a smooth solution $(\alpha, \underline{\alpha}, \rho, \sigma)$ to the Maxwell equation in the region $\mathcal{R}_1 \cup \mathcal{R}_{\mathcal{G}} \cup \mathcal{R}_2$ such that $(\alpha, \underline{\alpha}, \rho, \sigma)|_{\underline{\mathcal{A}}_1 \cup \underline{\mathcal{A}}_2}$ and $(\alpha, \underline{\alpha}, \rho, \sigma)|_{\mathcal{A}_1 \cup \mathcal{A}_2}$ agree with the data we have prescribed. Note that $\mathcal{R}_1, \mathcal{G}, \mathcal{R}_2$ is the (future) domain of dependence of the pairs $(\underline{\mathcal{A}}_1, \mathcal{A}_1)$, $(\underline{\mathcal{G}}, \mathcal{G})$, $(\underline{\mathcal{A}}_2, \mathcal{A}_2)$, respectively, where $\underline{\mathcal{G}} = \{v = v_0\} \cap \{1 \leq u \leq u_f\}$.

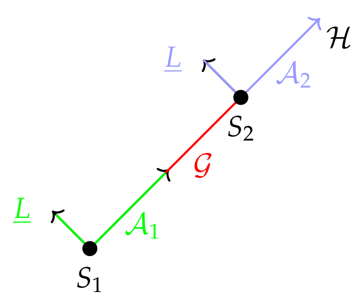
1.3 NULL GLUING PROBLEM FOR MAXWELL

We address the following version of the above gluing problem which will be sufficient for the classification of all null hypersurfaces admitting conservation laws. Consider the following definition.

Definition 1.2. We say that "we can perform kth-order gluing along $\mathcal{H}$" of characteristic data $(\underline{\mathcal{A}}_1, \mathcal{A}_1)$, $(\underline{\mathcal{A}}_2, \mathcal{A}_2)$ as defined above if there exists a smooth extension of the data in $\mathcal{G}$ such that the solutions

- $(\alpha_1, \underline{\alpha}_1, \rho_1, \sigma_1)$ with data given on $\underline{\mathcal{A}}_1, \mathcal{A}_1$, and
- $(\alpha_2, \underline{\alpha}_2, \rho_2, \sigma_2)$ with data given on $\underline{\mathcal{A}}_2, \mathcal{A}_2$

agree at S_2 to all orders tangential to $\mathcal{H}$ and up to first order in directions transversal to $\mathcal{H}$; that is, $(\alpha_1, \underline{\alpha}_1, \rho_1, \sigma_1) = (\alpha_2, \underline{\alpha}_2, \rho_2, \sigma_2)$ at S_2 to all orders tangential to $\mathcal{H}$ and $(\mathcal{L}^j_{\underline{L}}\alpha_1, \mathcal{L}^j_{\underline{L}}\underline{\alpha}_1, \underline{L}^j\rho_1, \underline{L}^j\sigma_1) = (\mathcal{L}^j_{\underline{L}}\alpha_2, \mathcal{L}^j_{\underline{L}}\underline{\alpha}_2, \underline{L}^j\rho_2, \underline{L}^j\sigma_2)$ at S_2 for $j = 1, ..., k$, where $\underline{L}$ is a smooth vector field transversal to $\mathcal{H}$ (see below diagram).



For zeroth-order gluing, we ignore any transversal information about $(\alpha, \underline{\alpha}, \rho, \sigma)$. Thus, our data for this problem is given by

$$\text{Data}^0(S_1) = \left\{(\alpha, \underline{\alpha}, \rho, \sigma) \mid_{S_1}\right\}$$

and

$$\text{Data}^0(S_2) = \left\{(\alpha, \underline{\alpha}, \rho, \sigma) \mid_{S_2}\right\}$$

Given that $(\alpha, \underline{\alpha}, \rho, \sigma)$ solves the Maxwell equations, the transversal derivatives $(\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)$ on $\mathcal{H}$ are completely determined by the data $(\alpha, \underline{\alpha}, \rho, \sigma)|_{\mathcal{H}}$ on $\mathcal{H}$ and the transversal derivatives $(\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)$ at a section S of $\mathcal{H}$. For this reason, we can "forget" about the incoming null hypersurfaces $\underline{\mathcal{H}}_1, \underline{\mathcal{H}}_2$ and thus just "keep" the following data

$$(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{\mathcal{A}_1}, \quad (\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)\ |_{S_1}$$

and

$$(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{\mathcal{A}_2}, \quad (\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)\ |_{S_2}.$$

We do not need to prescribe data on the entire subsets $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{H}$ and in fact we can simply think of the data as given at the two spheres S_1, S_2 as follows.

$$\left\{(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{S_1}, (\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)\ |_{S_1}\right\}$$

and

$$\left\{(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{S_2}, (\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)\ |_{S_2}\right\}$$

where L is tangential to the null generator of $\mathcal{H}$. However, looking at equations (1.2), (1.3) and (1.4), we see that $(\mathcal{L}_{\underline{L}}\alpha|_{S_i}, \underline{L}\rho|_{S_i}, \underline{L}\sigma|_{S_i})_{i=1,2}$ are determined by the initial data $(\alpha|_{S_i}, \underline{\alpha}|_{S_i}, \rho|_{S_i}, \sigma|_{S_i})_{i=1,2}$ and thus cannot be freely chosen. We can define our free data from the first-order gluing problem

$$\text{Data}^1(S_1) = \left\{(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{S_1}, \mathcal{L}_{\underline{L}}\underline{\alpha}\ |_{S_1}\right\}$$

and

$$\text{Data}^1(S_2) = \left\{(\alpha, \underline{\alpha}, \rho, \sigma)\ |_{S_2}, \mathcal{L}_{\underline{L}}\underline{\alpha}\ |_{S_2}\right\}$$

Our problem becomes smoothly extending $(\alpha, \underline{\alpha}, \rho, \sigma)$ on $\mathcal{H}$ between S_1 and S_2 such that $(\alpha, \underline{\alpha}, \rho, \sigma)(v = i) = (\alpha_i, \underline{\alpha}_i, \rho_i, \sigma_i)$ for $i = 1, 2$ and the transversal derivatives $(\mathcal{L}_{\underline{L}}\alpha, \mathcal{L}_{\underline{L}}\underline{\alpha}, \underline{L}\rho, \underline{L}\sigma)$ are continuous on $\mathcal{H} \cap \{1 \leq v \leq 2\}$ (and hence such that $(\alpha, \underline{\alpha}, \rho, \sigma)$ is C^1 on $\mathcal{H} \cap \{1 \leq v \leq 2\}$).
For the C^k case with $k > 1$ (where one needs to "glue" transversal derivatives up to the kth order), sphere data can be defined as well. Note that by a similar argument for the first-order sphere data, we can see that

$$\begin{aligned}
\mathcal{L}^k_{\underline{L}}\alpha &= F_1(\mathcal{L}^{k-1}_{\underline{L}}\alpha, ..., \alpha, \mathcal{L}^{k-1}_{\underline{L}}\underline{\alpha}, ..., \underline{\alpha}, \underline{L}^{k-1}\rho, ..., \rho, \underline{L}^{k-1}\sigma, ..., \sigma) \\
\underline{L}^k\rho &= F_2(\mathcal{L}^{k-1}_{\underline{L}}\underline{\alpha}, ..., \underline{\alpha}, \underline{L}^{k-1}\rho, ..., \rho) \\
\underline{L}^k\sigma &= F_3(\mathcal{L}^{k-1}_{\underline{L}}\underline{\alpha}, ..., \underline{L}^{k-1}\sigma, ..., \sigma)
\end{aligned}$$

In other words, the $(k-1)$th order data determines the kth order data for α, ρ and σ. We can now define our sphere data.

Definition 1.3 (kth-order sphere data)**.** Let $v \in [1,2]$. The *kth-order sphere data* of the 2-sphere S_v, equipped with metric $\not{g} := g|_{S_v}$, is the following tuple of S_v-tangent tensors

$$\text{Data}^k(S_v) = \left\{(\alpha, \underline{\alpha}, \rho, \sigma)|_{S_v}, \not{\mathcal{L}}_{\underline{L}}\underline{\alpha}|_{S_v}, ..., \not{\mathcal{L}}_{\underline{L}}^{k-1}\underline{\alpha}|_{S_v}, \not{\mathcal{L}}_{\underline{L}}^{k}\underline{\alpha}|_{S_v}\right\}$$

where

- ρ, σ are scalar functions
- $\alpha, \underline{\alpha}, \not{\mathcal{L}}_{\underline{L}}\underline{\alpha}, ..., \not{\mathcal{L}}_{\underline{L}}^{k}\underline{\alpha}$ are S_v-tangent 1-forms

Our approach for zeroth-order will be to first construct $\underline{\alpha}$ along $\mathcal{H}$ such that the necessary equations are satisfied. Note that our sphere data depends on the choice of foliation i.e. the choice of a positive function Ω. For simplicity, we will take $\Omega = 1$.

Before we state our first main theorem, we need to define the following: Let

$$\mathcal{U} = \Big\{\Theta \in \mathfrak{X}(\mathcal{H}) : \not{\mathcal{L}}_4\Theta = 0, \\ {}^*\not{\mathcal{D}} \circ \mathcal{O}((\Psi^{-1})^t(s) \cdot \Psi(2)^t \cdot \frac{\Theta}{\phi^2})(s) = (0,0), \forall s \in [1,2]\Big\} \quad (1.12)$$

where

$$\mathcal{O}(X) = -\frac{1}{2}\not{\mathcal{D}}^{\eta}(X) - J \cdot {}^*\not{\mathcal{D}}^{-1}(\phi\nabla_4\left(\frac{1}{\phi}\hat{\underline{\chi}}^{\sharp}\right) \cdot X) \\ + J \cdot {}^*\not{\mathcal{D}}^{-1}[\not{\mathcal{D}}, \mathbb{L}]^*\left({}^*\not{\mathcal{D}}^{-1}(\hat{\underline{\chi}}^{\sharp} \cdot X)\right) \quad (1.13)$$

and where $J = \text{diag}(-1,1)$ and $\mathbb{L} = \phi^{-1}\not{\nabla}_4(\phi^2 \cdot)$. We can now state our first main theorem.

Theorem 1.4 (Zeroth-order Gluing)**.** *Let $\mathcal{H}$ be a regular null hypersurface, free from conjugate or focal points, of a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$. Let $\mathcal{S} = (S_v)_{v\in\mathbb{R}}$ be a foliation of $\mathcal{H}$, such that each of the leaves S_v is diffeomorphic to $\mathbb{S}^2$. Finally, consider $\mathcal{O}$ to be the associated elliptic operator given by (1.13). Given zeroth-order sphere data $(\alpha_i, \underline{\alpha}_i, \rho_i, \sigma_i)$ for $i = 1,2$; zeroth-order gluing can be done given the following two conditions are met.*

1. *We first require*

$$\rho_0(1) = \rho_0(2) \quad (1.14)$$
$$\sigma_0(1) = \sigma_0(2) \quad (1.15)$$

Given this condition is met, we can construct $(\overline{\rho}, \overline{\sigma})$ smoothly on $\mathcal{H}$

2. *For all* $\Theta \in \mathcal{U}$

$$H_0^{\Theta}(2) = H_0^{\Theta}(1) \tag{1.16}$$

We can make a more precise statement in the special case where $\hat{\underline{\chi}} = 0$.

Theorem 1.5. *If* $\hat{\underline{\chi}} = 0$*, then* $\mathcal{U} = \{0\}$ *and the only obstruction to zeroth-order gluing is* $\rho_0(1) = \rho_0(2)$ *and* $\sigma_0(1) = \sigma_0(2)$*.*

We can also address particular cases for the first-order gluing problem and the general kth-order gluing problem. For the first-order gluing problem, we define

$$\begin{aligned}\mathcal{R}(\rho,\sigma) = \Big(-\frac{1}{2}{}^*\mathcal{D} + (\underline{\eta}, -\underline{\eta}^*)\Big)^{-1}\Bigg(&\frac{3}{4}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}(\rho,\sigma) + \hat{\underline{\chi}}^\sharp \cdot (\nabla\sigma)^* \\ &+ (\nabla_3\underline{\eta} - \mathrm{tr}\underline{\chi}\underline{\eta} + \hat{\underline{\chi}}^\sharp \cdot \underline{\eta} - \frac{1}{2}\nabla\mathrm{tr}\underline{\chi}, -\nabla_3\underline{\eta}^* \\ &+ \mathrm{tr}\underline{\chi}\underline{\eta}^* + \hat{\underline{\chi}}^\sharp \cdot \underline{\eta}^* + \frac{1}{2}(\nabla\mathrm{tr}\underline{\chi})^*) \cdot (\rho,\sigma)\Bigg)\end{aligned}$$

Let us also define the following set

$$\begin{aligned}\mathcal{U}_{Extremal} = \Big\{&\Theta \in \mathfrak{X}(\mathcal{H}) : \mathcal{L}_4\Theta = 0, \\ &-\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\Theta + {}^*\mathcal{D}(\underline{\eta}(\Theta), -\underline{\eta}^*(\Theta)) - 2\nabla_{\underline{\eta}}\Theta + \sigma_{(g)}\Theta^* = 0\Big\}\end{aligned} \tag{1.17}$$

Note that the operator

$$-\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\Theta + {}^*\mathcal{D}(\underline{\eta}(\Theta), -\underline{\eta}^*(\Theta)) - 2\nabla_{\underline{\eta}}\Theta + \sigma_{(g)}\Theta^* = (\mathcal{Q}^{(1)})^*(\Theta)$$

is the adjoint of $\mathcal{Q}^{(1)}$. We can show that for $\Theta \in \mathcal{U}_{Extremal} \neq \varnothing$, the quantities

$$\begin{aligned}H^{\Theta}_{Ext}(v) := \int_{S_v}\Big(&\nabla_3\underline{\alpha} + (\frac{3}{2}\mathrm{tr}\underline{\chi}\hat{\underline{\chi}}^\sharp + \underline{\alpha}^\sharp_{(g)}) \cdot \mathcal{D}^{-1}J(\rho,\sigma) \\ &- \mathcal{R}(\underline{\alpha} - \hat{\underline{\chi}}^\sharp \cdot \mathcal{D}^{-1}J(\rho,\sigma))\Big) \cdot \Theta d\mu_{S_v}\end{aligned}$$

are conserved along extremal horizons $\mathcal{H}$. We now state the second main theorem we will prove.

Theorem 1.6 (First-order Gluing)**.** *Let* $\mathcal{H}$ *be an Extremal Horizon of a four-dimensional Lorentzian manifold* $(\mathcal{M}, g)$*. Let* $\mathcal{S} = (S_v)_{v\in\mathbb{R}}$ *be a foliation of* $\mathcal{H}$*, such that each of leaves* S_v *diffeomorphic to* $\mathbb{S}^2$*. Finally, consider* $\mathcal{O}^{(1)}$ *to* *be the associated elliptic operator given by (1.3). Given first-order sphere data* $(\alpha_i, \underline{\alpha}_i, \rho_i, \sigma_i)$ *for* $i = 1, 2$*; zeroth order gluing can be done given the following two conditions are met.*

1. *We first require*

$$\rho_0(1) = \rho_0(2) \tag{1.18}$$

$$\sigma_0(1) = \sigma_0(2) \tag{1.19}$$

   Given this condition is met, we can construct $(\overline{\rho}, \overline{\sigma})$ *smoothly on* $\mathcal{H}$

2. *For all* $\Theta \in \mathcal{U}_{Extremal}$*,*

$$H^{\Theta}_{Ext}(2) = H^{\Theta}_{Ext}(1) \tag{1.20}$$

For the kth-order gluing problem on spherically symmetric extremal horizons we can state the following.

Theorem 1.7. *If* $\mathcal{H}$ *is extremal horizon, then the only obstruction to zeroth-order gluing is* $\rho_0(1) = \rho_0(2)$ *and* $\sigma_0(1) = \sigma_0(2)$*. If we further assume spherical symmetry, there are no further obstructions to kth-order gluing.*

1.4 PREVIOUS GLUING CONSTRUCTIONS

Gluing constructions for hyperbolic equations are important to understanding the Einstein equations. Our work is motivated by the result of Aretakis in [3, 4, 6, 9], where it was proven that the existence of conserved charges for the wave equation along extremal black holes showed that solutions to the wave equation did not disperse along the event horizon.

In more recent studies, Aretakis–Czimek–Rodnianski [10] introduced a gluing technique for the characteristic initial value problem for the Einstein equations. They showed the existence of a 10-dimensional family of gauge invariant charges and an infinite dimensional space of gauge-dependent charges that serve as an obstruction to gluing constructions of the null constraint equations. These gluing constructions were only shown to exist in a neighbourhood of Minkowski initial data. This development has applications such as a significant improvement over the Carlotto–Schoen result, summarized in [13]. Additionally, it provides an alternative proof for the Corvino–Schoen gluing results. Also, using a gluing construction, it was shown in [33] that the 3rd law of black hole dynamics was in fact false. We discuss the spacelike gluing problem for Maxwell's equations in Appendix A.1.

1.5 OUTLINE OF THE THESIS

An outline of the thesis is as follows: In Chapter 2 we introduce the basic geometric concepts relevant to a double null foliation and in Section 3 we derive the relation between the conservation laws and null zeroth-order gluing constructions on a general null hypersurface, as well as some special cases for first-order gluing. Finally in Chapter 4, we show how the change of foliation affects conservation laws and address the gauge invariance issue of the gluing problem.

2 THE GEOMETRIC SETUP

This chapter introduces the geometric setup for the null gluing problem. We will define the double null formulation and introduce the necessary operators required to prove our main theorems.

2.1 NULL FOLIATIONS AND OPTICAL FUNCTIONS

A foliation $\mathcal{S}$ of a null hypersurface $\mathcal{H}$ in a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$ is a set of sections S_v that vary smoothly with respect to the function v, such that $\cup_v S_v = \mathcal{H}$. We will assume for this paper that S_v are diffeomorphic to the 2-sphere, although higher genus surfaces can be considered in general as well. One can see that any foliation is uniquely determined by the choice of a single section, say S_1, the choice of the null tangential vector field to $\mathcal{H}$ vector field $L_{geod}\big|_{S_1}$ restricted on S_1, and a positive function Ω^2 defined on $\mathcal{H}$. We then extend the vector fields $L_{geod}\big|_{S_1}$ to a null vector field tangential to the null generators of $\mathcal{H}$, where

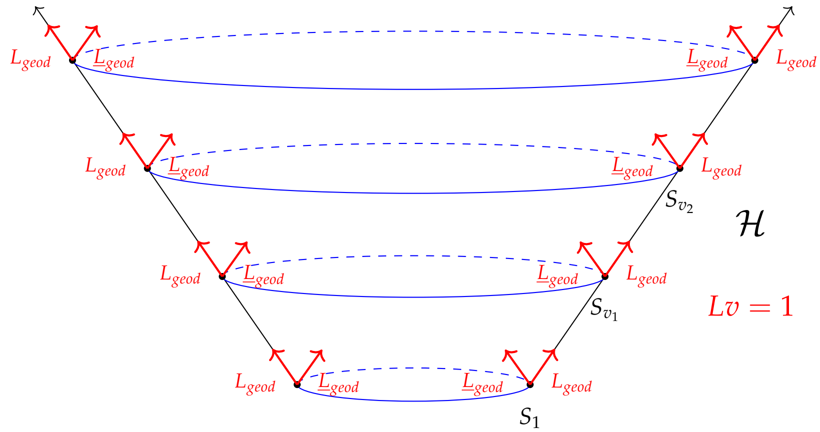
$$\nabla_{L_{geod}} L_{geod} = 0.$$

Looking at the procedure in [15] for a similar construction, we define the vector field

$$L = \Omega^2 \cdot L_{geod} \tag{2.1}$$

on $\mathcal{H}$ and consider the affine parameter v of L such that $Lv = 1$, with $v = 1$, on S_1. Let S_v denote the level sets of v on $\mathcal{H}$ which are the leaves of the foliaton $\mathcal{S}$. We use the notation

$$\mathcal{S} = \left\langle S_1, L_{geod}\big|_{S_1}, \Omega \right\rangle. \tag{2.2}$$



We will define $\mathcal{H}_1 = \mathcal{H}$ and denote $\underline{\mathcal{H}}_1$ to be the null hypersurface generated by all null hypersurfaces normal to S_1 and conjugate to $\mathcal{H}_1$.
Consider now the vector field $\underline{L}_{geod}|_{S_1}$ which we define at S_1. It is tangential to the null generators of $\underline{\mathcal{H}}_1$ and determined by

$$g(L_{geod}, \underline{L}_{geod}) = -\Omega^{-2}$$

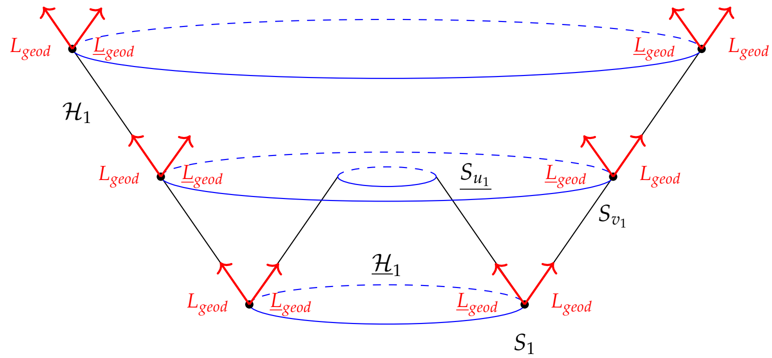
Next, we extend it on $\underline{\mathcal{H}}_1$ by solving the geodesic equation

$$\nabla_{\underline{L}_{geod}} \underline{L}_{geod} = 0.$$

We will also extend Ω to be a function on $\underline{\mathcal{H}}_1$ in a similar manner and consider the vector field

$$\underline{L} = \Omega^2 \cdot \underline{L}_{geod} \tag{2.3}$$

Let us also define the function u on $\underline{\mathcal{H}}_1$ which is given by $\underline{L}u = 1$, with $u = 0$ on S_1. Let $\underline{S}_\tau$ be the embedded 2-surface on $\underline{\mathcal{H}}_1$ such that $u = \tau$. We extend $\underline{L}_{geod}$ on $\mathcal{H}_1$ such that $g(L_{geod}, \underline{L}_{geod}) = -\Omega^{-2}$ and $\underline{L}_{geod}$ is a null normal vector field to S_τ. Similarly, we extend L_{geod} on $\underline{\mathcal{H}}_1$. Note that the affinely parametrized null geodesics, whose tangent we denote by $\underline{L}_{geod}$, span null hypersurfaces which we denote by $\underline{\mathcal{H}}_\tau$. Thus, we see that $\mathcal{H}_1 \cap \underline{\mathcal{H}}_\tau = S_\tau$. We define L_{geod} globally on the whole hypersurface and the hypersurfaces $\mathcal{H}_\tau$ so that their null normal is L_{geod}.



We then extend the vector field $L, \underline{L}$ to global vector fields such that

$$L = \Omega^2 \cdot L_{geod}, \quad \underline{L} = \Omega^2 \cdot \underline{L}_{geod}.$$

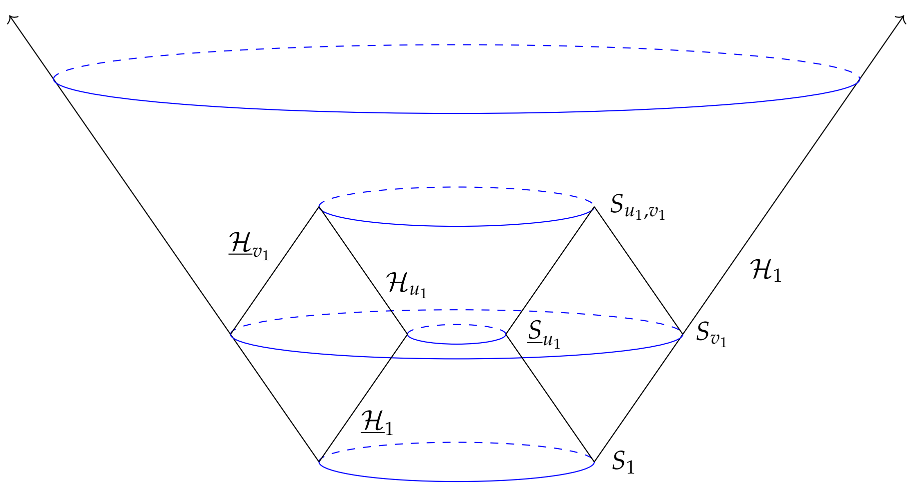
Let us call Ω the null lapse function. We extend the functions u, v globally as follows.

$$Lu = 0 \quad \underline{L}v = 0$$

Therefore, $\mathcal{H}_\tau = \{u = \tau\}$ and $\underline{\mathcal{H}}_\tau = \{v = \tau\}$. Note that u, v are optical functions and satisfy the following relations:

$$\nabla v = -\underline{L}_{geod} \quad \nabla u = -L_{geod}$$

and $Lv = 1, \underline{L}u = 1$. Since $g(\underline{L}, L_{geod}) = -1$, the vector field L_{geod} determines the optical function u on $\mathcal{H} = \mathcal{H}_1$ to first order.



