Nonlinear Modulation of Surface Water Waves over a Periodic Bottom

by

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Abstract

This thesis contributes to the study of the two-dimensional water wave problem in the presence of a variable bottom topography, which describes the motion of a free surface over a body of water under the influence of gravity. When the bottom topography is flat (with depth possibly infinite), it is known that, in a weakly nonlinear regime, the envelope of modulated surface waves is governed by the cubic nonlinear Schrödinger (NLS) equation. This result was derived in the case of infinity depth by Zakharov (1986), and later extended to finite depth domains over a flat bottom by Hasimoto and Ono (1978).

In this thesis, we extend this derivation to the two-dimensional water wave problem in the case of a variable bottom, assumed to be a smooth periodic function. Starting from the Zakharov/Craig-Sulem formulation of water waves, we use a multiple-scale method to write the surface wave in the form of a slowly modulated Bloch-Floquet wavepacket, which propagates at the group velocity. We show that the envelope of wave amplitude is governed by the NLS equation. A key step in this process is to investigate the actions of the Dirichlet-Neumann operator on multiple-scale functions of various forms.

We also present perturbative calculations of the Bloch-Floquet eigenvalues and eigenfunctions of the Dirichlet-Neumann operator when the variation of periodic bottom is small. These are used to study the effect of the variable bottom on the coefficients in the NLS equation.

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Contents

Chapter 1

Introduction

The water wave problem refers to a class of mathematical and physical problems related to the motion of waves on the surface of water bodies, such as oceans, lakes, rivers and canals. Water waves play a significant role in various natural and artificial phenomena, making their understanding crucial across scientific domains such as fluid dynamics, oceanography, coastal engineering, and environmental science. For instance, understanding water waves and their interactions with the environment contributes to safe marine transportation, enhances coastal engineering construction, facilitates harnessing wave energy for power generation, and improves weather forecasting and tsunami monitoring. Therefore, the study of the water wave problem has both scientific and practical importance.

The mathematical study of water wave equations can be traced back to 1781, when Lagrange wrote the basic equations to describe the motion of waves on the free surface of water and solved them in the case of small waves on shallow water [14]. Since then, the study of the water wave problem has maintained its significance as an active field in mathematics, physics and engineering for more than 200 years.

The water wave problem involves describing the motion of an incompressible, inviscid, and irrotational fluid in a d-dimensional fluid domain ($d = 2$ or 3), under the influence of gravity. The fluid domain $\mathbf{S}(\eta, b)$ is assumed to be infinite in the horizontal direction, bounded below by a fixed bottom $-h + b(x)$ and bounded above by a free surface $\eta(x,t)$, which acts as the interface between the fluid and the surrounding air (see Figure 1.1). As the free surface $\eta(x,t)$ varies over time, the domain $\mathbf{S}(\eta, b)$ is time-dependent and can be expressed as

$$\mathbf{S}(\eta, b) = \{(x,y) : x \in \mathbb{R}^{d-1}, -h + b(x) < y < \eta(x,t)\}, \tag{1.1}$$

with the boundary Γ_b at the bottom and the boundary Γ at the free surface, which are given by

$$\Gamma_b = \{(x,y) : x \in \mathbb{R}^{d-1}, y = -h + b(x)\}; \quad \Gamma = \{(x,y) : x \in \mathbb{R}^{d-1}, y = \eta(x,t)\}. \tag{1.2}$$

Here, h represents the average depth of the water, and $b(x)$ denotes the perturbation to a flat bottom.

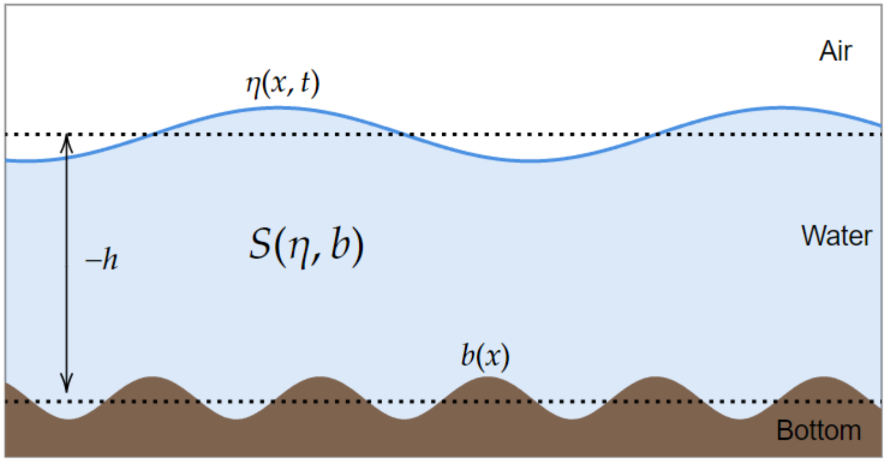

Figure 1.1: The fluid domain $\mathbf{S}(\eta,b)$

The case where $b(x)=0$ corresponds to the water wave problem over a flat bottom. This thesis concerns the case where the bottom is variable $b(x)\neq 0$.

The motion of water waves is governed by the Euler equations, with the unknowns being the velocity potential $\Phi(x,y,t)$ and the free surface elevation $\eta(x,t)$.

$$\begin{cases} \Delta\Phi = 0 & \text{in} \quad \mathbf{S}(\eta,b), \\ \partial_{\mathbf{n}}\Phi = 0 & \text{on} \quad \Gamma_b, \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0 & \text{on} \quad \Gamma, \\ \partial_t\eta + \partial_x\eta\cdot\partial_x\Phi - \partial_y\Phi = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.3}$$

where $\mathbf{n}$ is the unit exterior normal vector to the boundary Γ_b at bottom and g is the gravitational acceleration.

Zakharov [27] formulated the above boundary value problem in the form of a Hamiltonian system using canonical variables $\eta(x,t)$ and $\xi(x,t)$, where ξ is defined as the trace of the velocity potential on the free surface, expressed as $\xi(x,t)=\Phi(x,\eta(x,t),t)$. Following this, Craig and Sulem [11] made a significant contribution by introducing the Dirichlet–Neumann operator (DNO) $G[\eta,b]$, converting the original problem (1.3) with free boundary conditions into the problem (1.4), where unknowns η and ξ are evaluated only on the free surface:

$$\begin{cases} \partial_t\eta - G[\eta,b]\xi = 0 \\ \partial_t\xi + g\eta + \dfrac{1}{2}|\partial_x\xi|^2 - \dfrac{1}{2(1+|\partial_x\eta|^2)}(G[\eta,b]\xi + \partial_x\eta\cdot\partial_x\xi)^2 = 0. \end{cases} \tag{1.4}$$

This formulation of the gravity waves equations, known as the Zakharov/Craig-Sulem formulation, has been widely used to study the well-posedness of the water wave system [19], asymptotic models derived from the water wave system in various physical regimes [13], as well as numerical simulations [15].

Water wave equations (1.4) exhibit a particularly rich structure, allowing solutions with dramatically different properties depending on the physical characteristics of the flow. For instance, dispersive effects are more important in deep water than in shallow water, and nonlinear effects become more significant as the amplitude of the waves grows larger. Due to the complexity and variety of wave dynamics phenomena, it is useful to introduce non-dimensional parameters and derive approximate asymptotic equations to describe qualitative properties of the solutions to particular problem under consideration.

Let us introduce 2 typical length scales of the problem.

1. (Nonlinearity Parameter) $\varepsilon = \dfrac{a}{h}$, where h is the water depth and a is the amplitude of surface wave. ε describes the relative size of amplitude or equivalently strength of the nonlinearity.

2. (Shallowness Parameter) $\delta = \dfrac{h}{\lambda}$, where λ is the free surface wavelength. δ measures the shallowness of water.

In the asymptotic regime of shallow water with small amplitude, where ε, δ are small (i.e. $\varepsilon, \delta << 1$) and $\varepsilon = \delta^2$, the long-time dynamics of the surface wave η can be approximated by the Korteweg-de Vries (KdV) equation

$$\partial_\tau \tilde{\eta} + \partial_\mu^3 \tilde{\eta} + \tilde{\eta} \partial_\mu \tilde{\eta} = 0, \tag{1.5}$$

where free surface elevation $\eta = \varepsilon^2 \tilde{\eta}$, long-time variable $\tau = \varepsilon^3 t$ and $\mu = \varepsilon(x - c_g t)$ with $c_g = \sqrt{gh}$. Many works have been devoted to the derivation of the KdV equation and its rigorous justification [4, 19, 25].

Another important regime is the weakly nonlinear regime or modulational regime which provides a canonical description of small amplitude dispersive waves in various contexts. We first present this regime in the simple case of the Klein-Gordon equation (see Chapter 1 in [24] for more details)

$$v_{tt} - \Delta v + v = v^3. \tag{1.6}$$

If the nonlinear terms are neglected, v admits a solution in the form of a monochromatic wave

$$v(x,t) = \varepsilon u e^{i(kx - \omega(k)t)} + c.c. \quad (\varepsilon << 1), \tag{1.7}$$

where $c.c.$ denotes the complex conjugate of the preceding terms. The amplitude u is a constant and $\omega(k)$ is related to k by the dispersion relation

$$\omega^2 = k^2 + 1. \tag{1.8}$$

The method of multiple-scale consists in assuming that u is no longer a constant but depends on slow time and long spatial variables, and provides a canonical evolution equation for the envelope

u in terms of variables $\mu = \varepsilon\big(x - \omega'(k)t\big)$ and $\tau = \varepsilon^2 t$, which is the Nonlinear Schrödinger (NLS) equation

$$2i\,u_\tau + \omega''(k)\,u_{\mu\mu} + \frac{3}{\omega(k)}\,|u^2|u = 0. \tag{1.9}$$

This approach is very general and can be applied to many physical problems including the water wave problem.

In the case of water waves over a flat bottom of average depth h, the free surface displacement η is given by, at leading order,

$$\eta \sim \varepsilon\, u(\varepsilon x, \varepsilon t)\, e^{i(kx-\omega(k)t)} + c.c. + \cdots, \quad (\varepsilon << 1) \tag{1.10}$$

with the dispersion relation

$$\omega^2(k) = gk\tanh kh. \tag{1.11}$$

Then the slowly varying amplitude $u(\varepsilon x, \varepsilon t)$, in a reference frame moving with the group velocity $c_g = \omega'(k)$, satisfies the NLS equation

$$2iu_\tau + \omega''(k)u_{\mu\mu} + \chi|u|^2 u = 0, \tag{1.12}$$

where $\tau = \varepsilon^2 t$, $\mu = \varepsilon\big(x - \omega'(k)t\big)$ and χ is a coefficient depending on k and h.

The cubic NLS equation was derived from the two-dimensional water wave problem by Zakharov [27] in the case of infinitely deep water. Shortly thereafter, Hasimoto and Ono [18] obtained the NLS equation in the case of finite depth to describe the modulational regime of water waves. The above results are formal derivations. A partial rigorous derivation of the NLS equation from the two-dimensional and three-dimensional water wave system is given in [12] and [13] respectively. The authors proved that an approximate solution in the form of a wavepacket with an envelope satisfying the NLS equation (or the Davey–Stewartson system in three dimension) satisfies the water wave system up to a certain order. Later, Totz and Wu [26] obtained a bound for the error $(\eta - \tilde{\eta})$ between the exact solution and approximate one on time interval $O(\varepsilon^{-2})$ in the case of two-dimensional problem in infinite depth.

This thesis is devoted to the motion of a free surface of a two-dimensional fluid over a variable bottom. There is a large literature about the effect of a variable bottom on free surface waves [6–8, 22]. The usual mathematical assumptions is that the bottom variation are either periodic or describing by a random process. Here, we assume that the bottom variation $b(x)$ is a periodic function of x. The goal is to study the modulation regime in this case and derive an NLS equation for the amplitude of the wavepacket. In particular, we precisely calculate how the coefficients on the NLS equation depend on the function $b(x)$.

The organization of this thesis is outlined as follows.

In Chapter 2, we formally describe the mathematical formulation of the water wave problem. We start by introducing the Euler equations governing an incompressible, inviscid, and irrotational fluid in the domain $\mathbf{S}(\eta, b)$. From the Hamiltonian system of water waves equation, we define the

DNO $G[\eta, b]$ and derive the Zakharov/Craig-Sulem formulation (1.4), which is expressed in terms of the canonical variables η and ξ. We end this chapter by introducing the linearized water wave equations. In solving these linearized equations in the presence of a periodic bottom, we replace the monochromatic harmonic waves (with the dispersion relation (1.11))

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{i\omega(k)}{g} \\ 1 \end{pmatrix} u\, e^{i(kx-\omega(k)t)} + c.c. + \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \tag{1.13}$$

by the Bloch-Floquet plane waves of the form

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{i\Omega_n(\theta)}{g} \\ 1 \end{pmatrix} u\, e^{-i\Omega_n(\theta)t} \phi_n(x,\theta) + c.c. + \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \tag{1.14}$$

where $\Omega_n(\theta) = \sqrt{g\Lambda_n(\theta)}$. The eigenvalue $\Lambda_n(\theta)$ and eigenfunction $\phi_n(x,\theta)$ arise from the Bloch-Floquet spectral problem of the DNO $G[b] := G[0,b]$ with θ-periodic boundary conditions

$$\begin{cases} G[b]\phi(x,\theta) = \Lambda(\theta)\phi(x,\theta), \\ \phi(x+2\pi,\theta) = \phi(x,\theta)e^{2\pi i\theta}. \end{cases} \tag{1.15}$$

In Chapter 3, we focus on the Dirichlet-Neumann operator, which plays a central role in our modulation analysis of the water wave problem. We provide an overview of fundamental properties of the DNO used in this thesis, including the expressions of $G[b]$ and the Taylor expansions of $G[\eta, b]$ in powers of surface elevation η, as initially derived by Craig and Sulem [11] in the case of a flat bottom problem, and later extended to the case of a rough bottom problem by Craig, Guyenne, Nicholls and Sulem in [6] . Additionally, we review the Bloch-Floquet Theory for $G[b]$ developed in [1, 5, 20] for small bottom variations that are useful for our problem. Since $G[b]$ can be seen as a nonlocal pseudo-differential operator with periodic coefficients, the Bloch-Floquet spectral decomposition was introduced to describe its spectrum. According to the references mentioned above and under the assumption of small bottom variations, the spectrum of $G[b]$ is composed of bands separated by gaps, which arise due to the presence of a periodic bottom.

Chapter 4 is devoted to the case of a flat bottom problem, we present a formal modulation expansion of the water wave problem and derive the NLS equation as an envelope equation, serving as a review of previous work [13]. In contrast, Chapters 5 and 6 focus on the problem over a periodic bottom, representing the contribution of my study. In the weakly nonlinear modulation regime, the amplitude of surface deformations η is assumed to be small. Starting from the solutions to the linearized water wave equations, (1.13) or (1.14), the theory of modulation involves replacing the constants u (amplitude) and φ (mean potential) with slowly varying functions depending on variables $X = \varepsilon x$ and $T = \varepsilon t$. The multiple-scale analysis is introduced to avoid resonant terms arising from the cumulative effects of weak nonlinearities over long time intervals or large spatial distances. A crucial step in this analysis involves exploring the multiple-scale expansions of the

DNO when it acts on multiple-scale functions. In the flat bottom problem, the expansion of the DNO $G[0] = D \tanh hD$ acting on multiple-scale monochromatic harmonic waves is derived in [12]. However, in the case of a periodic bottom problem, it becomes more challenging because we lack explicit formulas for the Bloch-Floquet plane waves defined by Bloch-Floquet spectral problem (1.15), and these waves also depend on the Bloch-Floquet parameter θ. Therefore, in Chapter 5, we perform a detailed examination of the actions of $G[b]$ on multiple-scale functions of three various forms. Based on this preparatory work, we examine the solvability conditions at order ε^2 and ε^3 respectively in Chapter 6, and derive the cubic NLS equation for the envelope amplitude. The presence of a periodically varying bottom modifies the coefficients of the NLS equation. In particular, the coefficient of the nonlinear term becomes quite complicated.

In Chapter 7, to gain a better understanding of the NLS obtained in the periodic bottom problem, we perform perturbative calculations of the eigenvalues and eigenfunctions associated with the Bloch-Floquet spectral problem of $G[b]$ for $b(x) = \gamma\beta(x)$, where γ is assumed to be small and independent with the nonlinearity parameter ε. These calculations show in particular how the coefficient of the dispersion term in the NLS equation is affected by the presence of a small periodic bottom.

Chapter 8 is devoted to concluding remarks and open questions.

Chapter 2

Mathematical Formulation of the Problem

2.1 Free Surface Euler Equations

We begin by providing a mathematical description of the motion of two-dimensional surface gravity waves. The time-dependent fluid domain $\mathbf{S}(\eta, b)$, along with its boundaries Γ and Γ_b, is described in (1.1) and (1.2) with $d = 2$ (see Figure 1.1). The elevation of the free surface is denoted by $\eta(x,t)$, with the quiescent water level set at $y = 0$. The variable bottom is $\tilde{b}(x) = -h + b(x)$, where h denotes the finite average depth of water and $b(x)$ represents the bottom perturbation around $y = -h$. We assume, without loss of generality, that

$$\int_0^{2\pi} b(x)dx = 0. \tag{2.1.1}$$

The bottom perturbation $b(x)$ is assumed to be periodic with a period of 2π in the horizontal direction x. Specifically, we require $b(x) \in C^2(\mathbb{T}^1)$, where $\mathbb{T}^1$ is the periodized interval $[0, 2\pi)$.

To describe the motion of the fluid, we introduce the following physical assumptions:

- **(i)** The fluid is inviscid and incompressible.
- **(ii)** The flow is irrotational.
- **(iii)** There is no surface tension, and the pressure at Γ is equal to the air pressure $\mathscr{P}_{air}$. (Surface tension can be incorporated, but it is neglected in our study.)
- **(iv)** The fluid has constant density ρ.
- **(v)** The water bottom always remains below the water surface, that is, $\eta(x,t) + h - b(x) \geq h_0 > 0$ for some positive constant h_0.

Let $\mathbf{U}(x,y,t) \in \mathbb{R}^2$ be the velocity field of the fluid, and $\mathscr{P}(x,y,t) \in \mathbb{R}$ be the fluid pressure in $\mathbf{S}(\eta, b)$ at time t. The gravitational acceleration is denoted by g.

By the conservation of mass and momentum, the motion of an incompressible and inviscid flow is governed by the Euler equations

$$\nabla \cdot \mathbf{U} = 0 \quad \text{in} \quad \mathbf{S}(\eta, b), \tag{2.1.2}$$

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho}\nabla \mathscr{P} - g\mathbf{e_y} \quad \text{in} \quad \mathbf{S}(\eta, b), \tag{2.1.3}$$

where spatial gradient $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$ and vertical unit vector $\mathbf{e_y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In addition, the assumption of irrotational flow implies

$$\nabla \times \mathbf{U} = 0 \quad \text{in} \quad \mathbf{S}(\eta, b). \tag{2.1.4}$$

On the free surface Γ, the boundary conditions are given by the kinematic condition

$$\partial_t \eta = \sqrt{1 + |\partial_x \eta|^2}\,\mathbf{U} \cdot \mathbf{n} \quad \text{on} \quad \Gamma, \tag{2.1.5}$$

and the dynamic condition (in the absence of surface tension)

$$\mathscr{P} = \mathscr{P}_{air} \quad \text{on} \quad \Gamma, \tag{2.1.6}$$

where $\mathbf{n}$ is the unit exterior normal vector to the upper boundary Γ.

On the lower boundary Γ_b, the Euler boundary condition is

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_b \tag{2.1.7}$$

with the unit exterior normal vector $\mathbf{n}$ to the lower boundary Γ_b.

Equation (2.1.4) implies that we can define the velocity potential $\Phi(x, y, t) \in \mathbb{R}$ such that

$$\mathbf{U} = \nabla \Phi. \tag{2.1.8}$$

We can reformulate the Euler questions and boundary conditions in terms of Φ. From (2.1.2), the motion of fluid is described by Laplace equation

$$\Delta \Phi = 0 \quad \text{in} \quad \mathbf{S}(\eta, b). \tag{2.1.9}$$

We can rewrite (2.1.3) as

$$\nabla \Big(\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + \frac{1}{\rho}\mathscr{P} + gy \Big) = 0, \tag{2.1.10}$$

which implies

$$\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + gy = -\frac{1}{\rho}\mathscr{P} + f(t). \tag{2.1.11}$$

When evaluating (2.1.11) on free surface Γ, $\frac{1}{\rho}\mathscr{P}$ is a constant according to the dynamic boundary condition (2.1.6). Since $f(t)$ is an arbitrary function of t, without loss of generality, we can choose $f(t)$ such that the right-hand side of (2.1.11) is zero (when evaluating on Γ). Hence, the boundary condition (2.1.6) on Γ can be rewritten as

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0, \tag{2.1.12}$$

which is also known as Bernoulli condition. Furthermore, the boundary conditions (2.1.5) and (2.1.7) can also be rewritten in terms of Φ.

In summary, we obtain the following boundary value problem, known as the potential flow formulation of the Euler equations:

$$\Delta\Phi = 0 \quad \text{in} \quad \mathbf{S}(\eta, b). \tag{2.1.13}$$

On the bottom Γ_b, Φ obeys the Neumann boundary condition

$$\partial_{\mathbf{n}}\Phi = 0 \quad \text{on} \quad \Gamma_b, \tag{2.1.14}$$

where $\vec{\mathbf{n}}$ is the unit exterior normal vector and $\partial_{\mathbf{n}}\Phi$ at bottom is given by

$$\partial_{\mathbf{n}}\Phi\big|_{y=-h+b(x)} = \frac{1}{\sqrt{1+|\partial_x b(x)|^2}} \begin{pmatrix} \partial_x b(x) \\ -1 \end{pmatrix} \cdot \nabla\Phi.$$

The boundary conditions on free surface Γ are Bernoulli and kinematic conditions, namely

$$\begin{aligned} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta &= 0 \\ \partial_t \eta + \partial_x \eta \, \partial_x \Phi - \partial_y \Phi &= 0 \end{aligned} \quad \text{on} \quad \Gamma. \tag{2.1.15}$$

2.2 Hamiltonian System and Zakharov/Craig-Sulem Formulation

Another form of the water wave system is given by the Zakharov/Craig-Sulem formulation. This formulation will be used throughout our work because it has the advantage of evaluating all unknowns at the free surface only, making it well-suited for studying the asymptotic dynamics of water waves. Let us review the derivation of this formulation in detail.

We define $\xi(x,t) := \Phi(x, \eta(x,t), t)$ as the trace of the velocity potential on the free surface. Since the boundary value problem for Laplace's equation has a unique solution, the fluid flow is fully determined by quantities $\eta(x,t)$ and $\xi(x,t)$, which are defined on the free surface. In [27], Zakharov expressed the system (2.1.13)–(2.1.15) in the form of a Hamiltonian system, where the Hamiltonian function H of the canonical variables (η, ξ) represents the total energy

$$H = \int_{\mathbb{R}} \int_{-h+b(x)}^{\eta(x)} \frac{1}{2}|\nabla\Phi|^2 dy dx + \int_{\mathbb{R}} \frac{1}{2} g\eta^2(x) dx. \tag{2.2.1}$$

The first and second terms represent kinetic and potential energies, respectively.

Craig and Sulem [11] proposed a formulation involving the Dirichlet-Neumann Operator (DNO) $G[\eta, b]$ defined as follows:

$$G[\eta, b]\xi = \sqrt{1+|\partial_x \eta|^2}\partial_n \Phi\big|_{y=\eta}, \tag{2.2.2}$$

where Φ is the solution of the boundary value problem

$$\begin{cases} \Delta\Phi = 0 \quad \text{in} \quad \mathbf{S}(\eta, b), \\ \Phi|_{y=\eta} = \xi, \quad \partial_{\mathbf{n}}\Phi|_{y=-h+b(x)} = 0. \end{cases} \tag{2.2.3}$$

The normal derivative of Φ in (2.2.2) is given by

$$\partial_{\mathbf{n}}\Phi\big|_{y=\eta} = \frac{1}{\sqrt{1+|\partial_x \eta|^2}} \begin{pmatrix} -\partial_x \eta \\ 1 \end{pmatrix} \cdot \nabla\Phi(x, \eta(x), t). \tag{2.2.4}$$

The operator $G[\eta, b]$ is a linear map that associates ξ to the normal derivative $\partial_n \Phi$ on the free surface, multiplied by $\sqrt{1+|\partial_x \eta|^2}$. This non-local operator $G[\eta, b]$ nonlinearly depends on both η and b.

An application of Green's identity allows us to express the Hamiltonian (2.2.1) in terms of the canonical variables

$$H = \frac{1}{2}\int_{\mathbb{R}} \xi(x) G[\eta, b]\xi(x) + g\eta^2(x)dx. \tag{2.2.5}$$

The Euler's equations for water waves take the following representation

$$\begin{pmatrix} \partial_t \eta \\ \partial_t \xi \end{pmatrix} = \begin{pmatrix} \delta_\xi H \\ -\delta_\eta H \end{pmatrix}, \tag{2.2.6}$$

where the right-hand sides are variational derivatives of the functional H.

On the other hand, $\partial_t \eta$ and $\partial_t \xi$ can be expressed in terms of canonical variables η, ξ and the DNO $G[\eta, b]$. On the free surface, we apply chain rule to $\xi(x,t) = \Phi(x, \eta(x,t), t)$ to compute

$$\begin{cases} \partial_t \Phi = \partial_t \xi - \partial_t \eta\, \partial_y \Phi, \\ \partial_x \Phi = \partial_x \xi - \partial_x \eta\, \partial_y \Phi. \end{cases} \tag{2.2.7}$$

We additionally have on the free surface

$$\partial_y \Phi = \frac{G[\eta, b]\xi + \partial_x \eta\, \partial_x \xi}{1+|\partial_x \eta|^2}. \tag{2.2.8}$$

Combining equations (2.2.2), (2.2.4), (2.2.7) and (2.2.8), the boundary conditions (2.1.15) can be rewritten as

$$\begin{cases} \partial_t \eta = G[\eta, b]\xi, \\ \partial_t \xi = -g\eta - \frac{1}{2}(\partial_x \xi)^2 + \frac{1}{2(1+(\partial_x \eta)^2)} \Big(G[\eta, b]\xi + \partial_x \eta \, \partial_x \xi \Big)^2, \end{cases} \tag{2.2.9}$$

which is known as the Zakharov/Craig-Sulem formulation of the water wave problem. The right-hand side of (2.2.9) identifies to the variational derivative of H with respect to ξ and η in (2.2.6).

2.3 Linearized Water Wave Problem

When studying the modulation of weakly nonlinear surface waves, the surface deformations $\eta(x,t)$ are assumed to be small. It becomes crucial to analyze the linearized water wave equations near a surface at rest. Neglecting the nonlinear terms in (2.2.9), we derive the linearized water wave equations around the stationary solution $(\eta, \xi) = (0,0)$, given by

$$\begin{pmatrix} \partial_t & -G[b] \\ g & \partial_t \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.3.1}$$

When $\eta = 0$, we denote $G[0,b]$ by $G[b]$, which associates to the domain (see Figure 2.1)

$$\mathbf{S}(0,b) = \{(x,y) : x \in \mathbb{R}, -h + b(x) < y < 0\}. \tag{2.3.2}$$

The operator $G[b]$ is defined as

$$G[b]\xi = \partial_y \Phi\big|_{y=0}, \tag{2.3.3}$$

where Φ is the solution to the following boundary value problem

$$\begin{cases} \Delta \Phi = 0 \quad \text{in} \quad \mathbf{S}(0,b), \\ \Phi|_{y=0} = \xi, \quad \partial_{\mathbf{n}} \Phi|_{y=-h+b(x)} = 0. \end{cases} \tag{2.3.4}$$

Eliminating the variable ξ, the linearized system (2.3.1) can be simplified to a second-order evolution equation

$$\partial_{tt} \eta + g \, G[b] \eta = 0. \tag{2.3.5}$$

Equation (2.3.5) is analogous to the wave equation if we replace the operator $G[b]$ with the Laplacian. Thus, it can be solved using the method of separation of variables.

2.3.1 Linearized Problem over Flat Bottom

In the linearized water wave problem over a flat bottom $b(x) = 0$, the operator $G[0]$ can be explicitly expressed in terms of Fourier multiplier notation.

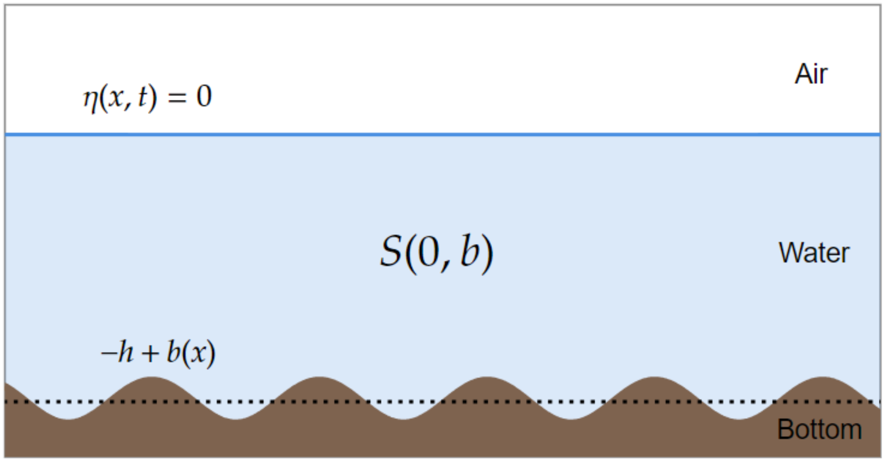

Figure 2.1: The fluid domain $\mathbf{S}(0,b)$

Definition. *(Fourier Multiplier). The Fourier multiplier $M(D): L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by altering its Fourier transform*

$$\widehat{M(D)f}(k) = M(k)\widehat{f}(k). \tag{2.3.6}$$

Using inverse Fourier transform, we have

$$M(D)f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}M(k)\widehat{f}(k)dk = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ik(x-x')}M(k)f(x')dx'dk. \tag{2.3.7}$$

Proposition 2.1. *When the bottom is flat, the operator $G[0]$ has a Fourier multiplier expression*

$$G[0] = D\tanh(hD) \tag{2.3.8}$$

with $D = -i\partial_x$.

Using (2.3.8), we can rewrite equation (2.3.5) as

$$\partial_{tt}\eta + gD\tanh(hD)\eta = 0, \tag{2.3.9}$$

which admits a traveling wave solution of the form $e^{i(kx-\omega t)}$. That is

$$\eta = ue^{i(kx-\omega t)} + c.c., \tag{2.3.10}$$

where u is an arbitrary constant and c.c. denotes the complex conjugate of the preceding terms. The wave number k is related to frequency ω by the dispersion relation for gravity waves on water of finite depth

$$\omega^2(k) = gk\tanh(hk). \tag{2.3.11}$$

Furthermore, we find that the linearized system (2.3.1) admits a solution in the form of monochromatic waves

$$\begin{cases} \eta(x,t) = \dfrac{i\omega(k)}{g} u\, e^{i(kx-\omega(k)t)} + c.c. \\ \xi(x,t) = u\, e^{i(kx-\omega(k)t)} + c.c. + \varphi. \end{cases} \tag{2.3.12}$$

The (complex) amplitude u and (real) mean potential φ are arbitrary constants. This solution describes a harmonic wave of wave number k which propagates in the direction of the positive x-axis at a phase speed of $\dfrac{\omega(k)}{k}$.

Due to the quadratic nonlinearities in the water wave problem, the mean velocity potential φ is incorporated into the solution (2.3.12). We introduce this quantity to balance non-oscillating resonant terms arising in the modulation of the water wave problem.

2.3.2 Linearized Problem over Periodic Bottom

In the water wave problem over a periodic bottom, $G[b]$ is a nonlocal operator depending on a 2π periodic function $b(x) \in C^2(\mathbb{T}^1)$. To solve the linearized water waves equations, we seek solutions of the form $e^{-i\omega t}\phi(x)$, which leads to the spectral problem:

$$G[b]\phi(x) = \lambda\phi(x) \tag{2.3.13}$$

with the dispersion relation $\omega^2 = g\lambda$. To solve it, we introduce the Bloch-Floquet theory, which is a classical tool used to study wave propagation in periodic media.

Definition. *(Bloch-Floquet Transform). The Bloch-Floquet transform defined by*

$$f(x) \mapsto f_\theta(x) := \sum_{n\in\mathbb{Z}} e^{-2\pi i\theta n} f(x+2\pi n) \tag{2.3.14}$$

is well-defined for $f \in \mathscr{S}(\mathbb{R})$, which is the Schwartz space, and can be uniquely extended to a unitary operator on $L^2(\mathbb{R})$.

This definition introduces a parameter $\theta \in [-\frac{1}{2}, \frac{1}{2})$, which is known as the Bloch-Floquet parameter. We say f_θ is θ-periodic in x, which means it satisfies

$$f_\theta(x+2\pi) = e^{2\pi i\theta} f_\theta(x). \tag{2.3.15}$$

In addition, we have

$$f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_\theta(x)\, d\theta. \tag{2.3.16}$$

For more details, we refer to Section XIII.16 in [23].

