HYPERBOLICITY AND RATIONAL POINTS ON COMPLEX BALL QUOTIENTS

by

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ABSTRACT

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Let $X = \Gamma\backslash\mathbb{B}^n$ be an n-dimensional complex ball quotient by a torsion-free non-uniform lattice Γ whose parabolic subgroups are unipotent. Let $\overline{X}$ be the unique toroidal complication of X constructed in [AMRT10, Mok12].

In the first part of this thesis we prove positivity properties of $\Omega^1_{\overline{X}}$ and $\Omega^1_{\overline{X}}(\log(D))$ depending intrinsically on X. We prove that $\Omega^1_{\overline{X}}(\log(D))\langle -rD\rangle$ is ample for all sufficiently small rational numbers $r > 0$, and $\Omega^1_{\overline{X}}(\log(D))$ is ample modulo D. Further, we conclude that if the cusps of X have uniform depth greater than 2π, then $\Omega^1_{\overline{X}}$ is semi-ample and is ample modulo D, and all subvarieties of X are of general type.

In the second part of this thesis we prove that the volumes of subvarieties of X are controlled by the systole of X, which is the length of the shortest closed geodesic of X. There are a number of arithmetic and geometric consequences: the systole of X controls the growth rate of rational points on X, uniformly in the field of definition of $\overline{X}$. Also, we obtain effective global generation and very ampleness results for multiples of the canonical bundle $K_{\overline{X}}$. These results follow from the bound we find for the Seshadri constant of $K_{\overline{X}}$ in terms of the systole.

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PUBLICATIONS

Chapter 1, Chapter 2, Chapter 4, Chapter 5, and Chapter 6 are based on [Mem23], and Chapter 3 of this thesis is based on [Mem22].

CONTENTS

1 INTRODUCTION

Let $\mathcal{H}$ be a bounded symmetric domain and $X := \Gamma\backslash\mathcal{H}$ be a quotient of $\mathcal{H}$ by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathcal{H})$. Various hyperbolicity properties of a smooth toroidal compactification $\overline{X}$ of X and curves in $\overline{X}$ have been explored by studying the positivity of $\mathbb{Q}$-line bundles in the form $K_{\overline{X}}(D) - rD$, where D is the boundary divisor, $K_{\overline{X}}(D)$ is the log-canonical bundle and r is a rational number varying on an interval, the size of which depends on X. For instance, see [Nad89, SB05] for the case of moduli space of principally polarized abelian varieties $\mathcal{A}_g$, [BT18a] for the case of Hilbert modular varieties and [BT18b, DCDC15] for the case of ball quotients.

Motivated by these results for the twists of $K_{\overline{X}}(D)$ and their implications for the hyperbolicity properties, in the first part of this thesis we study the positivity of the twists of $\Omega^1_{\overline{X}}(\log(D))$ and $\Omega^1_{\overline{X}}$ in the form of the $\mathbb{Q}$-vector bundles $\Omega^1_{\overline{X}}(\log(D))\langle -rD\rangle$ and $\Omega^1_{\overline{X}}\langle -rD\rangle$. As the vector bundles $\Omega^1_{\overline{X}}(\log(D))$ and $\Omega^1_{\overline{X}}$ restrict well to subvarieties (of arbitrary dimensions), the positivity of $\Omega^1_{\overline{X}}(\log(D))\langle -rD\rangle$ and $\Omega^1_{\overline{X}}\langle -rD\rangle$ reveals various hyperbolicity properties of subvarieties of $\overline{X}$.

We restrict ourselves to a complex ball quotient $X = \Gamma\backslash\mathbb{B}^n$, where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is a torsion-free lattice. We may refer to X as a complex hyperbolic manifold with cusps, where cusps correspond to parabolic fixed points of Γ. The complex ball has an intrinsic Hermitian metric (Bergman metric) which induces a Kähler form on X. This Kähler form also induces a Kähler form on a subvariety V of X. The volume of V with respect to the induced Kähler form on X will be called the induced Kähler volume. The systole of X is the length of the shortest closed geodesic of X with respect to the Bergamn metric.

In the second part this thesis, we find a uniform lower bound for the the induced Kähler volume of all subvarieties of X in terms of the systole. There are a number of arithmetic and geometric consequences with the assumption that the systole is sufficiently large:

- The systole of X bounds from above the growth rate of rational points on X, uniformly in the field of definition of $\overline{X}$.
- The systole of X bounds from below the canonical volume of subvarities of X. In particular, if the systole is sufficiently large, then all subvarities of X are of general type.
- The systole bounds from below the Seshadri constant of $K_{\overline{X}}$. This implies effective global generation and very ampleness results for multiples of the canonical bundle $K_{\overline{X}}$.

1.1 MAIN RESULTS

For cusps of X we associate a uniform depth d measuring the size of the largest embedded cusp neighborhoods which are disjoint from each other (see 2.1 for more precise definitions). Thanks to Parker's generalization of Shimizu's lemma [Par98, Proposition 2.4.] we know that the uniform depth of cusps is at least 2 for torsion-free lattices. Moreover, the uniform depth tends to infinity in the cofinal towers of normal covering (see Lemma 2.4).

Our first main result is the following theorem:

Theorem A. *(Theorem 3.2.6) Let X be a complex hyperbolic manifold with cusps whose toroidal compactification $\overline{X}$ has no orbifold points, and d be the uniform depth of cusps. Then, the $\mathbb{Q}$-vector bundle*

$$\Omega^1_{\overline{X}}\big(\log(D)\big)\langle -rD\rangle$$

is ample for all rational $r \in (0, d/2\pi)$.

In the case that X is an arithmetic locally symmetric domain, Brunebarbe's work ([Bru20b, section 3]) implies that for some highly ramified cover of X the $\mathbb{Q}$-vector bundle in Theorem A is big. However, Theorem A applies even in the case that X is not a cover of other locally symmetric domains and more importantly it implies stronger positivity.

When the dimension of X is greater than 1, the boundary divisor D is a union of étale quotients of abelian varieties. This implies that $\Omega^1_{\overline{X}|D}$ admits a numerically trivial quotient and therefore $\Omega^1_{\overline{X}}$ can't be ample. However, we showed that $\Omega^1_{\overline{X}}$ is semi-ample and ample modulo D (see Definition 2.5) provided that the uniform depth of cusps is sufficiently large:

Theorem B. *(Theorem 3.3.2) With X and $\overline{X}$ as in Theorem A, suppose that the uniform depth of cusps is greater than 2π. Then, the cotangent bundle $\Omega^1_{\overline{X}}$ is semi-ample and ample modulo D.*

Using results of L^2-estimates for a Kähler manifold, Wong [Won18] proved that there exist a constant $r(n)$ and $d(n)$ depending only on the dimension of X, such that if the injectivity radius is larger than $r(n)$ and the uniform depth of cups is larger than $d(n)$, then $\Omega^1_{\overline{X}}$ is ample modulo D. His method relies on the existence theorem of L^2-estimation ([AV65, Hör13]).

In Theorem A, $\Omega^1_{\overline{X}}\big(\log(D)\big)$ can not be replaced by $\Omega^1_{\overline{X}}$ without adding any extra assumptions about X because there are examples due to Hirzebruch [Hir84] of the toroidal compactification of a 2-dimensional ball quotient whose canonical bundle is not even nef. We showed that assuming the uniform depth is sufficiently large the replacement is possible as long as r varies on a smaller interval.

Theorem C. *(Theorem 3.3.1) With X and $\overline{X}$ as in Theorem A, suppose that the uniform depth of cusps d is greater than 2π. Then, the $\mathbb{Q}$-vector bundle*

$$\Omega^1_{\overline{X}}\langle -rD\rangle$$

is ample for all rational $r \in (0, -1 + d/2\pi)$.

Bakker and Tsimerman [BT18b] proved that $K_{\overline{X}}(D) - rD$ is ample for $r \in (0, (n+1)/2)$, where $n = dim\ X$ and concluded that if $n \geq 6$, $K_{\overline{X}}$ is ample. Similarly, Cadorel [Cad21a] showed that $V \subset \overline{X}$ with $dim\ V \geq 6$ is of general type if $V \not\subset D$.

In the second part of this thesis we find a uniform lower bound for the induced Kähler and canonical volumes of all subvarieties of a non-compact ball quotient X in terms of a geometric quantity of X :

Theorem D. *(Theorem 4.3.2) Let $X = \Gamma \backslash \mathbb{B}^n$ be a complex ball quotient by a torsion-free non-uniform lattice Γ whose parabolic stabilizers are unipotent. Let $V \subset X$ be an irreducible subvariety of dimension $m > 0$. Then,*

$$\mathrm{vol}_X(V) \geq \frac{(4\pi)^m}{m!} \sinh^{2m}\left(\mathrm{sys}(X)/2\right), \tag{1.1}$$

where $\mathrm{vol}_X(V)$ is the volume of V induced by Bergman metric on V and $\mathrm{sys}(X)$ is the length of the shortest closed geodesic in X.

The systole is anon-zero real number and can be estimated by the trace of the hyperbolic elements in a representation of Γ. When X is compact ball quotient, the inequality (1.1) is known by Hwang and To [HT99] and we generalized their inequalities for non-compact X with a mild assumption on the parabolic stabilizer of Γ. Note that every neat lattice Γ satisfies this assumption, and every Γ has a finite index subgroup which satisfies this assumption (see [Hum98]).

We prove in Theorem 4.1.9 that the systole of X bounds the uniform depth of cusps from below. Therefore, if $\mathrm{sys}(X)$ is sufficiently large, then $K_{\overline{X}}$ is ample.

For a subvariety $V \subset \overline{X}$ of dimension $m > 0$, we denote the degree of V with respect to the line bundle $K_{\overline{X}}$ by $\deg_{\overline{X}}(V)$:

$$\deg_{\overline{X}}(V) := K_{\overline{X}}^m \cdot V.$$

Also, we study the canonical volume of a subvariety V which is an intrinsic quantity of V and a priory does not depend on the ambient space $\overline{X}$. Let V' be a smooth variety birational to V with a canonical bundle $K_{V'}$. The canonical volume of the variety V is

$$\widetilde{\mathrm{vol}}_V := \limsup_{b \to \infty} \frac{h^0(V', bK_{V'})}{b^m/m!},$$

which does not depend on the choice of V'. In particular, if V is an integral curve, then the canonical volume of V is $2g - 2$. The canonical volume of V measures the asymptotic growth rate of the pluri-canonical linear series $|bK_{V'}|$. The canonical volume is a non-negative real number and it is positive if and only if the linear system $|bK_{V'}|$ embedds V' birationally in a projective space for a large enough b, i.e., V is of general type.

We prove that the systole controls both the canonical volume of V and its degree with respect to $K_{\overline{X}}$ in the following sense:

Theorem E. *(Theorem 4.3.5+Theorem 4.3.7) With the same assumption on X as Theorem D, let $\overline{X}$ be the toroidal compactification of X and $V \subset \overline{X}$ be a subvariety of dimensional $m > 0$*

with $X \cap V \neq \varnothing$*. Suppose that* $\mathrm{sys}(X) \geq 4 \ln\left(5n + (8\pi)^4\right)$*. Then the following inequalities hold:*

$$\widetilde{\mathrm{vol}}_V > \left(\frac{m}{4\pi}\right)^m e^{m\,\mathrm{sys}(X)/16},$$
$$\deg_{\overline{X}}(V) > \left(\frac{n}{4\pi}\right)^m e^{m\,\mathrm{sys}(X)/16}.$$

Note that systole cannot decrease in a cover and for every X there exists a finite cover X' such that $\mathrm{sys}(X')$ is sufficiently large. As a byproduct of Theorem E, we can observe that in a tower of covering of X where the systole increases, the canonical volume of subvarieties increases.

Application I: hyperbolicity of subvarities

As the cotangent bundle is well-behaved under the restriction to subvarieties, by virtue of Theorem A and Theorem C we get hyperbolicity of subvarieties of an arbitrary dimension in the following sense:

Corollary F. (Corollary 3.4.1) With the same notations as in Theorem A, suppose V is a smooth subvariety of $\overline{X}$ intersecting X with dimension $m > 0$. Then,

$$K_V - (r-1)D_{|V}$$

is ample for all rational $r \in (0, m\lfloor \frac{d-1}{2\pi} \rfloor)$. Moreover, if $d > 2\pi$, then K_V is ample.

Corollary G. (Corollary 3.4.2) With the same X as Theorem A, all subvarieties of X are of general type provided that the uniform depth of cusps is greater than 2π.

Given this result, Bombieri–Lang conjecture predicts that there are only finitely many rational points on X. However, this conjecture is widely open.

Application II: sparsity of rational points

A smooth toroidal compactification $\overline{X}$ of X can be defined over a number field F. (see [Fal84]) provided that Γ is neat and arithmetic. Combining Theorem E with [BM22, Theorem 3.4] on the growth rate of rational points, we get that $\mathrm{sys}(X)$ controls the growth rate of rational points:

Corollary H. (Corollary 6.2.2) Let $L = K_{\overline{X}}$ and ϵ be a positive number. Suppose that $\mathrm{sys}(X) \geq 4 \ln\left(5n + (4\pi)^4\right)$. Then, there exists a constant c depending on X, F and ϵ such that for every $B \geq \epsilon[F : \mathbb{Q}]$, one has:

$$\#\left\{x \in X(F) \mid \mathrm{H}_L(x) \leq B\right\} \leq cB^{\delta},$$

where

$$\delta = \frac{4\pi[F : \mathbb{Q}](n+3)}{e^{\mathrm{sys}(X)/16}}(1+\epsilon),$$

and H_L is the multiplicative height.

Corollary H tells us that the growth rate of F-rational points decreases as $\mathrm{sys}(X)$ gets larger.

Application III: effective very ampleness and seshadri constant

Combining Theorem E with the results in the adjunction theory proved by Angehrn-Siu [AS95], Kollar [Kol97] and Ein-Lazersfeld-Nakamaye [ELN96] gives effective results in global generation, very ampleness and separation of jets:

Corollary I. (Corollary 5.2.10) With the same X and $\overline{X}$ as Theorem E, suppose that

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left((1+2n+n!)(n+1)\right), \ln\left(5n+(8\pi)^4\right)\}.$$

Then, the following hold

1. $2K_{\overline{X}}$ is globally generated and very ample modulo D.
2. $3K_{\overline{X}}$ is very ample.

Another implication of Theorem E is the following bound on the Seshadri constant of $K_{\overline{X}}$:

Corollary J. (Theorem 5.2.5) Suppose that

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left((1+2n+n!)(n+s)\right), \ln\left(5n+(8\pi)^4\right)\}.$$

Then $2K_{\overline{X}}$ separates any s-jets and in particular for every $x \in X$, we have that

$$\epsilon(K_{\overline{X}}, x) \geq s/2.$$

Application IV : finite generation of symmetric differentials

Applying [Laz17, Example 2.1.29] to Theorem B yields that symmetric differentials over $\overline{X}$ forms a finitely generated $\mathbb{C}$-algebra:

Corollary K. With the same assumptions as Theorem B, the graded ring

$$\bigoplus_{n\geq 0} H^0(\overline{X}, S^n\Omega^1_{\overline{X}})$$

is finitely generated $\mathbb{C}$-algebra.

1.2 OUTLINE

In Chapter 2, we collect necessary background on the geometry of complex ball, the toroidal compactifications of ball quotients, the Siegel domain and we prove some basic lemmas about $\mathbb{Q}$-vector bundles. In Chapter 3, we study the properties of the Bergman metric induced on $O_{\mathbb{P}(\Omega^1_{\overline{X}}(\log(D))}(1)$ and we prove the results related to the positively of $\Omega^1_{\overline{X}}(\log(D))$ including Theorem A. Further, we prove the results related to the positivity of $\Omega^1_{\overline{X}}$, namely Theorem B and Theorem C, and we deduce the applications to hyperbolicity of subvarieties in Corollary F, and Corollary G.

In Chapter 4 we prove the bounds on the volume of subvarities, in particular Theorem D and Theorem E. In Chapter 5 we prove Corollary I and Corollary J on the effective very ampleness and global generation. Finally, in Chapter 6 we prove Corollary H on the sparsity of rational points.

2 BACKGROUND

In this chapter, we collect the necessary background and notations which will be used frequently in the sequel. We refer to [Gol99, Par98, Kap22, BT18b] for a much fuller account.

2.1 GEOMETRY OF COMPLEX CALL QUOTIENTS

The complex n-ball $\mathbb{B}^n$ is defined as

$$\mathbb{B}^n = \{z \in \mathbb{C}^n \mid |z|^2 < 1\}.$$

$\mathbb{B}^n$ has an intrinsic Hermitian metric called Bergman metric:

$$h = 4.\frac{(1 - |z|^2).\sum_i dz_i \otimes d\bar{z}_i + \sum_i \bar{z}_i dz_i \otimes \sum_i z_i d\bar{z}_i}{(1 - |z|^2)^2}.$$

The holomorphic isometry group of $\mathbb{B}^n$ with respect to this metric is the projective unitary group

$$G := \mathrm{PU}(n,1) = \frac{\mathrm{U}(n,1)}{Z(\mathrm{U}(n,1))},$$

where the centre $Z(\mathrm{U}(n,1))$ can be identified with the circle group $\{\mu I : |\mu| = 1\}$. The group G acts transitively on $\mathbb{B}^n$ and acts doubly transitively on the boundary sphere $\partial\mathbb{B}^n$. The stabilizer of the center of $\mathbb{B}^n$ is $\mathrm{U}(n)$. Every isometry $g \in G$ is continuous on the closed ball $\overline{\mathbb{B}^n}$ and it follows from Brouwer's fixed point theorem that g has a fixed point on the closed ball $\overline{\mathbb{B}^n}$. Moreover, if there is no fixed point on $\mathbb{B}^n$, there can be at most two fixed points on the boundary sphere $\partial\mathbb{B}^n$. Accordingly, an isometry $g \in G$ is classified as:

1. Elliptic: g has a fixed point z in $\mathbb{B}^n$. After conjugating g via $h \in G$ which sends z to 0,

   $$hgh^{-1} \in \mathrm{U}(n),$$

   and therefore all eigenvalues of g are roots of unity.

2. Parabolic: g has a unique fixed point in $\overline{\mathbb{B}^n}$ and this fixed point is on the boundary $\partial\mathbb{B}^n$. Equivalently,

   $$\inf_{z \in \mathbb{B}^n} d(z, gz) = 0,$$

   where $d(.,.)$ denotes the Bergman metric. This infimum is not realized for a parabolic g.

3. Hyperbolic: g has exactly two fixed points in $\overline{\mathbb{B}^n}$ and both are in $\partial\mathbb{B}^n$. In particular, g preserves the unique geodesic connecting these two fixed points in $\mathbb{B}^n$ and acts as a translation along this geodesic. This geodesic is called the axis of g. The length of a hyperbolic isometry $g \in G$ is

$$\ell(g) := \inf_{z\in\mathbb{B}^n} d(z, gz).$$

This infimum is not zero and is realized by any point on the axis of g. The work of Chen-Greenberg on the conjugacy classification of element of $\mathrm{U}(n,1)$ [CG74, Theorem 3.4.1] implies that a hyperbolic isometry g has two eigenvalues $re^{i\theta}$ and $r^{-1}e^{i\theta}$ with $r > 1$ and $n-1$ eigenvalues with norm 1.

Let $\Gamma \subset \mathrm{PU}(n,1)$ be a torsion-free lattice whose parabolic elements are unipotent. [Hum98] tells us that every lattice in $\mathrm{PU}(n,1)$ has a finite index subgroup with this property. With this property, an element $g \in \Gamma$ is hyperbolic if and only if g is semi-simple. Therefore, we will denote the set of the hyperbolic elements in Γ by Γ_s.

Let $X = \Gamma\backslash\mathbb{B}^n$. The systole of X is the length of the shortest closed geodesic with respect to the Bergman metric:

$$\mathrm{sys}(X) := \inf_{g\in\Gamma_s} \ell(g) = \inf_{g\in\Gamma_s}\{d(z,gz)|z \in \mathbb{B}^n\}.$$

Equivalently, the systole of X is the length of the shortest hyperbolic element in Γ.

Consider $x \in X$. Choose a fiber $\tilde{x} \in \mathbb{B}^n$ with stabilizer $\Gamma_{\tilde{x}}$ in Γ. The injectivity radius of x in X is defined to be

$$\mathrm{inj}_x(X) := \frac{1}{2}\inf_{\gamma\in\Gamma\backslash\Gamma_{\tilde{x}}} d(\tilde{x}, \gamma\cdot\tilde{x}),$$

which is independent of choice of $\tilde{x}$. The injectivity radius of X is $\mathrm{inj}(X) := \inf_{x\in X}\mathrm{inj}_x(X)$. In the case that X is compact, Γ only has semi-simple elements and hence $\mathrm{sys}(X) = inj(X)/2$. However, this relation does not hold for a non-compact X because of the parabolic elements in Γ.

2.2 SIEGEL DOMAIN MODEL

The half-plane model of 1-dimensional complex ball quotient is generalized by the Siegel domain model in higher dimensions. In horospherical coordinates, the Siegel domain of (complex) dimension n is $\mathbb{S} = \mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}^+$. The points of $\mathbb{S}$ will be written as $(\zeta, v, u) \in \mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}^+$. The boundary of $\mathbb{S}$ is $H_0 \cup \{\infty\}$, where ∞ is a distinguished point at infinity and $H_0 = \mathbb{C}^{n-1}\times\mathbb{R}\times\{0\}$. We will follow [Par98] to describe $\mathrm{PU}(n,1)$ as a matrix group by embedding the Siegel domain as a paraboloid in $\mathbb{P}(\mathbb{C}^{n,1})$. To do so, we should choose a Hermitian form of signature $(n,1)$ on $\mathrm{PU}(n,1)$. Let

$$J_0 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and $Q := z^* J_0 z$, where z is a column vector in $\mathbb{P}(\mathbb{C}^{n,1})$ and z^* is the Hermitian transpose of z. Consider the map $\psi : \overline{\mathbb{S}} \to \mathbb{P}(\mathbb{C}^{n,1})$ given by

$$\psi : (\zeta, v, u) \longrightarrow \begin{bmatrix} \frac{1}{2}(-||\zeta||^2 - u + iv) \\ \zeta \\ 1 \end{bmatrix}, \text{ for } (\zeta, v, u) \in \overline{\mathbb{S}} \setminus \{\infty\}; \ \psi : \infty \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (2.1)$$

The image of this map is the set of points in $\mathbb{P}(\mathbb{C}^{n,1})$, where the Hermitian form Q is negative. Also ψ is a homeomorphism of $\partial\mathbb{S}$ onto the set of points where Q is zero. We will refer to $q_\infty := \psi(\infty)$ as the cusp at ∞ and $q_0 := \psi(0)$ as the cusp at 0.