2.2 GAUGE FREEDOM

A double null foliation $\mathcal{D}$ can be completely determined by

$$\mathcal{D} = \left\langle S_1, L_{geod}\big|_{S_1}, \Omega|_{\mathcal{H}_1}, \Omega|_{\underline{\mathcal{H}}_1} \right\rangle .$$

We see that the freedom of choice we have for the vector field $L_{geod}\big|_{S_1}$ and the functions $\Omega|_{\mathcal{H}_1}, \Omega|_{\underline{\mathcal{H}}_1}$ reflects the freedom for the functions $u' = u'(u)$ and $v' = v'(v)$. Indeed, the optical functions u, v are determined by choosing Ω and L_{geod} (up to additive constants).

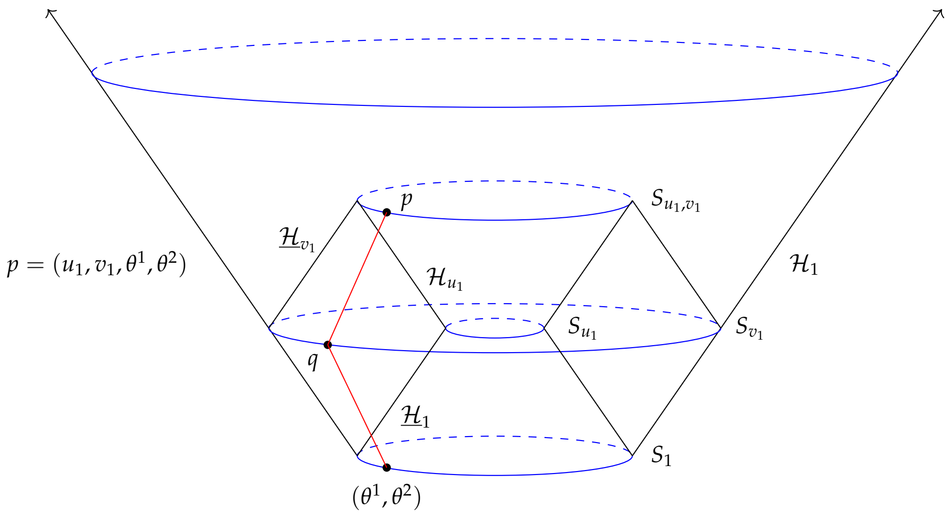
2.3 THE DOUBLE NULL COORDINATE SYSTEM

We describe how we construct a diffeomorphism $\Phi_{u,v}$ from any sphere $S_{u,v}$ to S_1. Given $p \in S_{u,v}$, take $q \in S_{0,v}$ via the intersection of $\mathcal{H}_1$ and the null generator of $\underline{\mathcal{H}}_u$ passing through p. The unique point of intersection of S_1 and the null generator of $\mathcal{H}_1$ passing through q is thus denoted by $\Phi_{u,v}(p)$. To obtain a diffeomorphism from $S_{u,v}$ to $\mathbb{S}^2$, we look at the diffeomorphism

$$\Phi : S_1 \to \mathbb{S}^2 \tag{2.4}$$

and compose $\Phi_{u,v}$ with Φ.

We shall develop a coordinate system appropriately fitted to the corresponding double null foliation of the spacetime, using the optical functions u, v with $u(p) = u_0, v(p) = v_0$ if $p \in \mathcal{M}$. Note, $p \in \mathcal{H}_{u_0} \cap \underline{\mathcal{H}}_{v_0}$. The following are the specified angular coordinates. Assume that a coordinate system on a domain of S_1 is (θ^1, θ^2). Then, the point $\Phi_{u_0,v_0}(p)$ and p have the same angular coordinates. The result of this construction is that $\frac{\partial}{\partial u} = \underline{L}$ and $\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2} \in TS_{u,v}$ everywhere and $\frac{\partial}{\partial v} = L$ on $\mathcal{H}_1$.



Note that, the latter equation will not always hold true: In general, $\partial_v = L + b^i \partial_{\theta^i}$. Since $\underline{L} = \partial_u$ and $[\partial_u, \partial_v] = [\partial_u, \partial_{\theta^i}] = 0$ we obtain $[L, \underline{L}] = -\frac{\partial b^i}{\partial u} \partial_{\theta^i} \in TS_{u,v}$, and thus,

$$\frac{\partial b^i}{\partial u} = -d\theta^i([L, \underline{L}]), \text{ and } b^i = 0 \text{ on } \mathcal{H}_1 = \{u = 0\}. \tag{2.5}$$

The metric g in the canonical null coordinate system defined above, is given by

$$g = -2\Omega^2 dudv + (b^i b^j \slashed{g}_{ij}) dvdv - 2(b^i \slashed{g}_{ij}) d\theta^j dv + \slashed{g}_{ij} d\theta^i d\theta^j, \tag{2.6}$$

where we denote $\slashed{g}$ to be the induced metric on the sphere $S_{u,v} = \mathcal{H}_u \cap \underline{\mathcal{H}}_v$. Moreover,

$$\det(g) = -\Omega^4 \cdot \det(\slashed{g}). \tag{2.7}$$

2.4 NULL FRAMES

For the remainder of the paper, we will denote $S_{u,v} = \mathcal{H}_u \cap \underline{\mathcal{H}}_v$. If we let $\{e_1, e_2\} = (e_A)_{A=1,2}$ be an arbitrary frame on the spheres $S_{(u,v)}$ then we can define the following null frames:

- **Geodesic frame:** $(e_1, e_2, L_{geod}, \underline{L}_{geod})$,
- **Equivariant frame:** $(e_1, e_2, L, \underline{L})$,
- **Normalized frame:** (e_1, e_2, e_3, e_4),
- **Coordinate frame:** $(\partial_{\theta^1}, \partial_{\theta^2}, \partial_v, \partial_u)$.

Here

$$e_3 = \Omega \underline{L}_{geod} = \frac{1}{\Omega}\underline{L}, \quad e_4 = \Omega L_{geod} = \frac{1}{\Omega} L.$$

2.5 THE CONFORMAL GEOMETRY AND CONFORMAL FACTOR

Consider the conformal class of metrics of $g\!\!\!/$. There exists a unique representative (metric) $\hat{g}$ such that $\sqrt{\hat{g}} = \sqrt{g\!\!\!/_{\mathbb{S}^2}}$. Here the diffeomorphism $\Phi \circ \Phi_v$ identifies the section S_v with $\mathbb{S}^2$. We can also see that $g\!\!\!/$ and $\hat{g}$ are such that the induced volume forms on S_v are equal. Note that since $g\!\!\!/$ and $\hat{g}$ are conformal there exists a conformal factor such that $g\!\!\!/ = \phi^2 \cdot \hat{g}$. Then, $\sqrt{g\!\!\!/} = \phi^2 \sqrt{\hat{g}} = \phi^2 \sqrt{g\!\!\!/_{\mathbb{S}^2}}$ and thus

$$\phi = \frac{\sqrt[4]{g\!\!\!/}}{\sqrt[4]{g\!\!\!/_{\mathbb{S}^2}}}. \tag{2.8}$$

Here, we see that ϕ is a smooth function defined on the sphere S_v and more importantly does not depend on the choice of the coordinate system. In spherically symmetric spacetimes, we have $\phi = r$, where r is the radius.

2.6 CONNECTION COEFFICIENTS

Consider the **normalized frame** (e_1, e_2, e_3, e_4) defined above. We define the connection coefficients with respect to this frame to be the smooth functions $\Gamma^{\lambda}_{\mu\nu}$ such that

$$\nabla_{e_\mu} e_\nu = \Gamma^{\lambda}_{\mu\nu} e_\lambda, \quad \lambda, \mu, \nu \in \{1, 2, 3, 4\}$$

Here ∇ denotes the connection of the spacetime metric g. Let us define the following

Definition 2.1. Let S_v be a leaf in our foliation $\mathcal{S}$. We define $g\!\!\!/$ and $\nabla\!\!\!\!/$ to be the induced Riemannian metric and covariant derivative respectively on S_v. Let T be an S_v-tangential tensor. We define $\nabla\!\!\!\!/_L T$ by

$$\nabla\!\!\!\!/_L T_{\alpha_1 \ldots \alpha_k} = \Pi^{\beta_1}_{\alpha_1} \ldots \Pi^{\beta_k}_{\alpha_k} \nabla_L T_{\beta_1 \ldots \beta_k}$$

where Π denotes the projection operator onto TS_v.

We are concerned mainly with the case where at least one of the indices λ, μ, ν is either 3 or 4 (otherwise, we simply get the Christoffel symbols with respect to the induced metric $\not{g}$). Following [15, 16], we define the *Ricci coefficients* of g with respect to the normalized frame as follows

$$\begin{aligned} \chi_{AB} &= g(\nabla_A e_4, e_B), & \underline{\chi}_{AB} &= g(\nabla_A e_3, e_B), \\ \eta_A &= g(\nabla_3 e_4, e_A), & \underline{\eta}_A &= g(\nabla_4 e_3, e_A), \\ \omega &= -g(\nabla_4 e_4, e_3), & \underline{\omega} &= -g(\nabla_3 e_3, e_4), \\ \zeta_A &= g(\nabla_A e_4, e_3) \end{aligned} \tag{2.9}$$

where $(e_A)_{A=1,2}$ is an arbitrary frame on the spheres S_v and $\nabla_\mu = \nabla_{e_\mu}$. The connection coefficient can be recovered from the Ricci coefficients by

$$\begin{aligned} \nabla_A e_B &= \not{\nabla}_A e_B + \chi_{AB} e_3 + \underline{\chi}_{AB} e_4 \\ \nabla_3 e_A &= \not{\nabla}_3 e_A + \eta_A e_3, & \nabla_4 e_A &= \not{\nabla}_4 e_A + \underline{\eta}_A e_4 \\ \nabla_A e_3 &= \underline{\chi}_A^B e_B + \zeta_A e_3, & \nabla_4 e_A &= \chi_A^B e_B - \zeta_A e_4 \\ \nabla_3 e_4 &= \eta^A e_A - \underline{\omega} e_4, & \nabla_4 e_3 &= \underline{\eta}^A e_A - \omega e_3 \\ \nabla_3 e_3 &= \underline{\omega} e_3, & \nabla_4 e_4 &= \omega e_4 \end{aligned}$$

We get the following identities.

$$\begin{aligned} \eta &= \zeta + \not{\nabla}(\log \Omega) & \underline{\eta} &= -\zeta + \not{\nabla}(\log \Omega) \\ \omega &= \nabla_4(\log \Omega) & \underline{\omega} &= \nabla_3(\log \Omega) \end{aligned}$$

Note that we can decompose the tensors χ and $\underline{\chi}$ into their trace and traceless parts as follows

$$\chi = \hat{\chi} + \frac{1}{2}(tr\chi)\not{g}\,, \quad \underline{\chi} = \hat{\underline{\chi}} + \frac{1}{2}(tr\underline{\chi})\not{g}\,. \tag{2.10}$$

We take the metric trace with respect to the induced metric $\not{g}$ of the S-tensor fields $\chi, \underline{\chi}$ (and more general S-tensor fields). Trace $tr\chi$ is defined to be the *expansion* and the component $\hat{\chi}$ is defined as the *shear* of S_v with respect to $\mathcal{H}$.

We can also show the following relations for χ and $\underline{\chi}$. Let $\not{\mathcal{L}}_L$ be the projection of the Lie derivative $\mathcal{L}_L$ onto the spheres S_v similar to how we defined $\not{\nabla}$ in 2.1. By the first variation formulas

$$\not{\mathcal{L}}_L \not{g} = 2\Omega\chi, \quad \not{\mathcal{L}}_L(\not{g}^{-1}) = -2\Omega\chi^{\sharp\sharp}. \tag{2.11}$$

We further obtain that

$$\partial_u \partial_v \sqrt[4]{\not{g}} = \left[\frac{1}{2}\partial_v(\Omega tr\underline{\chi}) + \frac{1}{4}(\Omega tr\chi)(\Omega tr\underline{\chi})\right] \cdot \sqrt[4]{\not{g}} \tag{2.12}$$

on $\mathcal{H}$. The S-tangent 1-form ζ is known as the *torsion*. Given the relation shown in [15],

$$[L, \underline{L}] = -2\Omega^2\zeta^\sharp, \tag{2.13}$$

we see that the torsion ζ is the obstruction to the integrability of the timelike planes given by $\langle e_3, e_4\rangle$ which are orthogonal to the spherical sections S_v.

2.7 TENSOR CALCULUS

Let us define the following operators acting on S-tangent vector fields and tensors.

Definition 2.2. Let X and Y be two S_v-tangential vector fields on S_v and ϵ^{AB} is defined with respect to the Levi-Civita symbol ϵ_{ABCD} as follows.

$$\epsilon_{AB} = \epsilon_{AB34}$$

We define

$$X \times Y = \epsilon^{AB} X_A Y_B$$

where $\not{g}$ denotes the induced metric on S_v. Further, for a vector field X, we define its dual X^* to be

$$X^*_A = \epsilon_{AB} X^B$$

Moreover, we define the $\mathrm{d\not{i}v}$ and $\mathrm{c\not{u}rl}$ of a vector field X as

$$\mathrm{d\not{i}v} X = \not{\nabla}^A X_A, \quad \mathrm{c\not{u}rl} X = \epsilon^{AB} \not{\nabla}_A X_B$$

We note that

$$\mathrm{d\not{i}v} X^* = \mathrm{c\not{u}rl} X$$
$$\mathrm{c\not{u}rl} X^* = -\mathrm{d\not{i}v} X$$

For a 2-tensor F defined on the Lorentzian 4-manifold $(\mathcal{M}, g)$, we define the Hodge dual ${}^*F$ of F as

$${}^*F_{\mu\nu} = \frac{1}{2} F^{\alpha\beta} \epsilon_{\alpha\beta\mu\nu}$$

where ϵ is the Levi-Civita symbol.

2.8 CURVATURE COMPONENTS

Following [16], we decompose the Riemann curvature R in terms of the normalized null frame. First, we define the following components, which contain at most two S-tangential components (and hence at least 2 null components):

$$\begin{aligned} \alpha^g_{AB} &= R_{A4B4}, \quad \underline{\alpha}^g_{AB} = R_{A3B3}, \\ \beta^g_A &= R_{A434}, \quad \underline{\beta}^g_A = R_{A334}, \\ \rho^g &= R_{3434}, \quad \sigma^g = (*R)_{3434}, \end{aligned} \tag{2.14}$$

where the Hodge star $*R$ is defined as follows: $(*R)_{\alpha\beta\gamma\delta} = \epsilon_{\mu\nu\alpha\beta} R^{\mu\nu}{}_{\gamma\delta}$.

2.9 HODGE OPERATORS

Consider the 2-sphere $(S, \not{g})$. We define the Hodge operators $\not{\mathcal{D}}$ and ${}^*\not{\mathcal{D}}$ as follows. For an S-tangent vector field X, $\not{\mathcal{D}}X = (\not{\mathrm{div}}X, \not{\mathrm{curl}}X)$ and for a pair of functions (ρ, σ) on S to the S-tangent vector field ${}^*\not{\mathcal{D}}(\rho, \sigma) = -\not{\nabla}\rho + (\not{\nabla}\sigma)^*$. The $\not{\mathcal{D}}$ operator's image is pair of functions with mean zero, with the mean of a function of S_v defined to be

$$\overline{F}(v) = \frac{1}{\mathrm{vol}(S_v)} \int_{S_v} F d\mu_{\not{g}}$$

These Hodge operators also satisfy the relations

$${}^*\not{\mathcal{D}}\not{\mathcal{D}} = -\not{\Delta}, \qquad \not{\mathcal{D}}{}^*\not{\mathcal{D}} = -\not{\Delta} + K \tag{2.15}$$

where K is the Gauss curvature of the spherical section S. We use the following from Proposition 4.22 in [35].

Lemma 2.3. *Consider the 2-sphere $(S, \not{g})$. Then the following hold*

1. *The operator $\not{\mathcal{D}}$ is invertible on its range and its inverse $\not{\mathcal{D}}^{-1}$ takes pairs of functions $f = (\rho, \sigma)$ (in the range of $\not{\mathcal{D}}$) into S-tangent vector fields X with $\not{\mathrm{div}}X = \rho$ and $\not{\mathrm{curl}}X = \sigma$, satisfying*

$$||\not{\nabla} \cdot \not{\mathcal{D}}^{-1} f||_{L^2(S)} + ||\not{\mathcal{D}}^{-1} f||_{L^2(S)} \lesssim ||f||_{L^2(S)} \tag{2.16}$$

2. *The operator ${}^*\not{\mathcal{D}}$ is invertible as an operator defined from pairs of H^1-functions with mean zero, and its inverse ${}^*\not{\mathcal{D}}^{-1}$ takes S-tangent L^2-bounded*

vector fields X (that is, the full range of ${}^*\mathcal{D}$*) into pairs of functions* (ρ, σ) *with mean zero such that* $-\nabla\rho + (\nabla\sigma)^* = X$ *, satisfying*

$$||\nabla \cdot^* \mathcal{D}^{-1} X||_{L^2(S)} \lesssim ||X||_{L^2(S)} \tag{2.17}$$

We will also require the following lemma, similar to Lemma 4.23 from [35].

Lemma 2.4. *The operators* $\mathcal{D}^{-1}, {}^*\mathcal{D}^{-1}$ *satisfy*

$$[\mathcal{D}^{-1}, \mathbb{L}] = \mathcal{D}^{-1}[\mathbb{L}, \mathcal{D}]\mathcal{D}^{-1}$$

where

$$[\mathbb{L}, \mathcal{D}]X = \Big(\mathbf{R}_{A4C}{}^{A}X^C + \chi_{AB}\left(\nabla^A X^B + \underline{\eta}^B X^A\right) - tr\chi\underline{\eta}\cdot X, \\ \epsilon^{AB}\mathbf{R}_{A4CB}X^C + \epsilon^{AB}\chi_{AC}\left(\nabla^C X_B + \underline{\eta}_B X^C\right)\Big)$$

and where $\mathbb{L}X = \phi^{-2}\nabla_L(\phi^2 X)$

We can now setup the null Maxwell equations.

2.10 MAXWELL'S EQUATIONS

It can be shown that the Maxwell equations defined in (1) are equivalent to

$$\mathbf{D}_{[\alpha}F_{\beta\gamma]} = 0 \quad \mathbf{D}_{[\alpha}{}^*F_{\beta\gamma]} = 0 \tag{2.18}$$

where ${}^*F_{AB} = \epsilon^{ABCD}F(e_C, e_D)$. In order to write these equations in the form of coupled null transport equations, we first need to define the following components of F.

Definition 2.5. Let F be the Maxwell tensor and consider the normalized frame $(e_A)_{A=1,\ldots,4}$ defined in Section 2.1. We define the null components of F as

$$\alpha(e_A) := F(e_4, e_A), \quad \underline{\alpha}(e_A) := F(e_3, e_A)$$
$$\rho := F(e_3, e_4), \quad \sigma := \frac{1}{2}F(e_A, e_B)\epsilon^{AB}$$

We also define ${}^{\circledast}\alpha, {}^{\circledast}\underline{\alpha}, {}^{\circledast}\rho, {}^{\circledast}\sigma$ to be the corresponding null components of ${}^*F$. We note that for $A = 1, 2$,

$$ {}^{\circledast}\alpha_A = -{}^*\alpha, \quad {}^{\circledast}\underline{\alpha} = {}^*\underline{\alpha} \tag{2.19} $$

$$ {}^{\circledast}\rho = \sigma, \quad {}^{\circledast}\sigma = -\rho \tag{2.20} $$

We want to derive (1.1) - (1.6) from (2.18). The entire content of Maxwell's equations are contained in the $[34A]$, $[3AB]$ and $[4AB]$ components of both equations (2.18).

The $[34A]$ **component**. Our first (2.18) equation tells us

$$ 0 = \mathbf{D}_{[3}F_{4A]} = \mathbf{D}_3 F_{4A} + \mathbf{D}_A F_{34} + \mathbf{D}_4 F_{A3} \tag{2.21} $$

By the Ricci coefficients defined in Section 2.6, we see that

$$ \begin{aligned} \mathbf{D}_3 F_{4A} &= e_3(\alpha_A) - \eta^B F_{BA} + \underline{\omega}\alpha_A + \eta_A \rho + \alpha(\nabla\!\!\!/_3 e_A) \\ \mathbf{D}_A F_{34} &= \nabla\!\!\!/_A(\rho) + \underline{\chi}_A^B \alpha_B - \chi_A^B \underline{\alpha}_B \\ \mathbf{D}_4 F_{A3} &= -e_4(\underline{\alpha}_A) + \underline{\eta}^B F_{BA} - \omega\alpha_A + \underline{\eta}_A \rho + \underline{\alpha}(\nabla\!\!\!/_4 e_A) \end{aligned} $$

Taking the sum of these terms, we see that (2.21) becomes

$$ \begin{aligned} 0 = (\nabla\!\!\!/_3\alpha)_A - (\nabla\!\!\!/_4\underline{\alpha})_A - (\eta^B - \underline{\eta}^B)\sigma\epsilon_{AB} + \underline{\omega}\alpha_A - \omega\underline{\alpha}_A + \nabla\!\!\!/_A\rho \\ + (\eta_A + \underline{\eta}_A)\rho + \underline{\chi}_A^B\alpha_B - \chi_A^B\underline{\alpha}_B \end{aligned} $$

where we used that $F_{BA} = \sigma\epsilon_{AB}$. Taking the same components for the equation for the dual tensor ${}^*F$ and using the relations in (2.19), we obtain the equation

$$ \begin{aligned} 0 = -(\nabla\!\!\!/_3\alpha)_A - (\nabla\!\!\!/_4\underline{\alpha})_A + (\eta^B + \underline{\eta}^B)\sigma\epsilon_{AB} - \underline{\omega}\alpha_A - \omega\underline{\alpha}_A + \epsilon_{AB}\nabla\!\!\!/^B\sigma \\ - (\eta_A - \underline{\eta}_A)\rho - \epsilon_{BA}\underline{\chi}^{BC}\epsilon_{CD}\alpha^D - \epsilon_{BA}\chi^{BC}\epsilon_{CD}\underline{\alpha}^D \end{aligned} $$

Taking the sum and difference of the previous two equations, we get that

$$ \begin{aligned} 0 =& -(\nabla\!\!\!/_4\underline{\alpha})_A + \underline{\eta}_A\rho - \omega\underline{\alpha}_A - \frac{1}{2}\epsilon_{AB}\nabla\!\!\!/^B\sigma + \underline{\eta}^B\epsilon_{AB}\sigma + \frac{1}{2}\nabla\!\!\!/_A\rho \\ &+ \frac{1}{2}\left(\underline{\chi}_{AD} + \epsilon_{BA}\underline{\chi}^{BC}\epsilon_{CD}\right)\alpha^D + \frac{1}{2}\left(-\chi_{AD} + \epsilon_{BA}\chi^{BC}\epsilon_{CD}\right)\underline{\alpha}^D \\ 0 =& -(\nabla\!\!\!/_3\alpha)_A + \eta_A\rho - \underline{\omega}\alpha_A + \frac{1}{2}\epsilon_{AB}\nabla\!\!\!/^B\sigma + \eta^B\epsilon_{AB}\sigma + \frac{1}{2}\nabla\!\!\!/_A\rho \\ &+ \frac{1}{2}\left(\underline{\chi}_{AD} - \epsilon_{BA}\underline{\chi}^{BC}\epsilon_{CD}\right)\alpha^D + \left(-\chi_{AD} - \epsilon_{BA}\chi^{BC}\epsilon_{CD}\right)\underline{\alpha}^D \end{aligned} $$

Finally, we arrive at equations (1.1) and (1.2) by using the following identities: For an arbitrary S-tangent 2-tensor V

$$V_{AD} - \epsilon_{BA} V^{BC} \epsilon_{CD} = \mathrm{tr}_{\not g} V \not g_{AD}$$
$$V_{AD} + \epsilon_{BA} V^{BC} \epsilon_{CD} = 2\hat{V}_{AD}$$

where $\hat{V}$ denotes the trace free part of V with respect to the section metric $\not g$.

The [3AB] component. We get

$$0 = \mathbf{D}_{[3} F_{AB]} = \mathbf{D}_3 F_{AB} + \mathbf{D}_B F_{3A} + \mathbf{D}_A F_{B3} \tag{2.22}$$

Using 2.6,

$$\mathbf{D}_3 F_{AB} = e_3(F_{AB}) - \eta_A \underline{\alpha}_B + \eta_B \underline{\alpha}_A + F(\not\nabla_3 e_A, e_B) + F(e_A, \not\nabla_3 e_B)$$
$$\mathbf{D}_B F_{3A} = \not\nabla_B \underline{\alpha}_A - \underline{\chi}_B^C F_{CA} - \zeta_B \underline{\alpha}_A - \underline{\chi}_{BA} \rho$$
$$\mathbf{D}_A F_{B3} = -\not\nabla_A \underline{\alpha}_B - \underline{\chi}_A^C F_{CB} + \zeta_A \underline{\alpha}_B - \underline{\chi}_{AB} \rho$$

Combining, we get

$$0 = e_3(\sigma)\epsilon_{AB} - (\not\nabla_B \underline{\alpha}_A - \not\nabla_A \underline{\alpha}_B) - (\eta_A \underline{\alpha}_B - \eta_B \underline{\alpha}_A) + (\zeta_A \underline{\alpha}_B - \zeta_B \underline{\alpha}_A) + \mathrm{tr}\underline{\chi} \epsilon_{AB} \sigma$$

where we used that $\underline{\chi}_A^C \epsilon_{CB} - \underline{\chi}_B^C \epsilon_{CA} = \mathrm{tr}\underline{\chi} \epsilon_{AB}$. This is equivalent to

$$0 = e_3 \sigma - \mathrm{c\not url} \underline{\alpha} - \eta \times \underline{\alpha} + \zeta \times \underline{\alpha} + \mathrm{tr}\underline{\chi} \sigma$$
$$= e_3 \sigma - \mathrm{c\not url} \underline{\alpha} - \not d \log \Omega \times \underline{\alpha} + \mathrm{tr}\underline{\chi} \sigma$$

Repeating the previous calculations for the dual tensor ${}^*F$ we get

$$0 = e_3 {}^{\circledast}\sigma - \mathrm{c\not url} {}^{\circledast}\underline{\alpha} - \not d \log \Omega \times {}^{\circledast}\underline{\alpha} + \mathrm{tr}\underline{\chi} {}^{\circledast}\sigma$$

By the following relations

$$\mathrm{c\not url} {}^{\circledast}\underline{\alpha} = \mathrm{c\not url}({}^*\underline{\alpha}) = \mathrm{d\not iv} \underline{\alpha}$$
$$\not d \log \Omega \times {}^{\circledast}\underline{\alpha} = -\not d \log \Omega \cdot \underline{\alpha}$$
$${}^{\circledast}\sigma = -\rho$$

we get

$$0 = -e_3 \sigma + \mathrm{d\not iv} \underline{\alpha} + \not d \log \Omega \cdot \underline{\alpha} - \mathrm{tr}\underline{\chi} \rho$$

Simplifying gives us both equations, (1.3) and (1.4).