Therefore, using the Bloch-Floquet transform, any function $f \in L^2(\mathbb{R})$ can be decomposed as an integral of a θ-periodic function. This θ-periodic function is uniquely determined by $f_\theta(x)$. Based on this, the spectral problem (2.3.13) can be analyzed using the Bloch-Floquet decomposition. The idea of this spectral decomposition is to parameterize a continuous spectrum and generalized eigenfunctions of $G[b]$ on $L^2(\mathbb{R})$ by a family of spectral problems for $G[b]_\theta$ in the interval $[0,2\pi)$, with periodic boundary conditions.

To this end, we consider the following eigenvalue problem

$$G[b]\phi(x,\theta) = \Lambda(\theta)\phi(x,\theta) \tag{2.3.17}$$

with θ-periodic boundary condition

$$\phi(x+2\pi,\theta) = e^{2\pi i\theta}\phi(x,\theta). \tag{2.3.18}$$

The principle of Bloch-Floquet theory suggests introducing the function

$$\psi(x,\theta) := e^{-i\theta x}\phi(x,\theta) \tag{2.3.19}$$

and the operator

$$G[b]_\theta := e^{-i\theta x}G[b]e^{i\theta x}. \tag{2.3.20}$$

Given any θ-periodic function $\phi(x,\theta)$, $\psi(x,\theta)$ defined in (2.3.19) is periodic in x with a period of 2π. Then the original problem (2.3.17)-(2.3.18) can be transformed to a spectral problem with periodic boundary condition

$$\begin{cases} G[b]_\theta\psi(x,\theta) = \Lambda(\theta)\psi(x,\theta), \\ \psi(x+2\pi,\theta) = \psi(x,\theta). \end{cases} \tag{2.3.21}$$

Applying the Bloch-Floquet theory of $G[b]$ as developed in [5, 10, 20], we know that for each fixed $\theta \in [-\frac{1}{2},\frac{1}{2})$, there exists a sequence of pairs $\big(\Lambda_n(\theta),\psi_n(x,\theta)\big)$ satisfying (2.3.21), where $\{\Lambda_n(\theta)\}$ are non-negative real eigenvalues and $\{\psi_n(x,\theta) = e^{-i\theta x}\phi_n(x,\theta)\}$ are the corresponding eigenfunctions. Moreover, $\{\psi_n(x,\theta)\}$ form an orthonormal basis of $H^1(\mathbb{T}^1)$, and each eigenvalue has finite multiplicity with

$$\Lambda_0(\theta) \le \Lambda_1(\theta) \le \Lambda_2(\theta) \le \cdots.$$

More details about the Bloch-Floquet theory will be discussed in Chapter 3.2.

Proposition 2.2. *The linearized system* $(2.3.1)$ *admits a solution in the form of Bloch-Floquet waves*

$$\begin{cases} \eta(x,t,\theta) = \dfrac{i\Omega_n(\theta)}{g}\,u\,e^{-i\Omega_n(\theta)t}\phi_n(x,\theta) + c.c., \\ \xi(x,t,\theta) = u\,e^{-i\Omega_n(\theta)t}\phi_n(x,\theta) + c.c. + \varphi, \end{cases} \tag{2.3.22}$$

where $\phi_n(x,\theta)$ is the normalized Bloch-Floquet eigenfunction with the corresponding Bloch-Floquet eigenvalue $\Lambda_n(\theta)$ satisfying (2.3.17) and (2.3.18), and the frequency $\Omega_n(\theta)$ satisfies the dispersion relation

$$\Omega_n^2(\theta) = g\Lambda_n(\theta). \tag{2.3.23}$$

The complex amplitude u and real mean potential φ are arbitrary constants.

Equivalently, (2.3.22) can be rewritten in terms of $\psi_n(x,\theta) = e^{-i\theta x}\phi_n(x,\theta)$ as

$$\begin{cases} \eta(x,t,\theta) = \dfrac{i\Omega_n(\theta)}{g}\, u\, e^{-(i\Omega_n(\theta)t+\theta x)}\psi_n(x,\theta) + c.c., \\ \xi(x,t,\theta) = u\, e^{-(i\Omega_n(\theta)t+\theta x)}\psi_n(x,\theta) + c.c. + \varphi. \end{cases} \tag{2.3.24}$$

Chapter 3

The Dirichlet–Neumann Operator (DNO)

3.1 Taylor Expansion of DNO in Powers of Surface Elevation

As the DNO plays a central role in the modulation analysis of the water wave problem, we review some fundamental properties in this chapter, which will be used in our subsequent work. For a more comprehensive presentation of the DNO, we refer to Chapter 3 in [19].

Property 3.1. *Given $\eta(x)$ and $b(x)$ in $C^1(\mathbb{R})$, $G[\eta, b] : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is a continuous operator and satisfies:*

1. *It is symmetric for the L^2-scalar product, and has a self-adjoint realization in $H^1(\mathbb{R})$.*

2. *It is positive semi-definite.*

3. *It depends analytically upon $\eta, b \in B_{R_0}(0) \subset C^1(\mathbb{R})$ for some positive value of R_0.*

See Chapter 3 and Appendix A2 in [19] for the first two properties in Property 3.1.

Recalling Proposition 2.1, $G[0] = D\tanh(hD)$ is a Fourier multiplier defined in a flat bottom problem. A similar result for pseudo-differential operator $G[b]$ in variable bottom problem was proved by Craig, Guyenne, Nicholls and Sulem [6].

Proposition 3.1. *[6] For $b(x) \in C^1(\mathbb{R})$, the operator $G[b]$, defined in (2.3.3), can be expressed as*

$$G[b] = G[0] + DL[b], \tag{3.1.1}$$

where $DL[b]$ is the correction term arising from the presence of bottom topography. The operator $L[b]$ can be implicitly expressed as

$$L[b] = -B[b]A[b] = -C[b]^{-1}A[b]. \tag{3.1.2}$$

Recalling that $D = -i\partial_x$ and $\tilde{b}(x) = -h + b(x)$, the operators $A[b]$ and $C[b]$ are defined as follows:

$$A[b]\xi(x) = \sinh\big(b(x)D\big)\operatorname{sech}(hD)\xi(x) := \int_{\mathbb{R}} e^{ikx}\sinh\big(b(x)k\big)\operatorname{sech}(hk)\hat{\xi}(k)dk, \tag{3.1.3}$$

and

$$C[b]\xi(x) = \cosh\big(\tilde{b}(x)D\big)\xi(x) := \int_{\mathbb{R}} e^{ikx}\cosh\big(\tilde{b}(x)k\big)\hat{\xi}(k)dk. \tag{3.1.4}$$

The operator $B[b]$ is defined as the inverse operator of $C[b]$.

This was proved as Proposition 2.1 in [6]. Furthermore, Craig et al. also proved that the inverse operator $B[b]$ is well defined (see Proposition 2.2 in [6]).

We now turn to the expansion of $G[\eta,b]$ for small but arbitrary perturbations $\eta(x)$ of the surface. The third property in Property 3.1 implies the existence of the Taylor expansions of $G[\eta,b]$ in powers of η. That property for the case of a flat bottom $b(x) = 0$ was proved by Coifman and Meyer in [9], and the Taylor expansion in powers of η was first derived by Craig and Sulem [11] in two dimensions. The case of three dimensions was proved in [13]. It was later extended to Lipschitz domains in [16] and precise estimates of radius of convergence of the Taylor series were given in [17]. In the case of a variable bottom, the analyticity of the DNO with respect to the elevation and the bottom was proved in [19] with explicit Taylor expansions given in [6].

It follows that $G[\eta,b]$ can be written in terms of a convergent Taylor expansion

$$G[\eta,b] = \sum_{j=0}^{\infty} G_j[\eta,b], \tag{3.1.5}$$

where $G_j[\eta,b]$ is a pseudo-differential operator homogeneous in η of degree j. In our discussion, we only require the explicit forms of the first three terms in this Taylor expansion.

Proposition 3.2. *The Taylor expansion of $G[\eta,b]$ about zero in powers of η takes the form*

$$G[\eta,b] = G[b] + G_1[\eta,b] + G_2[\eta,b] + O(\eta^3). \tag{3.1.6}$$

Here, $G[b] = D\tanh(hD) + DL(b)$ defined in Proposition 3.1, and the next two terms are given by

$$\begin{aligned} G_1[\eta,b] &= D\eta D - G[b]\eta G[b], \\ G_2[\eta,b] &= -\frac{1}{2}\Big(G[b]\eta^2 D^2 + D^2\eta^2 G[b] - 2G[b]\eta G[b]\eta G[b]\Big). \end{aligned} \tag{3.1.7}$$

The explicit recursion formulas of $G_j[\eta,b]$ for higher orders can be found in [6, 11, 13].

In the above Taylor expansion, $G[\eta,b]$ is not expanded in terms of b, and the operator $L[b]$ only appears in the leading-order term $G[b]$. In [6], Craig et al. also provided a Taylor expansion of $G[\eta,b]$ in powers of both η and b, which relies on a Taylor expansion of $L[b]$ in powers of b. We do not present the Taylor expansion of $G[\eta,b]$ in powers of both η and b here, as it is not used in our work. It can be found in [6] Appendix A. Instead, we are interested in the Taylor expansion of $L[b]$

in powers of $b(x)$, particularly focusing on its first three terms, which will be used in a perturbative calculation in Chapter 7.

Proposition 3.3. *[6] The Taylor expansion of $L[b]$ in powers of b takes the form*

$$L[b] = L_0 + L_1[b] + L_2[b] + O(b^3), \tag{3.1.8}$$

where $L_i[b]$ are homogeneous of degree i in powers of b. Furthermore, we have

$$\begin{aligned} L_0 &= 0, \\ L_1[b] &= -\operatorname{sech}(hD)\, b(x)\, D \operatorname{sech}(hD), \\ L_2[b] &= -\operatorname{sech}(hD)\, b(x)\, D \tanh(hD)\, b(x)\, D \operatorname{sech}(hD). \end{aligned} \tag{3.1.9}$$

3.2 Bloch-Floquet Theory

In Chapter 2.3.2, to address the linearized problem over a periodic bottom, we introduce the Bloch-Floquet eigenvalue problem of $G[b]$ with θ-periodic boundary condition

$$\begin{cases} G[b]\phi(x,\theta) = \Lambda(\theta)\phi(x,\theta), \\ \phi(x+2\pi,\theta) = e^{2\pi i\theta}\phi(x,\theta), \end{cases} \tag{3.2.1}$$

where the Bloch-Floquet eigenvalue $\Lambda(\theta)$ is real and the Bloch-Floquet eigenfunction $\phi(x,\theta)$ is θ-periodic.

Using the operator $G[b]_\theta$ in (2.3.20) and 2π periodic function $\psi(x,\theta)$ in (2.3.19), the Bloch-Floquet eigenvalue problem (3.2.1) can be rewritten as an eigenvalue problem of $G[b]_\theta$ with periodic boundary condition

$$\begin{cases} G[b]_\theta \psi(x,\theta) = \Lambda(\theta)\psi(x,\theta), \\ \psi(x+2\pi,\theta) = \psi(x,\theta). \end{cases} \tag{3.2.2}$$

In addition, we have

$$G[b](e^{i\theta x}\psi_n(x,\theta)) = e^{i\theta x}G[b]_\theta \psi_n(x,\theta) = e^{i\theta x}\Lambda_n(\theta)\psi_n(x,\theta). \tag{3.2.3}$$

We summarize some basic properties of $G[b]_\theta$ from [5] as follows.

Proposition 3.4. *[5] For any $\theta \in [-\frac{1}{2}, \frac{1}{2})$, $G[b]_\theta$ takes 2π periodic functions into 2π periodic functions. In particular, the operator $G[b]_\theta$ from $H^1(\mathbb{T}^1)$ to $L^2(\mathbb{T}^1)$ is symmetric for L^2-scalar product with periodic boundary conditions.*

Correspondingly, $G[b]$ preserves the class of θ-periodic functions. That means for a θ-periodic function $\xi(x)$ such that $e^{-i\theta x}\xi(x) \in H^1(\mathbb{T}^1)$, $G[b]\xi(x)$ is also a θ-periodic function, that is,

$$(G[b]\xi)(x+2\pi) = e^{2\pi i\theta}G[b]\xi(x). \tag{3.2.4}$$

We recall that $G[b] = G[0] + DL[b]$ with $L[b] = -B[b]A[b]$ defined in Proposition 3.1. These operators are also well defined for periodic, and more generally on θ-periodic functions. Therefore,

$$G[b]_\theta = G[0]_\theta + e^{-i\theta x} DL[b] e^{i\theta x} = G[0]_\theta - DB[b]_\theta A[b]_\theta, \tag{3.2.5}$$

where we define

$$\begin{aligned} G[0]_\theta &= e^{-i\theta x} G[0] e^{i\theta x}, \\ A[b]_\theta &= e^{-i\theta x} A[b] e^{i\theta x}, \\ DB[b]_\theta &= e^{-i\theta x} DB[b] e^{i\theta x}. \end{aligned} \tag{3.2.6}$$

These operators $G[0]_\theta, A[b]_\theta$ and $DB[b]_\theta$ map 2π periodic functions to 2π periodic functions. In particular,

$$G[0]_\theta e^{ikx} = (k+\theta)\tanh(h(k+\theta))e^{ikx}. \tag{3.2.7}$$

Proposition 3.5. *[5] The spectrum of $G[b]_\theta$ on the domain $H^1(\mathbb{T}^1) \subset L^2(\mathbb{T}^1)$ consists of a nondecreasing sequence of real eigenvalues*

$$0 \le \Lambda_0(\theta) \le \cdots \le \Lambda_n(\theta) \le \Lambda_{n+1}(\theta) \le \cdots, \tag{3.2.8}$$

which tend to infinity as n tends to infinity. The eigenvalues are continuous and periodic in θ of period 1. The corresponding eigenfunctions $\psi_n(x,\theta)$ of

$$G[b]_\theta \psi_n(x,\theta) = \Lambda_n(\theta)\psi_n(x,\theta) \tag{3.2.9}$$

are normalized 2π periodic in x and periodic in θ of period 1.

For any $\theta \in [-\frac{1}{2}, \frac{1}{2})$, $\{\psi_n(x,\theta)\}_n$ forms an orthonormal basis of $H^1(\mathbb{T}^1)$. Hence,

$$\langle \psi_n(x,\theta), \psi_k(x,\theta)\rangle := \int_0^{2\pi} \psi_n(x,\theta)\overline{\psi_k(x,\theta)}dx = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \tag{3.2.10}$$

The corresponding solutions $\phi_n(x,\theta)$ of $G[b]\phi_n(x,\theta) = \Lambda_n(\theta)\phi_n(x,\theta)$ are θ-periodic. In the case $\Lambda_{n-1}(\theta) < \Lambda_n(\theta) < \Lambda_{n+1}(\theta)$, the eigenvalue $\Lambda_n(\theta)$ is simple, and $\Lambda_n(\theta)$ and eigenfunctions $\phi_n(x,\theta)$ are locally analytic in θ.

We also notice that the ground state $\Lambda_0(\theta)$ satisfies $\Lambda_0(0) = 0$ for any $b(x)$, and its corresponding eigenfunction is $\phi_0(x,0) = \psi_0(x,0) = \dfrac{1}{\sqrt{2\pi}}$.

When the bottom is flat, the Bloch eigenvalues $\Lambda_n^{(0)}(\theta)$, where the superscript indicates flat bottom $b(x) = 0$, are given explicitly in terms of the classical dispersion relation for water waves over a constant depth:

$$\Lambda_n^{(0)}(\theta) = \omega^2(n+\theta) = (n+\theta)\tanh(h(n+\theta)). \tag{3.2.11}$$

We know $\Lambda_n^{(0)}(\theta)$ are simple when $-1/2<\theta<0$ and $0<\theta<1/2$; while eigenvalues have multiplicity two when $\theta=-\dfrac{1}{2},0,\dfrac{1}{2}$. Denoting

$$g_n(\theta)=(n+\theta)\tanh(h(n+\theta)), \tag{3.2.12}$$

we can relabel the eigenvalues $\Lambda_n^{(0)}(\theta)$ in order of increasing magnitude as following (see Figure 3.1):

$$\begin{cases}\Lambda_{2n}^{(0)}(\theta)=g_{-n}(\theta), & \psi_{2n}^{(0)}(x,\theta)=e^{-inx}, \qquad \text{for } -\frac{1}{2}\le\theta<0;\\ \Lambda_{2n}^{(0)}(\theta)=g_{n}(\theta), & \psi_{2n}^{(0)}(x,\theta)=e^{inx}, \qquad \text{for } 0\le\theta<\frac{1}{2};\end{cases} \tag{3.2.13}$$

and

$$\begin{cases}\Lambda_{2n-1}^{(0)}(\theta)=g_{n}(\theta), & \psi_{2n-1}^{(0)}(x,\theta)=e^{inx}, \qquad \text{for } -\frac{1}{2}\le\theta<0;\\ \Lambda_{2n-1}^{(0)}(\theta)=g_{-n}(\theta), & \psi_{2n-1}^{(0)}(x,\theta)=e^{-inx}, \qquad \text{for } 0\le\theta<\frac{1}{2}.\end{cases} \tag{3.2.14}$$

With this definition, both $\Lambda_n^{(0)}$ and $\psi_n^{(0)}$ are periodic in θ with period 1 and $\Lambda_n^{(0)}$ is continuous in θ, while $\psi_n^{(0)}$ has discontinuities at $\theta=-\frac{1}{2},0,\frac{1}{2}$.

The spectrum of $G[b]$ is the union over $-\frac{1}{2}\le\theta<\frac{1}{2}$ of the Bloch eigenvalues $\Lambda_n(\theta)$, that is,

$$\sigma_{L^2(\mathbb{R})}(G[b])=\bigcup_{n=0}^{+\infty}[\Lambda_n^-,\Lambda_n^+], \tag{3.2.15}$$

where $\Lambda_n^-=\min_\theta\Lambda_n(\theta)$ and $\Lambda_n^+=\max_\theta\Lambda_n(\theta)$. Different with the continuous spectrum of $G[0]$, the spectrum of $G[b]$ on the domain $H^1(\mathbb{R})$ is composed of bands and gaps. Due to the presence of the periodic bottom, the double eigenvalues may split, creating a spectral gap near $\theta=-\frac{1}{2},0,\frac{1}{2}$.

As shown in [5], Craig, Gazeau, Lacave and Sulem gave a sufficient condition for the opening of the first N gaps for small enough bottom perturbations $b(x)=\gamma\beta(x)$, where $\beta\in C^1(\mathbb{T}^1)$. Assume $\hat{\beta}_k\neq 0$ for all $k\le N$, where $\hat{\beta}_k$ is the k^{th} Fourier coefficient of $\beta(x)$. Then there exits small enough value $\gamma_0(N)$ such that for $\gamma<\gamma_0(N)$, the first N gaps open and gaps are of the size $O(\gamma)$. Moreover, gaps of smaller order are computed in [5] and [20] with different conditions.

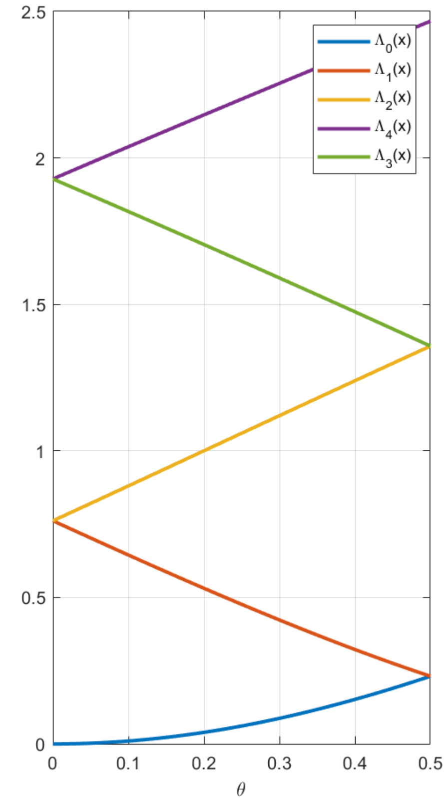

Figure 3.1: Graph of $\Lambda_n(\theta)$

Chapter 4

Construction of Nonlinear Modulated Solutions: Case of Flat Bottom

4.1 Multiple-Scale Expansion

We review the modulation expansion of the water wave problem and derive the NLS equation for the case of a flat bottom (see [24] Chapter 11). Throughout this chapter, the bottom perturbation $b(x)$ is assumed to be zero.

In a weakly nonlinear modulation of the water wave problem, we consider the small amplitude wave solution to (2.2.9), with $G[\eta,b]$ replaced by $G[\eta,0]$ in this chapter, that is

$$\begin{cases} \partial_t \eta - G[\eta,0]\xi = 0, \\ \partial_t \xi + g\eta + \frac{1}{2}(\partial_x \xi)^2 - \frac{1}{2(1+(\partial_x \eta)^2)}\Big(G[\eta,0]\xi + \partial_x \eta\, \partial_x \xi\Big)^2 = 0. \end{cases} \tag{4.1.1}$$

Recalling from Chapter 2.3.1, the linearized equations of (4.1.1) around the resting water state is

$$\begin{cases} \partial_t \eta - G[0]\xi = 0, \\ \partial_t \xi + g\eta = 0, \end{cases} \tag{4.1.2}$$

which possess a real-valued plane wave solution with wave number k, given by:

$$\begin{cases} \eta(x,t) = \dfrac{i\omega(k)}{g} u\, e^{i(kx-\omega(k)t)} + c.c., \\ \xi(x,t) = u\, e^{i(kx-\omega(k)t)} + c.c. + \varphi. \end{cases} \tag{4.1.3}$$

We recall that $G[0] = D\tanh(hD)$ is defined in Proposition 2.1, and the frequency $\omega(k)$ is determined by the dispersion relation

$$\omega^2(k) = gk\tanh(hk). \tag{4.1.4}$$

The constants u and φ represent the amplitude and mean potential of surface waves, respectively.

Assuming the solution has small amplitude, we expand it in the form

$$\begin{cases} \eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \cdots, \\ \xi = \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \cdots, \end{cases} \tag{4.1.5}$$

where ε is a small parameter and the superscripts indicate the orders in ε. The leading order $(\eta^{(1)}, \xi^{(1)})$ identifies with the solution (4.1.3) of the linear problem. To avoid secular terms arising from the accumulation of weakly nonlinear effects over long time spans or large distances, we perform a multiple-scale analysis. (An introduction of multiple-scale analysis applied to an example of nonlinear oscillator ODE problem is provided in [3].)

By introducing a large-scale spatial variable $X = \varepsilon x$ and a slow time $T = \varepsilon t$, we assume that the approximate solution (4.1.5) depends on the fast variables x and t, as well as the slow variables X and T. In the method of multiple-scale analysis, the fast variables and slow variables are treated as independent variables, although they are actually related by ε.

Notation 4.1. *After introducing new variables $X = \varepsilon x$ and $T = \varepsilon t$, the original partial derivatives ∂_x and ∂_t in (4.1.1) are now replaced by $\partial_x + \varepsilon\partial_X$ and $\partial_t + \varepsilon\partial_T$, respectively. Thus, the Fourier multiplier D now represents*

$$D = D_x + \varepsilon D_X \tag{4.1.6}$$

with $D_x = -i\partial_x$ and $D_X = -i\partial_X$.

In addition, we assume the operator $G[\eta, 0]$ acting on multiple-scale function $\xi(x, X)$ has an expansion of the form

$$G[\eta,0]\xi(x,X) = \big(G^{(0)} + \varepsilon G^{(1)} + \varepsilon^2 G^{(2)} + \cdots\big)\xi(x,X). \tag{4.1.7}$$

The superscript of $G^{(i)}$ indicates the orders in ε. In our analysis, we only require the first three terms of the expansion, as shown in the following proposition. For convenience, we omit multiple-scale function ξ.

Proposition 4.1. *The first three contributions in (4.1.7) are provided as*

$$G^{(0)} = D_x \tanh(hD_x), \tag{4.1.8}$$

$$G^{(1)} = \tanh(hD_x)D_X + hD_x\big(1 - \tanh^2(hD_x)\big)D_X + D_x\eta^{(1)}D_x - G^{(0)}\eta^{(1)}G^{(0)}, \tag{4.1.9}$$

and

$$\begin{aligned} G^{(2)} =& h\big(1-\tanh^2(hD_x)\big)D_X^2 - h^2 D_x\big(1-\tanh^2(hD_x)\big)\tanh(hD_x)D_X^2 \\ &+ D_x\eta^{(2)}D_x + D_x\eta^{(1)}D_X + D_X\eta^{(1)}D_x \\ &- \big(G^{(0)}\eta^{(2)}G^{(0)} + G^{(0)}\eta^{(1)}G^{(1)} + G^{(1)}\eta^{(1)}G^{(0)}\big) \\ &- \frac{1}{2}\big(G^{(0)}(\eta^{(1)})^2 D_x^2 + D_x^2(\eta^{(1)})^2 G^{(0)} - 2G^{(0)}\eta^{(1)}G^{(0)}\eta^{(1)}G^{(0)}\big). \end{aligned} \tag{4.1.10}$$

Proof. Firstly, we apply the Theorem 4.1 in [12] to obtain an asymptotic expansion of the operator $D\tanh(hD)$ in the multiple-scale regime:

$$\begin{aligned} D\tanh(hD) =& D_x\tanh(hD_x)+\varepsilon\Big(\tanh(hD_x)+hD_x\big(1-\tanh^2(hD_x)\big)\Big)D_X \\ &+\varepsilon^2\Big(h\big(1-\tanh^2(hD_x)\big)-h^2D_x\big(1-\tanh^2(hD_x)\big)\tanh(hD_x)\Big)D_X^2+O(\varepsilon^3). \end{aligned} \tag{4.1.11}$$

To calculate each $G^{(i)}$, we need to combine three expansions: the first is the Taylor expansion of $G[\eta,0]$ in terms of powers of η from Proposition 3.2 (taking $b(x)=0$); the second is the expansion of η in ε from (4.1.5); and the third is the expansion (4.1.11) derived above. We substitute all three expansions into the left-hand side of (4.1.7). By collecting powers of ε, we can find $G^{(i)}$ in the right-hand side of (4.1.7) for any i. In particular, $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$ yield (4.1.8), (4.1.9) and (4.1.10), respectively. □

4.2 Derivation of the Nonlinear Schrödinger Equation

Substituting (4.1.5) and (4.1.7) into (4.1.1), and rearranging terms according to the orders of ε, (4.1.1) can be expanded as follows:

$$\begin{pmatrix}\partial_t & -G^{(0)}\\ g & \partial_t\end{pmatrix}\left(\varepsilon\begin{pmatrix}\eta^{(1)}\\ \xi^{(1)}\end{pmatrix}+\varepsilon^2\begin{pmatrix}\eta^{(2)}\\ \xi^{(2)}\end{pmatrix}+\varepsilon^3\begin{pmatrix}\eta^{(3)}\\ \xi^{(3)}\end{pmatrix}+\cdots\right)=\varepsilon^2\begin{pmatrix}U^{(2)}\\ V^{(2)}\end{pmatrix}+\varepsilon^3\begin{pmatrix}U^{(3)}\\ V^{(3)}\end{pmatrix}+\cdots, \tag{4.2.1}$$

where $U^{(m)}$ and $V^{(m)}$ include $G^{(i)}$ (for $i<m$), and also the nonlinear terms arising at order ε^m. The expressions of $U^{(m)}$ and $V^{(m)}$ can be explicitly calculated at each order. The cases of $m=2$ and $m=3$ will be presented later as required.

Clearly, at leading order, we recover the linearized system from (4.2.1)

$$\begin{pmatrix}\partial_t & -G^{(0)}\\ g & \partial_t\end{pmatrix}\begin{pmatrix}\eta^{(1)}\\ \xi^{(1)}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{4.2.2}$$

with $G^{(0)}=D_x\tanh(hD_x)$ given in (4.1.8). Referring to (4.1.3), the solution can be written as modulated plane waves of the form

$$\begin{cases}\eta^{(1)}(x,t;X,T)=\dfrac{i\omega}{g}u_1(X,T)\,e^{i(kx-\omega t)}+c.c.,\\ \xi^{(1)}(x,t;X,T)=u_1(X,T)\,e^{i(kx-\omega t)}+c.c.+\varphi(X,T),\end{cases} \tag{4.2.3}$$

where the complex amplitude $u_1(X,T)$ and real mean potential $\varphi(X,T)$, instead of being constants, become functions dependent only on the large-scale variables X and T. This solution describes a slowly modulated monochromatic wave (with wave number k) propagating in the positive x direction. In particular, due to the quadratic nonlinearities in the water wave problem, the mean potential

$\varphi(X,T)$ is incorporated into the solution to balance nonoscillating resonant terms arising at higher order.

At higher order $O(\varepsilon^m)$ with $m>1$, (4.2.1) leads to an inhomogeneous linear system

$$\begin{pmatrix} \partial_t & -G^{(0)} \\ g & \partial_t \end{pmatrix} \begin{pmatrix} \eta^{(m)} \\ \xi^{(m)} \end{pmatrix} = \begin{pmatrix} U^{(m)} \\ V^{(m)} \end{pmatrix}, \tag{4.2.4}$$

which is not always solvable. The solvability conditions of (4.2.4) is that $\begin{pmatrix} U^{(m)} \\ V^{(m)} \end{pmatrix}$ must be orthogonal to the kernel of adjoint operator $\begin{pmatrix} -\partial_t & g \\ -G^{(0)} & -\partial_t \end{pmatrix}$. We know the kernel is spanned by $\begin{pmatrix} 1 \\ \frac{-i\omega}{g} \end{pmatrix} e^{i(kx-\omega t)}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, the solvability conditions can be further clarified as follows:

(S1) There is no term independent of $(kx-\omega t)$ in $U^{(m)}$.

(S2) Denoting the coefficients of $e^{i(kx-\omega t)}$ terms in $U^{(m)}$ and $V^{(m)}$ as $P^{(m)}$ and $Q^{(m)}$ respectively, they satisfy

$$P^{(m)} + \frac{i\omega}{g} Q^{(m)} = 0. \tag{4.2.5}$$

The solvability conditions at order $m=3$ lead to the cubic NLS equation, which governs the dynamics of the modulated wave amplitude. For clarity, we outline the derivation process in three steps.

- **Step 1: Finding the solvability conditions at $O(\varepsilon^2)$.**

At order ε^2, $U^{(2)}$ and $V^{(2)}$ in (4.2.1) can be computed as

$$\begin{aligned} U^{(2)} &= -\eta_T^{(1)} + G^{(1)}\xi^{(1)}, \\ V^{(2)} &= -\xi_T^{(1)} - \frac{1}{2}\big(\xi_x^{(1)}\big)^2 + \frac{1}{2}\big(G^{(0)}\xi^{(1)}\big)^2, \end{aligned} \tag{4.2.6}$$

where $G^{(0)}$ and $G^{(1)}$ are derived in Proposition 4.1.