The holomorphic isometry group of $\mathbb{S}$ with respect to the Bergman metric is the projective unitary group $\mathrm{PU}(Q)$ and it acts on $\mathbb{P}(\mathbb{C}^{n,1})$ by matrix multiplication. A matrix $h \in \mathrm{GL}(n+1, \mathbb{C})$ is in $\mathrm{PU}(Q)$ if and only if its inverse h^{-1} is given by $J_0 h^* J_0$, where * denotes the Hermitian transpose; that is, transpose the matrix and complex conjugate each of its entries. The general form for $h \in PU(Q)$ and its inverse is given in [Par98, pg 438] :

$$h = \begin{bmatrix} a & \tau^* & b \\ \alpha & A & \beta \\ c & \delta^* & e \end{bmatrix}, \quad h^{-1} = \begin{bmatrix} \bar{e} & \beta^* & \bar{b} \\ \delta & A^* & \tau \\ \bar{c} & \alpha^* & \bar{a} \end{bmatrix}, \qquad (2.2)$$

where A is an $(n-1) \times (n-1)$ matrix, $a, b, c, e \in \mathbb{C}$, and $\tau, \delta, \alpha, \beta$ are column vectors in $\mathbb{C}^{n-1}$.

The following lemma easily follows:

Lemma 2.1. Let h be an element of $PU(Q)$ written in form 2.2.

1. ([Par97, page 7]) If h swaps q_∞ and q_0, then it must have the following form:

$$h = \begin{bmatrix} 0 & 0 & 1/\bar{c} \\ 0 & A & 0 \\ c & 0 & 0 \end{bmatrix},$$

where $A \in U(n-1)$ and $c \in \mathbb{C}$. Therefore, h must be the following transformation:

$$h : (\zeta, u, v) \longrightarrow \Big(\frac{-2A\zeta}{c(||\zeta||^2 + u - iv)}, \frac{-4v}{|c|^2 |||\zeta||^2 + u - iv|^2}, \frac{4u}{|c|^2 |||\zeta||^2 + u - iv|^2} \Big).$$

2. If h fixes both the cusps q_∞ and q_0, then it must have the following form

$$h = \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1/\bar{a} \end{bmatrix},$$

where $A \in U(n-1)$ and $a \in \mathbb{C}$.

For any pair of points $z_1 = (\zeta_1, v_1, u_1)$ and $z_2 = (\zeta_2, v_2, u_2)$ in $\mathbb{S}$ the Bergman metric is given by:

$$d(z_1, z_2) = 2\cosh^{-1}\left(\frac{1}{4u_1u_2}\big|||\zeta_1 - \zeta_2||^2 + u_1 + u_2 + iv_1 - iv_2 + 2i\,\mathrm{im}\langle\zeta_1, \zeta_2\rangle\big|^2\right), \tag{2.3}$$

where $\langle.,.\rangle$ denotes the standard positive definite Hermitian form on $\mathbb{C}^{n-1}$. Since $\cosh^{-1}(x)$ is increasing, the following lower bound can be obtained for the metric :

$$d\big((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)\big) \geq 2\cosh^{-1}\left(\frac{|u_1 + u_2|^2}{4u_1u_2}\right) \tag{2.4}$$

The holomorphic sectional curvature of this metric is -1 and the sectional curvature of this metric varies on $[-1, -\frac{1}{4}]$ (see [Gol99]). It follows that the holomorphic bisectional curvature of this metric is bounded above by a negative constant since the holomorphic bisectional curvature always can be written as the sum of two sectional curvatures.

The Kähler form of the Bergman metric on $\mathbb{S}$ is given by

$$w_{\mathbb{S}} := -2i\partial\bar{\partial}\log(u) \tag{2.5}$$

(see, for example, [BT18b, Lemma 2.1]), and more explicitly we can write $\omega_{\mathbb{S}}$ in terms of (ζ, z)-coordinates

$$\begin{aligned} w_{\mathbb{S}} =& 2iu^{-2} \cdot \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}(u\delta_{ik} + \zeta_i\bar{\zeta}_k)d\zeta_k \wedge d\bar{\zeta}_i \\ &+ u^{-2} \cdot \sum_{j=1}^{n-1}(\bar{\zeta}_j d\zeta_j \wedge d\bar{z} - \zeta_j dz \wedge d\bar{\zeta}_j) \\ &- \frac{i}{2}u^{-2}dz \wedge d\bar{z}, \end{aligned} \tag{2.6}$$

where δ_{ik} is the Kronecker delta function. Using holomorphic coordinates ζ and z on $\mathbb{S}$, we can set a holomorphic coordinates (ζ, z, ξ, w) on $\Omega^1_{\mathbb{S}}$, where $\xi = (\frac{\partial}{\partial\zeta_1}, \frac{\partial}{\partial\zeta_2}, ..., \frac{\partial}{\partial\zeta_{n-1}})$ and $w = \frac{\partial}{\partial z}$. Using the frame $e_i = \xi_i$ for $1 \leq i \leq n-1$ and $e_n = w$, the hermitian matrix of the Bergman metric is

$$[h(e_i, \bar{e}_j)] = 2\begin{bmatrix} u^{-1} + u^{-2}|\zeta_1|^2 & u^{-2}\zeta_1\bar{\zeta}_2 & ... & u^{-2}\zeta_1\bar{\zeta}_{n-1} & \frac{1}{2i}u^{-2}\zeta_1 \\ u^{-2}\zeta_2\bar{\zeta}_1 & u^{-1} + u^{-2}|\zeta_2|^2 & ... & u^{-2}\zeta_2\bar{\zeta}_{n-1} & \frac{1}{2i}u^{-2}\zeta_2 \\ ... & ... & ... & ... & ... \\ u^{-2}\zeta_{n-1}\bar{\zeta}_1 & u^{-2}\zeta_{n-1}\bar{\zeta}_2 & ... & u^{-1} + u^{-2}|\zeta_{n-1}|^2 & \frac{1}{2i}u^{-2}\zeta_{n-1} \\ -\frac{1}{2i}u^{-2}\bar{\zeta}_1 & -\frac{1}{2i}u^{-2}\bar{\zeta}_2 & ... & -\frac{1}{2i}u^{-2}\bar{\zeta}_{n-1} & \frac{1}{4}u^{-2} \end{bmatrix}.$$

2.3 TOROIDAL COMPACTIFICATION

The complex ball quotient X has a unique toroidal compactification $\overline{X}$, which is a smooth projective variety (see [Mok12]). The boundary divisor of this compactification $D := \overline{X} \setminus X$ is a disjoint union of abelian varieties with ample conormal bundle.

It follows from Mumford's work on the singular Hermitian metric [Mum77] that the Bergman metric on X extends as a good Hermitian metric to $\overline{X}$. Integration against ω_X on the open part represents (as a current) a multiple of the first Chern class

$$c_1(K_{\overline{X}} + D) = \frac{1}{2\pi}\frac{n+1}{2}[\omega_X] \in H^{1,1}(\overline{X}, \mathbb{R}), \tag{2.7}$$

where $K_{\overline{X}}$ is the canonical bundle of $\overline{X}$ (see [BT18b]).

2.4 STABILIZER OF CUSPS

We denote the parabolic stabilizer of q_∞ in G by G_∞. With our choice of Hermitian form, the matrices corresponding to elements of G_∞ are upper triangular. There is an equivalent way to identify these matrices:

Lemma 2.1. ([Par98]) Let h be an element of $\mathrm{PU}(Q)$ written in the form 2.2. Then, h fixes the cusp q_∞ if and only if the c entry of h is 0.

Proof. Note that

$$h \cdot q_\infty = \begin{bmatrix} a & \tau^* & b \\ \alpha & A & \beta \\ c & \delta^* & e \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ \alpha \\ c \end{bmatrix},$$

and therefore h fixes q_∞ projectively if and only if $c = 0$ and $\alpha = 0$. Note that if the c entry of h is 0, the multiplication of the matrix of h and h^{-1} in the form 2.2 yields that α (and also δ) must be 0. □

The group G_∞ is generated by Heisenberg isometries I_∞ and a one-dimensional torus T. Heisenberg isometries consist of Heisenberg Rotations $U(n-1)$ and Heisenberg translations $\mathfrak{N}$. Heisenberg Rotations $U(n-1)$ acts on ζ-coordinates of $\mathbb{S}$ in the usual way and Heisenberg translations $\mathfrak{N} \cong \mathbb{C}^{n-1} \times \mathbb{R}$ acts on ζ and v coordinates of $\mathbb{S}$ via

$$(\tau, t) : (\zeta, v, u) \longrightarrow \big(\zeta + \tau, v + t + 2i\langle \tau, \zeta\rangle, u\big).$$

The element $(0, t) \in \mathfrak{N}$ will be called the vertical translation by t, and the subgroup generated by $(0, t)$ in G_∞ will be denoted by V_∞. The vertical translation V_∞ is the center of G_∞ and the quotient $V_\infty \backslash I_\infty$ is isomorphic to the unitary transformation of $\mathbb{C}^{n-1}$. We use $(A, \tau, t) \in U(n-1) \ltimes \mathfrak{N}$ to denote the transformation acting by

$$(A, \tau, t) \cdot (\zeta, z) \longrightarrow \big(A\zeta + \tau, z + t + i|\tau|^2 + 2i\langle A\zeta, \tau\rangle\big),$$

where the standard positive definite hermitian form $(.,.)$ chosen on $\mathbb{C}^{n-1}$. Using the chain rule we can observe that the action of (A,τ,t) induced on the cotangent bundle $\Omega^1_{\mathbb{S}}$ is

$$(A,\tau,t).(\zeta,z,\xi,w) = \big(A\zeta+\tau, z+t+i|\tau|^2+2i\langle A\zeta,\tau\rangle, A\xi+2i\langle A\xi,\tau\rangle, w\big). \tag{2.8}$$

A Heisenberg translation $(\tau,t)\in\mathfrak{N}$ fixing q_∞ corresponds to the matrix g_∞ and a Heisenberg translation $(\sigma,s)\in\mathfrak{N}$ fixing q_0 corresponds to the matrix g_0, where

$$g_\infty = \begin{bmatrix} 1 & -\tau^* & -(|\tau|+it)/2 \\ 0 & I & \tau \\ 0 & 0 & 1 \end{bmatrix}, \quad g_0 = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & I & 0 \\ -(|\sigma|+is)/2 & -\sigma^* & 1 \end{bmatrix}. \tag{2.9}$$

With our assumption on Γ, all parabolic stabilizers of q_∞ in Γ, i.e., $\Gamma_\infty := \Gamma\cap G_\infty$ are Heisenberg translations.

2.5 NEIGHBORHOOD OF CUSPS

A horoball centered at the cusp q_∞ with height $\tilde{u}$ is the open set

$$B_\infty(\tilde{u}) := \{(\zeta,v,u)\in\mathbb{S} \mid u>\tilde{u}\}.$$

The smallest $\tilde{u}$ such that $\Gamma_\infty\backslash B_\infty(\tilde{u})$ injects into X will be called the height of the cusp q_∞ and will be denoted by u_∞. The complex ball quotient X has a finitely many cusps and they are in one-to-one correspondence with the equivalence classes of parabolic fixed points of Γ. For a cusp q_i, there exists a $g\in\mathrm{PU}(Q)$ translating q_i to q_∞. The horoball based at cusp q_i with height $\tilde{u}$ is defined to be the image of the horoball based at cusp q_∞ with height $\tilde{u}$ translated by g :

$$B_i(\tilde{u}) = g\cdot B_\infty(\tilde{u})$$

Accordingly, the height of cusp q_i is the smallest $\tilde{u}$ such that $\Gamma_i\backslash B_i(\tilde{u})$ injects into X, where Γ_i is the parabolic stabilizer of q_i. We will denote the height of q_i by u_i. Thanks to the Parker's generalization of Shimuzu's lemma [Par98], $B_i(\tilde{u})$ must inject into X for a large enough $\tilde{u}$. Note that the height coordinate u on $\mathbb{S}$ is invariant under the action of Heisenberg rotations $\mathrm{U}(n-1)$ and Heisenberg translations $\mathfrak{N}$ and hence the horoball is invariant with respect to the action of these groups.

Let t_i be the shortest vertical translation in the parabolic stabilizer of q_i. The number $d_i = t_i/u_i$ is called the depth of cusp q_i which is invariant under conjugating Γ.

Definition 2.1. ([BT18b, Definition 3.7.]) The uniform depth of the cusps of X is the largest d satisfying the following properties:

1. for every i, $d\le d_i$ (this gives that $\Gamma_i\backslash B_i(t_i/d)$ injects into X).
2. all $B_i(t_i/d)$ are disjoint.

Remark 2.2. By passing to a cover of X, i.e., passing to a subgroup of Γ, the shortest vertical translations of parabolic subgroups can not decrease and the heights can not increase. Therefore, the canonical depth can not decrease by passing to a cover of X.

Definition 2.3. (see [Yeu94, Section 2]) A cofinal normal tower of X is a sequence $\{X_i\}_{i=1}^{\infty}$ of étale Galois coverings of $X = X_1$ given by a sequence of lattices $\{\Gamma_i\}_{i=1}^{\infty}$ such that for each subsequence $\{\Gamma_j\}_{j=1}^{\infty}$, one has $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$.

Similar to [Yeu94, Lemma 1.2.2.] and [HT06, Proposition 2.1.], we can prove that the uniform depth of cusps tends to infinity in towers:

Lemma 2.4. Let $\{X_i\}_{i=1}^{\infty}$ be a cofinal normal tower of X and $d(X_i)$ be the uniform depth of cusps of X_i. Then,

$$\lim_{i\to\infty} d(X_i) = \infty.$$

Proof. Let $p_i : X_i \to X$ be the covering map. Fix a cusp c_1 of X, and let c_i be a cusp of X_i such that $p_i(c_i) = c_1$. Note that as X_i is a normal cover of X, Γ acts transitively on cusps of X_i, and therefore the uniform depth $d(X_i)$ equals to the depth of each cusps. Hence, it is enough to show that

$$\lim_{i\to\infty} d(c_i) = \infty,$$

where $d(c_i)$ is the depth of c_i on X_i. Let $u(c_i)$ be the height of cusp c_i, and $t(c_i)$ be the shortest vertical translation in the stabilizer of c_i. Since $u(c_i) \leq u(c_1)$, it is sufficient to show that $t(c_i)$ tends to infinity. Note that as Γ is discrete, for every t_i there only exists finitely many elements in Γ with the length smaller than or equal to t_i. On the other hand, for every subsequence $\{\Gamma_j\}_{j=1}^{\infty}$, we have $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$. Putting these together, we conclude that the length of the shortest element except 1 in Γ_i tends to infinity. □

2.6 BASE LOCI

The goal of this section is to prove formal properties of $\mathbb{Q}$-vector bundles which will be used in §3.2 and §3.3 to study the twists of $\Omega^1_{\overline{X}}$ and $\Omega^1_{\overline{X}}(\log(D))$. To this end, we use the definitions and properties of the base loci of a vector bundle which has been systematically studied in [BKK+15]. We also use the definitions and properties of $\mathbb{Q}$-vector bundles from [Laz04, section 6].

Let E be a vector bundle over a projective variety $\overline{Y}$, and D be an integral divisor on $\overline{Y}$. The base locus of E is defined to be the subset

$$\mathbf{Bs}(E) = \{y \in \overline{Y} | H^0(\overline{Y}, E) \to E(y) \text{ is not surjective}\},$$

and the stable base loci of E is defined to be the algebraic subset

$$\mathbb{B}(E) = \bigcap_{m>0} \mathbf{Bs}(S^m E).$$

Let r be a rational number.

Definition 2.1. A $\mathbb{Q}$-vector bundle $E\langle rD\rangle$ on $\overline{Y}$ is a pair consisting of a vector bundle E on $\overline{Y}$ and a $\mathbb{Q}$-divisor $rD \in \mathrm{Div}_{\mathbb{Q}}(\overline{Y})$. A $\mathbb{Q}$-isomorphism of $\mathbb{Q}$-vector bundles is the equivalence relation generated by considering $E\langle D' + rD\rangle$ to be equivalent to $E \otimes O_{\overline{Y}}(D')\langle rD\rangle$, where D' is an integral divisor.

The formal symbol $E\langle rD\rangle$ is intended to say that we are twisting the vector bundle E by a $\mathbb{Q}$-divisor rD. The symmetric power $S^m\big(E\langle rD\rangle\big)$ denotes a $\mathbb{Q}$-vector bundle $S^m(E)\langle mrD\rangle$. It is easy to observe that every $\mathbb{Q}$-vector bundle has a symmetric power which is $\mathbb{Q}$-isomorphic to a vector bundle. Let a and b be integers such that b is positive. We will define the base loci of the $\mathbb{Q}$-vector bundle $E\langle \frac{a}{b}D\rangle$ to be the following set

$$\mathbb{B}(E\langle \frac{a}{b}D\rangle) = \bigcap_{m>0} \mathbf{Bs}\big(S^{mb}E \otimes O_{\overline{Y}}(maD)\big),$$

which does not depend on the choice of a, b. Let A be an ample divisor on $\overline{Y}$. The augmented base locus of E is the algebraic subset

$$\mathbb{B}_+(E) = \bigcap_{r\in\mathbb{Q}_+} \mathbb{B}(E\langle -rA\rangle),$$

which does not depend on the choice of the ample divisor A.

Definition 2.2. We say that $E\langle rD\rangle$ is ample (nef) if one of the following equivalent properties holds:

(i) The $\mathbb{Q}$-divisor $O_{\mathbb{P}(E)}(1) + \pi^*(rD)$ is ample (nef) on $\mathbb{P}(E)$.

(ii) $E\langle rD\rangle$ has a symmetric power which is $\mathbb{Q}$-isomorphic to an ample (a nef) vector bundle.

(iii) If $r = a/b$ and b is positive, then the vector bundle $S^bE \otimes O_{\overline{Y}}(aD)$ is ample (nef).

The equivalency of these definitions is checked in [Laz17, Lemma 6.2.8].

Lemma 2.3. If E is a nef vector bundle and rD is an ample $\mathbb{Q}$-divisor, then $E\langle rD\rangle$ is an ample $\mathbb{Q}$-vector bundle.

Proof. Take $a, b \in \mathbb{Z}$ such that $r = a/b$ and b is positive. As rD is ample, the integral divisor aD is ample. Since E is nef, S^bE is nef. Combing these two we get that $S^bE \otimes O_{\overline{Y}}(aD)$ is ample vector bundle. Hence, $E\langle rD\rangle$ is ample. □

Lemma 2.4. Suppose $0 \to G \to E \to F \to 0$ is an exact sequence of vector bundles on a projective variety $\overline{Y}$. The following hold:

(i) If $E\langle rD\rangle$ is ample (nef), then $F\langle rD\rangle$ is ample (nef).

(ii) If $G\langle rD\rangle$ and $F\langle rD\rangle$ are ample, then $E\langle rD\rangle$ is ample.

Proof. (i) Since $\mathbb{P}(E)$ parametrizes one-dimensional quotients, the surjection $E \to F$ corresponds to an inclusion $\mathbb{P}(E) \subset \mathbb{P}(F)$, such that the restriction of $O_{\mathbb{P}(E)}(1)$ is $O_{\mathbb{P}(F)}(1)$. Hence, the ampleness (nefness) of $O_{\mathbb{P}(E)}(1) + \pi^*(rD)$ implies the ampleness (nefness) of $O_{\mathbb{P}(F)}(1) + \pi^*(rD)$.

(ii) See [Laz04, Lemma 6.2.8]. □

Note that the above mentioned definitions for a $\mathbb{Q}$-vector bundle $E\langle rD\rangle$ agree with the usual definitions for a vector bundle when a vector bundle E is considered as the $\mathbb{Q}$-vector bundle $E\langle 0\rangle$.

Definition 2.5. A vector bundle E over $\overline{Y}$ is said to be ample modulo D if for every coherent sheaf $\mathscr{F}$ over $\overline{Y}$, there exists an $m_0 > 0$ such that if $m > m_0$, then for every $y \in \overline{Y} \setminus D$, the fiber $\mathscr{F} \otimes S^m(E)_{|y}$ is generated by $H^0\big(Y, \mathscr{F} \otimes S^m(E)\big)$.

There is an obvious relation between the notion of ampleness modulo a divisor and the augmented base loci:

Proposition 2.6. If the vector bundle E is ample modulo D, then the augmented base loci $\mathbb{B}_+(E)$ contained in D and $\mathbb{B}_+\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big)$ is contained in $\pi^*(D)$.

Proof. Pick an arbitrary point $y \in \overline{Y} \setminus D$. To show $\mathbb{B}_+(E) \subset D$, it is enough to show that $y \notin \mathbb{B}_+(E)$. Let A be an ample line bundle. By Definition 2.5, there exist n such that $(S^nE - A)_{|y}$ is generated by $H^0(Y, S^nE - A)$ and therefore $y \notin \mathbb{B}(E\langle -\frac{1}{n}A\rangle)$. Consequently, $y \notin \mathbb{B}_+(E)$.

[BKK+15, Proposition 3.2] tells us that $\mathbb{B}_+\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) \subset \pi^{-1}(\mathbb{B}_+(E))$ and therefore the previous part gives that $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1)) \subset \pi^*(D)$. □

Suppose L is a line bundle on a projective variety $\overline{Y}$. It is well-known that L is big if and only if $\mathbb{B}_+(L) \neq X$. This follows that if E is ample modulo D, then E is big in the sense that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big. Another well-know fact is that L is ample if and only if $\mathbb{B}_+(L) = \emptyset$. For vector bundles, we will use the following lemma to go from the ampleness modulo D to the ampleness:

Lemma 2.7. If E is ample modulo D and $E_{|D}$ is ample, then E is ample.