The [4AB] component. We get

$$0 = \mathbf{D}_{[4}F_{AB]} = \mathbf{D}_4 F_{AB} + \mathbf{D}_B F_{4A} + \mathbf{D}_A F_{B4} \tag{2.23}$$

Using 2.6,

$$\begin{aligned}
\mathbf{D}_4 F_{AB} &= e_4(F_{AB}) - \underline{\eta}_A \alpha_B + \underline{\eta}_B \alpha_A + F(\nabla\!\!\!/_4 e_A, e_B) + F(e_A, \nabla\!\!\!/_4 e_B) \\
\mathbf{D}_B F_{4A} &= \nabla\!\!\!/_B \alpha_A - \chi_B^C F_{CA} - \zeta_B \alpha_A - \chi_{BA}\rho \\
\mathbf{D}_A F_{B4} &= -\nabla\!\!\!/_A \alpha_B - \chi_A^C F_{CB} + \zeta_A \alpha_B + \chi_{AB}\rho
\end{aligned}$$

Combining, we get

$$0 = e_4\sigma - \mathrm{c}\!/\!\mathrm{url}\alpha - d\!\!/ \log\Omega \times \alpha + \mathrm{tr}\chi\sigma \tag{2.24}$$

Repeating the previous calculations for the dual tensor ${}^*F$ we get

$$0 = e_4{}^{\circledast}\sigma - \mathrm{c}\!/\!\mathrm{url}^{\circledast}\alpha - d\!\!/ \log\Omega \times {}^{\circledast}\alpha + \mathrm{tr}\chi^{\circledast}\sigma$$

By the following relations

$$\begin{aligned}
\mathrm{c}\!/\!\mathrm{url}^{\circledast}\alpha &= \mathrm{c}\!/\!\mathrm{url}({}^*\alpha) = \mathrm{d}\!/\!\mathrm{iv}\alpha \\
d\!\!/ \log\Omega \times {}^{\circledast}\alpha &= -d\!\!/ \log\Omega \cdot \alpha \\
{}^{\circledast}\sigma &= -\rho
\end{aligned}$$

we get

$$0 = -e_4\sigma + \mathrm{d}\!/\!\mathrm{iv}\alpha + d\!\!/ \log\Omega \cdot \alpha - \mathrm{tr}\chi\rho$$

Simplifying gives us equations (1.5) and (1.6). We will use various versions of these equations, defined below.

Covariant Formulation

We will take our gauge freedom $\Omega = 1$ for simplicity. We refer to the following set of equations as the covariant formulation of Maxwell's equations.

$$\frac{1}{\phi}\nabla\!\!\!/_4\left(\phi\underline{\alpha}\right) = +\frac{1}{2}\left(d\!\!\!/\rho + 2\underline{\eta}\rho\right) - \frac{1}{2}\left(d\!\!\!/\sigma + 2\underline{\eta}\sigma\right)^* + \hat{\underline{\chi}}\cdot\alpha \tag{2.25}$$

$$\frac{1}{\phi}\nabla\!\!\!/_3\left(\phi\alpha\right) = -\frac{1}{2}\left(d\!\!\!/\rho - 2\underline{\eta}\rho\right) - \frac{1}{2}\left(d\!\!\!/\sigma - 2\underline{\eta}\sigma\right)^* + \hat{\chi}\cdot\underline{\alpha} \tag{2.26}$$

$$\mathrm{d\!\!\!/iv}\underline{\alpha} = +\frac{1}{\phi^2}\underline{L}(\phi^2\rho) \tag{2.27}$$

$$\mathrm{cu\!\!\!/rl}\underline{\alpha} = +\frac{1}{\phi^2}\underline{L}(\phi^2\sigma) \tag{2.28}$$

$$\mathrm{d\!\!\!/iv}\alpha = -\frac{1}{\phi^2}L(\phi^2\rho) \tag{2.29}$$

$$\mathrm{cu\!\!\!/rl}\alpha = +\frac{1}{\phi^2}L(\phi^2\sigma) \tag{2.30}$$

Instead of writing our equations in terms of covariant derivatives, we can write them in terms of Lie derivatives.

Lie Formulation

We rewrite the equations by replacing the covariant derivatives with Lie derivatives to get the Lie Formulation of the Maxwell equations.

$$\mathcal{L\!\!\!/}_4\underline{\alpha} - \hat{\chi}^\sharp\cdot\underline{\alpha} = +\frac{1}{2}\left(\nabla\!\!\!/\rho + 2\underline{\eta}\rho\right) - \frac{1}{2}\left(\nabla\!\!\!/\sigma + 2\underline{\eta}\sigma\right)^* + \hat{\underline{\chi}}^\sharp\cdot\alpha \tag{2.31}$$

$$\mathcal{L\!\!\!/}_3\alpha - \hat{\underline{\chi}}^\sharp\cdot\alpha = -\frac{1}{2}\left(\nabla\!\!\!/\rho - 2\underline{\eta}\rho\right) - \frac{1}{2}\left(\nabla\!\!\!/\sigma - 2\underline{\eta}\sigma\right)^* + \hat{\chi}^\sharp\cdot\underline{\alpha} \tag{2.32}$$

$$\mathrm{d\!\!\!/iv}\underline{\alpha} = +\frac{1}{\phi^2}\underline{L}(\phi^2\rho) \tag{2.33}$$

$$\mathrm{cu\!\!\!/rl}\underline{\alpha} = +\frac{1}{\phi^2}\underline{L}(\phi^2\sigma) \tag{2.34}$$

$$\mathrm{d\!\!\!/iv}\alpha = -\frac{1}{\phi^2}L(\phi^2\rho) \tag{2.35}$$

$$\mathrm{cu\!\!\!/rl}\alpha = +\frac{1}{\phi^2}L(\phi^2\sigma) \tag{2.36}$$

Recall that for metric connections,

$$\nabla\!\!\!/_X Y - \nabla\!\!\!/_Y X = \mathcal{L\!\!\!/}_X Y$$

We first raise the indices of (2.25) by applying g^{-1} to both sides. Since the metric commutes with covariant derivatives.

$$\frac{1}{\phi}\nabla_4\left(\phi\underline{\alpha}^\sharp\right) = \frac{1}{\phi}\mathcal{L}_4\left(\phi\underline{\alpha}^\sharp\right) + \chi\cdot\underline{\alpha}^\sharp$$

We now have to lower $\underline{\alpha}^\sharp$ back to a 1-form. However, the metric does not commute with Lie derivatives. We see that

$$\begin{aligned} g\cdot\mathcal{L}_4\left(\phi\underline{\alpha}^\sharp\right) &= \mathcal{L}_4\left(\phi\underline{\alpha}\right) - \mathcal{L}_4(g)\cdot\phi\underline{\alpha}^\sharp \\ &= \mathcal{L}_4\left(\phi\underline{\alpha}\right) - 2\chi\cdot\phi\underline{\alpha}^\sharp \\ &= \mathcal{L}_4\left(\phi\underline{\alpha}\right) - 2\chi^\sharp\cdot\phi\underline{\alpha} \end{aligned}$$

Therefore,

$$g\cdot\frac{1}{\phi}\mathcal{L}_4\left(\phi\underline{\alpha}^\sharp\right) + \chi^\sharp\cdot\underline{\alpha} = \frac{1}{\phi}\mathcal{L}_4(\phi\underline{\alpha}) - \chi^\sharp\cdot\underline{\alpha}$$

Noting that $\hat{\chi}^\sharp = \chi^\sharp - \frac{1}{2}\mathrm{tr}\chi$, we arrive at the Lie formulation. The formulation of the zeroth-order gluing problem will be framed with respect to this formulation.

Complex Scalar formulation

We can also write a complex form of Maxwell's equations. While this form is not used in the proofs of the main theorems, it was a crucial step that helped clarify the Maxwell gluing problem. This formulation's main advantage was solving the zeroth-order gluing problem for $\underline{\hat{\chi}} = 0$, which is discussed in Remark 3.5. The complex Maxwell equations take the following form

$$\frac{1}{\phi}L(\phi\Psi) = \frac{1}{2}m(\Theta) - \zeta_m\Theta + \underline{\hat{\chi}}_1^m\Phi \tag{2.37}$$

$$\frac{1}{\phi}\underline{L}(\phi\Phi) + i\Gamma^2_{31}\Phi = -\frac{1}{2}\overline{m}(\Theta) - \zeta_{\overline{m}}\Theta + \hat{\chi}_1^{\overline{m}}\Psi \tag{2.38}$$

$$\mathcal{D}_m\Phi = -\phi^{-2}L(\phi^2\Theta) \tag{2.39}$$

$$\mathcal{D}_{\overline{m}}\Psi = \phi^{-2}\underline{L}(\phi^2\Theta) \tag{2.40}$$

where

$$\Psi = \underline{\alpha}_1 + i\underline{\alpha}_2 \tag{2.41}$$

$$\Theta = \rho + i\sigma \tag{2.42}$$

$$\Phi = \alpha_1 - i\alpha_2 \tag{2.43}$$

and

$$\begin{aligned} m &= e_1 + ie_2 & \overline{m} &= e_1 - ie_2 \\ \zeta_m &= \zeta(e_1) + i\zeta(e_2) & \zeta_{\overline{m}} &= \zeta(e_1) - i\zeta(e_2) \\ \underline{\hat{\chi}}_1^m &= \underline{\hat{\chi}}_1^1 + i\underline{\hat{\chi}}_1^2 & \hat{\chi}_1^{\overline{m}} &= \hat{\chi}_1^1 - i\hat{\chi}_1^2 \\ \not{D}_m &= m + i\Gamma^1_{m2} & \not{D}_{\overline{m}} &= \overline{m} + i\Gamma^1_{\overline{m}2} \\ \Gamma^1_{m2} &= \Gamma^1_{12} + i\Gamma^1_{22} & \Gamma^1_{\overline{m}2} &= \Gamma^1_{12} - i\Gamma^1_{22} \end{aligned} \tag{2.44}$$

We note that the indices of the 1-form are the evaluation of the form on a Fermi frame ($\not{\nabla}_L e_A = 0$). To derive equations (2.37) we evaluate (2.25) on vector field $m = e_1 + ie_2$ and rearrange to get the desired equations. First,

$$\frac{1}{\phi}\not{\nabla}_4(\phi\underline{\alpha}) = +\frac{1}{2}(\not{d}\rho - 2\zeta\rho) - \frac{1}{2}(\not{d}\sigma - 2\zeta\sigma)^* + \underline{\hat{\chi}}\cdot\alpha \tag{2.45}$$

Evaluating the left-hand side, we get

$$\begin{aligned} \frac{1}{\phi}\not{\nabla}_4(\phi\underline{\alpha})(e_1 + ie_2) &= \not{\nabla}_4\underline{\alpha}(e_1 + ie_2) + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}(e_1 + ie_2) \\ &= L(\underline{\alpha}_1 + i\underline{\alpha}_2) - \underline{\alpha}(\not{\nabla}_4 e_1 + i\not{\nabla}_4 e_2) + \frac{1}{2}\mathrm{tr}\chi(\underline{\alpha}_1 + i\underline{\alpha}_2) \\ &= L(\underline{\alpha}_1 + i\underline{\alpha}_2) + \frac{1}{2}\mathrm{tr}\chi(\underline{\alpha}_1 + i\underline{\alpha}_2) - \underline{\alpha}(\Gamma^2_{41}e_2 + i\Gamma^1_{42}e_1) \\ &= \frac{1}{\phi}L(\phi\Psi) + i\Gamma^2_{41}\Psi \end{aligned}$$

noting that $\nabla_4 e_1 = \Gamma^2_{41} e_2 = -\Gamma^1_{42} e_2$ and $\nabla_4 e_2 = \Gamma^1_{42} e_1$ and $\Gamma^2_{41} = -\Gamma^1_{42}$. Using the definitions in (2.44), we get the left-hand side of (2.37). For the right-hand side, we see that

$$\begin{aligned}
\frac{1}{2}\left(\not{d}\rho - 2\zeta\rho\right)(e_1 + ie_2) - \frac{1}{2}\left(\not{d}\sigma - 2\zeta\sigma\right)^* (e_1 + ie_2) + \underline{\hat{\chi}} \cdot \alpha(e_1 + ie_2) \\
&= \frac{1}{2}(e_1(\rho) + ie_2(\rho)) - \zeta(e_1 + ie_2)\rho - \frac{1}{2}\not{d}\sigma(\epsilon^{12}e_2 + i\epsilon^{21}e_1) - \\
&\quad \zeta(\epsilon^{12}e_2 + i\epsilon^{21}e_1)\sigma + \underline{\hat{\chi}} \cdot \alpha(e_1 + ie_2) \\
&= \frac{1}{2}m(\rho) - \zeta_m\rho + \frac{i}{2}m(\sigma) + i\zeta_m\sigma + \underline{\hat{\chi}}^1_1\alpha_1 + \underline{\hat{\chi}}^2_1\alpha_2 \\
&\quad + i(\underline{\hat{\chi}}^1_2\alpha_1 + \underline{\hat{\chi}}^2_2\alpha_2) \\
&= \frac{1}{2}m(\rho + i\sigma) - \zeta_m(\rho + i\sigma) + (\underline{\hat{\chi}}^1_1 + i\underline{\hat{\chi}}^2_1)(\alpha_1 - i\alpha_2)
\end{aligned}$$

where we used the fact that $\underline{\hat{\chi}}$ is a trace-free symmetric tensor and that, $\underline{\hat{\chi}}^2_1 = \underline{\hat{\chi}}^1_2$ and $\underline{\hat{\chi}}^1_1 = -\underline{\hat{\chi}}^2_2$. Equation (2.38) is derived similarly, evaluating the equation on the vector field $\overline{m}$ instead. To get the next equations, we com

$$\frac{1}{\phi^2}\underline{L}(\phi^2(\rho + i\sigma)) = \not{d}\text{iv}\underline{\alpha} + i\not{c}\text{url}\underline{\alpha} \tag{2.46}$$

$$= \nabla_1\underline{\alpha}_1 + \nabla_2\underline{\alpha}_2 + i\left(\nabla_1\underline{\alpha}_2 - \nabla_2\underline{\alpha}_1\right) \tag{2.47}$$

$$\begin{aligned}
&= e_1\underline{\alpha}_1 + e_2\underline{\alpha}_2 + i(e_1\underline{\alpha}_2 - e_2\underline{\alpha}_1) \\
&\quad + \underline{\alpha}(\nabla_1 e_1 + \nabla_2 e_2 + i\nabla_1 e_2 - i\nabla_2 e_1)
\end{aligned} \tag{2.48}$$

Since e_1, e_2 is an orthonormal frame, we get that

$$\nabla_1 e_1 = \Gamma^2_{11} e_2 \qquad \nabla_2 e_2 = \Gamma^1_{22} e_1 \qquad \nabla_1 e_2 = \Gamma^1_{12} e_1 \qquad \nabla_2 e_1 = \Gamma^2_{21} e_2 \tag{2.49}$$

Using the fact that $\Gamma^1_{22} = -\Gamma^2_{21}$ and $\Gamma^1_{12} = -\Gamma^2_{11}$ and using the definitions in (2.44), we can get the desired equation. We can derive (2.39) in a similar manner, by instead taking the sum $\not{d}\text{iv}\alpha - i\not{c}\text{url}\alpha$.

3

GLUING PROBLEMS AND CONSERVATION LAWS

In this chapter, we prove our main gluing theorems. We introduce the necessary elliptic structure to prove our general gluing theorems 1.4 and 1.6. We will also introduce methods to prove the invertibility of elliptic operators.

3.1 ZEROTH-ORDER GLUING CONSTRUCTIONS

Note that for zeroth-order gluing we only need to consider equations

$$\not{\mathcal{L}}_4\underline{\alpha} - \hat{\chi}^\sharp \cdot \underline{\alpha} = +\frac{1}{2}\left(\not\nabla\rho + 2\underline{\eta}\rho\right) - \frac{1}{2}\left(\not\nabla\sigma + 2\underline{\eta}\sigma\right)^* + \underline{\hat{\chi}}^\sharp \cdot \alpha \tag{3.1}$$

$$\not{\mathrm{d}}\mathrm{iv}\,\alpha = -\frac{1}{\phi^2}L(\phi^2\rho) \tag{3.2}$$

$$\not{\mathrm{c}}\mathrm{url}\,\alpha = +\frac{1}{\phi^2}L(\phi^2\sigma) \tag{3.3}$$

By the invertibility of $\not{\mathcal{D}}$ on its natural domain, the equations (3.1),(3.2) and (3.3) can be combined to get the following equation

$$\begin{aligned}\not{\mathcal{L}}_4 W(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma}) - \hat{\chi}^\sharp \cdot W(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma}) \\ = \mathcal{Q}(\rho - \overline{\rho}, \sigma - \overline{\sigma}) + \mathcal{V}(\overline{\rho}, \overline{\sigma})\end{aligned} \tag{3.4}$$

where,

$$
\begin{aligned}
W(\underline{\alpha}, \rho-\overline{\rho}, \sigma-\overline{\sigma}) &= \underline{\alpha} - \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-(\rho-\overline{\rho}), (\sigma-\overline{\sigma})) \\
\mathcal{Q}(\rho-\overline{\rho}, \sigma-\overline{\sigma}) &= -\frac{1}{2}{}^*\mathcal{D}^{\underline{\eta}}(\rho-\overline{\rho}, \sigma-\overline{\sigma}) \\
&\quad - \phi\nabla_4\left(\frac{1}{\phi}\underline{\hat{\chi}}^\sharp\right)\mathcal{D}^{-1}(-(\rho-\overline{\rho}), (\sigma-\overline{\sigma})) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}[\mathcal{D}, \mathbb{L}]\mathcal{D}^{-1}(-(\rho-\overline{\rho}), (\sigma-\overline{\sigma})) \\
\mathcal{V}(\overline{\rho}, \overline{\sigma}) &= \overline{\rho}(-\underline{\eta} - \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}((\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}), 0)) \\
&\quad + \overline{\sigma}(\underline{\eta} - \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}((\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}), 0))^* \\
&= \rho_0 Y_1 + \sigma_0 Y_2
\end{aligned}
$$

where $\mathbb{L} = \phi^{-2}\nabla_L(\phi^2\cdot)$. To prove this we first need the following Lemma

Lemma 3.1. *Operators $\mathcal{D}^{-1}, {}^*\mathcal{D}^{-1}$ satisfy*

$$
[\mathcal{D}^{-1}, \mathbb{L}] = \mathcal{D}^{-1}[\mathbb{L}, \mathcal{D}]\mathcal{D}^{-1}
$$

where

$$
\begin{aligned}
[\mathbb{L}, \mathcal{D}]X = \Big(&\mathbf{R}_{A4C}{}^A X^C + \chi_{AB}\left(\nabla^A X^B + \underline{\eta}^B X^A\right) - tr\chi\underline{\eta}\cdot X, \\
&\epsilon^{AB}\mathbf{R}_{A4CB}X^C + \epsilon^{AB}\chi_{AC}\left(\nabla^C X_B + \underline{\eta}_B X^C\right)\Big)
\end{aligned}
$$

Recalling the definition of $\mathcal{D}$, we see that

$$
\begin{aligned}
\underline{\hat{\chi}}^\sharp \cdot (\Omega\alpha) &= \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(\mathbb{L}(-\rho, \sigma)) \\
&= \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(\mathbb{L}(-(\rho-\overline{\rho}), \sigma-\overline{\sigma})) + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(\mathbb{L}(-\overline{\rho}, \overline{\sigma})) \\
&= \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(\mathbb{L}(-(\rho-\overline{\rho}), \sigma-\overline{\sigma})) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-\overline{\rho}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}), \overline{\sigma}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi})) \\
&= \underline{\hat{\chi}}^\sharp \cdot \mathbb{L}\mathcal{D}^{-1}(-(\rho-\overline{\rho}), \sigma-\overline{\sigma}) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}[\mathbb{L}, \mathcal{D}]\mathcal{D}^{-1}((-(\rho-\overline{\rho}), \sigma-\overline{\sigma})) \\
&\qquad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-\overline{\rho}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}), \overline{\sigma}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}))
\end{aligned}
$$

where we used that for any $g : \mathbb{S}^2 \to \mathbb{R}$

$$
\mathbb{L}\overline{g} = \overline{g}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi})
$$

We can further simplify the following operator. For any (f_1, f_2) with vanishing spherical mean

$$\begin{aligned}
\underline{\hat{\chi}}^\sharp \cdot \mathbb{L}\mathcal{D}^{-1}(f_1, f_2) &= \frac{1}{\phi^2}\underline{\hat{\chi}}^\sharp \cdot \nabla_4(\phi^2\mathcal{D}^{-1}(f_1, f_2)) \\
&= \frac{1}{\phi^2}\Omega\underline{\hat{\chi}}^\sharp \cdot \nabla_4(\phi^2\mathcal{D}^{-1}(f_1, f_2)) \\
&= \frac{1}{\phi^2}\nabla_4(\Omega\phi^2\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) - (\nabla_4\Omega\underline{\hat{\chi}}^\sharp) \cdot \mathcal{D}^{-1}(f_1, f_2) \\
&= \frac{1}{\phi}\nabla_4(\Omega\phi\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) + \frac{1}{2}\Omega\mathrm{tr}\chi(\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) \\
&\quad - (\nabla_4(\Omega\underline{\hat{\chi}}^\sharp)) \cdot \mathcal{D}^{-1}(f_1, f_2) \\
&= \frac{1}{\phi}\nabla_4(\Omega\phi\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) - \phi\nabla_4(\frac{\Omega}{\phi}\underline{\hat{\chi}}^\sharp) \cdot \mathcal{D}^{-1}(f_1, f_2) \\
&= \mathcal{L}_4(\Omega\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) - \hat{\chi}^\sharp \cdot (\Omega\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(f_1, f_2)) \\
&\quad - \phi\nabla_4(\frac{\Omega}{\phi}\underline{\hat{\chi}}^\sharp) \cdot \mathcal{D}^{-1}(f_1, f_2)
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\underline{\hat{\chi}}^\sharp \cdot (\Omega\alpha) &= \mathcal{L}_4(\Omega\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-(\rho - \overline{\rho}), \sigma - \overline{\sigma})) \\
&\quad - \hat{\chi}^\sharp \cdot (\Omega\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-(\rho - \overline{\rho}), \sigma - \overline{\sigma})) \\
&\quad - \phi\nabla_4(\frac{\Omega}{\phi}\underline{\hat{\chi}}^\sharp) \cdot \mathcal{D}^{-1}(-(\rho - \overline{\rho}), \sigma - \overline{\sigma}) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}[\mathbb{L}, \mathcal{D}]\mathcal{D}^{-1}((-(\rho - \overline{\rho}), \sigma - \overline{\sigma})) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-\overline{\rho}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}), \overline{\sigma}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}))
\end{aligned}$$

where we used that for any 1-form ω, $\phi^{-1}\nabla_4(\phi\omega) = \mathcal{L}_4\omega - \hat{\chi}^\sharp \cdot \omega$. Finally, we want to simplify the terms that only depend on $\overline{\rho}$ and $\overline{\sigma}$, which we will call $\mathcal{V}(\overline{\rho}, \overline{\sigma})$, so that it is of the form in (3.4). Note that

$$\frac{\Omega}{2}\left(\nabla\rho + 2\underline{\eta}\rho\right) - \frac{\Omega}{2}\left(\nabla\sigma + 2\underline{\eta}\sigma\right)^* = -\frac{\Omega}{2}\mathcal{D}^{\underline{\eta}}(\rho - \overline{\rho}, \sigma - \overline{\sigma}) - \Omega\underline{\eta}\overline{\rho} + \Omega\underline{\eta}^*\overline{\sigma}$$

and thus, we have

$$\mathcal{V}(\overline{\rho}, \overline{\sigma}) = -\underline{\eta}\overline{\rho} + \underline{\eta}^*\overline{\sigma} + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-\overline{\rho}(\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}), \overline{\sigma}(\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}))$$

