Ordering of the Tracy-Widom beta distributions and fractal dimension of the level sets of the directed landscape in the temporal direction

by

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Abstract

The first part of the thesis is related to the Tracy-Widom distribution. We give a stochastic comparison and ordering of the Tracy-Widom distribution with parameter β. In particular, we show that as β grows, the Tracy-Widom random variables get smaller modulo a multiplicative coefficient.

The second part of the thesis is related to the directed landscape. The directed landscape, $\mathcal{L}$, is a random 'metric' on $\mathbb{R}^2$ that arises as the rescaled limit of last passage percolation. We show that the level sets of last passage percolation converge to the level sets of the directed landscape in the Euclidean Hausdorff metric. We also describe the fractal nature of the level sets of the directed landscape. In particular, we prove that the level sets of $\mathcal{L}(0,0;0,t)$ have Hausdorff dimension of $2/3$ with positive probability. We prove this by finding matching upper and lower bounds. We provide an upper bound for the Hausdorff dimension in the usual way: by counting the number of squares that cover the level set. In the case of the lower bound, we provide sufficient conditions on the one and two-point density of any stochastic process to obtain a lower bound of the Hausdorff dimension of its level sets. This theorem generalizes for stochastic processes whose densities are not proved to exist. In that case, the conditions are on the one and two-point probability of being ε close to the level set. Then, we prove that the directed landscape satisfies the conditions on the two-point probability mentioned above. We conclude that $2/3$ is also the lower bound of the level set of $\mathcal{L}(0,0;0,t)$ with positive probability.

This thesis is dedicated to my mother, Rosa.

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Chapter 1

Introduction

This thesis contains my exploration of two questions in the area of probability. One question is related to understanding better Tracy-Widom β distributions and the other question concerns studying the level sets of the directed landscape. Both questions are related to models that belong to the Kardar-Parisi-Zhang (KPZ) universality class.

Universality refers to the phenomenon where probabilistic models with different characteristics produce the same limiting behaviour under rescaling. A classical example of universality is the Brownian motion. Consider a simple random walk, where a particle takes steps of fixed size in random directions at discrete time intervals. If we rescale appropriately the size of the steps as the number of steps increase, the trajectory of the random walk converges to a continuous path described by Brownian motion. This occurs regardless of the particular distribution of each step.

The KPZ universality class is supposed to represent the rescaled limit of a class of interface growth processes, where the growth rate at each point depends on the local slope of the interface. In particular, these growth interfaces present height fluctuation exponents of 1/3 and spatial correlation exponents of 2/3. Although there has not been a general universal result in the case of KPZ yet as there is for Brownian motion, the limiting behaviour of certain models expected to be in this universality class is well known and present the exponent behaviour explained above. One such case is Last Passage Percolation (LPP). This model will be related to both questions that we answer in this thesis. In the rest of the thesis, we will define LPP and its rescaled limit, the directed landscape and we will state the theorems that we obtained.

1.1 Last passage percolation

Last passage percolation is a model where one considers a grid, for example $\mathbb{Z}^2$, with random weights assigned to each vertex or edge. The goal is to find the maximum weight path from a starting point to an endpoint. The weight of each path is the sum of the weights of all of the vertices or edges along the way. Naturally if we do not add additional restrictions to the paths, every path has infinite weight so we usually ask that the paths be upwards and rightwards on the lattice. The weight of the maximally weighted path between two points, the passage time, can be considered a distance between the points and the maximal path turns into the geodesic in the LPP 'directed metric'.

In a sequence of papers, J. Baik, P. Deift, K. Johansson and E. Rains ([2], [21], [4], [3] and [5]) define several versions of a last passage percolation model on the $\mathbb{Z}^2$ lattice and take the rescaled limit of those models. They assign weights to each vertex of the lattice and they run random weighted walks on the square $[0, N]^2$. They study the paths with the largest weight from $(0, 0)$ to (N, N) and find that, in the rescaled limit, those paths converge to the Tracy-Widom distribution. They also apply certain symmetries to the lattice and obtain last passage paths that follow those symmetries and, rescaled, converge to Tracy-Widom distributions with different parameters (1, 2 or 4). In those papers, they imply that there is an interpolation of those last passage paths that provides an interpolation of the limits. In Chapter 2, we answer the question: can we stochastically order the Tracy-Widom β distributions not only for parameters $\beta = 1, 2$ or 4 but for a general parameter $\beta > 0$? In short, we prove and generalize the interpolation that was previously only implied. The motivation and statement of the result can be found in Subsection 1.2.

If instead of looking at the passage time from $(0, 0)$ to (N, N) we look at the passage time from $(0, 0)$ to any point $(x, y) \in \mathbb{N}^2$, we will still find that the rescaled limit follows a Tracy-Widom distribution. In fact, LPP is shift invariant so the same result would be obtained as the rescaled limit of the weight of the longest path between any two points as long as we keep the order. What would happen if we take the rescaled limit of the whole $\mathbb{Z}^2$ and the directed metric induced by LPP on it? This question was answered in [16]. The limiting 'directed metric' on $\mathbb{R}^2$ is called the directed landscape. The question that this thesis solves related to this topic is: what can be said about the level sets of the directed landscape metric? In Chapter 3, we prove that the level sets of the LPP metric converge to the level sets of the directed landscape metric with respect to the Euclidean Hausdorff metric. In Chapters 4, 5 and 6, we prove that the Hausdorff dimension of the level sets of the directed landscape from $(0, 0)$ to

$(0, t)$ with $t > 0$, is $\frac{2}{3}$. In Section 1.3, we define the directed landscape properly and state the results.

1.2 β Tracy-Widom distributions

In [4], Baik and Rains obtained the asymptotic fluctuations of the models mentioned above that we will now define. To each site $(i, j) \in \mathbb{Z}^2$ we assign a random variable $w(i, j)$. The random variables at each site are independent and identically distributed. We will denote a general up/right path as $\pi : (i, j) \nearrow (k, l)$, indicating its initial and final position. The weight or length of each path is the sum of the weights of the sites it visits. The goal is to describe the asymptotic length of the longest up/right path. We will apply three symmetries $T_{\boxast} : \mathbb{Z}^2 \to \mathbb{Z}^2$ onto the lattice. The identity symmetry will be named $T_{\square}$, the symmetry along the $y = x$ diagonal will be called $T_{\boxslash}$ and the symmetry along the other diagonal, $y = -x$ will be called $T_{\boxbslash}$.

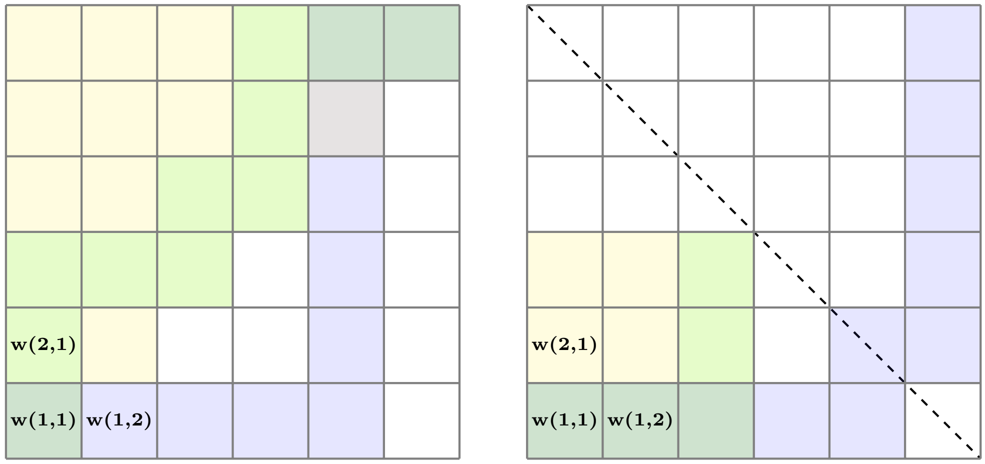
Then, the length of the longest up/right path on a square with side length N can be described as $G^{T_{\boxast}}(N) = \sup_{\pi:p\nearrow q} \sum_{(i,j)\in\pi} w(T_{\boxast}(i, j))$. We always take a square of size N but taking into account that the "diagonals" of the lattice have to coincide with the diagonals of the square, the initial and final points p and q might be different for each symmetry. However, the points p and q always represent the lower left and upper right points in the square (but we can not always take the square $[0, N] \times [0, N]$).

In this context, Baik and Rains proved that for each $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{G^{T_{\boxast}}(N) - aN}{bN^{1/3}} \leqslant x\right) = F_*(x),$$

where the constants a and b depend on the distribution of the weights $w(i, j)$ and for each symmetry $\boxast = \square, \boxslash$ or $\boxbslash$, the function $F_*(x)$ is the cumulative distribution function of the Tracy-Widom 2, 4 and 1 respectively, as originally defined in [29] and [30] by Tracy and Widom. We will name the random variable associated to F_* as $L^{\boxast}$.

We can compare $L^{\square}$ and $L^{\boxslash}$ by defining the last passage model as above on the square $[0, N] \times [0, N]$ and in the case of $T_{\boxslash}$ we symmetrize the half plane above the diagonal onto the lower half plane. This coupling gives us a simple comparison of $L^{\square}$ and $L^{\boxslash}$: in the case of the $\boxslash$ symmetry, we are taking the maximum of the up/right paths that stay in the upper half triangle while in the case of $\square$ the maximum is taken on all the up/right paths from the lower left corner to the upper right corner of the square. See Figure 1.1a below for an example. Since the weights in the upper half triangle are the same in both models, $G^{\square}$ is larger than $G^{\boxslash}$ almost surely. Therefore, $L^{\square} \overset{d}{\geqslant} L^{\boxslash}$.



(a) (b)

Figure 1.1: The pictures show the couplings and examples of possible last passage paths; the yellow region represents the points of the lattice that have been identified in the coupling and $w(i, j)$ are the weights. In the left picture (picture (b)), the green path represents an optimal path in the point-to-point last passage model in $[0, 3]^2$, ($\boxbslash$). The blue path is a last passage path in the point-to-line model from (1,1) to (6,6); that is why it is symmetrized along the dotted diagonal. On the right picture (picture (a)), the green path represents an optimal path in the point-to-point in the half space case ($\boxslash$). The blue path is a last passage path from (1,1) to (6,6).

Similarly, we can compare $L^{\square}$ with $L^{\boxbslash}$. In this case, we define the last passage model in the square $[-N, 0] \times [0, N]$. The model is shift invariant in the lattice and this square will allow us to couple both random variables. The symmetry $T_{\boxbslash}$ acts by copying the triangle below the $y = -x$ diagonal onto the upper triangle symmetrically. We can see that every up/right path from the lower left corner of the square to the upper right corner of the square in the symmetrized lattice consists of two symmetric paths: the path form the lower left corner to the diagonal is then repeated symmetrically in the upper triangle. Therefore, the weight of the longest path is exactly twice the weight of the longest path from the lower left corner to the diagonal. See Figure 1.1b above for an example. This path to the diagonal could be larger than the path from the lower left corner to the center of the square because the center of the square is in the diagonal, assuming that N is even. The weight of this new path is equal to $G^{\square}(N/2)$ so

$$\frac{1}{2} G^{\boxbslash}(N) \geqslant G^{\square}\left(\frac{N}{2}\right)$$

almost surely. After substracting the mean, rescaling and taking the limit, we obtain that

$$L^{\boxbslash} \overset{d}{\geqslant} 2^{2/3} L^{\square}.$$

As mentioned before, the random variables $L_{\boxast}$ are distributed according to the Tracy-Widom distributions as defined for the first time in [30] and [29]. These distributions are well known and have continuous and positive probability densities that can be expressed in terms of solutions of a differential equation. Moreover, $L_{\boxast}$ is the rescaled limit of the largest eigenvalue of a Gaussian random matrix. In [20] , Ramírez, Rider and Virág, propose a tridiagonal random matrix that depends on a parameter β and whose spectrum distribution, called the β-ensemble, coincides with the Gaussian Ensembles (GO/U/SE) in the cases where β is 1, 2 or 4. In that sense, they generalize the Tracy-Widom with parameter β by taking the rescaled limit of the largest eigenvalue. We call those random variables as TW_β. This new definition differs slightly from the original one for the cases where β is 1, 2 or 4. An explanation on the way the scaling differs in the two definitions can be found in the work of Bloemendal and Virág [9]. This slight difference means that, $L_{\boxbslash} \overset{d}{=} TW_1$, $L_{\square} \overset{d}{=} TW_2$ and $L_{\boxslash} \overset{d}{=} 2^{2/3} TW_4$.

From the coupling, we see a pattern in these stochastic comparisons:

$$TW_1 \overset{d}{\geqslant} 2^{2/3} TW_2 \quad \text{and} \quad TW_2 \overset{d}{\geqslant} 2^{2/3} TW_4.$$

We will prove that this generalizes for the Tracy-Widom β random variables defined originally, by Ramírez, Rider and Virág in [20].

The main result in this part is the following:

Theorem 1. *Let $\beta' > \beta > 0$ and $\alpha > 0$, then $TW_\beta \overset{d}{\geqslant} \alpha TW_{\beta'}$ if and only if*

$$\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}.$$

The proof of this theorem relies on the fact that there are two equivalent definitions of the notion of stochastic ordering. We say that the random variable X stochastically dominate the random variable Y, $X \overset{d}{\geqslant} Y$ if $\mathbb{P}(X \leqslant t) \leqslant \mathbb{P}(Y \leqslant t)$ for all $t \in \mathbb{R}$. Alternatively, if on the same probability space we can define random variables X' and Y' such that $X' \sim X$, $Y' \sim Y$ and such that $X' \geqslant Y'$ almost surely, then $X \overset{d}{\geqslant} Y$. We use each of these definitions to prove a direction of the if and only if argument in Theorem 1.

Chapter 2, will be devoted to the proper definition of the TW β-distributions, the heuristics of the result and the proper proof of this theorem. This part of the thesis is based on [26]. This work has already had some applications; see [6] where a similar stochastic comparison result is established for the largest eigenvalue of the finite matrices.

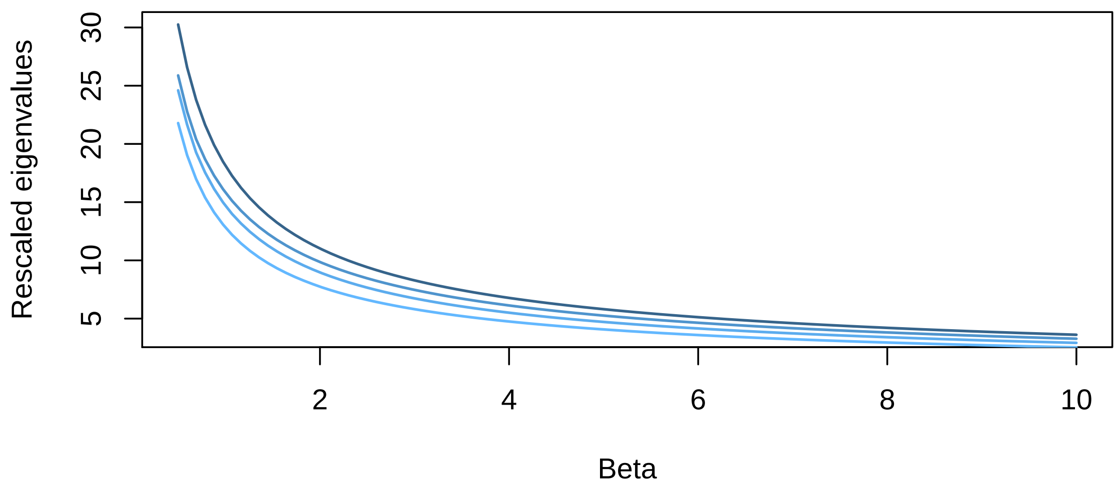


Figure 1.2: The first four rescaled eigenvalues $\frac{(\lambda_i(\beta)-2\sqrt{n})n^{1/6}}{\beta^{2/3}}$ of a 10x10 matrix distributed according to the β-ensemble plotted as functions of β. The colour gradient represents the order of the eigenvalues; the lighter the shade, the smaller the eigenvalue. The functions are decreasing.

1.3 Directed landscape

We will now consider the metric defined by LPP. To do that, we need to be more precise with the definition of last passage percolation. Consider the LPP model on $\mathbb{Z}^2$ where at each site (i, j), we assign a random independent exponential weight $\omega(i, j)$ of mean 1. Let $\vec{p}, \vec{q} \in \mathbb{Z}^2$ be two vectors such that $p_1 \leqslant q_1$ and $p_2 \leqslant q_2$. We define the passage time $G_{\vec{p},\vec{q}}$ to be the maximum sum of weights of all up-right paths from $\vec{p}$ to $\vec{q}$. Indeed,

$$G_{\vec{p},\vec{q}} = \sup_{\pi:\vec{p}\nearrow\vec{q}} \sum_{(i,j)\in\pi} \omega(i, j).$$

If $\vec{p}$ and $\vec{q}$ are not well ordered, meaning that there is no up-right path between them, then $G_{\vec{p},\vec{q}} = -\infty$. Consider the function

$$F(\vec{p}; \vec{q}) = G_{\vec{p},\vec{q}} - 2(q_1 - p_1) - 2(q_2 - p_2) - \omega(\vec{p}).$$

The function F defines a 'directed metric' on $\mathbb{Z}^2 \cup \{-\infty\}$. A directed metric on a set S is a function d such that $d(x, x) = 0$ for all $x \in S$ and such that the reverse triangle inequality is true:

$$d(x, z) \geqslant d(x, y) + d(y, z)$$

for all $x, y, z \in S$.