Proof. Let L be the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. First, we show that L is nef. Consider an irreducible curve C on $\mathbb{P}(E)$. If $C \subset \pi^*(D)$, then $C \cdot L > 0$ because $L_{|\pi^*(D)}$ is ample. If C is not contained in D, then there is a point $x \in C$ such that $x \notin \pi^*(D)$. Since E is ample modulo D, Proposition 2.6 gives that $\mathbb{B}_+(L) \subset \pi^*(D)$ and therefore $x \notin \mathbb{B}_+(L)$. It means that there exist $a, b \in \mathbb{Z}^+$ such that $x \notin \mathbf{Bs}(bL - aA)$, where A is an ample divisor on $\mathbb{P}(E)$. Consequently, the zero locus of $bL - aA$ is not contained in C and it follows that C intersects transversely with $bL - aA$. Hence. $C \cdot (bL - aA) \geq 0$ which gives that $C \cdot L > 0$.

To conclude ampleness it is sufficient to prove that L intersects positively with every subvariety of $\mathbb{P}(E)$. Let V be a an arbitrary subvariety of $\mathbb{P}(E)$. Since L is nef, thanks to [Bir17, Theorem 1.4], we know that

$$\mathbb{B}_+(L) = \bigcup_{L_{|V} \text{ not big}} V.$$

Since $\mathbb{B}_+(L) \subset \pi^*(D)$ and $E_{|D}$ is ample, we get $\mathbb{B}_+(L) = \emptyset$, that is, E is ample. □

Definition 2.8. We say that $E\langle rD\rangle$ is ample modulo D if one of the following two equivalent properties holds:

(i) $E\langle rD\rangle$ has a symmetric power which is $\mathbb{Q}$-isomorphic to an ample modulo D vector bundle.

(ii) If $r = a/b$ and b is positive, then the vector bundle $S^bE \otimes O_{\overline{Y}}(aD)$ is ample modulo D.

Definition 2.9. Consider two $\mathbb{Q}-$ vector bundles $E\langle \frac{a}{b}D\rangle$ and $F\langle \frac{a'}{b}D\rangle$, where r and s are two rational numbers. We say that $E\langle rD\rangle$ is a $\mathbb{Q}$-subsheaf of $F\langle sD\rangle$ if for every integer a, a' and every positive integer b such that $r = \frac{a}{b}$ and $s = \frac{a'}{b}$, the vector bundle $S^bE \otimes O_{\overline{Y}}(aD)$ is a subsheaf of $S^bF \otimes O_{\overline{Y}}(a'D)$.

Lemma 2.10. Let E and F be vector bundles on $\overline{Y}$ and they are isomorphic over $Y := \overline{Y} \setminus D$. Suppose $E\langle rD\rangle$ is a $\mathbb{Q}$-subsheaf of $F\langle sD\rangle$, and $E\langle rD\rangle$ is ample modulo D. Then $F\langle sD\rangle$ is ample modulo D.

Proof. Take $a, a', b \in \mathbb{Z}$ such that $a/b = r, a'/b' = s$ and b is positive. Since the $\mathbb{Q}$-vector bundle $E\langle \frac{a}{b}D\rangle$ is ample modulo D, there exists an integer $l \in \mathbb{Z}$ such that the vector bundle $S^{bl}E \otimes O_{\overline{Y}}(alD)$ is ample modulo D. It means that for a given coherent sheaves $\mathscr{F}$ over $\overline{Y}$ there exists $m_0 > 0$ such that for every $m > m_0$ and every $y \in Y$, $\big(\mathscr{F} \otimes S^{blm}E \otimes O_{\overline{Y}}(alD)\big)_{|y}$ is generated by $H^0(\overline{Y}, \mathscr{F} \otimes S^{blm}E \otimes O_{\overline{Y}}(alD))_{|y}$. Because $E\langle rD\rangle$ is a $\mathbb{Q}$-subsheaf of $F\langle sD\rangle$, we obtain that $\mathscr{F} \otimes S^{blm}E \otimes O_{\overline{Y}}(alD))$ is a subsheaf of $\mathscr{F} \otimes S^{blm}F \otimes O_{\overline{Y}}(a'lD))$. This implies that

$$H^0\big(\overline{Y}, \mathscr{F} \otimes S^{blm}E \otimes O_{\overline{Y}}(alD)\big) \subseteq H^0\big(\overline{Y}, \mathscr{F} \otimes S^{blm}F \otimes O_{\overline{Y}}(a'lD)\big).$$

Since these two sheaves restricts to the same sheaf on Y, for every $y \in Y$ and for every $m > m_0$, $\mathscr{F} \otimes S^{blm}E \otimes O_{\overline{Y}}(a'lD)$ is generated by $H^0\big(\overline{X}, F \otimes S^mE_2\big)$. □

Lemma 2.11. Let $r \in \mathbb{Q}^+$ be a positive rational number. If $E\langle -rD\rangle$ is ample modulo D, then $E\langle -sD\rangle$ is ample modulo D for any rational $s < r$.

Proof. Let $a, a', b \in \mathbb{Z}$ be positive integers such $a/b = r$ and $a'/b = s$. Since $s < r$, we get that $a' < a$ and therefore $S^bE \otimes_{\overline{Y}} (-aD)$ is a subsheaf of $S^bE \otimes_{\overline{Y}} (-a'D)$ and the claim follows from Lemma 2.10. □

Lemma 2.12. Suppose $f : Y' \longrightarrow Y$ is a finite surjective map and E is a vector bundle on Y. Then, f^*E is semi-ample, then E is so.

Proof. By passing to the projective bundles, the lemma follows from the corresponding facts for line bundles (see [Fuj83, 1.20]) □

3 POSITIVITY OF COTANGENT BUNDLE

3.1 POSITIVITY ON THE PROJECTIVE BUNDLE

Let E be a vector bundle of rank n on an algebraic variety or complex manifold Y and let

$$\pi : \mathbb{P}(E) \to Y$$

be the projective bundle of lines in the dual bundle E^*. The projective bundle $\mathbb{P}(E)$ carries a tautological quotient bundle line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)$ whose dual $\mathcal{O}_{\mathbb{P}(\mathbb{E})}(-1)$ is naturally a subbundle of $\pi^* E$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(-1) \longrightarrow \pi^* E.$$

In other words, a point $(y, [L])$ of $\mathbb{P}(E)$ is determined by a point $y \in Y$ together with $[L]$ in $\mathbb{P}^n = \mathbb{P}\big(E^*(y)\big)$. The fiber $\mathcal{O}_{\mathbb{P}(E)}(-1)_{(y,[L])}$ is the subvector space L and the fiber $\mathcal{O}_{\mathbb{P}(E)}(1)_{(y,[L])}$ is the one dimensional quotient of E corresponding to L. A vector bundle E is called ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample.

Let $X = \Gamma \backslash \mathbb{B}^n$ be a complex ball quotient by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ and $\hat{h}$ be the hermitian metric induced by the Bergman metric h on $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$.

In this section, we prove two properties of $\hat{h}$ in Proposition 3.2 and Proposition 3.4. These properties will be used to construct a singular hermitian metric on the line bundle $\mathcal{O}(1)$ over $\mathbb{P}(\Omega^1_X(\log(D)))$ to prove Theorem A.

Let $\hat{h}^*$ be the dual metric of $\hat{h}$ on $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(-1)$. Suppose U is an open subset of $\mathbb{P}(\Omega^1_X)$ such that for every $u \in U$, $\xi_{1,u} \neq 0$. After replacing ξ_i by $\frac{\xi_i}{\xi_1}$ and w by $\frac{w}{\xi_i}$, we can take a local section $\sigma = \sum_{i=1}^{n-1} \xi_i e_i + w e_n$ of $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(-1)$ on U. The first Chern form of $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(-1)$ on U is represented by

$$\begin{aligned} c_1(\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(-1), \hat{h}^*) &= \frac{i}{2\pi} \bar{\partial}\partial \log ||\sigma||^2_{\hat{h}^*} \\ &= \frac{i}{2\pi} \bar{\partial}\partial \log \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \bar{\xi}_j h(e_i, e_j) \Big) \\ &= \frac{i}{2\pi} (-2\bar{\partial}\partial \log u + \bar{\partial}\partial \log v), \end{aligned}$$

where

$$v = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left((u\delta_{ij} + \bar{\zeta}_i\zeta_j)\xi_i\bar{\xi}_j + \frac{1}{2i}\xi_j\bar{\xi}_n\bar{\zeta}_j - \frac{1}{2i}\bar{\xi}_j\xi_n\zeta_j\right) + \frac{1}{4}|\xi_n|^2. \tag{3.1}$$

As $O_{\mathbb{P}(\Omega^1_X)}(-1)$ is the dual of $O_{\mathbb{P}(\Omega^1_X)}(1)$, the first Chern form $c_1(O_{\mathbb{P}(\Omega^1_X)}(1), \hat{h})$ on U is given by

$$c_1\big(O_{\mathbb{P}(\Omega^1_X)}(1), \hat{h}\big) = 2i\bar{\partial}\partial \log u - i\bar{\partial}\partial \log v. \tag{3.2}$$

Lemma 3.1. For every $q \in \Omega^1_X$ there exists a point p in the orbit $I_\infty \cdot q$ such that in (ζ, z, ξ, w)-coordinates p is given by

$$\zeta_{i,p} = \xi_{i,p} = 0,$$

for $1 \leq i \leq n-2$, $\zeta_{n-1,p} = 0$ and $\xi_{n-1,p}$ being a positive real number.

Proof. Consider the action of $(A, \tau, t) \in U(n-1) \times \mathbb{C}^{n-1} \times \mathbb{R}$ described in 2.8 and take $(A, \tau, t) = (I, -\zeta_q, 0)$:

$$(I, -\zeta, 0).(\zeta_q, z_q, \xi_q, w_q) = (0, z', \xi_q + i|\zeta_q|^2 + 2i(\xi_q, -\zeta_q), w_q)$$

There exist $B \in U(n-1)$ such that

$$B\big(\xi + 2i(\xi, -\zeta)\big) = (0, ..., 0, |\xi_q + 2i(\xi_q, -\zeta_q)|).$$

The element $(B, 0, 0) \in U(n-1) \times \mathbb{C}^{n-1} \times \mathbb{R}$ sends $(0, z', \xi_q + i|\zeta_q|^2 + 2i(\xi_q, -\zeta_q), w_q)$ to a point p with the desired properties. □

Proposition 3.2. Let $X = \Gamma\backslash\mathbb{B}^n$ be a torsion-free ball quotient and $\hat{h}$ be the hermitian metric on $O_{\mathbb{P}(\Omega^1_X)}(1)$ induced by the Bergman metric on $\mathbb{B}^n$. Then,

(i) the first Chern form $c_1\big(O_{\mathbb{P}(\Omega^1_X)}(1), \hat{h}\big)$ is a Kähler form on $\mathbb{P}(\Omega^1_X)$;

(ii)
$$c_1\big(O_{\mathbb{P}(\Omega^1_X)}(1), \hat{h}\big) \geq \frac{1}{4\pi}\pi^*(w_X),$$
where w_X is the Kähler form of the Bergman metric on X.

Proof. First we prove part (ii) and then conclude part(i) from the inequality appears in the proof of (ii).

(ii) Since the Bergman metric and u is I_∞-invariant, its enough to check the inequality on I_∞-orbit. Consider an open set $U \subset \mathbb{P}(\Omega^1_X)$ such that $w \neq 0$. Replacing ξ_i by $\frac{\xi_i}{w}$ and w by 1, we can work on the affine coordinates on $\mathbb{P}(\Omega^1_X)_{|U}$. Thanks to Lemma 3.1, we can move every point in U by an element of I_∞ to point p such that

$$\zeta_{1,p} = \zeta_{2,p} = ... = \zeta_{n-1,p} = 0,$$

$$\xi_{1,p} = \xi_{2,p} = ... = \xi_{n-2,p} = 0,$$

$w = 1$ and $\xi_{n-1,p}$ is a real number. Note that the function v in 3.1 on U is

$$v = (\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}((u\delta_{ij} + \bar{\zeta}_i\zeta_j)\xi_i\bar{\xi}_j + \frac{1}{2i}\xi_j\bar{\zeta}_j - \frac{1}{2i}\bar{\xi}_j\zeta_j) + \frac{1}{4}.$$

Set $L = O_{\mathbb{P}(\Omega^1_X)}(1)$. The first Chern form at p is

$$c_1(L,\hat{h})(p) = \frac{i}{2\pi}\bar{\partial}\partial\log(v)(p) = \frac{i}{2\pi}(v(p))^{-2}\big(v(p)\cdot\bar{\partial}\partial v(p) - (\bar{\partial}v\wedge\partial v)(p)\big).$$

To find explicit formula note that

$$\begin{aligned} r := v(p) &= u\xi^2_{n-1} + \frac{1}{4}, \\ \partial v(p) &= u\xi_{n-1}d\xi_{n-1} + \frac{1}{2i}\xi^2_{n-1}dz - \frac{1}{2i}\xi_{n-1}d\zeta_{n-1}, \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}\partial v(p) = &\sum_{i=1}^{n-1}\big(-\xi^2_{n-1}d\bar{\zeta}_i\wedge d\zeta_i + ud\bar{\xi}_i\wedge d\xi_i + \frac{1}{2i}d\bar{\zeta}_i\wedge d\xi_i - \frac{1}{2i}d\bar{\xi}_i\wedge d\zeta_i\big) \\ &+ \xi^2_{n-1}d\bar{\zeta}_{n-1}\wedge d\zeta_{n-1} - \frac{1}{2i}\xi_{n-1}d\bar{z}\wedge d\xi_{n-1} + \frac{1}{2i}\xi_{n-1}d\bar{\xi}_{n-1}\wedge dz, \end{aligned}$$

giving that

$$\begin{aligned} c_1(L,\hat{h})(p) = \frac{i}{2\pi}r^{-2}\cdot\Big(&\sum_{i=1}^{n-2}\big(rud\bar{\xi}_i\wedge d\xi_i - r\xi^2_{n-1}d\bar{\zeta}_i\wedge d\zeta_i + \frac{1}{2i}rd\bar{\zeta}_i\wedge d\xi_i - \frac{1}{2i}rd\bar{\xi}_i\wedge d\zeta_i\big) \\ &+ \frac{1}{4}\cdot\big(ud\bar{\xi}_{n-1}\wedge d\xi_{n-1} + \frac{1}{2i}d\bar{\zeta}_{n-1}\wedge d\xi_{n-1} - \frac{1}{2i}d\bar{\xi}_{n-1}\wedge d\zeta_{n-1} \\ &- \frac{1}{2i}\xi_{n-1}d\bar{z}\wedge d\xi_{n-1} + \frac{1}{2i}\xi_{n-1}d\bar{\xi}_{n-1}\wedge dz - \xi^4_{n-1}d\bar{z}\wedge dz \\ &+ \xi^3_{n-1}d\bar{z}\wedge d\zeta_{n-1} + \xi^3_{n-1}d\bar{\zeta}_{n-1}\wedge dz - \xi^2_{n-1}d\bar{\zeta}_{n-1}\wedge d\zeta_{n-1}\big)\Big). \end{aligned}$$

Now, as $\omega_X = 2i\bar{\partial}\partial\log(u)$, using equation 3.2, we can write

$$c_1(L,\hat{h})(p) - \frac{1}{4\pi}\pi^*(w_X)(p) = \frac{i}{2\pi}\big(\bar{\partial}\partial\log(u)(p) - \bar{\partial}\partial\log(v)(p)\big),$$

and therefore denoting the form $c_1(L,\hat{h}) - \frac{1}{4\pi}\pi^*(w_X)$ by η, we obtain

$$\begin{aligned} \eta(p) = &\frac{i}{2\pi}\cdot\sum_{i=1}^{n-2}\big(ur^{-1}d\xi_i\wedge d\bar{\xi}_i + (u^{-1} - r^{-1}\xi^2_{n-1})d\zeta_i\wedge d\bar{\zeta}_i + \frac{1}{2i}r^{-1}d\xi_i\wedge d\bar{\zeta}_i - \frac{1}{2i}r^{-1}d\zeta_i\wedge d\bar{\xi}_i\big) \\ &+ \frac{i}{8}r^{-2}u^{-2}\Big((8ur^2 - 2u^2\xi^2_{n-1})d\zeta_{n-1}\wedge d\bar{\zeta}_{n-1} + 2u^3 d\xi_{n-1}\wedge d\bar{\xi}_{n-1} - iu^2 d\xi_{n-1}\wedge d\bar{\zeta}_{n-1} \\ &\qquad + iu^2 d\zeta_{n-1}\wedge d\bar{\xi}_{n-1} + iu^2\xi_{n-1}d\xi_{n-1}\wedge d\bar{z} - iu^2\xi_{n-1}dz\wedge d\bar{\xi}_{n-1} \\ &\qquad + (2r^2 - 2u^2\xi^4_{n-1})dz\wedge d\bar{z} + 2u^2\xi^3_{n-1}d\zeta_{n-1}\wedge d\bar{z} + 2u^2\xi^3_{n-1}dz\wedge d\bar{\zeta}_{n-1}\Big) \end{aligned}$$

To see that $\eta(p)$ is semi-positive, we choose the local frame β for L over U as follows:

$$\beta_{2i-1} = \frac{\partial}{\partial \xi_i}, \ \beta_{2i} = \frac{\partial}{\partial \zeta_i},$$

for $1 \leq i \leq n-1$ and $\beta_{2n-1} = \frac{\partial}{\partial z}$. In this frame,

$$[\eta(p)]_\beta = \frac{i}{2\pi} \begin{bmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{n-1} & 0 \\ 0 & 0 & \dots & 0 & B \end{bmatrix},$$

where $A_i = r^{-1} \begin{bmatrix} u & \frac{1}{2i} \\ -\frac{1}{2i} & \frac{1}{4u} \end{bmatrix}$ and $B = \frac{1}{8} r^{-2} u^{-2} \begin{bmatrix} 2u^3 & iu^2 & -iu^2 \xi_{n-1} \\ -iu^2 & 8ur^2 - 2u^2 \xi_{n-1}^2 & 2u^2 \xi^3 \\ iu^2 \xi_{n-1} & 2u^2 \xi^3 & 2r^2 - 2u^2 \xi_{n-1}^4 \end{bmatrix}$.

By computation we can see that $\det(A_i) = 0$, $\operatorname{tr}(A_i) > 0$ and determinate of all upper left sub-matrices of B are semi-positive. Hence,

$$c_1(L, \hat{h}) \geq \frac{1}{4\pi} w_X.$$

(i) Since $c_1(L, \hat{h})$ is a closed form, we only need to show that it is positive. As ω_X is zero only on the vertical directions and neither A_is nor B is zero on the vertical directions, $c_1(L, \hat{h})$ is a positive $(1,1)$-form.

□

In fact, if $\Gamma \subset PU(n,1)$ is a cocompact lattice, Proposition 3.2 implies that $\Gamma \backslash \mathbb{B}^n$ has ample cotangent bundle. This is well-know and the difficulty is, when Γ is not cocompact.

For the non-compact case, we will construct a hermitian metric on $O(1)$ over $\mathbb{P}(\Omega_X^1)$ which extends as singular hermitian metric to $O(1)$ over $\mathbb{P}(\Omega_{\overline{X}}^1(\log(D)))$.

To prove Theorem A, we will construct a singular hermitian metric whose curvature current is represented by a form. To this end, we prove the following Proposition which is inspired by [GK73, Lemma 6.18] and [Kol85, Proposition 5.16]:

Proposition 3.3. Suppose $\Phi(\zeta, |q|) : \mathbb{C}^{n-1} \times \mathbb{C} \to (0, \infty]$ is a function satisfying the following conditions:

(i) $\partial \log \left(\Phi(\zeta, |q|)\right)$ and $\partial\bar{\partial} \log \left(\Phi(\zeta, |q|)\right)$ are locally integrable on a neighborhood of the divisor $q = 0$;

(ii) $\lim_{q \to 0} \dfrac{\log(\Phi(\zeta, |q|))}{log|q|} = 0$, when $|\zeta|$ is bounded.

Then, the current $\left[\partial\bar{\partial} \log \left(\Phi(\zeta, |q|)\right)\right]$ is represented by the form $\partial\bar{\partial} \log \left(\Phi(\zeta, |q|)\right)$.