Proposition 4.2. *At this order, the solvability condition **(S2)** yields*

$$u_{1T}(X,T) + \omega'(k)\, u_{1X}(X,T) = 0. \tag{4.2.7}$$

Moreover, the solution $\eta^{(2)}$ and $\xi^{(2)}$ to (4.2.4) can be expressed in the form

$$\begin{cases} \eta^{(2)} = p_1(X,T)\, e^{i(kx-\omega t)} + p_2(X,T)\, e^{2i(kx-\omega t)} + c.c. + \overline{\eta}(X,T), \\ \xi^{(2)} = q_1(X,T)\, e^{i(kx-\omega t)} + q_2(X,T)\, e^{2i(kx-\omega t)} + c.c.. \end{cases} \tag{4.2.8}$$

Denoting $\sigma := \tanh(hk)$*, the coefficients in (4.2.8) are given by*

$$\begin{aligned} p_1 &= \frac{1}{g}u_{1T} + \frac{\sigma + hk(1-\sigma^2)}{\omega}u_{1X} + \frac{i\omega}{g}u_2, \qquad p_2 = \frac{(\sigma^2-3)k^2}{2g\sigma^2}u_1^2, \\ \overline{\eta} &= \frac{1}{g}\big(k^2(\sigma^2-1)\,|u_1|^2 - \varphi_T\big), \\ q_1 &= u_2, \qquad q_2 = \frac{i\omega k\big(4(1-\sigma^2)+(1+\sigma^2)^2\big)}{8g\sigma^3}u_1^2, \end{aligned} \tag{4.2.9}$$

where $u_2(X,T)$ *is an arbitrary complex function that depends only on the slow variables* X *and* T*.*

Proof. Firstly, we examine the solvability condition **(S2)**. To compute $U^{(2)}$ and $V^{(2)}$, we substitute $\eta^{(1)}$ and $\xi^{(1)}$ given in (4.2.3) into (4.2.6), and then apply the formulas of $G^{(0)}$ and $G^{(1)}$ derived in Proposition 4.1. We find that $U^{(2)}$ and $V^{(2)}$ can be represented in the following form

$$\begin{aligned} U^{(2)} &= P^{(2)}(X,T)\,e^{i(kx-\omega t)} + E^{(2)}(X,T)\,e^{2i(kx-\omega t)} + c.c., \\ V^{(2)} &= Q^{(2)}(X,T)\,e^{i(kx-\omega t)} + F^{(2)}(X,T)\,e^{2i(kx-\omega t)} + c.c. + K^{(2)}(X,T), \end{aligned} \tag{4.2.10}$$

with the coefficients

$$\begin{aligned} P^{(2)} &= -\frac{i\omega}{g}u_{1T} - i\big(\sigma + hk(1-\sigma^2)\big)\,u_{1X}, \qquad E^{(2)} = \frac{2i\omega}{g}k^2\big(1-\sigma\tanh(2hk)\big)\,u_1^2, \\ Q^{(2)} &= -u_{1T}, \qquad F^{(2)} = \frac{1}{2}k^2(1+\sigma^2)\,u_1^2, \qquad K^{(2)} = k^2(\sigma^2-1)\,|u_1|^2 - \varphi_T. \end{aligned} \tag{4.2.11}$$

From (4.2.10) and (4.2.11), the solvability condition **(S1)** is obviously satisfied at this order, while the solvability condition **(S2)** implies

$$P^{(2)} + \frac{i\omega}{g}Q^{(2)} = -\frac{2i\omega}{g}u_{1T} - i\big(\sigma + hk(1-\sigma^2)\big)\,u_{1X} = 0. \tag{4.2.12}$$

Using the dispersion relation (4.1.4), we have

$$\omega'(k) = \frac{g}{2\omega}\big(\sigma + kh(1-\sigma^2)\big), \tag{4.2.13}$$

and then (4.2.12) can be rewritten in terms of the group velocity $\omega'(k)$, which yields

$$u_{1T} + \omega'(k)\,u_{1X} = 0. \tag{4.2.14}$$

Next, we solve the inhomogeneous linear system (4.2.4) and find the solution $\eta^{(2)}$ and $\xi^{(2)}$ using the method of undetermined coefficients. From (4.2.10), we observe that $U^{(2)}$ and $V^{(2)}$ are composed of $e^{i(kx-\omega t)}$, $e^{2i(kx-\omega t)}$ (with their complex conjugates) and $(kx-\omega t)$-independent term. Hence, we assume $\eta^{(2)}$ and $\xi^{(2)}$ have the form provided in (4.2.8) with unknown coefficients p_i, q_i $(1 \le i \le 2)$ and $\overline{\eta}$. To determine these coefficients, we substitute (4.2.8) and (4.2.10) into the inhomogeneous linear system (4.2.4). By identifying the coefficients of various terms on two sides of (4.2.4), we obtain the following equations:

- For the coefficients of $e^{i(kx-\omega t)}$, we have

$$\begin{cases} -i\omega\, p_1 - k\sigma\, q_1 = P^{(2)}, \\ g\, p_1 - i\omega\, q_1 = Q^{(2)}. \end{cases} \tag{4.2.15}$$

- For the coefficients of $e^{2i(kx-\omega t)}$, we have

$$\begin{cases} -2i\omega\, p_2 - 2k\tanh(2hk)\, q_2 = E^{(2)}, \\ g\, p_2 - 2i\omega\, q_2 = F^{(2)}. \end{cases} \tag{4.2.16}$$

- For the term independent of $(kx-\omega t)$, we have

$$g\,\overline{\eta} = K^{(2)}. \tag{4.2.17}$$

Substituting coefficients in (4.2.11) and also using the fact

$$\tanh(2hk) = \frac{2\tanh(hk)}{1+\tanh^2(hk)} = \frac{2\sigma}{1+\sigma^2}, \tag{4.2.18}$$

all above equations can be solved, yielding the results in (4.2.9). □

- **Step 2: Finding the solvability conditions at $O(\varepsilon^3)$.**

At order ε^3, we compute $U^{(3)}$ and $V^{(3)}$ in (4.2.1) as

$$\begin{aligned} U^{(3)} =& -\eta_T^{(2)} + G^{(1)}\xi^{(2)} + G^{(2)}\xi^{(1)}, \\ V^{(3)} =& -\xi_T^{(2)} - \xi_x^{(1)}\xi_X^{(1)} - \xi_x^{(1)}\xi_x^{(2)} + \eta_x^{(1)}\xi_x^{(1)}G^{(0)}\xi^{(1)} \\ &+ \big(G^{(0)}\xi^{(1)}\big)\big(G^{(0)}\xi^{(2)} + G^{(1)}\xi^{(1)}\big), \end{aligned} \tag{4.2.19}$$

where $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$ are defined in Proposition 4.1.

Proposition 4.3. *At order ε^3, the solvability conditions **(S1)** and **(S2)** implies*

$$\overline{\eta}_T + h\varphi_{XX} + \frac{2\omega k}{g}\,|u_1|_X^2 = 0, \tag{4.2.20}$$

and

$$2i\big(u_{2T} + \omega'(k)\,u_{2X}\big) + \omega''(k)\,u_{1XX} = \delta_1\,|u_1|^2 u_1 - \Big(\frac{k^2(1-\sigma^2)}{\omega}\,\varphi_T - 2k\,\varphi_X\Big)u_1, \tag{4.2.21}$$

respectively. We have

$$\delta_1 = \frac{k^4}{2\omega}(-2\sigma^4 + 13\sigma^2 - 12 + 9\sigma^{-2}). \tag{4.2.22}$$

Proof. To examine the solvability conditions, we begin by substituting the expressions of $\eta^{(1)}, \xi^{(1)}$ from (4.2.3), and the expressions of $\eta^{(2)}, \xi^{(2)}$ from (4.2.8) into (4.2.19). Then, using the formulas of $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$ derived in Proposition 4.1, we can compute each term in $U^{(3)}$ and $V^{(3)}$.

- We find that all $(kx-\omega t)$-independent terms in $U^{(3)}$ consist of $(-\overline{\eta}_T)$ from $(-\eta_T^{(2)})$ and $(-h\varphi_{XX} - \frac{2\omega k}{g}\,|u_1|_X^2)$ from $G^{(2)}\xi^{(1)}$. Thus the solvability condition **(S1)** implies

$$-\overline{\eta}_T - h\varphi_{XX} - \frac{2\omega k}{g}\,|u_1|_X^2 = 0, \tag{4.2.23}$$

  which yields (4.2.20).

- To write down the solvability condition **(S2)**, we need to find $P^{(3)}$ and $Q^{(3)}$.

  Collecting all the $e^{i(kx-\omega t)}$ terms in $U^{(3)}$, we obtain the coefficient

$$\begin{aligned} P^{(3)} =& -p_{1T} - i\big(\sigma + hk(1-\sigma^2)\big)q_{1X} - h(1-\sigma^2)(1-hk\sigma)u_{1XX} \\ &+ k^2(1-\sigma^2)\overline{\eta}u_1 - \frac{2i\omega k^2}{g}\big(1-\sigma\tanh(2hk)\big)q_2\overline{u_1} - k^2(1+\sigma^2)p_2\overline{u_1} \\ &+ \frac{\omega k}{g}u_1\varphi_X + \frac{\sigma k^3\omega^2}{g^2}\big(-1+2\sigma\tanh(2hk)|u_1|u_1\big). \end{aligned} \tag{4.2.24}$$

  Similarly, we calculate the coefficient $Q^{(3)}$ in $V^{(3)}$ as

$$\begin{aligned} Q^{(3)} =& -q_{1T} - iku_1\varphi_X - 2k^2\big(1-\sigma\tanh(2hk)\big)q_2\overline{u_1} \\ &+ \frac{ik^3\sigma\omega}{g}\big(1-2\sigma\tanh(2hk)\big)|u_1|u_1. \end{aligned} \tag{4.2.25}$$

  Substituting $P^{(3)}$ and $Q^{(3)}$ into (4.2.5) results in (4.2.21). When computing the coefficients in (4.2.21), we use p_i, q_i from (4.2.9) and the fact from (4.2.18).

□

- **Step 3: Deriving the NLS eqution.**

To merge (4.2.14) at order $O(\varepsilon^2)$ with (4.2.21) at order $O(\varepsilon^3)$, we introduce $u := u_1 + \varepsilon u_2$, thereby obtaining (up to order ε)

$$2i\big(u_T + \omega'(k)u_X\big) + \varepsilon\omega''(k)u_{XX} = \varepsilon\Big(\delta_1|u|^2 - \big(\frac{k^2(1-\sigma^2)}{\omega}\,\varphi_T - 2k\,\varphi_X\big)\Big)u. \tag{4.2.26}$$

Recalling from (4.2.9), we know

$$\overline{\eta} = \frac{1}{g}(k^2(\sigma^2-1)\,|u_1|^2 - \varphi_T). \tag{4.2.27}$$

The quantity φ represents the mean velocity potential at the free surface, which is of order ε; while $\overline{\eta}$ denotes the mean elevation of the free surface, whose magnitude is of order ε^2. The dynamics of φ and $\overline{\eta}$ are governed by equations (4.2.20) and (4.2.27).

Substituting (4.2.27) into (4.2.20), we obtain

$$\left(k^2(\sigma^2-1)\,|u_1|_T^2-\varphi_{TT}\right)+gh\,\varphi_{XX}+2\omega k\,|u_1|_X^2=0. \tag{4.2.28}$$

Combining (4.2.14) and (4.2.28), we derive (up to leading order)

$$\varphi_{TT}-gh\varphi_{XX}=\delta_2|u|_X^2, \tag{4.2.29}$$

where

$$\delta_2=2k\omega(k)+(1-\sigma^2)k^2\omega'(k). \tag{4.2.30}$$

By introduce new variables $\mu=X-\omega'(k)T$ and $\tau=\varepsilon^2 t$, we can rewrite (4.2.26) and (4.2.29) in a reference frame moving at the group velocity $\omega'(k)$ over a longer time. Up to leading order, we derive

$$2iu_\tau+\omega''(k)u_{\mu\mu}=\delta_1|u|^2u+\frac{\delta_2}{\omega(k)}u\varphi_X. \tag{4.2.31}$$

and

$$(\omega'^2(k)-gh)\varphi_{\mu\mu}=\delta_2|u|_\mu^2. \tag{4.2.32}$$

Combining them leads to

$$2iu_\tau+\omega''(k)\,u_{\mu\mu}+\chi|u|^2u=0, \tag{4.2.33}$$

where

$$\chi=-\frac{k^3\omega(k)}{g}H(kh) \tag{4.2.34}$$

and

$$H(kh)=-\frac{1}{2\sigma}(-2\sigma^4+13\sigma^2-12+9\sigma^{-2})-\frac{\left(4\sigma+(1-\sigma^2)(\sigma+kh(1-\sigma^2))\right)^2}{\sigma\left((\sigma+kh(1-\sigma^2))^2-4kh\sigma\right)}. \tag{4.2.35}$$

Equation (4.2.33) is the cubic NLS equation for the modulation of a solution (with wave number k) to the water wave problem over a flat bottom. We will examine the coefficients $\omega''(k)$ and χ in Chapter 7.2.

Chapter 5

Modulation Analysis of The Water Wave Problem with Variable Bottom

5.1 Modulated Plane Wave Solutions

Starting from this chapter, our focus shifts towards the modulation analysis of the water wave problem in the presence of a periodic bottom. Following the same modulational Ansatz introduced in Chapter 4, we suppose that the solutions to the water wave system (2.2.9) have small amplitudes, and the effect of the weak nonlinearity of (2.2.9) will modulate the amplitude, making it a slowly varying function of space and time.

To eliminate secular terms over a time period of $O(\varepsilon^{-1})$, we introduce two large-scale variables, $X = \varepsilon x$ and $T = \varepsilon t$, and further assume the solution to (2.2.9) has a perturbation expansion of the form

$$\begin{cases} \eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \cdots, \\ \xi = \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \cdots. \end{cases} \tag{5.1.1}$$

As seen from the flat bottom problem in Chapter 4, we obtain the linearized system at leading order. Using Proposition 2.2, the leading order term takes the form of a modulated Bloch-Floquet wave

$$\begin{cases} \eta^{(1)}(x,t,\theta;X,T) = \dfrac{i\Omega_n(\theta)}{g}\, u_1(X,T)\, e^{-i\Omega_n(\theta)t}\phi_n(x,\theta) + c.c., \\ \xi^{(1)}(x,t,\theta;X,T) = u_1(X,T)\, e^{-i\Omega_n(\theta)t}\phi_n(x,\theta) + c.c. + \varphi(X,T). \end{cases} \tag{5.1.2}$$

The (complex) amplitude $u_1(X,T)$ and (real) mean potential $\varphi(X,T)$ are no longer constants but functions depending on the slow variables $X = \varepsilon x$ and $T = \varepsilon t$. The frequency $\Omega_n(\theta)$ is given by the dispersion relation

$$\Omega_n^2(\theta) = g\Lambda_n(\theta). \tag{5.1.3}$$

We recall from Chapter 2.3.2 that the Bloch-Floquet eigenfunction $\phi_n(x,\theta) = e^{i\theta x}\psi_n(x,\theta)$ of $G[b]$ with the corresponding Bloch-Floquet eigenvalue $\Lambda_n(\theta)$ is a θ-periodic function, while $\psi_n(x,\theta)$ is a periodic function in x with a period of 2π. Hence, (5.1.2) can be rewritten as

$$\begin{cases} \eta^{(1)} = \dfrac{i\Omega_n(\theta)}{g}\, u_1(X,T)\, e^{iS_n(x,t,\theta)}\, \psi_n(x,\theta) + c.c., \\ \xi^{(1)} = u_1(X,T)\, e^{iS_n(x,t,\theta)}\, \psi_n(x,\theta) + c.c. + \varphi(X,T), \end{cases} \tag{5.1.4}$$

where we denote

$$S_n := \theta x - \Omega_n(\theta) t. \tag{5.1.5}$$

In the modulation analysis, We choose a positive integer n and $\theta \in [-\frac{1}{2}, \frac{1}{2})$ such that the n^{th} Bloch-Floquet eigenvalue $\Lambda_n(\theta)$ is simple. For example, $\theta \in (\frac{1}{16}, \frac{1}{4} - \frac{1}{16})$.

Before considering higher orders in the modulation expansion, it is crucial to examine how $G[b]$ acts on multiple-scale functions of various forms, which we will encounter at higher orders. Because $G[b]$ only acts on the spatial variables x and X, for convenience, we temporarily omit the time variables t and T in multiple-scale functions.

We suppose $G[b]f(x,\theta;\varepsilon)$ can be expanded in powers of ε as follows:

$$G[b]f(x,\theta;\varepsilon) = \Big(G^{(0)}[b] + \varepsilon G^{(1)}[b] + \varepsilon^2 G^{(2)}[b] + \cdots\Big) f(x,\theta;\varepsilon). \tag{5.1.6}$$

Here, the notations $G^{(0)}[b], G^{(1)}[b]$ and $G^{(2)}[b]$ are somewhat ambiguous, and their specific expressions depend on the form of $f(x,\theta;\varepsilon)$. In the subsequent sub-chapter, we examine multiple-scale functions $f(x,\theta;\varepsilon)$ of three distinct forms: modulated θ-periodic functions $u(\varepsilon x)\,\phi_n(x,\theta)$, long-wave functions $\varphi(\varepsilon x)$, and modulated periodic functions $u(\varepsilon x)\, p(x)$.

5.2 Action of DNO on Multiple-Scale Functions

5.2.1 The Case of Modulated θ-Periodic Functions

We observe that the multiple-scale function $f(x,\theta;\varepsilon) = u(X)\phi_n(x,\theta)$ is present in $\xi^{(1)}$ as given in (5.1.2). Here, $u(X)$ denotes an arbitrary smooth function dependent only on the large-scale variable $X = \varepsilon x$, while the θ-periodic function $\phi_n(x,\theta)$ is the n^{th} Bloch-Floquet eigenfunction of $G[b]$ with the corresponding simple eigenvalue $\Lambda_n(\theta)$. For such modulated θ-periodic functions $f(x,\theta;\varepsilon)$, we have the following expansion when $G[b]$ acts on it.

Proposition 5.1. *The operator $G[b]$ acting on a multiple-scale function $f(x,\theta;\varepsilon) = u(X)\,\phi_n(x,\theta)$ has an asymptotic expansion*

$$G[b]f(x,\theta;\varepsilon) = \Big(G_I^{(0)}[b] + \varepsilon G_I^{(1)}[b] + \varepsilon^2 G_I^{(2)}[b] + \cdots\Big) u(X)\,\phi_n(x,\theta) \tag{5.2.1}$$

with

$$G_I^{(0)}[b]u(X)\phi_n(x,\theta) = \Lambda_n(\theta) u(X)\,\phi_n(x,\theta), \tag{5.2.2}$$

$$G_I^{(1)}[b]u(X)\,\phi_n(x,\theta) = D_X u(X)\,e^{i\theta x}\partial_\theta G[b,D_x+\theta]\,\psi_n(x,\theta), \tag{5.2.3}$$

and

$$G_I^{(2)}[b]u(X)\,\phi_n(x,\theta) = \frac{1}{2}D_{XX}u(X)\,e^{i\theta x}\partial_{\theta\theta}G[b,D_x+\theta]\,\psi_n(x,\theta). \tag{5.2.4}$$

We recall that $D_X = -i\partial_X$, $D_{XX} = (D_X)^2 = -\partial_{XX}$ *and* $\psi_n(x,\theta) = e^{-i\theta x}\phi_n(x,\theta)$.

Proof. Applying the Theorem A2.1 in [12], the pseudo-differential operator $G[b] = G[b,D]$ acting on multiple-scale function is given by

$$\begin{aligned} G[b]u(X)\phi_n(x,\theta) =& G[b,D]u(X)e^{i\theta x}\psi_n(x,\theta) \\ =& e^{i\theta x}G[b,D_x+\theta+\varepsilon D_X]u(X)\,\psi_n(x,\theta) \\ =& u(X)e^{i\theta x}G[b,D_x+\theta]\,\psi_n(x,\theta) \\ &+\varepsilon\, D_X u(X)e^{i\theta x}\partial_\theta G[b,D_x+\theta]\,\psi_n(x,\theta) \\ &+\frac{\varepsilon^2}{2}D_{XX}u(X)e^{i\theta x}\partial_{\theta\theta}G[b,D_x+\theta]\,\psi_n(x,\theta) \quad +O(\varepsilon^3). \end{aligned} \tag{5.2.5}$$

According to the order of ε, we denote

$$\begin{aligned} G_I^{(0)}[b]u(X)\phi_n(x,\theta) &:= u(X)e^{i\theta x}G[b,D_x+\theta]\,\psi_n(x,\theta), \\ G_I^{(1)}[b]u(X)\phi_n(x,\theta) &:= D_X u(X)e^{i\theta x}\partial_\theta G[b,D_x+\theta]\,\psi_n(x,\theta), \\ G_I^{(2)}[b]u(X)\phi_n(x,\theta) &:= \frac{1}{2}D_{XX}u(X)e^{i\theta x}\partial_{\theta\theta}G[b,D_x+\theta]\,\psi_n(x,\theta). \end{aligned} \tag{5.2.6}$$

From the eigenvalue problem (2.3.21), we have

$$G[b]\big(e^{i\theta x}\psi_n(x,\theta)\big) = \Lambda_n(\theta)\big(e^{i\theta x}\psi_n(x,\theta)\big), \tag{5.2.7}$$

which implies (in multiple-scale regime)

$$e^{i\theta x}G[b,D_x+\theta]\,\psi_n(x,\theta) = \Lambda_n(\theta)e^{i\theta x}\psi_n(x,\theta) = \Lambda_n(\theta)\phi_n(x,\theta), \tag{5.2.8}$$

which finishes the proof. □

The following lemmas illustrate some properties of $\partial_\theta G[b,D_x+\theta]\,\psi_n$ and $\partial_{\theta\theta}G[b,D_x+\theta]\,\psi_n$ in (5.2.3) and (5.2.4), respectively, and help us understand them. Recalling from Proposition 3.5, we have an orthonormal basis $\{\psi_k(x,\theta)\}_k$ for periodic functions.

Notation 5.1. *In this thesis, we define the* L^2*-inner product and* L^2*-norm for periodic functions* f *and* g *in* $H^1(\mathbb{T}^1)$ *as*

$$\langle f,g\rangle = \int_0^{2\pi} f(x)\,\overline{g(x)}dx \quad \textit{and} \quad ||f||_{L^2} = \sqrt{\langle f,f\rangle}. \tag{5.2.9}$$

Lemma 5.1. *For the periodic function $\partial_\theta G[b, D_x + \theta]\psi_n(x,\theta)$, its decomposition in terms of the basis $\{\psi_k(x,\theta)\}_k$ is given by*

$$\partial_\theta G[b,D_x+\theta]\psi_n(x,\theta) = \sum_{k=0}^{\infty} \langle \partial_\theta G[b,D_x+\theta]\psi_n(x,\theta), \psi_k(x,\theta)\rangle \psi_k(x,\theta), \tag{5.2.10}$$

where for $k = n$, we have

$$\langle \partial_\theta G[b,D_x+\theta]\psi_n(x,\theta), \psi_n(x,\theta)\rangle = \Lambda_n'(\theta), \tag{5.2.11}$$

and for $k \neq n$, we have

$$\langle \partial_\theta G[b,D_x+\theta]\psi_n(x,\theta), \psi_k(x,\theta)\rangle = \big(\Lambda_n(\theta) - \Lambda_k(\theta)\big)\langle \frac{\partial \psi_n}{\partial \theta}(x,\theta), \psi_k(x,\theta)\rangle. \tag{5.2.12}$$

Proof. For clarity, we simply write ψ_n instead of $\psi_n(x,\theta)$ when there is no confusion.

From (5.2.8), we have

$$G[b,D_x+\theta]\psi_n(x,\theta) = \Lambda_n(\theta)\psi_n(x,\theta). \tag{5.2.13}$$

Differentiating (5.2.13) with respect to θ gives

$$\partial_\theta G[b,D_x+\theta]\psi_n + G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta} = \Lambda_n'(\theta)\psi_n + \Lambda_n(\theta)\frac{\partial \psi_n}{\partial \theta}. \tag{5.2.14}$$

Taking the inner product of (5.2.14) with $\psi_n(x,\theta)$, we obtain

$$\langle \partial_\theta G[b,D_x+\theta]\psi_n, \psi_n\rangle + \langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta}, \psi_n\rangle = \Lambda_n'(\theta)\langle \psi_n, \psi_n\rangle + \Lambda_n(\theta)\langle \frac{\partial \psi_n}{\partial \theta}, \psi_n\rangle. \tag{5.2.15}$$

Since the DNO is symmetric for L^2-inner product with domain $H^1(\mathbb{T}^1)$, we have

$$\langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta}, \psi_n\rangle = \langle \frac{\partial \psi_n}{\partial \theta}, G[b,D_x+\theta]\psi_n\rangle = \Lambda_n(\theta)\langle \frac{\partial \psi_n}{\partial \theta}, \psi_n\rangle. \tag{5.2.16}$$

Using (5.2.16) and (3.2.10), we simplify (5.2.15) to obtain (5.2.11).

To prove (5.2.12), we take the inner product of (5.2.14) with $\psi_k(x,\theta)$, which yields

$$\langle \partial_\theta G[b,D_x+\theta]\psi_n, \psi_k\rangle + \langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta}, \psi_k\rangle = \Lambda_n'(\theta)\langle \psi_n, \psi_k\rangle + \Lambda_n(\theta)\langle \frac{\partial \psi_n}{\partial \theta}, \psi_k\rangle. \tag{5.2.17}$$

Similar to (5.2.16), we have

$$\langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta}, \psi_k\rangle = \langle \frac{\partial \psi_n}{\partial \theta}, G[b,D_x+\theta]\psi_k\rangle = \Lambda_k(\theta)\langle \frac{\partial \psi_n}{\partial \theta}, \psi_k\rangle. \tag{5.2.18}$$

From (3.2.10), the first term on the right-hand side of (5.2.17) vanishes. By substituting (5.2.18) into (5.2.17) and rearranging terms, we derive

$$\langle \partial_\theta G[b, D_x+\theta]\psi_n, \psi_k \rangle = \big(\Lambda_n(\theta) - \Lambda_k(\theta)\big)\langle \frac{\partial \psi_n}{\partial \theta}, \psi_k \rangle. \tag{5.2.19}$$

□

Lemma 5.2. *The operator $\partial_\theta G[D_x+\theta]$ is symmetric for L^2-inner product with domain $H^1(\mathbb{T}^1)$.*

Proof. Firstly, we notice that in the proof of (5.2.19), we can switch the roles of ψ_n and ψ_k to obtain

$$\langle \partial_\theta G[b, D_x+\theta]\psi_k, \psi_n \rangle = \big(\Lambda_k(\theta) - \Lambda_n(\theta)\big)\langle \frac{\partial \psi_k}{\partial \theta}, \psi_n \rangle. \tag{5.2.20}$$

Because eigenvalues $\Lambda_n(\theta)$ and $\Lambda_k(\theta)$ are real, (5.2.20) can be rewritten as

$$\langle \psi_n, \partial_\theta G[b, D_x+\theta]\psi_k \rangle = -\big(\Lambda_n(\theta) - \Lambda_k(\theta)\big)\langle \psi_n, \frac{\partial \psi_k}{\partial \theta} \rangle. \tag{5.2.21}$$

Subtracting (5.2.21) from (5.2.19), we obtain

$$\begin{aligned}
&\langle \partial_\theta G[D_x+\theta]\psi_n, \psi_k \rangle - \langle \psi_n, \partial_\theta G[D_x+\theta]\psi_k \rangle \\
=&\big(\Lambda_n(\theta) - \Lambda_k(\theta)\big)\big(\langle \frac{\partial \psi_n}{\partial \theta}, \psi_k \rangle + \langle \psi_n, \frac{\partial \psi_k}{\partial \theta} \rangle\big) \\
=&\big(\Lambda_n(\theta) - \Lambda_k(\theta)\big)\partial_\theta \langle \psi_n, \psi_k \rangle = 0.
\end{aligned} \tag{5.2.22}$$

Therefore, when $k \neq n$, we have

$$\langle \partial_\theta G[D_x+\theta]\psi_n, \psi_k \rangle = \langle \psi_n, \partial_\theta G[D_x+\theta]\psi_k \rangle. \tag{5.2.23}$$

When $k = n$, (5.2.23) is also valid because of (5.2.11).

Because $\{\psi_k(x,\theta)\}_k$ is an orthonormal basis of $H^1(\mathbb{T}^1)$, we conclude that $\partial_\theta G[D_x+\theta]$ is symmetric for L^2-inner product over $H^1(\mathbb{T}^1)$. □

Lemma 5.3. *For periodic function $\partial_{\theta\theta} G[b, D_x+\theta]\psi_n(x,\theta)$, its inner product with $\psi_n(x,\theta)$ gives $\Lambda_n''(\theta)$, with correction terms arising from the presence of bottom $b(x)$.*

$$\begin{aligned}
&\langle \partial_{\theta\theta} G[b, D_x+\theta]\psi_n(x,\theta), \psi_n(x,\theta) \rangle \\
=&\Lambda_n''(\theta) + 2\langle G[b, D_x+\theta]\frac{\partial \psi_n}{\partial \theta}(x,\theta), \frac{\partial \psi_n}{\partial \theta}(x,\theta) \rangle - 2\Lambda_n(\theta)||\frac{\partial \psi_n}{\partial \theta}(x,\theta)||^2_{L^2}.
\end{aligned} \tag{5.2.24}$$

Proof. Differentiating (5.2.13) twice with respect to θ, we get

$$\begin{aligned}
&\partial_{\theta\theta} G[b, D_x+\theta]\psi_n + 2\partial_\theta G[b, D_x+\theta]\frac{\partial \psi_n}{\partial \theta} + G[b, D_x+\theta]\frac{\partial^2 \psi_n}{\partial \theta^2} \\
=&\Lambda_n''(\theta)\psi_n + 2\Lambda_n'(\theta)\frac{\partial \psi_n}{\partial \theta} + \Lambda_n(\theta)\frac{\partial^2 \psi_n}{\partial \theta^2}.
\end{aligned} \tag{5.2.25}$$

Taking the inner product of (5.2.25) with $\psi_n(x,\theta)$ results in

$$\begin{aligned}&\langle \partial_{\theta\theta} G[b,D_x+\theta]\psi_n,\psi_n\rangle + 2\langle \partial_\theta G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + \langle G[b,D_x+\theta]\frac{\partial^2 \psi_n}{\partial \theta^2},\psi_n\rangle \\ =&\Lambda_n''(\theta)\langle \psi_n,\psi_n\rangle + 2\Lambda_n'(\theta)\langle \frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + \Lambda_n(\theta)\langle \frac{\partial^2 \psi_n}{\partial \theta^2},\psi_n\rangle\end{aligned} \tag{5.2.26}$$

Because $G[D_x+\theta]$ is symmetric for L^2-inner product, we have

$$\langle G[b,D_x+\theta]\frac{\partial^2 \psi_n}{\partial \theta^2},\psi_n\rangle = \langle \frac{\partial^2 \psi_n}{\partial \theta^2}, G[b,D_x+\theta]\psi_n\rangle = \Lambda_n(\theta)\langle \frac{\partial^2 \psi_n}{\partial \theta^2},\psi_n\rangle, \tag{5.2.27}$$

which implies that the third terms on both sides of (5.2.26) can be eliminated. Hence, we can simplify (5.2.26) as

$$\langle \partial_{\theta\theta} G[b,D_x+\theta]\psi_n,\psi_n\rangle + 2\langle \partial_\theta G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\psi_n\rangle = \Lambda_n''(\theta) + 2\Lambda_n'(\theta)\langle \frac{\partial \psi_n}{\partial \theta},\psi_n\rangle. \tag{5.2.28}$$

Similarly, taking the inner product of (5.2.14) with $\frac{\partial \psi_n}{\partial \theta}(x,\theta)$ gives

$$\begin{aligned}&\langle \partial_\theta G[b,D_x+\theta]\psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle + \langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\frac{\partial \psi_n}{\partial \theta}\rangle \\ =&\Lambda_n'(\theta)\langle \psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle + \Lambda_n(\theta)\langle \frac{\partial \psi_n}{\partial \theta},\frac{\partial \psi_n}{\partial \theta}\rangle.\end{aligned} \tag{5.2.29}$$

Then adding (5.2.28) with 2 times (5.2.29), we find

$$\begin{aligned}&\langle \partial_{\theta\theta} G[b,D_x+\theta]\psi_n,\psi_n\rangle + 2\langle G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\frac{\partial \psi_n}{\partial \theta}\rangle \\ &+2\langle \partial_\theta G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + 2\langle \partial_\theta G[b,D_x+\theta]\psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle \\ =&\Lambda_n''(\theta) + 2\Lambda_n(\theta)||\frac{\partial \psi_n}{\partial \theta}||_{L^2}^2 + 2\Lambda_n'(\theta)\Big(\langle \frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + \langle \psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle\Big).\end{aligned} \tag{5.2.30}$$

The third term on the right-hand side of (5.2.30) is zero because

$$\langle \frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + \langle \psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle = \partial_\theta\langle \psi_n,\psi_n\rangle = \partial_\theta 1 = 0. \tag{5.2.31}$$

Differentiating (5.2.11) with respect to θ, we obtain

$$\langle \partial_{\theta\theta} G[b,D_x+\theta]\psi_n,\psi_n\rangle + \langle \partial_\theta G[b,D_x+\theta]\frac{\partial \psi_n}{\partial \theta},\psi_n\rangle + \langle \partial_\theta G[b,D_x+\theta]\psi_n,\frac{\partial \psi_n}{\partial \theta}\rangle = \Lambda_n''(\theta). \tag{5.2.32}$$

Finally, combining (5.2.30) and (5.2.32) implies (5.2.24), which completes the proof. □

Lemma 5.4. *We have the following equality:*

$$\begin{aligned}&\langle G[b,D_x+\theta]\frac{\partial\psi_n}{\partial\theta},\frac{\partial\psi_n}{\partial\theta}\rangle-\Lambda_n(\theta)||\frac{\partial\psi_n}{\partial\theta}||^2_{L^2}\\=&\Lambda_n'(\theta)\langle\frac{\partial\psi_n}{\partial\theta},\psi_n\rangle-\langle\partial_\theta G[b,D_x+\theta]\frac{\partial\psi_n}{\partial\theta},\psi_n\rangle.\end{aligned}\tag{5.2.33}$$

Proof. Taking the inner product of $\frac{\partial\psi_n}{\partial\theta}$ with (5.2.14) gives

$$\begin{aligned}&\langle\frac{\partial\psi_n}{\partial\theta},\partial_\theta G[b,D_x+\theta]\psi_n\rangle+\langle\frac{\partial\psi_n}{\partial\theta},G[b,D_x+\theta]\frac{\partial\psi_n}{\partial\theta}\rangle\\=&\Lambda_n'(\theta)\langle\frac{\partial\psi_n}{\partial\theta},\psi_n\rangle+\Lambda_n(\theta)\langle\frac{\partial\psi_n}{\partial\theta},\frac{\partial\psi_n}{\partial\theta}\rangle.\end{aligned}\tag{5.2.34}$$

Since $G[b,D_x+\theta]$ and $\partial_\theta G[b,D_x+\theta]$ are symmetric for L^2-inner product, we have

$$\langle\partial_\theta G[b,D_x+\theta]\frac{\partial\psi_n}{\partial\theta},\psi_n\rangle+\langle G[b,D_x+\theta]\frac{\partial\psi_n}{\partial\theta},\frac{\partial\psi_n}{\partial\theta}\rangle=\Lambda_n'(\theta)\langle\frac{\partial\psi_n}{\partial\theta},\psi_n\rangle+\Lambda_n(\theta)||\frac{\partial\psi_n}{\partial\theta}||^2_{L^2}.\tag{5.2.35}$$

Rearranging terms in (5.2.35) completes the proof. □

It should be noted that the above analysis holds for different values of θ, except when θ is close to 0 and $\pm\frac{1}{2}$, where $\Lambda_n(\theta)$ could be a double eigenvalue. Based on the flat bottom problem, it is not surprising to observe $u(X)\phi_k(x,2\theta)$ or $u(X)e^{2i\theta x}\psi_k(x,2\theta)$ appearing in the subsequent modulation analysis. For such a multiple-scale function, we can choose θ to ensure $2\theta\neq 0,\pm\frac{1}{2}$ (for instance, $\theta\in\left(\frac{1}{16},\frac{1}{4}-\frac{1}{16}\right)$) and apply the above results by considering $\theta'=2\theta$.