The directed landscape is a directed metric on $\mathbb{R}^2$. So far, we have defined the directed metric defined by LPP on $\mathbb{Z}^2$. As it is done in [16], we can extend this directed metric to $\mathbb{R}^2$ as follows. Let $r : \mathbb{R}^2 \to \mathbb{Z}^2$ be the following function

$$r(x,y) = \begin{cases} (x,y) & x,y \in \mathbb{Z}; \\ (\lfloor x \rfloor, y) & x \notin \mathbb{Z}, y \in \mathbb{Z}; \\ (x, \lfloor y \rfloor) & x \in \mathbb{Z}, y \notin \mathbb{Z}; \\ (\lfloor x \rfloor, \lfloor y \rfloor) & x \notin \mathbb{Z}, y \notin \mathbb{Z}. \end{cases}$$

Then, for all $(x, y; w, z) \in \mathbb{R}^4$ such that $x \leqslant w$ and $y \leqslant z$, the function

$$F(x,y;w,z) = G_{(\lceil x \rceil, \lceil y \rceil), r(w,z)} - 2(w-x) - 2(z-y) - \omega(x,y)\mathbb{1}(x,y \in \mathbb{Z})$$

is an extension to the reals. Let

$$\mathbb{R}^4_\uparrow := \{(x,s;y,t) \in \mathbb{R}^4 : s < t\}.$$

Now, let $\mathcal{K}_n : \mathbb{R}^4_\uparrow \to \mathbb{R}$ be the rescaled version of F:

$$\mathcal{K}_n(x,s;y,t) = 2^{-4/3} n^{-1/3} F(ns + 2^{5/3} n^{2/3} x, ns; nt + 2^{5/3} n^{2/3} y, nt).$$

Notice that the rescaling follows 1:2:3 KPZ scaling.

We can now state the convergence result:

Theorem 2 (Dauvergne, Virág [16])**.** *There exists a directed metric $\mathcal{L}$ on $\mathbb{R}^2$ and a coupling of $\mathcal{K}_n$ and $\mathcal{L}$ for all n such that for any compact $K \subseteq \mathbb{R}^4_\uparrow$,*

$$\sup_K (\mathcal{K}_n - \mathcal{L}) \xrightarrow[n\to\infty]{} 0$$

almost surely.

The limiting metric $\mathcal{L}$ is the directed landscape.

We move on to the topic of the geometry of the level sets of the directed landscape. The comparison between the role of the Brownian motion as the universal limit of random walks and the directed landscape in the KPZ universality class also guided our questions. The study of the level sets of the Brownian motion was well developed, what can be said about the level sets of the directed landscape? The first result, shows that the level sets of LPP converge in the Euclidean Hausdorff metric to the level sets of the directed landscape.

Theorem 3. *Let d_H be the Euclidean Hausdorff metric on $\mathbb{R} \times (0, \infty)$. Let $h \in \mathbb{R}$ be a real number and let $K = [a,b] \times [c,d]$ with $a < b$ and $0 < c < d$ be a compact set* *on* $\mathbb{R}^2$. *Then,*

$$\lim_{n\to\infty} d_H(\mathcal{K}_n(0,0;\cdot,\cdot)^{-1}(h) \cap K, \mathcal{L}(0,0;\cdot,\cdot)^{-1}(h) \cap K) = 0$$

almost surely.

More details about this theorem and its proof can be found on Chapter 3.

The next question naturally arises when we further consider the comparison with Brownian motion. We know that the level sets of the Brownian motion have Hausdorff dimension 1/2 (see for example [25], Chapter 4), the question is what is the Hausdorff dimension of the level sets of the directed landscape in the temporal direction? Our theorem provides the answer.

Theorem 4. *Let* $\dim_H$ *be the Hausdorff dimension of a set. Let* $h \in \mathbb{R}$. *Then,*

$$\dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h)) = \frac{2}{3}$$

with positive probability.

More details about this theorem can be found on Chapter 4. The proof of this theorem relies on three other theorems that we proved: Theorem 5, 6 and 7. We state and discuss these results in the remainder of this section.

As usual, the Hausdorff dimension in Theorem 4 is proved by finding an upper bound and a lower bound to the Hausdorff density. The first theorem is in fact an upper bound for the Hausdorff dimension.

Theorem 5. *For each* $h \in \mathbb{R}$, *let* Z_h *be the set*

$$Z_h := \{t \in (0,\infty) : \mathcal{L}(0,0;0,t) = h\}.$$

Then, for any $h \in \mathbb{R}$,

$$\dim_H(Z_h) \leqslant \frac{2}{3}$$

almost surely.

More details about this result and its proof can be found in Chapter 4.

A second theorem provides a general template to obtain a lower bound for the Hausdorff dimension of stochastic processes. It is based on the energy method described for example in [25]. The theorem essentially says that if certain conditions on the one and two-point distributions of the stochastic process at the point h are met, then there is a lower bound to the Hausdorff dimension on the h-level set. However, the one or two-point distributions of the process need not be known (or even more their existence might not be proven) to apply this theorem.

Theorem 6. *Let $B(t)$ be a stochastic process on $\mathbb{R}$. Let h be a real number. Assume that there exists an ε_0, positive constants c_h, c'_h and c''_h and an exponent $0 < \beta < 1$ such that for all $t, s \in [1, 2]$,*

$$\mathbb{P}(B(t) \in (h - \varepsilon, h + \varepsilon)) \leqslant c_h 2\varepsilon \tag{1.1}$$

$$\mathbb{P}(B(t) \in (h - \varepsilon, h + \varepsilon)) \geqslant c'_h 2\varepsilon \tag{1.2}$$

$$\mathbb{P}(B(t) \in (h - \varepsilon, h + \varepsilon), B(s) \in (h - \varepsilon, h + \varepsilon)) \leqslant c''_h 4\varepsilon^2 |t - s|^{-\beta} \tag{1.3}$$

for all $\varepsilon \leqslant \varepsilon_0$. Then, we get a lower bound for the Hausdorff dimension of the level sets:

$$d_H(B^{-1}(h) \cap [1, 2]) \geqslant 1 - \beta$$

with positive probability p_h where

$$p_h = \frac{c'^2_h(\beta^2 - 3\beta + 2)}{8c''_h}.$$

The positive probability mentioned in the theorem goes to 0 when h goes to infinity, essentially because the level set might have a positive probability of being empty if $|h|$ is too large. This is the same positive probability mentioned in Theorem 4. More details about this theorem can be found on Chapter 5.

The final theorem proves the bound on the two-point density of $\mathcal{L}(0, 0; 0, t)$ that is needed to use Theorem 6.

Theorem 7. *Let $0 < \varepsilon \leqslant 1$, $h \in \mathbb{R}$ and $0 < s < t$. Then, there exists an absolute constant c such that*

$$\mathbb{P}(\mathcal{L}(0, 0; 0, s) \in (h - \varepsilon, h + \varepsilon), \mathcal{L}(0, 0; 0, t) \in (h - \varepsilon, h + \varepsilon)) \leqslant c|t - s|^{-1/3}\varepsilon^2. \tag{1.4}$$

The proof of this result relies heavily on the fact that the directed landscape is locally Gaussian. In the reminder we explain the key point of the proof of Theorem 7. A deeper explanation of this result and its proof can be found in Chapter 6.

There has been a lot of recent developments in the study of the fractal properties of the directed landscape mostly focused on fractacl properties of the geodesics. In particular, Ganguly and Zhang proved in [19] that the Hausdorff dimension of the level sets of the geodesics of the directed landscape in the temporal direction is 1/3 through building the geodesic local time. For more on this topic, see for example: [12], [13], [7], [8]. For a survey on these topics see [18].

To be able to understand the two-point distribution of $\mathcal{L}(0, 0; x, s)$ we have heavily relied on the fact that the parabolic Airy line ensemble is locally Brownian. The parabolic Airy line ensemble is a sequence of ordered random functions $\mathfrak{A}_1 > \mathfrak{A}_2 > \dots$. For more information about it see Section 3.2. The first line of the parabolic Airy ensemble is distributed like the directed landscape for a fixed time. More precisely, for each fixed $s, t \in \mathbb{R}$ such that $s < t$,

$$\mathcal{L}(x,s;y,t) \overset{d}{=} (t-s)^{1/3}\mathfrak{A}_1\Big(\frac{y-x}{(t-s)^{2/3}}\Big).$$

Therefore, the property of the parabolic Airy ensemble being locally Brownian translates to the directed landscape.

This property, first introduced by Corwin and Hammond in [11] as the Brownian Gibbs property, states that inside any region $K = \{1, \dots, k\} \times [a, b]$ conditionally on all values $\mathfrak{A}_i(x)$ for $(i,t) \notin K$ the parabolic Airy line ensemble on K is distributed according to a sequence of k independent Brownian bridges from $(a, \mathfrak{A}_i(a))$ to $(b, \mathfrak{A}_i(b))$ conditioned to not intersecting. This absolute continuity property is very strong and has been widely used to prove results related to the Airy process, the Airy line ensemble and other objects in the KPZ world, in particular it was crucial to the definition of the directed landscape itself ([15]). In this work we have used a similar result that provides more precision on the Radon-Nikodym derivative of the parabolic Airy line ensemble against the Wiener measure.

In [14], a similar result is introduced by Dauvergne. Now, inside any region $K = \{1, \dots, k\} \times [a, b]$ conditionally on all values $\mathfrak{A}_i(x)$ for $(i,t) \notin K$ the parabolic Airy line ensemble on K is distributed according to a sequence of k independent Brownian bridges conditioned to not intersecting. Moreover, this time, the Radon-Nikodym derivative is bounded by a function that only depends on the length of the interval $[a, b]$ and it also provides tail bound estimates for the boundary conditions. The drawback of this newer result is that the boundary is not longer the Airy process itself. However, it is a related process with tail bounds of the same order. We state the result that we will use later on:

Theorem 8 (Dauvergne, in [14])**.** *For any $T_0 \geqslant 1$, there exists an absolute constant $c > 0$ such that*

$$\text{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in[0,T_0]}\right) \leqslant e^{cT_0^3}\,\text{Law}\left(\left(L(r)+B(r)\right)_{r\in[0,T_0]}\right)$$

with B a diffusion parameter two Brownian bridge on $[0, T_0]$ from 0 to 0 and L an affine linear function which is independent of B. Moreover, there exist T_0-dependent *constants* $d_1, d_2 > 0$ *such that for all* $m > 0$, *we have that*

$$\mathbb{P}\Big(L(0) \vee L(T_0) > m\Big) \leqslant \exp\left(-\frac{4}{3}m^{\frac{3}{2}} + d_1 m^{\frac{5}{4}}\right) \tag{1.5}$$

$$\mathbb{P}\Big(L(0) \wedge L(T_0) < -m\Big) \leqslant 2\exp\left(-d_2 m^3\right) \tag{1.6}$$

$$\mathbb{P}\Big(|L(0) - L(T_0)| > m\Big) \leqslant \exp\left(-\frac{1}{4T_0}m^2 - \frac{2}{3}m^{\frac{3}{2}} + d_1 m^{\frac{5}{4}}\right). \tag{1.7}$$

The results stated in this subsection have been a collaboration with Lemonte Alie-Lamarche, unless otherwise stated.

Chapter 2

Stochastic comparison of β-GUE distributions

2.1 Stochastic Airy operator and Tracy-Widom beta random variables

For any $\beta > 0$, consider the probability distribution

$$\mathbb{P}_n^\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_n^\beta} e^{-\beta \sum_{k=1}^n \lambda_k^2/4} \prod_{j<i} |\lambda_j - \lambda_i|^\beta,$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. When $\beta = 1, 2$ and 4, this gives the joint distribution of the eigenvalues of Gaussian orthogonal, unitary and symplectic ensembles respectively, or G(O/U/S)E, of random matrix theory. In [20] (and more generally in [23]), Krishnapur, Ramírez, Rider and Virág obtain the point process limit of the spectral edge of the general β-ensemble. In fact, the eigenvalues of the β-ensemble converge in distribution to the eigenvalues of a stochastic operator called the Stochastic Airy Operator (SAO):

$$H_\beta := -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x \tag{2.1}$$

where b' is the white noise. The operator is defined on the Hilbert space L^*, the space of continuous functions f such that $f(0) = 0$ and

$$\int_0^\infty (f'(x))^2 + (1+x) f^2(x) dx < \infty.$$

The SAO acts on functions as a quadratic form in the following way: we decompose the Brownian motion in two terms $b = \bar{b} + (b - \bar{b})$ where $\bar{b}(x)$ is the average,

$$\bar{b}(x) = \int_x^{x+1} b_y dy.$$

For every function $f \in L^*$,

$$\begin{aligned} &\prec f, H_\beta f \succ \\ &\stackrel{d}{=} \int_0^\infty (f'(x))^2 + xf^2(x)dx + \frac{2}{\sqrt{\beta}} \left(\int_0^\infty f^2(x)\bar{b}'_x dx - 2\int_0^\infty f'(x)f(x)(\bar{b}_x - b_x)dx \right). \end{aligned}$$

The integrals are well defined and finite, see [20]. The definition looks more involved than it needs to be and that is because the last two integrals could be replaced, using integration by parts, with $\frac{-4}{\sqrt{\beta}} \int_0^\infty f(x)f'(x)b_x dx$ if this were a finite integral. If the function f is compactly supported, then this simpler definition of the quadratic form works.

In [10], Bloemendal provides an alternative definition of the Stochastic Airy operator, H_β as a generalized Sturm-Liouville operator. Additionally, in [24] Minami proves that H_β is a self-adjoint operator with probability one and has purely discrete spectrum.

To define the eigenvalues and eigenfunctions of H_β, we use their variational characterization. Then, the smallest eigenvalue, Λ_0, is defined as

$$\Lambda_0 := \inf\{\prec f, H_\beta f \succ : f \in L^*, \|f\|_2 = 1\}. \tag{2.2}$$

The infimum of the formula (2.2) is attained at an eigenfunction f_0 with corresponding eigenvalue Λ_0. Functions of compact support are dense in L^* (since functions in L^* have the boundary condition $f(0) = 0$) and the quadratic form $\prec \cdot, H_\beta \cdot\cdot \succ : (L^*)^2 \to \mathbb{R}$ is continuous as proved in [20] so we can take Λ_0 to be

$$\Lambda_0 = \inf\{\prec f, H_\beta f \succ : f \in L^*, \|f\|_2 = 1, f \text{ compactly supported}\}. \tag{2.3}$$

The rest of the eigenvalues are defined recursively as

$$\Lambda_k := \inf\{\prec f, H_\beta f \succ : f \in L^*, \|f\|_2 = 1, f \perp f_0, \dots, f_{k-1}\}$$

and the infimum is attained at an eigenfunction f_k. A proof that this variational characterization actually defines eigenvalues and that these eigenvalues are strictly ordered $\Lambda_0 < \Lambda_1 < \Lambda_2 < \dots$ can be found in [20], Corollary 2.6, Lemma 2.7 and Proposition 3.5.

Functions of compact support are also L^*-dense in the orthogonal complement of an eigenspace. In fact, if a function f in L^* is orthogonal to $f_0, \ldots, f_{k-1}$ we can choose a function g that is ε close to f in L^*. Then, the function $\tilde{g} = g - \sum_{i=0}^{k-1} \langle g, f_i \rangle f_i$ is a function in L^* orthogonal to the eigenfunctions f_i. Since the function f is also orthogonal to the eigenfunctions, we can rewrite f as $f = f - \sum_{i=0}^{k-1} \langle g, f_i \rangle f_i$. Then,

$$\begin{aligned} \|f - \tilde{g}\|_{L^*} &= \left\| f - g - \sum_{i=0}^{k-1} (\langle f, f_i \rangle - \langle g, f_i \rangle) f_i \right\|_{L^*} \\ &\leqslant \|f - g\|_{L^*} + \sum_{i=0}^{k-1} |\langle f, f_i \rangle - \langle g, f_i \rangle| \leqslant (k+1)\varepsilon \end{aligned}$$

using Cauchy-Schwarz inequality and the fact that $\|f\|_2 \leqslant \|f\|_{L^*}$. The continuity of the quadratic form in L^* means that we can restrict the definition to

$$\Lambda_k = \inf\{< f, H_\beta f >: f \in L^*, \|f\|_2 = 1, f \perp f_0, \ldots, f_{k-1}, f \text{ compactly supported}\}. \tag{2.4}$$

More details on this random operator and its eigenvalues can be found in [20].