Proof. Let U be an open set where the divisor D is given by $q = 0$. For a compactly supported $(n-1, n-1)-$form f on U,

$$\int_X \log\left(\Phi(\zeta, |q|)\right)\partial\bar{\partial}f = \lim_{\epsilon\to 0}\int_{X_\epsilon} \log\left(\Phi(\zeta, |q|)\right)\partial\bar{\partial}f,$$

where $X_\epsilon = \{(\zeta, q) \in \mathbb{C} \mid |q| > \epsilon\}$. Applying Stokes' theorem to the closed form $d\big(\log|\Phi(\zeta, |q|)\partial f\big)$, we get:

$$\int_{X_\epsilon} \log(\Phi(\zeta, |q|))\bar{\partial}\partial f = -\int_{X_\epsilon} \bar{\partial}\log\left(\Phi(\zeta, |q|)\right) \wedge \partial f + \int_{S_\epsilon} \log\left(\Phi(\zeta, |q|)\right)\partial f, \tag{3.3}$$

where $S_\epsilon = \{(\zeta, q) \in \mathbb{C}^n \mid |q| = \epsilon\}$ is oriented with its normal in the direction of decreasing $|q|$. Since $S_\epsilon = C_\epsilon \times \mathbb{C}^{n-1}$ where C_ϵ is the circle $\{q \in \mathbb{C} \mid |q| = \epsilon\}$, Fubini's theorem gives that

$$\begin{aligned}\int_{S_\epsilon} \log\left(\Phi(\zeta, |q|)\right)\partial f &= \int_{\mathbb{C}^{n-1}} \Big(\int_{C_\epsilon} \log(\Phi(\zeta, |q|)\tilde{f}(\zeta, q)dq\Big)\, d\zeta \wedge d\bar{\zeta} \\ &= \int_D \Big(\log\left(\Phi(\zeta, \epsilon)\right) \int_{C_\epsilon} \tilde{f}(\zeta, q)dq\Big)\, d\zeta \wedge d\bar{\zeta},\end{aligned} \tag{3.4}$$

where $d\zeta \wedge d\bar{\zeta} = d\zeta_1 \wedge d\bar{\zeta}_1 \wedge ...d\zeta_{n-1} \wedge d\bar{\zeta}_{n-1}$, $\tilde{f}(\zeta, q)dq \wedge d\zeta \wedge d\bar{\zeta}$ is $dq \wedge d\zeta \wedge d\bar{\zeta}$ part of ∂f and D is a compact set such that $D \times C_\epsilon$ contains the support of $\tilde{f}(\zeta, q)$. Since $\lim_{q\to 0} \dfrac{\log(\Phi(\zeta, |q|))}{log|q|} = 0$, there exists $\epsilon' > 0$ such that $|\log\left(\Phi(\zeta, \epsilon)\right)| < \epsilon'|log(\epsilon)|$. Furthermore, as $\tilde{f}(\zeta, q)$ is continuous and C_ϵ is compact, there exists a continuous function $M(\zeta)$ such that $\left|\displaystyle\int_{C_\epsilon} \tilde{f}(\zeta, q)dq\right| < M(\zeta) \cdot \epsilon$. It follows by compactness of D that

$$\left|\int_D \Big(\log\left(\Phi(\zeta, \epsilon)\right) \int_{C_\epsilon} \tilde{f}(\zeta, q)dq\Big)\, d\zeta \wedge d\bar{\zeta}\right| \leq R \cdot \epsilon \log(\epsilon), \tag{3.5}$$

where R is a constant. As such, 3.4 and 3.5 imply that $\lim_{\epsilon\to 0}\displaystyle\int_{S_\epsilon} log(\Phi(\zeta, |q|))\partial f = 0$ and by 3.3 we obtain that

$$\int_X \log\left(\phi(\zeta, |q|)\right)\partial\bar{\partial}f = \int_X \bar{\partial}\log\left(\phi(\zeta, |q|)\right) \wedge \partial f. \tag{3.6}$$

On the other hand, applying Stokes' theorem to the closed form $d\Big(f \wedge \bar{\partial}\log\left(\phi(\zeta, |q|)\right)\Big)$ yields that:

$$\int_{X_\epsilon} f \wedge \partial\bar{\partial}\log\left(\Phi(\zeta, |q|)\right) = -\int_{X_\epsilon} \partial f \wedge \bar{\partial}\log\left(\Phi(\zeta, |q|)\right) + \int_{S_\epsilon} f \wedge \bar{\partial}\log\left(\Phi(\zeta, |q|)\right). \tag{3.7}$$

It will be shown that $\lim_{\epsilon\to 0}\displaystyle\int_{S_\epsilon} f \wedge \bar{\partial}\log\left(\Phi(\zeta, |q|)\right) = 0$. Note that

$$\int_{S_\epsilon} f \wedge \bar{\partial} \log \left(\Phi(\zeta, |q|)\right) = \int_{r=\epsilon} f \wedge \frac{1}{\Phi(\zeta, r)} \cdot \frac{\partial \Phi(\zeta, r)}{\partial \bar{q}} + \int_{|q|=\epsilon} f \wedge \sum_{i=1}^{n-1} \frac{\partial \log \left(\Phi(\zeta, |q|)\right)}{\partial \bar{\zeta}_i}, \tag{3.8}$$

where $r = |q|$. Applying L'Hopital's rule to $\lim_{r \to 0} \frac{\log(\Phi(\zeta, r))}{log(r)} = 0$ yields

$$\lim_{r \to 0} \frac{r}{\Phi(\zeta, r)} \cdot \frac{\partial \Phi(\zeta, r)}{\partial r} = 0. \tag{3.9}$$

As $\frac{\partial r}{\partial \bar{q}} = \frac{1}{2}\left(\frac{q}{\bar{q}}\right)^{\frac{1}{2}}$ the chain rule gives

$$\frac{\partial \Phi(\zeta, |q|)}{\partial \bar{q}} = \frac{1}{2}\left(\frac{q}{\bar{q}}\right)^{\frac{1}{2}} \cdot \frac{\partial \Phi(\zeta, r)}{\partial r}.$$

Since f is compactly supported, there exists a constant M' such that

$$\left| \int_{r=\epsilon} f \wedge \frac{1}{\Phi(\zeta, r)} \cdot \frac{\partial \Phi(\zeta, |q|)}{\partial \bar{q}} \right| \leq M' \epsilon \cdot \max_{r=\epsilon} \left| \frac{1}{\Phi(\zeta, r)} \cdot \frac{\partial \Phi(\zeta, r)}{\partial r} \right|, \tag{3.10}$$

which tends to 0 as ϵ tends to 0 using 3.9. The other term in 3.8 can be written as

$$\begin{aligned} \int_{|q|=\epsilon} f \wedge \sum_{i=1}^{n-1} \frac{\partial \log \left(\Phi(\zeta, |q|)\right)}{\partial \bar{\zeta}_i} &= \int_{\mathbb{C}^n} \sum_{i=1}^{n-1} \frac{\partial \log \left(\Phi(\zeta, \epsilon)\right)}{\partial \bar{\zeta}_i} \int_{|q|=\epsilon} f \\ &= \int_{\mathbb{C}^n} \log \left(\Phi(\zeta, \epsilon)\right) \bar{\partial} \Big(\int_{|q|=\epsilon} f \Big), \end{aligned} \tag{3.11}$$

by Stokes' theorem. As f is compactly supported there exists a constant R' such that

$$\left| \int_{\mathbb{C}^n} \log \left(\Phi(\zeta, \epsilon)\right) \bar{\partial} \Big(\int_{|q|=\epsilon} f \Big) \right| \leq R' \epsilon' |\log(\epsilon)| \epsilon \tag{3.12}$$

tending to 0 as ϵ tends to 0. Therefore, combining 3.8, 3.10, 3.11 and 3.12 gives $\lim_{\epsilon \to 0} \int_{S_\epsilon} f \wedge \bar{\partial} \log \left(\Phi(\zeta, |q|)\right) = 0$ and by 3.7 we get

$$\int_X f \wedge \partial \bar{\partial} \log \left(\Phi(\zeta, |q|)\right) = - \int_X \partial f \wedge \bar{\partial} \log \left(\Phi(\zeta, |q|)\right). \tag{3.13}$$

The claim follows from 3.6 and 3.13. □

Proposition 3.4. Let s be a local section of $O_{\mathbb{P}(\Omega^1_X(\log D))}(1)$. Then, the current $[\partial \bar{\partial} \log(\hat{h}(s))]$ is represented by the form $\partial \bar{\partial} \log(\hat{h}(s))$.

Proof. Let D' be the pull back of $D = \overline{X}\backslash X$ to $\Omega^1_{\overline{X}}(\log D)$. Take an arbitrary point $a = (\zeta_1, ..., \zeta_{n-1}, q = 0; [\xi_1, ..., \xi_n]) \in D'$. At least one of $\xi_1, \xi_2, ..., \xi_n$ is non-zero. Let $\xi_k \neq 0$, for some $k \in \{1, 2, ..., n\}$. Suppose U is an open subset of $\mathbb{P}(\Omega^1_X(\log D))$ such that for every $u \in U$, $\xi_{k,u} \neq 0$ and $D' \cap U = (q = 0)$. Replacing ξ_i by $\frac{\xi_i}{\xi_k}$, $\eta = \sum_{i=1}^{n-1} \xi_i \frac{\partial}{\partial \zeta_i} + \xi_n q \frac{\partial}{\partial q}$ is a local section of $O_{\mathbb{P}(\Omega^1_X(\log D))}(-1)$ on U. Note that on U we have $h(s) = \frac{1}{||\eta||^2_{\hat{h}^*}}$, and therefore it is enough to check the conditions of Proposition 3.3 for $\log(||\eta||^2_{\hat{h}^*})$. We write this function in terms of the chosen coordinate. First, we have that

$$
\begin{aligned}
||\eta||^2_{\hat{h}^*} &= u^{-2}.\Big(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big((u\delta_{ij} + \bar{\zeta}_i\zeta_j)\xi_i\bar{\xi}_j + \frac{t_\infty}{4\pi}\xi_j\bar{\xi}_n\bar{\zeta}_j + \frac{t_\infty}{4\pi}\bar{\xi}_j\xi_n\zeta_j\big) + \frac{t_\infty^2}{16\pi}|\xi_n|^2\Big) \\
&= u^{-1}\cdot\sum_{i=1}^{n-1}|\xi_i|^2 + u^{-2}\cdot\Big(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big(\bar{\zeta}_i\zeta_j\xi_i\bar{\xi}_j + \frac{t_\infty}{4\pi}\xi_j\bar{\xi}_n\bar{\zeta}_j + \frac{t_\infty}{4\pi}\bar{\xi}_j\xi_n\zeta_j\big) + \frac{t_\infty^2}{16\pi}|\xi_n|^2\Big).
\end{aligned}
$$

Therefore,

$$
\log(||\eta||^2_{\hat{h}^*}) = -2\log(u) + \log\Big(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big((u\delta_{ij} + \bar{\zeta}_i\zeta_j)\xi_i\bar{\xi}_j + \frac{t_\infty}{4\pi}\xi_j\bar{\xi}_n\bar{\zeta}_j + \frac{t_\infty}{4\pi}\bar{\xi}_j\xi_n\zeta_j\big) + \frac{t_\infty^2}{16\pi}|\xi_n|^2\Big). \tag{3.14}
$$

To check that the 1-form $\partial\log(||\eta||^2_{\hat{h}^*})$ is locally integrable, we only need to check that the the function $\frac{1}{|q|\log|q|}$ is locally integrable around $q = 0$. This follows from

$$
\int_{|q|<\epsilon}\frac{dq\wedge d\bar{q}}{|q|\log|q|} = \int_0^{2\pi}\int_0^{\epsilon}\frac{2r^{\frac{1}{2}}drd\theta}{\log(r)} < \infty,
$$

for small enough $\epsilon > 0$.

To check that the $(1,1)$-form $\bar{\partial}\partial\log(||\eta||^2_{\hat{h}^*})$ is locally integrable around $q = 0$, we should only check that the function $\frac{1}{|q|^2\log^2|q|^2}$ is locally integrable around $q = 0$. This follows from

$$
\int_{|q|<\epsilon}\frac{dq\wedge d\bar{q}}{|q|^2\log^2|q|} = \int_0^{2\pi}\int_0^{\epsilon}\frac{drd\theta}{r\log^2(r)} < \infty,
$$

for small enough $\epsilon > 0$. When $\zeta_1, ..., \zeta_n$ and $\xi_1, ..., \xi_n$ are bounded, using (3.14) and $q \to 0$, we have the the asymptotic relation

$$
\frac{\log(||\eta||^2_{\hat{h}^*})}{\log(q)} \sim \frac{\log(u)}{\log|q|} \sim \frac{\log(\log|q|)}{\log|q|} \sim 0.
$$

Hence, we can apply Proposition 3.3 to $\log(||\eta||^2_{\hat{h}^*})$. $\square$

3.2 POSITIVITY OF LOGARITHMIC COTANGENT BUNDLE

Throughout this section, we adopt the convention that Γ is a torsion-free lattice in $PU(n,1)$, $X = \Gamma\backslash\mathbb{B}^n$ is a complex hyperbolic manifold with cusps, $\overline{X}$ is the toroidal compactification of X and d is the uniform depth of cusps. Additionally, we assume that $\overline{X}$ does not have any orbifold point.

The goal of this section is to prove the ampleness of the twisted logarithmic cotangent bundle as stated in Theorem A. To prove this theorem, we need to construct a singular hermitian metric using the properties of the Bergman metric proved in the previous section, namely Proposition 3.2 and Proposition 3.4.

Suppose $\overline{Y}$ is a smooth projective variety and D is divisor on $\overline{Y}$. As proved in [ADT22, BD18], if $dim\ \overline{Y} > 1$, the logarithmic cotangent bundle $\Omega^1_{\overline{Y}}(\log(D))$ is never ample because its restriction to D is an extension of the trivial bundle. Therefore, to describe a positivity properties of $\Omega^1_{\overline{Y}}(\log(D))$ we need a weaker positivity notion.

In the case of the toroidal compactification $\overline{X}$ of X, the first step toward understanding the positivity of the cotangent bundles is to study the positively of the cotangent bundle restricted to the boundary divisor D. The connected components of D are étale quotients of abelian varieties whose conormal bundles are ample and therefore considering the conormal bundle exact sequence

$$0 \longrightarrow O_D(-D) \longrightarrow \Omega^1_{\overline{X}|D} \longrightarrow \Omega^1_D \longrightarrow 0, \tag{3.15}$$

on D, one can observe that $\Omega^1_{\overline{X}|D}$ is an extension of a vector bundle with vanishing Chern classes, by an ample line bundle. Moreover, we will prove that $\Omega^1_{\overline{X}|D}$ is semi-ample in the sense that $O(1)$ on $\mathbb{P}(\Omega^1_{\overline{X}|D})$ is semi-ample. To this end, we need the following lemma:

Lemma 3.1. Suppose $0 \to G \to E \to F \to 0$ is an exact sequence of vector bundles on a projective variety $\overline{Y}$ such that $H^1(\overline{Y}, G) = 0$. If $S^m G$ and F are globally generated, then $O_{\mathbb{P}(E)}(m)$ is globally generated.

Proof. $H^1(\overline{Y}, G) = 0$ gives the exact sequence of global sections:

$$0 \longrightarrow H^0(\overline{Y}, G) \longrightarrow H^0(\overline{Y}, E) \longrightarrow H^0(\overline{Y}, F) \longrightarrow 0.$$

This implies that $H^0(\overline{Y}, S^m G)$ injects into $H^0(\overline{Y}, S^m F)$ and $H^0(\overline{Y}, E)$ surjects onto $H^0(\overline{Y}, F)$.

To prove that $O_{\mathbb{P}(E)}(m)$ is globally generated, we need to show that for every $p \in \mathbb{P}(E)$, the fiber $O_{\mathbb{P}(E)}(m)_{|p}$ is generated by global sections of $O_{\mathbb{P}(E)}(m)$. Suppose $p \in \mathbb{P}(E)$ is determined by $y \in \overline{Y}$ and an one-dimensional quotient $E_p \to L_p$. Consider the map $\eta_y : G_y \to L_y$. We may have two cases:

1. $\eta_y : G_y \to L_y$ is the zero map. In this case, η_y factors through F_y. On the other hand, $H^0(\overline{Y}, E)$ surjects onto $H^0(\overline{Y}, F)$, and F is globally generated. Since

$$H^0(\overline{Y}, E) \cong H^0(\mathbb{P}(E), O_{\mathbb{P}(E)}(1)),$$

the fiber $O_{\mathbb{P}(E)}(1)_{|p}$ is generated by the global sections and therefore $O_{\mathbb{P}(E)}(m)_{|p}$ is generated by global sections.

2. $\eta_y : G_y \to L_y$ is a non-zero map. Since $H^0(\overline{Y}, S^m G)$ injects into $H^0(\overline{Y}, S^m E)$ and $S^m G$ is globally generated, the global sections

   $H^0(\mathbb{P}(E), O_{\mathbb{P}(E)}(m)) \cong H^0(\overline{Y}, S^m E)$ generates the fiber $O_{\mathbb{P}(E)}(m)_{|p}$.

   Hence, $O_{\mathbb{P}(E)}(m)$ is globally generated, as desired.

□

Now, we can study the positivity of the twisted cotangent bundle restrict to the boundary divisor:

Proposition 3.2. Suppose that the dimension of X is greater than 1 and r is a rational number. Consider the $\mathbb{Q}$-vector bundles

$$E_r := \Omega^1_{\overline{X}|D}\langle -rD_{|D}\rangle.$$

The following hold:

(i) If $r = 0$, then $E_r = \Omega^1_{\overline{X}|D}$ is semi-ample, but not ample.

(ii) If $r > 0$, then E_r is ample.

Proof. (i) Let D_i be a connected component of D. We can write the exact sequence 3.15 on D_i :

$$0 \longrightarrow O_{D_i}(-D_i) \longrightarrow E_{0|D_i} \longrightarrow \Omega^1_{D_i} \longrightarrow 0.$$

As D_i is an étale quotient of abelian variety, there is a finite étale map $f : D' \longrightarrow D_i$, where D' is an abelian variety. We pull back the previous exact sequence to D' :

$$0 \longrightarrow f^*O_{D_i}(-D_i) \xrightarrow{\phi} f^*E_{0|D_i} \longrightarrow f^*\Omega^1_{D_i} \longrightarrow 0.$$

As $O_{D_i}(-D_i)$ is ample and f is finite, $f^*O_{D_i}(-D_i)$ is ample. As f is an étale map $f^*\Omega^1_{D_i} \cong \Omega^1_{D'}$ which is trivial because D' is an abelian variety. Since $f^*E_{0|D_i}$ has a trivial quotient, $E_{0|D_i}$ can not be ample.

To show semi-ampleness, note that $\Omega^1_{D_i}$ is in particular globally generated. Also, $f^*O_{D_i}(-D_i)$ is ample because $-D_{i|D_i}$ is ample and f is finite. Therefore, there exist $m \in \mathbb{Z}^+$ such that $mf^*O_{D_i}(-D_i)$ is globally generated on $f^{-1}(D_i)$. Additionally, the Kodaira's vanishing theorem gives that $H^1(f^{-1}(D_i), f^*O_{D_i}(-D_i)) = 0$. Thanks to Lemma 3.1, we can conclude that $f^*E_{0|D_i}$ is semi-ample and therefore Lemma 2.12, implies that $E_{0|D_i}$ is semi-ample.

(ii) Since $E_{0|D_i}$ is in particular nef by (i) and the conormal bundle $-D_D$ is ample, we can conclude from Lemma 2.3 that E_r is ample.

□

Proposition 3.3. Suppose that the dimension of X is greater than 1 and r is a rational number. Consider the $\mathbb{Q}$-vector bundles

$$F_r := \Omega^1_{\overline{X}}\big(\log(D)\big)_{|D}\langle -rD_{|D}\rangle.$$

The following hold:

(i) If $r = 0$, then $F_r = \Omega^1_{\overline{X}}(\log(D))_{|D}$ is nef, but not ample.

(ii) If $r > 0$, then F_r is ample.

Proof. Let D_i be a connected component of D. The vector bundle $F_{0|D_i} = \Omega^1_{\overline{X}}\big(\log(D)\big)_{|D_i}$ on D_i fits into the exact sequence:

$$\Omega^1_{\overline{X}|D_i} \xrightarrow{\phi_1} F_{0|D_i} \xrightarrow{\phi_2} \mathcal{O}_{D_i} \longrightarrow 0,$$

where ϕ_1 is the inclusion and ϕ_2 is the residue map sending $\sum_{i=1}^{n-1} f_i dz_i + g\dfrac{dq}{q}$ to $g_{|D_i}$ on an open set U on which the local coordinate of D_i is $(q = 0)$. Therefore, we get the exact sequence

$$0 \longrightarrow \operatorname{Im}(\phi_1) \longrightarrow F_{0|D_i} \xrightarrow{\phi_2} \mathcal{O}_{D_i} \longrightarrow 0.$$

Note that $\operatorname{Im}(\phi_1)$ is a quotient of $\Omega^1_{\overline{X}|D_i}$ which is in particular nef by Proposition 3.2. Therefore, $\operatorname{Im}(\phi_1)$ is nef.

(i) Since $F_{0|D_i}$ admits a trivial quotient, it is not ample. However, it is squeezed between two nef bundles, and therefore it is nef.

(ii) Since $F_{0|D_i}$ is nef and $-rD_{i|D_i}$ is ample for $r > 0$, it follows from Lemma 2.3 that F_r is ample when $r > 0$.

□

Another ingredient we need in order to construct a singular hermitian metric is an appropriate weight function. Roughly speaking, this weight function will be a plurisubharmonic function, supported on a horoball with the largest possible Lelong number on the boundary. The desired wight function on the horoball around the cusp c_i will be constructed using the following Lemma:

Lemma 3.4. For a positive real number u_i and sufficiently small $\epsilon > 0$, there exists a C^2 function $\rho_i : \mathbb{R}_{>0} \to \mathbb{R}$ and a constant number c satisfying the following properties:

1. $\rho_i(u) = -\log(u) + c$ on $(0, u_i + \epsilon']$, where ϵ' is a positive number depending on ϵ.

2. $i\partial\bar{\partial}\rho_i \geq 0$

Let $u = -\frac{t_i}{2\pi} \cdot \log|q| - |\zeta|^2$, where $t_i \in \mathbb{R}^+$, and $(\zeta, q) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

3. When (ζ, q) varies on a compact set with non-empty intersection with $q = 0$,

$$\lim_{q \to 0} \frac{\rho_i(u)}{\log|q|} = \frac{1}{2\pi}(d_i - \epsilon),$$

where $d_i = \frac{t_i}{u_i}$.

4. The forms $\bar{\partial}\big(\rho(u) - \frac{1}{2\pi}(d_i - \epsilon)\log|q|\big)$ and $\partial\bar{\partial}\big(\rho(u)\big)$ are locally integrable on the set $\{(\zeta, q) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid u > u_0 + \epsilon'\}$.

Proof. Let $\epsilon > 0$ be small enough such that there exists $\epsilon' \in (0, \frac{t_i}{4\pi})$ satisfying $\frac{t_i}{u_i} - \frac{t_i}{u_i+2\epsilon'} = \epsilon$ and define $\rho_i : \mathbb{R}_{>0} \to \mathbb{R}$ be the function given by

$$\rho_i(u) = \begin{cases} -\log(u) + c_0 & 0 < u < u_i + \epsilon', \\ \dfrac{-u}{u_i + 2\epsilon'} + c_1 e^{-a(u-u_i-\epsilon')} & u \geq u_i + \epsilon', \end{cases}$$

where $a = \frac{1}{\epsilon'} + \frac{1}{u_i+\epsilon'}, c_1 = \left(\frac{\epsilon'}{u_i+2\epsilon'}\right)^2$, and $c_0 = -\log(u_i + \epsilon') - c_1 - \dfrac{u_i + \epsilon'}{u_i + 2\epsilon'}$. Since $a \cdot c_1 = \frac{\epsilon'}{(u_i+\epsilon')(u_i+2\epsilon')}$, and $a^2 c_1 = \frac{1}{(u_i+\epsilon')^2}$ its straightforward to see that the function ρ_i is C^2.