For multiple-scale function $u(X)\psi_k(x,0)$ when $\theta=0$, not all above results are valid. We still have

$$G_I^{(0)}[b]u(X)\psi_k(x,0)=\Lambda_k(0)u(X)\psi_k(x,0).\tag{5.2.36}$$

Especially, as $\Lambda_0(0)=0$, we have

$$G_I^{(0)}[b]u(X)\psi_0(x,0)=0.\tag{5.2.37}$$

However, The $G_I^{(1)}[b]u(X)\psi_k(x,0)$ and $G_I^{(2)}[b]u(X)\psi_k(x,0)$ in (5.2.3) and (5.2.4) respectively are not valid for $\theta=0$ because $\Lambda_k(0)$ may be not differentiable at $\theta=0$. Therefore, we need to find alternative expansions of $G[b]f(x,\theta=0;\varepsilon)$, where $f(x,\theta=0;\varepsilon)$ takes the form $u(X)$ or the form $u(X)p(x)$.

5.2.2 The Case of Long-Wave Functions

From (5.1.2), we know that the large-scale function $\varphi(X)$ is present in $\xi^{(1)}$, where $\varphi(X)$ is an arbitrary smooth real-valued function depending on the large-scale spatial variable X. Therefore, it is important to understand the action of $G[b]$ on such a function.

Proposition 5.2. *The operator $G[b]$ acting on a long-wave function of the form $f(x;\varepsilon) = \varphi(X)$ has the asymptotic expansion*

$$G[b]f(x;\varepsilon) = G_{II}^{(0)}[b]\varphi(X) + \varepsilon G_{II}^{(1)}[b]\varphi(X) + \varepsilon^2 G_{II}^{(2)}[b]\varphi(X) + O(\varepsilon^3), \tag{5.2.38}$$

where

$$G_{II}^{(0)}[b]\varphi(X) = 0, \tag{5.2.39}$$

$$G_{II}^{(1)}[b]\varphi(X) = -D_X\varphi(X)\Big(D_x B_0[b]b(x)\Big), \tag{5.2.40}$$

and

$$G_{II}^{(2)}[b]\varphi(X) = D_{XX}\varphi(X)\Big(h - B_0[b]b(x) + D_x B_0[b]\tilde{b}(x)\sinh\big(\tilde{b}(x)D_x\big)B_0[b]b(x)\Big). \tag{5.2.41}$$

Here, $B_0[b]$ stands for the inverse of the operator $\cosh\big(\tilde{b}(x)D_x\big)$ acting on functions of x, and we recall $\tilde{b}(x) = -h + b(x)$.

The inverse operator $B_0[b]$ is defined in Proposition 3.1 with operator D is now replaced by D_x in multiple scale regime.

Proof. From Proposition 3.1, we have $G[b] = D\tanh(hD) + DL[b]$. We examine the expansions of $D\tanh(hD)\varphi(X)$ and $DL[b]\varphi(X)$ separately.

Firstly, using the result from (4.1.11), we get the expansion

$$\begin{aligned} &D\tanh(hD)\varphi(X) \\ =&D_x\tanh(hD_x)\varphi(X) + \varepsilon\Big(\tanh(hD_x) + hD_x\big(1-\tanh^2(hD_x)\big)\Big)D_X\varphi(X) \\ &+\varepsilon^2\Big(h\big(1-\tanh^2(hD_x)\big) - h^2D_x\big(1-\tanh^2(hD_x)\big)\tanh(hD_x)\Big)D_{XX}\varphi(X) + O(\varepsilon^3) \\ =&\varepsilon^2 hD_{XX}\varphi(X) + O(\varepsilon^3). \end{aligned} \tag{5.2.42}$$

Secondly, using equation (4.8) in [6], $DL[b]$ acting on functions of the long-scale variables can be approximated by

$$\begin{aligned} &DL[b]\varphi(X) \\ =&-\varepsilon D_X\varphi(X)\big(D_xB_0[b]b(x)\big) \\ &-\varepsilon^2 D_{XX}\varphi(X)\Big(B_0[b]b(x) - D_xB_0[b]\tilde{b}(x)\sinh\big(\tilde{b}(x)D_x\big)B_0[b]b(x)\Big) + O(\varepsilon^3), \end{aligned} \tag{5.2.43}$$

where $\tilde{b}(x) = -h + b(x)$ and $B_0[b]$ is the inverse of operator $\cosh(\tilde{b}(x)D_x)$.

Finally, we combine (5.2.42) and (5.2.43) to obtain the asymptotic expansion of $G[b]\varphi(X)$. Specifically, we denote the first three orders in this expansion as $G_{II}^{(0)}[b]\varphi(X)$, $G_{II}^{(1)}[b]\varphi(X)$ and $G_{II}^{(2)}[b]\varphi(X)$, respectively. □

5.2.3 The Case of Modulated Periodic Functions

Now we consider the action of $G[b]$ on multiple-scale functions of the form $f(x;\varepsilon) = u(X)p(x)$, where $u(X)$ is an arbitrary smooth function of large-scale variable X and $p(x)$ represents an arbitrary 2π periodic function. In our work, we only require the first two terms in the expansion, as shown in the following proposition.

Proposition 5.3. *The operator $G[b]$ acting on a modulated periodic function $u(X)p(x)$ has the asymptotic expansion*

$$G[b]\,f(x;\varepsilon) = G_{III}^{(0)}[b]u(X)p(x) + \varepsilon G_{III}^{(1)}[b]u(X)p(x) + O(\varepsilon^2), \tag{5.2.44}$$

where

$$G_{III}^{(0)}[b]u(X)p(x) = u(X)\Big(D_x \tanh(hD_x) - D_x B_0[b]\sinh(b(x)D_x)\,\mathrm{sech}(hD_x)\Big)p(x), \tag{5.2.45}$$

and

$$\begin{aligned}
&G_{III}^{(1)}[b]u(X)p(x)\\
=&D_X u(X)\Big(hD_x\big(1-\tanh^2(hD_x)\big) + \tanh(hD_x) - B_0[b]\sinh(b(x)D_x)\,\mathrm{sech}(hD_x)\\
&+ D_x B_0[b]b(x)\cosh(b(x)D_x)\,\mathrm{sech}(hD_x) - D_x B_0[b]\sinh(b(x)D_x)\tanh(hD_x)\,\mathrm{sech}(hD_x)\\
&+ D_x B_0[b]\tilde{b}(x)\sinh(\tilde{b}(x)D_x)B_0[b]\sinh(b(x)D_x)\,\mathrm{sech}(hD_x)\Big)p(x).
\end{aligned} \tag{5.2.46}$$

Proof. From Proposition 3.1, we know $G[b] = D\tanh(hD) + DL[b] = D\tanh(hD) - DB[b]A[b]$. For clarity, we divide the calculation of $G[b]u(X)p(x)$ into 3 steps.

- **Step 1**. Using the result from (4.1.11), we obtain the expansion of $D\tanh(hD)u(X)p(x)$

$$\begin{aligned}
&D\tanh(hD)u(X)p(x)\\
=&u(X)D_x\tanh(hD_x)p(x) + \varepsilon D_X u(X)\Big(hD_x\big(1-\tanh^2(hD_x)\big) + \tanh(hD_x)\Big)p(x) + O(\varepsilon^2).
\end{aligned} \tag{5.2.47}$$

- **Step 2**. We calculate the expansions of $A[b]u(X)p(x)$ and $B[b]u(X)p(x)$.

Applying the Theorem 4.1 in [12] to $A[b]$, we obtain the asymptotic expansion of $A[b]u(X)p(x)$ as

$$\begin{aligned}
&A[b]u(X)p(x)\\
=&\sinh(b(x)D)\,\mathrm{sech}(hD)u(X)p(x)\\
=&u(X)\big(\sinh(b(x)D_x)\,\mathrm{sech}(hD_x)\big)p(x)\\
&+\varepsilon D_X u(X)\Big(b(x)\cosh(b(x)D_x)\,\mathrm{sech}(hD_x) - \sinh(b(x)D_x)\tanh(hD_x)\,\mathrm{sech}(hD_x)\Big)p(x)\\
&+O(\varepsilon^2).
\end{aligned} \tag{5.2.48}$$

Similarly, using equation (4.7) in [6], we obtain the asymptotic expansion of $B[b]u(X)p(x)$ as

$$\begin{aligned} &B[b]u(X)p(x) \\ =&u(X)B_0[b]p(x) - \varepsilon D_X u(X)\Big(B_0[b]\tilde{b}(x)\sinh\big(\tilde{b}(x)D_x\big)B_0[b]\Big)p(x) + O(\varepsilon^2). \end{aligned} \tag{5.2.49}$$

- **Step 3**. Combining the results from step 1 and step 2 to obtain the expansion of $G[b]u(X)p(x)$.

We notice that from (5.2.48), the expansion of $A[b]u(X)p(x)$ is also a modulated periodic function. Hence, we can combine (5.2.48) and (5.2.49) to calculate $B[b]A[b]u(X)p(x)$ as

$$\begin{aligned} &B[b]A[b]u(X)p(x) \\ =&u(X)\Big(B_0[b]\sinh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &- \varepsilon D_X u(X)\Big(B_0[b]b(x)\cosh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &+ \varepsilon D_X u(X)\Big(B_0[b]\sinh(b(x)D_x)\tanh(hD_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &- \varepsilon D_X u(X)\Big(B_0[b]\tilde{b}(x)\sinh(\tilde{b}(x)D_x)B_0[b]\sinh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) + O(\varepsilon^2). \end{aligned} \tag{5.2.50}$$

Then we obtain

$$\begin{aligned} &DL[b]u(X)p(x) \\ =&-(D_x+\varepsilon D_X)B[b]A[b]u(X)p(x) \\ =&-u(X)\Big(D_xB_0[b]\sinh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &+ \varepsilon D_X u(X)\Big(D_xB_0[b]b(x)\cosh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &- \varepsilon D_X u(X)\Big(D_xB_0[b]\sinh(b(x)D_x)\tanh(hD_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &+ \varepsilon D_X u(X)\Big(D_xB_0[b]\tilde{b}(x)\sinh(\tilde{b}(x)D_x)B_0[b]\sinh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) \\ &- \varepsilon D_X u(X)\Big(B_0[b]\sinh(b(x)D_x)\operatorname{sech}(hD_x)\Big)p(x) + O(\varepsilon^2). \end{aligned} \tag{5.2.51}$$

Finally, adding (5.2.47) and (5.2.51) completes the proof.

□

In conclusion, depending on the function on which $G[b]$ acts, the expressions of $G^{(0)}[b], G^{(1)}[b]$ and $G^{(2)}[b]$ in the expansion of $G[b]f(x,\theta;\varepsilon)$ in powers of ε are different.

- When $f(x,\theta;\varepsilon) = u(\varepsilon x)\,\phi_n(x,\theta)$, we apply the expressions of $G^{(0)}[b], G^{(1)}[b]$ and $G^{(2)}[b]$ obtained in (5.2.2)-(5.2.4).
- When $f(x;\varepsilon) = \varphi(\varepsilon x)$, we apply the expressions of $G^{(0)}[b], G^{(1)}[b]$ and $G^{(2)}[b]$ obtained in (5.2.39)-(5.2.41).
- When $f(x;\varepsilon) = u(\varepsilon x)\,p(x)$, we apply the expressions of $G^{(0)}[b]$ and $G^{(1)}[b]$ obtained in (5.2.45) and (5.2.46).

Chapter 6

Construction of Nonlinear Modulated Solutions: Case of Variable Bottom

6.1 Analysis at Order ε^2

When expanding the water wave equations (2.2.9) in powers of ε, we obtain a system similar to (4.2.1), but with $G^{(0)}$ replaced by $G^{(0)}[b]$. The linearized system is obtained at leading order

$$\begin{pmatrix} \partial_t & -G^{(0)}[b] \\ g & \partial_t \end{pmatrix} \begin{pmatrix} \eta^{(1)} \\ \xi^{(1)} \end{pmatrix} = 0, \tag{6.1.1}$$

which admits a solution of the form

$$\begin{cases} \eta^{(1)} = \dfrac{i\Omega_n(\theta)}{g} u_1(X,T)\, e^{iS_n(x,t,\theta)}\, \psi_n(x,\theta) + c.c., \\ \xi^{(1)} = u_1(X,T)\, e^{iS_n(x,t,\theta)}\, \psi_n(x,\theta) + c.c. + \varphi(X,T). \end{cases} \tag{6.1.2}$$

The amplitude u and mean potential φ are slowly modulated in space and time.

At higher order $O(\varepsilon^m)$, the expansion of (2.2.9) in powers of ε leads to an inhomogeneous linear system

$$\begin{pmatrix} \partial_t & -G^{(0)}[b] \\ g & \partial_t \end{pmatrix} \begin{pmatrix} \eta^{(m)} \\ \xi^{(m)} \end{pmatrix} = \begin{pmatrix} U^{(m)} \\ V^{(m)} \end{pmatrix}. \tag{6.1.3}$$

To guarantee that (6.1.3) is solvable, we need to remove the secular terms on the right-hand side of (6.1.3), which implies the following two solvability conditions:

- **(S1)** $U^{(m)}$ does not contain any 'constant term', where the 'constant term' refers to a term that is independent of x and t, though it may depend on the slow variables X and T.
- **(S2)** Denoting the coefficients of $e^{iS_n(x,t,\theta)}\psi_n(x,\theta)$ terms in $U^{(m)}$ and $V^{(m)}$ as $P^{(m)}$ and $Q^{(m)}$ respectively, they satisfy

$$P^{(m)} + \frac{i\Omega_n(\theta)}{g} Q^{(m)} = 0. \tag{6.1.4}$$

6.1.1 Solvability Conditions

Notation 6.1. *For clarity, we will use the following abbreviations when there is no confusion:*

$$\phi_n = \phi_n(x,\theta),\ u_1 = u_1(X,T),\ \varphi = \varphi(X,T) \text{ and } S_n = S_n(x,t,\theta) = \theta x - \Omega_n(\theta)t.$$

Similar to the flat bottom problem, $U^{(2)}$ and $V^{(2)}$ can be computed as

$$\begin{cases} U^{(2)} = -\eta_T^{(1)} + \Big(G^{(1)}[b]\xi^{(1)} + D_x\eta^{(1)}D_x\xi^{(1)} - G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}\Big), \\ V^{(2)} = -\xi_T^{(1)} - \frac{1}{2}\big(\xi_x^{(1)}\big)^2 + \frac{1}{2}\big(G^{(0)}[b]\xi^{(1)}\big)^2. \end{cases} \tag{6.1.5}$$

To write down the solvability condition **(S2)**, we need to compute $P^{(2)}$ and $Q^{(2)}$ using (6.1.5).

Proposition 6.1. *At order $O(\varepsilon^2)$, the solvability condition **(S2)** yields the following equation*

$$u_{1T} + \Omega_n'(\theta)\, u_{1X} = 0. \tag{6.1.6}$$

This expresses that the wavepacket travels at group velocity $\Omega_n'(\theta)$.

Before proving Proposition 6.1, it is worth noting that $\partial_x\big(e^{i\theta x}\psi_n(x,\theta)\big)$ recurs frequently in our computations. For convenience, we introduce the following notation ℓ_θ.

Notation 6.2. *For any 2π periodic function $p(x)$, ℓ_θ is defined by*

$$\ell_\theta\big(p(x)\big) = i\theta p(x) + \frac{dp}{dx}(x). \tag{6.1.7}$$

Then $\partial_x\big(e^{i\theta x}\psi_n(x,\theta)\big)$ can be conveniently computed as

$$\partial_x\big(e^{i\theta x}\psi_n(x,\theta)\big) = e^{i\theta x}\big(i\theta\psi_n(x,\theta) + \partial_x\psi_n(x,\theta)\big) = e^{i\theta x}\ell_\theta\big(\psi_n(x,\theta)\big). \tag{6.1.8}$$

It is obvious that

$$\overline{\ell_\theta\big(p(x)\big)} = l_{-\theta}\big(\overline{p(x)}\big), \tag{6.1.9}$$

where the overline represents the complex conjugate of the corresponding term.

Moreover, for any two 2π periodic functions $p(x)$ and $q(x)$, we have (using integration by parts)

$$\langle \ell_\theta\big(p(x)\big), q(x)\rangle = -\langle p(x), \ell_\theta\big(q(x)\big)\rangle. \tag{6.1.10}$$

Similarly, we can define $\ell_{2\theta}$ for 2π periodic functions by

$$\ell_{2\theta}\big(p(x)\big) = 2i\theta p(x) + \frac{dp}{dx}(x), \tag{6.1.11}$$

which also satisfies (6.1.9) and (6.1.10) for 2θ.

Proof. **Proof of Proposition 6.1.**

For clarity, we divide the proof into 3 steps.

Step 1. We calculate each term of $U^{(2)}$ and $V^{(2)}$ appearing in (6.1.5).

- From the expressions of $\eta^{(1)}$ and $\xi^{(1)}$ in (6.1.2), we compute $-\eta_T^{(1)}$ in $U^{(2)}$, $-\xi_T^{(1)}$ and $-\frac{1}{2}\big(\xi_x^{(1)}\big)^2$ in $V^{(2)}$ as follows.

$$-\eta_T^{(1)} = -\frac{i\Omega_n(\theta)}{g}\, u_{1T}\, e^{iS_n}\, \psi_n + c.c., \tag{6.1.12}$$

$$-\xi_T^{(1)} = -u_{1T}\, e^{iS_n}\, \psi_n + c.c. - \varphi_T, \tag{6.1.13}$$

$$-\frac{1}{2}\big(\xi_x^{(1)}\big)^2 = -\frac{1}{2} u_1^2\, e^{2iS_n}\big(\ell_\theta(\psi_n)\big)^2 + c.c. - |u_1|^2\, |\ell_\theta(\psi_n)|^2. \tag{6.1.14}$$

- Recalling $D_x = -i\partial_x$, we calculate in $U^{(2)}$

$$D_x\eta^{(1)}D_x\xi^{(1)} = -\frac{i\Omega_n(\theta)}{g}\, u_1^2\, e^{2iS_n}\alpha_1(x,\theta) + c.c. - \frac{i\Omega_n(\theta)}{g}\, |u_1|^2\, \alpha_2(x,\theta), \tag{6.1.15}$$

where

$$\alpha_1(x,\theta) := \ell_{2\theta}\big(\psi_n\, \ell_\theta(\psi_n)\big) \tag{6.1.16}$$

and

$$\alpha_2(x,\theta) := \partial_x\big(\psi_n\, \overline{\ell_\theta(\psi_n)} - \overline{\psi_n}\, \ell_\theta(\psi_n)\big). \tag{6.1.17}$$

Both $\alpha_1(x,\theta)$ and $\alpha_2(x,\theta)$ are 2π periodic functions of x. In particular, $\alpha_2(x,\theta)$ is purely imaginary.

- Using $G_I^{(1)}[b]$ defined in (5.2.3) and $G_{II}^{(1)}[b]$ defined in (5.2.40), we compute $G^{(1)}[b]\xi^{(1)}$ in $U^{(2)}$ as

$$\begin{aligned} G^{(1)}[b]\xi^{(1)} &= G_I^{(1)}[b](u_1\, e^{iS_n}\, \psi_n) + c.c. + G_{II}^{(1)}[b]\varphi \\ &= -i\, u_{1X}\, e^{iS_n}\big(\partial_\theta G[b, D_x+\theta]\, \psi_n\big) + c.c. + \varphi_X\, \big(\partial_x B_0[b]b(x)\big). \end{aligned} \tag{6.1.18}$$

- Using $G_I^{(0)}[b]$ defined in (5.2.2) and $G_{II}^{(0)}[b]$ defined in (5.2.39), we compute

$$G^{(0)}[b]\xi^{(1)} = G_I^{(0)}[b](u_1\, e^{iS_n}\, \psi_n) + c.c. + G_{II}^{(0)}[b]\varphi = \Lambda_n(\theta)\, u_1\, e^{iS_n}\, \psi_n + c.c.. \tag{6.1.19}$$

Then in $V^{(2)}$ we have

$$\frac{1}{2}\big(G^{(0)}[b]\xi^{(1)}\big)^2 = \frac{1}{2}\Lambda_n^2(\theta)\, u_1^2\, e^{2iS_n}\, \psi_n^2 + c.c. + \Lambda_n^2(\theta)\, |u_1|^2\, |\psi_n|^2. \tag{6.1.20}$$

Using (6.1.19) and $G_I^{(0)}[b]$ defined in (5.2.6) (replacing θ in (5.2.6) by 2θ), we find in $U^{(2)}$

$$\begin{aligned}
-G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)} &= -G^{(0)}[b]\big(\eta^{(1)}(\Lambda_n(\theta)\,u_1\,e^{iS_n}\,\psi_n + c.c.)\big) \\
&= -G_I^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)\,u_1^2\,e^{2iS_n}\,\psi_n^2 + c.c.\big) \\
&= -\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)\,u_1^2\,e^{2iS_n}\big(G[b,D_x+2\theta]\,\psi_n^2\big) + c.c..
\end{aligned} \tag{6.1.21}$$

Combining all the terms computed above, $U^{(2)}$ and $V^{(2)}$ have the forms

$$\begin{cases} U^{(2)} = U_1^{(2)}(x,\theta;X,T)\,e^{iS_n} + U_2^{(2)}(x,\theta;X,T)\,e^{2iS_n} + c.c. + U_0^{(2)}(x,\theta;X,T), \\ V^{(2)} = V_1^{(2)}(x,\theta;X,T)\,e^{iS_n} + V_2^{(2)}(x,\theta;X,T)\,e^{2iS_n} + c.c. + V_0^{(2)}(x,\theta;X,T), \end{cases} \tag{6.1.22}$$

where

$$\begin{cases} U_1^{(2)} := -\dfrac{i\Omega_n(\theta)}{g}\,u_{1T}\,\psi_n - iu_{1X}\,\big(\partial_\theta G[b,D_x+\theta]\,\psi_n\big), \\ U_2^{(2)} := -\dfrac{i\Omega_n(\theta)}{g}\,u_1^2\,\big(\alpha_1(x,\theta) + \Lambda_n(\theta)(G[b,D_x+2\theta]\,\psi_n^2)\big), \\ U_0^{(2)} := -\dfrac{i\Omega_n(\theta)}{g}\,|u_1|^2\,\alpha_2(x,\theta) + \varphi_X\,\big(\partial_x B_0[b]b(x)\big), \end{cases} \tag{6.1.23}$$

and

$$\begin{cases} V_1^{(2)} := -u_{1T}\,\psi_n, \\ V_2^{(2)} := \dfrac{1}{2}u_1^2\,\big(\Lambda_n^2(\theta)\,\psi_n^2 - (\ell_\theta(\psi_n))^2\big), \\ V_0^{(2)} := |u_1|^2\big(\Lambda_n^2(\theta)\,|\psi_n|^2 - |\ell_\theta(\psi_n)|^2\big) - \varphi_T. \end{cases} \tag{6.1.24}$$

All $U_i^{(2)}$ and $V_i^{(2)}$ $(j=0,1,2)$ are 2π periodic functions of x and are independent of variable t. Hence, $U_1^{(2)}\,e^{iS_n}$ and $V_1^{(2)}\,e^{iS_n}$ are the θ-periodic components of $U^{(2)}$ and $V^{(2)}$, respectively. $U_2^{(2)}\,e^{2iS_n}$ and $V_2^{(2)}\,e^{2iS_n}$ are the 2θ-periodic components of $U^{(2)}$ and $V^{(2)}$. The periodic components, $U_0^{(2)}$ and $V_0^{(2)}$, are real.

Step 2. We examine the solvability condition **(S2)**.

From Chapter 3.2, we know that $\{\psi_k(x,\theta)\}_{k\geq 0}$ is an orthonormal basis for periodic functions. Hence, the coefficient $P^{(2)}$ of $e^{iS_n}\psi_n(x,\theta)$ in $U^{(2)}$ is given by

$$\begin{aligned}
P^{(2)} &= \langle U_1^{(2)}, \psi_n(x,\theta)\rangle \\
&= -\frac{i\Omega_n(\theta)}{g}\,u_{1T} - iu_{1X}\,\langle \partial_\theta G[b,D_x+\theta]\,\psi_n(x,\theta), \psi_n(x,\theta)\rangle \\
&= -\frac{i\Omega_n(\theta)}{g}\,u_{1T} - i\Lambda_n'(\theta)\,u_{1X}.
\end{aligned} \tag{6.1.25}$$

When decomposing the periodic function $\partial_\theta G[b,D_x+\theta]\,\psi_n(x,\theta)$ in terms of $\{\psi_k(x,\theta)\}_{k\geq 0}$, the coefficient $\Lambda_n'(\theta)$ of $\psi_n(x,\theta)$ is given by (5.2.11) in Lemma 5.1.

In addition, the coefficient $Q^{(2)}$ in $V^{(2)}$ is

$$Q^{(2)} = -u_{1T}. \tag{6.1.26}$$

Thus, the solvability condition **(S2)** can be expressed as

$$-i\Lambda_n'(\theta)\,u_{1X} - \frac{2i\Omega_n(\theta)}{g}\,u_{1T} = 0. \tag{6.1.27}$$

From the dispersion relation (5.1.3), we have

$$\Lambda_n'(\theta) = 2\Omega_n'(\theta)\frac{\Omega_n(\theta)}{g}. \tag{6.1.28}$$

Substituting $\Lambda_n'(\theta)$ into (6.1.27) yields (6.1.6).

Step 3. We examine the solvability condition **(S1)**.

In $U^{(2)}$, the term independent of x and t is only contained in $U_0^{(2)}$. It is given by the zero-mode of the periodic function $U_0^{(2)}$, which we can compute as

$$\begin{aligned}\int_0^{2\pi} U_0^{(2)}dx &= -\frac{i\Omega_n(\theta)}{g}\,|u_1|^2\int_0^{2\pi}\alpha_2(x,\theta)dx + \varphi_X\int_0^{2\pi}\partial_x B_0[b]b(x)dx\\ &= -\frac{i\Omega_n(\theta)}{g}|u_1|^2\left(\overline{\psi_n}\ell_\theta(\psi_n) - \psi_n\overline{\ell_\theta(\psi_n)}\right)\Big|_0^{2\pi} + \varphi_X\left(B_0[b]b(x)\right)\Big|_0^{2\pi}\\ &= 0,\end{aligned} \tag{6.1.29}$$

because $\left(\overline{\psi_n}\ell_\theta(\psi_n) - \psi_n\overline{\ell_\theta(\psi_n)}\right)$ and $B_0[b]b(x)$ are both 2π periodic functions. Therefore, the solvability condition **(S1)** is satisfied naturally, and does not provide us any additional equation.

□

6.1.2 Approximation of Solution at $O(\varepsilon^2)$

To find the solvability conditions at next order, we need to solve the inhomogeneous linear system (6.1.3) for $m = 2$ and precisely calculate $\eta^{(2)}, \xi^{(2)}$.

We begin by decomposing the different components of $U^{(2)}$ and $V^{(2)}$ obtained in (6.1.22) using various orthonormal bases: $\{\psi_k(x,\theta)\}_{k\geq 0}, \{\psi_k(x,2\theta)\}_{k\geq 0}$ and $\{\psi_k(x,0)\}_{k\geq 0}$. This is shown in the next lemma.

Notation 6.3. *For clarity, we will omit the variables x in eigenfunctions and only indicate the variable θ to avoid confusion:*

$$\psi_k = \psi_k(x,\theta),\ \psi_k(2\theta) = \psi_k(x,2\theta) \textit{ and } \psi_k(0) = \psi_k(x,0).$$

Lemma 6.1. *$U^{(2)}$ and $V^{(2)}$ can be reformulated as*

$$\begin{aligned}U^{(2)} =& -\big(\frac{i\Omega_n(\theta)}{g}\,u_{1T} + i\Lambda_n'(\theta)\,u_{1X}\big)\,e^{iS_n}\,\psi_n - \sum_{\substack{k\geq 0,\\ k\neq n}} i\big(\Lambda_n(\theta)-\Lambda_k(\theta)\big)\langle\frac{\partial\psi_n}{\partial\theta},\psi_k\rangle\,u_{1X}\,e^{iS_n}\,\psi_k + c.c.\\ &-\sum_{k\geq 0}\frac{i\Omega_n(\theta)}{g}\,p_k(\theta)\,u_1^2\,e^{2iS_n}\,\psi_k(2\theta) + c.c.\\ &+\sum_{k>0}\frac{1}{2}\Big(-\frac{i\Omega_n(\theta)}{g}\langle\alpha_2(x,\theta),\psi_k(0)\rangle\,|u_1|^2 + \langle\partial_x B_0[b]b(x),\psi_k(0)\rangle\,\varphi_X\Big)\,\psi_k(0) + c.c.\end{aligned} \tag{6.1.30}$$

and

$$\begin{aligned}V^{(2)} =& -u_{1T}\,e^{iS_n}\,\psi_n + c.c.\\ &+\sum_{k\geq 0}\frac{1}{2}q_k(\theta)\,u_1^2\,e^{2iS_n}\,\psi_k(2\theta) + c.c.\\ &+\sum_{k>0}\frac{1}{2}\langle\Lambda_n^2(\theta)\,|\psi_n|^2 - |\ell_\theta(\psi_n)|^2,\psi_k(0)\rangle\,|u_1|^2\,\psi_k(0) + c.c.\\ &-\frac{1}{2\pi}\kappa_1(\theta)|u_1|^2 - \varphi_T,\end{aligned} \tag{6.1.31}$$

where we introduce

$$p_k(\theta) := \langle\alpha_1(x,\theta) + \Lambda_n(\theta)\Lambda_k(2\theta)\,\psi_n^2,\psi_k(2\theta)\rangle, \tag{6.1.32}$$

$$q_k(\theta) := \langle\Lambda_n^2(\theta)\,\psi_n^2 - \big(\ell_\theta(\psi_n)\big)^2,\psi_k(2\theta)\rangle, \tag{6.1.33}$$

and

$$\kappa_1(\theta) := ||\ell_\theta(\psi_n)||^2 - \Lambda_n^2(\theta). \tag{6.1.34}$$

We recall that $\alpha_1(x,\theta)$ and $\alpha_2(x,\theta)$ are defined in (6.1.16) and (6.1.17) respectively.

Proof. From (6.1.22), $U^{(2)}$ and $V^{(2)}$ consist of θ-periodic, 2θ-periodic (including their complex conjugates), and (real) periodic components:

$$\begin{cases} U^{(2)} = U_1^{(2)}\,e^{iS_n} + U_2^{(2)}\,e^{2iS_n} + c.c. + U_0^{(2)},\\ V^{(2)} = V_1^{(2)}\,e^{iS_n} + V_2^{(2)}\,e^{2iS_n} + c.c. + V_0^{(2)}.\end{cases} \tag{6.1.35}$$

From Chapter 3.2, we know $\{\psi_k(x,\theta)\}_{k\geq 0}$, $\{\psi_k(x,2\theta)\}_{k\geq 0}$ and $\{\psi_k(x,0)\}_{k\geq 0}$ are different orthonormal bases for periodic functions. Hence, we can decompose periodic functions $U_i^{(2)}$ and $V_i^{(2)}$ $(i=0,1,2)$ in terms of these orthonormal bases.

Specifically, we decompose $U_1^{(2)}$ in (6.1.23) and $V_1^{(2)}$ in (6.1.24) in terms of the basis $\{\psi_k(x,\theta)\}_{k\geq 0}$

as follows:

$$\begin{aligned} U_1^{(2)} &= \sum_{k\geq 0} \langle U_1^{(2)}, \psi_k \rangle \, \psi_k \\ &= -\big(\frac{i\Omega_n(\theta)}{g} \, u_{1T} + i\Lambda_n'(\theta) \, u_{1X} \big) \psi_n - \sum_{k\neq n} i \langle \partial_\theta G[b, D_x + \theta] \psi_n, \psi_k \rangle \, u_{1X} \, \psi_k, \end{aligned} \tag{6.1.36}$$

and

$$V_1^{(2)} = \sum_{k\geq 0} \langle V_1^{(2)}, \psi_k \rangle \, \psi_k = -u_{1T} \, \psi_n. \tag{6.1.37}$$

We recall that the coefficient of ψ_n is $P^{(2)}$, which has been computed in (6.1.25).

In addition, using (5.2.12) in Lemma 5.1, we can replace $\langle \partial_\theta G[b, D_x + \theta] \psi_n, \psi_k \rangle$ in $U_1^{(2)}$ by $\big(\Lambda_n(\theta) - \Lambda_k(\theta)\big) \langle \frac{\partial \psi_n}{\partial \theta}, \psi_k \rangle$.