Then, (from [20]) $\Lambda_0 < \Lambda_1 < \cdots < \Lambda_{k-1}$ are the k lowest elements of the set of eigenvalues of the operator H_β and the vector

$$(n^{1/6}(2\sqrt{n} - \lambda_l))_{l=1,\ldots,k}$$

converges in distribution to $(\Lambda_0, \Lambda_1, \ldots \Lambda_{k-1})$ as $n \to \infty$.

The rescaled limit of the largest eigenvalue of the β-ensembles mentioned earlier is distributed according the Tracy-Widom β, so we define the Tracy-Widom β distribution as the distribution of $-\Lambda_0$. In fact,

$$TW_\beta \overset{d}{=} -\Lambda_0$$

There is a deterministic operator associated with the SAO which is the Airy Operator

$$A := -\frac{d^2}{dx^2} + x.$$

We can think of the Airy operator as the SAO with parameter $\beta = \infty$.

2.2 Stochastic comparison of β-GUE distributions

Theorem 9. *Let $\beta' > \beta > 0$ and $\alpha > 0$, then $TW_\beta \overset{d}{\geqslant} \alpha TW_{\beta'}$ if and only if*

$$\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}.$$

Proof. The goal is to stochastically compare the eigenvalues of the SAO$_\beta$. Recall that the parameter β only appears in the operator (2.1) as part of the coefficient of the random term, so the coupling used to obtain the comparison will consist of keeping the same source of randomness for all β.

There is a natural partial order on the space of self adjoint operators, the Loewner order: we say that two operators A and B are ordered $A \geqslant B$ if the operator $A - B$ is positive definite. We would like to establish an order on $\{H_\beta\}_{\beta \geqslant 1}$.

Assume that $\beta' > \beta > 0$ and $(\beta'/\beta)^{1/3} \leqslant \alpha \leqslant (\beta'/\beta)^{2/3}$. Then, we will show that we can couple $H_{\beta'}$ and H_β such that $\alpha H_{\beta'} - H_\beta$ is positive definite and $\alpha H_{\beta'} \overset{d}{\geqslant} H_\beta$.

We start by making the following remark. Let b be a standard Brownian and $s > 0$. Then, the stochastic process $\tilde{b}(x) = s^{1/2} b(x/s)$ is also distributed as a standard Brownian motion by the Brownian scaling. Then

$$\tilde{H}_\beta = -\partial_x^2 + x + \frac{2}{\sqrt{\beta}} \tilde{b}'_x$$

has the same distribution as H_β. In particular, if Λ_k^β is the kth eigenvalue of H_β and $\tilde{\Lambda}_k^\beta$ is the kth eigenvalue of $\tilde{H}_\beta$, then $\Lambda_k^\beta \overset{d}{=} \tilde{\Lambda}_k^\beta$.

Recall the variational definition of the eigenvalues of SAO$_\beta$ (2.3) and (2.4). Let $f(x) \in L^*$, $\|f\|_2 = 1$ be a test function and $s > 0$. Then, $g(x) = \frac{1}{\sqrt{s}} f(x/s)$ is also a suitable test function since $g \in L^*$ and $\|g\|_2 = 1$. We will compare $\prec f, \tilde{H}_\beta f \succ$ with $\prec g, H_\beta g \succ$.

By making the change of variables $y = sx$ we get

$$\int_0^\infty (f'(x))^2\, dx = s^2 \int_0^\infty \left(\frac{1}{s^{3/2}} f'\left(\frac{y}{s}\right)\right)^2 dy = s^2 \int_0^\infty (g'(y))^2\, dy$$
$$\int_0^\infty x f^2(x) dx = \frac{1}{s} \int_0^\infty y \left(\frac{1}{\sqrt{2}} f\left(\frac{y}{s}\right)\right)^2 dy = \frac{1}{s} \int_0^\infty y g^2(y) dy$$
$$\int_0^\infty f(x) f'(x) \tilde{b}_x dx = \sqrt{s} \int_0^\infty \frac{1}{\sqrt{s}} f\left(\frac{y}{s}\right) \frac{1}{s^{3/2}} f'\left(\frac{y}{s}\right) s^{1/2} b_{y/s} dy = \sqrt{s} \int_0^\infty g(y) g'(y) b_y dy.$$

Therefore, we have that the spectrum of $\tilde{H}_\beta$ is the same as the same as the spectrum of the operator

$$H_\beta^s = -s^2 \partial_y^2 + \frac{1}{s} y + 2\sqrt{\frac{s}{\beta}} b'_y.$$

Recall that the spectrum of $\tilde{H}_\beta$ is equal in distribution to the spectrum of H_β.

Let $s = \frac{\alpha^2 \beta}{\beta'}$. Then,

$$H_\beta^s = -\frac{\alpha^4 \beta^2}{\beta'^2} \partial_y^2 + \frac{\beta'}{\alpha^2 \beta} y + \frac{2\alpha}{\sqrt{\beta'}} b'_y.$$

Notice that the random term of this operator is the same as the random term of the operator $\alpha H_{\beta'}$. We can couple H_β^s and $\alpha H_{\beta'}$ by using the same Brownian motion in each operator. By subtracting both operators, we get that

$$\alpha H_{\beta'} - H_\beta^s = -\left(\alpha - \frac{\alpha^4 \beta^2}{\beta'^2}\right) \partial_y^2 + \left(\alpha - \frac{\beta'}{\alpha^2 \beta}\right) y.$$

Notice that the Airy operator is positive definite since

$$\langle Af, f \rangle = \int_0^\infty -f''(x) f(x) + x f^2(x) dx = \int_0^\infty f'^2(x) + x f^2(x) dx \geqslant 0.$$

In fact, if we take the deterministic operator $A_{a,b} = -a\partial_x^2 + bx$, we know that $A_{a,b}$ is positive definite if and only if both a and b are positive. Then, we need that

$$\alpha \geqslant \frac{\alpha^4 \beta^2}{\beta'^2}$$
$$\alpha \geqslant \frac{\beta'}{\alpha^2 \beta}$$

which happens if and only if

$$\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}.$$

Notice that this is the hypothesis condition.

We conclude that $\alpha H_{\beta'} \overset{d}{\geqslant} H_\beta^s$. Let $\Lambda_k^{s,\beta}$ be the kth eigenvalue of H_β^s. Since the positive definite partial order implies an ordering of the eigenvalues, we have that $\alpha \Lambda_k^{\beta'} \overset{d}{\geqslant} \Lambda_k^{\beta,s} \overset{d}{=} \Lambda_k^\beta$.

We have proved that if $\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}$, then $TW_\beta \overset{d}{\geqslant} \alpha TW_{\beta'}$. (Here, the inequality reverses because the $-\Lambda_0$ is distributed according to TW_β).

Notice that this proof gives a comparison of the whole spectrum of the Stochastic Airy Process and not only on the smallest eigenvalue which is distributed according to the Tracy-Widom distribution. In fact, if, as before, $\Lambda_0^\beta, \Lambda_1^\beta, \ldots$ are the eigenvalues of H_β in increasing order,

$$\alpha\Lambda_k^{\beta'} \overset{d}{\geqslant} \Lambda_k^{\beta},$$

for any k, given that $\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}$.

In the opposite direction, we can look at the tails of the TW_β distribution and get from there a possible range of αs. From Ramírez, Rider, Virág [20] we get that for $a \to \infty$,

$$\begin{aligned} \mathbb{P}(TW_\beta > a) &= \exp\left(-\frac{2}{3}\beta a^{3/2}(1+o(1))\right) \\ \mathbb{P}(TW_\beta < -a) &= \exp\left(-\frac{1}{24}\beta a^{3}(1+o(1))\right). \end{aligned}$$

If $\alpha TW_{\beta'} \overset{d}{\leqslant} TW_\beta$, then $\mathbb{P}(\alpha TW_{\beta'} > a) \leqslant \mathbb{P}(TW_\beta > a)$ which means that for a large enough

$$\exp\left(-\frac{2}{3}\beta'\frac{a^{3/2}}{\alpha^{3/2}}(1+o(1))\right) \leqslant \exp\left(-\frac{2}{3}\beta a^{3/2}(1+o(1))\right)$$

or

$$\exp\left(-\frac{2}{3}a^{3/2}(\beta'/\alpha^{3/2}-\beta)(1+o(1))\right) \leqslant 1$$

so $\beta'/\alpha^{3/2} - \beta \geqslant 0$ or equivalently, $\alpha \leqslant \left(\frac{\beta'}{\beta}\right)^{2/3}$. Doing a similar calculation with the left-hand tail, gives us that $\left(\frac{\beta'}{\beta}\right)^{1/3} \leqslant \alpha$ which is the same range that we found through the other method. This concludes the proof.

$\square$

Chapter 3

Convergence of the level sets of LPP

This chapter will be devoted to proving the that the h-level sets of the "metric" defined in the Last Passage Percolation model converges in the Euclidean Hausdorff metric to the h-level sets of the directed landscape on compacts.

3.1 LPP converges to directed landscape

Recall from Section 1.3, that LPP converges to the directed landscape in the Euclidean Hausdorff metric.

Theorem 10 (Dauvergne, Virág [16])**.** *There exists a directed metric $\mathcal{L}$ on $\mathbb{R}^2$ and a coupling of $\mathcal{K}_n$ and $\mathcal{L}$ for all n such that for any compact $K \subseteq \mathbb{R}^4_\uparrow$,*

$$\sup_K (\mathcal{K}_n - \mathcal{L}) \xrightarrow[n\to\infty]{} 0$$

almost surely.

The Hausdorff distance is the standard way to define distance between subsets of a metric space. In this case, the underlying metric space is $\mathbb{R} \times (0, \infty)$ with the Euclidean distance. The definition of the Hausdorff distance that we use in this section is the following: for each pair of non-empty subsets $A, B \subseteq \mathbb{R} \times (0, \infty)$,

$$d_H(A, B) = \inf\{\delta > 0 : A \subseteq (B)_\delta \text{ and } B \subseteq (A)_\delta\}$$

where $(A)_\delta = \bigcup_{x\in A} \{z \in \mathbb{R} \times (0, \infty) : \|x - z\| \leqslant \delta\}$.

Let $h \in \mathbb{R}$. The proof also relies on the fact that the directed lansdcape has no local extrema with value h on any rectangle on $\mathbb{R} \times (0, \infty)$. We say that a function

f has a local maximum (or minimum) with value h on a compact K if there exist a point $x \in K$ and an open set $B \subset \mathbb{R} \times (0, \infty)$ such that $f(y) \leqslant f(x) = h$ (or $f(y) \geqslant f(x) = h$) for all $y \in B \cap K$.

3.2 Properties of the directed landscape

The proof requires some properties of the directed landscape. These properties are called to use several times in this thesis but we will write them down on this section.

Recall from the introduction, that we defined the directed landscape $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$

$$(x, s; y, t) \longmapsto \mathcal{L}(x, s; y, t)$$

as the rescaled limit of the directed metric defined by exponential LPP. We make a point here to note that we call the second coordinate (s, t in this case) as time and the first coordinate (x, y) as space. The directed landscape was first obtained as the rescaled limit of Brownian last passage percolation by Dauvergne, Ortmann and Virág in [15]. Since exponential LPP is a discrete model, it is not clear if the directed landscape $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ is continuous. Fortunately, there is a great description of the modulus of continuity of the directed landscape; it turns out that $\mathcal{L}$ grows almost as $t^{1/3}$ in time and as $x^{1/2}$ in space.

Theorem 11 (Dauvergne, Ortmann, Virág, [15])**.** *Let $\mathcal{K}(x, s; y, t) = \mathcal{L}(x, s; y, t) + \frac{(x-y)^2}{(t-s)}$ be the stationary version of the directed landscape for each $(x, s; y, t) \in \mathbb{R}^4_\uparrow$. Let $n \geqslant 2$ and $0 < \delta \leqslant 1$ and*

$$K_n^\delta = [-n, n]^4 \cap \Big\{(x, s; y, t) \in \mathbb{R}^4_\uparrow : t - s \geqslant \delta\Big\}.$$

For two points $u_i = (x_i, s_i; y_i, t_i), i = 1, 2,$ let

$$\xi = \xi(u_1, u_2) = \|(x_2, y_2) - (x_1, y_1)\|, \tau = \tau(u_1, u_2) = \|(s_2, t_2) - (s_1, t_1)\|.$$

Then, there exists a random constant $C_{K_n^\delta}$ such that

$$|\mathcal{K}(u_2) - \mathcal{K}(u_1)| \leqslant C_{K_n^\delta}(\tau^{1/3} \log^{2/3}(\tau^{-1} + 1) + \xi^{1/2} \log^{1/2}(\xi^{-1} + 1)),$$

for every $u_1, u_2 \in K_n^\delta$ such that $0 < \tau < \delta^3/n^3$ and $0 < \xi$. Moreover, there exist universal positive constants c and d such that for all $M > 0$, we have that

$$\mathbb{P}(C_{K_n^\delta} > M) \leqslant cn^{10}\delta^{-6}e^{-dM^{3/2}}.$$

Remark: Notice that even though the modulus of continuity is expressed for the case when $\xi > 0$, it is clear that if $\xi = 0$ and $0 < \tau < \delta^3/n^3$ then there exists a constant $C_{K_n^\delta}$ such that

$$|\mathcal{K}(u_2) - \mathcal{K}(u_1)| \leqslant C_{K_n^\delta} \tau^{1/3} \log^{2/3}(\tau^{-1} + 1)). \tag{3.1}$$

In fact, let $\varepsilon > 0$. We define $u_2^\varepsilon := u_2 + (\varepsilon, \varepsilon, \varepsilon, \varepsilon)$. Then,

$$|\mathcal{K}(u_2) - \mathcal{K}(u_1)| \leqslant |\mathcal{K}(u_2) - \mathcal{K}(u_2^\varepsilon)| + |\mathcal{K}(u_2^\varepsilon) - \mathcal{K}(u_1)|.$$

By Theorem 11, there exists a random constant $C_{K_n^\delta}$ such that

$$\begin{aligned}
|\mathcal{K}(u_2)-&\mathcal{K}(u_2^\varepsilon)| \\
&\leqslant C_{K_n^\delta}(\tau(u_2^\varepsilon, u_2)^{1/3} \log^{2/3}(\tau(u_2^\varepsilon, u_2)^{-1} + 1) + \xi(u_2^\varepsilon, u_2)^{1/2} \log^{1/2}(\xi(u_2^\varepsilon, u_2)^{-1} + 1)) \\
|\mathcal{K}(u_2^\varepsilon)-&\mathcal{K}(u_1)| \\
&\leqslant C_{K_n^\delta}(\tau(u_2^\varepsilon, u_1)^{1/3} \log^{2/3}(\tau(u_2^\varepsilon, u_1)^{-1} + 1) + \xi(u_2^\varepsilon, u_2)^{1/2} \log^{1/2}(\xi(u_2^\varepsilon, u_2)^{-1} + 1)).
\end{aligned}$$

Notice that this is true for all $\varepsilon > 0$. Recall that the constant $C_{K_n^\delta}$ does not depend on the points u_1 and u_2 but only on the compact set K_n^δ. It is clear that $\tau(u_2^\varepsilon, u_2) \to 0$, $\xi(u_2^\varepsilon, u_2) \to 0$ and $\tau(u_2^\varepsilon, u_1) \to \tau$ when $\varepsilon \to 0$. Moreover,

$$\lim_{x\to 0} x^{1/3} \log^{2/3}(x^{-1} + 1) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^{1/2} \log^{1/2}(x^{-1} + 1) = 0.$$

Then,

$$|\mathcal{K}(u_2) - \mathcal{K}(u_1)| \leqslant \lim_{\varepsilon\to 0} |\mathcal{K}(u_2) - \mathcal{K}(u_2^\varepsilon)| + |\mathcal{K}(u_2^\varepsilon) - \mathcal{K}(u_1)| \leqslant C_{K_n^\delta} \tau^{1/3} \log^{2/3}(\tau^{-1} + 1).$$

So we can state a temporal modulus of continuity as follows:

Theorem 12 (Dauvergne, Ortmann, Virág, [15])**.** *Let $n \geqslant 2$ and $0 < \delta \leqslant 1$ and*

$$K_n^\delta = [\delta, n].$$

Then, there exists a random constant $C_{K_n^\delta}$ such that

$$|\mathcal{L}(0, 0; 0, t) - \mathcal{L}(0, 0; 0, s)| \leqslant C_{K_n^\delta}(|t - s|^{1/3} \log^{2/3}(|t - s|^{-1} + 1),$$

for every $s, t \in K_n^\delta$ such that $0 < |t - s| < \delta^3/n^3$. Moreover, there exist universal positive constants c and d such that for all $M > 0$, we have that

$$\mathbb{P}(C_{K_n^\delta} > M) \leqslant cn^{10}\delta^{-6}e^{-dM^{3/2}}.$$

The marginals, or one-point distribution of the landscape are well known. They are related to the first line of the parabolic airy line ensemble, $\mathfrak{A}_1$. The parabolic Airy line ensemble is a sequence of ordered random functions $\mathfrak{A}_1 > \mathfrak{A}_2 > \mathfrak{A}_3 > \dots$. The ensemble is called parabolic because each line is centered around a negative parabola. The related ensemble $\{\mathfrak{A}_i(t) + t^2\}_{i\in\mathbb{N}}$ is stationary. The stationary Airy ensemble was first constructed in [27] through a determinantal formula. The first line of the stationary ensemble $\mathfrak{A}_1(t) + t^2$ is sometimes called the Airy process. This process arises naturally as a rescaled limit in the context of the KPZ universality environment. The marginals of the parabolic Airy line ensemble are well known. In particular, for any $t \in \mathbb{R}$,

$$\mathfrak{A}_1(t) + t^2 \overset{d}{=} TW_2. \tag{3.2}$$

We state some properties that will be useful.