As $-u$ is a plurisubharmonic function, and c_1 and a are positive, $-log(u) + c_0$, $\frac{-u}{u_i+2\epsilon'} + c_1 e^{-(u-u_i-\epsilon')}$ are plurisubharmonic because the function $-log(-x)$ when $x < 0$ and the function e^{bx} for every $b > 0$ are monotonically increasing and convex. Hence, $\rho_i(u)$ satisfies the second properties.

$\rho_i(u)$ satisfies the third condition since

$$\lim_{q \to 0} \frac{\rho_i(u)}{\log|q|} = \frac{1}{2\pi} \frac{t_i}{u_i + 2\epsilon'} = \frac{1}{2\pi}\left(\frac{t_i}{u_i} - \epsilon\right) = \frac{1}{2\pi}(d_i - \epsilon).$$

To check the forth condition note that

$$e^{-a(u-u_i-\epsilon')} = c' \cdot e^{-a|\zeta|^2} \cdot |q|^l,$$

where $l = \dfrac{a \cdot t_i}{2\pi}$ and c' is a constant. The inequality $a > \frac{1}{\epsilon'} > \frac{4\pi}{t_i}$ gives that $l > 2$, the function $|q|^l$, and the forms $\bar{\partial}(|q|^l), \partial(|q|^l)$ and $\partial\bar{\partial}(|q|^l)$ are locally integrable and therefore letting $c'' = \frac{1}{u_0+2\epsilon'}$,

$$\partial\bar{\partial}\big(\rho_i(u)\big) = \partial\bar{\partial}(c'' \cdot |\zeta|^2) + \partial\bar{\partial}(c' e^{-a|\zeta|^2}|q|^l)$$

is a locally integrable form. On the other hand, $\frac{-u}{u_i+2\epsilon'} = (d_i - \epsilon)\log|q| + \frac{|\zeta|^2}{u_i+2\epsilon'}$, in other words,

$$\bar{\partial}\Big(\frac{-u}{u_i + 2\epsilon'} - (d - \epsilon)log|q|\Big) = \bar{\partial}(c'' \cdot |\zeta|^2),$$

which is a locally integrable form. Hence, $\bar{\partial}\big(\rho(u) - (d_i - \epsilon)\log|q|\big)$ is locally integrable on $\{(\zeta, q) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid u > u_0 + \epsilon'\}$ as desired. □

The main difficulty to prove Theorem A is to prove the following proposition. In the proof, we construct a singular hermitian metric by Preposition 3.3 and Lemma 3.4:

Proposition 3.5. For a sufficiently small $\epsilon > 0$, with $r = (d-\epsilon)/2\pi$ being rational, the $\mathbb{Q}$-vector bundle

$$\Omega^1_{\overline{X}}\big(\log(D)\big)\langle -rD\rangle$$

is ample.

Proof. Denote $\Omega^1_{\overline{X}}\big(\log(D)\big)$ by F. Suppose $\tilde{D}$ is the pullback of D by the natural projection

$$\pi : \mathbb{P}(F) \longrightarrow \overline{X}.$$

The goal is to show that the $\mathbb{Q}$-line bundle $O_{\mathbb{P}(F)}(1) - r\tilde{D}$ is ample, i.e., it intersects positively with every subvariety of $\mathbb{P}(F)$. Since $-r = -(d-\epsilon) < 0$ for a sufficiently small $\epsilon > 0$, by Proposition 3.3, it is enough to show that $O_{\mathbb{P}(F)}(1) - r\tilde{D}$ intersects positively with every subvariety of $\mathbb{P}(F)$ that is not contained in $\tilde{D}$ but possibly intersects with $\tilde{D}$. Let V be such a subvariety.

Let a and b be positive integer such that $a/b = r$. We show that the line bundle $L = O_{\mathbb{P}(F)}(b) \otimes O_{\mathbb{P}(F)}(-a\tilde{D})$ intersects positively with V by constructing a singular hermitian metric $\bar{h}$ on L such that $c_1(L,\bar{h})^{dim(V)} \cdot V > 0$.

Suppose D_i is the component of D compactifing the cusp c_i and D_i is given by $q_i = 0$ on the horoball $B_i(u_i)$, where $u_i = t_i/d$. Let $\tilde{D}_i = \pi^*(D_i)$ and $\tilde{B}_i(u_i) = \pi^{-1}\big(B_i(u_i)\big)$. Taking ϵ to be small enough, by Shimuzu's lemma the horoballs are disjoint and therefore $L_{|\tilde{B}_i(u_i)}$ naturally isomorphic to $O_{\mathbb{P}(F)}(b) \otimes O_{\mathbb{P}(F)}(-a\tilde{D}_i)$.

Substituting u_i and ϵ to Lemma 3.4, we obtain $\rho_i(u)$ and constant c satisfying the properties in Lemma 3.4. Now, we define a singular hermitian metric

$$\bar{h}(s^b \otimes \beta) := \exp\Big(b \cdot \big(-\rho_i(u) - \log(u) - c\big)\Big)\hat{h}(s)^b$$

on $O_{\mathbb{P}(F)}(b) \otimes O_{\mathbb{P}(F)}(-a\tilde{D}_i)$, where s^b is a section of $O_{\mathbb{P}(F)}(b)$ and β is the canonical rational section of $-a\tilde{D}_i$ corresponding to 1. As horoballs are disjoint, Lemma 3.4 implies that $\bar{h}$ is a well-defined metric on L and on the complements of horoballs it equals to $\hat{h}^b$. Evaluating $\bar{h}$ at a local generator of L on $\tilde{H}_i(u_i)$ gives

$$\bar{h}(s^b \otimes \beta q_i^a) = \exp\Big(b \cdot \big(-\rho_i(u) + r \cdot log|q_i| - \log(u) - c\big)\Big)\hat{h}(s).$$

Now, we can compute the curvature current on the horoball $\tilde{B}_i(u_i)$:

$$\begin{aligned}\big[c_1(L,\bar{h})\big] = \frac{i}{2\pi}\Big(&\big[\partial\bar{\partial}\big(\rho_i(u) - r \cdot \log(q_i)\big)\big] \\ &+ \big[\partial\bar{\partial}\log(u)\big] - \big[\partial\bar{\partial}\log(\hat{h}(s))\big]\Big)\end{aligned}$$

By Proposition 3.4, the current $[\partial\bar{\partial}\log(\hat{h}(s))]$ is represented by the form $\partial\bar{\partial}\log(\hat{h}(s))$. Applying Proposition 3.3 to $\Phi_1(\zeta, |q_i|) = \exp\big(\rho_i(u) - r\log(q_i)\big)$ and $\Phi_2(\zeta, |q_i|) = \exp\big(-\log(u)\big)$ gives that

$$\big[\partial\bar{\partial}\big(\rho_i(u) - r \cdot \log(q_i))\big)\big] + \big[\partial\bar{\partial}\log(u)\big]$$

is represented by the form $\partial\bar{\partial}\big(\rho_i(u) - r \cdot \log(q_i)\big) + \partial\bar{\partial}\log(u)$. By Proposition 3.2, we observe that $\partial\bar{\partial}\log(u) - \partial\bar{\partial}\log(\hat{h}(s))$ is a semi-positive form. On the other hand, Lemma 3.4 gives that $\partial\bar{\partial}\big(\rho_i(u) - r \cdot \log(q_i)\big) = i\partial\bar{\partial}\rho_i(u)$ is semi-positive. Putting these together, we obtain that $[c_1(L,\bar{h})]$ is represented by a semi-positive form on $\tilde{B}_i(u_i)$.

Note that by Lemma 3.4 we killed the Lelong number of the current $\big[\partial\bar{\partial}\big(\rho_i(u) - r \cdot \log(q_i)\big)\big]$ on $q = 0$, and the currents $[\partial\bar{\partial}\log(u)]$ and $[\partial\bar{\partial}\log(\hat{h}(s))]$ also have 0 Lelong number on $q = 0$ (see the last part of the proof of Proposition 3.4). Therefore, we can apply [Dem92, Corollary 7.6.] and obtain that

$$\int_V [c_1(L,\bar{h})]^{\dim(V)} \geq \int_{V_{|\pi^{-1}(X)}} c_1(L,\bar{h})^{\dim(V)} \geq 0.$$

By Proposition 3.2, we have the strict positivity of $c_1(L,h) > 0$ on X, therefore

$$\int_{V_{|\pi^{-1}(X)}} c_1(L,\bar{h})^{\dim(V)} > 0.$$

Since $\bar{h}$ does not depend on V, we can conclude that every subvariety $V \subset \mathbb{P}(E)$ that is not entirely contained in $\tilde{D}$ intersects positively with L. □

Using Proposition 3.3 together with Proposition 3.5, we can prove Theorem A:

Theorem 3.2.6. *(Theorem A) For every rational $r \in (0, d/2\pi)$, the $\mathbb{Q}$-vector bundle*

$$\Omega^1_{\overline{X}}\big(\log(D)\big)\langle -rD\rangle$$

is ample.

Proof. Fix $r = a/b \in (0, d/2\pi)$, and choose $\epsilon > 0$ such that $r < (d-\epsilon)/2\pi$. Putting Proposition 3.5 and Lemma 2.11 together, we get that $\Omega^1_{\overline{X}}\big(\log(D)\big)\langle -rD\rangle$ is ample module D, i.e. the vector bundle $S^b\Omega^1_{\overline{X}}\big(\log(D)\big) \otimes \mathcal{O}_{\overline{X}}(-aD)$ is ample modulo D. Moreover, it follows from Proposition 3.3 that the restricted $\mathbb{Q}$-vector bundle $\Omega^1_{\overline{X}}\big(\log(D)\big)_{|D}\langle -rD_{|D}\rangle$ is ample and therefore by Lemma 2.7 we can conclude that the vector bundle $S^b\Omega^1_{\overline{X}}\big(\log(D)\big) \otimes \mathcal{O}_{\overline{X}}(-aD)$ is ample.

□

Theorem 3.2.6 implies that $\Omega^1_{\overline{X}}\big(\log(D)\big)$ is a limit of ample $\mathbb{Q}$-vector bundles. Moreover, we show in Corollary 3.2.7 that $\Omega^1_{\overline{X}}\big(\log(D)\big)$ is ample modulo D and nef. It is previously proved by Cadorél that $\Omega^1_{\overline{X}}\big(\log(D)\big)$ is big and nef ([Cad21b, Theorem 3]).

Corollary 3.2.7. The logarithmic cotangent bundle $\Omega^1_{\overline{X}}\big(\log(D)\big)$ is ample modulo D and nef.

Proof. Denote $\Omega^1_{\overline{X}}\big(\log(D)\big)$ by F. Putting Proposition 3.5 and Lemma 2.11 together, we get that $\Omega^1_{\overline{X}}\big(\log(D)\big)$ is ample module D.

It is easy to conclude from Theorem 3.2.6 that F is nef. If F were not nef, then there would be a curve $C \subset \mathbb{P}(F)$ such that the intersection $I := C \cdot \mathcal{O}_{\mathbb{P}(F)}(1)$ is negative. Let $\tilde{D} = \pi^* D$. Choose a small enough $r > 0$ such that $I - r(C \cdot \tilde{D}) < 0$ and therefore

$$C \cdot \left(\mathcal{O}_{\mathbb{P}(F)}(1) - r\tilde{D}\right) < 0,$$

which contradicts the ampleness of $F\langle -rD \rangle$.

$\square$

3.3 POSITIVITY OF COTANGENT BUNDLE

Throughout this section, we denote a complex hyperbolic manifold with cusps by X, and its smooth toroidal compactification by $\overline{X}$. Additionally, we assume that $\overline{X}$ does not have any orbifold point.

The goal of this section is to prove the results on the positivity of the cotangent bundle, namely Theorem B and Theorem C. Further, we conclude that if the canonical depth of cusps is sufficiently large, then the symmetric differentials on $\overline{X}$ is finitely generated $\mathbb{C}$-algebra.

To pass from positivity on the log-cotangent bundle in the previous section to positivity of the cotangent bundle in this section we consider the following exact sequence of coherent sheaves over $\overline{X}$:

$$0 \longrightarrow \Omega^1_{\overline{X}}\left(\log(D)\right) \otimes \mathcal{O}_{\overline{X}}(-D) \xrightarrow{\phi} \Omega^1_{\overline{X}} \longrightarrow i_* \Omega^1_D \to 0, \tag{3.16}$$

where ϕ sends $\left(\sum_{i=1}^{n} f_i d\zeta_i + g\frac{dq}{q}\right) \otimes q$ to $\sum_{i=1}^{n} q f_i d\zeta_i + g dq$ on an open set U on which the local coordinate of D is given by $q = 0$.

Theorem 3.3.1. *(Theorem C) Suppose that the uniform depth of cusps d is greater than 2π. Then,*

$$\Omega^1_{\overline{X}}\langle -rD \rangle$$

is ample for all rational $r \in (0, -1 + d/2\pi)$.

Proof. Let $r = a/b$ with a positive b be a rational number in $(0, -1 + d/2\pi)$. Since $d > 2\pi$, Theorem 3.2.6 gives that the $\mathbb{Q}$-vector bundle $\Omega^1_{\overline{X}}\left(\log(D)\right)\langle(-1-r)D\rangle$ is ample. In particular, $\Omega^1_{\overline{X}}\left(\log(D)\right)\langle(-1-r)D\rangle$ is ample modulo D. Note that the exact sequence 3.16 gives that the vector bundle $\Omega^1_{\overline{X}}\left(\log(D)\right) \otimes \mathcal{O}_{\overline{X}}(-D)$ is a subbundle of $\Omega^1_{\overline{X}}$. Therefore, the $\mathbb{Q}$-vector bundle $\Omega^1_{\overline{X}}\left(\log(D)\right)\langle(-1-r)D\rangle$ is a $\mathbb{Q}$-subsheaf of $\Omega^1_{\overline{X}}\langle -rD \rangle$. It follows from Lemma 2.10 that $\Omega^1_{\overline{X}}\langle -rD \rangle$ is ample modulo D, that is, the vector bundle $S^b\Omega^1_{\overline{X}} \otimes \mathcal{O}_{\overline{X}}(-aD)$ is ample modulo D. On the other hand, the restricted bundle $\Omega^1_{\overline{X}|D}\langle -rD_{|D} \rangle$ is ample thanks to Proposition 3.2. This means that $\left(S^b\Omega^1_{\overline{X}} \otimes \mathcal{O}_{\overline{X}}(-aD)\right)_{|D}$ is ample. Hence, it follows from Lemma 2.7 that $\Omega^1_{\overline{X}}\langle -rD \rangle$ is ample.

$\square$

Theorem 3.3.2. *(Theorem B) Suppose that the uniform depth of cusps d is greater than 2π. Then, $\Omega^1_{\overline{X}}$ is ample modulo D and semi-ample.*

Proof. Set $E = \Omega^1_{\overline{X}}$. Since the uniform depth of all cusps is greater then 2π, Theorem 3.2.6 implies that $\Omega^1_{\overline{X}}(\log(D)) \otimes O_{\overline{X}}(-D)$ is ample. It therefore follows from 3.16 and Lemma 2.10 that E is ample modulo D.

Let $Y = \mathbb{P}(E)$, $\pi : \mathbb{P}(E) \to \overline{X}$ be the natural projection and $\tilde{D} = \pi^*(D)$. Given that E is ample modulo D and $E_{|D}$ is semi-ample by Proposition 3.2, to show that E is semi-ample, we only need to show that there is a large enough n such that setting $L = O_Y(n)$

$$H^0(Y, L) \xrightarrow{\phi} H^0(\tilde{D}, L_{|\tilde{D}})$$

is surjective. Choose $r \in (0, -1 + d/2\pi)$. By Theorem 3.3.1, $\Omega^1_{\overline{X}}\langle -rD\rangle$ is ample, i.e., $O_Y(1) - r\tilde{D}$ is ample. We can choose n large enough so that $H^1(Y, L - \tilde{n}\tilde{D}) = 0$, where $\tilde{n} = nr \in \mathbb{Z}$. Therefore, denoting the restriction of L to the $\tilde{n}$th order thickening of $\tilde{D}$ by $L_{|\tilde{n}\tilde{D}}$, we obtain

$$H^0(Y, L) \xrightarrow{\psi} H^0(Y, L_{|\tilde{n}\tilde{D}})$$

is surjective. Considering the commutative diagram

$$\begin{tikzcd} & H^0(\tilde{D}, L_{|\tilde{n}\tilde{D}}) \arrow[d, "\eta"] \\ H^0(Y, L) \arrow[ur, "\psi"] \arrow[r, "\phi"] & H^0(\tilde{D}, L_{|\tilde{D}}), \end{tikzcd}$$

to conclude that ϕ is surjective, it is enough to show that η is surjective. To this end, consider the following exact sequence on $\tilde{D}$:

$$0 \longrightarrow L(-m\tilde{D})_{|\tilde{D}} \longrightarrow L_{|(m+1)\tilde{D}} \longrightarrow L_{|m\tilde{D}} \longrightarrow 0,$$

where m is a positive integer. We prove that for every $m > 0$,

$$H^0(\tilde{D}, L_{|(m+1)\tilde{D}}) \longrightarrow H^0(\tilde{D}, L_{|m\tilde{D}})$$

is surjective by showing that $H^1(\tilde{D}, L(-m\tilde{D})_{|\tilde{D}}) = 0$. This implies that η is surjective and therefore ϕ is surjective which finishes the proof.

Let $\varphi : D \longrightarrow Spec(\mathbb{C})$. Applying the exact sequence of low degrees to the composition of the push-forward functor

$$\pi_* : Sh(\tilde{D}) \longrightarrow Sh(D)$$

and the global section functor φ_* yields

$$0 \longrightarrow H^1\big(D, \pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) \longrightarrow R^1(\varphi \circ \pi)_*(L(-m\tilde{D})_{|\tilde{D}}) \longrightarrow H^0\big(D, R^1\pi_*(L(-m\tilde{D})_{|\tilde{D}})\big).$$

Note that $(\varphi \circ \pi)_*$ is the global section functor $H^0(\tilde{D}, -)$ and therefore

$$R^1(\varphi \circ \pi)_*(L(-m\tilde{D})_{|\tilde{D}}) = H^1(\tilde{D}, L(-m\tilde{D})_{|\tilde{D}}).$$

Hence, it is sufficient to prove that

$$H^1\big(D, \pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = H^0\big(D, R^1\pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = 0.$$

To prove $H^0\big(D, R^1\pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = 0$, note that π is a flat morphism because the fibers of π are projective spaces of the same dimension. Since for every $x \in D$, the dimension $h^1(x, \pi_*(L - m\tilde{D})_x) = h^1\big(\pi^{-1}(x), \mathcal{O}_{\mathbb{P}^{n-1}}(n)\big)$ is constant, Grauert's theorem ([Har13, Corollary 12.9]) gives the isomorphism

$$R^1\pi_*(L(-m\tilde{D})_{|\tilde{D}}) \cong H^1(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(n)) = 0.$$

Hence $H^0\big(D, R^1\pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = 0$.

To prove $H^1\big(D, \pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = 0$, note that $\pi_*(L(-m\tilde{D})_{|\tilde{D}}) = S^n\Omega^1_{\overline{X}|D}(-mD_{|D})$. Consider the filtration of $S^n\Omega^1_{\overline{X}|D}$ obtained by the exact sequence 3.15:

$$S^n\Omega^1_{\overline{X}|D} = F^0 \supseteq F^1 \supseteq \dots \supseteq F^m \supseteq F^{m+1},$$

with quotients

$$F^i/F^{i+1} \cong S^{m-i}\Omega^1_D(-iD_{|D})$$

for each i. Tensoring the filtration by $-mD_{|D}$, we get a filtration for $S^n\Omega^1_{\overline{X}|D}(-mD_{|D})$ whose successive quotients are $S^{m-i}\Omega^1_D(-jD_{|D})$ for some $j > 0$. As D is an étale quotient of abelian variety, there exists a finite étale map $f : D' \to D$, where D' is an abelian variety. Since $f^*\Omega^1_D \cong \Omega^1_{D'}$, and $\Omega^1_{D'}$ is trivial, we get that $f^*S^{m-i}\Omega^1_D(-jD_{|D})$ is a power of $f^*(-jD_{|D})$. Since the canonical bundle $K_{D'}$ is trivial, f is finite and $-D_{|D}$ is ample, Kodaira vanishing theorem gives that $H^1(D', f^*(-jD_{|D})) = 0$ for every positive integer j.

It follows that the successive quotients of the filtration of $S^n\Omega^1_{\overline{X}|D}(-mD_{|D})$ have vanishing first cohomology. Hence, $H^1\big(D, S^n\Omega^1_{\overline{X}|D}(-mD_{|D})\big) = 0$, i.e.,

$$H^1\big(D, \pi_*(L(-m\tilde{D})_{|\tilde{D}})\big) = 0.$$

□

Applying [Laz17, Example 2.1.29] to Theorem 3.3.2, we get that symmetric differentials over $\overline{X}$ forms a finitely generated $\mathbb{C}$-algebra provided that the uniform depth is sufficiently large. More precisely, we get:

Corollary 3.3.3. (Corollary K) With the same assumption as Theorem 3.3.2, the graded ring

$$\bigoplus_{n>0} H^0(\overline{X}, S^n\Omega^1_{\overline{X}})$$

is finitely generated $\mathbb{C}$-algebra.

3.4 APPLICATION TO HYPERBOLICITY OF SUBVARIETIES

We follow the same notations as previous section and denote a ball quotient $\Gamma\backslash\mathbb{B}^n$ with a torsion-free lattice Γ by X, the boundary divisor $\overline{X} \setminus X$ by D, the toroidal compactification of X by $\overline{X}$ and the uniform depth of cusps by d. In addition, we assume that $\overline{X}$ does not have any orbifold point. Let V be an irreducible subvariety of $\overline{X}$ intersecting X.

The purpose of this section is to prove the result related to hyperbolicity of V. We first prove Corollary F and Corollary G. Further, we prove that the hyperbolicity increases in towers of normal covering in the sense that the minimum volume of subvarieties of $\overline{X}$ intersecting both X and D tends to infinity. Note that, as the components of D are étale quotient of abelian varieties, we can not expect that the hyperbolic volume of a variety entirely contained in D tends to infinity.