Let $f = \Omega\text{tr}\chi - \overline{\Omega\text{tr}\chi}$ and $X = \not{\mathcal{D}}^{-1}(-\overline{\rho}f, \overline{\sigma}f)$. We see that

$$\begin{aligned}\not{\mathcal{D}}((\overline{\rho} - \epsilon\overline{\sigma})X) &= \overline{\rho}\not{\mathcal{D}}(X) - \overline{\sigma}\not{\mathcal{D}}(X^*) \\ &= \overline{\rho}(-\overline{\rho}f, \overline{\sigma}f) - \overline{\sigma}(\overline{\sigma}f, \overline{\rho}f) \\ &= -(\overline{\rho}^2 + \overline{\sigma}^2)(f, 0)\end{aligned}$$

Thus, we get that

$$\begin{aligned}X = \not{\mathcal{D}}^{-1}&(-\overline{\rho}f, \overline{\sigma}f) \\ &- (\overline{\rho}^2 + \overline{\sigma}^2)(\overline{\rho} - \epsilon\overline{\sigma})^{-1}\not{\mathcal{D}}^{-1}(f, 0) \\ &= -(\overline{\rho} + \epsilon\overline{\sigma})\not{\mathcal{D}}^{-1}(f, 0)\end{aligned}$$

Now we can write $\mathcal{V}(\overline{\rho}, \overline{\sigma})$ as follows

$$\begin{aligned}\mathcal{V}(\overline{\rho}, \overline{\sigma}) &= \overline{\rho}(-\Omega\underline{\eta} - \underline{\hat{\chi}}^\sharp \cdot \not{\mathcal{D}}^{-1}((\Omega\text{tr}\chi - \overline{\Omega\text{tr}\chi}), 0)) \\ &\quad + \overline{\sigma}(\Omega\underline{\eta} - \underline{\hat{\chi}}^\sharp \cdot \not{\mathcal{D}}^{-1}((\Omega\text{tr}\chi - \overline{\Omega\text{tr}\chi}), 0))^* \\ &= \rho_0 Y_1 + \sigma_0 Y_2\end{aligned}$$

where the 1-forms Y_1, Y_2 only depend on the background metric. Combining these terms, and setting $\Omega = 1$, we arrive at (3.4). To find a necessary condition for our initial data to satisfy, we need to 'integrate' (3.4) in the v direction. Thus, we need to define the dv integral of a tensor. Let $(e_A)_{A=1,2}$ be a Lie transported frame along $\mathcal{H}$ and let $(e^A)_{A=1,2}$ be its corresponding dual frame. Let T be an arbitrary $(1,1)$ tensor such that $T(v)$ is S_v-tangent for all $v \in [1,2]$. We define the integral of T as follows

$$\left(\int_a^b T(s)ds\right)_A^B = \int_a^b T_A^B(s)ds$$

Let $M(v) = \int_a^v T(s)ds$. Note that since the frame is Lie transported i.e. $[L, e_A] = 0$, we see that by the above definition

$$\begin{aligned}(\not{\mathcal{L}}_4 M)_A^B &= L(M_A^B) - M([L, e_A], e^B) - M(e_A, \not{\mathcal{L}}_4 e^B) \\ &= \partial_v \int_1^v T_A^B(s)ds \\ &= T_A^B\end{aligned}$$

Note that this naturally generalizes to any S-tangent (r,s) tensor. We finally need to define the Ψ tensor. Let $W_0(v)$ be the S_v-tangent 1-form solution to the following transport equation

$$\not{\mathcal{L}}_4 W_0 - \hat{\chi}^\sharp \cdot W_0 = 0 \qquad W_0(1) = W(1) \tag{3.5}$$

Let us denote the solution to the equation (which exists since it is simply a system of ODEs), as follows

$$W_0(v) = \Psi(v) \cdot W(1) \tag{3.6}$$

where $\Psi(v)$ is a $(1,1)$ tensor that only depends on the background metric. We see that $(\underline{\alpha}, \rho, \sigma)$ satisfies (3.4) if and only if

$$W(v) = \Psi(v) \cdot W(1) + \Psi(v) \cdot \int_1^v \Psi^{-1}(s) \cdot \mathcal{Q}(\rho - \overline{\rho}, \sigma - \overline{\sigma})(s)ds \\ + \rho_0 Z_1 + \sigma_0 Z_2$$

where $Z_i(v) = \Psi(v) \int_1^v \Psi^{-1}(s) \cdot Y_i(s)ds$. $\Psi(v)$ is invertible by uniqueness theorems for ODEs. Our gluing construction will rely on Fredholm theory. We will use the following theorem, stated in [23].

Theorem 3.2 (Fredhom Alternative)**.** *Suppose $T : X \to X$ is a compact operator, where X is a Banach space and let $\lambda \in \mathbb{C}$ be non-zero. Then, exactly one of the following hold:*

- *(Eigenvalue) There is a non-trivial solution to the equation $Tx = \lambda x$*
- *(Bounded resolvent) The operator $T - \lambda$ has a bounded inverse on X*

It follows that the spectrum of a compact operator can only accumulate at 0 and all nonzero elements of the spectrum are eigenvalues of finite multiplicity. Note here that we will think of the adjoint operator acting on the dual space of 1-forms on $\mathbb{S}^2$ i.e. vector fields on $\mathbb{S}^2$. Given $X \in \mathfrak{X}(\mathbb{S}^2)$ and $\omega \in \Omega^1(\mathbb{S}^2)$, we define the following dual coupling

$$< \omega, X >_{S_v} = \int_{S_v} \omega(X) d\mu_g$$

Thus, we first define the adjoint $\mathcal{O}_v$ with respect each $\mathcal{Q}_v$ and then define $\mathcal{O}$ such that $O|_{S_v} = \mathcal{O}_v$. Our proof requires our operator to be elliptic with its spectrum only accumulating at infinity. However, our operator $\mathcal{Q}$ in its current form will not satisfy the required conditions. Instead we define the following operators

$$\widetilde{\mathcal{Q}} = \mathcal{Q} \circ \mathcal{D} \\ \widetilde{\mathcal{O}} = \widetilde{\mathcal{Q}}^* = {}^*\mathcal{D} \circ \mathcal{O}$$

We see that $\widetilde{\mathcal{Q}}$ is an operator that takes 1-forms to 1-forms. Also, from Section 2 of [35], we have the following bounds

$$||\not{\mathcal{D}}^{-1}X||_{L^2(S)} \leq C_1||X||_{L^2(S)}$$
$$||\not{\mathcal{D}}^{-1}[\mathbb{L},\not{\mathcal{D}}]\not{\mathcal{D}}^{-1}X||_{L^2(S)} \leq C_2||X||_{L^2(S)}$$

and thus for any $X \in \mathfrak{X}(\mathbb{S}^2)$

$$\begin{aligned}|\int_{S_v}(-\underline{\hat{\chi}}^\sharp\cdot\not{\mathcal{D}}^{-1}[L+\mathrm{tr}\chi,\not{\mathcal{D}}]\not{\mathcal{D}}^{-1}+\phi\nabla_4(\frac{1}{\phi}\underline{\hat{\chi}})\not{\mathcal{D}}^{-1})(J\not{\mathcal{D}}(X))\cdot Xd\mu_g| \\ \leq C||\not{\mathcal{D}}X||_{L^2(S)}||X||_{L^2(S)} \\ \leq \int_{S_v} C\epsilon_2|\not{\mathcal{D}}X|^2+\frac{C}{\epsilon_2}|X|^2d\mu_g\end{aligned}$$

for some $C>0$ and for all $\epsilon_2>0$. Here $J(f_1,f_2)=(-f_1,f_2)$. Indeed if we take the operator, $\widetilde{\mathcal{Q}}_{temp}=\widetilde{\mathcal{Q}}-\frac{1}{\epsilon}$, then

$$\begin{aligned}\int_{S_v}\widetilde{\mathcal{Q}}_{temp}X\cdot Xd\mu_g &= \int_{S_v}-\frac{1}{2}|\not{\mathcal{D}}X|^2-\not{\mathcal{D}}X\cdot(-\zeta(X),\zeta^*(X)) \\ &\quad -\underline{\hat{\chi}}^\sharp\cdot\not{\mathcal{D}}^{-1}[L+\mathrm{tr}\chi,\not{\mathcal{D}}]\not{\mathcal{D}}^{-1}(J\not{\mathcal{D}}(X))\cdot X \\ &\quad +\phi\nabla_4(\frac{1}{\phi}\underline{\hat{\chi}})\not{\mathcal{D}}^{-1}(J\not{\mathcal{D}}(X))\cdot X-\frac{1}{\epsilon}|X|^2d\mu_g \\ &\leq -\int_{S_v}(\frac{1}{2}-\epsilon_1-C\epsilon_2)|\not{\mathcal{D}}X|^2+(-\frac{|\zeta|^2}{\epsilon_1}-\frac{C}{\epsilon_2}+\frac{1}{\epsilon})|X|^2\end{aligned}$$

If we choose ϵ_1 and ϵ_2 small enough to make the first $\frac{1}{2}-\epsilon_1-C\epsilon_2$ positive and then choose ϵ sufficiently small to make the last term positive, we can conclude that $\widetilde{\mathcal{Q}}_{temp}$ has a trivial kernel. By the Fredholm alternative $\widetilde{\mathcal{Q}}_{temp}$ is an invertible operator. Since $H^2(\mathbb{S}^2)$ compactly embed into $L^2(\mathbb{S}^2)$ by Rellichs Theorem, $\widetilde{\mathcal{Q}}_{temp}$ has a compact inverse by Poincare's inequality. The spectrum of this operator is thus bounded and accumulates at zero. Thus $\widetilde{\mathcal{Q}}_{temp}$ is an elliptic operator with discrete eigenvalues with the only accumulation point at infinity and

$$im(\widetilde{\mathcal{Q}}_v)=ker(\widetilde{\mathcal{O}}_v))^\perp \tag{3.7}$$

Thus, since the domain of $\widetilde{\mathcal{Q}}$ are smooth 1-forms, we can rewrite our necessary condition (3.8) as follows

$$\begin{aligned}W(2)-\Psi(2)W(1)-\rho_0Z_1(2)-\sigma_0Z_2(2) \\ =\int_1^2\Psi(2)\cdot\Psi^{-1}(s)\cdot\widetilde{\mathcal{Q}}(\Xi)(s)ds\end{aligned} \tag{3.8}$$

where $\Xi := \mathcal{D}^{-1}(-(\rho - \overline{\rho}), \sigma = \overline{\sigma})$. Thus we instead want to find a 1-form F on $\mathcal{H}$ such that for any $s \in [1,2]$

$$F(s) \in \mathrm{im}(\Psi(2) \cdot \Psi^{-1}(s) \cdot \widetilde{\mathcal{Q}}(s)) \tag{3.9}$$

and

$$\int_1^2 F(s)ds = W(2) - \Psi(2)W(1) - \rho_0 Z_1(2) - \sigma_0 Z_2(2)$$

which is equivalent to

$$F(s) \in \ker(\widetilde{\mathcal{O}} \circ (\frac{1}{\phi^2}(\Psi^{-1})^t \circ \Psi(2)^t)(s))^{\perp}$$

To construct F, we will use the following lemmas from [9]

Lemma 3.3. *There is an upper bound for the dimension of the kernel $K(v) \subset L^2(\mathbb{S}^2)$ of the operator $\widetilde{\mathcal{O}}_v$ for $v \in [0,1]$. Moreover, there is a dense set $\mathcal{Z} \subset [0,1]$ such that for any $x \in \mathcal{Z}$, there is an open neighbourhood $V_x \subset \mathcal{Z}$ such that $K(v)$ varies smoothly for $v \in V_x$.*

Lemma 3.4. *Let $I = \bigcup_{k=1}^n I_k$ be a union of compact intervals of $\mathbb{R}$. For each $k = 1, ..., n$ consider $\Pi_k(v), v \in I_k$, to be smoothly varying n_k-dimensional subspace of $L^2(\mathbb{S}^2)$ spanned by k smooth functions on $\mathbb{S}^2$. Define subspaces $V_k \subset L^2(\mathbb{S}^2)$ as follows*

$$V_k = \bigcap_{v \in I_k} \Pi_k(v) \tag{3.10}$$

Given a function $\rho \in C^\infty(\mathbb{S}^2)$, there is a function $F_\rho \in C^\infty(I \times \mathbb{S}^2)$ which vanishes to infinite order at $\partial I \times \mathbb{S}^2$ and is such that

$$\int_I F_\rho(v, \cdot)dv = \rho(\cdot) \tag{3.11}$$

and

$$F_\rho(v, \cdot) \in (\Pi_k(v))^{\perp} \text{ for all } v \in I_k \text{ and } k = 1, 2, ..., n \tag{3.12}$$

if and only if

$$\rho \in (V_1 \cap V_2 \cap ... \cap V_n)^{\perp} \subset L^2(\mathbb{S}^2) \tag{3.13}$$

We can now prove our main theorems.
Proof of Theorem 1.4: To see the necessity of the first condition, we see that the quantities $\rho_0(v)$ and $\sigma_0(v)$ are conserved since

$$\begin{aligned} L\rho_0 &= L\int_{S_v} \rho d\mu_g \\ &= L\int_{S_v} \phi^2 \rho d\mu_{S^2} \\ &= \int_{S_v} \frac{1}{\phi^2} L(\phi^2\rho) d\mu_g \\ &= \int_{S_v} (\operatorname{d\!\!/iv}\alpha) d\mu_g = 0 \end{aligned}$$

since the divergence and curl of forms/vector fields have a spherical average of 0. Therefore, given initial data for (ρ, σ), if $\rho_0(1) = \rho_0(2)$ and $\sigma_0(1) = \sigma_0(2)$, we can construct $\overline{\rho}(v)$ and $\overline{\sigma}(v)$ along the whole null hypersurface. The agreement of these quantities on initial spheres S_1 and S_2 is a necessary condition for gluing. Let $J = J_1 \cup ... \cup J_k \subset [1,2]$ such that the kernel $K(v)$ of our adjoint $\widetilde{\mathcal{O}} \circ (\frac{1}{\phi^2}(\Psi^{-1})^t \circ \Psi^t(2))$ varies smoothly on each open subinterval J_i and such that

$$\bigcap_{v \in J} K(v) = \mathcal{U}$$

The existence of the intervals is given by Lemma 3.3. Let $\Xi_i = \mathcal{D\!\!\!/}^{-1}(-(\rho_i - \overline{\rho_i}), \sigma_i - \overline{\sigma_i})$. We extend Ξ from S_1 and S_2 smoothly on $([1,2] - J) \times \mathbb{S}^2$ such that $\Xi|_{\partial J}$ vanishes to infinite order. We additionally impose that the extension near S_1 and S_2 satisfies

$$L\mathcal{D\!\!\!/}(\Xi)|_{S_i} + L(-\overline{\rho}, \overline{\sigma})|_{S_i} = (-L\rho, L\sigma)|_{S_i} = \mathcal{D\!\!\!/}(\alpha_i) - \mathrm{tr}\chi|_{S_i} \cdot (-\rho_i, \sigma_i)$$

for $i = 1, 2$. This is to ensure the extension is compatible with the initial data of α. Let us define ξ as follows

$$\begin{aligned} \xi := (W(2) - \Psi(2)W(1) - \rho_0 Z_1(2) - \sigma_0 Z_2(2)) \\ - \Psi(2) \int_{[1,2]-J} \Psi^{-1}(s) \widetilde{\mathcal{Q}}(\Xi)(s) dv \end{aligned}$$

To complete the gluing construction, we want to construct Ξ on $[1,2] \times \mathbb{S}^2$ such that

$$\xi = \Psi(2) \int_J \Psi^{-1}(s) \widetilde{\mathcal{Q}}(\Xi)(s) ds$$

In other words, we would like to construct a vector field F on $\mathbb{S}^2 \times J$ that vanishes identically to infinite order (with respect to $\mathcal{L}_4$ derivatives) on $\mathbb{S}^2 \times \partial J$ such that $F(v, \cdot) \in \mathrm{im}(\Psi(2) \cdot \Psi^{-1}(s) \cdot \widetilde{\mathcal{Q}}(s))$ and

$$\xi = \int_J F(\cdot, v) dv$$

Using Lemma (3.4) and the ellipticity of $\widetilde{\mathcal{Q}}$, we can construct F if $\xi \in \mathcal{U}^{\perp}$. Let us verify this. Note that for all $\Theta \in \mathcal{U}$, if Condition 2 in 1.4 holds, then

$$\begin{aligned}\int_{\mathbb{S}^2} \xi(\Theta) d\mu_{\mathbb{S}^2} &= \int_{\mathbb{S}^2} (W(2) - \Psi(2)W(1) - \rho_0 Z_1(2) - \sigma_0 Z_2(2))(\Theta) d\mu_{\mathbb{S}^2} \\ &\quad - \int_{\mathbb{S}^2} \left(\int_{[1,2]-J} \Psi(2)\Psi^{-1}(s) \widetilde{\mathcal{Q}}(\Xi)(s) ds \right) (\Theta) d\mu_{\mathbb{S}^2} \\ &= - \int_{[1,2]-J} \int_{\mathbb{S}^2} \left(\Psi(2)\Psi^{-1}(s) \widetilde{\mathcal{Q}}(\Xi) \right) (\Theta) d\mu_{\mathbb{S}^2} ds \\ &= 0\end{aligned}$$

Where we used that $\mathcal{L}_4\Theta = 0$ to move Θ into the ds integral and that

$$\begin{aligned}\int_{\mathbb{S}^2} \Psi(2)\Psi^{-1}(s) \widetilde{\mathcal{Q}}(\Xi)(s)(\Theta) d\mu_{\mathbb{S}^2} &= \int_{S_v} \Xi \cdot \mathcal{O} \left(\frac{1}{\phi^2} (\Psi^{-1})^t(s) \Psi(2)^t \Theta \right) (s) d\mu_{g} \\ &= 0\end{aligned}$$

We get that $\mathcal{L}_4\Theta = 0$ since $\Theta \in \mathcal{U} = \bigcap_{v \in J} K(v)$ and therefore does not depend on v. It can be seen from the fact that for any vector field $X = X^1(v)e_1 + X_2(v)e_2$, since the frame $(e_A)_{A=1,2}$ is Lie transported, then

$$\mathcal{L}_4 X = \partial_v X^1 e_1 + \partial_v X^2 e_2$$

Thus, if X is independent of v, $\partial_v X^A = 0 \implies \mathcal{L}_4 X = 0$. We also used $H_0^{\Theta}(2) = H_0^{\Theta}(1)$ in the first step to show

$$\begin{aligned}0 &= \Psi(2)(H_0^{\Theta}(2) - H_0^{\Theta}(1)) \\ &= \int_{\mathbb{S}^2} (W(2) - \Psi(2)W(1) - \rho_0 Z_1(2) - \sigma_0 Z_2(2))(\Theta) d\mu_{\mathbb{S}^2}\end{aligned}$$

Thus, $\xi \in \mathcal{U}^{\perp}$. We construct F using Lemma (3.4) and complete the gluing construction of (ρ, σ) by taking $\mathcal{D}\Xi$ and then adding back the spherical averages and by constructing α as follows.

$$\alpha := \mathcal{D}^{-1} \left(\frac{1}{\phi^2} (-L(\phi^2 \rho), L(\phi^2 \sigma)) \right)$$

□

Let us look at the particular case $\underline{\hat{\chi}} = 0$ and show that only Condition 1 of Theorem 1.4 is an obstruction to gluing.
Proof of Theorem 1.5 Consider Maxwell's equations for zeroth-order gluing given $\underline{\hat{\chi}} = 0$

$$\frac{1}{\phi}\mathcal{L}_4(\phi\underline{\alpha}) + \chi \cdot \underline{\alpha} = -\frac{1}{2}{}^*\mathcal{D}(\rho,\sigma) + \underline{\eta}\rho - \underline{\eta}^*\sigma \tag{3.14}$$

$$\text{d}\not{\text{i}}\text{v}\alpha = -\frac{1}{\phi^2}L(\phi^2\rho) \tag{3.15}$$

$$\text{c}\not{\text{u}}\text{rl}\alpha = +\frac{1}{\phi^2}L(\phi^2\sigma) \tag{3.16}$$

The initial data $(\Theta_i, \Phi_i, \Psi_i)$ combined with (3.14),(3.15) and (3.16) gives us $\mathcal{L}_4\overline{\alpha}|_{S_i}, L\rho|_{S_i}, L\sigma|_{S_i}$. We define

$$\begin{aligned}
\mathcal{L}_4\underline{\alpha}_i &:= -\frac{1}{2}{}^*\mathcal{D}(\rho_i,\sigma_i) + \underline{\eta}\rho_i - \underline{\eta}^*\sigma_i + \chi \cdot \underline{\alpha}_i - \frac{1}{2}\text{tr}\chi \cdot \underline{\alpha}_i \\
L\rho_i &:= -\text{d}\not{\text{i}}\text{v}\alpha_i + \text{tr}\chi\rho_i \\
L\sigma_i &:= \text{c}\not{\text{u}}\text{rl}\alpha_i - \text{tr}\chi\sigma_i
\end{aligned}$$

Taking an $\mathcal{L}_4$ derivative of these equations gives us, $\mathcal{L}_4^2\overline{\alpha}|_{S_i}, L^2\rho|_{S_i}, L^2\sigma|_{S_i}$ as well. We can define

$$\begin{aligned}
\mathcal{L}_4\mathcal{L}_4\underline{\alpha}_i := &-\frac{1}{2}{}^*\mathcal{D}(L\rho_i, L\sigma_i) - \frac{1}{2}[\mathcal{L}_4, {}^*\mathcal{D}](\rho_i,\sigma_i) + \underline{\eta}L\rho_i + (\mathcal{L}_4\underline{\eta})\rho_i - \underline{\eta}^*L\sigma_i \\
&- (\mathcal{L}_4\underline{\eta}^*)\sigma_i + \mathcal{L}_4\chi \cdot \underline{\alpha}_i + \chi \cdot \mathcal{L}_4\underline{\alpha}_i - \frac{1}{2}L(\text{tr}\chi) \cdot \underline{\alpha}_i - \frac{1}{2}\text{tr}\chi \cdot \mathcal{L}_4\underline{\alpha}_i
\end{aligned}$$

Note that $[\mathcal{L}_4, {}^*\mathcal{D}](f_1, f_2)$ will be an $S-$tangent 1-form (See Appendix) and thus we do not need L derivatives of (ρ,σ) to compute the term in the definition. Thus, any gluing of these functions will not only have to agree with the initial data but also with the derivatives defined above as well. Note that if $\underline{\hat{\chi}} \neq 0$, then $\mathcal{L}_4^2\underline{\alpha}$ cannot be determined from initial data since $\mathcal{L}_4\alpha$ would appear on the right-hand side of (3.14). We can now do our gluing construction. We first construct $\underline{\alpha}$ along $\mathcal{H}$ such that

$$\begin{aligned}
\underline{\alpha}|_{S_i} &= \underline{\alpha}_i \\
\mathcal{L}_4\underline{\alpha}|_{S_i} &= \mathcal{L}_4\underline{\alpha}_i \\
\mathcal{L}_4^2\underline{\alpha}|_{S_i} &= \mathcal{L}_4^2\underline{\alpha}_i
\end{aligned}$$

To construct (ρ,σ), we need to invert the following operator

$${}^*\mathcal{D}^{\underline{\eta}}(\rho,\sigma) := {}^*\mathcal{D}(\rho,\sigma) - 2\underline{\eta}\rho + 2\underline{\eta}^*\sigma$$

For convenience, we will use the Levi-Civita operator ϵ to refer to the * operator where $(^*\xi)_A = \epsilon_{AB}\xi^B$. Note the Levi-Civita symbol normally maps vector fields to 1-forms but with a slight abuse of notation it will just map vector fields to vector fields i.e. $\epsilon\xi =^* \xi$. Note that $\epsilon^2 = -1$. With this notation, we say $^*\mathcal{D}(f_1, f_2) = -\nabla f_1 + \epsilon\nabla f_2$. In order to invert the operator $^*\mathcal{D}^{\underline{\eta}}$, we note that $\underline{\eta} = \nabla_4 e_3$ is an L^2 vector field and thus in the image of $^*\mathcal{D}$. Thus there exists (g_1, g_2) such that

$$^*\mathcal{D}(g_1, g_2) = -2\underline{\eta}$$

We define the following functions

$$\begin{aligned} h_1 &= e^{g_1}\cos g_2 \\ h_2 &= e^{g_1}\sin g_2 \end{aligned}$$

Note that $\nabla h_1 = h_1\nabla g_1 - h_2\nabla g_2$ and $\nabla h_2 = h_2\nabla g_1 + h_1\nabla g_2$. We see that

$$\begin{aligned} ^*\mathcal{D}(h_1\rho - h_2\sigma, h_2\rho + h_1\sigma) &= -\nabla(h_1\rho - h_2\sigma) + \epsilon\nabla(h_2\rho + h_1\sigma) \\ &= h_1\big(\rho(-\nabla g_1 + \epsilon\nabla g_2) + \sigma(\nabla g_2 + \epsilon\nabla g_1) \\ &\quad + (-\nabla\rho + \epsilon\nabla\sigma)\big) + h_2\big(\rho(\nabla g_2 + \epsilon\nabla g_1) \\ &\quad + \sigma(\nabla g_2 - \epsilon\nabla g_1) + (\epsilon\nabla\rho + \nabla\sigma)\big) \\ &= (h_1 - \epsilon h_2)((\rho - \epsilon\sigma)(-2\underline{\eta}) + {}^*\mathcal{D}(\rho, \sigma)) \\ &= (h_1 - \epsilon h_2)^*\mathcal{D}^{\underline{\eta}}(\rho, \sigma) \end{aligned}$$

We can now rewrite (3.14) as follows

$$\frac{1}{\phi}\mathcal{L}_4(\phi\underline{\alpha}) + \chi\cdot\underline{\alpha} = -\frac{1}{2}\frac{1}{h_1^2 + h_2^2}(h_1 + \epsilon h_2)^*\mathcal{D}(h_1\rho - h_2\sigma, h_2\rho + h_1\sigma)$$

where we used $(h_1 - \epsilon h_2)^{-1} = (h_1^2 + h_2^2)^{-1}(h_1 + \epsilon h_2)$. Thus we have written $^*\mathcal{D}^{\underline{\eta}}$ in terms of $^*\mathcal{D}$, which we can invert. However, the image of the inverse of $^*\mathcal{D}$ are pairs of functions with mean zero. Thus we see that if $\frac{1}{\phi}\mathcal{L}_4(\phi\underline{\alpha}) + \chi\cdot\underline{\alpha} = X$, then

$$^*\mathcal{D}^{-1}(-2((h_1 - \epsilon h_2))X) = (h_1\rho - h_2\sigma - \overline{h_1\rho - h_2\sigma}, h_2\rho + h_1\sigma - \overline{h_2\rho + h_1\sigma})$$

We have to now recover (ρ, σ) from the right-hand side of the above equation. Let $(F_1, F_2) = {}^*\mathcal{D}^{-1}(-2((h_1 - \epsilon h_2))X)$. We define

$$H = \begin{pmatrix} h_1 & -h_2 \\ h_2 & h_1 \end{pmatrix}$$

Note that $\det H = h_1^2 + h_2^2 = \exp(2g_1) \neq 0$, thus H is invertible for all (h_1, h_2). We can now see that

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = H \begin{pmatrix} \rho \\ \sigma \end{pmatrix} - \overline{H \begin{pmatrix} \rho \\ \sigma \end{pmatrix}}$$

Rearranging, and taking an average,

$$\overline{H \begin{pmatrix} \rho \\ \sigma \end{pmatrix}} = \overline{H^{-1}}^{-1} \overline{\begin{pmatrix} \rho \\ \sigma \end{pmatrix}} - \overline{H^{-1}}^{-1} \overline{H^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}} \tag{3.17}$$

Here the spherical average of a matrix is the average over each component. We also used the fact that

$$\overline{H^{-1}\overline{H^{-1} \begin{pmatrix} \rho \\ \sigma \end{pmatrix}}} = \overline{H^{-1}}\,\overline{H^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}$$

The right-hand side of (3.17) is completely determined by the background metric as well as $(\overline{\rho}, \overline{\sigma})$, which can be constructed along $\mathcal{H}$ given initial data. We can now complete our gluing construction by defining the following

$$\begin{aligned} \begin{pmatrix} \rho \\ \sigma \end{pmatrix} &:= H^{-1} \left(\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \overline{H^{-1}}^{-1} \overline{\begin{pmatrix} \rho \\ \sigma \end{pmatrix}} - \overline{H^{-1}}^{-1} \overline{H^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}} \right) \\ \alpha &:= \not{\mathcal{D}}^{-1} \left(\frac{1}{\phi^2} (-L(\phi^2 \rho), L(\phi^2 \sigma)) \right) \end{aligned}$$

Thus there are no obstructions to zeroth-order gluing when $\underline{\hat{\chi}} = 0$.