Proposition 13 (Dauvergne, Ortmann, Virág, [15])**.** *The directed landscape is a random function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ that satisfies the following properties.*

1. *(Continuity) The function $\mathcal{L}(x, s; y, t)$ is continuous (see Theorem 11).*

2. *(Marginals) For each $s, t \in \mathbb{R}$ fixed such that $s < t$,*

$$\mathcal{L}(0, s; y, t) \overset{d}{=} (t - s)^{1/3}\mathfrak{A}_1\Big(\frac{y}{(t-s)^{2/3}}\Big)$$

   as a function of y.

3. *(Independent increments) For any disjoint time intervals $\{(s_i, t_i) : i = 1, \dots, k\}$, the random functions*

$$\mathcal{L}(\cdot, s_i; \cdot, t_i), \qquad i = 1, \dots, k$$

   are independent.

4. *(Metric composition law) Almost surely, for any $s < r < t$ and $x, y \in \mathbb{R}$ we have that*

$$\mathcal{L}(x, s; y, t) = \sup_{z\in\mathbb{R}} \mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t).$$

The metric composition law in particular will be very significant in the proof of Theorem 7. Notice that the two terms in the supremum above are independent. When thinking about the two-point distribution density of $\mathcal{L}$ it will be convenient to think that the distance from time 0 to time t goes through time s at some point z and the metric composition law provides that result. Also, the metric composition law implies a reverse triangle inequality for the directed landscape.

Lastly, we will state some symmetry properties of the directed landscape.

Lemma 14 (Dauvergne, Ortmann, Virág, [15])**.** *Let $\mathcal{L}$ be the directed landscape. Let $r, c \in \mathbb{R}$ and $q > 0$.*

1. *(Time stationarity)*

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x, s + r; y, t + r)$$

2. *(Spatial stationarity)*

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x + c, s; y + c, t)$$

3. *(Flip symmetry)*

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(-y, -t; -x, -s)$$

4. *(Skew stationarity)*

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} \mathcal{L}(x + cs, s; y + ct; t) + (t - s)^{-1}\big(x - y - c(t - s)^2 - (x - y)^2\big)$$

5. *(Rescaling)*

$$\mathcal{L}(x, s; y, t) \stackrel{d}{=} q\mathcal{L}(q^{-2}x, q^{-3}s; q^{-2}y, q^{-3}t)$$

3.3 Strategy of the proof

For ease of reading, we rewrite the statement of the theorem.

Theorem 15. *Let d_H be the Euclidean Hausdorff metric on $\mathbb{R} \times (0, \infty)$. Let $h \in \mathbb{R}$ be a real number and let $K = [a, b] \times [c, d]$ with $a < b$ and $0 < c < d$ be a compact set on $\mathbb{R}^2$. Then,*

$$\lim_{n\to\infty} d_H(\mathcal{K}_n(0, 0; \cdot, \cdot)^{-1}(h) \cap K, \mathcal{L}(0, 0; \cdot, \cdot)^{-1}(h) \cap K) = 0$$

almost surely.

The strategy of the proof is the following:

1. We will prove a general, deterministic result for the convergence of h-level sets when a sequence of functions converges uniformly on compacts. The only requirement for this convergence is that the limiting function, f, doesn't have local extrema for any point $\vec{x}$, such that $f(\vec{x}) = h$. See Subsection 3.4.1.

2. We need to prove that the directed landscape doesn't have local extrema on the h-level set. To do that we will use the fact that, for fixed times the landscape

is equal in distribution to the top line of the parabolic Airy line ensemble and that the parabolic Airy line ensemble is locally Brownian. This will allow us to express $\mathcal{L}(0,0,y,t)$ as the sum of a Brownian bridge and an independent random variable. See Subsection 3.4.2.

3. We will show that we can express a Brownian bridge B as the sum of a Gaussian X and an independent stochastic process. Then, $\mathcal{L}(0,0,y,t) = X + D(y,t)$ where D is a random function independent of X. See Proposition 17.

4. We will prove that any random function of the form $X + D(y,t)$ has no atoms with probability 1.

3.4 Convergence of level sets of LPP to directed landscape

In this section we will prove Theorem 3.

3.4.1 Convergence of level sets when a sequence of functions converges

We start by proving a deterministic version of our theorem.

Lemma 16. *Let $K = [a,b] \times [c,d]$ with $a < b$ and $0 < c < d$ be a compact set on $\mathbb{R}^2$ and $h \in \mathbb{R}$ be a real number. Let $g_n : K \to \mathbb{R}$ be a sequence of continuous functions that converges to a function $g : K \to \mathbb{R}$ uniformly. Suppose that g has no local extrema with value h on K. Then $g_n^{-1}(h) \cap K$ converges to $g^{-1}(h) \cap K$ with respect to the (Euclidean) Hausdorff metric.*

Proof. Let $K = [a,b] \times [c,d]$ with $a < b$ and $0 < c < d$ be a compact set on $\mathbb{R}^2$. We need to prove that

$$\lim_{n\to\infty} d_H(g_n^{-1}(h) \cap K, g^{-1}(h) \cap K) = 0.$$

Recall that for any pair of sets $A, B \subseteq K$,

$$d_H(A,B) = \inf\{\delta > 0 : A \subseteq (B)_\delta \text{ and } B \subseteq (A)_\delta\}$$

where $(A)_\delta = \bigcup_{x\in A} \{z \in K : \|x - z\| \leqslant \delta\}$.

We claim that there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$, such that that $g^{-1}(h) \cap K \subseteq (g_n^{-1}(h)) \cap K)_\delta$ for all $\delta < \delta_0$ and $n \geqslant n_0$. Take any $x \in g^{-1}(h) \cap K$. Since x cannot be a local maxima or minima of g on K, there must exist two sequences, $(x_j)_j$ and $(y_j)_j$ in K such that

$$g(x_j) < g(x) = h < g(y_j)$$

such that for all $j \in \mathbb{N}$,

$$\|x_j - x\| \leqslant \frac{1}{j} \qquad \text{and} \qquad \|y_j - x\| \leqslant \frac{1}{j} \tag{3.3}$$

Since we also assumed that g_n converges to g uniformly, for each $\varepsilon > 0$, there exists a natural number $N_1(\varepsilon)$ such that for $n \geqslant N_1(\varepsilon)$,

$$|g_n(w) - g(w)| < \varepsilon$$

for any $w \in K$. For each $j \in \mathbb{N}$, we take $N_1(j)$ such that for all $n \geqslant N_1(j)$,

$$|g_n(w) - g(w)| < g(x) - g(x_j)$$

for any $w \in K$. Then, evaluating at the point x_j, we get

$$g_n(x_j) - g(x_j) < g(x) - g(x_j).$$

This proves that for all $n \geqslant N_1(j)$, $g_n(x_j) < g(x) = h$. Similarly, there must exist an integer $N_2 = N_2(j)$ such that for $n \geqslant N_2$, $h < g_n(y_j)$. So for $n \geqslant N = N(j) := \max\{N_1, N_2\}$, we will have that

$$g_n(x_j) - h < 0 < g_n(y_j) - h.$$

For each $n \geqslant N(j)$, by the Intermediate Value Theorem, there is a point x_n in the segment that starts at x_j and ends at y_j such that $g_n(x_n) = h$. Since K is a rectangle and $x_j, y_j \in K$, the point x_n is in K too. By (3.3), we know that $\|x_j - x\| \leqslant \frac{1}{j}$ and $\|y_j - x\| \leqslant \frac{1}{j}$ so

$$\|x_n - x\| \leqslant \|x_n - x_j\| + \|x_j - x\| \leqslant \|y_j - x_j\| + \|x_j - x\| < \frac{3}{j}.$$

This means that $x_n \in B(x, 3/j) \cap g_n^{-1}(h)$ for $n \geqslant N(j)$. So for each $j \in \mathbb{N}$, $x \in (g_n^{-1}(h) \cap K)_{3/j}$ for all $n \geqslant N(j)$.

We now show the reverse inclusion: that for each $j \in \mathbb{N}$,

$$g_n^{-1}(h) \cap K \subseteq (g^{-1}(h) \cap K)_{3/j}$$

for n sufficiently large. Fix $j \in \mathbb{N}$ and suppose that this is not true. Then there exists a sequence of points on the h-level set, $z_{n_k} \in g_{n_k}^{-1}(h) \cap K$ such that for any $x \in g^{-1}(h) \cap K$, $|z_{n_k} - x| > \frac{3}{j}$.

Fix any one such $x \in g^{-1}(h) \cap K$. By the compactness of K, $\{z_{n_k}\}$ must have a convergent subsequence which we will without loss of generality take to be $\{z_{n_k}\}$ itself with a limit z. We claim that $z \in g^{-1}(h)$. To see this, observe that by the triangle inequality,

$$|g(z) - h| \leqslant |g(z) - g_{n_k}(z)| + |g_{n_k}(z) - g_{n_k}(z_{n_k})| + |g_{n_k}(z_{n_k}) - h|.$$

The third summand is 0 by definition of z_{n_k}. The first summand tends to 0 as $k \to \infty$ by uniform convergence of g_n to g. The second summand goes to zero as $k \to \infty$ by continuity of g_{n_k} for each k, and so we see that $|g(z) - h|$ must be 0 as the upper bound becomes arbitrarily small as $k \to \infty$. This means that $g(z) = h$ as claimed, and this is a contradiction. We chose z_{n_k} so that $|x - z_{n_k}| > \frac{3}{j}$ for any $x \in g^{-1}(0)$ but $|z - z_{n_k}| < \frac{3}{j}$ eventually by definition of z.

We conclude that for all $j \in \mathbb{N}$, there exists M, such that

$$g_n^{-1}(0) \cap K \subseteq (g^{-1}(0) \cap K)_{3/j} \text{ and } g^{-1}(0) \cap K \subseteq (g_n^{-1}(0) \cap K)_{3/j}$$

for all $n \geqslant M$. This proves that

$$\lim_{n\to\infty} d_H(g_n^{-1}(h) \cap K, g^{-1}(h) \cap K) = 0.$$

□

3.4.2 Proof of Theorem 3

By Lemma 16, to prove Theorem 3 we need to show that for any compact set $K = [a,b] \times [c,d]$ with $a < b$ and $0 < c < d$, the directed landscape $\mathcal{L}(0,0;y,t) : K \to \mathbb{R}$ has no extrema on the h-level set with probability 1. To prove that, we will use the fact that for fixed times, the directed landscape is locally Brownian to express $\mathcal{L}$ as the sum of an absolutely continuous random variable and an independent random function. Then, we will prove that any stochastic process expressed in that way has probability 0 of having an extrema on the h-level set over any rectangle.

Proof of Theorem 3. Let $h \in \mathbb{R}$. By Lemma 16, it is enough to prove that for any compact $K = [a,b] \times [c,d]$ with $a < b$ and $0 < c < d$,

$$\mathbb{P}\Big(\sup_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\Big) = 0 \tag{3.4}$$

and

$$\mathbb{P}\Big(\inf_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\Big) = 0. \tag{3.5}$$

Notice that Lemma 16, requires that there is no local extrema on a compact on the

h-level set and we are only proving that there is no absolute maximum or minimum on the h-level set. This is not a problem because the probabilities (3.4) and (3.4) are 0 for any rectangle, not only the rectangle on the statement of Theorem 3.

We begin by observing that for any $y \in \mathbb{R}$ and $0 < s < t$, we can use the metric composition property of the directed landscape from Proposition 13 to express

$$\mathcal{L}(0,0;y,t) = \sup_{x\in\mathbb{R}} \mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;y,t).$$

Here, $\mathcal{L}(0,0;x,s)$ and $\mathcal{L}(x,s;y,t)$ are independent. Moreover, we know that for fixed s, by Proposition 13

$$\mathcal{L}(0,0;\cdot,s) \overset{d}{=} s^{1/3}\mathfrak{A}_1\Big(\frac{\cdot}{s^{2/3}}\Big),$$

where $\mathfrak{A}_1$ is the top line of the parabolic Airy ensemble. For any function f, we will denote $f^{(s)}(x) = s^{1/3} f\big(\frac{x}{s^{2/3}}\big)$. Then, by this equality in distribution and the independence previously mentioned, we know that

$$\mathcal{L}(0,0;y,t) \overset{d}{=} \sup_{x\in\mathbb{R}} s^{1/3}\mathfrak{A}_1\Big(\frac{x}{s^{2/3}}\Big) + \mathcal{L}(x,s;y,t) = \sup_{x\in\mathbb{R}} \mathfrak{A}_1^{(s)}(x) + \mathcal{L}(x,s;y,t)$$

where $\mathfrak{A}_1$ and $\mathcal{L}$ are independent.

In turn, we can translate the probabilities (3.4) and (3.5) as

$$\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in\mathbb{R}} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}(x,c/2;y,t) = h\Big) \tag{3.6}$$

and

$$\mathbb{P}\Big(\inf_{(y,t)\in K}\sup_{x\in\mathbb{R}} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}(x,c/2;y,t) = h\Big) \tag{3.7}$$

respectively since $K = [a,b]\times[c,d]$ and $c/2 \leqslant t$ for all $t \in K$. We would like to use the absolute continuity of $\mathfrak{A}_1$ with respect to a Brownian bridge on a compact (see Theorem 8) but we are looking at the supremum of $\mathfrak{A}_1$ over the whole line. First, we need to break the supremum over $\mathbb{R}$ into smaller pieces. It is clear that there exists a random number $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}\big(x,\frac{c}{2};y,t\big) = \sup_{x\in[-N,N]} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}\big(x,\frac{c}{2};y,t\big).$$

Then,

$$\begin{aligned}\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in\mathbb{R}}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big)\\ &=\mathbb{P}\Big(\bigcup_{n\in\mathbb{N}}\Big\{\sup_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big\}\Big)\\ &\leqslant\sum_{n\in\mathbb{N}}\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\Big(\inf_{(y,t)\in K}\sup_{x\in\mathbb{R}}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big)\\ &=\mathbb{P}\Big(\bigcup_{n\in\mathbb{N}}\Big\{\inf_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big\}\Big)\\ &\leqslant\sum_{n\in\mathbb{N}}\mathbb{P}\Big(\inf_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big).\end{aligned}$$

The result will follow from proving that for each $n\in\mathbb{N}$,

$$\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big)=0 \tag{3.8}$$

and

$$\mathbb{P}\Big(\inf_{(y,t)\in K}\sup_{x\in[-n,n]}\mathfrak{A}_1^{(c/2)}(x)+\mathcal{L}\big(x,\frac{c}{2};y,t\big)=h\Big)=0. \tag{3.9}$$

To study these random variables, we are finally going to use the fact that the Airy line ensemble is locally Brownian. By the stationarity of the stationary Airy process, we have the equality in distribution

$$\Big(\mathfrak{A}_1^{(\sigma)}(r)\Big)_{r\in[a,b]}\overset{d}{=}\Big(\mathfrak{A}_1^{(\sigma)}(r)+\frac{r^2}{\sigma}-\frac{(r+a-1)^2}{\sigma}\Big)_{r\in[1,b-a+1]}.$$

The more natural thing would be to shift the parabolic Airy process to the origin but it will become clear later why we need a margin far from 0.