Corollary 3.4.1. (Corollary F) Suppose V is smooth with dimension $m > 0$. Then, $\mathbb{Q}$-line bundle

$$K_V - (r-1)D_{|V}$$

is ample for all rational $r \in (0, m\lfloor \frac{d-1}{2\pi} \rfloor)$. Moreover, if $d > 2\pi$, then K_V is ample.

Proof. Let b be a positive integer strictly less than $d/2\pi$ and D' be $D_{|V}$. Since the vector bundle $\Omega^1_V(\log(D')) \otimes \mathcal{O}_V(-bD')$ is a quotient of

$$\left(\Omega^1_{\overline{X}}(\log(D)) \otimes \mathcal{O}_{\overline{X}}(-bD)\right)_{|V},$$

Theorem 3.2.6 gives that $\Omega^1_V(\log(D')) \otimes \mathcal{O}_V(-bD')$ is ample and therefore its determinant $K_V - (mb-1)D'$ is ample. Now, for any positive rational number $r \leq mb$, the $\mathbb{Q}$-line bundle $K_V - (r-1)D'$ is ample modulo D' by Lemma 2.10. Similarly, we can conclude from 3.2.7 that $(K_V + D')_{|D'}$ is nef. On the other hand, the conormal bundle of D is ample and therefore $(K_V - (r-1)D')_{|D'}$ must be ample. Hence, it follows from Lemma 2.7 that $K_V - (r-1)D'$ is ample.

If $d > 2\pi$, we can choose $b = 1$, which gives that K_V is ample. □

Corollary 3.4.2. (Corollary G) All subvarieties of X are of general type provided that the uniform depth of cusps is greater than 2π.

Proof. Let V_0 be m-dimensional (possibly non-smooth) subvariety of $\overline{X}$ not entirely contained in D and let $\mu : V' \to V_0$ be a resolution of singularities. There is a generically surjective homomorphism $\mu^*\Omega^m_{\overline{X}} \to \Omega^m_{V'} = K_{V'}$. Thanks to Theorem 3.3.2, $\Omega^m_{\overline{X}}$ is ample modulo D and therefore, the pull back $\mu^*\Omega^m_{\overline{X}}$ is ample modulo μ^*D. This implies that $K_{V'}$ is in particular big, i.e., V_0 is of general type. □

The volume of a line bundle L on an m-dimensional projective variety V is defined as the non-negative real number

$$\mathrm{vol}_V(L) := \limsup_{b\to\infty} \frac{h^0(V, bL)}{b^m/m!},$$

which measures the positivity of L from the point of view of birational geometry. If L is a nef line bundle on V, then $vol_V(L) = L^n$. Let V' be a smooth variety birational to V with a canonical bundle $K_{V'}$. The canonical volume of the subvariety V is

$$\widetilde{\text{vol}}_V := \limsup_{b\to\infty} \frac{h^0(V', bK_{V'})}{b^m/m!},$$

which does not depend on the choice of V'.

The volume of a line bundle is a birational invariant and it is positive if and only if the line bundle is big. It turns out that if L is nef, then $vol(L) = L^n$.

In the setting of the compactification of a locally symmetric domain, a natural quantity reflecting the hyperbolicity behaviour is the volume of the log-canonical bundle. In particular, $vol_V(K_V + D_{|V}) > 0$ if and only if V is of log-general type. In the case of ball quotients, Corollary 3.2.7 in particular implies $\Omega^1_V(\log D_{|V})$ is nef and therefore $K_V + D_{|V}$ is nef on V. Hence,

$$\text{vol}_V(K_V + D_{|V}) = (K_V + D_{|V})^m.$$

We show that the minimum volume of log-canonical bundle of subvarieties of $\overline{X}$ is controlled by the uniform depth of cusps:

Corollary 3.4.3. Suppose V is a m-dimensional smooth subvariety of $\overline{X}$ not entirely contained in D. If l is the number of component of D intersecting with V, then

$$\text{vol}_V(K_V + D_{|V}) > m^m \lfloor \frac{d-1}{2\pi} \rfloor^m l(m-1)!.$$

Proof. Let $r = \lfloor \frac{d-1}{2\pi} \rfloor$ and D' be $D_{|V}$. Since D is a union of étale quotient of abelian varieties with ample conormal bundle $O_D(-D)$, we get

$$l \leq \frac{D' \cdot (-D')^{m-1}}{(m-1)!} = \frac{-(-D')^m}{(m-1)!}. \tag{3.17}$$

On the other hand, as $0 \leq r < d/2\pi$, Theorem 3.2.6 and Corollary 3.2.7 imply that $\Omega^1_V(\log(D')) \otimes O_V(-rD')$ is ample modulo D and nef as it is a quotient of $\left(\Omega^1_{\overline{X}}(\log(D)) \otimes O_{\overline{X}}(-rD)\right)_{|V}$ and therefore

$$K_V - (mr-1)D' \cong \det\left(\Omega^1_V(\log(D')) \otimes O_V(-rD')\right)$$

is in particular big and nef. Thus, $\left(K_V - (mr-1)D'\right)^m > 0$. On the other hand, $(K_V + D')_{|D'}$ is nef and $-D'_{|D'}$ is ample. Consequently, for every $0 < i < m$,

$$(K_V + D')^{m-i}(-D')^i = -\left((K_V + D')_{|D'}\right)^{m-i}(-D'_{|D'})^{i-1}$$

is non-positive. Hence, $\left(K_V - (mr-1)D'\right)^m > 0$ yields that

$$(K_V + D')^m > -(rm)^m(-D')^m.$$

Combining this with the inequality 3.17 gives the desired inequality. □

Remark 3.4. Parker's generalization of Shimizu's lemma [Par98, Proposition 2.4.] gives that the uniform depth of cusps is at least 2. Plugging in this result to Corollary 3.4.3 gives that if V intersects D, then V is of log general type. This can be concluded from the recent result of Guenancia [Gue22, Theorem B] as well.

We also get a uniform lower bound for the volume of the canonical bundles of subvarieties:

Corollary 3.4.5. If the canonical depth of cusps d is greater than 2π, then

$$\widetilde{\mathrm{vol}}_V \geq (m\lfloor\frac{d-1}{2\pi}\rfloor - 1)^m l(m-1)!$$

Proof. Since $d > 2\pi$, by Corollary 3.4.1 we get that K_V is ample and therefore $vol_V(K_V) = K_V^m$. Let $r = \lfloor\frac{d-1}{2\pi}\rfloor$ and D' be $D_{|V}$. As sated in the proof of Corollary 3.4.3, $(K_V - (mr-1)D')^m > 0$. Since $K_{V|D'}$ and $-D'_{D'}$ are ample, for every $0 < i < m$,

$$K_V^{m-i} \cdot (-D')^i \cong -(K_{V|D'})^{m-i} \cdot (-D'_{|D'})^{i-1}$$

is negative. Hence, $(K_V - (mr-1)D')^m > 0$ together with the inequality 3.17 yields that

$$K_V^m \geq -(rm-1)^m(-D')^m \geq (rm-1)^m l(m-1)!$$

□

As a generalization of Brunebarbe's work [Bru20a], for towers of ball quotients we show that the minimum volume of subvarieties of X containing a cusp of X, tends to infinity in towers:

Corollary 3.4.6. Let $\{X_i\}_{i=1}^{\infty}$ be a tower of $X = X_1$. Suppose that toroidal compactification of X_i does not have any orbifold points. Then, given a positive number v, for all but finitely many i, every subvariety V of X_i containing a cusp of X_i have $vol(K_V) > v$.

Proof. Combining Lemma 2.4 and Corollary 3.4.5 gives the result.

□

4 VOLUME ESTIMATE

Let $X = \Gamma \backslash \mathbb{B}^n$, where $\Gamma \subset \mathrm{PU}(n,1)$ is a torsion-free lattice whose parabolic stabilizers are unipotent.

4.1 SYSTOLE AND DEPTH OF CUSPS

The main goal of this chapter is to prove Theorem 4.1.9, where we show that the systole $\mathrm{sys}(X)$ bounds the uniform depth of cusps d of X from below.

To see the relation between the systole and depth of cusps, we first prove that the length of a hyperbolic element in Γ only depends on its non-unit eigenvalues:

Proposition 4.1. Suppose $h \in \Gamma$ is a hyperbolic element whose non-unit eigenvalues are $re^{i\theta}$ and $r^{-1}e^{i\theta}$. Then,

$$\ell(h) = 2\cosh^{-1}\Big(\frac{1}{4}\cdot(r+\frac{1}{r})^2\Big).$$

Proof. Since h is hyperbolic, it fixes two distinct points x_1 and x_2 on the boundary $\partial \mathrm{S}$. As $\mathrm{PU}(Q)$ acts doubly transitive on the boundary, there exists $P \in \mathrm{PU}(Q)$ such that $P(x_1) = q_0$ and $P(x_2) = q_\infty$. Now we can write

$$d(x, hx) = d(Px, PhP^{-1}Px) = d(x', PhP^{-1}x'),$$

where $x' = Px$. Suppose $x' = (\zeta_1, v_1, u_1)$, and $PhP^{-1}x' = (\zeta_2, v_2, u_2)$. Since PhP^{-1} fixes both q_0 and q_∞, it follows from Lemma 2.1 that for a complex number a and $A \in \mathrm{U}(n-1)$

$$PhP^{-1}x' = \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1/\bar{a} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(-||\zeta_1||^2 - u_1 + iv_1) \\ \zeta_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a}{2}(-||\zeta_1||^2 - u_1 + iv_1) \\ A\zeta_1 \\ 1/\bar{a} \end{bmatrix}.$$

This gives that $\zeta_2 = \bar{a}A\zeta_1$ and

$$\frac{1}{2}(-||\zeta_2||^2 - u_2 + iv_2) = \frac{|a|^2}{2}(-||\zeta_1||^2 - u_1 + iv_1). \tag{4.1}$$

Therefore, $u_2 = |a|^2 u_1$ and $v_2 = |a|^2 v_1$. Note that as conjugation does not change the eigenvalues, we have that $|a|^2 = r^2$ or $\frac{1}{r^2}$. On the other hand, the inequality 2.4 yields that

$$\begin{aligned} d(x', PhP^{-1}x') &\geq 2\cosh^{-1}\Big(\frac{|u_1+u_2|^2}{4u_1u_2}\Big) \\ &\geq 2\cosh^{-1}\Big(\frac{1}{4}\cdot(r+\frac{1}{r})^2\Big). \end{aligned}$$

Since this lower bound is realized on $\zeta_1 = 0, v = 0$, we can conclude the desired equality. □

Consider $\gamma \in \Gamma$ and the corresponding element in the matrix group $g_\gamma \in \mathrm{PU}(Q)$. When we talk about $|\operatorname{tr}(\gamma)|$, we mean $|\operatorname{tr}(g_\gamma)|$. Proposition 4.1 tells us if for every hyperbolic element $\gamma \in \Gamma$ we have that $|\operatorname{tr}(\text{fl})|$ is sufficiently large, then systole $\operatorname{sys}(X)$ will be large, i.e., the systole $\operatorname{sys}(X)$ can be estimated just be the trace of the hyperbolic elements. In particular, the systole goes to infinity in a cofinal tower of any ball quotient.

Consider the set associated to Γ

$$S_\Gamma := \big\{\gamma \in \Gamma \big| |\operatorname{tr}(\gamma)| > n+1\big\}.$$

It follows from the classification of isometries that if $\gamma \in \Gamma$ has $|\operatorname{tr}(\gamma)| > n+1$, then γ must be hyperbolic. Hence, all elements of S_Γ are hyperbolic. We associate the number

$$\lambda_\Gamma := \inf_{\gamma \in S_\Gamma} |\operatorname{tr}(\gamma)|,$$

to Γ which will play a role as an intermediate quantity to relate the systole of X to the depth of cusps of X. Specifically, we can see how $\operatorname{sys}(X)$ bounds λ_Γ from below:

Proposition 4.2. The following inequality holds:

$$\lambda_\Gamma \geq 1 - n + \sqrt{2}e^{\operatorname{sys}(X)/4}.$$

Proof. Consider $\gamma \in S_\Gamma$. Let $re^{i\theta}$ and $r^{-1}e^{i\theta}$ be eigenvalues of γ which are not units. As $\operatorname{sys}(X)$ is the length of the shortest geodesic, Proposition 4.1 implies that

$$r + r^{-1} \geq \sqrt{4\cosh\big(\frac{1}{2}\operatorname{sys}(X)\big)}.$$

Since the other $n-1$ eigenvalues of γ have norm 1, and $\cosh(x) > \frac{1}{2}e^x$, by the triangle inequality we can conclude the desired inequality.

□

We prove a lemma which will help us to see the relation between the height of cusp q_∞ in terms of the trace of the hyperbolic elements in Γ :

Lemma 4.3. Suppose $\gamma \in \Gamma \setminus \Gamma_\infty$ written in from 2.2. For every $z \in \mathbb{B}^n$ the following inequality holds:

$$u(z)u(\gamma \cdot z) \leq |\frac{2}{c}|^2.$$

Proof. There are unique Heisenberg transformations h_1, h_2 such that $h_1(0) = \gamma(\infty)$ and $h_2^{-1}(0) = \gamma^{-1}(\infty)$. Consider $\tilde{\gamma} = h_1^{-1}\gamma h_2^{-1}$ and note that as the Heisenberg translations are stabilizers of the $u-$coordinate, we have that $u(\gamma z) = u(\tilde{\gamma} z)$. Since $\tilde{\gamma}$ swaps ∞ and 0, Lemma 2.1 tells us that $\tilde{\gamma}$ must have the form:

$$\tilde{\gamma} : (\zeta, v, u) \longrightarrow \Big(\frac{A\zeta r_\gamma^2}{||\zeta||^2 + u - iv}, \frac{-vr_\gamma^4}{\big|||\zeta||^2 + u - iv\big|^2}, \frac{ur_\gamma^4}{\big|||\zeta||^2 + u - iv\big|^2} \Big)$$

where $A \in U(n-1)$ and $r_\gamma = \sqrt{\frac{2}{|c|}}$. This gives that

$$u(z)u(\gamma \cdot z) = u(z)u(\tilde{\gamma} \cdot z) = \frac{u^2}{|||\zeta||^2 + u - iv|^2} \cdot \Big|\frac{2}{c}\Big|^2 \leq \Big|\frac{2}{c}\Big|^2.$$

□

Let c_m be the minimal value of $|c|$ among all $\gamma \in \Gamma \setminus \Gamma_\infty$ written in the form 2.2:

$$\gamma = \begin{bmatrix} a & \tau^* & b \\ \alpha & A & \beta \\ c & \delta^* & e \end{bmatrix}. \tag{4.2}$$

Note that as $\gamma \notin \Gamma_\infty$, the c entry cannot be zero (see Remark 2.1). Lemma 4.3 implies that:

Proposition 4.4. The horoball $B_\infty(\frac{2}{c_m})$ injects into X, i.e., the height of the cusp q_∞ is at least $2/c_m$ and therefore the depth of q_∞ is at least $\frac{c_m \cdot t_\infty}{2}$.

We recall a lemma from Parker's version of Shimizu's lemma which tells us how the c entry shows up in the trace of (some of) hyperbolic elements:

Lemma 4.5. (see [Par98, Lemma 2.6]) Let $g_\infty = (0, t)$, be the vertical translation centered at q_∞ and h be an element of $\mathrm{PU}(n, 1)$ written in the form 2.2. Then,

$$\mathrm{tr}[g_\infty, h] = n + 1 + \Big|\frac{tc}{2}\Big|^2.$$

Proof. The matrix representations of g_∞, h and h^{-1} are given by:

$$g_\infty = \begin{bmatrix} 1 & 0 & i\frac{t}{2} \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; h = \begin{bmatrix} a & \tau^* & b \\ \alpha & A & \beta \\ c & \delta^* & e \end{bmatrix} \; h^{-1} = \begin{bmatrix} \bar{e} & \beta^* & \bar{b} \\ \delta & A^* & \tau \\ \bar{c} & \alpha^* & \bar{a} \end{bmatrix},$$

To find $\mathrm{tr}[g_\infty, h]$, note that

$$g_\infty h_0 = \begin{bmatrix} a + \frac{i}{2}ct & \tau^* + \frac{i}{2}t\delta^* & b + \frac{i}{2}te \\ \alpha & A & \beta \\ c & \delta^* & e \end{bmatrix}, \; g_\infty^{-1} h_0^{-1} = \begin{bmatrix} \bar{e} - \frac{i}{2}\bar{c}t & \beta^* - \frac{i}{2}t\alpha^* & \bar{b} - \frac{i}{2}t\bar{a} \\ \delta & A^* & \tau \\ \bar{c} & \alpha^* & \bar{a} \end{bmatrix}$$

Therefore, using the relation given by 2.2 it follows that

$$\begin{aligned}\operatorname{tr}[g_\infty, h] &= a\bar{e} + \frac{1}{4}|ct|^2 + \frac{it}{2}(c\bar{e} - \bar{c}a) + \delta\tau^* + \frac{i}{2}t|\delta|^2 + b\bar{c} + \frac{i}{2}te\bar{c} \\ &\quad + \alpha\beta^* - \frac{i}{2}t|\alpha|^2 + AA^* + \beta\alpha^* + c\bar{b} - \frac{i}{2}t\bar{a}c + \delta^*\tau + e\bar{a} \\ &= n + 1 + \frac{1}{4}|ct|^2\end{aligned}$$

□

Now, we can show the relation between the quantity λ_Γ and depth of each cusps of X :

Proposition 4.6. The depth of each cusp of $X = \Gamma\backslash\mathbb{B}^n$ is at least $\sqrt{\lambda_\Gamma - n - 1}$.

Proof. Since λ_Γ is invariant under the conjugation by an element of $\mathrm{PU}(n, 1)$, it is sufficient to prove the lemma for the cusp q_∞. Let $g_\infty = (0, t_\infty)$ be the shortest vertical translation in Γ_∞. Suppose that $h \in \Gamma$ is an element which does not fix q_∞ and written in the form 2.2. It follows from Lemma 4.5 that

$$\operatorname{tr}[g_\infty, h] = n + 1 + |\frac{t_\infty c}{2}|^2.$$

Since h does not fix ∞, we have that $c \neq 0$, and therefore $[g_\infty, h] \in S_\lambda$. This implies that

$$|\frac{t_\infty c}{2}| \geq \sqrt{\lambda_\Gamma - n - 1}.$$

Since this inequality holds for every $h \in \Gamma \setminus \Gamma_\infty$, we can conclude that

$$|\frac{t_\infty c_m}{2}| \geq \sqrt{\lambda_\Gamma - n - 1}.$$

Hence, Proposition 4.4 implies

$$d_\infty \geq \sqrt{\lambda_\Gamma - n - 1},$$

where d_∞ is the depth of cusp q_∞. □

To pass from the individual depth of cups to the uniform depth of cusps we will use this lemma:

Lemma 4.7. ([Par98, Lemma 2.5]) Let $B_0(\tilde{u}_0)$ be the horoball of height $\tilde{u}_0$ based at q_0, and let $B_\infty(\tilde{u}_\infty)$ be the horoball of height $\tilde{u}_\infty$ based at q_∞. These two horoballs are disjoint if and only if

$$\tilde{u}_0 \cdot \tilde{u}_\infty \geq 4.$$

Proposition 4.8. Let d be the uniform depth of cusps of X. Then,

$$d \geq \min\{(\lambda_\Gamma - n - 1)^{\frac{1}{4}}, (\lambda_\Gamma - n - 1)^{\frac{1}{2}}\}.$$

Proof. By Proposition 4.6, we only need to show that if $d > (\lambda_\Gamma - n - 1)^{\frac{1}{4}}$, then the horoballs are disjoint. Let $g_0 = (0, t_0)$ be the shortest vertical translation based at q_0

and $g_\infty = (0, t_\infty)$ be the shortest vertical translation based at q_∞. We write g_∞ and g_0 in the form 2.9:

$$g_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ -it_0/2 & 0 & 0 \end{bmatrix}, \quad g_\infty = \begin{bmatrix} 1 & 0 & -it_\infty/2 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 4.5 implies that $\mathrm{tr}[g_\infty, g_0] = n + 1 + \left|\frac{t_0 t_\infty}{4}\right|^2$. Therefore, $[g_\infty, g_0] \in S_\lambda$ and it follows that

$$t_0 t_\infty \geq 4\sqrt{\lambda_\Gamma - n - 1}.$$

Consider $\tilde{u}_0 := \frac{t_0}{(\lambda_\Gamma - n - 1)^{\frac{1}{4}}}$ and $\tilde{u}_\infty := \frac{t_\infty}{(\lambda_\Gamma - n - 1)^{\frac{1}{4}}}$. The inequality above implies that

$$\tilde{u}_0 \cdot \tilde{u}_\infty \geq 4,$$

and therefore it follows from Lemma 4.7 that the horoballs are disjoint. □

Combining Proposition 4.2 with Proposition 4.8, we finally conclude that the systole bounds the uniform depth from below:

Theorem 4.1.9. *Let d be the uniform depth of cusps of X. Then,*

$$d \geq \min\{(-2n + s')^{\frac{1}{4}}, (-2n + s')^{\frac{1}{2}}\},$$

where $s' = \sqrt{2}e^{\frac{1}{4}\,\mathrm{sys}(X)}$.

Direct computation gives the following Corollary which will be used later to bound the uniform depth of cusps in terms of $\mathrm{sys}(X)$:

Corollary 4.1.10. If $\mathrm{sys}(X) \geq 4\ln\left(5n + (4\pi)^4\right)$, then

$$d > e^{\mathrm{sys}(X)/16} > 4\pi. \tag{4.3}$$

4.2 THIN-THICK DECOMPOSITION

In this section, we introduce a version of thin-thick decomposition relative to the systole. This is not the same as Margulis' thin-thick decomposition because our decomposition depends on the lattice Γ. The main goal of this section is to prove Theorem 4.2.4 which tells us that the thin part of X does not have any subvariety.