□

Remark 3.5. The complex Maxwell's equations for zeroth order gluing under this assumption are given by

$$\frac{1}{\phi} L(\phi \Psi) = \frac{1}{2} m(\Theta) - \zeta_m \Theta \tag{3.18}$$

$$\not{\mathcal{D}}_m \Phi = -\phi^{-2} L(\phi^2 \Theta) \tag{3.19}$$

Since m, as an operator, maps onto all of L^2, there exists a function g such that $m(g) = -2\zeta_m$. Letting $h = e^g$, we get $m(h)/h = m(g)$. Plugging this into (3.18)

$$\frac{1}{\phi}L(\phi\Psi) = \frac{1}{2}m(\Theta) + \frac{1}{2}m(g)\Theta \tag{3.20}$$

$$= \frac{1}{2}m(\Theta) + \frac{1}{2}\frac{m(h)}{h}\Theta \tag{3.21}$$

$$= \frac{1}{2h}(m(\Theta)h + m(h)\Theta) \tag{3.22}$$

$$= \frac{1}{2h}m(h\Theta) \tag{3.23}$$

It is much easier to see how to factor the operator in this formulation than in the Lie Formulation, which was how it was originally proven.

3.2 FIRST-ORDER GLUING - SPHERICAL SYMMETRY

Before looking at Minkowski, we look at the general first-order problem in spherical symmetry. Note that in this case, our Maxwell equations take the following form

$$\mathcal{L}_4\underline{\alpha} = -\frac{1}{2}{}^*\mathcal{D}(\rho,\sigma) \tag{3.24}$$

$$\mathcal{L}_3\alpha = -\frac{1}{2}{}^*\mathcal{D}(-\rho,\sigma) \tag{3.25}$$

$$\mathcal{D}\underline{\alpha} = \frac{1}{\phi^2}\underline{L}(\phi^2\rho,\phi^2\sigma) \tag{3.26}$$

$$\mathcal{D}\alpha = \frac{1}{\phi^2}L(-\phi^2\rho,\phi^2\sigma) \tag{3.27}$$

The nonexistence of zeroth-order obstructions to the gluing problem follows from Theorem 1.5. For the first-order problem, we get the following equation

$$
\begin{aligned}
\mathcal{L}_4\mathcal{L}_3\underline{\alpha} &= \mathcal{L}_3\mathcal{L}_4\underline{\alpha} \\
&= -\frac{1}{2}\mathcal{L}_3{}^*\mathcal{D}(\rho,\sigma) \\
&= -\frac{1}{2}\nabla_3{}^*\mathcal{D}(\rho,\sigma) - \frac{1}{2}\underline{\chi}^{\sharp *}\mathcal{D}(\rho,\sigma) \\
&= -\frac{1}{2}{}^*\mathcal{D}(\underline{L}\rho,\underline{L}\sigma) - \frac{1}{2}[\nabla_3,{}^*\mathcal{D}](\rho,\sigma) - \frac{1}{2}\underline{\chi}^{\sharp *}\mathcal{D}(\rho,\sigma) \\
&= -\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\underline{\alpha} + \frac{1}{2}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}(\rho,\sigma) \\
&= -\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\underline{\alpha} + L(\mathrm{tr}\underline{\chi})\underline{\alpha} - \mathcal{L}_4(\mathrm{tr}\underline{\chi}\underline{\alpha})
\end{aligned}
$$

where we used $[\mathcal{L}_4,\mathcal{L}_3] = \mathcal{L}_{[e_4,e_3]} = 0$ under the assumption of spherical symmetry. Thus, the obstructions to gluing are given by integrals of the form

$$H^{\Theta}_{sph}(v) = \int_{S_v}(\mathcal{L}_3\underline{\alpha} + \mathrm{tr}\underline{\chi}\underline{\alpha})\Theta d\mu_{\mathbb{S}^2}$$

where $\Theta \in \mathcal{U}_{sph}$ is given by

$$\mathcal{U}_{sph} = \{X \in \mathfrak{X}(\mathbb{S}^2) : (-\frac{1}{2}{}^*\mathcal{D}\mathcal{D} + L(\mathrm{tr}\underline{\chi}))(X) = 0, \mathcal{L}_4 X = 0\}$$

Note that the elliptic operator on the right-hand side is also self-adjoint.

Minkowski spacetime

In spherical symmetry, we can replace $\phi = r$, where r is the area radius of the spherical section S, given by

$$r = \sqrt{\frac{1}{4\pi}\int_S d\mu_S}$$

In Minkowski, the area radius coincides with the coordinate r in the polar coordinate representation. Using the fact that $\mathrm{tr}\chi = \frac{\sqrt{2}}{r}$ and $\mathrm{tr}\underline{\chi} = -\frac{\sqrt{2}}{r}$, and that $L = \frac{1}{\sqrt{2}}(\partial_t + \partial_r)$ and $\underline{L} = \frac{1}{\sqrt{2}}(\partial_t - \partial_r)$ given our gauge choice $\Omega = 1$, we get that

$$\mathcal{L}_4(\mathcal{L}_3\underline{\alpha} + \mathrm{tr}\underline{\chi}\underline{\alpha}) = \frac{1}{2r^2}(\mathring{\Delta} + 1) \tag{3.28}$$

where we used the identity ${}^*\mathcal{D}\mathcal{D} = -\Delta + K = -\frac{1}{r^2}(\mathring{\Delta} - 1)$ on Minkowski. The operator on the right-hand side has a 6-dimensional kernel defined in terms of vector spherical harmonics $E^{(1m)}, H^{(1m)}$ (defined in Appendix A.3) with $m = -1, 0, 1$. We see that the quantities

$$\int_{S_v} (\not{\mathcal{L}}_3 \underline{\alpha} + \mathrm{tr}\underline{\chi}\underline{\alpha}) E^{(1m)} d\mu_{\mathbb{S}^2}, \quad \int_{S_v} (\not{\mathcal{L}}_3 \underline{\alpha} + \mathrm{tr}\underline{\chi}\underline{\alpha}) H^{(1m)} d\mu_{\mathbb{S}^2} \tag{3.29}$$

are conserved. Since the operator $\frac{1}{2r^2}(\mathring{\Delta} + 1)$ has a discrete spectrum of eigenvalues with finite multiplicity accumulating at ∞, we can repeat the proof of 1.4, *mutatis mutandis*, to solve the gluing problem for spherically symmetric metrics, including Minkowski.

Schwarzschild Horizon

By the transversal propagation equation for $\underline{\chi}$, and assuming $\mathcal{H} = \{r = 2M\}$ is the Schwarschild event horizon, we get the following equation

$$\partial_v(\mathrm{tr}\underline{\chi}) = -K$$

Thus, we get the following first-order equation along $\mathcal{H}$

$$\not{\mathcal{L}}_4(\not{\mathcal{L}}_3 \underline{\alpha} + \mathrm{tr}\underline{\chi}\underline{\alpha}) = \frac{1}{2}(-{}^*\not{\mathcal{D}}\not{\mathcal{D}} - 2K)\underline{\alpha} = \mathcal{Q}_{sph}(\underline{\alpha})$$

We can see that for all $v \in [1, 2]$ and $\forall X \in \mathfrak{X}(\mathbb{S}^2)$

$$\int_{S_v} X \cdot \mathcal{Q}_{sph}(X) d\mu_{S_v} = -\frac{1}{2} \int_{S_v} |\not{\mathcal{D}} X|^2 + 2K|X|^2 d\mu_{S_v} < 0$$

where we used that $K > 0$ for round spheres. Thus, the operator $\mathcal{Q}_{sph}$ has an empty kernel and therefore no first-order conservation laws along $\mathcal{H}$. We will now consider the case where $\mathcal{H}$ is a Killing Horizon, simplifying the Maxwell equations and allowing us to address the problem of higher-order gluing constructions.

3.3 BLACK HOLE SPACETIMES

Let $(\mathcal{M}, g)$ be a stationary spacetime admitting a black hole region (defined in [53]). For such a spacetime, the event horizon $\mathcal{H}$ is a Killing Horizon, meaning that there exists a vector field ξ that is tangent and normal to $\mathcal{H}$ such that

$$\mathbf{D}_\xi \xi = \kappa \cdot \xi \quad \text{on} \quad \mathcal{H} \tag{3.30}$$

where κ is constant along the null generators of $\mathcal{H}$. By the zeroth of law of black hole dynamics, we will take κ to be constant on all of $\mathcal{H}$, in which case, we call κ the *surface gravity* of $\mathcal{H}$. Killing Horizons with $\kappa = 0$ are called *extremal horizons*. The following properties of Killing Horizons are proven in [7].

Lemma 3.6. *Let $\mathcal{H}$ be a Killing horizon and let $\mathcal{D} = \langle S_1, L_{geod}, \Omega = 1 \rangle$ be foliation of $\mathcal{H}$, defined in 2. Let ξ be a Killing vector field normal to $\mathcal{H}$ satisfying (3.30). Then, we get the following relations on $\mathcal{H}$*

1. $\chi = 0$
2. $\not{\mathcal{L}}_L \not{g} = 0$
3. $\not{d}\kappa = g(\xi, \underline{L}) \cdot \beta$
4. $\not{\mathcal{L}}_L \underline{\eta} = \not{\nabla}_L \underline{\eta} = \beta$
5. *If we take $L_{geod}|_{S_1} = \xi|_{S_1}$, and κ is constant on $\mathcal{H}$, then*

$$\not{\mathcal{L}}_L \underline{\chi} = \not{\nabla}_L \underline{\chi} = \frac{\kappa}{f} \underline{\chi} \tag{3.31}$$

where f is such that $\xi = f \cdot L_{geod}$ on $\mathcal{H}$

Note that if we assume the Einstein vacuum equations $\mathrm{Ric}(g) = 0$, we get from the Codazzi equations that

$$\not{\mathrm{d}}\mathrm{iv}\chi - \not{d}\mathrm{tr}\chi + \chi^\sharp \cdot \zeta - (\mathrm{tr}\chi) \cdot \zeta = \beta$$

We see that by result 1) in Lemma 3.6, $\beta = 0$. If we do not assume the Einstein vacuum equations but assume constant surface gravity, it also follows that $\beta = 0$ from result 3) of Lemma 3.6. Along Killing Horizons, we get a transport equation for $\not{\nabla}_3 \underline{\alpha}$ which we derive as follows.

$$\begin{aligned}
\not{\nabla}_4 \not{\nabla}_3 \underline{\alpha} &= \not{\nabla}_3 \not{\nabla}_4 \underline{\alpha} + [\not{\nabla}_4, \not{\nabla}_3] \underline{\alpha} \\
&= -\frac{1}{2} {}^* \not{\mathcal{D}} \not{\mathcal{D}} \underline{\alpha} + (\underline{\eta}, -\underline{\eta}^*) \not{\mathcal{D}} \underline{\alpha} + 2 \not{\nabla}_{\underline{\eta}} \underline{\alpha} - \sigma_{(g)} \underline{\alpha}^* + \frac{3}{4} \mathrm{tr} \underline{\chi}^* \not{\mathcal{D}}(\rho, \sigma) \\
&\quad + \hat{\underline{\chi}}^\sharp \cdot (\not{\nabla} \sigma)^* + \big(\not{\nabla}_3 \underline{\eta} - \mathrm{tr} \underline{\chi} \underline{\eta} + \hat{\underline{\chi}}^\sharp \cdot \underline{\eta} - \frac{1}{2} \not{\nabla} \mathrm{tr} \underline{\chi}, -\not{\nabla}_3 \underline{\eta}^* + \mathrm{tr} \underline{\chi} \underline{\eta}^* \\
&\quad + \hat{\underline{\chi}}^\sharp \cdot \underline{\eta}^* + \frac{1}{2} (\not{\nabla} \mathrm{tr} \underline{\chi})^* \big) \cdot (\rho, \sigma) - \frac{3}{2} \mathrm{tr} \underline{\chi} \hat{\underline{\chi}}^\sharp \cdot \alpha - \underline{\alpha}^\sharp_{(g)} \cdot \alpha
\end{aligned}$$

where $\underline{\alpha}_{(g)}(e_A, e_B) = R_{A3B3}$ and $\sigma_{(g)} = \frac{1}{2} \not{\epsilon}^{AB} R_{AB34}$. For extremal Killing horizons, we get that the Killing field $\xi = L$. We also get that on extremal horizons, $\not{\mathcal{L}}_\xi R = \not{\mathcal{L}}_\xi Ric = \not{\mathcal{L}}_\xi R_{sc} = \not{\mathcal{L}}_\xi \underline{\chi} = 0$ (proven in [7]). Let us assume that $(\rho(2), \sigma(2)) = (\rho(1), \sigma(1))$. Thus for any pairs of function

(f,g) defined on $\mathcal{H}$ with conserved spherical averages, we define the inverse $of \mathcal{D}$ as follows

$$\mathcal{D}^{-1}(f,g) := \mathcal{D}^{-1}(f-\overline{f}, g-\overline{g})$$

Thus, along extremal horizons, we see that

$$\begin{aligned}
&\alpha = \nabla_4 \mathcal{D}^{-1} J(\rho,\sigma) \\
&(\rho,\sigma) = \nabla_4 \left(\left(-\frac{1}{2}{}^*\mathcal{D} + (\underline{\eta}, -\underline{\eta}^*) \right)^{-1} (\underline{\alpha} - \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1} J(\rho,\sigma)) \right) \\
&(-\frac{1}{2}{}^*\mathcal{D}\mathcal{D} + (\underline{\eta}, -\underline{\eta}^*)\mathcal{D} + 2\nabla_{\underline{\eta}} - \sigma_{(g)}\not{\epsilon})\underline{\alpha} = \\
&\quad \nabla_4 \left(\nabla_3 \underline{\alpha} + (\frac{3}{2}\mathrm{tr}\underline{\chi}\underline{\hat{\chi}}^\sharp + \underline{\alpha}^\sharp_{(g)}) \cdot \mathcal{D}^{-1} J(\rho,\sigma) - \mathcal{R}(\underline{\alpha} - \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1} J(\rho,\sigma)) \right)
\end{aligned}$$

where we used that $[\nabla_4, (-\frac{1}{2}{}^*\mathcal{D} + (\underline{\eta}, -\underline{\eta}^*))] = 0$ and thus holds for the inverse by the relation $[A, B^{-1}] = B^{-1}[B, A]B^{-1}$ and where the operator $\mathcal{R}$ is as follows.

$$\mathcal{R}(\rho,\sigma) = \left(-\frac{1}{2}{}^*\mathcal{D} + (\underline{\eta}, -\underline{\eta}^*) \right)^{-1} \left(\frac{3}{4}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}(\rho,\sigma) + \underline{\hat{\chi}}^\sharp \cdot (\nabla\sigma)^* \right.$$

$$\left. + (\nabla_3\underline{\eta} - \mathrm{tr}\underline{\chi}\underline{\eta} + \underline{\hat{\chi}}^\sharp \cdot \underline{\eta} - \frac{1}{2}\nabla \mathrm{tr}\underline{\chi}, -\nabla_3\underline{\eta}^* + \mathrm{tr}\underline{\chi}\underline{\eta}^* + \underline{\hat{\chi}}^\sharp \cdot \underline{\eta}^* + \frac{1}{2}(\nabla \mathrm{tr}\underline{\chi})^*) \cdot (\rho,\sigma) \right)$$

Let us now prove our second main theorem.

Proof of Theorem 1.6: If the conserved charges of the prescribed data $H^{\Theta}_{Ext}(1)$ at S_1 and $H^{\Theta}_{Ext}(2)$ at S_2 agree, we should be able to construct a solution along $\mathcal{H}$. Note the following estimate

$$\begin{aligned}
\int_{S_v} X \cdot (\mathcal{Q}^{(1)} - \frac{1}{\epsilon_1})(X) d\mu_{S_v} &= \int_{S_v} -\frac{1}{2}|\mathcal{D}X|^2 + (\underline{\eta}(X), -\underline{\eta}^*(X))\mathcal{D}X \\
&\qquad + 2\nabla_{\underline{\eta}} X \cdot X - \frac{1}{\epsilon_1}|X|^2 d\mu_{S_v} \\
&= \int_{S_v} -\frac{1}{2}|\mathcal{D}X|^2 + (\underline{\eta}(X), -\underline{\eta}^*(X))\mathcal{D}X \\
&\qquad - \mathrm{d}\!\!/\mathrm{iv}\underline{\eta}|X|^2 - \frac{1}{\epsilon_1}|X|^2 d\mu_{S_v}
\end{aligned}$$

Note that along Extremal Horizons

$$L(\mathrm{tr}\underline{\chi}) = -\mathrm{d}\!\!/\mathrm{iv}\underline{\eta} + |\underline{\eta}|^2 - K - \mathrm{tr}\chi\mathrm{tr}\underline{\chi} = 0 \tag{3.32}$$

Hence

$$0 = -\text{d}\!\!/\text{iv}\underline{\eta} + |\underline{\eta}|^2 - K$$

Therefore, we get that

$$\begin{aligned}\int_{S_v} X \cdot (\mathcal{Q}^{(1)} - \frac{1}{\epsilon_1})(X) d\mu_{S_v} &= -\frac{1}{2}\int_{S_v} |\not{\mathcal{D}}X|^2 - 2(\underline{\eta}(X), -\underline{\eta}^*(X))\not{\mathcal{D}}X \\ &\qquad (-2|\underline{\eta}|^2 + 2K + \frac{1}{\epsilon_1})|X|^2 d\mu_{S_v} \\ &\leq -\frac{1}{2}\int_{S_v} (1-\epsilon)|\not{\mathcal{D}}X|^2 \\ &\qquad + ((-2 - \frac{2}{\epsilon})|\underline{\eta}|^2 + 2K + \frac{1}{\epsilon_1})|X|^2 d\mu_{S_v} \\ &\leq -\frac{1}{2}\int_{S_v} (1-\epsilon)|\nabla\!\!\!/ X|^2 \\ &\qquad + ((-2 - \frac{2}{\epsilon})|\underline{\eta}|^2 + (3-\epsilon)K + \frac{1}{\epsilon_1})|X|^2 d\mu_{S_v}\end{aligned}$$

Choosing ϵ, ϵ_1 sufficiently small, $(\mathcal{Q}^{(1)} - \frac{1}{\epsilon_1})$ is invertible and by the Fredholm alternative and must have discrete eigenvalues with an accumulation point at infinity. Note that the propagation equation (3.32) and Gauss equation do not need to be assumed for this estimate but make it slightly easier. Thus, if $H_\Theta(2) = H_\Theta(1)$ for all $\Theta \in \mathcal{U}_{extremal}$, we can use a similar argument as we did in Theorem 1.4 for zeroth-order gluing to construct $\underline{\alpha}$. We then construct (ρ, σ) such that

$$(\int_1^2 \rho dv, \int_1^2 \sigma dv) = \left(\left(-\frac{1}{2}{}^*\not{\mathcal{D}} + (\underline{\eta}, -\underline{\eta}^*) \right)^{-1} (\underline{\alpha} - \underline{\hat{\chi}}^\sharp \cdot \not{\mathcal{D}}^{-1} J(\rho, \sigma)) \right) \Big|_{v=1}^{v=2}$$

and agrees with the prescribed data at $v = 1, 2$. Finally, we construct α and $\nabla\!\!\!/_3 \underline{\alpha}$ from the remaining null Maxwell equations to complete the gluing construction.