Then, we can rewrite equations (3.8) and (3.9) as

$$\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in[1,2n+1]}\mathfrak{A}_1^{(c/2)}(x)+\frac{2x^2}{c}-\frac{2(x-n-1)^2}{c}+\mathcal{L}\big(x-n-1,\frac{c}{2};y,t\big)=h\Big)=0$$

and

$$\mathbb{P}\Big(\inf_{(y,t)\in K}\sup_{x\in[1,2n+1]}\mathfrak{A}_1^{(c/2)}(x)+\frac{2x^2}{c}-\frac{2(x-n-1)^2}{c}+\mathcal{L}\big(x-n-1,\frac{c}{2};y,t\big)=h\Big)=0.$$

By Theorem 8 for any interval $[0,T]$ with $T \geqslant 1$, there exist a standard Brownian bridge B and an independent random linear shift L and an absolute constant $c_1 > 0$ such that for any subset S of $C([0,T],\mathbb{R})$,

$$\mathbb{P}(\mathfrak{A}_1(x) \in S) \leqslant e^{c_1 T^2}\mathbb{P}(B(2x) + L(x) \in S).$$

We will use this result. Notice that the natural interval to apply this result to is the interval $[c^{2/3}/2^{2/3}, 2^{1/3}nc^{2/3} + c^{2/3}/2^{2/3}]$. However, to make sure that the interval has length larger than 1 and give us a little margin, we will use a slightly larger interval. Then, on the interval $[0, 2^{1/3}nc^{2/3} + c^{2/3}] =: [0, T_0]$, we get that

$$\begin{aligned}\mathbb{P}\Big(&\sup_{(y,t)\in K}\sup_{x\in[1,2n+1]} \mathfrak{A}_1^{(c/2)}(x) + \frac{2x^2}{c} - \frac{2(x-n-1)^2}{c} + \mathcal{L}\big(x-n-1, \frac{c}{2}; y,t\big) = h\Big)\\ &\leqslant e^{c_1 c^{4/3}(1+2^{1/3}n)^2}\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in[1,2n+1]} B^{(c/2)}(2x) + L^{(c/2)}(x) + g(x,y,t) = h\Big).\end{aligned} \tag{3.10}$$

where $g(x,y,t) = +\frac{2x^2}{c} - \frac{2(x-n-1)^2}{c} + \mathcal{L}\big(x-n-1, \frac{c}{2}; y, t\big)$. Here we have used again that $\mathfrak{A}_1$ and $\mathcal{L}$ and therefore $\mathfrak{A}_1$ and g are independent in the expression above.

We will now prove that one can 'pull out' a random variable from a Brownian bridge. Meaning, that for any Brownian bridge, B, one can write B as the sum of a random variable and an independent stochastic process. This proposition will also be applied in Section 6.3 to find the Hausdorff dimension of the level sets.

Proposition 17. *Let $a \in \mathbb{R}$, $T > 0$, and let $(B(r))_{r\in[a,a+T]}$ be a Brownian bridge with arbitrary start and end values. Then for any $\delta \in (0, \frac{1}{2})$, we may write*

$$(B(r))_{r\in[a+\delta T, a+(1-\delta)T]} = N + (B(r) - N)_{r\in[a+\delta T, a+(1-\delta)T]}$$

where $N \sim \mathcal{N}(0, \frac{1}{2}\delta T)$ and is independent of the process $(B(r) - N)_{r\in[a+\delta T, a+(1-\delta)T]}$.

Proof. By subtracting a deterministic linear function, we may assume that $B(a) = B(a+T) = 0$ without loss of generality. Define

$$N = \frac{B(a+\delta T) + B(a+(1-\delta)T)}{2}$$

and let $r \in [a + \delta T, a + (1-\delta)T]$. We then compute that

$$\begin{aligned}
\mathrm{Cov}(N, B(r)) &= \mathrm{Cov}(\frac{B(a+\delta T)}{2}, B(r)) + \mathrm{Cov}(\frac{B(a+(1-\delta)T)}{2}, B(r)) \\
&= \frac{1}{2}\frac{((a+T)-r)((a+\delta T)-a)}{T} + \frac{1}{2}\frac{((a+T)-(a+(1-\delta)T))(r-a)}{T} \\
&= \frac{1}{2}\frac{((a+T)-r)\delta T}{T} + \frac{1}{2}\frac{\delta T(r-a)}{T} \\
&= \frac{1}{2}\delta T.
\end{aligned}$$

We also see immediately that

$$\begin{aligned}
\mathrm{Var}(N) &= \frac{1}{4}\mathrm{Var}(B(a+\delta T)) + \frac{1}{4}\mathrm{Var}(B(a+(1-\delta)T) + \frac{1}{2}\mathrm{Cov}(B(a+\delta T), B(a+(1-\delta)T)) \\
&= \frac{1}{4}\frac{(1-\delta)T(\delta T)}{T} + \frac{1}{4}\frac{\delta T(1-\delta)T}{T} + \frac{1}{2}\frac{((a+T)-(a+(1-\delta)T))((a+\delta T)-a)}{T} \\
&= \frac{1}{4}(\delta T - \delta^2 T) + \frac{1}{4}(\delta T - \delta^2 T) + \frac{1}{2}\delta^2 T \\
&= \frac{1}{2}\delta T.
\end{aligned}$$

Therefore, for each $r \in [a + \delta T, a + (1-\delta)T]$ we have that

$$\mathrm{Cov}(N, B(r) - N)) = \mathrm{Cov}(N, B(r)) - \mathrm{Var}(N) = 0.$$

Thus the process $(B(r) - N)_{r\in[a+\delta T, a+(1-\delta)T]}$ is uncorrelated with N, and hence the two are independent as claimed. □

Recall that $T_0 = c^{2/3}(1 + 2^{1/3}n)$. Let δ be a positive real number such that $\delta < \min\{1/2, (c/2)^{2/3}\}$. Going back to the proof of the convergence of the level sets, by Proposition 17, we know that there exists a random variable $X \sim \mathcal{N}(0, \frac{1}{2}\delta T_0)$ such that

$$B(2x) = X + (B(2x) - X)$$

for any $x \in [\delta T_0, (1-\delta)T_0]$ such that X and $(B(2x) - X)_x$ are independent. Then, we can rewrite probability (3.10) as

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{(y,t)\in K}\sup_{x\in[1,2n+1]} B^{(c/2)}(2x) + L^{(c/2)}(x) + g(x,y,t) = h\Big) \\
&\qquad\qquad = \mathbb{P}\Big(\big(\frac{c}{2}\big)^{1/3} X + \sup_{(y,t)\in K}\sup_{x\in[1,2n+1]} f(x,y,t) = h\Big)
\end{aligned}$$

where f is the random function defined as

$$f(x,y,t) := B^{(c/2)}(2x) - \big(\frac{c}{2}\big)^{1/3} X + L^{(c/2)}(x) + g(x,y,t)$$

and X is independent of the stochastic function $(f(x,y,t))_{[1,2n+1]\times K}$.

To conclude we need to prove that any random variable expressed as

$$Z := \Big(\frac{c}{2}\Big)^{1/3} X + \sup_{(y,t)\in K} \sup_{x\in[1,2n+1]} f(x,y,t)$$

has no atoms on the h-level set. In fact, let h_X be the density of $\big(\frac{c}{2}\big)^{1/3} X$ and

$$Y_{K,n} := \sup_{(y,t)\in K} \sup_{x\in[1,2n+1]} f(x,y,t),$$

then the cumulative distribution function of Z, F_Z is

$$F_Z(w) = \int_{\mathbb{R}} F_{Y_{K,n}}(w-z) h_X(z) dz$$

where $F_{Y_{K,n}}$ is the cumulative distribution function of the random variable $Y_{K,n}$. Since h_X is C^1, we know that the convolution of $F_{Y_{K,n}}$ and h_X is also C^1. This proves that F_Z is differentiable with continuous derivative and that Z is an absolutely continuous random variable. Then,

$$\mathbb{P}\Big(\big(\frac{c}{2}\big)^{1/3} X + \sup_{(y,t)\in K} \sup_{x\in[1,2n+1]} f(x,y,t) = h\Big) = \mathbb{P}(Z=h) = 0.$$

We have proved that

$$\mathbb{P}\Big(\sup_{(y,t)\in K} \sup_{x\in[-n,n]} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}\big(x, \frac{c}{2}; y, t\big) = h\Big) = 0$$

for all $n \in \mathbb{N}$. Therefore,

$$\mathbb{P}\Big(\sup_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\Big) \leqslant \sum_{n\in\mathbb{N}} \mathbb{P}\Big(\sup_{(y,t)\in K} \sup_{x\in[-n,n]} \mathfrak{A}_1^{(c/2)}(x) + \mathcal{L}\big(x, \frac{c}{2}; y, t\big) = h\Big) = 0.$$

In exactly the same way, one can prove that

$$\mathbb{P}\Big(\inf_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\Big) = 0.$$

This concludes the proof. □

Chapter 4

Hausdorff dimension of the level sets of the directed landscape

This chapter contains the proof of two results. In the first section, we will assume that Theorems 5, 6 and 7 are true. Under that assumption, the proof of the Hausdorff dimension of the level sets of the directed landscape, Theorem 4, is simple. In the second section, we will prove the upper bound on the Hausdorff dimension of the level sets of the directed landscape, Theorem 5. The proof of Theorems 6 and 7 can be found on Chapters 5 and 6 respectively.

4.1 The Hausdorff dimension

In this section we will introduce the Hausdorff dimension and relate it to the directed landscape. More details on the specific techniques used to calculate the Hausdorff dimension of the level sets of the directed landscape can be found on 4.3.1 and in Chapter 5.

The Hausdorff dimension is a measure of the 'size' of a mathematical object that extends the traditional integer dimension notion (a point has dimension 0, a line has dimension 1, etc.) to allow to give a measure to the complexity of self-similar objects that present a lot of 'roughness'. It is the natural notion of dimension for fractals.

Let X be a metric space. For any $\alpha \geqslant 0$ and subset $U \subset X$, the α-dimensional Hausdorff measure of U is defined as

$$\lim_{\delta \searrow 0} \inf \left\{ \sum_i (\operatorname{diam} U_i)^\alpha : \{U_i\} \text{ is a covering of } U \text{ with } 0 < \operatorname{diam}(U_i) < \delta \right\}. \tag{4.1}$$

The Hausdorff dimension of U is

$$\dim_H(U) = \inf\{\alpha > 0 : \alpha - \text{dimensional Hausdorff measure of } U \text{ is zero}\}.$$

Typically, to find the Hausdorff dimension of a set, one would obtain matching upper and lower bounds.

The idea is a sophisticated version of the following reasoning: take the square $[0,1]^2$. Split it smaller squares $\{S_i\}$ of size $\frac{1}{n^2}$. We need n^2 squares S_i to cover $[0,1]^2$. Intuitively, that exponent is the dimension. In fractal objects, the coverings need to include sets of different sizes to cover the whole object at all scalings.

Similarly to the scaling properties of the Brownian motion, the scaling properties of the directed landscape (see Lemma 14) mean that it is a fractal object. Therefore, its level sets are too.

The intuition behind the specific number $\frac{2}{3}$ is the scaling of the directed landscape. By Lemma 14, for any $q > 0$,

$$\mathcal{L}(0,0;0,t) \overset{d}{=} q^{1/3}\mathcal{L}(0,0;0,q^{-1/3}t).$$

This means that the number of zeroes on $[0,\varepsilon]$ scales like $\varepsilon^{2/3}$ heuristically.

The Hausdorff dimension of the level sets of $\mathcal{L}(0,0;x,t)$ is $\frac{5}{3}$ and can be found using the same strategy. This work was done by Lemonte Alie-Lamarche in collaboration with me.

The Hausdorff dimension has a property that we will use a couple of times: it is stable under countable unions. In fact, if $X = \bigcup_{i\in\mathbb{N}} X_i$, then

$$\dim_H(X) = \sup_{i\in\mathbb{N}} \dim_H(X_i).$$

From now on, the set $\mathbb{N}$ does not contain the number 0.

4.2 Hausdorff dimension of the level sets of the directed landscape

For ease of readability, we will rewrite the statement of the result from the introduction.

Theorem 18. *Let* $\dim_H$ *be the Hausdorff dimension of a set. Let* $h \in \mathbb{R}$*. Then,*

$$\dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h)) = \frac{2}{3}$$

with positive probability.

Remark. The positive probability is the one obtained in Theorem 6 for the stochastic process $\mathcal{L}(0,0;0,t)$ on $t \in (0,\infty)$ and $\beta = \frac{1}{3}$.

Proof. By Theorem 5, we know that

$$\dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h)) \leqslant \frac{2}{3}$$

almost surely.

By Theorem 6, we know that if there exists an $\varepsilon_0 > 0$ and positive constants c_h, c_h and c''_h such that for any $s, t \in [1,2]$ with $s < t$ and for all $\varepsilon \leqslant \varepsilon_0$ the three following 'density' bounds are true

$$\mathbb{P}(\mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \leqslant c_h 2\varepsilon \tag{4.2}$$

$$\mathbb{P}(\mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \geqslant c'_h 2\varepsilon \tag{4.3}$$

$$\mathbb{P}(\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon), \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \leqslant c''_h 4\varepsilon^2 |t-s|^{-1/3}, \tag{4.4}$$

then

$$\dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h) \cap [1,2]) \geqslant 1 - \frac{1}{3} = \frac{2}{3}$$

with positive probability $p_h = \frac{5}{36}\frac{c'^2_h}{c''_h}$.

Take $\varepsilon_0 = 1$. Recall from Proposition 13 that for each t fixed,

$$\mathcal{L}(0,0;0,t) \stackrel{d}{=} t^{1/3}\mathfrak{A}_1(0) \stackrel{d}{=} t^{1/3}X$$

where X is distributed according to the Tracy-Widom 2 distribution. Recall that the Tracy-Widom 2 distribution, sometimes also called the GUE Tracy-Widom distribution, has a continuous positive density, f_{TW_2}, on $\mathbb{R}$ (see for example, the survey [28]). Then, we can take

$$c_h = \max_{[h-1,h+1]} f_{TW_2}(x) \qquad \text{and} \qquad c'_h = \min_{[h-1,h+1]} f_{TW_2}(x)$$

that exist by continuity of the density and are positive because $f_{TW_2}(x) > 0$ for all $x \in \mathbb{R}$. These constants satisfy inequalities (4.2) and (4.3).

Inequality (4.4) is proved in Theorem 7.

We have proved that

$$\dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h) \cap [1,2]) \geqslant \frac{2}{3} \tag{4.5}$$

with positive probability $p_h = \frac{5}{36}\frac{c'^2_h}{c''_h}$. By the countable stability of the Hausdorff dimension, we know that if a set $Y = \bigcup_{n\in\mathbb{N}} Y_n$ is a countable union of subsets then

$$\dim_H(Y) = \sup_{n\in\mathbb{N}} \dim_H(Y_n).$$

The directed landscape $\mathcal{L}(0,0;0,t)$ is defined on $(0,\infty)$ and can be expressed as a countable union as follows:

$$\mathcal{L}(0,0;0,t)^{-1}(h) = \bigcup_{n\in\mathbb{N}} \mathcal{L}(0,0;0,t)^{-1} \cap [n,n+1](h) \bigcup \bigcup_{k\in\mathbb{N}} \mathcal{L}(0,0;0,t)^{-1} \cap \Big[\frac{1}{k+1}, \frac{1}{k}\Big](h)$$

Using the countable stability of the Hausdorff dimension, we get that

$$\dim_H(\mathcal{L}(0,0;0,t)^{-1}(h)) \geqslant \dim_H(\mathcal{L}^{-1}(0,0;0,\cdot)(h) \cap [1,2])$$

which as we know by (4.5) is larger than $\frac{2}{3}$ with probability p_h. This concludes the proof.

□

4.3 Upper bound on the Hausdorff dimension of level sets

This section will be devoted to proving Theorem 5. We restate it for readability purposes.

Theorem 19. *For each $h \in \mathbb{R}$, let Z_h be the set*

$$Z_h := \{t \in (0,\infty) : \mathcal{L}(0,0;0,t) = h\}.$$

Then, for any $h \in \mathbb{R}$,

$$\dim_H(Z_h) \leqslant \frac{2}{3}$$

almost surely.

4.3.1 Strategy for the upper bound

Before we prove this result we will explain briefly our reasoning. Suppose that we need to find an upper bound for the Hausdorff dimension of a subset E of $\mathbb{R}^n$. We can split a compact region $C \subset \mathbb{R}^n$ that contains E in squares whose side has length of $1/m$. If we count the number of squares needed to cover E and we find it is of order m^d squares, then d is an upper bound because we would have found a covering that upper bounds equation 4.1.

Of course, the level sets of the directed landscape are not deterministic. However, suppose that E is random and say we can show that for any square R of diameter r,

$$\mathbb{P}(E \cap R \neq \varnothing) \leqslant r^{n-d}, \tag{4.6}$$

possibly multiplied by a constant and logarithmic terms. Then, splitting the compact C in approximately r^{-n} squares of diameter r, the expected number needed to cover E is bounded by

$$\# \text{ of squares } \times \mathbb{P}(E \cap R \neq \varnothing) \leqslant r^{-d}.$$

This would prove that the α-Hausdorff measure of E is 0 for every $\alpha > d$.