Let q_i be a cusp of X with unipotent stabilizer Γ_i. Fix $\epsilon > 0$. Consider the set

$$\tilde{U}_{i,\epsilon} = \{x \in \mathbb{B}^n | \exists g \in \Gamma_i, d(x, g \cdot x) < \epsilon\}.$$

We define the ϵ-thin neighborhood around the cusp q_i as the set $U_{i,\epsilon} := \Gamma \backslash \tilde{U}_{i,\epsilon}$. Also, we fix $\rho = \mathrm{sys}(X)/2$ and define the thin part of X as union of all ρ-thin neighborhood around cusps of X :

$$X_{thin} := \cup_{i=1}^{k} U_{i,\rho}.$$

The following Proposition shows that X_{thin} is actually disjoint union of the ρ-thin neighborhood around cusps:

Proposition 4.1. If $\epsilon < sys(X)/2$, then $U_{i,\epsilon} \cap U_{j,\epsilon} = \varnothing$ for $i \neq j$.

Proof. For the sake of the contradiction assume that $x \in U_{i,\epsilon} \cap U_{j,\epsilon}$. It means that there exist $\gamma_1 \in \Gamma_i$ and $\gamma_2 \in \Gamma_j$ such that $d(\tilde{x}, \gamma_1 \cdot \tilde{x}) < \epsilon$ and $d(\tilde{x}, \gamma_2 \cdot \tilde{x}) < \epsilon$, where $\tilde{x} \in \mathbb{B}^n$ is a fiber of x. This in particular implies that $d(\tilde{x}, \gamma_1^{-1} \cdot \tilde{x}) < \epsilon$.

Suppose $\gamma_1 = (\tau, t)$ is a Heisenberg translation based at q_∞ and $\gamma_2 = (\sigma, s)$ is a Heisenberg translation based at q_0. We represent γ_1 and γ_2 by the matrices $g_\infty, g_0 \in \mathrm{PU}(Q)$ respectively, where

$$g_\infty = \begin{bmatrix} 1 & -\tau^* & -(|\tau| + it)/2 \\ 0 & I_{n-1} & \tau \\ 0 & 0 & 1 \end{bmatrix}, \quad g_0 = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & I_{n-1} & 0 \\ -(|\tau| + is)/2 & -\sigma^* & 1 \end{bmatrix}.$$

Note that $\gamma_1^{-1} = (-\tau, -t)$ corresponds to g_∞^{-1}. We can write:

$$\begin{aligned} \left|\operatorname{tr}(g_\infty g_0)\right| + \left|\operatorname{tr}(g_\infty^{-1} g_0)\right| &\geq \left|\operatorname{tr}\left((g_\infty + g_\infty^{-1}) g_0\right)\right| \\ &= \left|\operatorname{tr}\left(\begin{bmatrix} 2 & 0 & -|\tau|^2 \\ 0 & 2I_{n-1} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \sigma & I_{n-1} & 0 \\ (-|\sigma|^2 + is)/2 & -\sigma^* & 1 \end{bmatrix}\right)\right| \\ &= \left|2(n+1) + \frac{1}{2}|\tau|^2(|\sigma|^2 - is)\right| \\ &\geq 2(n+1) + \frac{1}{2}|\tau|^2|\sigma|^2. \end{aligned}$$

Hence, either $|\operatorname{tr}(g_\infty g_0)| \geq n + 1 + \frac{1}{4}|\tau|^2|\sigma|^2$ or $|\operatorname{tr}(g_\infty^{-1} g_0)| \geq n + 1 + \frac{1}{4}|\tau|^2|\sigma|^2$ and therefore either $\gamma_1\gamma_2$ or $\gamma_1^{-1}\gamma_2$ must be hyperbolic. But this implies that either $d(\tilde{x}, \gamma_1\gamma_2 \cdot \tilde{x}) \geq \mathrm{sys}(X)$ or $d(\tilde{x}, \gamma_1^{-1}\gamma_2 \cdot \tilde{x}) \geq \mathrm{sys}(X)$, which is a contradiction because $d(\tilde{x}, \gamma_2 \cdot \tilde{x}) < \mathrm{sys}(X)/2$, $d(\tilde{x}, \gamma_1 \cdot \tilde{x}) < \mathrm{sys}(X)/2$, and $d(\tilde{x}, \gamma_1^{-1} \cdot \tilde{x}) < \mathrm{sys}(X)/2$. □

We define the thick part of X as the complement of the thin part:

$$X_{thick} := X \setminus \cup_{i=1}^{k} U_{i,\rho}.$$

Since every point in a thin part of X has a displacement less than $\mathrm{sys}(X)/2$, the following Proposition tells us that $X_{thcik} \neq \varnothing$.

Proposition 4.2. There exists $x \in X$ such that

$$\mathrm{inj_x(X)} \geq \mathrm{sys}(X)/2.$$

Proof. Note that if $\gamma \in \Gamma$ is not unipotent, then it is semi-simple and for every $x \in \mathbb{B}^n$, we have $d(x, \gamma \cdot x) \geq \mathrm{sys}(X)$. Now, assume for the sake of contradiction that for all $x \in X$ we have $\mathrm{inj}_x(X) < \mathrm{sys}(X)/2$. Therefore, the thin part of X covers all X. However, this is not possible because the thin part of X is disjoint union of open sets by Proposition 4.1 and X is connected.

□

Now we show that the monodromy of ρ-thin part of X around each cusp is in the stabilizer of that cusp:

Lemma 4.3. Suppose that $\epsilon \leq \mathrm{sys}(X)/2$. Let U_i' be a connected component of $U_{i,\epsilon}$ and $\iota : U_i' \to X$ be the identity map. Then, $\iota(\pi_1(U_i'))$ is a subgroup of Γ_i.

Proof. As $\epsilon < \mathrm{sys}(X)/2$, Proposition 4.1 implies that $U_{i,\epsilon}$s are disjoint. Fix $x \in U_{i,\epsilon}'$ and let $\gamma : [0,1] \to X$ be a loop at x which is representative of a class in $\iota(\pi_1(U', x))$. Let $\tilde{x}$ be a lift of x to the universal cover $\mathbb{B}^n$. As γ is fully contained in $U_{i,\epsilon}'$, it lifts to a loop $\tilde{\gamma}(t) : [0,1] \to \tilde{U}_{i,\epsilon}$ which starts at $\tilde{x}$. Therefore, $\tilde{y} := \gamma \cdot \tilde{x} = \tilde{\gamma}(1)$ is in $\tilde{U}_{i,\epsilon}$ which means that $d(\tilde{x}, \tilde{y}) < \epsilon$. Let $\gamma' \in \Gamma_i$ such that $d(\tilde{x}, \gamma' \cdot \tilde{x}) < \epsilon$. By homogeneity, we have

$$d(\tilde{y}, \gamma\gamma'\gamma^{-1} \cdot \tilde{y}) = d(\gamma \cdot \tilde{x}, \gamma\gamma'\gamma^{-1}\gamma \cdot \tilde{x}) = d(\tilde{x}, \gamma' \cdot \tilde{x}) < \epsilon.$$

Since $\gamma\gamma'\gamma^{-1}$ fixes $\gamma(q_i)$ and $\tilde{y} \in U_i$, we get that $\gamma(q_i) = q_i$ and therefore $\gamma \in \Gamma_i$. □

Finally, we show that every subvariety of X intersects with X_{thick}, that is, every subvariety of X contains a point whose injectivity radius in X is larger than $\mathrm{sys}(X)/2$:

Theorem 4.2.4. *Every subvariety of $\overline{X}$ either intersects with X_{thick} or fully contained in D.*

Proof. For the sake of the contradiction assume that there exists a connected subvariety V which is fully contained in X_{thin}. Without loss of generality we can assume that V is fully contained in a connected component of a thin part around the cusp q_∞. Consider the function $-u$ which is a plurisubharmonic function on the Siegel domain $\mathbb{S}$ and invariant under the action of stabilizer Γ_∞(see [BT18b, §2]). Therefore, it follows from Lemma 4.3 that $-u$ is a well-defined function on every component of the thin part around q_∞. Hence, $-u$ is a well-defined plurisubharmonic function on V. Notice that if a plurisubharmonic function achieves its maximum, it has to be constant. Since V is compact, $-u$ must be constant on V. However, it is not possible because the Kähler form on X is induced from $-2i\partial\bar{\partial}\log(u)$(see [BT18b, Lemma 2.1]) and if $-\log(u)$ were constant, the volume of V would be zero. □

4.3 VOLUMES OF SUBVARIETIES

In this section, we prove Theorem D, and Theorem E. We first state the Hwang and To's theorem in the following way:

Theorem 4.3.1. *([HT02, Theorem 1.1]) Take $x \in X$ with injectivity radius $r = \mathrm{inj}_x(X)$. Let $B(x,r)$ be Bergman ball of radius r centered at x. Suppose V is an m-dimensional subvariety of X passing through x. Then, the following inequality holds:*

$$\mathrm{vol}_X(V \cap B(x,r)) \geq \frac{(4\pi)^m}{m!} \sinh^{2m}(r) \cdot \mathrm{mult}_x(V). \tag{4.4}$$

Hwang and To generalized the above-mentioned theorem for a general Hermitian symmetric domain in [HT00, HT02].

In the compact case, Theorem 4.3.1 gives the lower bound on volume of subvarieties in terms of the injectivity radius of X, however, in the case that X is not compact, the injectivity radius of X escapes to zero as we get closer to the cusps. So we use the systole, the length of the shortest closed geodesic in X, as a geometric invariant of X to uniformly bound the volumes of all subvarieties of X. For a compact ball quotient, the systole is twice the injectivity radius. However, for a non-compact X the systole is still not zero and can be estimated by the trace of the hyperbolic elements in a representation of Γ.

Theorem 4.2.4 tells us that every subvariety of X has a point with injectivity as large as $\mathrm{sys}(X)/2$. Hence, we will get the following theorem:

Theorem 4.3.2. *Let V be an m-dimensional subvariety of $\overline{X}$ which is not contained in D. Then,*

$$\mathrm{vol}_X(V) \geq \frac{(4\pi)^m}{m!}\sinh^{2m}\big(\mathrm{sys}(X)/2\big). \tag{4.5}$$

Proof. Theorem 4.2.4 implies that $V \cap X_{thick} \neq \varnothing$. By Proposition 4.2 there exits a point $x \in V$ with $\mathrm{inj}_x(X) \geq \mathrm{sys}(X)/2$. Now, Theorem 4.3.1 gives:

$$\mathrm{vol}_X(V) \geq \frac{(4\pi)^m}{m!}\sinh^{2m}\big(\mathrm{sys}(X)/2\big).$$

□

Corollary 4.3.3. With the same notation as Theorem 4.3.2, we have that

$$(K_{\overline{X}} + D)^m \cdot V \geq (n+1)^m \sinh^{2m}\big(\mathrm{sys}(X)/2\big).$$

Proof. Theorem 4.3.2 together with 2.7 gives:

$$\begin{aligned}(K_{\overline{X}} + D)^m \cdot V &= \Big(\frac{n+1}{4\pi}\Big)^m m!\,\mathrm{vol}_X(V) \text{ (by 2.7)}\\ &\geq (n+1)^m \sinh^{2m}\big(\mathrm{sys}(X)/2\big) \text{ (by Theorem 4.3.2)}\end{aligned}$$

□

We recall Bakker and Tsimerman's theorem which tells us that the uniform depth of cusps of X bounds the intersection numbers of $K_{\overline{X}}$ with subvarieties of $\overline{X}$ which is not contained in D.

Theorem 4.3.4. *([BT18b, Corrolary 3.8]) Suppose d is the uniform depth of cusps. Then,*

$$K_{\overline{X}} + (1-\lambda)D$$

is ample for $\lambda \in (0, d(n+1)/4\pi)$.

Now, putting together Theorem 4.3.2, Theorem 4.3.4 and what we proved for the uniform depth of cusps, Theorem 4.1.9, yields a lower bound for the degree of $K_{\overline{X}}$ on V in terms of $\mathrm{sys}(X)$:

Theorem 4.3.5. *Let V be an m-dimensional subvariety of $\overline{X}$ which is not contained in D. If* $\mathrm{sys}(X) \geq 4\ln\left(5n + (4\pi)^4\right)$, *then*

$$K_{\overline{X}}^m \cdot V > (\frac{n}{4\pi})^m \cdot e^{m\,\mathrm{sys}(X)/16}. \tag{4.6}$$

Proof. We deal separately with the following two cases:

1. $V \cap D = \varnothing$: In this case we have $K_{\overline{X}}^m \cdot V = (K_{\overline{X}} + D)^m \cdot V$ and from Corollary 4.3.3 we get that

$$K_{\overline{X}}^m \cdot V \geq (n+1)^m \sinh^{2m}\left(\mathrm{sys}(X)/2\right) \tag{4.7}$$

$$\geq (\frac{n}{4\pi})^m e^{m\,\mathrm{sys}(X)} \quad \text{(becasue of the bound on the systole).} \tag{4.8}$$

2. $V \cap D \neq \varnothing$: Since $\mathrm{sys}(X) \geq 4\ln\left(5n + (4\pi)^4\right)$), 4.3 gives that the uniform depth of cusp is at least 4π and it follows from Bakker-Tsimerman's theorem (Theorem 4.3.4) that $K_{\overline{X}}$ is ample. In particular, this implies that $K_{\overline{X}|D}$ is ample. On the other hand, we know that the conormal bundle $-D_{|D}$ is ample. Therefore, for every $i > 1$, we have

$$K_{\overline{X}}^{m-i} \cdot (-D)^i \cdot V = -(K_{\overline{X}|D})^{m-i} \cdot (-D_{|D})^{i-1} \cdot V_{|D} < 0. \tag{4.9}$$

By Bakker-Tsimerman's theorem, Theorem 4.3.4, we get that

$$\left(K_{\overline{X}} - \left((n+1)d/4\pi - 1\right)D\right)^m \cdot V \geq 0.$$

Expanding this and combining with 4.9 gives:

$$\begin{aligned} K_{\overline{X}}^m \cdot V &\geq \left((n+1)d/4\pi - 1\right)^m \cdot -(-D)^m \cdot V \\ &\geq \left((n+1)d/4\pi - 1\right)^m \text{(by ampleness of } -D) \\ &\geq \left(nd/4\pi\right)^m \text{ (by 4.3)} \\ &\geq \left(n/4\pi\right)^m \cdot e^{m\,\mathrm{sys}(X)/16} \text{ (by 4.3).} \end{aligned} \tag{4.10}$$

Combining 4.7 and 4.10 gives that:

$$K_{\overline{X}}^m \cdot V > (\frac{n}{4\pi})^m \cdot e^{m\,\mathrm{sys}(X)/16}.$$

□

To prove the bound on the canonical volume for a subvariety which does not intersect the boundary D, we will use the following lemma proved by Brunebarbe:

Lemma 4.6. [Bru20a, Proposition 3.2] Let Y be a Kähler manifold with Kähler form ω. Assume that its holomorphic bisectional curvature is non-positive and that its holomorphic sectional curvature is bounded from above by $-c < 0$. Then,

$$\frac{1}{c}\operatorname{Ric}_\omega \geq \omega.$$

Theorem 4.3.7. *Let V be an m-dimensional subvariety of $\overline{X}$ which is not contained in D. If* $\operatorname{sys}(X) \geq 4\ln\left(5n + (8\pi)^4\right)$*, then*

$$\widetilde{\operatorname{vol}}_V > \left(\frac{m}{4\pi}\right)^m e^{m\operatorname{sys}(X)/16}. \tag{4.11}$$

Proof. Let $V' \longrightarrow V$ be a desingularization such that the set-theoretic preimage of the boundary divisor D' is a normal crossing divisor. We consider two cases:

1. $V \cap D = \varnothing$: Let ω be the Kähler form induced by the Bergman metric on V'. Since the holomorphic sectional curvature of the Bergman metric is -1 and the curvature decreases on a subvariety, we can apply Lemma 4.6 to get that $K_{V'} - K_{\overline{X}|V'}$ is nef on V' and therefore

$$\widetilde{\operatorname{vol}}_V = \operatorname{vol}_{V'}(K_{V'}) \geq \operatorname{vol}_{V'}(K_{\overline{X}|V'}).$$

Since the uniform depth of cusp is sufficiently large, by Theorem 4.3.4 we get that $K_{\overline{X}}$ is ample and therefore

$$\operatorname{vol}_{V'}(K_{\overline{X}|V'}) = K^m_{\overline{X}|V'} = K^m_{\overline{X}} \cdot V = \deg_{\overline{X}}(V).$$

We can conclude the desired inequality for this case by Theorem 4.3.5.

2. $V \cap D \neq \varnothing$: Since $\operatorname{sys}(X) \geq 4\ln\left(5n + (4\pi)^4\right)$), 4.3 gives that the uniform depth of cusp is at least 4π. Theorem A implies that twisted log-cotangent bundle $\Omega^1_{V'}(\log(D))\langle -rD'\rangle$ is ample for every $r \in (0, d/2\pi)$. Taking determinate gives that $K_{V'} + (1 - mr)D'$ is ample for every $r \in (0, d/2\pi)$. Therefore, with the bound on the depth we get that $K_{V'}$ is ample. On the other hand, we have that the bundle $-D'_{|D'}$ is ample. Therefore, for every $i > 1$, we have

$$K^{m-i}_{V'} \cdot (-D')^i \cdot V' = -(K_{V'|D})^{m-i} \cdot (-D'_{|D'})^{i-1} \cdot V_{|D'} < 0. \tag{4.12}$$

Set $r' = d/2\pi - \epsilon$, for a small $\epsilon > 0$. Expanding $\left(K_{V'} + (1 - mr')D'\right)^m \geq 0$ and using 4.12 gives that

$$\begin{aligned} K^m_{V'} &\geq \left(mr' - 1\right)^m (-D'_{|D'})^m \\ &\geq \left(mr' - 1\right)^m \text{ (by the ampleness of } -D'_{|D'}) \\ &\geq (md/4\pi)^m \\ &\geq \left(\frac{m}{4\pi}\right)^m e^{m\operatorname{sys}(X)/16} \text{ (by 4.3).} \end{aligned}$$

By the ampleness of $K_{V'}$ we get $\widetilde{\mathrm{vol}}_V = \mathrm{vol}_{V'}(K_{V'}) = K_{V'}^m$ and hence the claim follows.

$\square$

5 EFFECTIVE GLOBAL GENERATION AND VERY AMPLENESS

In this chapter, we prove Corollary I, Corollary J based on the bound we found for $\deg_{\overline{X}}(V)$ in Theorem 4.3.5. First, we analyze the problem on the boundary divisor D.

5.1 BASE-POINT FREENESS AND VERY AMPLENESS ON D.

In this section, we prove that if the uniform depth of cusps is sufficiently large, then $2K_{\overline{X}}$ does not have a base point on D, and moreover $3K_{\overline{X}}$ can separate any two points, and any tangent direction on D. We first prove that the restricted bundles on the boundary satisfy these properties. Consider the decomposition of the boundary divisor D to the connected components $D = \sqcup_{i=1}^{k} D_i$. Due to [Mok12], we know that each D_i is an abelian variety with ample conormal bundle $O_{D_i}(-D_i)$.

Lemma 5.1. The line bundle $2K_{\overline{X}|D_i}$ is base-point free and $3K_{\overline{X}|D_i}$ is very ample for every i.

Proof. The adjunction formula gives that $K_{\overline{X}|D_i} \cong -D_{i|D_i}$. As the conormal bundle is ample and D_i is an abelian variety, $-2D_{i|D_i}$ is base-point free and $-3D_{i|D_i}$ is very ample (see [Ohb87]).

□

In the next two lemmas, we see how we can lift the sections from the restricted bundle to $\overline{X}$. The base locus of a line bundle L on $\overline{X}$ will be denoted by $\mathrm{Bs}(L)$.

Lemma 5.2. Suppose that the uniform depth of cusps is larger than 4π, Then, the following hold:

1. $\mathrm{Bs}\left(2K_{\overline{X}}\right) \cap D = \varnothing$
2. For any two points on different components of D, there exists a global section of $2K_{\overline{X}}$ which separate them.

Proof. Let L be $2K_{\overline{X}}$.

1. By Lemma 5.1, $L_{|D}$ is base-point free and therefore it is enough to show that we can lift the global section from D to $\overline{X}$, that is, $H^0(\overline{X}, L) \longrightarrow H^0(D, L_{|D})$ is surjective. Consider the following exact sequence on $\overline{X}$:

$$0 \longrightarrow L - D \longrightarrow L \longrightarrow L_{|D} \longrightarrow 0$$

Writing the long exact sequence we can see that it is sufficient to show $H^1(\overline{X}, L - D) = 0$. As $L - D = K_{\overline{X}} + (K_{\overline{X}} - D)$, if the uniform depth is sufficiently large, then by Theorem 4.3.4 $K_{\overline{X}} - D$ is ample. Therefore, the vanishing of $H^1(\overline{X}, L - D)$ follows from Kodaira's vanishing theorem.

2. Suppose that we want to separate $x \in D_i$ and $y \in D_j$ with $i \neq j$. It is sufficient to find a global section of $L - D_i$ which does not vanish at y. To this end, we can repeat the argument of the first part, but instead of using Theorem 4.3.4, we should use [BT18b, Proposition 3.6].

□

Lemma 5.3. If the uniform depth of cusps is larger than 2π, then $3K_{\overline{X}}$ can separate any two points, and any tangent direction on every connect component of D.

Proof. By Lemma 5.2 and Lemma 5.1, it is enough to show that we can lift the sections from the boundary, i.e.,

$$H^0(\overline{X}, 3K_{\overline{X}}) \longrightarrow H^0(D, 3K_{\overline{X}|D}) \longrightarrow 0.$$

Hence, it is enough to show that $H^1(\overline{X}, 3K_{\overline{X}} - D) = 0$. Since $d > 2\pi$, it follows from Theorem 4.3.4 that $2K_{\overline{X}} - D$ is ample. Therefore, by Kodaira's vanishing theorem we get that $H^1(\overline{X}, 3K_{\overline{X}} - D) = 0$. □

5.2 GLOBAL GENERATION AND VERY AMPLENESS ON $\overline{X}$

In this section, we see how we can conclude effective global generation and effective very ampleness results by using Theorem 4.3.5. We first recall the famous theorem of Angehrn and Siu on pointwise base point freeness:

Theorem 5.2.1. *[AS95, Theorem 0.1] Let Y be a smooth projective variety of dimension n, and let L be an ample line bundle on Y. Fix a point $y \in Y$, and assume that*

$$L^m \cdot V > \left(\frac{n(n+1)}{2}\right)^m \tag{5.1}$$

for every subvariety V of dimension m passing through y. Then, $K_Y + L$ has a section that does not vanish at y.