The choice of the basis $\{\psi_k(x,\theta)\}_{k\geq 0}$ to decompose $U_1^{(2)}$ and $V_1^{(2)}$ is motivated by two facts: one is that $U_1^{(2)} e^{iS_n}, V_1^{(2)} e^{iS_n}$ are θ-periodic functions; another is that $\phi_k(x,\theta) = e^{i\theta x} \psi_k(x,\theta)$ satisfies the spectral problems of $G[b]$ with θ-periodic boundary condition, that is

$$\begin{cases} G[b]\big(e^{i\theta x} \psi_k(x,\theta)\big) = \Lambda_k(\theta) \big(e^{i\theta x} \psi_k(x,\theta)\big), \\ e^{i\theta(x+2\pi)} \psi_k(x+2\pi, \theta) = e^{i\theta x} \psi_k(x,\theta). \end{cases} \tag{6.1.38}$$

Later, we can use (6.1.38) to compute the actions of $G[b]$ on $U_1^{(2)} e^{iS_n}$ and $V_1^{(2)} e^{iS_n}$ based on the decomposition in terms of $\{\psi_k(x,\theta)\}_{k\geq 0}$.

Similarly, we decompose $U_2^{(2)}$ and $V_2^{(2)}$ in terms of the basis $\{\psi_k(x, 2\theta)\}_{k\geq 0}$ as follows:

$$\begin{aligned} U_2^{(2)} &= \sum_{k\geq 0} \langle U_2^{(2)}, \psi_k(2\theta) \rangle \, \psi_k(2\theta) \\ &= -\sum_{k\geq 0} \frac{i\Omega_n(\theta)}{g} \langle \alpha_1(x,\theta) + \Lambda_n(\theta) \Lambda_k(2\theta) \, \psi_n^2, \psi_k(2\theta) \rangle \, u_1^2 \, \psi_k(2\theta), \end{aligned} \tag{6.1.39}$$

$$\begin{aligned} V_2^{(2)} &= \sum_{k\geq 0} \langle V_2^{(2)}, \psi_k(2\theta) \rangle \, \psi_k(2\theta) \\ &= \sum_{k\geq 0} \frac{1}{2} \langle \Lambda_n^2(\theta) \psi_n^2 - \big(\ell_\theta(\psi_n)\big)^2, \psi_k(2\theta) \rangle \, u_1^2 \, \psi_k(2\theta). \end{aligned} \tag{6.1.40}$$

When computing $\langle G[b, D_x + 2\theta] \psi_n^2, \psi_k(2\theta) \rangle$ in $U_2^{(2)}$, we use $G_I^{(0)}[b]$ defined in (5.2.2) (replacing θ in (5.2.2) by 2θ) and we know $G[b, D_x + 2\theta]$ is symmetric for the L^2-inner product, as shown in Lemma 5.2. Hence, we have

$$\langle G[b, D_x + 2\theta] \psi_n^2, \psi_k(2\theta) \rangle = \langle \psi_n^2, G[b, D_x + 2\theta] \psi_k(2\theta) \rangle = \Lambda_k(2\theta) \langle \psi_n^2, \psi_k(2\theta) \rangle. \tag{6.1.41}$$

Finally, we decompose $U_0^{(2)}$ in (6.1.23) and $V_0^{(2)}$ in (6.1.24) in terms of the basis $\{\psi_k(x,0)\}_{k\geq 0}$.

When $k = 0$, we have

$$\psi_0(x,0) = \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \Lambda_0(0) = 0. \tag{6.1.42}$$

Therefor, the coefficients of $\psi_0(x,0)$ are related to the zero-mode of $U_0^{(2)}$ and $V_0^{(2)}$. In (6.1.29), we have proven that the zero-mode of $U_0^{(2)}$ is 0.

Because $U_0^{(2)}$ and $V_0^{(2)}$ are real, we can decompose them as

$$\begin{aligned} U_0^{(2)} &= \sum_{k\geq 0} \langle \frac{1}{2} U_0^{(2)}, \psi_k(0)\rangle\, \psi_k(0) + c.c. \\ &= \sum_{k>0} \frac{1}{2}\Big(-\frac{i\Omega_n(\theta)}{g} \langle \alpha_2(x,\theta), \psi_k(0)\rangle\, |u_1|^2 + \langle\, \partial_x B_0[b]b(x), \psi_k(0)\rangle\, \varphi_X \Big)\psi_k(0) + c.c. \end{aligned} \tag{6.1.43}$$

$$\begin{aligned} V_0^{(2)} &= \sum_{k\geq 0} \langle \frac{1}{2} V_0^{(2)}, \psi_k(0)\rangle\, \psi_k(0) + c.c. \\ &= \sum_{k>0} \frac{1}{2} \langle \Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2, \psi_k(0)\rangle\, |u_1|^2\, \psi_k(0) + c.c. \\ &\quad - \frac{1}{2\pi}\big(||\ell_\theta(\psi_n)||^2 - \Lambda_n^2(\theta)\big)|u_1|^2 - \varphi_T. \end{aligned} \tag{6.1.44}$$

Combining all the above decomposition completes the proof.

□

Next, we use the method of undetermined coefficients to solve system (6.1.3) and find $\eta^{(2)}$ and $\xi^{(2)}$ as follows.

Proposition 6.2. *The system (6.1.3) at order $m = 2$ is solved in the form*

$$\begin{aligned} \eta^{(2)} &= \sum_{k\geq 0} a_k e^{iS_n}\psi_k(x,\theta) + \sum_{k\geq 0} c_k e^{2iS_n}\psi_k(x,2\theta) + \sum_{k>0} e_k \psi_k(x,0) + c.c. + \bar{\eta}, \\ \xi^{(2)} &= \sum_{k\geq 0} b_k e^{iS_n}\psi_k(x,\theta) + \sum_{k\geq 0} d_k e^{2iS_n}\psi_k(x,2\theta) + \sum_{k>0} f_k \psi_k(x,0) + c.c., \end{aligned} \tag{6.1.45}$$

where coefficients $a_k, b_k, c_k, d_k, e_k, f_k$ and $\bar{\eta}$ are functions depending on large-scale variables X and T, and possibly on parameter θ.

$$\begin{aligned} a_n(\theta,X,T) =& \frac{1}{g} u_{1T}(X,T) + \frac{\Lambda_n'(\theta)}{\Omega_n(\theta)} u_{1X}(X,T) + \frac{i\Omega_n(\theta)}{g} u_2(X,T), \\ b_n(X,T) =& u_2(X,T), \\ a_k(\theta,X,T) =& \frac{\Omega_n(\theta)}{g} \langle \frac{\partial \psi_n}{\partial\theta}(x,\theta), \psi_k(x,\theta)\rangle\, u_{1X}(X,T) \quad \textit{for } k \neq n, \\ b_k(\theta,X,T) =& -i\langle \frac{\partial \psi_n}{\partial\theta}(x,\theta), \psi_k(x,\theta)\rangle\, u_{1X}(X,T) \quad \textit{for } k \neq n. \end{aligned} \tag{6.1.46}$$

$$\begin{aligned} c_k(\theta,X,T) =& \widetilde{c_k}(\theta)\, u_1^2(X,T), \qquad d_k(\theta,X,T) = \widetilde{d_k}(\theta)\, u_1^2(X,T), \\ f_k(\theta,X,T) =& \widetilde{f_k}(\theta)\, |u_1(X,T)|^2 + \breve{f}_k\, \varphi_X(X,T), \qquad e_k(\theta,X,T) = \widetilde{e_k}(\theta)\, |u_1(X,T)|^2, \\ \bar{\eta}(\theta,X,T) =& -\frac{1}{g}\big(\frac{1}{2\pi}\kappa_1(\theta)|u_1(X,T)|^2 + \varphi_T(X,T)\big). \end{aligned} \tag{6.1.47}$$

A new arbitrary function $u_2(X,T)$ of the slow variables is introduced into the solution at order $O(\varepsilon^2)$. In addition, the explicit formulas of coefficients $\widetilde{c_k}(\theta), \widetilde{d_k}(\theta), \widetilde{f_k}(\theta), \breve{f}_k$, and $\widetilde{e_k}(\theta)$ are provided in (6.1.57) and (6.1.61).

Proof. We are looking for the solution, $\eta^{(2)}$ and $\xi^{(2)}$, to the inhomogeneous system

$$\begin{cases} \partial_t \eta^{(2)} - G^{(0)}[b]\xi^{(2)} & = U^{(2)}, \\ g\eta^{(2)} + \partial_t \xi^{(2)} & = V^{(2)}. \end{cases} \tag{6.1.48}$$

According to Lemma 6.1, we assume that $\eta^{(2)}$ and $\xi^{(2)}$ have the forms given in (6.1.45) with undetermined coefficients $a_k, b_k, c_k, d_k, e_k, f_k$, and $\bar{\eta}$. These coefficients depend on the slow variables X and T, but they remain independent of x and t. To determine these coefficients, we substitute (6.1.45) into (6.1.48), and then identify the coefficients of different terms on two sides of (6.1.48).

Substituting $\eta^{(2)}$ and $\xi^{(2)}$ into the left-hand side of (6.1.48), we obtain

$$\begin{aligned} &\partial_t \eta^{(2)} - G^{(0)}[b]\xi^{(2)} \\ =& -\big(i\Omega_n(\theta)a_n + \Lambda_n(\theta)b_n\big)\, e^{iS_n}\psi_n - \sum_{\substack{k\geq 0,\\ k\neq n}} \big(i\Omega_n(\theta)a_k + \Lambda_k(\theta)b_k\big)\, e^{iS_n}\psi_k \\ &- \sum_{k\geq 0} \big(2i\Omega_n(\theta)c_k + \Lambda_k(2\theta)d_k\big)\, e^{2iS_n}\psi_k(2\theta) - \sum_{k>0} \Lambda_k(0) f_k\, \psi_k(0) + c.c. \end{aligned} \tag{6.1.49}$$

and

$$\begin{aligned} g\eta^{(2)} + \partial_t \xi^{(2)} =& \big(ga_n - i\Omega_n(\theta)b_n\big)\, e^{iS_n}\psi_n + \sum_{\substack{k\geq 0,\\ k\neq n}} \big(ga_k - i\Omega_n(\theta)b_k\big)\, e^{iS_n}\psi_k \\ &+ \sum_{k\geq 0} \big(gc_k - 2i\Omega_n(\theta)d_k\big)\, e^{2iS_n}\psi_k(2\theta) + \sum_{k>0} ge_k\, \psi_k(0) + c.c. + g\bar{\eta}. \end{aligned} \tag{6.1.50}$$

In the above computation, we use (5.2.2) (replacing θ with 2θ and 0, respectively) to compute $G^{(0)}[b]e^{2iS_n}\psi_k(2\theta)$ and $G^{(0)}[b]\psi_k(0)$.

On the other hand, $U^{(2)}$ and $V^{(2)}$ on the right-hand side of (6.1.48) are provided in (6.1.30) and (6.1.31), respectively. By identifying the coefficients of various terms on two sides of (6.1.48), we derive the following equations, which can be solved to find all undetermined coefficients.

1. **Coefficients of $e^{iS_n}\psi_n(x,\theta)$:**

Identifying the coefficients of $e^{iS_n}\psi_n(x,\theta)$ results in the equations

$$\begin{aligned} -i\Omega_n(\theta)a_n - \Lambda_n(\theta)b_n &= -\frac{i\Omega_n(\theta)}{g}\,u_{1T} - i\Lambda_n'(\theta)\,u_{1X}, \\ ga_n - i\Omega_n(\theta)b_n &= -u_{1T}, \end{aligned} \tag{6.1.51}$$

which can be solved as

$$\begin{cases} a_n = \dfrac{1}{g}\,u_{1T} + \dfrac{\Lambda_n'(\theta)}{\Omega_n(\theta)}\,u_{1X} + \dfrac{i\Omega_n(\theta)}{g}\,u_2, \\ b_n = u_2. \end{cases} \tag{6.1.52}$$

Here, $u_2 = u_2(X,T)$ is an arbitrary smooth function that depends on X and T.

2. **Coefficients of $e^{iS_n}\psi_k(x,\theta)$ for $k \neq n$:**

   Identifying the coefficients of $e^{iS_n}\psi_k(x,\theta)$ for any $k \geq 0$ except n leads to the equations

$$\begin{aligned} -i\Omega_n(\theta)a_k - \Lambda_k(\theta)b_k &= -iu_{1X}\left(\Lambda_n(\theta) - \Lambda_k(\theta)\right)\langle\frac{\partial\psi_n}{\partial\theta},\psi_k\rangle, \\ ga_k - i\Omega_n(\theta)b_k &= 0, \end{aligned} \tag{6.1.53}$$

   which can be solved as

$$\begin{cases} a_k = \dfrac{\Omega_n(\theta)}{g}\langle\dfrac{\partial\psi_n}{\partial\theta},\psi_k\rangle\,u_{1X}, \\ b_k = -i\langle\dfrac{\partial\psi_n}{\partial\theta},\psi_k\rangle\,u_{1X}. \end{cases} \tag{6.1.54}$$

3. **Coefficients of $e^{2iS_n}\psi_k(x,2\theta)$:**

   Identifying the coefficients of $e^{2iS_n}\psi_k(x,2\theta)$ for any $k \geq 0$ leads to the equations

$$\begin{aligned} -2i\Omega_n(\theta)c_k - \Lambda_k(2\theta)d_k &= -\frac{i\Omega_n(\theta)}{g}p_k(\theta)\,u_1^2, \\ gc_k - 2i\Omega_n(\theta)d_k &= \frac{1}{2}q_k(\theta)\,u_1^2, \end{aligned} \tag{6.1.55}$$

   where $p_k(\theta)$ and $q_k(\theta)$ are defined in (6.1.32) and (6.1.33). Then the solution to (6.1.55) is

$$\begin{cases} c_k = \widetilde{c_k}(\theta)\,u_1^2, \\ d_k = \widetilde{d_k}(\theta)\,u_1^2, \end{cases} \tag{6.1.56}$$

   with

$$\begin{aligned} \widetilde{c_k}(\theta) &:= \frac{1}{2g}\frac{4\Lambda_n(\theta)p_k(\theta) - \Lambda_k(2\theta)q_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)}, \\ \widetilde{d_k}(\theta) &:= \frac{i\Omega_n(\theta)}{g}\frac{q_k(\theta) - p_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)}. \end{aligned} \tag{6.1.57}$$

In Appendix A, we show that the denominator $\big(4\Lambda_n(\theta)-\Lambda_k(2\theta)\big)$ may vanish for certain values of n and θ. We exclude them in the present analysis.

4. **Coefficients of** $\psi_k(x,0)$**:**

Identifying the coefficients of $\psi_k(x,0)$ for $k>0$ results in the following equations

$$\begin{aligned} -\Lambda_k(0) f_k &= -\frac{i\Omega_n(\theta)}{2g}\langle \alpha_2(x,\theta), \psi_k(0)\rangle\, |u_1|^2 + \frac{1}{2}\langle \partial_x B_0[b]b(x), \psi_k(0)\rangle\, \varphi_X, \\ g e_k &= \frac{1}{2}\langle \Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2, \psi_k(0)\rangle\, |u_1|^2, \end{aligned} \tag{6.1.58}$$

and identifying the coefficients of $\psi_0(x,0)=\frac{1}{\sqrt{2\pi}}$ gives

$$g\bar{\eta} = -\frac{1}{2\pi}\kappa_1(\theta)\,|u_1|^2 - \varphi_T. \tag{6.1.59}$$

Solving above equations, we find

$$\begin{cases} f_k = \widetilde{f}_k(\theta)\,|u_1|^2 + \breve{f}_k\,\varphi_X, \\ e_k = \widetilde{e}_k(\theta)\,|u_1|^2, \\ \bar{\eta} = -\frac{1}{g}\big(\frac{1}{2\pi}\kappa_1(\theta)\,|u_1|^2 + \varphi_T\big), \end{cases} \tag{6.1.60}$$

where we denote

$$\begin{aligned} \widetilde{f}_k(\theta) &:= \frac{i\Omega_n(\theta)}{2g\Lambda_k(0)}\,\langle \alpha_2(x,\theta), \psi_k(0)\rangle, \\ \breve{f}_k &:= -\frac{1}{2\Lambda_k(0)}\,\langle \partial_x B_0[b]b(x), \psi_k(0)\rangle, \\ \widetilde{e}_k(\theta) &:= \frac{1}{2g}\langle \Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2, \psi_k(0)\rangle. \end{aligned} \tag{6.1.61}$$

Since $\Lambda_k(0)\neq 0$ when $k\neq 0$, all $\widetilde{f}_k(\theta)$ and $\breve{f}_k$ are well-defined.

□

6.2 Analysis at Order ε^3: Solvability Condition (S2)

At order ε^3, $U^{(3)}$ and $V^{(3)}$ in (6.1.3) can be computed as

$$\begin{aligned}
U^{(3)} =& -\eta_T^{(2)} + \Big(G^{(1)}[b]\xi^{(2)} + D_x\eta^{(1)}D_x\xi^{(2)} - G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(2)}\Big) \\
&+ \Bigg[G^{(2)}[b]\xi^{(1)} + D_x\eta^{(2)}D_x\xi^{(1)} - G^{(0)}[b]\eta^{(2)}G^{(0)}[b]\xi^{(1)} + D_x\eta^{(1)}D_X\xi^{(1)} \\
&- G^{(0)}[b]\eta^{(1)}G^{(1)}[b]\xi^{(1)} + D_X\eta^{(1)}D_x\xi^{(1)} - G^{(1)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)} \\
&- \frac{1}{2}\Big(G^{(0)}[b](\eta^{(1)})^2D_x^2\xi^{(1)} + D_x^2(\eta^{(1)})^2G^{(0)}[b]\xi^{(1)} - 2G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}\Big)\Bigg], \\
V^{(3)} =& -\xi_T^{(2)} - \xi_x^{(1)}\xi_X^{(1)} - \xi_x^{(1)}\xi_x^{(2)} + \eta_x^{(1)}\xi_x^{(1)}G^{(0)}[b]\xi^{(1)} \\
&+ \big(G^{(0)}[b]\xi^{(1)}\big)\Big(G^{(0)}[b]\xi^{(2)} + G^{(1)}[b]\xi^{(1)} + D_x\eta^{(1)}D_x\xi^{(1)} - G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}\Big).
\end{aligned} \tag{6.2.1}$$

The formulas of $U^{(3)}$ and $V^{(3)}$ in (6.2.1) are similar to those in (4.2.19) from Chapter 4 (after substituting $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$ into (4.2.19)).

Now we examine the solvability conditions **(S1)** and **(S2)** at order $O(\varepsilon^3)$, which lead to the NLS equation. Because $U^{(3)}$ and $V^{(3)}$ consist of numerous terms, for the clarity of computation, we label different terms in $U^{(3)}$ as $U_i^{(3)}$ and in $V^{(3)}$ as $V_j^{(3)}$:

$$\begin{aligned}
&U_1^{(3)} := -\eta_T^{(2)}, \qquad U_2^{(3)} := G^{(2)}[b]\xi^{(1)}, \qquad U_3^{(3)} := G^{(1)}[b]\xi^{(2)}, \\
&U_4^{(3)} := D_x\eta^{(2)}D_x\xi^{(1)} - G^{(0)}[b]\eta^{(2)}G^{(0)}[b]\xi^{(1)}, \\
&U_5^{(3)} := D_x\eta^{(1)}D_X\xi^{(1)} - G^{(0)}[b]\eta^{(1)}G^{(1)}[b]\xi^{(1)}, \\
&U_6^{(3)} := D_X\eta^{(1)}D_x\xi^{(1)} - G^{(1)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}, \\
&U_7^{(3)} := -\frac{1}{2}\big(G^{(0)}[b](\eta^{(1)})^2D_x^2\xi^{(1)} + D_x^2(\eta^{(1)})^2G^{(0)}[b]\xi^{(1)}\big), \\
&U_8^{(3)} := G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}, \\
&U_9^{(3)} := D_x\eta^{(1)}D_x\xi^{(2)}, \quad U_{10}^{(3)} := -G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(2)}.
\end{aligned} \tag{6.2.2}$$

$$\begin{aligned}
&V_1^{(3)} := -\xi_T^{(2)}, \qquad V_2^{(3)} := -\xi_x^{(1)}\xi_X^{(1)}, \qquad V_3^{(3)} := -\xi_x^{(1)}\xi_x^{(2)}, \\
&V_4^{(3)} := (G^{(0)}[b]\xi^{(1)})(G^{(0)}[b]\xi^{(2)}), \qquad V_5^{(3)} := (G^{(0)}[b]\xi^{(1)})(G^{(1)}[b]\xi^{(1)}), \\
&V_6^{(3)} := (G^{(0)}[b]\xi^{(1)})(D_x\eta^{(1)}D_x\xi^{(1)}), \quad V_7^{(3)} := -(G^{(0)}[b]\xi^{(1)})(G^{(0)}[b]\eta^{(1)}G^{(0)}[b]\xi^{(1)}), \\
&V_8^{(3)} := \eta_x^{(1)}\xi_x^{(1)}G^{(0)}[b]\xi^{(1)}.
\end{aligned} \tag{6.2.3}$$

Furthermore, we denote $P_i^{(3)}$ as the coefficients of $e^{iS_n}\psi_n(x,\theta)$ in $U_i^{(3)}$, and $Q_i^{(3)}$ as the coefficients of $e^{iS_n}\psi_n(x,\theta)$ in $V_j^{(3)}$.

By computing and combining all $P_i^{(3)}$ and $Q_i^{(3)}$, we obtain $P^{(3)} + \dfrac{i\Omega_n(\theta)}{g}Q^{(3)}$ in condition **(S2)**, leading us to the following proposition.

Proposition 6.3. *At order $O(\varepsilon^3)$, the solvability condition **(S2)** for (6.1.3) reads as*

$$2i\big(u_{2T}+\Omega_n'(\theta)\,u_{2X}\big)+\Omega_n''(\theta)\,u_{1XX}=\chi_1(\theta)\,u_1|u_1|^2-\frac{1}{\Omega_n(\theta)}\big(\kappa_1(\theta)\,\varphi_T+\kappa_2(\theta)\,\varphi_X\big)u_1, \quad (6.2.4)$$

where $\kappa_1(\theta)$ is given in (6.1.34),

$$\kappa_2(\theta):=\Omega_n(\theta)\Big(2i\langle\ell_\theta(\psi_n),\psi_n\rangle-\big(\sum_{k>0}\frac{1}{2\Lambda_k(0)}\langle\partial_x B_0[b]b(x),\psi_k(0)\rangle\langle\psi_k(0),i\alpha_2(x,\theta)\rangle+c.c.\big)\Big), \quad (6.2.5)$$

and

$$\chi_1(\theta):=\frac{\Omega_n(\theta)}{g}\Big(\frac{1}{\Lambda_n(\theta)}r_3(\theta)+\Lambda_n(\theta)r_4(\theta)+2\Lambda_n^2(\theta)r_5(\theta)+2r_6(\theta)\Big). \quad (6.2.6)$$

Coefficients r_3,r_4,r_5 and r_6 are defined in (6.2.27), (6.2.35), (6.2.44) and (6.2.54), respectively. Furthermore, coefficients $\kappa_1(\theta)$, $\kappa_2(\theta)$ and $\chi_1(\theta)$ are all real.

To simplify the computation of $P^{(3)}+\frac{i\Omega_n(\theta)}{g}Q^{(3)}$, we will group $U_i^{(3)}$ and $V_j^{(3)}$ by their characteristics, and divide the entire proof into 6 steps for clarity.

- **Step 1: Combination of terms $U_1^{(3)}$, $U_2^{(3)}$, $U_3^{(3)}$ and $V_1^{(3)}$.**

All $U_1^{(3)}$, $U_2^{(3)}$, $U_3^{(3)}$ and $V_1^{(3)}$ are the linear terms in $U^{(3)}$ and $V^{(3)}$, which contribute to the coefficients of linear terms in the NLS equation.

Lemma 6.2. *The linear terms $U_1^{(3)}$, $U_2^{(3)}$, $U_3^{(3)}$ and $V_1^{(3)}$ contribute*

$$\sum_{i=1}^{3}P_i^{(3)}+\frac{i\Omega_n(\theta)}{g}Q_1^{(3)}=-\frac{\Omega_n(\theta)}{g}\Big(2i\big(u_{2T}+\Omega_n'(\theta)\,u_{2X}\big)+\Omega_n''(\theta)\,u_{1XX}\Big) \quad (6.2.7)$$

to $P^{(3)}+\frac{i\Omega_n(\theta)}{g}Q^{(3)}$.

Proof. Firstly, we compute coefficients $P_1^{(3)}$, $P_2^{(3)}$, $P_3^{(3)}$ and $Q_1^{(3)}$.

1. From the expressions of $\eta^{(2)}$ and $\xi^{(2)}$ in (6.1.45), we have in $U_1^{(3)}$ and $V_1^{(3)}$

$$\begin{cases} P_1^{(3)}=-a_{nT}=-\frac{1}{g}u_{1TT}-\frac{\Lambda_n'(\theta)}{\Omega_n(\theta)}u_{1XT}-\frac{i\Omega_n(\theta)}{g}u_{2T}, \\ Q_1^{(3)}=-b_{nT}=-u_{2T}. \end{cases} \quad (6.2.8)$$

2. Using $\xi^{(1)}$ in (6.1.2) and $G_I^{(2)}[b]$ defined in (5.2.4), the $e^{iS_n}\psi_n$ term in $U_2^{(3)}$ comes from

$$G_I^{(2)}[b]\big(u_1e^{iS_n}\psi_n\big)=\frac{1}{2}D_{XX}u_1\,e^{iS_n}\partial_{\theta\theta}G[b,D_x+\theta]\psi_n.$$

Applying Lemma 5.3, we have

$$\begin{aligned} P_2^{(3)} &= -\frac{1}{2} u_{1XX} \langle \partial_{\theta\theta} G[b, D_x+\theta]\psi_n, \psi_n\rangle \\ &= -\frac{1}{2}\Lambda_n''(\theta)\, u_{1XX} - u_{1XX}\Big(\langle G[b, D_x+\theta]\frac{\partial\psi_n}{\partial\theta}, \frac{\partial\psi_n}{\partial\theta}\rangle - \Lambda_n(\theta)||\frac{\partial\psi_n}{\partial\theta}||^2\Big). \end{aligned} \tag{6.2.9}$$

3. Using $\xi^{(2)}$ in (6.1.45) and $G_I^{(1)}[b]$ defined in (5.2.3), the $e^{iS_n}\psi_n$ term in $U_3^{(3)}$ comes from

$$\begin{aligned} &G_I^{(1)}[b]\Big(b_n e^{iS_n}\psi_n + \sum_{k\neq n} b_k e^{iS_n}\psi_k\Big) \\ &= -ib_{nX}\, e^{iS_n}\partial_\theta G[b, D_x+\theta]\psi_n - i\sum_{k\neq n} b_{kX}\, e^{iS_n}\partial_\theta G[b, D_x+\theta]\psi_k. \end{aligned}$$

Then the coefficient of $e^{iS_n}\psi_n$ is

$$P_3^{(3)} = -ib_{nX}\langle\partial_\theta G[b, D_x+\theta]\psi_n, \psi_n\rangle - i\sum_{k\neq n} b_{kX}\langle\partial_\theta G[b, D_x+\theta]\psi_k, \psi_n\rangle. \tag{6.2.10}$$

We substitute b_n and b_k from (6.1.46) into (6.2.10), and then use (5.2.11) to rewrite

$$\begin{aligned} P_3^{(3)} &= -i\Lambda_n'(\theta)\, u_{2X} - u_{1XX}\sum_{k\neq n}\langle\frac{\partial\psi_n}{\partial\theta}, \psi_k\rangle\langle\partial_\theta G[b, D_x+\theta]\psi_k, \psi_n\rangle \\ &= -i\Lambda_n'(\theta)\, u_{2X} - u_{1XX}\langle\partial_\theta G[b, D_x+\theta]\big(\sum_{k\neq n}\langle\frac{\partial\psi_n}{\partial\theta}, \psi_k\rangle\psi_k\big), \psi_n\rangle. \end{aligned} \tag{6.2.11}$$

When decomposing $\dfrac{\partial\psi_n}{\partial\theta}$ in terms of the basis $\{\psi_k(x,\theta)\}_{k\geq 0}$, we have

$$\frac{\partial\psi_n}{\partial\theta} - \langle\frac{\partial\psi_n}{\partial\theta}, \psi_n\rangle\psi_n = \sum_{k\neq n}\langle\frac{\partial\psi_n}{\partial\theta}, \psi_k\rangle\psi_k. \tag{6.2.12}$$

Substituting (6.2.12) into (6.2.11) and using (5.2.11) again, we obtain

$$P_3^{(3)} = -i\Lambda_n'(\theta)\, u_{2X} - u_{1XX}\Big(\langle\partial_\theta G[b, D_x+\theta]\frac{\partial\psi_n}{\partial\theta}, \psi_n\rangle - \Lambda_n'(\theta)\langle\frac{\partial\psi_n}{\partial\theta}, \psi_n\rangle\Big). \tag{6.2.13}$$

Applying Lemma 5.4, we observe that the second term in (6.2.13) is canceled with the second term in (6.2.9), which implies

$$P_2^{(3)} + P_3^{(3)} = -\frac{1}{2}\Lambda_n''(\theta)\, u_{1XX} - i\Lambda_n'(\theta)\, u_{2X}. \tag{6.2.14}$$

Therefore, combining (6.2.8) and (6.2.14) gives

$$\begin{aligned}&\sum_{i=1}^{3} P_i^{(3)} + \frac{i\Omega_n(\theta)}{g} Q_1^{(3)} \\ &= -\frac{1}{g} u_{1TT} - \frac{\Lambda_n'(\theta)}{\Omega_n(\theta)} u_{1XT} - \frac{1}{2}\Lambda_n''(\theta) u_{1XX} - \frac{2i\Omega_n(\theta)}{g} u_{2T} - i\Lambda_n'(\theta) u_{2X}.\end{aligned} \tag{6.2.15}$$

By differentiating the dispersion relation (5.1.3) twice, we find

$$\Lambda_n'(\theta) = \frac{2}{g}\Omega_n(\theta)\Omega_n'(\theta) \quad \text{and} \quad \Lambda_n''(\theta) = \frac{2}{g}\big(\Omega_n(\theta)\Omega_n''(\theta) + \Omega_n'^2(\theta)\big). \tag{6.2.16}$$

Substituting (6.2.16) into (6.2.15) and also using (6.1.6), we can simplify the terms in (6.2.15) as follows:

$$\begin{aligned}&-\frac{1}{2}\Lambda_n''(\theta)u_{1XX} - \frac{1}{g}u_{1TT} - \frac{\Lambda_n'(\theta)}{\Omega_n(\theta)}u_{1XT} \\ &= -\frac{\Omega_n(\theta)}{g}\Big(\Omega_n''(\theta)u_{1XX} + \frac{1}{\Omega_n(\theta)}\big(u_{1T} + \Omega_n'(\theta)u_{1X}\big)_T + \frac{\Omega_n'(\theta)}{\Omega_n(\theta)}\big(u_{1T} + \Omega_n'(\theta)u_{1X}\big)_X\Big) \\ &= -\frac{\Omega_n(\theta)}{g}\Omega_n''(\theta)u_{1XX},\end{aligned} \tag{6.2.17}$$

and

$$-\frac{2i\Omega_n(\theta)}{g}u_{2T} - i\Lambda_n' u_{2X} = \frac{-2i\Omega_n}{g}\big(u_{2T} + \Omega_n' u_{2X}\big), \tag{6.2.18}$$

which completes the proof of Lemma 6.2.

□

Remark 1. *Different from the linear terms in Lemma 6.2, identifying the* $e^{iS_n}\psi_n$ *terms in the nonlinear* $U_i^{(3)}$ *and* $V_j^{(3)}$ *is more challenging. The strategy is considering all combinations that yield* e^{iS_n}*, because operators* D *and* $G[b]$ *preserve* e^{iS_n}*.*

Specifically, we first find all combinations in $U_i^{(3)}$ *or* $V_j^{(3)}$ *that contain* e^{iS_n}*, and these combinations provide us* θ*-periodic functions. Next, we multiply these* θ*-periodic functions by* e^{-iS_n} *to obtain periodic functions, which can be decomposed in terms of the basis* $\{\psi_k(x,\theta)\}_k$*. Then the coefficients* $P_i^{(3)}$ *or* $Q_j^{(3)}$ *can be computed as the inner product of the periodic functions with* $\psi_n(x,\theta)$*.*

I will explain this computational process through an example later.

- **Step 2: Combination of terms** $U_5^{(3)}$**,** $U_6^{(3)}$**,** $V_2^{(3)}$ **and** $V_5^{(3)}$**.**

Lemma 6.3. *The terms* $U_5^{(3)}$*,* $U_6^{(3)}$*,* $V_2^{(3)}$ *and* $V_5^{(3)}$ *contribute coefficient*

$$P_5^{(3)} + P_6^{(3)} + \frac{i\Omega_n(\theta)}{g}\big(Q_2^{(3)} + Q_5^{(3)}\big) = -r_1(\theta)\frac{2i\Omega_n(\theta)}{g} u_1\, \varphi_X, \tag{6.2.19}$$

to $P^{(3)} + \dfrac{i\Omega_n(\theta)}{g}Q^{(3)}$*, where*

$$r_1(\theta) = \langle \ell_\theta(\psi_n), \psi_n\rangle. \tag{6.2.20}$$

Proof. We compute each coefficient as follows.