Now the question is how to find the estimate in (4.6). The modulus of continuity is of help with this in our case. Take a random function $f : \mathbb{R} \to \mathbb{R}$ that is α-Holder continuous. Then, if f has a point on the h-level set at $y \in [a, b]$ then for any $x \in [a, b]$,

$$|f(x) - f(y)| = |f(x) - h| \leqslant |x - y|^{\alpha} \leqslant (b - a)^{\alpha}.$$

Then,

$$\mathbb{P}(Z \cap [a, a + \varepsilon] \neq \varnothing) \leqslant \mathbb{P}(|f(a + \varepsilon)| \leqslant \varepsilon^{\alpha}). \tag{4.7}$$

To bound the right hand side of the equation above, we can use a density bound on f and we would obtain the desired estimate.

The modulus of continuity of the directed landscape is Theorem 11 and essentially means that if $K \subset \{(s,t) \in \mathbb{R}^2 : s < t\}$ is a compact, then

$$|\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s)| \leqslant C_K (t-s)^{1/3} \log^{2/3}((t-s)^{-1} + 1)$$

for all $s, t \in K$ close enough. Here, the constant C is a random variable. This explains why the upper bound to the Hausdorff dimension is $\frac{2}{3}$.

The proof is very similar to the proof of the same result but in the case of the Brownian motion (see [25]). The constant C_K adds a challenge since it is random and so the simple argument expressed in equation (4.7) of using the density of the random variable, in this case the one-point density of the directed landscape, won't work. But we know from Theorem 11, that C_K decays exponentially. In fact,

$$\mathbb{P}(C_K > M) = O(e^{-M^{3/2}}) \quad \text{as } M \to \infty.$$

So we can use that either $|\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s)| \leqslant M(t-s)^{1/3} \log^{2/3}((t-s)^{-1} + 1)$ or $C_K > M$. Both events have very little probability which makes the probability of having an element of the Z_h level set very small if K is small too. This argument would work the same for other stochastic processes with a modulus of continuity that have a random constant with fast decay.

We move on to the proof of Theorem 5.

4.3.2 Proof of Theorem 5

Proof. We start by giving a heuristic idea about the (fractal) dimension of a set. Start by splitting a compact region K in $\mathbb{R}^n$ that contains Z_h into squares whose side has length $1/m$. If we count the number of squares needed to cover the set Z_h and find that we need $O(m^d)$ squares, then d is an upper bound of the dimension. So, we start by estimating the number of boxes of small size that are needed to cover the set Z_h in a compact. Since $Z_h \subset \mathbb{R}$, our boxes are closed intervals. To estimate this number we will find an upper bound of

$$\mathbb{P}(Z_h \cap [a, a+\varepsilon] \neq \varnothing) \tag{4.8}$$

for arbitrary $a > 0$ and $\varepsilon > 0$ small enough.

Let $n \in \mathbb{N}$. Consider the random function $\mathcal{L}(0,0;0,\cdot) : \left[\frac{1}{n}, n\right] \longrightarrow \mathbb{R}$. The interval $\left[\frac{1}{n}, n\right]$ is the compact set that we mentioned above. At the end of the proof we will argue, using the countable stability of the Hausdorff dimension, that $\dim_H(Z_h) \leqslant \frac{2}{3}$ and not just over this compact. For now and until the very end of the proof, n will be fixed. Fix $a > 0$ and $0 < \varepsilon < n^{-6}$ such that

$$[a, a+\varepsilon] \subseteq \left[\frac{1}{n}, n\right].$$

Suppose that for some $s \in [a, a+\varepsilon]$, we have that $\mathcal{L}(0,0;0,s) = h$. Then,

$$|\mathcal{L}(0,0;0,t) - h| = |\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s)| \tag{4.9}$$

for all $t \in [a, a+\varepsilon]$. By applying Theorem 12 with $\delta = \frac{1}{n}$, we know that there exists a random constant C_n, that only depends on n, such that

$$|\mathcal{L}(0,0;0,t) - h| \leqslant C_n \tau^{1/3} \log^{2/3}(\tau^{-1} + 1) \tag{4.10}$$

for all $t \in [a, a+\varepsilon]$. We will use this fact to bound probability (4.8). However, first we will take a look at the right hand side of the inequality above with the hopes of simplifying the expression.

Notice that for any $0 < \gamma < \frac{1}{3}$, we have that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{1/3} \log^{2/3}(\varepsilon^{-1} + 1)}{\varepsilon^{1/3-\gamma}} = 0.$$

Then, there exists $\varepsilon_0 > 0$ such that

$$\varepsilon^{1/3} \log^{2/3}(\varepsilon^{-1} + 1) \leqslant \varepsilon^{1/3-\gamma}$$

for all $\varepsilon \leqslant \varepsilon_0$. In fact, if $\varepsilon \leqslant \min\{\varepsilon_0, n^{-6}\}$, using equation (4.10) we get that for all $s, t \in [a, a+\varepsilon]$,

$$|\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s)| \leqslant C_n |t-s|^{1/3-\gamma} \leqslant C_n \varepsilon^{1/3-\gamma}.$$

We can conclude that if $\varepsilon > 0$ is small enough and there exists $s \in [a, a+\varepsilon]$ such that $\mathcal{L}(0,0;0,s) = h$, then

$$|\mathcal{L}(0,0;0,a) - h| \leqslant C_n \varepsilon^{1/3-\gamma}$$

for any $0 < \gamma < 1/3$. Translating this into probabilities, we get that

$$\mathbb{P}(Z_h \cap [a, a+\varepsilon] \neq \varnothing) \leqslant \mathbb{P}(|\mathcal{L}(0,0;0,a) - h| \leqslant C_n \varepsilon^{1/3-\gamma}) \tag{4.11}$$

for any $0 < \gamma < 1/3$. Recall that our first goal was to bound the term on the left hand side of the inequality above. The only thing left to do is to understand the distribution of $\mathcal{L}(0,0;0,a)$ and of C_n.

Recall from Proposition 13, that the one-point distribution of the directed landscape is related to the distribution of $\mathfrak{A}_1$ in the following way:

$$\mathcal{L}(0,0;0,a) \stackrel{d}{=} a^{1/3} \mathfrak{A}_1(0),$$

where $\mathfrak{A}_1(0)$ is the top line of the parabolic Airy line ensemble $\mathfrak{A}$ evaluated at 0. We know that $\mathfrak{A}(0)$ is distributed according to Tracy-Widom 2 distribution, TW_2. Let f_{TW_2} be the density of TW_2 and $\lambda := \|f_{TW_2}\|_\infty$. Then, we know that

$$\begin{aligned}
\mathbb{P}(|\mathcal{L}(0,0;0,a) - x| \leqslant k) &= \mathbb{P}(|a^{1/3}\mathfrak{A}_1(0) - x| \leqslant k) \\
&= \mathbb{P}(|\mathfrak{A}_1(0) - xa^{-1/3}| \leqslant a^{-1/3}k) \\
&\leqslant 2a^{-1/3}k\lambda
\end{aligned} \tag{4.12}$$

for all $x \in \mathbb{R}$, $k > 0$.

Of course, we don't know the distribution of C_n but we know that it decays fast. In fact, by Theorem 12, we know that there exist absolute positive constants c, d such that

$$\mathbb{P}(C_n > M) \leqslant c n^{16} e^{-dM^{3/2}}. \tag{4.13}$$

Now we are ready, to bound equation (4.11). We are going to use the idea explained before: either C_n is small, in which case we can bound (4.11) using the uniform norm of the Tracy-Widom 2 density or C_n is large, in which case, we will use its fast decay to bound (4.11). By the law of total probability and the fact that the probability intersection of two events is smaller than the probability of each of them, we get that

$$\mathbb{P}\Big(|\mathcal{L}(0,0;0,a)-h| \leqslant C_n\varepsilon^{1/3-\gamma}\Big) \leqslant \mathbb{P}\Big(|\mathcal{L}(0,0;0,a)-h| \leqslant C_n\varepsilon^{1/3-\gamma}, C_n \leqslant M\Big) + \mathbb{P}\Big(C_n > M\Big). \quad (4.14)$$

Using the bound (4.12), we get that

$$\mathbb{P}\Big(|\mathcal{L}(0,0;0,a)-h| \leqslant C_n\varepsilon^{1/3-\gamma}, C_n \leqslant M\Big) \leqslant \mathbb{P}\Big(|\mathcal{L}(0,0;0,a)-h| \leqslant M\varepsilon^{1/3-\gamma}\Big) \leqslant 2a^{-1/3}M\varepsilon^{1/3-\gamma}.$$

Joining this last bound with (4.11), (4.13) and (4.14), we obtain that

$$\mathbb{P}\Big(Z_h \cap [a, a+\varepsilon] \neq \varnothing\Big) \leqslant 2a^{-1/3}M\varepsilon^{1/3-\gamma} + cn^{16}e^{-dM^{3/2}} \quad (4.15)$$

for every $a, M > 0$, ε sufficiently small and $n \in \mathbb{N}$.

Now we will focus on counting the number of small intervals in $[\frac{1}{n}, n]$ needed to cover the h-level set. We will split $[\frac{1}{n}, n]$ in intervals of length $\frac{1}{2^m}$, with m large enough to make $\frac{1}{2^m} \leqslant \min\{\varepsilon_0, n^{-6}\}$. Notice that for all $m \in \mathbb{N}$,

$$\Big[\frac{1}{n}, n\Big] \subseteq \bigcup_{j=1}^{\lceil n-\frac{1}{n}\rceil 2^m} \Big[\frac{1}{n} + \frac{j-1}{2^m}, \frac{1}{n} + \frac{j}{2^m}\Big].$$

We define the (random) number of intervals of length $1/2^m$ needed to cover $Z_h \cap [\frac{1}{n}, n]$ as

$$N_m := \sum_{j=1}^{\lceil n-\frac{1}{n}\rceil 2^m} \mathbb{1}_{\left\{Z_h \cap [\frac{1}{n}+\frac{j-1}{2^m}, \frac{1}{n}+\frac{j}{2^m}] \neq \varnothing\right\}}.$$

We are going to estimate this number, by finding a bound on its expectation.

Looking at $\mathbb{E}[N_m]$, we observe that

$$\mathbb{E}[N_m] \leqslant \sum_{j=1}^{\lceil n-\frac{1}{n}\rceil 2^m} \mathbb{P}\Big(Z_h \cap \Big[\frac{1}{n} + \frac{j-1}{2^m}, \frac{1}{n} + \frac{j}{2^m}\Big] \neq \varnothing\Big).$$

By (4.15), we can bound the terms in the sum above and we obtain that

$$\begin{aligned}\mathbb{E}[N_m] \leqslant & \sum_{j=1}^{\lceil n-\frac{1}{n}\rceil 2^m} 2n^{1/3}M2^{-m/3+m\gamma} + cn^{16}e^{-dM^{3/2}} \\ =& \left\lceil n - \frac{1}{n}\right\rceil 2^m \left(2n^{1/3}M2^{-m/3+m\gamma} + cn^{16}e^{-dM^{3/2}}\right) \\ \leqslant & 2n^2 M 2^{m(2/3+\gamma)} + cn^{17}2^m e^{-dM^{3/2}} \end{aligned} \tag{4.16}$$

for any $M > 0$. We will choose M wisely, so that $\frac{\mathbb{E}[N_m]}{2^{m(2/3+\eta)}}$ is summable on m. In that way, we will have estimated the number of small intervals needed to cover the level set and in fact it will be of the correct order.

Let $M = (\frac{m}{3d})^{2/3}$ and $\eta > 0$. Let m_0 be large enough so that $2^{-m} \leqslant \min\{\varepsilon_0, n^{-6}\}$. By the Monotone Convergence Theorem and the inequality on (4.16), we see that

$$\begin{aligned}\mathbb{E}\left[\sum_{m=m_0}^{\infty} \frac{N_m}{2^{m(2/3+\gamma+\eta)}}\right] \leqslant & \sum_{m=m_0}^{\infty} \frac{\mathbb{E}[N_m]}{2^{m(2/3+\gamma+\eta)}} \\ \leqslant & \frac{2n}{(3d)^{2/3}} \sum_{m=m_0}^{\infty} \frac{m^{2/3}2^{m(2/3+\gamma)}}{2^{m(2/3+\gamma+\eta)}} + cn^{17} \sum_{m=m_0}^{\infty} \frac{2^m e^{-m/3}}{2^{m(2/3+\gamma+\eta)}} \\ =& \frac{2n}{(3d)^{2/3}} \sum_{m=m_0}^{\infty} m^{2/3}2^{-m\eta} + cn^{17} \sum_{m=m_0}^{\infty} \left(\frac{2}{e}\right)^{m/3} \frac{1}{2^{m(\gamma+\eta)}}\end{aligned}$$

which is finite for all $\eta > 0$ and $\gamma \in (0, \frac{1}{3})$. This implies that for any $\eta > 0$ and $\gamma \in (0, \frac{1}{3})$,

$$\limsup_{m\to\infty} \frac{N_m}{2^{m(2/3+\gamma+\eta)}} = 0$$

almost surely. This is true for all $n \in \mathbb{N}$.

We will now finish the proof. Recall that by the definition of the Hausdorff dimension

$$\dim_H\left(Z_h \cap \left[\frac{1}{n}, n\right]\right) = \inf\left\{\alpha : \mathcal{H}^\alpha\left(Z_h \cap \left[\frac{1}{n}, n\right]\right) = 0\right\}$$

where the α-Hausdorff measure is defined as

$$\mathcal{H}^\alpha\left(Z_h \cap \left[\frac{1}{n}, n\right]\right) = \lim_{\delta\to 0^+} \mathcal{H}^\alpha_\delta\left(Z_h \cap \left[\frac{1}{n}, n\right]\right)$$

and

$$\mathcal{H}^\alpha_\delta\left(Z_h \cap \left[\frac{1}{n}, n\right]\right) = \inf\left\{\sum_{i=1}^{\infty} |E_i|^\alpha : E_1, E_2, E_3, \ldots, \text{ cover } Z_h \cap \left[\frac{1}{n}, n\right] \text{ and } |E_i| \leqslant \delta\right\}.$$

Then, the union of each dyadic interval that intersects the level set is a covering of the h-level set by sets of diameter $\frac{1}{2^m}$ and such that the sum of the diameter of each set to the power of $\frac{2}{3}+\gamma+\eta$ is equal to $\frac{N_m}{2^{m(2/3+\gamma+\eta)}}$. Then,

$$\mathcal{H}_{1/2^m}^{\frac{2}{3}+\gamma+\eta}\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) \leqslant \frac{N_m}{2^{m(2/3+\gamma+\eta)}}$$

for every $\eta > 0$, $\gamma \in (0, \frac{1}{3})$ almost surely. In turn, this means that the α-Hausdorff measure of the level set Z_h on $\big[\frac{1}{n}, n\big]$ can be bounded as

$$\mathcal{H}^{\alpha}\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) \leqslant \limsup_{m\to\infty} \frac{N_m}{2^{m(2/3+\gamma+\eta)}} = 0$$

for all $\eta > 0$ and all $\gamma \in (0, \frac{1}{3})$ almost surely. Then,

$$\dim_H\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) = \inf\Big\{\alpha : \mathcal{H}^{\alpha}\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) = 0\Big\} \leqslant \frac{2}{3} + \gamma + \eta$$

almost surely. Taking $\eta \to 0$ and $\gamma \to 0$, we get that

$$\dim_H\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) \leqslant \frac{2}{3}$$

almost surely for all $n \in \mathbb{N}$.

Finally, using the countable stability of the Hausdorff dimension we get that

$$\dim_H(Z_h) = \dim_H \bigcup_{n=1}^{\infty}\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) = \sup_{n\in\mathbb{N}} \dim_H\Big(Z_h \cap \big[\frac{1}{n}, n\big]\Big) \leqslant \frac{2}{3}.$$

□

Chapter 5

Strategy for the lower bound of the Hausdorff dimension of level sets

In this chapter we will give some conditions for the existence of a lower bound to the Hausdorff dimension of the level sets of any stochastic process $(B(t))_t$ with positive probability. The conditions are equivalent to density bounds for the one and two-point distributions of the process. However, it does not require the proven existence of the density.

5.1 Lower bound of the Hausdorff dimension

The goal is to find the lower bound of the Hausdorff dimension of the level sets of a stochastic process. The real end goal is to have a template to find the Hausdorff dimension of the level sets of the directed landscape $\mathcal{L}(0,0;0,t)$. The proof is based on a similar early proof of the Hausdorff dimension of the level sets of the Brownian motion by Adler available in [1] (Theorem 8.4.2).