Combining Angehrn and Siu's result with our Theorem 4.3.5 gives that if $\mathrm{sys}(X)$ is sufficiently large relative to n, then $2K_{\overline{X}}$ is globally generated:

Theorem 5.2.2. *If* $\mathrm{sys}(\mathrm{X}) \geq 20 \ln\left(5n + (4\pi)^4\right)$, *then $2K_{\overline{X}}$ is globally generated.*

Proof. Using 4.3 we get that $d > 4\pi$. Therefore, by Lemma 5.2, $2K_{\overline{X}}$ does not have any base-point on D. On the other hand, Theorem 4.3.5 implies that for every m-dimensional subvariety $V \subset \overline{X}$ which is not contained in D, we have

$$\begin{aligned} K_{\overline{X}}^{m} \cdot V &\geq (\frac{n}{4\pi})^{m} \cdot e^{m\,\mathrm{sys}(X)/16} \\ &\geq n^{m}\big(5n + (4\pi)^{4}\big)^{m} (\text{by the bound on } \mathrm{sys}(X)) \\ &> (\frac{n(n+1)}{2})^{m}. \end{aligned}$$

Therefore, Theorem 5.2.1 implies that for every point $x \in \overline{X} \setminus D$, there is a section of $2K_{\overline{X}}$ which does not vanish at x. Hence, $2K_{\overline{X}}$ is globally generated. □

Now, we prove a proposition which will be used to show that $2K_{\overline{X}}$ can separate a point in X from a point in D :

Proposition 5.3. If $\mathrm{sys(X)} \geq 20 \ln\big(5n + (8\pi)^{4}\big)$, then for every $x \in X$ there exits $s \in H^{0}(\overline{X}, 2K_{\overline{X}} - D)$ such that s does not vanish at x.

Proof. Since $\mathrm{sys(X)} \geq 20 \ln\big(5n + (8\pi)^{4}\big)$, the uniform depth of cusps is larger than 4π(see 4.3) and therefore by Bakker-Tsimerman's result, Theorem 4.3.4, $K_{\overline{X}} + (1-\lambda)D$ is ample for $\lambda \in (0, \frac{(n+1)d}{4\pi})$. On the other hand, as $K_{\overline{X}|D} \cong -D_{D}$ and $-D_{|D}$ is ample, for every subvariety V of dimension m and every $1 \leq i \leq m$ we have

$$(K_{\overline{X}} - 2D)^{m-i}(-D)^{i} \cdot V_{|D} = -(K_{\overline{X}} - 2D)^{i}_{|D}(-D_{|D})^{j-1} \cdot V = -3^{i}(-D_{|D})^{n-1} \cdot V_{|D} \leq 0. \tag{5.2}$$

Expanding $(K_{\overline{X}} - (1 - \frac{(n+1)d}{4\pi})D)^{m} \cdot V \geq 0$ and using 5.2 we get:

$$\begin{aligned} (K_{\overline{X}} - 2D)^{m} \cdot V &\geq (\frac{(n+1)d}{4\pi} + 1)^{m}(-D_{|D})^{m-1} \cdot V_{|D} \\ &\geq (\frac{(n+1)d}{4\pi})^{m} (\text{by the ampleness of } -D_{|D}) \\ &\geq (\frac{n+1}{4\pi})^{m} \cdot e^{m\,\mathrm{sys}(X)/16} \quad (\text{by } 4.3) \\ &> (n+1)^{m} n^{m} \quad (\text{by the bound on } \mathrm{sys}(X)) \end{aligned}$$

Hence, Theorem 5.2.1 gives that $2K_{\overline{X}} - D$ has a global section which does not vanish at x.

□

We recall the result of Ein-Lazarsfeld-Nakamay on the pointwise separation of jets:

Theorem 5.2.4. *([ELN96, Theorem 4.4]) Let Y be a smooth projective variety of dimension n and let L be an ample line bundle on Y satisfying $L^{n} > (n+s)^{n}$. Let b be a non-negative number such that $K_{Y} + bL$ is nef. Suppose that m_{0} is a positive integer such that $m_{0}L$ is free. Then, for any point $y \in Y$ either*

(a) $K_{Y} + L$ separates s-jets at y, or

(b) there exists a dimension m subvariety V containing y and satisfying

$$\deg_L(V) \leq \left(b + m_0 \cdot m + \frac{n!}{(n-m)!}\right)^{n-m} (n+s)^n \tag{5.3}$$

Let Y be a smooth projective variety and L be a nef line bundle on Y. Fix a point $y \in Y$. The Seshadri constant of L at y is the real number

$$\varepsilon(L, y) = \inf \frac{L \cdot C}{\mathrm{mult}_y(C)},$$

where the infimum is taken over all curves C passing through y.

Plugging in our Theorem 4.3.5 and Theorem 5.2.2 to the result of Ein-Lazarsfeld-Nakamay allows us to separates s-jets of $2K_{\overline{X}}$ on X if $\mathrm{sys}(X)$ is sufficiently large with respect to n and s :

Theorem 5.2.5. *Let s be a positive integer. Suppose that*

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left((1+2n+n!)(n+s)\right), \ln\left(5n + (8\pi)^4\right)\}.$$

Then for every $x \in X$, the line bundle $2K_{\overline{X}}$ separates s-jets at x. In particular, for every x we have that

$$\epsilon(K_{\overline{X}}, x) \geq s/2.$$

Proof. Since $\mathrm{sys}(X) > 20 \ln\left(5n + (8\pi)^4\right)$, Theorem 5.2.2 implies that $2K_{\overline{X}}$ is globally generated. Also, as $\mathrm{sys}(X) \geq 20 \ln(n+s)$, Theorem 4.3.5 implies that

$$K_{\overline{X}}^n > (n+s)^n.$$

Note that plugging in the lower bounds on $\mathrm{sys}(X)$ in Theorem 4.3.5 gives that for a subvariety V of dimension m which does not contained in D, the following inequality holds:

$$K_{\overline{X}}^m \cdot V \geq n^m(1+2n+n!)^n(n+s)^n \geq (b + 2m + \frac{n!}{(n-m))!})^{n-m}(n+s)^n.$$

Now, applying Theorem 5.2.4 to $L = K_{\overline{X}}, m_0 = 2$ and $b = 1$ gives that $2K_{\overline{X}}$ separates s-jest at every $x \in X$.

Combining the separation of jets with [BRH+09, Proposition 2.2.5] gives that $\epsilon(2K_{\overline{X}}, x) \geq s$. Since $\epsilon(2K_{\overline{X}}, x) = 2\epsilon(K_{\overline{X}}, x)$, we get the desired inequality. □

We recall a result of Kollar which tells us that a line can separate two points if the degree of every subvariety passing through either of the points with respect to the line bundle is sufficiently large relative to the dimension of the ambient space:

Theorem 5.2.6. *([Kol97, Theorem 5.9]) Let L be a nef and big divisor on a smooth projective variety Y. Let x_1, x_2 be closed points and assume that there are positive numbers $c(k)$ with the following properties:*

1. *If $V \subset Y$ is an irreducible m-dimensional subvariety which contains x_1 or x_2 then*

$$L^m \cdot V > c(m)^m.$$

2. *The numbers $c(k)$ satisfy the inequality*

$$\sum_{k=1}^{\dim(x)} \sqrt[k]{2}\frac{k}{c(k)} \leq 1.$$

Then, $K_Y + L$ separates x_1 and x_2.

Definition 5.7. ([Tak93]) Let L be a line bundle on a smooth projective variety Y and D be a divisor on Y. The line bundle L is said to be very ample modulo D if the rational map $\Phi_L : Y \dashrightarrow \mathbb{P}(H^0(Y, O_Y(L))$ is an embedding of $Y \setminus D$.

Note that Theorem 5.2.2 says that the rational map $\Phi_{2K_{\overline{X}}} : Y \dashrightarrow \mathbb{P}(H^0(Y, O_Y(2K_{\overline{X}}))$ is globally defined map on Y. Moreover, the following theorem gives that this map is in particular injective on X and can separate any two tangent directions at whole $\overline{X}$:

Theorem 5.2.8. *Suppose that*

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left((1 + 2n + n!)(n + 1)\right), \ln\left(5n + (8\pi)^4\right)\}.$$

Then the map $\Phi_{2K_{\overline{X}}} : \overline{X} \to \mathbb{P}\left(H^0(\overline{X}, 2K_{\overline{X}})\right)$ satisfies the following properties:

1. *If $\phi_{2K_{\overline{X}}}(x_1) = \phi_{2K_{\overline{X}}}(x_2)$ for some $x_1, x_2 \in \overline{X}$, then $x_1, x_2 \in D_i$, where D_i is some connected component of D.*

2. *$\Phi_{2K_{\overline{X}}}$ separates tangent directions at every $x \in X$.*

Proof. Separation of points: Note that by Lemma 5.2 if $\phi_{2K_{\overline{X}}}(x_1) = \phi_{2K_{\overline{X}}}(x_2)$ and $x_1, x_2 \in D$, then they both lie on the same component of D. Hence, we only need to deal with the following two cases:

1. $x_1, x_2 \in X$: Let $V \subset \overline{X}$ be a subvariety of dimension m which passes through either x_1 or x_2. Fix $c = ne^{\mathrm{sys}(X)/20}$. By Theorem 4.3.5 we have that

$$K_{\overline{X}}^m \cdot V \geq c^m.$$

Therefore, by Kollar's Theorem (5.2.6) we can separate any two points $x_1, x_2 \in X$.

2. $x_1 \in X, x_2 \in D$: By Proposition 5.3, there is a section $s \in H^0(\overline{X}, 2K_{\overline{X}} - D)$ which does not vanish at x_1. Therefore, as $2K_{\overline{X}} - D$ is a subbundle of $2K_{\overline{X}}$, we get a section of $2K_{\overline{X}}$ which does not vanish at x_1, but vanishes on D and in particular at x_2.

Separation of tangent directions: For $x \in X$, the separation of tangent direction follows from Theorem 5.2.5 when $s = 1$.

□

In particular, Theorem 5.2.8 implies that $2K_{\overline{X}}$ is very ample modulo D.

Theorem 5.2.9. *With the same assumption on* $\mathrm{sys}(X)$ *as Theorem 5.2.8, $3K_{\overline{X}}$ is very ample.*

Proof. By Theorem 5.2.8, we only need to show that $3K_{\overline{X}}$ can separate any two points and any tangent direction on any connected component of D, which follows from Lemma 5.3. □

Putting all of these together, we get:

Corollary 5.2.10. Suppose that

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left((1+2n+n!)(n+1)\right), \ln\left(5n+(8\pi)^4\right)\}.$$

Then, the following hold

1. $2K_{\overline{X}}$ is globally generated and very ample modulo D.

2. $3K_{\overline{X}}$ is very ample.

Proof. The global generation of $2K_{\overline{X}}$ follows from Theorem 5.2.2. The very ampleness modulo D follows from Theorem 5.2.8. The very ampleness of $3K_{\overline{X}}$ follows from Theorem 5.2.9. □

5.3 SESHADRI CONSTANT

The goal of this section is to study the relation between the Seshadri constant and the systole of X and in particular we prove Corollary **??** in this section.

In addition to what we obtained on Theorem 5.2.5 about the Seshadri constant $\epsilon(2K_{\overline{X}}, x)$ for $x \in X$, we can obtain the following result which has smaller bound on $\mathrm{sys}(X)$:

Corollary 5.3.1. Suppose that $\mathrm{sys}(X) \geq 20 \ln\left(5n+(8\pi)^4\right)$. Let

$$E := \{x \in X | \epsilon(K_{\overline{X}}, x) < e^{\mathrm{sys}(X)/20}\}.$$

Then, E satisfies the following properties:

1. $E \cap X_{thick} = \varnothing$.

2. E does not contain any positive dimensional subvariety.

3. E is contained in a Zariski closed proper subset of X.

Proof. 1. Fix $x \in X_{thick}$. Let $C \subset \overline{X}$ be a curve passing through x. Since $x \in X_{thick}$ we have $\mathrm{inj}_x(X) \geq \mathrm{sys}(X)/2$. On the other hand, the bound on the systole gives that $d \geq 8\pi$, therefore by Theorem 4.3.4 $K_{\overline{X}} - D$ is ample. We can write:

$$\begin{aligned} 2K_{\overline{X}} \cdot C &\geq (K_{\overline{X}} + D) \cdot C \text{ (By ampleness of } K_{\overline{X}} - D) \\ &\geq \frac{n+1}{4\pi} \mathrm{vol}_X(C) \\ &\geq \frac{n+1}{4\pi} \sinh^2\left(\mathrm{sys}(X)/2\right) \cdot \mathrm{mult}_x(C) \text{ (by 4.3.1).} \end{aligned}$$

Therefore,

$$\epsilon(x, K_{\overline{X}}) \geq \frac{n+1}{8\pi} \sinh^2\left(\mathrm{sys}(X)/2\right) > e^{\mathrm{sys}(X)/20},$$

and this gives the first property.

2. Combining (i) with Theorem 4.2.4 we conclude that E does not have any positive dimensional subvariety.

3. Note that Theorem 4.3.5 implies that for every m-dimensional subvariety $V \not\subset D$, we have

$$(K_{\overline{X}}^m \cdot V)^{\frac{1}{m}} \geq \frac{n+1}{4\pi} e^{\mathrm{sys}(X)/16}.$$

Putting this in [EKL95, Theorem 3.1] gives

$$\epsilon(K_{\overline{X}}, x) \geq \frac{1}{4\pi} e^{\mathrm{sys}(X)/16} > e^{\mathrm{sys}(X)/20}. \tag{5.4}$$

for all $x \in \overline{X}$ off the union of countably many proper subvarieties of $\overline{X}$. On the other hand as $K_{\overline{X}}$ is ample by using [EKL95, Lemma 1.4] we can conclude that the inequality 5.4 holds on Zarisiky open set, i.e, E is contained in a proper subvariety of $\overline{X}$.

□

Consider the decomposition of the boundary divisor D to the connected components $D = \sqcup_{i=1}^k D_i$. Due to [Mok12], we know that each D_i is an abelian variety with ample conormal bundle $O_{D_i}(-D_i)$. The adjunction formula gives that $K_{\overline{X}|D_i}$ is isomorphic to the conormal bundle $O_{D_i}(-D_i)$. Suppose that $D_i = \Lambda_i \backslash W_i$, where $W_i \cong \mathbb{C}^{n-1}$ is a complex vector space of dimension $n-1$, and $\Lambda_i \cong \mathbb{Z}^{n-1}$ is a lattice in W_i. It is classical that every ample line bundle on D_i determines a positive definite Hermitian form on W_i. Suppose H_i is the positive definite Hermitian form determined by $K_{\overline{X}|D_i}$ on W_i. The real part

$$B_i = \mathrm{Re}(H_i)$$

defines a Euclidean inner product on W_i (see [Laz17, sec 5.3.A] for more details). Let l_i be the length of the shortest vector of Λ_i with respect to B_i. We define the systole of the boundary as

$$\mathrm{sys}(D) := \min_{i=1}^k l_i.$$

The following lemma gives a lower bound for the Seshadri constant of $K_{\overline{X}|D}$ in terms of the systole of the boundary:

Lemma 5.2. Let x be a point on a connected component of the boundary, D_i. Then,

$$\epsilon(K_{\overline{X}|D_i}, x) \geq \frac{\pi}{4} \cdot \mathrm{sys}(D)^2.$$

Proof. This follows from [Laz17, Theorem 5.3.6]. □

Combining this lemma with the previous results gives that if the systole of $\overline{X}$ and $\mathrm{sys}(D)$ are sufficiently large, then the Seshadri constant $\epsilon(K_{\overline{X}}, x)$ is large and in particular $2K_{\overline{X}}$ is very ample:

Corollary 5.3.3. Suppose that $\mathrm{sys}(D) > 2\sqrt{2n/\pi}$ and that

$$\mathrm{sys}(X) \geq 20 \max\{n \ln\left(5n(1+2n+n!)\right), \ln\left(5n+(8\pi)^4\right)\}.$$

Then, for every $x \in \overline{X}$ we have

$$\epsilon(K_{\overline{X}}, x) \geq 2n,$$

and in particular $2K_{\overline{X}}$ is very ample.

Proof. Let $C \subset \overline{X}$ be a connected curve passing through a point $x \in \overline{X}$. We consider there cases:

1. $x \in D$ and C fully contained in a D : Let D_i be the connected component of D which contains x. Lemma 5.2 implies that

$$K_{\overline{X}} \cdot C = K_{\overline{X}|D_i} \cdot C \geq \frac{\pi}{4} \mathrm{sys}(D)^2 \cdot \mathrm{mult}_x(C) \geq 2n \cdot \mathrm{mult}_x(C).$$

2. $x \in D$ and C is not contained in D : Plugging in the bound on the systole in Theorem 4.1.9 gives that the uniform depth of cusps d is at least 8π. By the theorem of Bakker-Tsimerman, Theorem 4.3.4, the line bundle $K_{\overline{X}} + (1-\lambda)D$ is ample for $\lambda \in (0, (n+1)d/4\pi)$. Hence, we can write

$$\begin{aligned} K_{\overline{X}} \cdot C &\geq \left(\frac{(n+1)d}{4\pi} - 1\right) D \cdot C \\ &\geq \frac{nd}{4\pi} \mathrm{mult}_x(C) \\ &\geq 2n \, \mathrm{mult}_x(C). \end{aligned}$$

3. $x \in X$: For this case we will use Theorem 5.2.5. Plugging in $s = 2n$ to this theorem gives that:

$$K_{\overline{X}} \cdot C \geq 2n \cdot \mathrm{mult}_x(C).$$

   Hence, for every $x \in X$ we get that $\epsilon(K_{\overline{X}}, x) \geq 2n$. Combining this with Demailly's theorem [Dam92, Proposition 6.8).] implies that $2K_{\overline{X}}$ is very ample.

□

6 RATIONAL POINTS

6.1 BOMBIERI–LANG CONJECTURE

Let C be a smooth projective curve defined over $\mathbb{Q}$ and let g be the geometric genus of C. In 1983 Faltings proved in his celebrated paper [Fal83] that if $g > 1$, i.e., C is a compact ball quotient of complex dimension 1, then $C(\mathbb{Q})$ is finite. Bombieri and Lang stated a conjecture on a generalization of Faltings' theorem to higher-dimensional varieties:

Conjecture 6.1. *(Bombieri–Lang) Let Y be a smooth projective variety defined over a number field K. Suppose Y is a variety of general type. Then, there exists a proper algebraic subset $D \subset Y$ that contains all but finitely many points of $Y(K)$.*

This conjecture is known for curves and subvarieties of abelian varieties by the seminal work of Faltings [Fal83, Fal91] in 1983 and 1991. Up to the cases that can be reduced to Falting's results, this conjecture has remained largely open since 1971.

Given Corollary 3.4.2, Bombieri–Lang conjecture predicts that there are only finitely many rational points on ball quotient with uniform depth greater 2π. While this conjecture is widely open, we can bound the growth rate of rational points in terms of systole.

6.2 SPARSITY OF RATIONAL POINTS

The goal of this section is to prove Corollary H which is based on fundamental idea of Bombieri-Pila:

Theorem 6.2.1. *([BM22, Theorem 3.4]) Let D be a closed subvariety of $\mathbb{P}^N_F$, $\epsilon > 0$ be a real number, and $n \geq 0$ and $e \geq 1$ be integers.*

Then, there is a real number $C = c(n, e, N, F, D, \epsilon)$ with the following property: For an integral n-dimensional closed subvariety Y of $\mathbb{P}^N_F$ of degree $\leq e$ such that each positive-dimensional integral closed subvariety in Y not contained in D has degree $\geq d^{\dim(X)}$ for some integer $d \geq 1$, and a real number $B > [F : \mathbb{Q}]\epsilon$, the following inequality holds:

$$\#\{x \in Y(F) \setminus D \mid \mathrm{H}(x) \leq B\} \leq CB^{(1+\epsilon)[F:\mathbb{Q}]n(n+3)/d}.$$

Now, combining our effective estimate (Theorem 4.3.5) on the degree of the subvarieties with Theorem 6.2.1 we can conclude:

Corollary 6.2.2. Suppose that $\overline{X}$ is defined over a number field F and $\mathrm{sys}(X) \geq 4\ln\left(5n + (4\pi)^4\right)$. Let $L = K_{\overline{X}}$ and ϵ be a positive number. Then, there exists a constant C depending on X, F and ϵ such that for every $B \geq \epsilon[F : \mathbb{Q}]$ one has :

$$\#\left\{x \in X(F) \mid \mathrm{H}_L(x) \leq B\right\} \leq CB^{\delta},$$

where

$$\delta = \frac{4\pi[F:\mathbb{Q}](n+3)}{e^{\mathrm{sys}(X)/16}}(1+\epsilon).$$

Proof. With the bound on the systole, Theorem 4.3.5 tells us that L is ample bundle as it has positive intersection with all subvarieties. Let b be an integer such that bL is very ample. Now, we can embed $\overline{X}$ into some projective space $\mathbb{P}^N$ by bL. Applying Theorem 4.3.5 gives us that for every subvariety of $\overline{X}$ not contained in D one has:

$$((bL)^m \cdot V)^{1/m} \geq (\frac{nb}{4\pi})e^{\mathrm{sys}(X)/16}.$$

Because of the bound on the systole, we know that the left hand side of the inequality is greater than 1. Hence, applying Theorem 6.2.1 gives us that:

$$\#\{x \in X(F) \mid \mathrm{H}_{bL}(x) \leq B\} \leq CB^{4\pi[F:\mathbb{Q}](n+3)(1+\epsilon)/bs}, \tag{6.1}$$

where $s = e^{\mathrm{sys}(X)/16}$, and C is constant depending on X, F and ϵ (note that N, n and e is fixed when we fixed $\overline{X}$ and L. Also, the toroidal compactification is unique for a ball quotient, therefore all of theses data only depend on X). To conclude, note that $\mathrm{H}_L(x) \leq B$ if and only if $\mathrm{H}_{bL}(x) \leq B^b$. Therefore replacing B with B^b implies the claim. □

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