1. The $e^{iS_n}\psi_n$ term in $U_5^{(3)}$ comes from a combination of the first term $(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n)$ in $\eta^{(1)}$ and the third term φ in $\xi^{(1)}$ given in (6.1.2), that is,

$$D_x\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)D_X(\varphi)-G_I^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)G_{II}^{(1)}[b](\varphi).$$

Remark 2. *As the first example of nonlinear term, here we present the detailed computation process, which will be omitted in the future. For the θ-periodic function*

$$D_x\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)D_X(\varphi)-G_I^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)G_{II}^{(1)}[b](\varphi),$$

we first multiple e^{-iS_n} to get a periodic function, and then we can take the inner product with ψ_n to compute the coefficient $P_5^{(3)}$ of $e^{iS_n}\psi_n$ term in $U_5^{(3)}$:

$$\begin{aligned}
&\langle e^{-iS_n}D_x\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)D_X(\varphi),\psi_n\rangle-\langle e^{-iS_n}G_I^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\big)G_{II}^{(1)}[b](\varphi),\psi_n\rangle\\
=&\frac{i\Omega_n(\theta)}{g}\langle e^{-iS_n}D_x\big(u_1e^{iS_n}\psi_n\big)(-i\varphi_X),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}\langle e^{-iS_n}G_I^{(0)}[b]u_1e^{iS_n}\big(\psi_nG_{II}^{(1)}[b](\varphi)\big),\psi_n\rangle\\
=&-i\frac{i\Omega_n(\theta)}{g}\langle e^{-iS_n}D_x\big(u_1\varphi_Xe^{iS_n}\psi_n\big),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}u_1\langle e^{-iS_n}e^{iS_n}G[b,D_x+\theta]\big(\psi_nG_{II}^{(1)}[b](\varphi)\big),\psi_n\rangle\\
=&(-i)^2\frac{i\Omega_n(\theta)}{g}u_1\varphi_X\langle e^{-iS_n}e^{iS_n}\ell_\theta(\psi_n),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}u_1\langle\psi_nG_{II}^{(1)}[b](\varphi),G[b,D_x+\theta]\psi_n\rangle\\
=&(-i)^2\frac{i\Omega_n(\theta)}{g}u_1\varphi_X\langle\ell_\theta(\psi_n),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}u_1\langle\psi_nG_{II}^{(1)}[b](\varphi),\Lambda_n(\theta)\psi_n\rangle\\
=&-\frac{i\Omega_n(\theta)}{g}u_1\varphi_X\langle\ell_\theta(\psi_n),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1\langle G_{II}^{(1)}[b]\varphi,|\psi_n|^2\rangle.
\end{aligned}$$

Since $G_{II}^{(1)}[b]\varphi$ will be eliminated later, there is no need to substitute (5.2.40) here.

Therefore, the coefficient $P_5^{(3)}$ is

$$P_5^{(3)}=-\frac{i\Omega_n(\theta)}{g}u_1\varphi_X\langle\ell_\theta(\psi_n),\psi_n\rangle-\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1\langle G_{II}^{(1)}[b]\varphi,|\psi_n|^2\rangle. \tag{6.2.21}$$

2. $U_6^{(3)}$ does not contain any term of the form $e^{iS_n}\psi_n$, which implies $P_6^{(3)}=0$.

3. Using the first term $(u_1e^{iS_n}\psi_n)$ and the third term φ in $\xi^{(1)}$, the $e^{iS_n}\psi_n$ term in $V_2^{(3)}$ comes from the combination

$$-\big(\partial_x(u_1e^{iS_n}\psi_n)\big)\big(\varphi_X\big),$$

which has the coefficient

$$Q_2^{(3)}=-u_1\varphi_X\langle\ell_\theta(\psi_n),\psi_n\rangle. \tag{6.2.22}$$

4. Using the first term $(u_1 e^{iS_n}\psi_n)$ and the third term φ in $\xi^{(1)}$, the $e^{iS_n}\psi_n$ term in $V_5^{(3)}$ comes from the combination

$$(G_I^{(0)}[b]u_1 e^{iS_n}\psi_n)(G_{II}^{(1)}[b]\varphi).$$

Using $G_I^{(0)}[b]$ defined in (5.2.2), we compute the coefficient $Q_5^{(3)}$ as

$$Q_5^{(3)} = \Lambda_n(\theta)u_1\langle \psi_n G_{II}^{(1)}[b]\varphi, \psi_n\rangle = \Lambda_n(\theta)u_1\langle G_{II}^{(1)}[b]\varphi, |\psi_n|^2\rangle.$$

In summary, combining all the computations above yields

$$P_5^{(3)} + P_6^{(3)} + \frac{i\Omega_n(\theta)}{g}\left(Q_2^{(3)} + Q_5^{(3)}\right) = -\frac{2i\Omega_n(\theta)}{g}u_1\varphi_X\langle \ell_\theta(\psi_n), \psi_n\rangle. \tag{6.2.23}$$

□

- **Step 3: Term $U_4^{(3)}$.**

Notation 6.4. *For convenience, we introduce the following notations:*

$$\begin{cases} \gamma_{1k}(\theta) := \langle \ell_{2\theta}\left(\psi_n \ell_\theta(\psi_n)\right), \psi_k(2\theta)\rangle, \\ \gamma_{2k}(\theta) := \langle (\ell_\theta(\psi_n))^2, \psi_k(2\theta)\rangle, \\ \gamma_{3k}(\theta) := \langle \psi_n^2, \psi_k(2\theta)\rangle. \end{cases} \tag{6.2.24}$$

Then we can rewrite $p_k(\theta)$ in (6.1.32) and $q_k(\theta)$ in (6.1.33) as

$$\begin{cases} p_k(\theta) = \gamma_{1k} + \Lambda_n(\theta)\Lambda_k(2\theta)\,\gamma_{3k}, \\ q_k(\theta) = \Lambda_n^2(\theta)\,\gamma_{3k} - \gamma_{2k}. \end{cases} \tag{6.2.25}$$

Lemma 6.4. *$U_4^{(3)}$ contributes the following coefficient to $P^{(3)} + \dfrac{i\Omega_n(\theta)}{g}Q^{(3)}$:*

$$P_4^{(3)} = -\frac{1}{g}\left(\kappa_1(\theta)\,u_1\varphi_T - r_3(\theta)\,u_1|u_1|^2\right), \tag{6.2.26}$$

where κ_1 is defined in (6.1.34) and

$$\begin{aligned} r_3(\theta) = &-||\Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2||^2 \\ &+ \frac{1}{2}\sum_{k\geq 0}\frac{4\Lambda_n(\theta)p_k(\theta) - \Lambda_k(2\theta)q_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)}\left(\overline{\gamma_{2k}} - \Lambda_n^2(\theta)\overline{\gamma_{3k}}\right). \end{aligned} \tag{6.2.27}$$

Proof. From the expressions of $\eta^{(2)}$ in (6.1.45) and $\xi^{(1)}$ in (6.1.2), we find two combinations in $U_4^{(3)}$ that contain $e^{iS_n}\psi_n$ terms.

1. One combination is

$$D_x\big(\sum_{k\geq 0} c_k e^{2iS_n}\psi_k(2\theta)\big)D_x\big(\overline{u_1 e^{iS_n}\psi_n}\big) - G^{(0)}[b]\big(\sum_{k\geq 0} c_k e^{2iS_n}\psi_k(2\theta)\big)G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big),$$

which contributes coefficient

$$(i) = -\overline{u_1}\sum_{k\geq 0} c_k\langle \ell_\theta\big(\psi_k(2\theta)\overline{\ell_\theta(\psi_n)}\big),\psi_n\rangle - \overline{u_1}\Lambda_n^2(\theta)\sum_{k\geq 0} c_k\langle \psi_k(2\theta)\overline{\psi_n},\psi_n\rangle \tag{6.2.28}$$

to $P_4^{(3)}$.

Remark 3. *Here we explain the computation of* $\langle \ell_\theta\big(\psi_k(2\theta)\overline{\ell_\theta(\psi_n)}\big),\psi_n\rangle$ *as follows:*

$$\langle \ell_\theta\big(\psi_k(2\theta)\overline{\ell_\theta(\psi_n)}\big),\psi_n\rangle = -\langle \psi_k(2\theta)\overline{\ell_\theta(\psi_n)},\ell_\theta(\psi_n)\rangle = -\langle \psi_k(2\theta),(\ell_\theta(\psi_n))^2\rangle = \overline{\gamma_{2k}},$$

where we use (6.1.10) and (6.2.24) in the above computation.

Substituting c_k in (6.1.47), (i) can be expressed as

$$(i) = \frac{u_1|u_1|^2}{2g}\sum_k \frac{4\Lambda_n(\theta)p_k(\theta) - \Lambda_k(2\theta)q_k(\theta)}{4\Lambda_n(\theta)-\Lambda_k(2\theta)}\Big(\overline{\gamma_{2k}} - \Lambda_n^2(\theta)\overline{\gamma_{3k}}\Big) \tag{6.2.29}$$

If we also substitute $p_k(\theta)$ and $q_k(\theta)$ in (6.2.25), (i) can be expressed in terms of γ_{ik} as

$$\begin{aligned}(i) = &\frac{u_1|u_1|^2}{2g}\sum_k \frac{1}{4\Lambda_n(\theta)-\Lambda_k(2\theta)}\Big(4\Lambda_n(\theta)\gamma_{1k}\overline{\gamma_{2k}} - 4\Lambda_n^3(\theta)\gamma_{1k}\overline{\gamma_{3k}} + \Lambda_k(2\theta)|\gamma_{2k}|^2 \\ &- \Lambda_n^2(\theta)\Lambda_k(2\theta)\gamma_{2k}\overline{\gamma_{3k}} + 3\Lambda_n^2(\theta)\Lambda_k(2\theta)\overline{\gamma_{2k}}\gamma_{3k} - 3\Lambda_n^4(\theta)\Lambda_k(2\theta)|\gamma_{3k}|^2\Big).\end{aligned} \tag{6.2.30}$$

2. Another combination is

$$D_x\big(\sum_{k>0} e_k\psi_k(0) + c.c. + \bar\eta\big)D_x\big(u_1 e^{iS_n}\psi_n\big) - G^{(0)}[b]\big(\sum_{k>0} e_k\psi_k(0) + c.c. + \bar\eta\big)G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big),$$

which contributes coefficient

$$(ii) = u_1\langle\sum_{k>0} e_k\psi_k(0) + c.c. + \bar\eta, |\ell_\theta(\psi_n)|^2\rangle - u_1\Lambda_n^2(\theta)\langle\sum_{k>0} e_k\psi_k(0) + c.c. + \bar\eta, |\psi_n|^2\rangle, \tag{6.2.31}$$

to $P_4^{(3)}$.

Substituting e_k and $\bar\eta$ given in (6.1.47), we obtain

$$(ii) = -\frac{1}{g}u_1|u_1|^2||\Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2||^2 - \frac{1}{g}u_1\varphi_T\big(||\ell_\theta(\psi_n)||^2 - \Lambda_n^2(\theta)\big). \tag{6.2.32}$$

In summary,

$$\begin{aligned} P_3^{(4)} =& \frac{1}{2g} u_1 |u_1|^2 \sum_k \frac{4\Lambda_n(\theta) p_k(\theta) - \Lambda_k(2\theta) q_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)} \big(\overline{\gamma_{2k}} - \Lambda_n^2(\theta) \overline{\gamma_{3k}} \big) \\ & - \frac{1}{g} u_1 |u_1|^2 || \Lambda_n^2(\theta) |\psi_n|^2 - |\ell_\theta(\psi_n)|^2 ||^2 \\ & - \frac{1}{g} u_1 \varphi_T \big(||\ell_\theta(\psi_n)||^2 - \Lambda_n^2(\theta) \big). \end{aligned} \tag{6.2.33}$$

□

- **Step 4: Combination of terms $U_7^{(3)}$, $V_6^{(3)}$ and $V_8^{(3)}$.**

Lemma 6.5. *The terms $U_7^{(3)}$, $V_6^{(3)}$ and $V_8^{(3)}$ contribute the coefficient*

$$P_7^{(3)} + \frac{i\Omega_n(\theta)}{g} \big(Q_6^{(3)} + Q_8^{(3)} \big) = r_4(\theta) \frac{\Omega_n^2(\theta)}{g^2} \Lambda_n(\theta)\, u_1 |u_1|^2, \tag{6.2.34}$$

to $P^{(3)} + \dfrac{i\Omega_n(\theta)}{g} Q^{(3)}$, where

$$r_4(\theta) = \langle -2\theta^2 |\psi_n|^2 + 2i\theta (\overline{\psi_n} \psi_n' - \psi_n \overline{\psi_n'}) + \overline{\psi_n} \psi_n'' + \psi_n \overline{\psi_n''}, |\psi_n|^2 \rangle. \tag{6.2.35}$$

In Lemma 6.5, we use $'$ to denote ∂_x for convenience, although it is not rigorous. Thus $\psi_n' = \partial_x \psi_n$ and $\psi_n'' = \partial_{xx} \psi_n$.

Proof. 1. Substituting $\eta^{(1)}$ and $\xi^{(1)}$ in (6.1.2) into $U_7^{(3)}$, we find two different combinations in $U_7^{(3)}$ that contain $e^{iS_n}\psi_n$ terms. One combination involves selecting $(\frac{2\Omega_n^2(\theta)}{g^2} |u_1|^2 |\psi_n|^2)$ from $(\eta^{(1)})^2$ and $(u_1 e^{iS_n} \psi_n)$ from $\xi^{(1)}$:

$$-\frac{1}{2} G^{(0)}[b] \big(\frac{2\Omega_n^2(\theta)}{g^2} |u_1|^2 |\psi_n|^2 \big) D_x^2 \big(u_1 e^{iS_n} \psi_n \big) - \frac{1}{2} D_x^2 \big(\frac{2\Omega_n^2(\theta)}{g^2} |u_1|^2 |\psi_n|^2 \big) G^{(0)}[b] \big(u_1 e^{iS_n} \psi_n \big).$$

This θ-periodic function contributes coefficient (i) to $P_7^{(3)}$, where

$$(i) = \frac{\Omega_n^2(\theta)}{g^2} u_1 |u_1|^2 \Lambda_n(\theta) \Big(\langle |\psi_n|^2 \ell_\theta^2(\psi_n), \psi_n \rangle + \langle \ell_\theta(\ell_\theta(|\psi_n|^2 \psi_n)), \psi_n \rangle \Big). \tag{6.2.36}$$

Another combination involves selecting $(\frac{i\Omega_n(\theta)}{g} u_1^2 e^{2iS_n} \psi_n)^2$ from $(\eta^{(1)})^2$ and $(\overline{u_1 e^{iS_n} \psi_n})$ from $\xi^{(1)}$:

$$-\frac{1}{2} G^{(0)}[b] \big(\frac{-\Omega_n^2(\theta)}{g^2} u_1^2 e^{2iS_n} \psi_n^2 \big) D_x^2 \big(\overline{u_1 e^{iS_n} \psi_n} \big) - \frac{1}{2} D_x^2 \big(\frac{-\Omega_n^2(\theta)}{g^2} u_1^2 e^{2iS_n} \psi_n^2 \big) G^{(0)}[b] \big(\overline{u_1 e^{iS_n} \psi_n} \big).$$

The coefficient of $e^{iS_n}\psi_n$ in above θ-periodic function is

$$(ii) = -\frac{\Omega_n^2(\theta)}{2g^2}u_1|u_1|^2\Lambda_n(\theta)\Big(\langle \psi_n^2\overline{\ell_\theta^2(\psi_n)}, \psi_n\rangle + \langle \ell_\theta(\ell_\theta(|\psi_n|^2\psi_n)), \psi_n\rangle\Big). \tag{6.2.37}$$

Therefore,

$$P_7^{(3)} = (i) + (ii) = \frac{\Omega_n^2(\theta)}{g^2}\Lambda_n(\theta)u_1|u_1|^2\langle |\psi_n|^2\ell_\theta^2(\psi_n), \psi_n\rangle. \tag{6.2.38}$$

Remark 4. *We provide some details about the computation in (6.2.38).*

Using (6.1.10), we compute

$$\begin{aligned}
&\langle |\psi_n|^2\ell_\theta^2(\psi_n), \psi_n\rangle + \langle \ell_\theta(\ell_\theta(|\psi_n|^2\psi_n)), \psi_n\rangle - \frac{1}{2}\langle \psi_n^2\overline{\ell_\theta^2(\psi_n)}, \psi_n\rangle - \frac{1}{2}\langle \ell_\theta(\ell_\theta(|\psi_n|^2\psi_n)), \psi_n\rangle \\
=&\langle |\psi_n|^2\ell_\theta^2(\psi_n), \psi_n\rangle + \frac{1}{2}\langle \ell_\theta(\ell_\theta(|\psi_n|^2\psi_n)), \psi_n\rangle - \frac{1}{2}\langle \psi_n^2\overline{\psi_n}, \ell_\theta^2(\psi_n)\rangle \\
=&\langle |\psi_n|^2\ell_\theta^2(\psi_n), \psi_n\rangle + \frac{(-1)^2}{2}\langle |\psi_n|^2\psi_n, \ell_\theta^2(\psi_n)\rangle - \frac{1}{2}\langle |\psi_n|^2\psi_n, \ell_\theta^2(\psi_n)\rangle \\
=&\langle |\psi_n|^2\ell_\theta^2(\psi_n), \psi_n\rangle.
\end{aligned}$$

2. Turning to $V_6^{(3)}$, we identify 3 combinations that include $e^{iS_n}\psi_n$ term, which are as follows:

$$\begin{aligned}
&\Big(G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big)\Big)\Big(D_x\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)D_x\big(u_1 e^{iS_n}\psi_n\big)\Big), \\
&\Big(G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big)\Big(D_x\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big)D_x\big(u_1 e^{iS_n}\psi_n\big)\Big), \\
&\Big(G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big)\Big(D_x\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)D_x\big(\overline{u_1 e^{iS_n}\psi_n}\big)\Big).
\end{aligned}$$

Calculating their coefficients of $e^{iS_n}\psi_n$ leads to

$$\begin{aligned}
Q_6^{(3)} =& -\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle \overline{\psi_n}\ell_{2\theta}(\psi_n\ell_\theta(\psi_n)), \psi_n\rangle \\
&+\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle \psi_n(\overline{\psi_n}\ell_\theta(\psi_n))', \psi_n\rangle \\
&-\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle \psi_n(\psi_n(\overline{\ell_\theta\psi_n}))', \psi_n\rangle.
\end{aligned} \tag{6.2.39}$$

3. Because $V_8^{(3)}$ has a similar structure with $V_6^{(3)}$, the $e^{iS_n}\psi_n$ terms in $V_8^{(3)}$ are also provided by

three combinations.

$$\overline{\Big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\Big)'}\big(u_1e^{iS_n}\psi_n\big)'G^{(0)}[b]\big(u_1e^{iS_n}\psi_n\big),$$
$$\Big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\Big)'\big(\overline{u_1e^{iS_n}\psi_n}\big)'G^{(0)}[b]\big(u_1e^{iS_n}\psi_n\big),$$
$$\Big(\frac{i\Omega_n(\theta)}{g}u_1e^{iS_n}\psi_n\Big)'\big(u_1e^{iS_n}\psi_n\big)'G^{(0)}[b]\big(\overline{u_1e^{iS_n}\psi_n}\big),$$

which contribute to coefficient

$$\begin{aligned}Q_3^{(8)}=&-\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle|\ell_\theta(\psi_n)|^2\psi_n,\psi_n\rangle\\&+\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle|\ell_\theta(\psi_n)|^2\psi_n,\psi_n\rangle\\&+\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle(\ell_\theta(\psi_n))^2\overline{\psi_n},\psi_n\rangle\\=&\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1|u_1|^2\langle(\ell_\theta(\psi_n))^2\overline{\psi_n},\psi_n\rangle.\end{aligned}\tag{6.2.40}$$

In summary, $U_7^{(3)}$, $V_6^{(3)}$ and $V_8^{(3)}$ contribute the following coefficient

$$\begin{aligned}&P_7^{(3)}+\frac{i\Omega_n(\theta)}{g}(Q_6^{(3)}+Q_8^{(3)})\\=&\frac{\Omega_n^2(\theta)}{g^2}\Lambda_n(\theta)u_1|u_1|^2\Big(\langle|\psi_n|^2\ell_\theta^2(\psi_n),\psi_n\rangle+\langle\overline{\psi_n}\ell_{2\theta}(\psi_n\ell_\theta(\psi_n)),\psi_n\rangle\\&-\langle\psi_n(\overline{\psi_n}\ell_\theta(\psi_n))',\psi_n\rangle+\langle\psi_n(\psi_n(\overline{\ell_\theta\psi_n}))',\psi_n\rangle\\&-\langle(\ell_\theta(\psi_n))^2\overline{\psi_n},\psi_n\rangle\Big).\end{aligned}\tag{6.2.41}$$

Using the definition of ℓ_θ in (6.1.7), we can rewrite (6.2.41) as

$$\frac{\Omega_n^2(\theta)}{g^2}\Lambda_n(\theta)u_1|u_1|^2\langle-2\theta^2|\psi_n|^2+2i\theta(\overline{\psi_n}\psi_n'-\psi_n\overline{\psi_n'})+(\overline{\psi_n}\psi_n''+\psi_n\overline{\psi_n''}),|\psi_n|^2\rangle.\tag{6.2.42}$$

□

- **Step 5: Combination of terms $U_8^{(3)}$ and $V_7^{(3)}$.**

Lemma 6.6. *$U_8^{(3)}$ and $V_7^{(3)}$ contribute the following coefficient to $P^{(3)}+\frac{i\Omega_n(\theta)}{g}Q^{(3)}$:*

$$P_8^{(3)}+\frac{i\Omega_n(\theta)}{g}Q_7^{(3)}=r_5(\theta)\frac{2\Omega_n^2(\theta)}{g^2}\Lambda_n^2(\theta)u_1|u_1|^2.\tag{6.2.43}$$

Recalling γ_{3k} defined in (6.2.24), we have

$$r_5(\theta) = \sum_{k\geq 0} \Lambda_k(2\theta)|\gamma_{3k}|^2. \tag{6.2.44}$$

Proof. 1. Substituting $\eta^{(1)}$ and $\xi^{(1)}$ in (6.1.2) into $U_8^{(3)}$, there are 3 combinations that contains $e^{iS_n}\psi_n$ terms. The first combination is

$$G^{(0)}[b]\big(\overline{\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n}\big)G^{(0)}[b]\big(\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big),$$

where the coefficient of $e^{iS_n}\psi_n$ in it is

$$(i) = \frac{\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\langle\overline{\psi_n}(G[b,D_x+2\theta]\psi_n^2),\psi_n\rangle. \tag{6.2.45}$$

The second combination is

$$G^{(0)}[b]\big(\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(\overline{\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n}\big)G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big),$$

with the coefficient

$$(ii) = \frac{\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\langle\psi_n(G[b,D_x+0]|\psi_n|^2),\psi_n\rangle. \tag{6.2.46}$$

The last combination is given as

$$G^{(0)}[b]\big(\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(\frac{i\Omega_n}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big),$$

with the coefficient

$$(iii) = -\frac{\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\langle\psi_n(G[b,D_x+0]|\psi_n|^2),\psi_n\rangle. \tag{6.2.47}$$

We notice that $(ii)+(iii)=0$, then we conclude that

$$P_8^{(3)} = \frac{\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\langle\overline{\psi_n(x,\theta)}(G[b,D_x+2\theta]\psi_n^2(x,\theta)),\psi_n(x,\theta)\rangle. \tag{6.2.48}$$

2. Turing to $V_7^{(3)}$, it has similar structure to $U_8^{(3)}$. Hence, a similar calculation shows that the

$e^{iS_n}\psi_n$ terms in $V_7^{(3)}$ are contained in

$$-\Big(G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big)\Big)\Big(G^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big),$$
$$-\Big(G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big)\Big(G^{(0)}[b]\overline{\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)}G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big),$$
$$-\Big(G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big)\Big)\Big(G^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big)\Big).$$

Then the coefficient $Q_3^{(7)}$ is calculated as

$$\begin{aligned} Q_3^{(7)} =& -\frac{i\Omega_n(\theta)}{g}u_1|u_1|^2\Lambda_n^2(\theta)\langle\overline{\psi_n}(G[b,D_x+2\theta]\psi_n^2),\psi_n\rangle \\ &+\frac{i\Omega_n(\theta)}{g}u_1|u_1|^2\Lambda_n^2(\theta)\langle\psi_n(G[b,D_x+0]|\psi_n|^2),\psi_n\rangle \\ &-\frac{i\Omega_n(\theta)}{g}u_1|u_1|^2\Lambda_n^2(\theta)\langle\psi_n(G[b,D_x+0]|\psi_n|^2),\psi_n\rangle \\ =& -\frac{i\Omega_n(\theta)}{g}u_1|u_1|^2\Lambda_n^2(\theta)\langle\overline{\psi_n}(G[b,D_x+2\theta]\psi_n^2),\psi_n\rangle. \end{aligned} \tag{6.2.49}$$

In summary, we have

$$P_8^{(3)}+\frac{i\Omega_n(\theta)}{g}Q_7^{(3)} = \frac{2\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\langle G[b,D_x+2\theta]\psi_n^2(x,\theta),\psi_n^2(x,\theta)\rangle. \tag{6.2.50}$$

We can decompose the periodic function $\psi_n^2(x,\theta)$ in terms of basis $\{\psi_k(x,2\theta)\}_k$.

$$\psi_n^2(x,\theta) = \sum_k\langle\psi_n^2(\theta),\psi_k(2\theta)\rangle\psi_k(2\theta). \tag{6.2.51}$$

Substituting (6.2.51) into (6.2.50) and using (5.2.13) (with θ in (5.2.13) replaced by 2θ), we compute (6.2.50) as

$$P_8^{(3)}+\frac{i\Omega_n(\theta)}{g}Q_7^{(3)} = \frac{2\Omega_n^2(\theta)}{g^2}u_1|u_1|^2\Lambda_n^2(\theta)\sum_k\Lambda_k(2\theta)|\langle\psi_n^2(x,\theta),\psi_k(x,2\theta)\rangle|^2. \tag{6.2.52}$$

Substituting γ_{3k} defined in (6.2.24) completes the proof. □

- **Step 6: Combination of terms $U_9^{(3)}$, $U_{10}^{(3)}$, $V_3^{(3)}$ and $V_4^{(3)}$.**

Lemma 6.7. *$U_9^{(3)}$, $U_{10}^{(3)}$, $V_3^{(3)}$ and $V_4^{(3)}$ contribute the following coefficient to $P^{(3)}+\frac{i\Omega_n(\theta)}{g}Q^{(3)}$:*

$$P_9^{(3)}+P_{10}^{(3)}+\frac{i\Omega_n(\theta)}{g}(Q_3^{(3)}+Q_4^{(3)}) = r_6(\theta)\frac{2\Omega_n^2(\theta)}{g^2}u_1|u_1|^2 - r_7(\theta)\frac{\Omega_n(\theta)}{g}u_1\varphi_X, \tag{6.2.53}$$

where

$$r_6(\theta) = -\sum_k \frac{q_k(\theta) - p_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)} \big(\Lambda_n(\theta)\Lambda_k(2\theta)\overline{\gamma_{3k}} + \overline{\gamma_{1k}}\big) - \frac{1}{2}\sum_{k\neq 0} \frac{1}{\Lambda_k(0)} |\langle \alpha_2(x,\theta), \psi_k(0)\rangle|^2 \tag{6.2.54}$$

and

$$r_7(\theta) = -\frac{1}{2}\sum_{k\neq 0} \frac{1}{\Lambda_k(0)} \langle \partial_x B_0[b]b(x), \psi_k(0)\rangle \langle \psi_k(0), i\alpha_2(x,\theta)\rangle + c.c.. \tag{6.2.55}$$

We recall that α_2 is defined in (6.1.17), p_k, q_k are defined in (6.2.25), and γ_{ik} (for i=1,2,3) are defined in (6.2.24).

Proof. 1. From the expressions of $\xi^{(2)}$ in (6.1.45) and $\eta^{(1)}$ in (6.1.2), we find there are two combinations in $U_9^{(3)}$, which contain $e^{iS_n}\psi_n$ terms. The first combination is

$$D_x\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big) D_x\big(\sum_{k\neq 0} f_k\psi_k(0) + c.c.\big),$$

which contributes coefficient

$$(i) = -\frac{i\Omega_n(\theta)}{g}u_1 \sum_{k\neq 0}\langle \ell_\theta\big(\psi_n\,\partial_x(f_k\psi_k(0)+c.c.)\big), \psi_n\rangle. \tag{6.2.56}$$

Another combination is

$$D_x\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big) D_x\big(\sum_k d_k e^{2iS_n}\psi_k(2\theta)\big),$$

with coefficient

$$(ii) = \frac{i\Omega_n(\theta)}{g}\overline{u_1}\sum_k d_k \langle \ell_\theta\big(\overline{\psi_n}\ell_{2\theta}(\psi_k(2\theta))\big), \psi_n\rangle. \tag{6.2.57}$$

Therefore,

$$\begin{aligned} P_9^{(3)} =& -\frac{i\Omega_n(\theta)}{g}u_1 \sum_{k\neq 0}\langle f_k\psi_k(0) + c.c, \partial_x\big(\ell_\theta(\psi_n)\overline{\psi_n}\big)\rangle \\ &+ \frac{i\Omega_n(\theta)}{g}\overline{u_1}\sum_k d_k\langle \ell_\theta\big(\overline{\psi_n}\ell_{2\theta}(\psi_k(2\theta))\big), \psi_n\rangle. \end{aligned} \tag{6.2.58}$$

2. Because $U_9^{(3)}$ and $U_{10}^{(3)}$ have similar structure, with D_x is replaced by $G^{(0)}[b]$, the $e^{iS_n}\psi_n$ terms in $U_{10}^{(3)}$ are contained in

$$\begin{aligned} &- G^{(0)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big) G^{(0)}[b]\big(\sum_{k\neq 0} f_k\psi_k(0) + c.c.\big) \\ &- G^{(0)}[b]\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big) G^{(0)}[b]\big(\sum_k d_k e^{2iS_n}\psi_k(2\theta)\big) \end{aligned}$$

with the coefficient

$$\begin{aligned} P_{10}^{(3)} =& -\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)u_1\sum_{k\neq 0}\Lambda_k(0)\langle f_k\psi_k(0)+c.c,|\psi_n|^2\rangle \\ &+\frac{i\Omega_n(\theta)}{g}\Lambda_n(\theta)\overline{u_1}\sum_k d_k\Lambda_k(2\theta)\langle\overline{\psi_n}\psi_k(2\theta),\psi_n\rangle. \end{aligned} \tag{6.2.59}$$

3. Turning to $V_3^{(3)}$, from the expressions of $\xi^{(2)}$ in (6.1.45) and $\xi^{(1)}$ in (6.1.2), we find there are two combinations contain $e^{iS_n}\psi_n$ terms:

$$-\partial_x\big(u_1e^{iS_n}\psi_n\big)\partial_x\big(\sum_{k\neq 0}f_k\psi_k(0)+c.c.\big)-\partial_x\big(\overline{u_1e^{iS_n}\psi_n}\big)\partial_x\big(\sum_k d_ke^{2iS_n}\psi_k(2\theta)\big)$$

with coefficient

$$Q_3^{(3)} = u_1\sum_{k\neq 0}\langle f_k\psi_k(0)+c.c,\partial_x\big(\psi_n\overline{\ell_\theta(\psi_n)}\big)\rangle-\overline{u_1}\sum_k d_k\langle\overline{\ell_\theta(\psi_n)}\ell_{2\theta}(\psi_k(2\theta)),\psi_n\rangle. \tag{6.2.60}$$

4. Since $V_4^{(3)}$ has similar structure to $V_3^{(3)}$, we can easily find the two combinations contain $e^{iS_n}\psi_n$ are

$$\begin{aligned} &\Big(G^{(0)}[b]\big(u_1e^{iS_n}\psi_n\big)\Big)\Big(G^{(0)}[b]\big(\sum_{k\neq 0}f_k\psi_k(0)+c.c.\big)\Big), \\ &\Big(G^{(0)}[b]\big(\overline{u_1e^{iS_n}\psi_n}\big)\Big)\Big(G^{(0)}[b]\big(\sum_k d_ke^{2iS_n}\psi_k(2\theta)\big)\Big), \end{aligned}$$

and the corresponding coefficient is

$$\begin{aligned} Q_4^{(3)} =& u_1\Lambda_n(\theta)\sum_{k\neq 0}\Lambda_k(0)\langle f_k\psi_k(0)+c.c.,|\psi_n|^2\rangle \\ &+\overline{u_1}\Lambda_n(\theta)\sum_k d_k\Lambda_k(2\theta)\langle\overline{\psi_n}\psi_k(2\theta),\psi_n\rangle. \end{aligned} \tag{6.2.61}$$

In summary, $U_9^{(3)}$, $U_{10}^{(3)}$, $V_3^{(3)}$ and $V_4^{(3)}$ contribute the following coefficient to $P_3+\frac{i\Omega_n(\theta)}{g}Q_3$.

$$\begin{aligned} &P_9^{(3)}+P_{10}^{(3)}+\frac{i\Omega_n(\theta)}{g}(Q_3^{(3)}+Q_4^{(3)}) \\ =&\frac{2i\Omega_n(\theta)}{g}\overline{u_1}\Lambda_n(\theta)\sum_k d_k\Lambda_k(2\theta)\langle\overline{\psi_n}\psi_k(2\theta),\psi_n\rangle \\ -&\frac{2i\Omega_n(\theta)}{g}\overline{u_1}\sum_k d_k\langle\ell_{2\theta}(\psi_k(2\theta)),\psi_n\ell_\theta(\psi_n)\rangle \\ +&\frac{i\Omega_n(\theta)}{g}u_1\sum_{k\neq 0}\langle f_k\psi_k(0)+c.c.,\alpha_2(x,\theta)\rangle, \end{aligned} \tag{6.2.62}$$

where α_2 is defined in (6.1.17).