Let $B(t)$ be a stochastic process. The goal is to prove a lower bound for the Hausdorff dimension of $B^{-1}(h) \cap [1,2]$ of $1-\beta$ where β is the "scale" of the stochastic process. We will use the energy method. Suppose that we need to find a lower bound for the set $X \subset \mathbb{R}^n$. Suppose that there exists a measure μ, supported on X and such that $0 < \mu(X) < \infty$. Let $\alpha \geqslant 0$. We define the α-energy of μ is

$$I_\alpha(\mu) = \iint \frac{d\mu(x)d\mu(y)}{\|x-y\|^\alpha}.$$

We define

$$\mathcal{H}^\alpha_\varepsilon(X) = \inf\left\{\sum_i (\operatorname{diam} U_i)^\alpha : \{U_i\} \text{ is a covering of } X \text{ with } 0 < \operatorname{diam}(U_i) < \varepsilon\right\}$$

Theorem 20 (Energy method, see for example [25], Theorem 4.27)**.** *Let $\alpha \geqslant 0$ and μ a measure supported on X, finite and non-zero. Then, for every $\varepsilon > 0$,*

$$\mathcal{H}_\varepsilon^\alpha(X) \geqslant \frac{\mu(X)^2}{\iint\limits_{\{\|x-y\|<\varepsilon\}} \frac{d\mu(x)d\mu(y)}{\|x-y\|^\alpha}}.$$

So if $I_\alpha(\mu) < \infty$, then the α-dimensional Hausdorff measure of X is infinity so $\dim_H(X) \geqslant \alpha$.

The measure μ is sometimes called the Frostman measure. The idea is that if $I_\alpha(\mu) < \infty$, the measure μ spreads the mass in such a way that at each point, it compensates for zero in the denominator of order $\|x-y\|^{-\beta}$. This means that the measure is spreading the mass at the right scaling or less.

If the set is random, as it is for us, it is enough to show that

$$\mathbb{E}I_\alpha(\mu) < \infty$$

for a suitable random measure μ on X.

In our case, we need to find a random measure μ supported on the h-level set of $\mathcal{L}(0,0;0,t)$ such that

$$\mathbb{E}\left[\iint \frac{1}{|s-t|^\alpha} d\mu(s)d\mu(t)\right] < \infty \tag{5.1}$$

for all $\alpha = \frac{1}{3}$. The measure μ needs to be random because already its support is random. We will define μ through a limit of finite measures.

5.2 Remarks about the statement

We begin by rewriting the statement for ease of readability.

Theorem 21. *Let $B(t)$ be a stochastic process on $\mathbb{R}$. Let h be a real number. Assume that there exists an ε_0, positive constants c_h, c_h' and c_h'' and an exponent $0 < \beta < 1$ such that for all $t, s \in [1,2]$,*

$$\mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon)) \leqslant c_h 2\varepsilon \tag{5.2}$$

$$\mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon)) \geqslant c_h' 2\varepsilon \tag{5.3}$$

$$\mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon), B(s) \in (h-\varepsilon, h+\varepsilon)) \leqslant c_h'' 4\varepsilon^2 |t-s|^{-\beta} \tag{5.4}$$

for all $\varepsilon \leqslant \varepsilon_0$. Then, we get a lower bound for the Hausdorff dimension of the level sets:

$$d_H(B^{-1}(h) \cap [1,2]) \geqslant 1-\beta$$

with positive probability p_h *where*

$$p_h = \frac{c_h'^2(\beta^2 - 3\beta + 2)}{8c_h''}.$$

Try to get these remarks numbered

Remark. The same argument works for any interval of length 1. The choice of the interval $[1, 2]$ is to simplify the notation of the theorem.

Remark. Conditions (5.2) and (5.3) are immediate if the stochastic process B has a continuous positive density on $\mathbb{R}$. See Section 4.2 for an example of this.

Remark. The theorem can be easily used to prove that the Hausdorff dimension of the h-level set of B has the same lower bound. In fact, since

$$B^{-1}(h) = \bigcup_{n\in\mathbb{Z}} B^{-1}(h) \cap [n, n+1],$$

then by the countable stability of the Hausdorff dimension,

$$\dim_H(B^{-1}(h)) = \sup_{n\in\mathbb{Z}} \dim_H(B^{-1}(h) \cap [n, n+1]) \geqslant \dim_H(B^{-1}(h) \cap [1, 2]) \geqslant 1 - \beta.$$

Remark. Notice that naturally, $\lim_{h\to\pm\infty} c_h' = 0$ so $\lim_{h\to\pm\infty} p_h = 0$ too. In the proof, the probability p_h is related the probability of the level set not being empty.

5.3 Proof of Theorem 4

This section contains the proof of Theorem 3. The idea is that we are going to find a sequence of measures that is supported almost on the h-level set with finite energy integral and such that: it is tight and therefore converges, the limit is supported on the h-level set of B, the limit is not the zero measure and the limit has finite energy integral. The limit is indeed, our Frostman measure.

Proof. We will split the proof in subsections.

5.3.1 Defining a sequence of measures $\mu_{h,\varepsilon}$

For any subset $A \subset [1, 2]$, we define the random measure of A as

$$\mu_{h,\varepsilon}(A) = \frac{1}{2\varepsilon}\lambda(t : (t, B(t)) \in A \times (h - \varepsilon, h + \varepsilon)),$$

where λ is the Lebesgue measure on $\mathbb{R}$. This would be the occupation time of the stochastic process $B(t)$ on the set A in an interval around h, which is the level set.

Notice that we can rewrite the measure as an integral:

$$\mu_{h,\varepsilon}(A) = \frac{1}{2\varepsilon}\int_A \mathbb{1}(B(t) \in (h-\varepsilon, h+\varepsilon))dt. \tag{5.5}$$

5.3.2 Proof that μ_{h,ε_n} has a subsequence that converges in distribution as $n \to \infty$

A natural way to prove precompactness of a sequence of random measures is to prove tightness of such sequence. In fact, we will use the following form of Prokhorov's Theorem (see for example [22], Lemma 14.15):

Theorem 22. *Let $\mu_1, \mu_2, \ldots$ be random measures on a locally compact simply connected Hausdorff space S. Then the sequence (μ_n) is relatively compact in distribution iff $(\mu_n(A))$ is tight in $\mathbb{R}_+$ for every $A \in S$.*

In our case, $S = [1,2]$ and $\mu_n = \mu_{h,\varepsilon}$. Take A a subset of $[1,2]$. To prove that $(\mu_{h,\varepsilon}(A))$ is tight in $\mathbb{R}_+$, it is enough to prove that the mean of $(\mu_{h,\varepsilon}(A))$ is uniformly bounded. Given that $(\mu_{h,\varepsilon}(A))$ is non-zero for all h and ε, we just need to prove a uniform upper bound for the expectation of $\mu_{h,\varepsilon}(A)$.

Using (5.5), we get that

$$\mathbb{E}[\mu_{h,\varepsilon}(A)] = \frac{1}{2\varepsilon}\mathbb{E}\left[\int_A \mathbb{1}(B(t) \in (h-\varepsilon, h+\varepsilon))dt\right].$$

By Fubini's theorem, we exchange the order of integration and we get that the expectation above is equal to

$$\frac{1}{2\varepsilon}\int_A \mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon))dt. \tag{5.6}$$

The probability in the integrand can be bounded using (5.2) by $c\varepsilon$ where c is a constant so we end up with

$$\mathbb{E}[\mu_{h,\varepsilon}(A)] \leqslant c_h \int_A dt \leqslant c_h,$$

where in the last inequality we used that A is a subset of $[1,2]$. This proves that $(\mu_{h,\varepsilon}(A))$ is tight and therefore that there exists a subsequence, (μ_{h,ε_n}), that converges to a measure μ in distribution. The measure μ_h is our candidate to be the non-zero measure supported in the level set that has finite energy. We will prove these claims in the following subsections.

5.3.3 Proof that the limiting measure is non-zero with positive probability

By the Paley-Zygmund inequality we know that for a non-negative random variable X,

$$\mathbb{P}(X > \theta\mathbb{E}[X]) \geqslant (1-\theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]},$$

where $\theta \in [0,1]$. We want to prove that $\mu_{h,\varepsilon}([1,2])$ is not zero with positive probability. Clearly, $\mu_{h,\varepsilon}([1,2])$ is non-negative. Notice that if we prove that $\mathbb{E}[\mu_{h,\varepsilon}([1,2])] \geqslant K$ and $\mathbb{E}[\mu_{h,\varepsilon}([1,2])^2] \leqslant C$ where K and C are uniform constants then taking θ positive and $\theta < K$, we have that

$$\begin{aligned}\mathbb{P}(\mu_{h,\varepsilon}([1,2]) > \theta_h) &\geqslant \left(1 - \frac{\theta_h}{\mathbb{E}[\mu_{h,\varepsilon}([1,2])]}\right)^2 \frac{\mathbb{E}[\mu_{h,\varepsilon}([1,2])]^2}{\mathbb{E}[\mu_{h,\varepsilon}([1,2])^2]} \\ &\geqslant \left(1 - \frac{\theta_h}{K}\right)^2 \frac{K^2}{C} =: p_h > 0. \end{aligned} \tag{5.7}$$

We will now focus on the uniform bounds on the first and second moments of $\mu_{h,\varepsilon}([1,2])$. We need a lower bound for the mean of $\mu_{h,\varepsilon}([1,2])$. First, notice that, as in (5.6)

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])] = \frac{1}{2\varepsilon}\int_1^2 \mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon))dt.$$

Using the bound (5.3) we obtain,

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])] \geqslant c'_h \int_1^2 dt \geqslant c'_h.$$

This proves that

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])] \geqslant c'_h > 0.$$

For upper bound of the second moment, a similar calculation shows that:

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])^2] = \frac{1}{4\varepsilon^2} \iint\limits_{[1,2]^2} \mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon), B(s) \in (h-\varepsilon, h+\varepsilon))dtds.$$

Notice that this expression is symmetric with respect to s and t so we can assume that $s < t$ and multiply by a factor of 2. Using the bound (5.4),

$$\mathbb{P}(B(t) \in (h-\varepsilon, h+\varepsilon), B(s) \in (h-\varepsilon, h+\varepsilon)) \leqslant c''_h 4\varepsilon^2 |t-s|^{-\beta}$$

we get:

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])^2] \leqslant \frac{2}{4\varepsilon^2}\int_1^2\int_s^2 c_h'' 4\varepsilon^2 (t-s)^{-\beta} dtds.$$

Performing the change of variable $t-s=u$, we obtain

$$\mathbb{E}[\mu_{h,\varepsilon}([1,2])^2] \leqslant 2c_h''\int_0^1\int_0^{2-s} u^{-\beta} duds = 2c_h''\int_1^2 \frac{(2-s)^{1-\beta}}{1-\beta} ds = \frac{2c_h''}{\beta^2-3\beta+2} < \infty,$$

where we have used that $\beta<1$.

For each $h\in\mathbb{R}$, we use the Paley-Zygmund result from (5.7) with $\theta_h = \frac{c_h'}{2}$, we obtain that

$$\mathbb{P}(\mu_{h,\varepsilon}([1,2]) > \theta_h) \geqslant \frac{c_h'^2(\beta^2-3\beta+2)}{8c_h''} := p_h$$

This proves that for all $\varepsilon \leqslant \varepsilon_0$,

$$\mathbb{P}(\mu_{h,\varepsilon}([1,2]) > \theta_h) \geqslant p_h,$$

where θ_h and p_h depend on the level set h but not on ε. Now, we need to prove that this is also true for the limit. But using the properties of convergence in distribution, we get

$$\mathbb{P}(\mu_h([1,2]) \geqslant \theta_h) \geqslant \limsup_{n\to\infty}\mathbb{P}(\mu_{h,\varepsilon_n}([1,2]) \geqslant \theta_h) \geqslant \limsup_{n\to\infty}\mathbb{P}(\mu_{h,\varepsilon_n}([1,2]) > \theta_h) \geqslant p_h.$$

The conclusion is that the limiting measure is non-negative with probability at least p_h.

5.3.4 Proof that the limiting measure is supported on the level set

The purpose of this section is to establish that the limiting measure μ_h is indeed supported on (a subset of) the $h-$level set of the random process $B(t)$ on the interval $[1,2]$. To begin, take $(\Omega,\mathcal{F},\mathbb{P})$ to be the underlying probability space on which B and our measures $\mu_{h,\varepsilon}$ are defined, i.e.

$$B:\Big(\Omega,\mathcal{F},\mathbb{P}\Big)\to (C([1,2]),\sigma(\tau_{unif}))$$
$$\mu_{h,\varepsilon}:\Big(\Omega,\mathcal{F},\mathbb{P}\Big)\to (M_{[1,2]},\sigma(\tau_{vague}))$$

where $\sigma(\tau_{unif}))$ is the sigma algebra generated by the topology of uniform convergence, $M_{[1,2]}$ is the space of finite (positive) measures on $[1,2]$, and $\sigma(\tau_{vague})$ is the sigma algebra generated by the vague topology on $M_{[1,2]}$. Note that $M_{[1,2]}$ is a Polish space and that $C([1,2])$ is a complete metric space under the sup-norm by the Stone-Weierstrass theorem. This will be important momentarily.

Next, let d_{vague} be the metric generating the vague topology on $M_{[1,2]}$ and define a metric d_{prod} on the product of these two spaces by

$$d_{prod} : (M_{[1,2]} \times C([1,2])) \times (M_{[1,2]} \times C([1,2])) \to \mathbb{R}_+ \tag{5.8}$$

$$((f,\mu),(g,\nu)) \mapsto \sqrt{d_{vague}(\mu,\nu)^2 + (\sup_{x\in[1,2]} |f(x) - g(x)|)^2}. \tag{5.9}$$

Under this metric we can see immediately that the product space

$$(M_{[1,2]} \times C([1,2]), d_{prod})$$

is again a separable complete metric space. Let τ_{prod} denote the topology generated by the metric d_{prod} and $\sigma(\tau_{prod})$ be the Borel sigma algebra generated by this topology. In turn this means that any finite measure on the product space endowed with $\sigma(\tau_{prod})$ will automatically have compact support.

We will now focus our attention on the aforementioned convergent in law sequence of random measures $(\mu_{h,\varepsilon_n}(\cdot))_{n=1}^{\infty}$. With the conventions in this subsection thus far, we can view the pairs $(\mu_{h,\varepsilon_n}(\cdot), B(\cdot))$ as random elements

$$(\mu_{h,\varepsilon_n}(\cdot), B(\cdot)) : \left(\Omega, \mathcal{F}, \mathbb{P}\right) \to (M_{[1,2]} \times C([1,2]), \sigma(\tau_{prod})). \tag{5.10}$$

As such if for each $n \in \mathbb{Z}_{>0}$ we define the probability measure

$$Q_n = Law((\mu_{h,\varepsilon_n}(\cdot), B(\cdot))) \tag{5.11}$$

on $(M_{[1,2]\times} C([1,2]), \sigma(\tau_{prod}))$ then the sequence of probability measures (Q_n) has a weak limit Q_∞, and each probability measure Q_n (including $n = \infty$) has separable support. Thus we may use the Skorokhod Representation Theorem to construct a new probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ and random elements

$$Y_n(\cdot) : \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right) \to (M_{[1,2]} \times C([1,2]), \sigma(\tau_{prod})) \tag{5.12}$$

for each $n > 0$ (including $n = \infty$) such that $Y_n \to Y_\infty$ $\tilde{\mathbb{P}}-$almost surely and with $Law(Y_n) = Q_n$ for each n. For each $n \in \mathbb{Z}_{>0}$ write

$$Y_n(\cdot) =: (\tilde{\mu}_{h,\varepsilon_n}(\cdot), B^{(n)}(\cdot)) \tag{5.13}$$

and write

$$Y_\infty(\cdot) =: (\mu_h(\cdot), B^{(\infty)}(\cdot)). \tag{5.14}$$

As the almost-sure convergence $Y_n \to Y_\infty$ is with respect to the metric d_{prod}, we have by 5.8 that for $\tilde{\mathbb{P}}-$almost every $\omega \in \tilde{\Omega}$ and as $n \to \infty$,

$$\tilde{\mu}_{h,\varepsilon_n}(\omega) \to \mu_h(\omega) \tag{5.15}$$

in the vague topology on $M_{[1,2]}$ and that

$$B^{(n)}(\omega) \to B^{(\infty)}(\omega) \tag{5.16}$$

with respect to the sup-norm on $C([1,2])$.

Now observe that each $B^{(n)}$ (including $n = \infty$) is a copy of the random process B on $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, though they are not necessarily the same realization of B. We will show that this is not an issue for our purposes. To that end, we make several more elementary observations before addressing our original problem. First we establish that for each finite n

$$\operatorname{supp} \tilde{\mu}_{h,\varepsilon_n}(\omega) \subseteq (B^{(n)}(\omega))^{-1}([h - \varepsilon_n, h + \varepsilon_n]) \tag{5.17}$$

$\tilde{\mathbb{P}}-$almost surely. Letting π_1 and π_2 be the usual projection maps on $M_{[1,2]} \times C([1,2])$, this follows immediately from the fact that

$$\tilde{\mathbb{P}}(\pi_1(Y_n)(\pi_2(Y_n)^{-1}([h-\varepsilon_n, h+\varepsilon_n]^C)) > 0) = \mathbb{P}(\mu_{h,\varepsilon_n}(B^{-1}([h-\varepsilon_n, h+\varepsilon_n]^C)) > 0) = 0 \tag{5.18}$$

by our original definition of μ_{h,ε_n} in terms of B.