□

Proof of Theorem 1.7: Let us introduce a method of v-weighted integrals to prove gluing constructions along extremal horizons. On extremal horizons, we get that for any S-tangent tensor T, $\nabla\!\!\!/_4 T = \not{\mathcal{L}}_4 T$. Since $\chi = \not{\mathcal{L}}_4 \not{g} = 0$, we get that for any S-tangent 1 -form ω on $\mathcal{H}$

$$\int_1^v \not{\mathcal{D}}\omega(v')dv' = \not{\mathcal{D}} \int_1^v \omega(v')dv'$$

since our frame (e_A) is Lie transported ($[\partial_v, e_A] =$) and $\chi = 0$. In sphere coordinates, letting $e_A = e^i_A \partial_i$, we see that

$$\begin{aligned}\int_1^v \mathrm{d}\!\!/\mathrm{iv}\,\omega(v')dv' &= \int_1^v g^{11} e_1(\omega_1) + g^{22} e_2(\omega_2) dv' \\ &= g^{11} \int_1^v e_1(\omega_1) dv' + g^{22} \int_1^v e_2(\omega_2) dv' \\ &= g^{11} \int_1^v e^i_1 \partial_i(\omega_1) dv' + g^{22} \int_1^v e^i_2 \partial_i(\omega_2) dv' \\ &= g^{11} e^i_1 \partial_i \int_1^v (\omega_1) dv' + g^{22} e^i_2 \partial_i \int_1^v (\omega_2) dv' \\ &= \mathrm{d}\!\!/\mathrm{iv} \int_1^v \omega(v') dv'\end{aligned}$$

where we used that

$$\begin{aligned}\partial_v(g^{AB}) &= (\mathcal{L}\!\!\!/_4 g^{-1})^{AB} + g^{-1}(\mathcal{L}\!\!\!/_4 e^A, e^B) + g^{-1}(\mathcal{L}\!\!\!/_4 e^B, e^A) \\ &= -2\chi^{AB} = 0\end{aligned}$$

and the fact that $0 = [\partial_v, e_A] = \partial_v(e^i_A)\partial_i \implies \partial_v(e^i_A) = 0$. We can rewrite equations (2.35) and (2.36)

$$(\rho, \sigma) = (\rho(1), \sigma(1)) + J\mathcal{D}\!\!\!/ \int_1^v \alpha(v') dv' \tag{3.33}$$

where we used that $L\phi = 0$. Plugging this expression into (2.31), and using that $\mathcal{L}\!\!\!/_4 \underline{\eta} = 0$

$$\begin{aligned}\mathcal{L}\!\!\!/_4 \underline{\alpha} = -\frac{1}{2}{}^*\mathcal{D}\!\!\!/ J \mathcal{D}\!\!\!/ (\int_1^v \alpha(v')dv') + (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D}\!\!\!/ (\int_1^v \alpha(v')dv') + \underline{\hat{\chi}} \cdot \alpha \\ - \frac{1}{2}{}^*\mathcal{D}\!\!\!/ J\mathcal{D}\!\!\!/ (\rho(1), \sigma(1)) + (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D}\!\!\!/ (\rho(1), \sigma(1))\end{aligned}$$

Integrating this equation from $v = 1$ to $v = 2$, we get,

$$\begin{aligned}\underline{\alpha}(2) - \underline{\alpha}(1) - \int_1^2 -\frac{1}{2}{}^*\mathcal{D}\!\!\!/ J\mathcal{D}\!\!\!/ (\rho(1), \sigma(1)) + (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D}\!\!\!/ (\rho(1), \sigma(1)) dv \\ = -\frac{1}{2}{}^*\mathcal{D}\!\!\!/ J\mathcal{D}\!\!\!/ (\int_1^2 \int_1^v \alpha(v')dv'dv) + (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D}\!\!\!/ (\int_1^2 \int_1^v \alpha(v')dv'dv) \\ + \int_1^2 \underline{\hat{\chi}}^\sharp \cdot \alpha dv\end{aligned}$$

Note that the left hand side only depends on the background geometry and the initial data. By changing the order of integration of v and v' we get that

$$\int_1^2 \int_1^v \alpha(v')dv'dv = 2\int_1^2 \alpha(v)dv - \int_1^2 v\alpha(v)dv \tag{3.34}$$

In fact, in general, we see that, for any integrable object $f(v)$

$$\int_1^2 \int_1^{v_1} ... \int_1^{v_n} f(v_{n+1})dv_{n+1}...dv_1 dv = \int_1^2 \frac{1}{n!}(2-v)^n f(v)dv \tag{3.35}$$

Note that by setting $v = 2$ in (3.33) we get the following equation

$$(\rho(2), \sigma(2)) - (\rho(1), \sigma(1)) = J\mathcal{D}\int_1^v \alpha(v')dv' \tag{3.36}$$

In addition, for the left-hand side to be in the image of $J\mathcal{D}$, the spherical average of the left-hand side has to equal 0. This is related to the quantities (ρ_0, σ_0) being conserved. Assuming this, we see that the integral $\int_1^2 \alpha dv$ is completely determined by initial data. Thus, we replace the integral $\int_1^2 \int_1^v \alpha(v')dv'dv$ with the integral $\int_1^2 v\alpha dv$ and move the initial data terms to the right. We have hence reduced the gluing problem along Killing Horizons to solving the following system

$$\begin{aligned} F &= \int_1^2 \alpha(v)dv \\ G' &= \frac{1}{2}{}^*\mathcal{D}J\mathcal{D}(\int_1^2 v\alpha(v)dv) - (\underline{\eta}, -\underline{\eta}^*)\cdot J\mathcal{D}(\int_1^2 v\alpha(v)dv) + \int_1^2 \underline{\hat{\chi}}^\sharp \cdot \alpha dv \end{aligned}$$

where

$$\begin{aligned} F &= \mathcal{D}^{-1}(-\rho(2) + \rho(1), \sigma(2) - \sigma(1)) \\ G' &= \underline{\alpha}(2) - \underline{\alpha}(1) - \int_1^2 -\frac{1}{2}{}^*\mathcal{D}J\mathcal{D}(\rho(1), \sigma(1)) + (\underline{\eta}, -\underline{\eta}^*)\cdot J\mathcal{D}(\rho(1), \sigma(1))dv \\ &\quad + {}^*\mathcal{D}J\mathcal{D}(F) - 2(\underline{\eta}, -\underline{\eta}^*)\cdot J\mathcal{D}(F) \end{aligned}$$

Note that along a Killing Horizon, if the Killing vector field $\xi|_{S_1} = L$, then

$$\mathcal{L}_4\underline{\chi} = \nabla_4\underline{\chi} = \kappa\cdot\underline{\chi}|_{S_1} \tag{3.37}$$

where κ is the surface gravity. We see that $\underline{\hat{\chi}}^\sharp = (v-1)\kappa\underline{\hat{\chi}}^\sharp|_{S_1} + \underline{\hat{\chi}}^\sharp|_{S_1}$ and we can write

$$F = \int_1^2 \alpha(v)dv \tag{3.38}$$

$$\begin{aligned} G &= \frac{1}{2}{}^*\mathcal{D}J\mathcal{D}(\int_1^2 v\alpha(v)dv) - (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D}(\int_1^2 v\alpha(v)dv) + \kappa\underline{\hat{\chi}}^\sharp|_{S_1} \cdot \int_1^2 v\alpha dv \\ &= (\frac{1}{2}{}^*\mathcal{D}J\mathcal{D} - (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D} + \kappa\underline{\hat{\chi}}^\sharp|_{S_1})(\int_1^2 v\alpha dv) \end{aligned} \tag{3.39}$$

where again F, G simply represent 1-forms that only depend on prescribed data and the background geometry. Therefore, in order to do zeroth-order gluing on Killing Horizons, we want the operator $\frac{1}{2}{}^*\mathcal{D}J\mathcal{D} - (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D} + \kappa\underline{\hat{\chi}}^\sharp|_{S_1}$ to be surjective. Let us look at the case where $\kappa = 0$ i.e. an Extremal Horizon. From Theorem 1.5 and the fact that $\mathcal{D}$ is invertible, we see that the operator $\frac{1}{2}{}^*\mathcal{D}J\mathcal{D} - (\underline{\eta}, -\underline{\eta}^*)J\mathcal{D} = \frac{1}{2}{}^*\mathcal{D}^{\underline{\eta}}J\mathcal{D}$ is invertible. The problem is therefore reduced to the following lemma.

Lemma 3.7. *Given $n+2$ arbitrary smooth 1-forms on the sphere $W_1, ..., W_{n+2}$, can we find an S-tangent 1-form α on $\mathbb{S}^2 \times [1,2]$ such that*

$$\begin{aligned} W_1 &= \alpha(1), \\ W_2 &= \alpha(2) \\ W_3 &= \int_1^2 \alpha(v)dv \\ W_4 &= \int_1^2 v\alpha(v)dv \\ &\vdots \\ W_{n+2} &= \int_1^2 v^n\alpha(v)dv \end{aligned}$$

The proof of this uses an orthogonality argument and is identical to the proof of Lemma 4.18 in [10]. While we were able to prove gluing for the extremal case, for the $\kappa \neq 0$, we are interested in whether or not $\mathcal{Q}_{sub} = \frac{1}{2}{}^*\mathcal{D}J\mathcal{D} - (\underline{\eta}, -\underline{\eta}^*) \cdot J\mathcal{D} + \kappa\underline{\hat{\chi}}^\sharp|_{S_1}$ is invertible.

Question 3.8. *Is $\mathcal{Q}_{sub}$ invertible? If not, how do we use its kernel to construct conservation laws?*

For the nth-order gluing problem for spherically symmetric extremal horizons, similar to the first-order case, the nth-order derivatives of ρ, σ and α can be determined by the lower-order transversal derivatives. Thus, for 2nd order gluing, we only need to consider the following transport equations for the transversal derivatives of $\underline{\alpha}$

$$\begin{aligned}\nabla_4\underline{\alpha} &= -\frac{1}{2}{}^*\mathcal{D}(\rho,\sigma)\\ \nabla_4\nabla_3\underline{\alpha} &= -\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\underline{\alpha} + \frac{3}{4}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}(\rho,\sigma)\\ \nabla_4\nabla_3\nabla_3\underline{\alpha} &= -\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\nabla_3\underline{\alpha} + \frac{3}{4}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}\mathcal{D}\underline{\alpha} - \frac{9}{8}(\mathrm{tr}\underline{\chi})^{2*}\mathcal{D}(\rho,\sigma)\end{aligned}$$

By induction, one can see that,

$$\nabla_4\nabla_3^k\underline{\alpha} = -\frac{1}{2}{}^*\mathcal{D}\mathcal{D}\nabla_3^{(k-1)}\underline{\alpha} + G(\nabla_3^{(k-2)}\underline{\alpha},...,\underline{\alpha}) + C(\mathrm{tr}\underline{\chi})^{k*}\mathcal{D}(\rho,\sigma)$$

where G is a smooth operator satisfying

$$\int_1^v G(\nabla_3^{(k-2)}\underline{\alpha},...,\underline{\alpha})dv = G(\int_1^v \nabla_3^{(k-2)}\underline{\alpha}dv,...,\int_1^v \underline{\alpha}dv)$$

where we used the following commutation relations along spherically symmetric Killing horizons

$$\begin{aligned}[\nabla_3, {}^*\mathcal{D}] &= -\frac{1}{2}\mathrm{tr}\underline{\chi}{}^*\mathcal{D}\\ [\nabla_3, \mathcal{D}] &= \frac{1}{2}\mathrm{tr}\underline{\chi}\mathcal{D}\end{aligned}$$

which implies

$$[\nabla_3, {}^*\mathcal{D}\mathcal{D}] = 0$$

Integrating in v, assuming $\mathcal{H}$ is an extremal horizon, we get the following system

$$\begin{aligned}F_1 &= \int_1^2 \alpha(v)dv\\ F_2 &= ({}^*\mathcal{D}J\mathcal{D})(\int_1^2 v\alpha dv)\\ F_3 &= ({}^*\mathcal{D}\mathcal{D}{}^*\mathcal{D}J\mathcal{D})(\int_1^2 v^2\alpha dv)\\ &\vdots\\ F_k &= (({}^*\mathcal{D}\mathcal{D})^{k-2*}\mathcal{D}J\mathcal{D})(\int_1^2 v^{k-1}\alpha dv)\end{aligned}$$

Where F_i only depends on initial data. Since the operators $\not{\mathcal{D}}, {}^*\not{\mathcal{D}}$ are invertible, there are no obstructions to gluing. By applying Lemma 3.7, we finish the proof of 1.6.

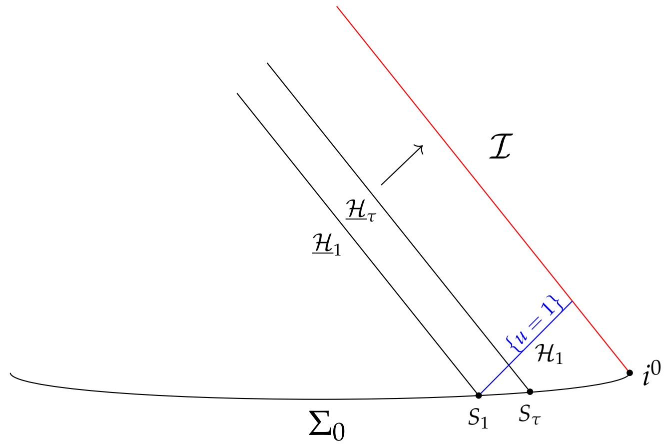
3.4 CONSERVATION LAWS AT NULL INFINITY

In this section, we are interested in constructing conservation laws along null infinity $\mathcal{I}$ by constructing conservation laws along ingoing null hypersurfaces $\underline{\mathcal{H}}$, and then taking the appropriate limit to infinity. In [9], Aretakis similarly constructs conservation laws for the wave equation along ingoing null hypersurfaces and by studying the limiting case, recovers the Newman-Penrose constants, defined and discussed in [43] and [44]. We discuss the geometric structure of null infinity of a $(3+1)$-dimensional asymptotically flat Lorentzian manifold.

Structure of null infinity $\mathcal{I}$

Let $(\mathcal{M}, g)$ be a regular globally hyperbolic four-dimensional Lorentzian manifold and let S_1 be a sphere embedded in an initial Cauchy hypersurface Σ. We assume that the outgoing null geodesics normal to S_1 are future complete and thus generate a future complete null hypersurface $\mathcal{H}_1$. We take $\mathcal{S} = \langle S_1, L_{geod}, \Omega = 1\rangle$ to be a foliation of $\mathcal{H}_1$. Let τ be the affine parameter of L_{geod} on $\mathcal{H}_1$, in other words $\tau|_{S_1} = 1$ and $L_{geod}(\tau) = 1$, and let S_τ be the corresponding spherical sections on $\mathcal{H}_1$. Let $\underline{\mathcal{H}}_\tau$ denote the ingoing null hypersurface of $\mathcal{M}$ generated by incoming null geodesics normal to S_τ. We consider the collection $\{\mathcal{D}_\tau, \tau \geq 1\}$ of the double null foliations generated by $\mathcal{D}_\tau = \left\langle S_\tau, L_{geod}\big|_{S_\tau}, \Omega|_{\mathcal{H}_1} = 1, \Omega|_{\underline{\mathcal{H}}_\tau} = 1\right\rangle$ where $L_{geod}\big|_{S_\tau}$ is normalized such that

$$tr\chi + tr\underline{\chi} = 0 \ : \quad \text{on } S_\tau. \tag{3.40}$$



Let us now consider the induced foliation $\underline{\mathcal{S}}_\tau = \left\langle S_\tau, \underline{L}_{geod}\Big|_{S_\tau}, \Omega|_{\underline{\mathcal{H}}_\tau} = 1 \right\rangle$ on $\underline{\mathcal{H}}_\tau$. Let $g\!\!\!/$ be the induced metric, $\nabla\!\!\!\!/$ the induced covariant derivative with respect to $g\!\!\!/$ and $\Delta\!\!\!\!/$ the induced Laplacian. We also define $A(S)$ to be the area and $r(S) = \sqrt{A(S)/4\pi}$ the radius function of a particular spherical section S. Suppose that as $\tau \to +\infty$ we have $r(S_\tau) \to +\infty$. If $(\mathcal{M}, g)$ is asymptotically flat in the sense of [53] then the null infinity $\mathcal{I}$ is defined to be the limit as $\tau \to +\infty$ of the hypersurfaces $\underline{\mathcal{H}}_\tau$. Quantities associated to $\mathcal{I}$ can only be understood in a limiting, appropriately rescaled sense with respect to the radius variable r (note that the limit of the foliations $\underline{\mathcal{S}}_\tau$, as $\tau \to +\infty$, induces a foliation of $\mathcal{I}$). The precise decay rates for the metric of asymptotically flat spacetimes vary in the literature. In our setting, it suffices to have the following rates which were used to understand the Newman-Penrose constants in [9] [1] (which are consistent with [14]):

$$\begin{aligned} &\frac{1}{r^2}g\!\!\!/ \to g\!\!\!/_{\mathbb{S}^2}, \quad \frac{1}{r^2}\sqrt{g\!\!\!/} \to \sin\theta, \quad r^2\nabla\!\!\!\!/ \to \nabla\!\!\!\!/_{\mathbb{S}^2}, \quad r^2\Delta\!\!\!\!/ \to \Delta\!\!\!\!/_{\mathbb{S}^2}, \quad \frac{1}{r}\phi \to 1, \\ &\partial_v r \to \frac{1}{\sqrt{2}}, \quad \partial_u r \to -\frac{1}{\sqrt{2}}, \quad r\zeta \to Z, \\ &r tr\chi \to \sqrt{2}, \quad r tr\underline{\chi} \to -\sqrt{2}, \quad r^2\partial_v tr\underline{\chi} \to 1, \quad r^2\partial_u tr\chi \to 1, \end{aligned} \tag{3.41}$$

where Z is a 1-form defined on $\mathbb{S}^2$. The definition of the above quantities can be found in Section 2.1. The above limits should be understood in terms of the pullback of the induced tensor fields to the standard sphere via the diffeomorphism $\Phi_{u,v}$ (see Section 2.1).

1 No other assumptions are required for the metric; no constraint equations need to be satisfied on null infinity.

Newman-Penrose constants - Wave equation

Before deriving the conservation laws for Maxwell, we look at the wave equation to understand a simpler model for conservation laws. We derive the conserved Newman-Penrose constants, as was shown in [9]. In spherical symmetry, the wave equation takes the following form

$$-2\partial_u\partial_v(r\psi) = \not{\Delta}(r\psi) + \left[\partial_u(tr\chi) + \frac{1}{2}(tr\underline{\chi})\cdot(tr\chi)\right]\cdot(r\psi)$$

Projecting the equation onto spherical harmonics Y^{lm}, and by defining $\psi^{lm} = \int_{\mathbb{S}^2}\psi\cdot Y^{lm}d\mu_{\mathbb{S}^2}$ we get

$$\begin{aligned} &-2\partial_u\partial_v(r\psi^{lm}) = \\ &\qquad \frac{1}{r^2}\left[-l(l+1) + r^2\partial_u(tr\chi) + \frac{r^2}{2}(tr\underline{\chi})\cdot(tr\chi)\right]\cdot(r\psi^{lm}) \end{aligned} \tag{3.42}$$

If we use the ingoing Eddington-Finkelstein coordinates $(u, r, \theta^1, \theta^2)$, we obtain

$$\psi(u,r,\theta) = \frac{a_1(u,\theta^1,\theta^2)}{r} + \frac{a_2(u,\theta^1,\theta^2)}{r^2} + O(\frac{1}{r^3})$$

and thus

$$\psi^{lm}(u,r) = \frac{a_1^{lm}(u)}{r} + \frac{a_2^{lm}(u)}{r^2} + O(\frac{1}{r^3})$$

From this, we note the following limits

$$\lim_{r\to+\infty} r\psi^{lm} = a_1^{lm}$$
$$\lim_{r\to+\infty} r^2\partial_v(r\psi^{lm}) = -\sqrt{2}a_2^{lm}$$

Note that

$$\begin{aligned} r^2\partial_u\partial_v(r\psi^{lm}) &= \partial_u(r^2\partial_v(r\psi^{lm})) - 2r(\partial_u r)\partial_v(r\psi^{lm})) \\ &\to \partial_u(r^2\partial_v(r\psi^{lm})) \end{aligned}$$

Therefore, in order to get a nonzero finite limit at infinity, we need to multiply equation (3.42) by r^2. The right-side limit is given by

$$\lim_{r\to+\infty}\frac{r^2}{r^2}\left[-l(l+1) + r^2\partial_u(tr\chi) + \frac{r^2}{2}(tr\underline{\chi})\cdot(tr\chi)\right]\cdot(r\psi^{lm}) = -l(l+1)a_1^{lm}(u)$$

where we used that $r^2(\partial_u(tr\chi) + \frac{1}{2}(tr\underline{\chi}) \cdot (tr\chi)) \to (1-1) = 0$. Hence, we get a conserved quantity when $l = m = 0$ and we recover the Newman-Penrose constant

$$\lim_{r \to +\infty} \int_{S_u} r^2 \partial_v (r\psi) d\mu_{\mathbb{S}^2} = -\sqrt{2} \int_{S_u} a_2(u, \theta^1, \theta^2) d\mu_{\mathbb{S}^2}$$

Note the importance of having second-order asymptotics to derive the conservation law. Let us now look at the case for Maxwell.

Conservation laws at null infinity - Maxwell

Let us look at how the Maxwell equations propagate along ingoing null hypersurfaces. For first-order gluing along ingoing null hypersurfaces, the roles of the vector fields L and $\underline{L}$ reverse, with $\underline{L}$ being the null normal to the hypersurface $\underline{\mathcal{H}}_\tau$ for a fixed τ and L the transversal direction. Our notion of kth-order gluing now involves gluing L derivatives of our null Maxwell components. We also let (S_u) be a foliation of this ingoing hypersurface. One can see that the Maxwell equations are invariant under the frame transformation $(e_3, e_4) \mapsto (e_4, e_3)$ since under this transformation

$$\alpha \mapsto \underline{\alpha}$$
$$\underline{\alpha} \mapsto \alpha$$
$$\rho \mapsto -\rho$$
$$\sigma \mapsto \sigma$$

Thus, we can apply this transformation to any outgoing transport equation to get the corresponding ingoing equation. In order to get a transport equation for the transversal derivative of α, we will use the covariant formulation of Maxwell's equations in spherical symmetry, which are given as

$$\frac{1}{\phi} \nabla\!\!\!/_4 (\phi \underline{\alpha}) = -\frac{1}{2} {}^* \mathcal{D}\!\!\!/ (\rho, \sigma) \tag{3.43}$$

$$\frac{1}{\phi} \nabla\!\!\!/_3 (\phi \alpha) = -\frac{1}{2} {}^* \mathcal{D}\!\!\!/ (-\rho, \sigma) \tag{3.44}$$

$$\mathcal{D}\!\!\!/ \underline{\alpha} = \frac{1}{\phi^2} \underline{L} (\phi^2 \rho, \phi^2 \sigma) \tag{3.45}$$

$$\mathcal{D}\!\!\!/ \alpha = \frac{1}{\phi^2} L (-\phi^2 \rho, \phi^2 \sigma) \tag{3.46}$$

Our first-order equation can be derived from Equations (3.24) - (3.27) as follows

$$\begin{aligned}\nabla\!\!\!/_3\nabla\!\!\!/_4(\phi\alpha) &= \nabla\!\!\!/_4\nabla\!\!\!/_3(\phi\alpha)\\ &= \nabla\!\!\!/_4\left(-\frac{\phi}{2}{}^*\mathcal{D}\!\!\!/(-\rho,\sigma)\right)\\ &= -\frac{\phi}{2}\nabla\!\!\!/_4{}^*\mathcal{D}\!\!\!/(-\rho,\sigma) - \frac{\phi}{4}\mathrm{tr}\chi^*\mathcal{D}\!\!\!/(-\rho,\sigma)\\ &= -\frac{\phi}{2}{}^*\mathcal{D}\!\!\!/(-L\rho,L\sigma) - \frac{\phi}{2}[\nabla\!\!\!/_4,{}^*\mathcal{D}\!\!\!/](-\rho,\sigma) - \frac{\phi}{4}\mathrm{tr}\chi^*\mathcal{D}\!\!\!/(-\rho,\sigma)\\ &= -\frac{\phi}{2}{}^*\mathcal{D}\!\!\!/\mathcal{D}\!\!\!/\alpha + \frac{1}{2}\mathrm{tr}\chi^*\mathcal{D}\!\!\!/(-\rho,\sigma)\\ &= -\frac{\phi}{2}{}^*\mathcal{D}\!\!\!/\mathcal{D}\!\!\!/\alpha + \partial_u(\mathrm{tr}\chi)\phi\alpha - \nabla\!\!\!/_3(\phi\mathrm{tr}\chi\alpha)\end{aligned}$$

Using that $\phi = r$, we get

$$\begin{aligned}\nabla\!\!\!/_3(\nabla\!\!\!/_4(r\alpha) + \mathrm{tr}\chi(r\alpha)) &= -\frac{1}{2}{}^*\mathcal{D}\!\!\!/\mathcal{D}\!\!\!/(r\alpha) + \partial_u(\mathrm{tr}\chi)(r\alpha)\\ &= -\frac{1}{2r^2}(\Delta\!\!\!/_{S^2} - 1 + 2r^2\partial_u(\mathrm{tr}\chi))(r\alpha) := \underline{\mathcal{Q}}(r\alpha)\end{aligned}$$

Since the metric commutes with covariant derivatives, we get the same equations for $\alpha^\sharp$ and $\underline{\alpha}^\sharp$. In order to find the appropriate equation along null infinity $\mathcal{I}$, we project the vector equations onto the vector spherical harmonics (E^{lm}, H^{lm}), defined in Appendix A.3. Let rK^{lm} refer to either vector harmonic (rE^{lm}, rH^{lm}). Then,

$$\int_{S_u}\nabla\!\!\!/_3(\nabla\!\!\!/_4(r\alpha^\sharp) + \mathrm{tr}\chi(r\alpha))\cdot rK^{lm}d\mu_{S^2} = \int_{S_u}\underline{\mathcal{Q}}(r\alpha^\sharp)\cdot rK^{lm}d\mu_{S^2}$$

Note that the volume measure for the spherical section is $d\mu_{S_u} = r^2 d\mu_{S^2}$ where $d\mu_{S^2}$ is the volume measure for the round unit sphere. We rewrite the above equation in more convenient notation as follows

$$\partial_u(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) = \frac{1}{2r^2}(-l(l+1) + 2r^2\partial_u(\mathrm{tr}\chi))(r\alpha_K^{lm}) \tag{3.47}$$

where we used that $\nabla\!\!\!/_4(rK^{lm}) = 0 = \nabla\!\!\!/_3(rK^{lm})$ by our definition in Appendix A.3 and defined

$$\alpha_K^{lm} = \int_{S_u}\alpha^\sharp\cdot(rK^{lm})d\mu_{S^2} \tag{3.48}$$

Note that in spherical symmetry $[L, \underline{L}] = 0$ and thus $L = e_4 = \partial_v$ and $\underline{L} = e_3 - \partial_u$. Suppose the leading order asymptotics for α near null infinity are given by

$$\alpha = \frac{C(u, \theta^1, \theta^2)}{r^p} + o(r^{-p-\delta})$$

and thus for α_K^{lm}

$$\alpha_K^{lm} = \frac{C_K^{lm}(u)}{r^p} + o(r^{-p-\delta})$$

It is interesting to note that the precise late-time asymptotics of the null Maxwell equations are not known. However, as shown in [12] and [31], the null component α satisfies $|\alpha| \lesssim r^{-5/2}$ near null infinity and thus we must have $p \geq 5/2$. One can see that hence,

$$\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm}) = ((1-p)\partial_v r + \mathrm{tr}\chi r)\frac{C_K^{lm}(u)}{r^p} + o(r^{-p-\delta})$$

which implies that

$$\lim_{r\to\infty} r^p(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) = \frac{3-p}{\sqrt{2}} C_K^{lm}(u) \tag{3.49}$$