Substituting d_k and f_k in (6.1.47), we can compute (6.2.62) and simplify it to

$$\begin{aligned}
&P_3^{(9)} + P_3^{(10)} + \frac{i\Omega_n(\theta)}{g}(Q_3^{(3)} + Q_3^{(4)}) \\
=& -\frac{2\Omega_n^2(\theta)}{g^2} u_1|u_1|^2 \sum_k \frac{q_k(\theta) - p_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)} \big(\Lambda_n(\theta)\Lambda_k(2\theta)\overline{\gamma_{3k}} + \overline{\gamma_{1k}}\big) \\
&- \frac{\Omega_n^2(\theta)}{g^2} u_1|u_1|^2 \sum_{k\neq 0} \frac{1}{\Lambda_k(0)} |\langle i\alpha_2(x,\theta), \psi_k(0)\rangle|^2 \\
&+ \frac{\Omega_n(\theta)}{2g} u_1 \varphi_X \sum_{k\neq 0} \frac{1}{\Lambda_k(0)} \langle \partial_x B_0[b]b(x), \psi_k(0)\rangle \langle \psi_k(0), i\alpha_2(x,\theta)\rangle + c.c.,
\end{aligned} \tag{6.2.63}$$

where α_2 defined in (6.1.17) is purely imaginary. □

Now we proceed to prove Proposition 6.3 by combining all the previously established lemmas.

Proof. **Proof of Proposition 6.3.**

In order to to get $P_3 + \frac{i\Omega_n(\theta)}{g} Q_3$, we add all the coefficients obatined in Lemma 6.2 - 6.7, and then the solvability condition **(S2)** (i.e. $P_3 + \frac{i\Omega_n(\theta)}{g} Q_3 = 0$) reads

$$\begin{aligned}
0 =& -\frac{\Omega_n(\theta)}{g}\Omega_n''(\theta) u_{1XX} - \frac{2i\Omega_n(\theta)}{g}\big(u_{2T} + \Omega_n' u_{2X}\big) \\
&- \frac{\Omega_n(\theta)}{g}\big(2ir_1(\theta) + r_7(\theta)\big) u_1\varphi_X - \frac{1}{g}\kappa_1(\theta) u_1 \varphi_T \\
&+ \Big(\frac{1}{g} r_3(\theta) + r_4(\theta)\frac{\Omega_n^2(\theta)}{g^2}\Lambda_n(\theta) + r_5(\theta)\frac{2\Omega_n^2(\theta)}{g^2}\Lambda_n^2(\theta) + r_6(\theta)\frac{2\Omega_n^2(\theta)}{g^2}\Big) u_1|u_1|^2.
\end{aligned} \tag{6.2.64}$$

After multiplying $(-\frac{g}{\Omega_n(\theta)})$ to (6.2.64), we obtain

$$2i\big(u_{2T} + \Omega_n'(\theta) u_{2X}\big) + \Omega_n''(\theta) u_{1XX} = \chi_1(\theta) u_1|u_1|^2 - \frac{u_1}{\Omega_n(\theta)}\big(\kappa_2\varphi_X + \kappa_1\varphi_T\big), \tag{6.2.65}$$

where $\kappa_1(\theta)$ and $\kappa_2(\theta)$ are introduced in (6.1.34) and (6.2.5). $\kappa_1(\theta)$ is obviously real by definition. $\kappa_2(\theta)$ is also real because both $\langle \ell_\theta(\psi_n), \psi_n\rangle$ and α_2 are purely imaginary.

The coefficient of the nonlinear term $u_1|u_1|^2$ is

$$\chi_1(\theta) = \frac{\Omega_n(\theta)}{g}\big(\frac{1}{\Lambda_n(\theta)} r_3(\theta) + \Lambda_n(\theta) r_4(\theta) + 2\Lambda_n^2(\theta) r_5(\theta) + 2r_6(\theta)\big), \tag{6.2.66}$$

which is also real. This is proved in Appendix B. □

6.3 Analysis at Order ε^3: Solvability Condition (S1)

Now we examine the solvability condition **(S1)** at order ε^3. We begin with a lemma that will be useful later.

Lemma 6.8. *For orthonormal basis $\{\psi_k(x,0)\}_{k\geq 0}$ for periodic functions, we have*

1. *When $k \neq 0$ and $\theta = 0$, we have*

$$\int_0^{2\pi} \psi_k(x,0)dx = 0. \tag{6.3.1}$$

2. *For any 2π periodic function $f(x)$, we have*

$$\int_0^{2\pi} G^{(0)}[b]f(x)dx = 0. \tag{6.3.2}$$

Proof. 1. When $\theta = 0$ and $k \neq 0$, we know $\psi_k(x,0)$ and $\psi_0(x,0) = \frac{1}{\sqrt{2\pi}}$ are orthonormal. Therefore,

$$\int_0^{2\pi} \psi_k(x,0)dx = \sqrt{2\pi}\langle \psi_k(x,0), \psi_0(x,0)\rangle = 0. \tag{6.3.3}$$

2. As we know $\Lambda_0(0) = 0$, for any 2π periodic function $f(x)$, we have

$$\begin{aligned}\int_0^{2\pi} G^{(0)}[b]f(x)dx &= \sqrt{2\pi}\langle G^{(0)}[b]f(x), \psi_0(x,0)\rangle \\ &= \sqrt{2\pi}\langle f, G^{(0)}[b]\psi_0(x,0)\rangle \\ &= \sqrt{2\pi}\Lambda_0(0)\langle f, \psi_0(x,0)\rangle = 0.\end{aligned} \tag{6.3.4}$$

□

Computing the "constant terms" (terms that are independent of x and t) in $U^{(3)}$ from (6.2.1) gives the following equation.

Proposition 6.4. *At order ε^3, the solvability condition **(S1)** of (6.1.3) implies*

$$\bar{\eta}_T + (h_b + \rho_2)\varphi_{XX} - \Big(\frac{i\Omega_n(\theta)}{g\pi} r_1(\theta) - \rho_1(\theta)\Big)|u_1|_X^2 = 0, \tag{6.3.5}$$

where we define

$$h_b := h - \frac{1}{2\pi}\int_0^{2\pi} B_0[b]b(x)dx, \tag{6.3.6}$$

$$\rho_1(\theta) := \frac{i}{2\pi}\sum_{k>0} \widetilde{f}_k(\theta)\int_0^{2\pi} B_0[b]A_0[b]\psi_k(x,0)dx + c.c., \tag{6.3.7}$$

and

$$\rho_2 := \frac{i}{2\pi}\sum_{k>0} \breve{f}_k \int_0^{2\pi} B_0[b]A_0[b]\psi_k(x,0)dx + c.c.. \tag{6.3.8}$$

The operators $B_0[b]$ and $A_0[b]$ are defined in Proposition 3.1 with operator D replaced by D_x in multiple scale regime:

$$B_0[b]A_0[b]\psi_k(x,0) = B_0[b]\sinh\big(b(x)D_x\big)\operatorname{sech}(hD_x)\psi_k(x,0). \tag{6.3.9}$$

We recall that $\bar{\eta}$, $\widetilde{f}_k(\theta)$, $\breve{f}_k$, and $r_1(\theta)$ are defined in (6.1.60), (6.1.61), and (6.2.20) respectively.

It is worth noting that h_b defined in (6.3.6) is a modified depth that also appears in the analysis of the effect of a periodic bottom in the KdV limit (see [6], page 854).

Proof. Recalling $U_i^{(3)}$ defined in (6.2.2), to find the "constant terms" in $U^{(3)}$, we need to compute the "constant term" in each $U_i^{(3)}$.

We observe that the variable t appears exclusively in either e^{iS_n} or e^{2iS_n} at this order. Therefore, terms independent of t must be those that do not contain e^{iS_n} or e^{2iS_n}. To find the "constant term", we only need to compute the zero-mode of t-independent term in $U_i^{(3)}$.

Our computation reveals that only $U_1^{(3)}$, $U_2^{(3)}$, $U_3^{(3)}$ and $U_6^{(3)}$ contain non-zero "constant terms", while the rest $U_i^{(3)}$ contribute nothing. The four non-zero "constant terms" are computed as follows:

1. Using the expression of $\eta^{(2)}$ in (6.1.45), we observe that the "constant term" in $U_1^{(3)}$ is

$$-\bar{\eta}_T. \tag{6.3.10}$$

2. Using the expression of $\xi^{(1)}$ in (6.1.2), we find the t-independent term in $U_2^{(3)}$ is $G_{II}^{(2)}[b]\varphi$.

   Using $G_{II}^{(2)}[b]$ defined in (5.2.41), the "constant term" is given by the the zero-mode of $G_{II}^{(2)}[b]\varphi$, which is

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} G_{II}^{(2)}[b]\varphi dx\\ =&-\frac{\varphi_{XX}}{2\pi}\int_0^{2\pi} h - B_0[b]b(x) + D_xB_0[b]b(x)\sinh(b(x)D_x)B_0[b]b(x)dx\\ =&-\varphi_{XX}\Big(h-\frac{1}{2\pi}\int_0^{2\pi}B_0[b]b(x)dx\Big) = -h_b\varphi_{XX}.\end{aligned} \tag{6.3.11}$$

   Because $B_0[b]\tilde{b}(x)\sinh(\tilde{b}(x)D_x)B_0[b]b(x)$ is a periodic function of x,

$$\int_0^{2\pi} D_xB_0[b]\tilde{b}(x)\sinh(\tilde{b}(x)D_x)B_0[b]b(x)dx = 0. \tag{6.3.12}$$

3. Using the expression of $\xi^{(2)}$ in (6.1.45), we find the t-independent term in $U_3^{(3)}$ is

$$G^{(1)}[b]\big(\sum_{k>0} f_k\psi_k(x,0) + c.c.\big),$$

which contains the "constant term"

$$\frac{1}{2\pi}\Big(\sum_{k>0}\int_0^{2\pi} G_{III}^{(1)}[b] f_k \psi_k(x,0)dx + c.c.\Big). \tag{6.3.13}$$

Using $G_{III}^{(1)}[b]$ defined in (5.2.46), we obtain

$$\begin{aligned}\int_0^{2\pi} G_{III}^{(1)}[b] f_k \psi_k(x,0)dx =& i f_{kX}\int_0^{2\pi} B_0[b]A_0[b]\psi_k(x,0)dx \\ =& i\big(\widetilde{f}_k(\theta)|u_1|_X^2 + \check{f}_k\varphi_{XX}\big)\int_0^{2\pi} B_0[b]A_0[b]\psi_k(x,0)dx\end{aligned} \tag{6.3.14}$$

Hence, the "constant term" in $U_3^{(3)}$ can be represented as

$$\begin{aligned}&\frac{1}{2\pi}\Big(\sum_{k>0}\int_0^{2\pi} G_{III}^{(1)}[b] f_k \psi_k(x,0)dx + c.c.\Big) \\ =&\frac{i}{2\pi}\sum_{k>0}\big(\widetilde{f}_k(\theta)|u_1|_X^2 + \check{f}_k\varphi_{XX}\big)\int_0^{2\pi} B_0[b]A_0[b]\psi_k(x,0)dx + c.c. \\ =&\rho_1|u_1|_X^2 + \rho_2\varphi_{XX},\end{aligned} \tag{6.3.15}$$

where $\rho_1(\theta)$ and $\rho_2(\theta)$ are defined in (6.3.7) and (6.3.8), respectively.

4. In $U_6^{(3)}$, the t-independent terms arise from the following two combinations of the terms in $\eta^{(1)}$ and $\xi^{(1)}$:

$$\begin{aligned}&D_X\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big)D_x\big(u_1 e^{iS_n}\psi_n\big) - G^{(1)}[b]\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big)G^{(0)}[b]\big(u_1 e^{iS_n}\psi_n\big) \\ &+ D_X\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)D_x\big(\overline{u_1 e^{iS_n}\psi_n}\big) - G^{(1)}[b]\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)G^{(0)}[b]\big(\overline{u_1 e^{iS_n}\psi_n}\big)\end{aligned} \tag{6.3.16}$$

Then the "constant term" is given as

$$\frac{i\Omega_n(\theta)}{\pi g}|u_1|_X^2\langle \ell_\theta(\psi_n),\psi_n\rangle. \tag{6.3.17}$$

Remark 5. *When computing (6.3.17), we have*

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} D_X\big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\big)D_x\big(u_1 e^{iS_n}\psi_n\big) + D_X\big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\big)D_x\big(\overline{u_1 e^{iS_n}\psi_n}\big)dx \\ =&\frac{1}{2\pi}\frac{i\Omega_n(\theta)}{g}|u_1|_X^2\int_0^{2\pi}\ell_\theta(\psi_n)\overline{\psi_n} - \psi_n\overline{\ell_\theta(\psi_n)}dx \\ =&\frac{1}{2\pi}\frac{i\Omega_n(\theta)}{g}|u_1|_X^2\big(\langle \ell_\theta(\psi_n),\psi_n\rangle - \langle \psi_n,\ell_\theta(\psi_n)\rangle\big) \\ =&\frac{i\Omega_n(\theta)}{\pi g}|u_1|_X^2\langle \ell_\theta(\psi_n),\psi_n\rangle\end{aligned}$$

Because $\langle \ell_\theta(\psi_n), \psi_n\rangle$ is purely imaginary, so we have

$$\langle \ell_\theta(\psi_n), \psi_n\rangle - \langle \psi_n, \ell_\theta(\psi_n)\rangle = 2\langle \ell_\theta(\psi_n), \psi_n\rangle.$$

In addition, we compute

$$\begin{aligned}
&-\frac{1}{2\pi}\int_0^{2\pi} G^{(1)}[b]\Big(\overline{\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n}\Big)G^{(0)}[b]\Big(u_1 e^{iS_n}\psi_n\Big)dx\\
&-\frac{1}{2\pi}\int_0^{2\pi} G^{(1)}[b]\Big(\frac{i\Omega_n(\theta)}{g}u_1 e^{iS_n}\psi_n\Big)G^{(0)}[b]\Big(\overline{u_1 e^{iS_n}\psi_n}\Big)dx\\
&=\frac{i\Omega_n(\theta)}{2\pi g}\Lambda_n(\theta)\int_0^{2\pi} G^{(1)}[b]\Big(|u_1|^2\overline{\psi_n}\psi_n\Big)dx - G^{(1)}[b]\Big(|u_1|^2\psi_n\overline{\psi_n}\Big)dx = 0
\end{aligned}$$

In total, combining all "constant terms" find in $U_1^{(3)}$, $U_2^{(3)}$, $U_3^{(3)}$ and $U_6^{(3)}$ yields (6.3.5). Using Lemma 6.8, it is easy to check that the "constant terms" in $U_i^{(3)}$ are all zero for $i = 4,5,7,8,9,10$. We do not present the details here.

6.4 Derivation of NLS Equation

We introduce $u := u_1 + \varepsilon u_2$ to combine (6.1.6) (at order ε^2) and (6.2.4) (at order ε^3) to obtain the following equation up to order ε

$$2i\big(u_T + \Omega_n'(\theta)u_X\big) + \varepsilon\Omega_n''(\theta)u_{XX} = \varepsilon\Big(\chi_1(\theta)|u|^2 - \frac{1}{\Omega_n(\theta)}\big(\kappa_2(\theta)\varphi_X + \kappa_1(\theta)\varphi_T\big)\Big)u. \tag{6.4.1}$$

On the other hand, the dynamics of $\overline{\eta}$ and φ are governed by equation (6.3.5) and the following one.

$$\bar{\eta} = -\frac{1}{g}\big(\frac{1}{2\pi}\kappa_1(\theta)|u_1|^2 + \varphi_T\big). \tag{6.4.2}$$

Substituting (6.4.2) into (6.3.5) to eliminate $\bar{\eta}$, we obtain (up to leading order)

$$\varphi_{TT} - g\big(h_b + \rho_2\big)\varphi_{XX} = -\big(\frac{i\Omega_n(\theta)}{\pi}r_1(\theta) - g\rho_1(\theta)\big)|u|_X^2 - \frac{1}{2\pi}\kappa_1(\theta)|u|_T^2. \tag{6.4.3}$$

Using (6.1.6), we can simplify (6.4.3) as

$$\varphi_{TT} - g\big(h_b + \rho_2\big)\varphi_{XX} = \kappa_3(\theta)|u|_X^2, \tag{6.4.4}$$

where

$$\kappa_3(\theta) := -\frac{i\Omega_n(\theta)}{\pi}r_1(\theta) + g\rho_1(\theta) + \frac{\Omega_n'(\theta)}{2\pi}\kappa_1(\theta). \tag{6.4.5}$$

$r_1(\theta)$, $\rho_1(\theta)$ and $\kappa_1(\theta)$ are defined in (6.2.20), (6.3.7) and (6.1.34).

We introduce $\mu = X - \Omega_n'(\theta)T$ and the longer time $\tau = \varepsilon^2 t$ to use a reference frame moving at the group velocity $\Omega_n'(\theta)$. Then (6.4.1) and (6.4.4) lead to (up to leading order)

$$2i\,u_\tau + \Omega_n''(\theta)\,u_{\mu\mu} = \chi_1(\theta)\,|u|^2u + \kappa_4(\theta)\,u\varphi_\mu \tag{6.4.6}$$

and

$$\Big(\big(\Omega_n'(\theta)\big)^2 - g(h_b+\rho_2)\Big)\varphi_{\mu\mu} = \kappa_3(\theta)\,|u|^2_\mu, \tag{6.4.7}$$

where

$$\kappa_4(\theta) := \frac{1}{\Omega_n(\theta)}\big(\Omega_n'(\theta)\kappa_1(\theta) - \kappa_2(\theta)\big). \tag{6.4.8}$$

For wave number $(n+\theta)$ such that $\big(\Omega_n'(\theta)\big)^2 - g(h_b+\rho_2) = 0$, (6.4.7) is singular and a different scaling needs to be considered [2] (see also Chapter 11.1 in [24]). We exclude this situation in our work.

Substituting (6.4.7) into (6.4.6), we derive the cubic NLS equation for the modulation of a solution with wave number $(n+\theta)$ to the water wave problem over a periodically variable bottom.

$$2i\,u_\tau + \Omega_n''(\theta)\,u_{\mu\mu} + \chi_b(\theta)|u|^2u = 0, \tag{6.4.9}$$

where

$$\chi_b(\theta) = -\chi_1(\theta) - \chi_2(\theta) \tag{6.4.10}$$

and

$$\chi_2(\theta) = \frac{\kappa_3(\theta)\kappa_4(\theta)}{(\Omega_n'(\theta))^2 - g(h_b+\rho_2)}. \tag{6.4.11}$$

In Appendix B, we check that the coefficient $\chi_b(\theta)$ of nonlinear term in the NLS equation is real.

Chapter 7

Perturbation Analysis for Small Bottom Variations

7.1 Approximation of Bloch-Floquet Eigenvalues and Eigenfunctions

7.1.1 Perturbation of Simple Eigenvalues

Recalling from Chapter 3, the Bloch-Floquet spectral problem of $G[b]_\theta = e^{-i\theta x}G[b]e^{i\theta x}$ with periodic boundary condition is

$$\begin{cases} G[b]_\theta\, \psi_n(x,\theta) = \Lambda_n(\theta)\, \psi_n(x,\theta), \\ \psi_n(x+2\pi,\theta) = \psi_n(x,\theta). \end{cases} \tag{7.1.1}$$

To perform a perturbative calculation of eignevalue $\Lambda_n(\theta)$ and eigenfunction $\psi_n(x,\theta)$, we express the periodic bathymetry as

$$b(x) = \gamma\beta(x), \tag{7.1.2}$$

and we assume that $\beta(x)$ has the Fourier expansion

$$\beta(x) = \sum_{k\in\mathbb{Z}} \widehat{\beta}_k e^{ikx} \tag{7.1.3}$$

with Fourier coefficients $\widehat{\beta}_k$. For clarity of computation, γ is assumed to be small and independent of the nonlinearity parameter ε.

By applying Proposition 3.1 and Proposition 3.3, we conclude that

$$G[b] = G[0] + DL[\gamma\beta] = D\tanh(hD) + \gamma DL_1[\beta] + \gamma^2 DL_2[\beta] + O(\gamma^3), \tag{7.1.4}$$

where

$$\begin{aligned} DL_1[\beta] &= -D\,\mathrm{sech}(hD)\,\beta\,D\,\mathrm{sech}(hD),\\ DL_2[\beta] &= -D\,\mathrm{sech}(hD)\,\beta\,D\tanh(hD)\,\beta\,D\,\mathrm{sech}(hD). \end{aligned} \tag{7.1.5}$$

Notation 7.1. *For convenience, we introduce the following notations:*

$$g_n(\theta) = (n+\theta)\tanh\big(h(n+\theta)\big) \quad \text{and} \quad s_n(\theta) = (n+\theta)\,\mathrm{sech}\big(h(n+\theta)\big). \tag{7.1.6}$$

In the case of a flat bottom $\beta = 0$, the eigenvalues and normalized eigenfunctions are easily computed as

$$\begin{cases} \lambda_n^{(0)}(\theta) = (n+\theta)\tanh(h(n+\theta)) = g_n(\theta),\\ \widetilde{\psi_n}^{(0)}(x,\theta) = \dfrac{1}{\sqrt{2\pi}}e^{inx}. \end{cases} \tag{7.1.7}$$

Here, the superscript $^{(0)}$ indicates the flat bottom $\beta(x) = 0$. $(\lambda_n, \widetilde{\psi_n})$ denote the eigenvalues and normalized eigenfunctions without relabeling. Later, we will relabel them in order of increasing magnitude of $\Lambda_n(\theta)$, as explained in Chapter 3.2, to obtain (Λ_n, ψ_n).

When θ is not too close to 0 and $\pm\frac{1}{2}$, $\lambda_n(\theta)$ is simple eigenvalue and can be approximated as follows.

Proposition 7.1. *In the case of a periodic bottom $\beta \neq 0$, the eigenvalue $\lambda_n(\theta)$ and eigenfunction $\widetilde{\psi_n}(x,\theta)$ of (7.1.1) have the expansions in γ around $\beta = 0$ as follows:*

$$\begin{cases} \lambda_n(\theta) = g_n(\theta) + \gamma\lambda_n^{(1)}(\theta) + \gamma^2\lambda_n^{(2)}(\theta) + \cdots,\\ \widetilde{\psi_n}(x,\theta) = \alpha_n^{(0)}e^{inx} + \gamma\sum_{k\neq n}\alpha_k^{(1)}e^{ikx} + \gamma^2\sum_{k\neq n}\alpha_k^{(2)}e^{ikx} + \cdots, \end{cases} \tag{7.1.8}$$

where

$$\begin{aligned} \lambda_n^{(1)}(\theta) =& 0,\\ \lambda_n^{(2)}(\theta) =& -s_n^2(\theta)\Big(\sum_{k\in\mathbb{Z}}|\widehat{\beta}_k|^2 g_{n+k}(\theta) + \sum_{\substack{k\in\mathbb{Z},\\ k\neq n}}|\widehat{\beta_{n-k}}|^2\frac{s_k^2(\theta)}{g_k(\theta)-g_n(\theta)}\Big), \end{aligned} \tag{7.1.9}$$

and

$$\begin{aligned} \alpha_k^{(1)} =& \alpha_n^{(0)}\frac{\widehat{\beta_{k-n}}s_n(\theta)s_k(\theta)}{g_k(\theta)-g_n(\theta)},\\ \alpha_k^{(2)} =& \alpha_n^{(0)}\frac{s_n(\theta)s_k(\theta)}{g_k(\theta)-g_n(\theta)}\Big(\sum_{l\in\mathbb{Z}}\widehat{\beta_{k-n-l}}\widehat{\beta}_l g_{n+l}(\theta) + \sum_{\substack{l\in\mathbb{Z},\\ l\neq n}}\widehat{\beta_{k-l}}\widehat{\beta_{l-n}}s_l^2(\theta)\frac{1}{g_l(\theta)-g_n(\theta)}\Big). \end{aligned} \tag{7.1.10}$$

Coefficient $\alpha_n^{(0)}$ is a constant used to normalize the eigenfunction $\widetilde{\psi_n}(x,\theta)$.

Proof. We assume that the expansions of $\lambda_n(\theta)$ and $\widetilde{\psi_n}(x,\theta)$ in powers of γ have the forms described in (7.1.8). Then we substitute $G[b]$ from (7.1.4) and $(\lambda_n, \widetilde{\psi_n})$ from (7.1.8) into (7.1.1). By rearranging and collecting terms according to the orders of γ, we get at $O(\gamma)$

$$\begin{aligned}&\big(e^{-i\theta x}G[0]e^{i\theta x}\big)\sum_{k\neq n}\alpha_k^{(1)}e^{ikx}+\big(e^{-i\theta x}DL_1[\beta]e^{i\theta x}\big)\alpha_n^{(0)}e^{inx}\\ =&g_n(\theta)\sum_{k\neq n}\alpha_k^{(1)}e^{ikx}+\lambda_n^{(1)}(\theta)\alpha_n^{(0)}e^{inx},\end{aligned} \tag{7.1.11}$$

and at $O(\gamma^2)$

$$\begin{aligned}&\big(e^{-i\theta x}G[0]e^{i\theta x}\big)\sum_{k\neq n}\alpha_k^{(2)}e^{ikx}+\big(e^{-i\theta x}DL_2[\beta]e^{i\theta x}\big)\alpha_n^{(0)}e^{inx}+\big(e^{-i\theta x}DL_1[\beta]e^{i\theta x}\big)\sum_{k\neq n}\alpha_k^{(1)}e^{ikx}\\ =&g_n(\theta)\sum_{k\neq n}\alpha_k^{(2)}e^{ikx}+\lambda_n^{(2)}(\theta)\alpha_n^{(0)}e^{inx}+\lambda_n^{(1)}\sum_{k\neq n}\alpha_k^{(1)}e^{ikx}.\end{aligned} \tag{7.1.12}$$

At $O(\gamma)$, we compute the two terms in the left-hand side of (7.1.11) as

$$\big(e^{-i\theta x}G[0]e^{i\theta x}\big)\sum_{k\neq n}\alpha_k^{(1)}e^{ikx}=\sum_{k\neq n}\alpha_k^{(1)}g_k(\theta)e^{ikx}, \tag{7.1.13}$$

and

$$\begin{aligned}\big(e^{-i\theta x}DL_1[\beta]e^{i\theta x}\big)\alpha_n^{(0)}e^{inx}=&\big(-e^{-i\theta x}D\,\mathrm{sech}(hD)\,\beta\,D\,\mathrm{sech}(hD)e^{i\theta x}\big)\alpha_n^{(0)}e^{inx}\\ =&\big(-e^{-i\theta x}D\,\mathrm{sech}(hD)\big)\big(\sum_{k\in\mathbb{Z}}\widehat{\beta}_ke^{ikx}\big)\alpha_n^{(0)}s_n(\theta)e^{i(n+\theta)x}\\ =&\big(-e^{-i\theta x}D\,\mathrm{sech}(hD)\big)\sum_{k\in\mathbb{Z}}\widehat{\beta}_k\alpha_n^{(0)}s_n(\theta)e^{i(n+k+\theta)x}\\ =&-e^{-i\theta x}\sum_{k\in\mathbb{Z}}\widehat{\beta}_k\alpha_n^{(0)}s_n(\theta)s_{n+k}(\theta)e^{i(n+k+\theta)x}\\ =&-\sum_{k\in\mathbb{Z}}\widehat{\beta}_k\alpha_n^{(0)}s_n(\theta)s_{n+k}(\theta)e^{i(n+k)x}\\ =&-\alpha_n^{(0)}s_n(\theta)\sum_{p\in\mathbb{Z}}\widehat{\beta_{p-n}}s_p(\theta)e^{ipx},\end{aligned} \tag{7.1.14}$$

where we use $DL_1[\beta]$ in (7.1.5) and β in (7.1.3) in the above calculation. Hence, the left-hand side of (7.1.11) can be expressed as

$$\sum_{k\neq n}\alpha_k^{(1)}g_k(\theta)e^{ikx}-\alpha_n^{(0)}s_n(\theta)\sum_{p\in\mathbb{Z}}\widehat{\beta_{p-n}}s_p(\theta)e^{ipx}. \tag{7.1.15}$$

Identifying (7.1.15) with the right-hand side of (7.1.11), the coefficient of e^{inx} satisfies

$$-\alpha_n^{(0)}s_n^2(\theta)\widehat{\beta}_0=\lambda_n^{(1)}(\theta)\alpha_n^{(0)}, \tag{7.1.16}$$

as well as the coefficient of e^{ikx} (for $k \neq n$) satisfies

$$\alpha_k^{(1)} g_k(\theta) - \alpha_n^{(0)} s_n(\theta) s_k(\theta) \widehat{\beta_{k-n}} = g_n(\theta) \alpha_k^{(1)}. \tag{7.1.17}$$

Under the assumption $\int_0^{2\pi} \beta(x)dx = 0$, we require $\widehat{\beta}_0 = 0$. Hence, (7.1.16) is solved as

$$\lambda_n^{(1)}(\theta) = 0. \tag{7.1.18}$$

Moreover, we solve (7.1.17) to get

$$\alpha_k^{(1)} = \alpha_n^{(0)} \frac{\widehat{\beta_{k-n}} s_n(\theta) s_k(\theta)}{g_k(\theta) - g_n(\theta)}. \tag{7.1.19}$$

At $O(\gamma^2)$, we compute three terms on the left-hand side of (7.1.12) in a manner analogous to our approach at the order γ, yielding

$$\left(e^{-i\theta x} G[0] e^{i\theta x}\right) \sum_{k\neq n} \alpha_k^{(2)} e^{ikx} = \sum_{k\neq n} \alpha_k^{(2)} g_k(\theta) e^{ikx}, \tag{7.1.20}$$

$$\left(e^{-i\theta x} DL_2[\beta] e^{i\theta x}\right) \alpha_n^{(0)} e^{inx} = -\alpha_n^{(0)} s_n(\theta) \sum_{p,k\in\mathbb{Z}} \widehat{\beta_{p-n-k}} \widehat{\beta}_k g_{n+k}(\theta) s_p(\theta) e^{ipx}, \tag{7.1.21}$$

and

$$\left(e^{-i\theta x} DL_1[\beta] e^{i\theta x}\right) \sum_{k\neq n} \alpha_k^{(1)} e^{ikx} = -\sum_{\substack{k,q\in\mathbb{Z},\\ k\neq n}} \widehat{\beta_{q-k}} \alpha_k^{(1)} s_k(\theta) s_q(\theta) e^{iqx}. \tag{7.1.22}$$

Therefore, the left-hand side of (7.1.12) can be expressed as

$$\sum_{k\neq n} \alpha_k^{(2)} g_k(\theta) e^{ikx} - \alpha_n^{(0)} s_n(\theta) \sum_{p,k\in\mathbb{Z}} \widehat{\beta_{p-n-k}} \widehat{\beta}_k g_{n+k}(\theta) s_p(\theta) e^{ipx} - \sum_{\substack{k,q\in\mathbb{Z},\\ k\neq n}} \widehat{\beta_{q-k}} \alpha_k^{(1)} s_k(\theta) s_q(\theta) e^{iqx} \tag{7.1.23}$$

Identifying the coefficients of e^{inx} on both sides of (7.1.12) leads to

$$-\alpha_n^{(0)} s_n^2(\theta) \sum_{k\in\mathbb{Z}} \widehat{\beta_{-k}} \widehat{\beta}_k g_{n+k}(\theta) - s_n(\theta) \sum_{k\neq n} \widehat{\beta_{n-k}} \alpha_k^{(1)} s_k(\theta) = \lambda_n^{(2)}(\theta) \alpha_n^{(0)}. \tag{7.1.24}$$

By substituting $\alpha_k^{(1)}$ from (7.1.19), we solve (7.1.24) as

$$\lambda_n^{(2)}(\theta) = -s_n^2(\theta) \Big(\sum_{k\in\mathbb{Z}} |\widehat{\beta}_k|^2 g_{n+k}(\theta) + \sum_{k\neq n} |\widehat{\beta_{n-k}}|^2 \frac{s_k^2(\theta)}{g_k(\theta) - g_n(\theta)} \Big). \tag{7.1.25}$$

Similarly, identifying the coefficients of e^{ikx} (for $k \neq n$) on both sides of (7.1.12) implies that

$$\alpha_k^{(2)} g_k(\theta) - \alpha_n^{(0)} s_n(\theta) \sum_{l\in\mathbb{Z}} \widehat{\beta_{k-n-l}} \widehat{\beta}_l g_{n+l}(\theta) s_k(\theta) - \sum_{l\neq n} \widehat{\beta_{k-l}} \alpha_l^{(1)} s_l(\theta) s_k(\theta) = g_n(\theta) \alpha_k^{(2)}, \tag{7.1.26}$$

which implies that

$$\alpha_k^{(2)} = \alpha_n^{(0)} \frac{s_n(\theta)s_k(\theta)}{g_k(\theta)-g_n(\theta)} \Big(\sum_{l\in\mathbb{Z}} \widehat{\beta_{k-n-l}}\widehat{\beta_l} g_{n+l}(\theta) + \sum_{l\neq n} \widehat{\beta_{k-l}}\widehat{\beta_{l-n}} s_l^2(\theta) \frac{1}{g_l(\theta)-g_n(\theta)} \Big). \quad (7.1.27)$$

In above computation, $\alpha_n^{(0)}$ can be an arbitrary constant; however, we choose its value such that the eigenfunction $\widetilde{\psi_n}(x,\theta)$ is normalized. □

In summary, we observe that

$$\lambda_n(\theta) = g_n(\theta) - \gamma^2 s_n^2(\theta) \Big(\sum_{k\in\mathbb{Z}} |\widehat{\beta_k}|^2 g_{n+k}(\theta) + \sum_{\substack{k\in\mathbb{Z},\\ k\neq n}} |\widehat{\beta_{n-k}}|^2 \frac{s_k^2(\theta)}{g_k(\theta)-g_n(\theta)} \Big) + O(\gamma^3), \quad (7.1.28)$$

and $\lambda_n(\theta)$ contributes zero at $O(\gamma)$.