Remark 3.9. *It is important to note that the radius function r is not constant along ingoing null hypersurfaces $\underline{\mathcal{H}}_\tau$ but can be taken to be constant along $\mathcal{I}$ in the limiting sense. We also note that*

$$\begin{aligned} r^p\partial_u(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) &= \partial_u(r^p(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm}))) \\ &\quad - pr^{p-1}(\partial_u r)(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) \\ &\to \partial_u(r^p(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm}))) \end{aligned}$$

as $r \to \infty$

We see that in order to get a non-vanishing limit at null infinity, we need to multiply Equation (3.47) by the function r^p before taking the limit. However, evaluating the same limit on the right-hand side, we get

$$\begin{aligned} \frac{r^p}{2r^2}(-l(l+1) + 2r^2\partial_u(\mathrm{tr}\chi))(r\alpha_K^{lm}) &= \frac{1}{2}(-l(l+1) + 2r^2\partial_u(\mathrm{tr}\chi))(r^{p-1}\alpha_K^{lm}) \\ &\to (\frac{1}{2}(-l(l+1) + 2)\cdot 0) = 0 \end{aligned}$$

and therefore $\lim_{r\to\infty} r^p(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm}))$ is conserved along null infinity. This implies the existence of an infinite number of conserved quantities along null infinity. Let us look at the case when $p = 3$, in which case the limit (3.49) goes to 0. In this case, we require second-order asymptotics of α. Suppose that we have the following expansion, near null infinity

$$\alpha_K^{lm} = \frac{C_K^{lm}(u)}{r^p} + \frac{D_K^{lm}(u)}{r^{p+q}} + o(r^{-p-q-\delta})$$

We observe

$$\lim_{r\to\infty} r^{q+1}(\partial_v(r^p\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) = \frac{-q}{\sqrt{2}}D_K^{lm}(u) \tag{3.50}$$

We note that for general $p, q > 0$

$$\begin{aligned} r^{p+q}\partial_u(\partial_v(r\alpha_K^{lm}) + \mathrm{tr}\chi(r\alpha_K^{lm})) &= r^{p+q}\partial_u(r^{-2}\partial_v(r^3\alpha_K^{lm})) \\ &= r^{q+1}(\partial_u\partial_v(r^p\alpha_K^{lm}) - \partial_u(\partial_v r(p-3)r^{p-1}\alpha_K^{lm}) \\ &\quad - \partial_u r(p-1)r^{p-1}\partial_v(r^3\alpha_K^{lm})) \\ &= \partial_u(r^{q+1}\partial_v(r^p\alpha_K^{lm})) - (q+1)r^q\partial_u r\partial_v(r^p\alpha_K^{lm}) \\ &\quad - r^q(\partial_u(\partial_v r(p-3)r^{p-1}\alpha_K^{lm}) \\ &\quad - \partial_u r(p-1)r^{p-4}\partial_v(r^3\alpha_K^{lm})) \\ &\to \partial_u(r^{q+1}\partial_v(r^p\alpha_K^{lm})) \end{aligned}$$

The right-hand side of the equation is given as follows

$$\begin{aligned} \frac{r^{p+q}}{2r^2}(-l(l+1) + 2r^2\partial_u(\mathrm{tr}\chi))(r\alpha_K^{lm}) &= \frac{r^{q+1}}{2r^2}(-l(l+1) + 2r^2\partial_u(\mathrm{tr}\chi))(r^p\alpha_K^{lm}) \\ &\to (\frac{1}{2}(-l(l+1)+2)\cdot C_K^{lm})\cdot \lim_{r\to\infty} r^{q-1} \end{aligned}$$

In order for the limit to exist and be nonzero, we see that $q = 1$. In this particular case, we get six conservation quantities, similar to Minkowski, given by

$$r^2\partial_v(r^3\alpha_E^{1m}), \qquad r^2\partial_v(r^3\alpha_H^{1m})$$

with $m \in \{-1, 0, 1\}$. From our earlier calculation, we saw that for there to not be an infinite number of conservation laws along null infinity, that $p = 3$.

Question 3.10. *Is $p = 3$ the leading order asymptotic near null infinity of the null component α given that it prevents the existence of an infinite number of conservation laws along $\mathcal{I}$ and is consistent with decay rate of $r^{-5/2}$ derived in [12]?*

3.5 FIRST-ORDER GLUING IN COMPLEX FORMULATION

In this section, we discuss the first-order gluing problem in the complex Maxwell Formulation. We prove a gluing theorem in the case where $\underline{\hat{\chi}} = 0$. While the proof can be repeated using the standard Lie Formulation as was done for the main theorems, the benefits of using the complex formulation are shown here, as many of the proofs of the previous gluing constructions were done originally in the Complex Formulation and converted into the Lie Formulation.

Before deriving the first-order equations and proving our theorem, we must first discuss the Hodge operators and the spaces they act on. Note that the properties of the Hodge operators shown in Lemma 3.3 can be extended to the analogous Hodge operators $m, \mathcal{D}_m, \overline{m}$ and $\mathcal{D}_{\overline{m}}$. For any 1-form ω the map $\omega \mapsto \omega_m = \omega_1 + i\omega_2$ is an invertible map, and since

$$\begin{aligned} \mathcal{D}_1\omega &= (\text{d}\!\!/\text{iv}\,\omega, \text{c}\!\!/\text{url}\,\omega) \\ &= (\mathbf{Re}\mathcal{D}_m\omega_m, \mathbf{Im}\mathcal{D}_m\omega_m) \end{aligned}$$

The operator $\mathcal{D}_m$ is an invertible operator onto complex functions with mean zero. We also see that the operator ${}^*\mathcal{D}$ is an invertible operator from pairs of functions with mean zero onto L^2. We see that for two functions (ρ, σ) with mean zero and by defining $\Theta = \rho + i\sigma$, we get

$$\begin{aligned} g(-{}^*\mathcal{D}_1(\rho,\sigma), (e_1 + ie_2)) &= (\not{d}\rho - (\not{d}\sigma)^*)(e_1 + ie_2) \\ &= (e_1 + ie_2)(\rho + i\sigma) \\ &= m(\Theta) \end{aligned}$$

Since the imaginary and real parts of the right-hand side recover the vector field, the derivative operator $m = e_1 + ie_2$ is invertible on mean zero functions onto a subspace $\mathcal{D}(S)$ given by.

$$\begin{aligned} \mathcal{D}(S) = \{f \in C^\infty(\mathbb{S}^2) : X = \mathbf{Re}(f)e_1 + \mathbf{Im}(f)e_2 \\ \text{extends smoothly to } \widetilde{X} \in \mathfrak{X}(S)\} \end{aligned}$$

The $L^2(S)$ space we define for our complex functions is given the by inner product

$$< f, g >_{L^2} = \mathbf{Re} \int_S f\overline{g} d\mu_g$$

Note that the real part of any inner product is an inner product and does not change the norm, hence the space of functions does not change. The real part is taken so that the inner product between functions is equal to the corresponding inner product between their 1-form and vector field counterparts. In other words, given a vector field X and a 1-form ω, we see that for functions $f = \omega_1 + i\omega_2, g = X^1 + iX^2$

$$\int_S \omega(X) d\mu_g = \int_S g(X, \omega^\sharp) d\mu_g = \mathbf{Re} \int_S f\overline{g} d\mu_g$$

Note that this defines an isometry between the Hilbert spaces $L^2(\mathfrak{X}(S_v))$ and $\mathcal{D}(S_v)$. We define $\mathcal{D}(\mathcal{H}) \subset C^\infty(\mathcal{H})$ to be that space of all f such that $f(v, \cdot) \in \mathcal{D}(\mathbb{S}^2)$. With respect to this inner product, we get the following identities for the adjoints

$$\mathcal{D}_m^* = -\overline{m} \quad \mathcal{D}_{\overline{m}}^* = -m$$

By the invertibility of $\mathcal{D}_m$ on its natural domain, the zeroth-order equations can be combined to get the following equation

$$\frac{1}{\phi} L(\phi(\Psi + \underline{\hat{\chi}}_1^m \mathcal{D}_m^{-1}(\Theta - \Theta_0) + QZ_m)) = \mathcal{Q}(\Theta - \Theta_0) \tag{3.51}$$

where,

$$\begin{aligned} \mathcal{Q}(\Theta - \Theta_0) &= \frac{1}{2} m(\Theta - \Theta_0) + \underline{\eta}_m(\Theta - \Theta_0) \\ &\quad - \underline{\hat{\chi}}_1^m \mathcal{D}_m^{-1} [L + \mathrm{tr}\chi, \mathcal{D}_m] \mathcal{D}_m^{-1}(\Theta - \Theta_0) + \phi L(\frac{\underline{\hat{\chi}}_1^m}{\phi}) \mathcal{D}_m^{-1}(\Theta - \Theta_0) \\ Z_m &= \frac{1}{\phi} \int_1^v \phi \left((\mathrm{vol}(S_v))^{-1} \underline{\hat{\chi}}_1^m \mathcal{D}_m^{-1}(\mathrm{tr}\chi - (\mathrm{tr}\chi)_0)(s) - \underline{\eta}_m(s) \right) ds \end{aligned}$$

To do first-order gluing, we note that 0th-order gluing gives us a gluing construction for $\underline{L}\Theta, \underline{L}\Phi$ from Maxwell's equations. Thus for first-order gluing, we must do the 0th-order gluing and $\underline{L}\Psi$ gluing simultaneously. To perform gluing, we need an equation for $\underline{L}\Psi$. We see that

$$\frac{1}{\phi} L\underline{L}(\phi\Psi) = \frac{1}{\phi} \underline{L}L(\phi\Psi) + \frac{1}{\phi}[L, \underline{L}](\phi\Psi) \tag{3.52}$$

$$= \underline{L}(\frac{1}{\phi} L(\phi\Psi)) + \frac{\mathrm{tr}\underline{\chi}}{2} \frac{1}{\phi} L(\phi\Psi) + \frac{1}{\phi}[L, \underline{L}](\phi\Psi) \tag{3.53}$$

Using equations (2.37),(2.38),(2.39),(2.40), as well as the commutation relation,

$$[\underline{L}, m] = (-\frac{1}{2}\mathrm{tr}\underline{\chi} - i\Gamma_{31}^2) m - \underline{\hat{\chi}}_1^m \overline{m} \tag{3.54}$$

we get the following equation

$$\frac{1}{\phi}L(\underline{L}(\phi\Psi) = \frac{1}{2}m\mathcal{D}_{\overline{m}}(\Psi) + a_1 m(\Psi) + a_2\mathcal{D}_{\overline{m}}(\Psi) + a_3(\Psi) + b_1 m(\Theta) + b_2\mathcal{D}_{\overline{m}}(\Theta) + b_3\Theta + c_1\Phi$$

where

$$\begin{aligned}
a_1 &= -\zeta_{\overline{m}} + \frac{i}{2}\Gamma^1_{\overline{m}2} \\
a_2 &= -2\zeta_m \\
a_3 &= -\phi^{-1}m(\phi) - \phi^{-1}\overline{m}(\phi) + i\zeta_m\Gamma^1_{\overline{m}2} + \hat{\underline{\chi}}^m_1\hat{\chi}^{\overline{m}}_1 \\
b_1 &= -\frac{i}{2}\Gamma^2_{31} \\
b_2 &= -\hat{\underline{\chi}}^m_1 \\
b_3 &= \frac{1}{2}m(\mathrm{tr}\underline{\chi}) + \underline{L}(\zeta_m) + \hat{\underline{\chi}}^m_1\zeta_m + i\hat{\underline{\chi}}^m_1\Gamma^1_{\overline{m}2} \\
c_1 &= \frac{\mathrm{tr}\underline{\chi}}{2}\hat{\underline{\chi}}^m_1 + L(\hat{\underline{\chi}}^m_1) - i\Gamma^2_{31}
\end{aligned}$$

We replace the Φ term by inverting $\mathcal{D}_m$ in (2.40). Hence, we can rewrite our equation in the following form

$$\frac{1}{\phi}L((\underline{L}(\phi\Psi)) + \phi c_1\mathcal{D}_m^{-1}(\Theta - \Theta_0) + QW_m) = \mathcal{Q}_2(\Psi) + \mathcal{Q}_3(\Theta - \Theta_0) \quad (3.55)$$

where

$$Q(v) := \int_{S_v}\Theta d\mu_g$$

and

$$\begin{aligned}
\mathcal{Q}_2(\Psi) &= \frac{1}{2}m\mathcal{D}_{\overline{m}}(\Psi) + a_1 m(\Psi) + a_2\mathcal{D}_{\overline{m}}(\Psi) + a_3(\Psi) \\
\mathcal{Q}_3(\Theta - \Theta_0) &= b_1 m(\Theta - \Theta_0) + b_2\mathcal{D}_{\overline{m}}(\Theta - \Theta_0) + b_3(\Theta - \Theta_0) \\
&\quad + \phi L(\frac{c_1}{\phi})\mathcal{D}_m^{-1}(\Theta - \Theta_0) - c_1\mathcal{D}_m^{-1}[L + \mathrm{tr}\chi, \mathcal{D}_m]\mathcal{D}_m^{-1}(\Theta - \Theta_0) \\
W_m &= \frac{1}{\phi}\int_1^v \phi(\mathrm{vol}(S_v))^{-1}\left(c_1\mathcal{D}_m^{-1}(\mathrm{tr}\chi - (\mathrm{tr}\chi)_0)(s) - b_3(s)\right)ds
\end{aligned}$$

We define $\mathcal{Q}_1$ and $\mathcal{V}_1$ to be the $\mathcal{Q}$ and $\mathcal{V}$ defined in the Section (3.4). For metrics where $\underline{\hat{\chi}} = 0$ along $\mathcal{H}$, we get that (3.55) can be rewritten in the following form the following form

$$\frac{1}{\phi}L(\underline{L}(\phi\Psi) + (\frac{3}{2}\mathrm{tr}\underline{\chi} + 2i\Gamma^2_{31})\Psi) = \frac{1}{2}m\mathcal{D}_{\overline{m}}\Psi - 2\zeta_m\overline{m}(\Psi) - \zeta_{\overline{m}}m(\Psi)$$
$$+ (-i\zeta_m\Gamma^1_{\overline{m}2} - 2\zeta(\nabla\!\!\!/ \log\phi) - L(\frac{3}{2}\mathrm{tr}\underline{\chi} + 2i\Gamma^1_{31}))\Psi$$
$$+ (-\frac{1}{2}m(\mathrm{tr}\underline{\chi}) - 2i\Gamma^2_{31}\zeta_m + 3\zeta_m\mathrm{tr}\underline{\chi})\Theta \quad (3.56)$$

Recall that when $\underline{\hat{\chi}}^m_1 = 0$, we can write, using the zeroth-order equation

$$\Theta = h^{-1}m^{-1}(\frac{2h}{\phi}L(\phi\Psi)) + (h\Theta)_0 h^{-1}$$

where $m(h) = -2\zeta_m$. Let $f = (-\frac{1}{2}m(\mathrm{tr}\underline{\chi}) - 2i\Gamma^2_{31}\zeta_m + 3\zeta_m\mathrm{tr}\underline{\chi})$. We can now write

$$f\Theta = \frac{1}{\phi}L(f\phi h^{-1}m^{-1}(2h\Psi)) - \frac{1}{\phi}L(f\phi h^{-1})m^{-1}(2h\Psi)$$
$$+ fh^{-1}(m^{-1}[L,m]m^{-1})(2h\Psi)$$
$$+ fh^{-1}m^{-1}((2\mathrm{tr}\chi - L(2h))\Psi) + f(h\Theta)_0 h^{-1} \quad (3.57)$$

We can thus write equation (3.56) as follows

$$\frac{1}{\phi}L(\underline{L}(\phi\Psi) + (\frac{3}{2}\mathrm{tr}\underline{\chi} + 2i\Gamma^1_{31})\Psi - f\phi h^{-1}m^{-1}(2h\Psi))$$
$$= \mathcal{Q}^{(1)}_{temp}(\Psi) + (h\Theta)_0 h^{-1} \quad (3.58)$$

In order to replace the $(h\Theta)_0$ term, we can use equation (3.57) without the f term. By taking the spherical average and commuting out the L derivative, we get that

$$\Theta_0 - (h\Theta)_0(h^{-1})_0 = L((m^{-1}(2h\Psi))_0) + ((\mathrm{tr}\chi)_0 - \frac{1}{\phi}L(\phi^2 h))(m^{-1}(2h\Psi))_0$$
$$+ (h^{-1}m^{-1}[L,m]m^{-1}(2h\Psi))_0 - ((m^{-1}(\phi L(\frac{2h}{\phi})\Psi)))_0$$

Hence, we can replace $(h\Theta)_0 h^{-1} = ((\Theta_0 - (h\Theta)_0(h^{-1})_0) + \Theta_0)(h^{-1})((h^{-1})_0)^{-1}$ in (3.58). Since taking spherical averages is a linear operator, we can absorb all Ψ terms into the $\mathcal{Q}^{(1)}$ operator and we are finally left with the following equation

$$\frac{1}{\phi}L(\underline{L}(\phi\Psi)+\phi(\frac{3}{2}\mathrm{tr}\underline{\chi}+2i\Gamma^1_{31})\Psi-\phi fh^{-1}m^{-1}(2h\Psi)$$
$$+\phi(h^{-1})((h^{-1})_0)^{-1}(m^{-1}(2h\Psi))_0+QY_m)=\mathcal{Q}^{(1)}(\Psi)$$

where

$$Y_m=\frac{1}{\phi}\int_1^v\phi(\mathrm{vol}(S_v))^{-1}(h^{-1})((h^{-1})_0)^{-1}(s)ds$$

By Jensens inequality, the operator $F\mapsto(F)_0$ for a fixed sphere S_v satisfies the following bound

$$||(F)_0||_{L^2(\mathbb{S}^2)}\leq||F||_{L^2(\mathbb{S}^2)}$$

Thus, if the volumes are bounded in v, spherical averaging is a bounded operator. We see that $\mathcal{Q}^{(1)}$ can be written in the following form

$$\mathcal{Q}^{(1)}=\frac{1}{2}m\mathcal{D}_{\overline{m}}+g_1\mathcal{D}_{\overline{m}}+g_2m+A$$

where A is a sum of compositions of operators m^{-1}, $m^{-1}[L,m]m^{-1}$ and $F\mapsto(F)_0$ which are all $L^2\to L^2$ bounded operators and thus A is an L^2 bounded operator. Suppose $||AF||\leq C||F||$ where the norm is the L^2 norm. Then we see that

$$\mathbf{Re}\int_{S_v}((\mathcal{Q}^{(1)}-\frac{1}{\epsilon})F)\overline{F}d\mu_g\geq-\mathbf{Re}\int_{S_v}\frac{1}{2}|\mathcal{D}_{\overline{m}}F|^2-|g_1||\mathcal{D}_{\overline{m}}F||F|-g_2m(F)\overline{F}$$
$$-|AF||F|+\frac{1}{\epsilon}|F|^2d\mu_g$$
$$\geq-\int_{S_v}(\frac{1}{2}-\epsilon_1-\epsilon_2)|\mathcal{D}_{\overline{m}}F|^2$$
$$+(-\frac{(|g_1|)}{\epsilon_1}-\frac{|m(g_2)|}{\epsilon_2}-C+\frac{1}{\epsilon})|F|^2d\mu_g$$

where we used that $m^*=-\mathcal{D}_{\overline{m}}$. Choose $\epsilon_1+\epsilon_2<1/2$ and then choose ϵ sufficiently small so the coefficient in front of $|F|^2$ is positive. Thus the operator $\mathcal{Q}^{(1)}-\frac{1}{\epsilon}$ has a trivial kernel. We can argue that $\mathcal{Q}^{(1)}$ has a discrete spectrum accumulating at infinity and thus $\ker(\mathcal{O}^{(1)})^{\perp}=im(\mathcal{Q}^{(1)})$ where $\mathcal{O}^{(1)}$ is the adjoint of $\mathcal{Q}^{(1)}$. Let us define the following set

$$\mathcal{U}^{(1)}=\left\{f\in C^\infty(\mathcal{H}):Lf=0,\mathcal{O}^{(1)}(\frac{1}{\phi}f)=0\ on\ \mathcal{H}\right\}$$

Given this set, we can define first-order gluing condition. Note that by equations (2.38) and (2.40), prescribing $(\Theta_i, \Phi_i, \Psi_i)$ determines $(\underline{L}\Theta_i, \underline{L}\Phi_i)$ Therefore, we only need to prescribe $\underline{L}\Psi_i$ on the initial spheres.

Theorem 3.11. *Given initial data $(\Theta_i, \Phi_i, \Psi_i, \underline{L}\Psi_i)$ for $i = 1, 2$ and and assuming that $\hat{\underline{\chi}} = 0$ along $\mathcal{H}$, first-order gluing can be done given the following two conditions are met*

1. *We first require*
$$Q(1) = Q(2)$$
Given this condition is met, we can construct the function
$$\Theta_0(v) = \frac{Q}{vol(S_v)}$$
The first condition ensures the continuity of the function $\Theta_0(v)$

2. *Given the first condition is satisfied, we require for all $f^{(1)} \in \mathcal{U}^{(1)}$*
$$\int_{S^2} \left(F(\underline{L}\Psi, \Psi, Q)\big|_{S_2} \right) \overline{f^{(1)}} d\mu_{S^2} = \int_{S^2} \left(F(\underline{L}\Psi, \Psi, Q)\big|_{S_1} \right) \overline{f^{(1)}} d\mu_{S^2}$$
where
$$F(\underline{L}\Psi, \Psi, Q) = \underline{L}(\phi\Psi) + \phi(\frac{3}{2} tr\underline{\chi} + 2i\Gamma^1_{31})\Psi - \phi f h^{-1} m^{-1}(2h\Psi) + \phi(h^{-1})((h^{-1})_0)^{-1}(m^{-1}(2h\Psi))_0 + QY_m$$

If both conditions hold, we can solve first-order gluing

Proof. Following the same proof as Theorems 1.4 and using the Hilbert space structure introduced in the beginning of this section, we construct Ψ on $\mathcal{H}$. We use Maxwell's equations to construct the remaining equations. □

4

FOLIATION COVARIANCE

In this chapter we address the problem of *foliation covariance*. In our proofs of Theorems 1.4 and 1.6, we fixed our gauge freedom to be $\Omega = 1$. We saw that our frame was dependent on this choice as our vector fields e_1 and e_2 were tangent to the S_v leaves of the foliation. We would like to determine how changing the foliation of $\mathcal{H}$ would affect the sets (1.12) and (1.17) i.e. can we "gauge away" the kernels of our elliptic operators. We will show that the conservation laws for charges (1.7) cannot be "gauged away" and we will address the difficulties in answering the same question for (1.12).

4.1 GAUGE TRANSFORMATIONS

In order see how our conserved charges change under gauge transformations, we first must see how geometric quantities change. Consider two different foliations

$$\mathcal{S} = (S_v)_{v\in\mathbb{R}} = \left\langle S_1, L_{geod}\big|_{S_1}, \Omega_{\mathcal{S}} = 1 \right\rangle$$

and

$$\mathcal{S}' = (S'_{v'})_{v'\in\mathbb{R}} = \left\langle S_1, L_{geod}\big|_{S_1}, \Omega_{\mathcal{S}'} = \Omega \right\rangle$$

of $\mathcal{H}$. Note that our initial null outwards normals L_{geod}, L'_{geod} are defined to be equal on our initial sphere S_1, thus

$$e_4 = L = \partial_v = L_{geod}$$

and

$$e'_4 = \Omega e_4, \quad L' = \partial_{v'} = \Omega^2 L$$

on $\mathcal{H}$. Hence, we see that

$$\chi' = \Omega \chi$$

Let us take $p \in S_v \cap S'_{v'}$ and let $X \in T_p S_v$ be tangent to spherical sections under the original gauge $\Omega = 1$. There is a unique point along the line $X + < L > \subset \mathcal{H}$ that intersects $T_p S'_{v'}$. We therefore get that the new vector $X' \in T_P S'_{v'}$ takes the form

$$X' = X + \gamma(X) L$$

To find γ from Ω we note the following. Recall that $v = v' = 1$ on S_1. We also note that

$$1 = \partial_{v'} v' = \Omega^2 \partial_v v' \implies \partial_v v' = \Omega^{-2}$$

And thus,

$$v'(v) = 1 + \int_1^v \Omega^{-2}(s) ds$$

For an arbitrary vector X we obtain the following

$$\begin{aligned} 0 &= X'(v') = X(v') + \gamma(X)\partial_v v' \\ \implies \gamma(X) &= -\Omega^2 \cdot \not{d} v'(X) \end{aligned}$$

Thus we get that

$$\gamma = -\Omega^2 \cdot \not{d} \left(\int_1^v \Omega^{-2} d\overline{v} \right)$$

Note that since $S_1' = S_1$ and $S_2' = S_2$ we have that $v'(v=1) = 1$ and that $\not{d}(\int_1^2 -\Omega^{-2} d\overline{v}) = 0$. Let us see how the equations we have seen change under gauge transformations.

If we do not fix our gauge freedom to be $\Omega = 1$, our zeroth order Maxwell's equations take the following form,

$$\begin{aligned} \not{\mathcal{L}}_4 (\Omega \underline{\alpha}) - \hat{\chi}^\sharp \cdot (\Omega \underline{\alpha}) &= +\frac{\Omega}{2} \left(\not{\nabla} \rho + 2\underline{\eta} \rho \right) - \frac{\Omega}{2} \left(\not{\nabla} \sigma + 2\underline{\eta} \sigma \right)^* + \underline{\hat{\chi}}^\sharp \cdot (\Omega \alpha) \\ \not{\mathrm{div}}(\Omega \alpha) &= -\frac{1}{\phi^2} L(\phi^2 \rho) \\ \not{\mathrm{curl}}(\Omega \alpha) &= +\frac{1}{\phi^2} L(\phi^2 \sigma) \end{aligned}$$

Note that all geometric tensors also change under the gauge transformation. In this gauge, we get that (3.4) takes the form

$$\begin{aligned} &\not{\mathcal{L}}_4 W_\Omega(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma}) - \hat{\chi}^\sharp \cdot W_\Omega(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma}) \\ &\qquad = \mathcal{Q}_\Omega(\rho - \overline{\rho}, \sigma - \overline{\sigma}) + \mathcal{V}_\Omega(\overline{\rho}, \overline{\sigma}) \end{aligned} \tag{4.1}$$

where,

$$\begin{aligned}
W_\Omega(\underline{\alpha}, \rho - \overline{\rho}, \sigma - \overline{\sigma}) &= \Omega\underline{\alpha} - \Omega\underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-(\rho - \overline{\rho}), (\sigma - \overline{\sigma})) \\
\mathcal{Q}_\Omega(\rho - \overline{\rho}, \sigma - \overline{\sigma}) &= -\frac{\Omega}{2}{}^*\mathcal{D}^{\underline{\eta}}(\rho - \overline{\rho}, \sigma - \overline{\sigma}) \\
&\quad - \phi\nabla_4\left(\frac{\Omega}{\phi}\underline{\hat{\chi}}^\sharp\right)\mathcal{D}^{-1}(-(\rho - \overline{\rho}), (\sigma - \overline{\sigma})) \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}[\mathcal{D}, \mathbb{L}]\mathcal{D}^{-1}(-(\rho - \overline{\rho}), (\sigma - \overline{\sigma})) \\
\mathcal{V}_\Omega(\overline{\rho}, \overline{\sigma}) &= -\Omega\underline{\eta}\overline{\rho} + \Omega\underline{\eta}^*\overline{\sigma} \\
&\quad + \underline{\hat{\chi}}^\sharp \cdot \mathcal{D}^{-1}(-\overline{\rho}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}), \overline{\sigma}(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}))
\end{aligned}$$

The adjoint $\mathcal{O}_\Omega$ of $\mathcal{Q}_\Omega$ is

$$\begin{aligned}
\mathcal{O}_\Omega(X) = -\frac{1}{2}\mathcal{D}^\eta(\Omega X) - J \cdot {}^*\mathcal{D}^{-1}(\phi\nabla_4\left(\frac{\Omega}{\phi}\underline{\hat{\chi}}^\sharp\right) \cdot X) \\
+ J \cdot {}^*\mathcal{D}^{-1}[\mathcal{D}, \mathbb{L}]^*\left({}^*\mathcal{D}^{-1}(\underline{\hat{\chi}}^\sharp \cdot X)\right)
\end{aligned}$$

We would like to write our new gauge tensors in terms of the $\Omega = 1$ operators as was done in [7]. Let us see how $(1,1)$ tensor $\Psi(v)$ changes under this transformation. Recall that $W = \Psi' \cdot W_0$ solves the following Lie transport equation

$$\mathcal{L}_{4'}W_\Omega - \hat{\chi}^{\sharp\prime} \cdot W_\Omega = 0, \qquad W_\Omega(1) = W_0$$

Note that S_1, S_2 stay unchanged after the change of foliation, and thus, W_0 remains the same. Let W be the solution to the transport equation for $\Omega = 1$. To see how the solution changes under the gauge transformation, it is easier to convert the equation to a transport equation for $W^\sharp$.

$$\nabla_{4'}W_\Omega^\sharp + \frac{1}{2}\mathrm{tr}\chi^{\sharp\prime} \cdot W_\Omega^\sharp = 0, \qquad W_\Omega(1)^\sharp = W_0^\sharp \tag{4.2}$$

We claim that $W_\Omega^\sharp = W^{\sharp\prime} = W^\sharp + \gamma(W^\sharp)L$. Under the gauge transformation, we see that

$$\begin{aligned}
e_4' &= \Omega e_4 \\
\chi' &= \Omega\chi
\end{aligned}$$

Plugging this into (4.2), we see that

$$\begin{aligned}
\not{g}(\nabla\!\!\!/_{4'} W^\sharp_\Omega + \frac{1}{2}\mathrm{tr}\chi' \cdot W^\sharp_\Omega, e'_A) &= \not{g}(\nabla\!\!\!/_{4'} W^{\sharp\prime} + \frac{1}{2}\mathrm{tr}\chi' \cdot W^{\sharp\prime}, e'_A) \\
&= \Omega g(\nabla\!\!\!/_L (W^\sharp + \gamma(W^\sharp)L) \\
&\quad + \frac{1}{2}\mathrm{tr}\chi(W^\sharp + \gamma(W^\sharp)L), e_A + \gamma_A L) \\
&= \Omega g(\nabla\!\!\!/_L W^\sharp + \frac{1}{2}\mathrm{tr}\chi W^\sharp, e_A) \\
&= 0
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
\Psi'(v')W_0 &= \Psi(v)W_0 + \gamma(\Psi(v)W_0)L \\
\Psi' &= \Psi + \gamma \cdot \Psi \otimes L
\end{aligned}$$

We now compute $\mathrm{c\!\!/url}'\xi'$ for a section S' and vector field ξ'

$$\begin{aligned}
\mathrm{c\!\!/url}'\xi' &= \sum_{A,B} \epsilon^{AB} g(\nabla\!\!\!/'_{E'_A}\xi', E'_B) = \sum_{A,B} \epsilon^{AB} g(\nabla_{E'_A}\xi', E'_B) \\
&= \sum_{A,B} \epsilon^{AB} g(\nabla_{E_A + \gamma_A L}(\xi + \gamma(\xi)L), E_B + \gamma_B L) \\
&= \sum_{A,B} \epsilon^{AB} \Big(g(\nabla_{E_A}\xi, E_B) + g(\nabla_{E_A}\xi, \gamma_B L) + \gamma_A g(\nabla_L \xi, E_B) \\
&\quad + \gamma(\xi) g(\nabla_{E_A} L, E_B) \Big) \\
&= \mathrm{c\!\!/url}\xi + \epsilon^{AB}\gamma_B g(\nabla_{E_A}\xi, L) + g(\nabla\!\!\!/_L \xi, -(\gamma^*)^\sharp) + \gamma(\xi)\epsilon^{AB}\chi_{AB} \\
&= \mathrm{c\!\!/url}\xi - \epsilon^{AB}\gamma_B g(\xi, \nabla_{E_A} L) + g(\nabla\!\!\!/_\xi L, -(\gamma^*)^\sharp) + g(\mathcal{L\!\!\!/}_L \xi, -(\gamma^*)^\sharp) \\
&= \mathrm{c\!\!/url}\xi - 2\chi((\gamma^*)^\sharp, \xi) - \gamma^*(\mathcal{L\!\!\!/}_L \xi)
\end{aligned}$$

Similar calculations done in [7] show us

$$\mathrm{d\!\!/iv}'\xi' = \mathrm{d\!\!/iv}\xi + \gamma(\xi)tr\chi + \gamma(\mathcal{L\!\!\!/}_L \xi) \tag{4.3}$$

$$\mathrm{c\!\!/url}'\xi' = \mathrm{c\!\!/url}\xi - 2\chi((\gamma^*)^\sharp, \xi) - \gamma^*(\mathcal{L\!\!\!/}_L \xi) \tag{4.4}$$

$$\underline{\eta}' = \underline{\eta} - (\chi \cdot \gamma^\sharp) + d\!\!/ \log\Omega + 2\gamma L \log\Omega \tag{4.5}$$

Let us show that the conserved quantities (ρ_0, σ_0) remained conserved under the change of foliation. Recall that

$$\begin{aligned}\alpha &:= F(e_4, \cdot)\\ \underline{\alpha} &:= F(e_3, \cdot)\\ \rho &:= F(e_3, e_4)\\ \sigma &:= \frac{1}{2}\epsilon^{AB}F(e_A, e_A)\end{aligned}$$

Thus, under the change of foliation,

$$\begin{aligned}\rho' &= F(e'_3, e'_4) = F(\Omega^{-1}\underline{L}', \Omega L)\\ &= F(\frac{1}{2}(\gamma, \gamma)L + \underline{L} + \gamma^\sharp, L)\\ &= \rho + \alpha(\gamma^\sharp) = \rho + \gamma(\alpha^\sharp)\\ \sigma' &= F(e'_1, e'_2)\\ &= F(e_1 + \gamma_1 L, e_2 + \gamma_2 L)\\ &= \sigma + \gamma \times \alpha\end{aligned}$$

Proposition 4.1. The conserved quantities

$$Q_0(v) = \int_{S_v} \rho d\mu_g, \quad Q_1(v) = \int_{S_v} \sigma d\mu_g$$

are gauge invariant.

Proof: Let us show that $L'\rho'_0 = \int_{S_v} \phi^{-2}L'(\phi^2\rho')d\mu_g = 0$. We see that

$$\begin{aligned}\frac{1}{\phi^2}L'(\phi^2\rho') &= \frac{\Omega^2}{\phi^2}L(\phi^2(\rho + \gamma(\alpha^\sharp)))\\ &= \frac{\Omega^2}{\phi^2}L(\phi^2\rho) + \frac{\Omega^2}{\phi^2}L(\phi^2\gamma(\alpha^\sharp))\\ &= \frac{\Omega^2}{\phi^2}L(\phi^2\rho) + \Omega^2\mathrm{tr}\chi\gamma(\alpha^\sharp) + \Omega^2 L\gamma(\alpha^\sharp)\\ &= \frac{\Omega^2}{\phi^2}L(\phi^2\rho) + \Omega^2\mathrm{tr}\chi\gamma(\alpha^\sharp) + \Omega^2(\not{\mathcal{L}}_L\gamma)(\alpha^\sharp) + \Omega^2\gamma(\not{\mathcal{L}}_L\alpha^\sharp)\\ &= \frac{\Omega^2}{\phi^2}L(\phi^2\rho) + \Omega^2(\not{\mathcal{L}}_L\gamma)(\alpha^\sharp) + \Omega^2\gamma(\frac{1}{\phi^2}\not{\mathcal{L}}_L(\phi^2\alpha^\sharp))\end{aligned}$$

Recall that from the definition of γ and the Cartan formula

$$\begin{aligned}\mathcal{L}_L\gamma &= d(\iota_L(\gamma)) + \iota_L(d\gamma)\\ &= d\left(-\Omega^2 L\int_1^v \Omega^{-2}(\overline{v})d\overline{v}\right) + \iota_L\left(-d\Omega^2\wedge d\left(\int_1^v \Omega^{-2}(\overline{v})d\overline{v}\right)\right)\\ &= -\Omega^{-2}L(\Omega^2)\gamma + \Omega^{-2}d\Omega^2\end{aligned}$$

Plugging this in and using (2.29) and(4.3)

$$\begin{aligned}\frac{1}{\phi^2}L'(\phi^2\rho') &= \frac{\Omega^2}{\phi^2}L(\phi^2\rho) + \Omega^2\gamma(\frac{1}{\phi^2}\not{\mathcal{L}}_L(\phi^2\alpha^\sharp)) - L(\Omega^2)\gamma(\alpha^\sharp) + \not\nabla\Omega^2\cdot\alpha^\sharp\\ &= \Omega^2 \not{\mathrm{d}}\mathrm{iv}\,\alpha^\sharp + \not\nabla\Omega^2\cdot\alpha^\sharp + \gamma(\frac{\Omega^2}{\phi^2}\not{\mathcal{L}}_L(\phi^2\alpha^\sharp) - L(\Omega^2)\alpha^\sharp)\\ &= \not{\mathrm{d}}\mathrm{iv}(\Omega^2\alpha^\sharp) + \gamma(\frac{1}{\phi^2}\not{\mathcal{L}}_L(\phi^2\Omega^2\alpha^\sharp))\\ &= \not{\mathrm{d}}\mathrm{iv}'(\Omega\alpha'^\sharp)\end{aligned}$$

Note that the right-hand side is a divergence term so the integral over the sphere vanishes. Thus the charge remains conserved.

□

For the second gluing condition in Theorem 1.4, we are interested in how the set $\mathcal{U}$ under a different gauge choice and how this affects the gluing condition. We note that for any function $\psi : \mathbb{S}^2 \to \mathbb{R}$

$$\not\nabla'\psi = (\not\nabla\psi + \gamma^\sharp\cdot L\psi) + \gamma(\not\nabla\psi + \gamma^\sharp\cdot L\psi)L$$

And thus we see that

$$\begin{aligned}{}^*\not{\mathcal{D}}'(f_1,f_2) &= {}^*\not{\mathcal{D}}(f_1,f_2) - \gamma^\sharp\cdot Lf_1 + (\gamma^\sharp)^*\cdot Lf_2\\ &\quad + \gamma({}^*\not{\mathcal{D}}(f_1,f_2) - \gamma^\sharp\cdot Lf_1 + (\gamma^\sharp)^*\cdot Lf_2)L \end{aligned}\tag{4.6}$$

In order to address the gauge transformation of the set (1.12), we first would have to understand the following question.

Question 4.2. *How do the inverses of $\not{\mathcal{D}}, {}^*\not{\mathcal{D}}$ change under foliation? More importantly, how do the inverses change when restricted to vector fields X where $\not{\mathcal{L}}_4 X = 0$.*

A APPENDICES

A.1 THE SPACELIKE GLUING PROBLEM

The Cauchy problem for Maxwell's equations in Minkowski can be formulated as follows, as stated by Carlotto in [13]. Note that given the antisymmetric 2-tensor F, we can define the electric field $\mathbf{E}^i = F^{0i}$ and magnetic field $\mathbf{B}^i = \epsilon^{ijk}F_{jk}$ in Euclidean coordinates. The Maxwell equations take the following form

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0 \\ \operatorname{div}\mathbf{B} &= 0 \\ \vec{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \vec{\nabla} \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

where $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$. Suppose we prescribe $(\mathbf{E}(t=0), \mathbf{B}(t=0)) = (E, B)$ on $\Sigma = \{t = 0\}$. We see that on Σ, (E, B) must satisfy

$$\begin{aligned} \operatorname{div}E &= 0 \\ \operatorname{div}B &= 0 \end{aligned}$$

These are the *constraint equations* for the Maxwell equations. Suppose we have two disjoint regions $\Sigma_1 \subset \Sigma$ and $\Sigma_2 \subset \Sigma$ and prescribed data (E_i, B_i) on Σ_i for $i = 1, 2$. We only require that one is compact and the other is closed. This assumption allows us to construct a smooth function $\varphi(x)$ where $\varphi(x)|_{\Sigma_1} = 1, \varphi(x)|_{\Sigma_2} = 0$ by Urysohn's Lemma. By the Helmholtz decomposition, we can construct vector fields A_i^E and A_i^B on Σ such that

$$\vec{\nabla} \times A_i^E = E_i, \quad \vec{\nabla} \times A_i^B = B_i$$

on Σ_i for $i = 1, 2$. We extend A_i^E and A_i^B smoothly to all of Σ such that

$$\begin{aligned} A_1^E|_{\Sigma_2} &= 0 = A_1^B|_{\Sigma_2} \\ A_2^E|_{\Sigma_1} &= 0 = A_2^B|_{\Sigma_1} \end{aligned}$$

We can finally define our gluing construction for (E, B) as follows

$$E = \vec{\nabla} \times (\varphi A_1^E + (1-\varphi) A_2^E)$$
$$B = \vec{\nabla} \times (\varphi A_1^B + (1-\varphi) A_2^B)$$

Thus there are no obstructions for spacelike gluing of Maxwell's equations in Minkowski. For the Cauchy problem for Maxwell, we look at the formulation given by Strichartz in [51]: Recall the geometric form of the Maxwell equations

$$\mathbf{d}F = 0, \qquad \delta F = 0 \tag{A.1}$$

where F is our Maxwell 2-tensor, $\mathbf{d}$ is the exterior derivative on the entire spacetime and δ is the codifferential. Consider a Lorentzian manifold $(\mathcal{M}^{3+1}, g)$ where $\mathcal{M}^{3+1} \simeq M^3 \times \mathbb{R}$ and where M^3 is a 3-manifold diffeomorphic to $\mathbb{R}^3$. The metric takes the form $-dt^2 + h_{ij}(x)dx^i dx^j$, where $h_i j$ is a Riemannian metric defined on M^3 slices and $x = (x^1, x^2, x^3)$ are local coordinates for M^3. In this formulation, we can rewrite F in terms of electric and magnetic fields E and H. For fixed t, $E(x,t)$ and $H(x,t)$ are 1-forms defined on M^3, which are related to F by relation $F = E \wedge dt + *H$. We can now rewrite Maxwell's equations (A.1)

$$-\frac{\partial E}{\partial t} + *dH = 0, \qquad \frac{\partial H}{\partial t} + *dE = 0, \tag{A.2}$$
$$*d*E = 0, \qquad *d*H = 0 \tag{A.3}$$

Here d and $*$ are M^3 tangent operators. We see that equations (A.3) become our constraint equations. Suppose $\Sigma_1 \subset \Sigma$ and $\Sigma_2 \subset \Sigma$, where $\Sigma = M^3 \times \{0\}$ and prescribed data (E_i, H_i) on Σ_i for $i = 1, 2$ satisfying (A.3) in the regions. Assuming Σ_1 is compact and Σ_2 is closed, we can extend (E_i, H_i) smoothly to the entire region Σ such that their supports are disjoint. Since $(*E_i, *H_i)$ are closed, by the Poincare Lemma, they are exact so we get functions $(\varphi_i^E, \varphi_i^H)$ such that

$$d\varphi_i^E = *E_i, \qquad d\varphi_i^H = *H_i$$

on Σ_i. We proceed to interpolate between these functions as we did for Minkowski to get our spacelike gluing construction. Thus, the general spacelike gluing problem for Maxwell has no obstructions.

A.2 TENSOR NORMS

In this section we define tensor norms and prove an estimate on the Ψ tensor defined in (3.5).

Definition 1. (Tensor norms). Let $n \geq 1$ and $w \geq 0$ be integers. For an n-tensor T on $\mathbb{S}^2$, we define its $H^w(\mathbb{S}^2)$ norm to be

$$\|T\|_{H^w(\mathbb{S}^2)} = \sqrt{\sum_{k \leq w} \int_{\mathbb{S}^2} |\mathring{\nabla}^k T|^2_{\mathring{\gamma}} d\mu_{\mathbb{S}^2}}$$

where $\mathring{\gamma}$ is the standard round metric on the 2-sphere and

$$|\mathring{\nabla}^k T|^2_{\mathring{\gamma}} = |\mathring{\gamma}^{I_1 J_1} ... \mathring{\gamma}^{I_n J_n} \mathring{\gamma}^{A_1 B_1} ... \mathring{\gamma}^{A_k B_k} \mathring{\nabla}_{A_1} ... \mathring{\nabla}_{A_k} T_{I_1 ... I_n} \mathring{\nabla}_{B_1} ... \mathring{\nabla}_{B_k} T_{J_1 ... J_n}|$$

where $\mathring{\nabla}$ is covariant derivative with respect to $\mathring{\gamma}$

We also define the $W^{w,\infty}(\mathbb{S}^2)$ norm as follows

$$\|T\|_{W^{w,\infty}(\mathbb{S}^2)} = \sup_{k \leq w} \sup_{A_1,...,A_k,I_1,...,I_n} \|\mathring{\nabla}_{A_1} ... \mathring{\nabla}_{A_k} T_{I_1 ... I_n}\|_{L^\infty(\mathbb{S}^2)}$$

Any tensor with finite norm is said to be an element of the respective space.

Let $W(v)$ be a 1-form satisfying (3.5). Then,

$$\partial_v \|W(v)\|^2_{H^k(\mathbb{S}^2)} = \partial_v \sum_{k \leq w} \int_{\mathbb{S}^2} |\nabla^k W|^2_{\mathring{\gamma}} d\mu_{\mathbb{S}^2} \tag{A.4}$$

$$\lesssim \sum_{k \leq w} \int_{\mathbb{S}^2} |\mathring{\gamma}^{IJ} \mathring{\gamma}^{A_1 B_1} ... \mathring{\gamma}^{A_k B_k} \partial_v (\mathring{\nabla}_{A_1} ... \mathring{\nabla}_{A_k} W_I) \mathring{\nabla}_{B_1} ... \mathring{\nabla}_{B_k} W_J| d\mu_{\mathbb{S}^2} \tag{A.5}$$

Note that for any 1-form V, given a Lie transported frame

$$[\nabla_4, \mathring{\nabla}_B] V_I = (e_B(\chi_I^K) + \chi_I^J \mathring{\Gamma}^K_{BJ} - \mathring{\Gamma}^J_{BI} \chi_J^K) V_K$$
$$:= D^K_{BI} V_K$$

Since the round metric is smooth, and assuming smoothness of the background metric, the symbol D^C_{AB} defined above is smooth as well. Thus, by commuting the ∂_v derivative in (A.5) and using the fact that $\nabla\!\!\!/_4 = \mathcal{L}\!\!\!/_4 - \chi^\sharp$ when acting on 1-forms, we get that

$$\begin{aligned}\partial_v \|W(v)\|^2_{H^k(\mathbb{S}^2)} &\lesssim \|\mathcal{L}\!\!\!/_4 W(v)\|^2_{H^k(\mathbb{S}^2)} + \|\chi \cdot W(v)\|^2_{H^k(\mathbb{S}^2)} \\ &\quad + \left(\sum_{A,B,C} \|D^C_{AB}\|_{W^{k,\infty}(\mathbb{S}^2)}\right) \|W(v)\|^2_{H^k(\mathbb{S}^2)} \\ &\lesssim \Bigg(\|\hat{\chi}\|_{W^{k,\infty}(\mathbb{S}^2)} + \|\chi\|^2_{W^{k,\infty}(\mathbb{S}^2)} \\ &\quad + \sum_{A,B,C} \|D^C_{AB}\|_{W^{k,\infty}(\mathbb{S}^2)} + 1 \Bigg) \|W(v)\|^2_{H^k(\mathbb{S}^2)}\end{aligned}$$

where we used (3.5) to replace Lie derivative term. Thus, by Gronwall's inequality, we get

$$\begin{aligned}\|W(v)\|_{H^k(\mathbb{S}^2)} \lesssim \|W(1)\|_{H^k(\mathbb{S}^2)} \exp\Bigg(\int_1^v \Big(\|\hat{\chi}\|_{W^{k,\infty}(\mathbb{S}^2)} + \|\chi\|_{W^{k,\infty}(\mathbb{S}^2)} \\ + \sum_{A,B,C} \|D^C_{AB}\|_{W^{k,\infty}(\mathbb{S}^2)} + 1 \Big) ds \Bigg)\end{aligned}$$

and therefore

$$\|\Psi\|_{L^1_v W^{k,\infty}_{\mathbb{S}^2}([1,2]\times\mathbb{S}^2)} \lesssim 1$$

We see that the H^k norm of $W(v)$ is finite assuming that $W(1)$ has finite norm and $\chi, D^C_{AB} \in L^1_v W^{k,\infty}_{\mathbb{S}^2}([1,2] \times \mathbb{S}^2)$

A.3 TENSOR HARMONICS

Here we follow the definitions provided by Czimek in [18]. We will define operators on round Euclidean spheres $(S_r, \mathring{\gamma})$ of radius r. We also assume that all tensors will be S_r-tangent. For $r > 0$, let

$$\left\{ Y^{(lm)}(r,\theta,\phi) : l \geq 0, m \in \{-l,...,l\} \right\}$$

denote the set of normalized real spherical harmonics. They form a complete orthonormal basis $L^2(S_r, \mathring{\gamma})$ and satisfy

$$\Delta\!\!\!/ Y^{(lm)} = -\frac{l(l+1)}{r^2} Y^{(lm)}$$

Using these functions, we can define the vector spherical harmonics $E^{(lm)}$ and $H^{(lm)}$ on S_r. For $l \geq 1$ and $m \in \{-l, ..., l\}$,

$$E^{(lm)} := \frac{r}{\sqrt{l(l+1)}} {}^*\mathcal{D}(Y^{(lm)}, 0)$$
$$H^{(lm)} := \frac{r}{\sqrt{l(l+1)}} {}^*\mathcal{D}(0, Y^{(lm)})$$

By using relations in (2.15), we can show that for $l \geq 1, m \in \{-l, ..., l\}$,

$$\Delta\!\!\!/ E^{(lm)} = \frac{1 - l(l+1)}{r^2} E^{(lm)}$$
$$\Delta\!\!\!/ H^{(lm)} = \frac{1 - l(l+1)}{r^2} H^{(lm)}$$

The set of functions

$$\left\{ E^{(lm)}, H^{(lm)} : l \geq 1, m \in \{-l, ..., l\} \right\}$$

form a complete orthonormal basis for the set of L^2-integrable vector fields on S_r.

Remark A.1. For all $r > 0$, the vector fields with $l = 1$

$$\left\{ E^{(1m)}, H^{(1m)} : m \in \{-1, 0, 1\} \right\}$$

form an orthonormal basis of the six-dimensional space of conformal Killing fields on $(S_r, \mathring{\gamma})$. Note that these are exactly the vector fields that appear in the conserved charges (3.29).

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