As explained in Chapter 3.2, we reorder $\lambda_n(\theta)$ appropriately in order of increasing magnitude and relabel them by $\Lambda_n(\theta)$ as follows:

$$\begin{cases} \Lambda_{2n}(\theta) = \lambda_{-n}(\theta), & \psi_{2n}(x,\theta) = \widetilde{\psi_{-n}}(x,\theta), \quad \text{for } -\frac{1}{2} \leq \theta < 0; \\ \Lambda_{2n}(\theta) = \lambda_n(\theta), & \psi_{2n}(x,\theta) = \widetilde{\psi_n}(x,\theta), \quad \text{for } 0 \leq \theta < \frac{1}{2}; \end{cases} \quad (7.1.29)$$

and

$$\begin{cases} \Lambda_{2n-1}(\theta) = \lambda_n(\theta), & \psi_{2n-1}(x,\theta) = \widetilde{\psi_n}(x,\theta), \quad \text{for } -\frac{1}{2} \leq \theta < 0; \\ \Lambda_{2n-1}(\theta) = \lambda_{-n}(\theta), & \psi_{2n-1}(x,\theta) = \widetilde{\psi_{-n}}(x,\theta), \quad \text{for } 0 \leq \theta < \frac{1}{2}. \end{cases} \quad (7.1.30)$$

7.1.2 Approximation of the Spectral Gaps

Another perturbative calculation is provided in [5]. W Craig *et al*. compute the two eigenvalues $\Lambda_{2n-1}(\theta)$ and $\Lambda_{2n}(\theta)$ perturbatively and thus give conditions for the opening of the nth spectral gap created near $\theta = 0$ in the presence of a periodic bottom $b(x) = \gamma\beta(x)$. Starting by briefly recalling their calculation near $\theta = 0$, we extend their work and present a similar result around $\theta = \frac{1}{2}$.

Case 1: Around $\theta = 0$.

When $b(x) = 0$, the eigenvalue is double at $\theta = 0$. We know

$$\Lambda_{2n-1}^{(0)}(0) = \Lambda_{2n}^{(0)}(0) = n\tanh(hn) \quad (7.1.31)$$

with corresponding eigenfunctions $\psi_{2n-1}^{(0)}(x,\theta) = e^{-inx}$ and $\psi_{2n}^{(0)}(x,\theta) = e^{inx}$ respectively, as shown in (3.2.13) and (3.2.14).

To calculate the two eigenvalues $\Lambda_{2n-1}(\theta)$ and $\Lambda_{2n}(\theta)$ near $\theta = 0$ in the presence of a periodic bottom $b(x)$, we restrict our calculation to the subspace spanned by $\{e^{-inx}, e^{inx}\}$, neglecting the contribution of the other modes. In addition, we simplify the operator $DL[b]$ in $G[b]$ by replacing it by its first term $DL_1[b]$ of the Taylor expansion in $b(x)$.

We examine the action of $e^{-i\theta x}\big(G[0]+\gamma DL_1[\beta]\big)e^{i\theta x}$ on periodic function $\alpha_n e^{inx}+\alpha_{-n}e^{-inx}$, and consider the following spectrum problem

$$e^{-i\theta x}\big(G[0]+\gamma DL_1[\beta]\big)e^{i\theta x}\big(\alpha_n e^{inx}+\alpha_{-n}e^{-inx}\big)=\Lambda(\theta)\big(\alpha_n e^{inx}+\alpha_{-n}e^{-inx}\big). \tag{7.1.32}$$

Substituting the Fourier expansion of $\beta(x)$ in (7.1.3) and $DL_1[\beta]$ in (7.1.5), we obtain a matrix equation

$$\begin{pmatrix} g_n(\theta) & -\gamma\widehat{\beta_{2n}}s_n(\theta)s_n(-\theta) \\ -\gamma\overline{\widehat{\beta_{2n}}}s_n(\theta)s_n(-\theta) & g_{-n}(\theta)\end{pmatrix}\begin{pmatrix}\alpha_n\\ \alpha_{-n}\end{pmatrix}=\Lambda(\theta)\begin{pmatrix}\alpha_n\\ \alpha_{-n}\end{pmatrix}. \tag{7.1.33}$$

Here, the eigenvalues $\Lambda(\theta)$ are explicitly computed as

$$\Lambda(\theta)=\frac{1}{2}\Big(g_n(\theta)+g_n(-\theta)\pm\sqrt{\big(g_n(\theta)-g_n(-\theta)\big)^2+4\gamma^2|\widehat{\beta_{2n}}|^2 s_n^2(\theta)s_n^2(-\theta)}\Big), \tag{7.1.34}$$

where the different signs lead to two eigenvalues $\Lambda_{2n-1}(\theta)\leq\Lambda_{2n}(\theta)$:

$$\begin{aligned}\Lambda_{2n-1}(\theta)=&\frac{1}{2}\Big(g_n(\theta)+g_n(-\theta)-\sqrt{\big(g_n(\theta)-g_n(-\theta)\big)^2+4\gamma^2|\widehat{\beta_{2n}}|^2 s_n^2(\theta)s_n^2(-\theta)}\Big),\\ \Lambda_{2n}(\theta)=&\frac{1}{2}\Big(g_n(\theta)+g_n(-\theta)+\sqrt{\big(g_n(\theta)-g_n(-\theta)\big)^2+4\gamma^2|\widehat{\beta_{2n}}|^2 s_n^2(\theta)s_n^2(-\theta)}\Big).\end{aligned} \tag{7.1.35}$$

Especially, at $\theta=0$, we have

$$\begin{cases}\Lambda_{2n-1}(0) & =g_n(0)-\gamma|\widehat{\beta_{2n}}|s_n^2(0)\\ \Lambda_{2n}(0) & =g_n(0)+\gamma|\widehat{\beta_{2n}}|s_n^2(0)\end{cases} \tag{7.1.36}$$

Therefore, if $\widehat{\beta_{2n}}\neq 0$, the eigenvalues $\Lambda_{2n-1}(0)$ and $\Lambda_{2n}(0)$ split and the gap between them at $\theta=0$ is

$$\Lambda_{2n}(0)-\Lambda_{2n-1}(0)=2\gamma|\widehat{\beta_{2n}}|s_n^2(0), \tag{7.1.37}$$

which exponentially decays as n approaches infinity.

Case 2: Around $\theta=\frac{1}{2}$.

As shown in (3.2.13) and (3.2.14), we have double eigenvalues

$$\Lambda^{(0)}_{2n+1}(\frac{1}{2})=\Lambda^{(0)}_{2n}(\frac{1}{2})=g_n(\frac{1}{2}) \tag{7.1.38}$$

with corresponding eigenfunctions $\psi^{(0)}_{2n+1}(x,\theta)=e^{-i(n+1)x}$ and $\psi^{(0)}_{2n}(x,\theta)=e^{inx}$ respectively. Similar to the case abound 0, we consider the following spectral problem

$$e^{-i\theta x}\big(G[0]+\gamma DL_1[\beta]\big)e^{i\theta x}\big(\alpha_n e^{inx}+\alpha_{-n-1}e^{-i(n+1)x}\big)=\Lambda(\theta)\big(\alpha_n e^{inx}+\alpha_{-n-1}e^{-i(n+1)x}\big). \tag{7.1.39}$$

Examining the action of $e^{-i\theta x}\big(G[0]+\gamma DL_1[\beta]\big)e^{i\theta x}$ on periodic function $\alpha_n e^{inx}+\alpha_{-n-1}e^{-i(n+1)x}$ leads to

$$\begin{pmatrix} g_n(\theta) & \gamma\overline{\widehat{\beta_{2n+1}}}s_n(\theta)s_{-n-1}(\theta) \\ \gamma\widehat{\beta_{2n+1}}s_n(\theta)s_{-n-1}(\theta) & g_{-n-1}(\theta) \end{pmatrix} \begin{pmatrix} \alpha_n \\ \alpha_{-n-1} \end{pmatrix} = \Lambda(\theta) \begin{pmatrix} \alpha_n \\ \alpha_{-n-1} \end{pmatrix}. \tag{7.1.40}$$

The eigenvalues $\Lambda_{2n}(\theta) \le \Lambda_{2n+1}(\theta)$ are explicitly computed as

$$\begin{aligned} \Lambda_{2n}(\theta) &= \frac{1}{2}\Big(g_n(\theta)+g_{-k-1}(\theta)-\sqrt{\big(g_n(\theta)-g_{-k-1}(\theta)\big)^2+4\gamma^2|\widehat{\beta_{2n+1}}|^2 s_n^2(\theta)s_{-n-1}^2(\theta)}\Big), \\ \Lambda_{2n+1}(\theta) &= \frac{1}{2}\Big(g_n(\theta)+g_{-k-1}(\theta)+\sqrt{\big(g_n(\theta)-g_{-k-1}(\theta)\big)^2+4\gamma^2|\widehat{\beta_{2n+1}}|^2 s_n^2(\theta)s_{-n-1}^2(\theta)}\Big). \end{aligned} \tag{7.1.41}$$

When $\theta = \frac{1}{2}$, we have

$$\begin{aligned} g_n(\frac{1}{2}) &= (n+\frac{1}{2})\tanh(n+\frac{1}{2}) = (-n-1+\frac{1}{2})\tanh(-n-1+\frac{1}{2}) = g_{-n-1}(\frac{1}{2}), \\ s_n(\frac{1}{2}) &= (n+\frac{1}{2})\,\mathrm{sech}(n+\frac{1}{2}) = (-n-1+\frac{1}{2})\,\mathrm{sech}(-n-1+\frac{1}{2}) = -s_{-n-1}(\frac{1}{2}), \end{aligned} \tag{7.1.42}$$

because $x\tanh(x)$ is even and $x\,\mathrm{sech}(x)$ is odd. Therefore, we conclude that

$$\begin{cases} \Lambda_{2n+1}(\frac{1}{2}) & = g_n(\frac{1}{2})+\gamma|\widehat{\beta_{2n+1}}|s_n^2(\frac{1}{2}), \\ \Lambda_{2n}(\frac{1}{2}) & = g_n(\frac{1}{2})-\gamma|\widehat{\beta_{2n+1}}|s_n^2(\frac{1}{2}), \end{cases}$$

and the nonzero Fourier coefficients $\widehat{\beta_{2n+1}}$ leads to a gap at $\theta = \frac{1}{2}$

$$\Lambda_{2n+1}(\frac{1}{2})-\Lambda_{2n}(\frac{1}{2}) = 2\gamma|\widehat{\beta_{2n+1}}|s_n^2(\frac{1}{2}). \tag{7.1.43}$$

in the presence of a periodic bottom.

7.2 Effect of Bottom Topography on NLS Coefficients

We recall that the cubic NLS equation derived from the flat bottom problem in Chapter 4 is

$$2iu_\tau + \omega''(k)\,u_{\mu\mu} + \chi(k)|u|^2 u = 0. \tag{7.2.1}$$

Since $\omega(k)$ is determined by the dispersion relation

$$\omega(k) = \sqrt{gk\tanh(hk)}, \tag{7.2.2}$$

the coefficient $\omega''(k)$ in (7.2.1) is negative for all k.

The coefficient $\chi(k)$ of the nonlinear term in (7.2.1) is defined as

$$\chi(k) = -\frac{k^3\omega(k)}{g}H(kh), \tag{7.2.3}$$

with $\sigma = \tanh(hk)$ and

$$H(kh) = -\frac{1}{2\sigma}(-2\sigma^4 + 13\sigma^2 - 12 + 9\sigma^{-2}) - \frac{\left(4\sigma + (1-\sigma^2)(\sigma + kh(1-\sigma^2))\right)^2}{\sigma\left((\sigma + kh(1-\sigma^2))^2 - 4kh\sigma\right)}. \tag{7.2.4}$$

A numerical evidence (see Figure 7.1) shows that the function $H(kh)$ is monotonically decreasing, with values that are positive when $kh < 1.363$ and negative when $kh > 1.363$. The horizontal asymptote of $H(kh)$ is $y = -4$, as shown in Figure 7.1.

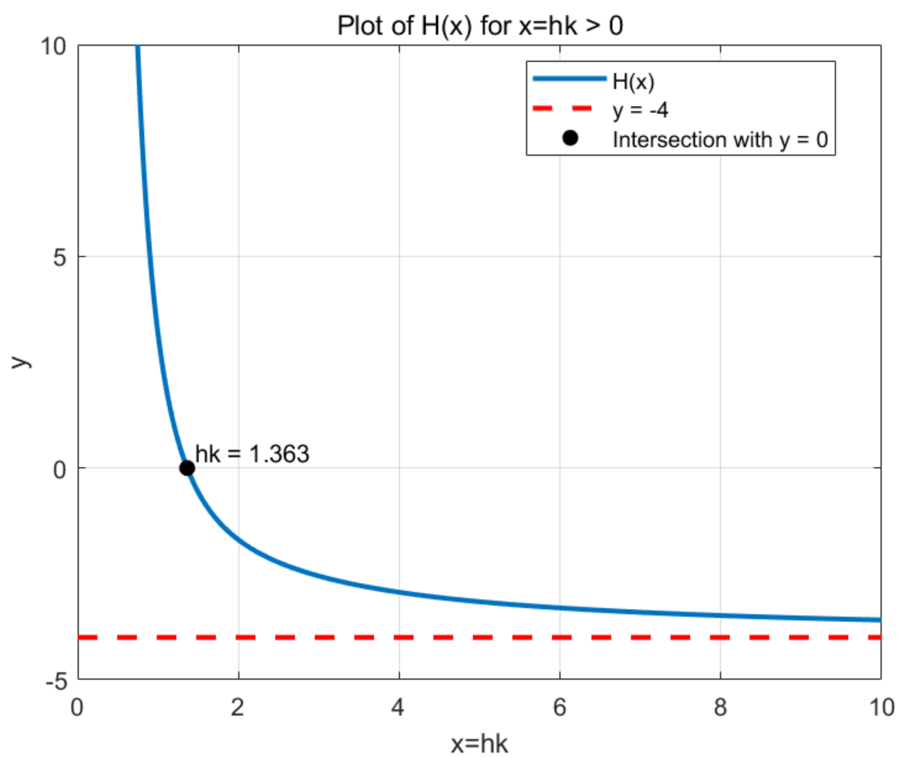

Figure 7.1: Graph of $H(kh)$

Depending on the values of the coefficients $\omega''(k)$ and $H(kh)$, there are two types of the NLS equation. If $\omega''(k)H(kh) > 0$, it is referred to as the focusing case. Conversely, if $\omega''(k)H(kh) < 0$, it is termed the defocusing case.

Furthermore, the modulational instability, also know as Benjamin-Feir instability, of a monochromatic wave depends on hk. For the deep water case ($kh > 1.363$), we have the focusing NLS equation and the solutions are modulationally unstable. For the shallow water case ($kh < 1.363$), we have the defocusing NLS equation and the solutions are modulationally stable.

Unlike $\omega''(k) < 0$ in the flat bottom problem, we show that the presence of the bottom $b(x)$ could result in a change in the sign of $\Omega_n''(\theta)$ by providing a specific example. Specifically, we choose a depth $h = 1$ and a bottom $b(x) = \gamma\beta(x)$ with $\gamma = 0.1$ and $\beta(x) = cosx + cos2x$. The Fourier expansion of this $\beta(x)$ is

$$\beta(x) = cosx + cos2x = \sum_{k=-2}^{2} \frac{1}{2} e^{ikx}. \tag{7.2.5}$$

As $\widehat{\beta_1} = \widehat{\beta_2} = \frac{1}{2} \neq 0$, the first gap occurs at $\theta = \pm\frac{1}{2}$, and the second gap occurs at $\theta = 0$. Both of these gaps are of order $O(\gamma)$. Substituting all these parameters into the different formulas of $\Lambda_n(\theta)$ obtained in (7.1.8), (7.1.35) and (7.1.41), we can create a graph of $\Omega_n(\theta) = \sqrt{g\Lambda_n(\theta)}$ for $0 \leq n \leq 4$.

Specifically, around $\theta = 0$ and $\frac{1}{2}$, we employ the eigenvalues provided in (7.1.35) and (7.1.41) respectively. These eigenvalues are obtained from the perturbation of double eigenvalues near $\theta = 0$ and $\frac{1}{2}$. However, for θ far away from 0 and $\frac{1}{2}$, we use the eigenvalues in (7.1.8), which are obtained from the perturbation of simple eigenvalues.

From the Figure 7.2, we can see the concavity of $\Omega_1(\theta)$ changes around $\frac{1}{2}$, and the concavity of $\Omega_2(\theta)$ changes around 0 because of the presence of a periodic bottom b(x). This is different with case in the flat bottom problem that $\omega''(k)$ is always negative.

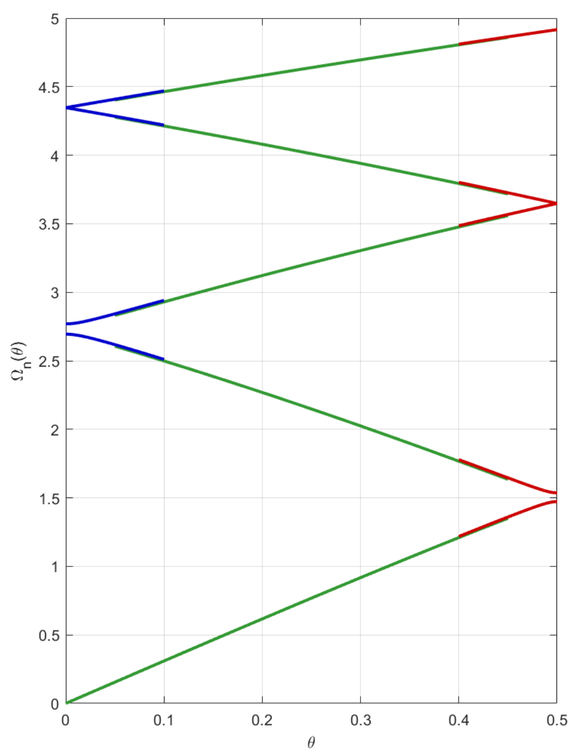

Figure 7.2: Graph of $\Omega_n(\theta)$

Chapter 8

Conclusion and Future Work

In this thesis, we performed a modulational analysis of the two-dimensional water wave problem in the presence of a periodic bottom. We express the solution of the water wave problem in the form of slowly modulated Bloch-Floquet waves and derive the cubic NLS equation (6.4.9) that governs the dynamics of the amplitude of wavepackets.

Furthermore, to investigate the effect of the periodic bottom on the NLS equation, we perform perturbative calculations to the Bloch-Floquet eigenvalues $\Lambda_n(\theta)$ and eigenfunctions $\psi_n(x,\theta)$. We find that the coefficient $\Omega_n''(\theta)$ of linear term in (6.4.9) may change the sign near $\theta = 0, \frac{1}{2}$. Due to the presence of the periodic bottom, the double eigenvalues $\Lambda_n(\theta)$ may split, creating a spectral gap near $\theta = -\frac{1}{2}, 0, \frac{1}{2}$. For values of n and θ near such gaps, we observe from Figure 7.2 that $\Omega_n(\theta)$ changes the concavity, indicating that $\Omega_n''(\theta)$ changes to positive for these values of n and θ.

However, because of the complexity of the formula giving the coefficient of nonlinear term in (6.4.9), we are unable to explicitly calculate how it is modified due to the presence of a variable bottom. The presence of the bottom could have the effect of changing the sign of the linear and nonlinear terms in the NLS equation, making it focusing or defocusing. In particular, in the focusing case, the Benjamin-Feir instability may occur.

In this thesis, we construct an approximate solution of the water wave problem in the form

$$\begin{cases} \eta^{app} = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + \cdots, \\ \xi^{app} = \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \varepsilon^3\xi^{(3)} + \cdots. \end{cases} \tag{8.1}$$

with explicit expressions for $\eta^{(1)}, \xi^{(1)}, \eta^{(2)}$ and $\xi^{(2)}$. Inspired by the works of [12] and [26], another potential work in the future is to estimate how well these constructed approximate solutions satisfy the water wave problem.

In a first step towards a justification of the validity of this approximation, we propose the following analysis. Denoting the original water wave system as

$$W(\eta, \xi) = 0, \tag{8.2}$$

one computes $W(\eta^{app}, \xi^{app})$ and estimate it in Sobolev norms. The goal is to find that for some appropriate s and p, an estimate of the form $||W(\eta^{app}, \xi^{app})||_{W^{s,p}} = O(\varepsilon^q)$ for some $q > 2$ holds. In the case of infinite depth, Totz and Wu proved a stronger result. They compare the leading order of the constructed approximate solution to an exact solution of the water wave problem, namely establishing an estimate of the form $||(\eta, \xi) - \varepsilon(\eta^{(1)}, \xi^{(1)})||_{H^s}$ over time $t = O(\varepsilon^{-2})$, corresponding to a carefully selected initial condition close to a modulated wavepacket.

Appendix A

Evaluation of the coefficient $(4\Lambda_n(\theta) - \Lambda_k(2\theta))$

We examine whether the denominator $4\Lambda_n(\theta) - \Lambda_k(2\theta) = 0$ that appear in the expressions of c_k and d_k in (6.1.56) may vanish. In such cases, the analysis breaks down and a new scaling may be required. We exclude these cases in our work. However, it is crucial to ensure that for some selected (n,θ), $4\Lambda_n(\theta) \neq \Lambda_k(2\theta)$ for any k.

Recalling the perturbation calculation (7.1.28) we conducted in Chapter 7, the effect of small $b(x) = \gamma\beta(x)$ on the simple eigenvalues is:

$$\Lambda_n(\theta) = \Lambda_n^{(0)}(\theta) + O(\gamma^2). \tag{A.1}$$

Therefore, if we check $4\Lambda_n^{(0)}(\theta) \neq \Lambda_k^{(0)}(2\theta)$ for some values of (n,θ) when $b(x) = 0$, then by continuity, $4\Lambda_n(\theta) \neq \Lambda_k(2\theta)$ remains true for small enough γ.

When $b(x) = 0$, the eigenvalues $\Lambda_n^{(0)}(\theta)$ are reordered by their magnitude as follows (see Figure 3.1):

$$\begin{cases} \Lambda_{2n}^{(0)}(\theta) = g_{-n}(\theta), & \psi_{2n}^{(0)}(x,\theta) = e^{-inx}, \quad \text{for } -\frac{1}{2} \leq \theta < 0; \\ \Lambda_{2n}^{(0)}(\theta) = g_n(\theta), & \psi_{2n}^{(0)}(x,\theta) = e^{inx}, \quad \text{for } 0 \leq \theta < \frac{1}{2}; \end{cases} \tag{A.2}$$

and

$$\begin{cases} \Lambda_{2n-1}^{(0)}(\theta) = g_n(\theta), & \psi_{2n-1}^{(0)}(x,\theta) = e^{inx}, \quad \text{for } -\frac{1}{2} \leq \theta < 0; \\ \Lambda_{2n-1}^{(0)}(\theta) = g_{-n}(\theta), & \psi_{2n-1}^{(0)}(x,\theta) = e^{-inx}, \quad \text{for } 0 \leq \theta < \frac{1}{2}, \end{cases} \tag{A.3}$$

where $g_n(\theta) = (n+\theta)\tanh(h(n+\theta))$.

Without loss of generality, we can choose $h = 1$, $\theta \in (0, 1/2)$ and n to be an even number, that is, $n = 2n'$ for some n'. Then the eigenvalue

$$\Lambda_n^{(0)}(\theta) = g_{n'}(\theta) = (n'+\theta)\tanh(n'+\theta). \tag{A.4}$$

For $\Lambda_k^{(0)}(2\theta)$, there are two cases could happen:

1. k is even (i.e. $k=2k'$).

$$\Lambda_k^{(0)}(2\theta)=g_{k'}(2\theta)=(k'+2\theta)\tanh(k'+2\theta). \tag{A.5}$$

For relatively large n (for example, $n\geq 4$), $\tanh(n'+\theta)\approx 1$. Therefore,

$$\begin{aligned}4\Lambda_n^{(0)}(\theta)-\Lambda_k^{(0)}(2\theta)&=4(n'+\theta)\tanh(n'+\theta)-(k'+2\theta)\tanh(k'+2\theta)\\&\approx 4(n'+\theta)-(k'+2\theta)\\&=(4n'-k')+2\theta.\end{aligned} \tag{A.6}$$

It is non-zero if $2\theta\neq 0$ and 1, which means $\theta\neq 0$ and $\frac{1}{2}$.

2. k is odd (i.e. $k=2k'+1$).

$$\Lambda_k^{(0)}(2\theta)=g_{-k'}(2\theta)=(k'-2\theta)\tanh(k'-2\theta). \tag{A.7}$$

For relatively large n, we have

$$\begin{aligned}4\Lambda_n^{(0)}(\theta)-\Lambda_k^{(0)}(2\theta)&=4(n'+\theta)\tanh(n'+\theta)-(k'-2\theta)\tanh(k'-2\theta)\\&\approx 4(n'+\theta)-(k'-2\theta)\\&=(4n'-k')+6\theta.\end{aligned} \tag{A.8}$$

It is non-zero if $6\theta\neq 0,1,2$, which means $\theta\neq 0,\frac{1}{6}$ and $\frac{1}{3}$.

Therefore, if we choose an even number n and θ not close to $0,1/6,1/3$ and $1/2$, we have $4\Lambda_n^{(0)}(\theta)-\Lambda_k^{(0)}(2\theta)\neq 0$. For instance, we choose $\theta\in(2/9,5/18)$, which is the middle third from the interval $(1/6,1/3)$.

Here, we focus solely on the case that n is even and θ is positive. However, it is worth noting that the analysis leads to the same conclusion when n is odd. Additionally, if θ is negative, θ should be situated away from 0, $-1/6$, $-1/3$, and $-1/2$ because eigenvalues Λ_n are symmetric with respect to θ.

Appendix B

Checking $\chi_b(\theta)$ in (6.4.10) is Real

We examine the coefficient χ_b of the nonlinear term in the NLS equation (6.4.9), and show it is real. From (6.4.10), we know $\chi_b = -(\chi_1 + \chi_2)$. We check χ_b is real by checking both χ_1 and χ_2 are real.

Firstly, the coefficient χ_1 is defined as

$$\chi_1(\theta) = \frac{\Omega_n(\theta)}{g}\Big(\frac{1}{\Lambda_n(\theta)}r_3(\theta) + \Lambda_n(\theta)r_4(\theta) + 2\Lambda_n^2(\theta)r_5(\theta) + 2r_6(\theta)\Big), \tag{B.1}$$

where r_3, r_4, r_5 and r_6 are given in (6.2.27), (6.2.35), (6.2.44) and (6.2.54), respectively. Substituting them into $\chi_1(\theta)$, we have

$$\begin{aligned}
&\chi_1(\theta)\\
=&\frac{\Omega_n(\theta)}{g\Lambda_n(\theta)}\Big(-||\Lambda_n^2(\theta)|\psi_n|^2 - |\ell_\theta(\psi_n)|^2||^2 + \frac{1}{2}\sum_{k\geq 0}\frac{4\Lambda_n(\theta)p_k(\theta) - \Lambda_k(2\theta)q_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)}\big(\overline{\gamma_{2k}} - \Lambda_n^2(\theta)\overline{\gamma_{3k}}\big)\Big)\\
&+\frac{\Lambda_n(\theta)\Omega_n(\theta)}{g}\Big(\langle -2\theta^2|\psi_n|^2 + 2i\theta(\overline{\psi_n}\psi_n' - \psi_n\overline{\psi_n'}) + \overline{\psi_n}\psi_n'' + \psi_n\overline{\psi_n''}, |\psi_n|^2\rangle\Big)\\
&+\frac{2\Lambda_n^2(\theta)\Omega_n(\theta)}{g}\Big(\sum_{k\geq 0}\Lambda_k(2\theta)|\gamma_{3k}|^2\Big)\\
&+\frac{2\Omega_n(\theta)}{g}\Big(-2\sum_{k\geq 0}\frac{q_k(\theta) - p_k(\theta)}{4\Lambda_n(\theta) - \Lambda_k(2\theta)}\big(\Lambda_n(\theta)\Lambda_k(2\theta)\overline{\gamma_{3k}} + \overline{\gamma_{1k}}\big) + \sum_{k\neq 0}\frac{1}{\Lambda_k(0)}|\langle\alpha_2(x,\theta), \psi_k(0)\rangle|^2\Big)
\end{aligned}$$

For clarity, we separate all terms in $\chi_1(\theta)$ into 3 parts:

1. We examine all terms in $\chi_1(\theta)$ that involve $(4\Lambda_n(\theta)-\Lambda_k(2\theta))$:

$$\begin{aligned}\chi_{11}(\theta)=&\frac{2\Omega_n(\theta)}{g}\sum_{k\geq 0}\frac{1}{4\Lambda_n(\theta)-\Lambda_k(2\theta)}\Big(\big|\Lambda_n(\theta)\Lambda_k(2\theta)\gamma_{3k}-\gamma_{1k}\big|^2\\&+\frac{\Lambda_n(\theta)\Lambda_k(2\theta)}{4}\big|\frac{1}{\Lambda_n(\theta)}\gamma_{2k}-\Lambda_n(\theta)\gamma_{3k}\big|^2+\Lambda_n^2(\theta)\big(\gamma_{1k}\overline{\gamma_{3k}}+\overline{\gamma_{1k}}\gamma_{3k}\big)\\&+\Lambda_n(\theta)\Lambda_k(2\theta)\big(\gamma_{2k}\overline{\gamma_{3k}}+\overline{\gamma_{2k}}\gamma_{3k}\big)-\big(\gamma_{1k}\overline{\gamma_{2k}}+\overline{\gamma_{1k}}\gamma_{2k}\big)\\&-2\Lambda_n^3(\theta)\Lambda_k(2\theta)\big|\gamma_{3k}\big|^2\Big),\end{aligned}\tag{B.2}$$

   which is a real value.

2. We examine the term in $\chi_1(\theta)$ that involves $\Lambda_k(0)$:

$$\chi_{12}(\theta)=\frac{2\Omega_n(\theta)}{g}\sum_{k\neq 0}\frac{1}{\Lambda_k(0)}|\langle\alpha_2(x,\theta),\psi_k(0)\rangle|^2,\tag{B.3}$$

   which is real.

3. The remaining terms in $\chi(\theta)$:

$$\begin{aligned}\chi_{13}(\theta)=&-\frac{\Omega_n(\theta)}{g\Lambda_n(\theta)}||\Lambda_n^2(\theta)|\psi_n|^2-|\ell_\theta(\psi_n)|^2||^2\\&+\frac{\Lambda_n(\theta)\Omega_n(\theta)}{g}\langle-2\theta^2|\psi_n|^2+2i\theta(\overline{\psi_n}\psi_n'-\psi_n\overline{\psi_n'})+\overline{\psi_n}\psi_n''+\psi_n\overline{\psi_n''},|\psi_n|^2\rangle,\end{aligned}\tag{B.4}$$

   which is real.

Therefore, $\chi_1(\theta)=\chi_{11}(\theta)+\chi_{12}(\theta)+\chi_{13}(\theta)$ is real.

Next, we check $\chi_2(\theta)$ is real. We have

$$\chi_2(\theta)=\frac{\kappa_3(\theta)\kappa_4(\theta)}{(\Omega_n'(\theta))^2-g(h_b+\rho_2)},\tag{B.5}$$

with

$$\kappa_3(\theta)=-\frac{i\Omega_n(\theta)}{\pi}r_1(\theta)+g\rho_1(\theta)+\frac{\Omega_n'(\theta)}{2\pi}\kappa_1(\theta).\tag{B.6}$$

and

$$\kappa_4(\theta)=\frac{1}{\Omega_n(\theta)}\big(\Omega_n'(\theta)\kappa_1(\theta)-\kappa_2(\theta)\big).\tag{B.7}$$

From (6.2.20), we can check $r_1(\theta)$ is purely imaginary. $\rho_1(\theta)$ and $\rho_2(\theta)$ defined in (6.3.7) and (6.3.8) are real because terms come with their complex conjugates. $\kappa_1(\theta)$ in (6.1.34) and $\kappa_2(\theta)$ in (6.2.5) are real. Combining all these implies that $\chi_2(\theta)$ is real.

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