Next we fix $\delta > 0$ and recall that $\tilde{\mathbb{P}}-$almost surely there exists $N = N(\delta, \omega)$ such that for all $n > N$, $\varepsilon_n < \delta/2$ and

$$\sup_{x \in [1,2]} |B^{(n)}(\omega)(t) - B^{(\infty)}(\omega)(t)| < \delta/2. \tag{5.19}$$

As an immediate consequence of (5.19) and the fact that $\varepsilon_n + \delta/2 < \delta$ we have that for all $n > N$

$$B^{(n)}(\omega)^{-1}([h - \varepsilon_n, h + \varepsilon_n]) \subseteq B^{(\infty)}(\omega)^{-1}([h - \delta, h + \delta]) \tag{5.20}$$

$\tilde{\mathbb{P}}-$almost surely. In conjunction with (5.17), this means that

$$\operatorname{supp} \tilde{\mu}_{h,\varepsilon_n}(\omega) \subseteq B^{(\infty)}(\omega)^{-1}([h-\delta, h+\delta]) \tag{5.21}$$

$\tilde{\mathbb{P}}-$almost surely. With this we are now ready to establish that $\tilde{\mathbb{P}}-$almost surely,

$$\operatorname{supp} \mu_h(\omega) \subseteq B^{(\infty)}(\omega)^{-1}([h-\delta, h+\delta]) \tag{5.22}$$

for any $\delta > 0$, i.e. that

$$\operatorname{supp} \mu_h(\omega) \subseteq B^{(\infty)}(\omega)^{-1}(h) \tag{5.23}$$

as desired, which would allow us to take this concrete construction of $\mu_h(\cdot)$ in terms of $B^{(\infty)}$ as the definition of our Frostman measure. To prove that (5.22) is true, it suffices to show that for any continuous

$$f : [1, 2] \to \mathbb{R}$$

vanishing on an arbitrary open neighbourhood of $B^{(\infty)}(\omega)^{-1}([h-\delta, h+\delta])$ that

$$\int_1^2 f(x) d\mu_h(\omega)(x) = 0. \tag{5.24}$$

By (5.15), we know that for any such function f,

$$\int_1^2 f(x) d\mu_h(\omega)(x) = \lim_{n\to\infty} \int_1^2 f(x) d\tilde{\mu}_{h,\varepsilon_n}(\omega)(x). \tag{5.25}$$

By (5.20) and the hypothesis about the support of f we know that for $n > N = N(\delta, \omega)$ each integral on the righthand side above will be exactly 0. This in turn establishes (5.22) for any fixed $\delta > 0$ and by letting $\delta \to 0$, this finally proves that (5.23) is true. Thus we have now obtained a legitimate well-defined construction of our Frostman measure.

5.3.5 Proof that the energy integral is finite

As explained in (5.1), we need to prove that

$$\iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_h(s) d\mu_h(t) < \infty$$

almost surely. Notice that since this is a random integral, to prove that this integral is finite, it is enough to show that its expectation is finite. First, we will prove that the mean of the energy integral for $\mu_{h,\varepsilon}$ is uniformly bounded. By the definition of

$\mu_{h,\varepsilon}$,

$$\mathbb{E}\left[\iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon}(s) d\mu_{h,\varepsilon}(t)\right]$$
$$= \mathbb{E}\left[\frac{1}{4\varepsilon^2} \iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} \mathbb{1}(B(s) \in (h-\varepsilon, h+\varepsilon)) \mathbb{1}(B(t) \in (h-\varepsilon, h+\varepsilon)) ds dt\right].$$

By Fubini's Theorem, the expression on the right hand side is equal to

$$\frac{1}{4\varepsilon^2} \iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} \mathbb{P}(B(s) \in (h-\varepsilon, h+\varepsilon), B(t) \in (h-\varepsilon, h+\varepsilon)) ds dt.$$

As before, we can use the bound (5.4) on the two-point distribution, and get the bound

$$\mathbb{E}\left[\iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon}(s) d\mu_{h,\varepsilon}(t)\right] \leqslant 2c_h \int_1^2 \int_s^2 \frac{1}{|s-t|^{\alpha+\beta}} dt ds.$$

By the change of variables $t - s = u$, the right hand side of the expression above can be rewritten as

$$2c_h \int_1^2 \int_0^{2-s} \frac{1}{|u|^{\alpha+\beta}} du ds.$$

If $\alpha + \beta < 1$, the integral above converges and we get

$$\mathbb{E}\left[\iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon}(s) d\mu_{h,\varepsilon}(t)\right] \leqslant 2c_h \int_1^2 \frac{(2-s)^{1-\alpha-\beta}}{1-\alpha-\beta} ds < \infty. \tag{5.26}$$

The goal is to prove that

$$\iint\limits_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_h(s) d\mu_h(t) < \infty \tag{5.27}$$

almost surely. To do that, we will relate the sequence (whose mean we have just proved that is uniformly bounded) with that integral.

Assume that

$$\mu_{h,\varepsilon_n}(\cdot, \omega) \longrightarrow \mu_h(\cdot, \omega) \tag{5.28}$$

in distribution (in the space of random measures) a.s. for $\omega \in \Omega$ (the probability space where the Brownian Motion(s) are defined). Let $\omega \in \Omega$ be such that the convergence on (5.28) occurs. Since the integral on the left hand side of (5.27) is a double integral, we should prove that

$$\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n}(\cdot,\omega) \longrightarrow \mu_h \times \mu_h(\cdot,\omega) \tag{5.29}$$

in distribution on the space of measures on $[1,2]^2$. Notice that for a fixed ω, $\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n}(\cdot,\omega)$ is a product measure so the convergence of the subsequence $\mu_{h,\varepsilon_n}(\cdot,\omega)$ in distribution implies the convergence of the product measure as described in (5.29).

To relate the energy integral of the sequence to the energy integral of the limit we will make use of Fatou's Lemma. In particular, Fatou's Lemma for weakly convergent measures. From Theorem 2.4 in [17] we know that since $\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n}(\cdot,\omega)$ converges weakly to $\mu_{h,\varepsilon_n}(\cdot,\omega)$ and $f(s,t) = |s-t|^{-\alpha}$ is a positive measurable function that takes values in $\mathbb{R} \cup \{\pm\infty\}$, then

$$\iint_{[1,2]^2} \liminf_{(s',t')\to(s,t)} \frac{1}{|s'-t'|^\alpha} d\mu_h(s) d\mu_h(t) \leqslant \liminf_{n\to\infty} \iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon_n}(s) d\mu_{h,\varepsilon_n}(t).$$

Since $\liminf_{(s',t')\to(s,t)} \frac{1}{|s'-t'|^\alpha} = \frac{1}{|s-t|^\alpha}$, we have

$$\iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_h(s) d\mu_h(t) \leqslant \liminf_{n\to\infty} \iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon_n}(s) d\mu_{h,\varepsilon_n}(t).$$

Taking expectation on both sides we get

$$\mathbb{E}\left[\iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_h(s) d\mu_h(t)\right] \leqslant \mathbb{E}\left[\liminf_{n\to\infty} \iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon_n}(s) d\mu_{h,\varepsilon_n}(t)\right].$$

Using the regular version of Fatou's Lemma on the right hand expression, we obtain

$$\mathbb{E}\left[\iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_h(s) d\mu_h(t)\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\iint_{[1,2]^2} \frac{1}{|s-t|^\alpha} d\mu_{h,\varepsilon_n}(s) d\mu_{h,\varepsilon_n}(t)\right] < \infty,$$

where we have used the uniform bound found in (5.26). □

Chapter 6

Two-point 'density' bound for the directed landscape in the temporal direction

This section will be devoted to proving Theorem 7. This provides a bound for the event that two point are very close to be on the h-level set of the directed landscape. This the key condition (5.4) required by Theorem 6 to get a lower bound for the level set. For ease of readability, we will rewrite the theorem here.

Theorem 23. *Let $0 < \varepsilon \leqslant 1$, $h \in \mathbb{R}$ and $0 < s < t$. Then, there exists an absolute constant c such that*

$$\mathbb{P}(\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon), \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \leqslant c|t-s|^{-1/3}\varepsilon^2. \quad (6.1)$$

6.1 Organization of the proof

Before embarking on a proof we will give a short explanation on how it will go. Clearly, there is no formula for the distribution of the directed landscape but we still need to find a specific bound on its density as seen on the statement of the theorem above. Essentially, we will use the fact that, when time is fixed, the directed landscape follows the distribution of the top line of a parabolic Airy line ensemble which is the same as the distribution of the Airy process minus a parabola. The Airy process does have a known two-point density, given by a Fredholm determinant. However, these determinental formulas are hard to use. We will use another strategy. We will use that the Airy process is locally Brownian. This means that the Airy process on compacts is absolutely continuous with respect to a Brownian bridge that only depends on the process at the boundary of the compact. Crucially, the Radon-

Nikodym derivative only depends on the size of the compact. The structure of the proof is as follows. In Subsection 6.2.1, we will use the metric composition law to split the metric space derived from the directed landscape in two independent metric spaces: whatever happens from time 0 to time s and whatever happens from time s to time t. In fact,

$$\mathcal{L}(0,0;0,t) = \sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z,s;0,t).$$

Moreover,

$$\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s) = \sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z,s;0,t) - \mathcal{L}(0,0;0,s).$$

Understanding this random variable will be the key to our proof. To allow us to use the absolute continuity with respect to the Brownian bridge of the Airy process as explained earlier, we will break the space coordinate, represented by z in the equation above, in small intervals. This will create two regimes, the density of the supremum above when the supremum is achieved in an interval close to 0 or on an interval far for 0. Heuristically, the supremum above can be seen, modulo a random linear function, as the supremum of the sum of a parabolic Airy process, $\mathcal{L}(z,s;0,t)$, and a Brownian motion, $\mathcal{L}(0,0;z,s)-\mathcal{L}(0,0;0,s)$, through the locally Brownian condition of the landscape for fixed times. Naturally, since $\mathcal{L}(z,s;0,t)$ is distributed according to a stationary Airy process minus a parabola centered at 0, the likelihood of achieving the supremum in an interval close to 0 is larger. The strategy is to make the intervals small enough to compensate for this larger probability. In Subsection 6.2.2 we use the property of the directed landscape to be locally Brownian to express the two-point distribution as the supremum of an expression that only involves well known stochastic processes, and in subsection 6.2.3 we study the distribution of the supremum of the sum of the parabolic Airy process and the Brownian motion when that supremum occurs in intervals close to 0 rigorously. In Subsections 6.2.4 and 6.2.5, we will do the same with the intervals far from the origin. The last section 6.3 contains the proof of the absolute continuity lemmas used in the proof.

6.2 Proof of Theorem 7

This section contains the proof of Theorem 7.

Proof. We will split the proof in subsections.

6.2.1 Setting the stage

The goal is to find a bound as in equation (6.1).

We begin by noticing that if both $\mathcal{L}(0,0;0,s)$ and $\mathcal{L}(0,0;0,t)$ are close to h then they are also quite close to each other. Recall that $s < t$. Therefore, we have the following inequality:

$$\begin{aligned}\mathbb{P}(\mathcal{L}(0,0;0,s) &\in (h-\varepsilon, h+\varepsilon), \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \\ &\leqslant \mathbb{P}(\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon), |\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,s)| \leqslant 2\varepsilon). \quad (6.2)\end{aligned}$$

Using the metric composition law of the directed landscape proved in Section 10 of [15] (see Theorem 13), we have that

$$\mathcal{L}(0,0;0,t) = \sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z,s;0,t).$$

Then, the right hand side of inequality (6.2) is equal to

$$\mathbb{P}\left(\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon), \left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z,s;0,t) - \mathcal{L}(0,0;0,s)\right| \leqslant 2\varepsilon\right). \tag{6.3}$$

The independent increments property of the directed landscape implies that $\mathcal{L}(0,0;z,s)$ and $\mathcal{L}(z,s;0,t)$ are independent. Also, by temporal and spatial stationarity we have that for all $z \in \mathbb{R}$,

$$\mathcal{L}(z,s;0,t) \stackrel{d}{=} \widetilde{\mathcal{L}}(0,0;-z,t-s)$$

where the equality is in distribution and $\widetilde{\mathcal{L}}$ is an independent copy of the directed landscape. We conclude that the probability (6.3) above can be rewritten again as

$$\mathbb{P}\left(|\mathcal{L}(0,0;0,s) - h| \leqslant \varepsilon, \left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \widetilde{\mathcal{L}}(0,0;-z,t-s) - \mathcal{L}(0,0;0,s)\right| \leqslant 2\varepsilon\right). \tag{6.4}$$

Notice that, by the definition of the landscape (see Theorem 13), we have the following equality in distribution as a function of $z \in \mathbb{R}$:

$$\widetilde{\mathcal{L}}(0,0;-z,t-s) \stackrel{d}{=} (t-s)^{1/3}\mathfrak{A}_1\left(\frac{-z}{(t-s)^{2/3}}\right) \tag{6.5}$$

where $\mathfrak{A}_1$ is the parabolic Airy process (or the first line of the parabolic Airy line ensemble). Also, we know that

$$\mathcal{L}(0,0;\cdot,s) \stackrel{d}{=}: s^{1/3}\mathfrak{A}_1\left(\frac{\cdot}{s^{2/3}}\right). \tag{6.6}$$

From now on we adopt the notation $f^{(s)}(x) = s^{1/3} f\left(\frac{x}{s^{2/3}}\right)$. Then, we can rewrite probability (6.3) as

$$\mathbb{P}\left(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, \left|\sup_{z\in\mathbb{R}} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \tilde{\mathfrak{A}}_1^{(t-s)}(-z)\right| \leqslant 2\varepsilon\right), \tag{6.7}$$

where $\mathfrak{A}_1$ and $\tilde{\mathfrak{A}}_1$ are independent parabolic Airy processes.

The strategy to prove the intended inequality is to divide the real line where the supremum is taken in intervals; since the supremum must happen in an interval, then we can bound the probability (6.7) by a countable sum of probabilities each one refering to the supremum occurring in the interval. Then we will focus on that sum being summable, the sum being uniformly bounded by a constant times $\varepsilon^2|t-s|^{1/3}$ as in (6.1). By the previous discussion, the line will be split in intervals of length $(t-s)^{2/3} =: \sigma^{2/3}$ such that $\frac{z}{\sigma^{2/3}} \in [i-1/2, i+1/2]$ for $i \in \mathbb{Z}$. This makes the Airy process in (6.5) of order 1. From now on, we will use the convention that $[i \pm 1/2] := [i-1/2, i+1/2]$. This gives us the following decomposition:

$$\begin{aligned}&\mathbb{P}\Big(|\mathcal{L}(0,0;0,s) - h| \leqslant \varepsilon, \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)\Big)\\ &\leqslant \sum_{i\in\mathbb{Z}} \mathbb{P}\left(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, \left|\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm1/2]} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \tilde{\mathfrak{A}}_1^{(t-s)}(-z)\right| \leqslant 2\varepsilon\right).\end{aligned} \tag{6.8}$$

The behaviour of the supremum will have two regimes: one when i is small and another when i is large. This is because when i is smaller than $7\sigma^{-2/3}$, we can find an interval of order 1 that contains both any $z \in [\sigma^{2/3} i \pm \sigma^{2/3}/2]$ and 0. In that case, we can use Theorem 8 to get an absolute continuity result simultaneously for $\mathfrak{A}_1^{(s)}(z)$ and $\mathfrak{A}_1^{(s)}(0)$. Recall that $\tilde{\mathfrak{A}}_1^{(t-s)}$ is independent so a similar absolute continuity result will be used to treat it as a Brownian bridge but independently than $\mathfrak{A}_1^{(s)}$. If i is larger, then 0 and the interval $[\sigma^{2/3} i \pm \sigma^{2/3}/2]$ will be far and we will have to use a slightly different result also available in [14] by Dauvergne to prove that we can resample the parabolic Airy process in disjoint intervals independently on each interval. This will be done in Subsection 6.2.4. We will start by looking at the case when $|i| \leqslant \frac{7}{\sigma^{2/3}}$. Recall that σ is bounded above but it can be 0 since it is the distance between s and t to the power of 2/3.

In the next section, we will use the absolute continuity of the Airy process with respect to the Brownian bridge from Theorem 8 to bound this probability

$$\mathbb{P}\left(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, \left|\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm1/2]} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \tilde{\mathfrak{A}}_1^{(t-s)}(-z)\right| \leqslant 2\varepsilon\right) \tag{6.9}$$

with a more manageable one.

6.2.2 Absolute continuity of the Airy process on a compact

Let $|i| \leqslant \frac{7}{\sigma^{2/3}}$. We now introduce a key lemma. This lemma will give us a key decomposition of the rescaled Airy processes in (6.9) into a sum of a Gaussian random variable and an independent random function on $\mathbb{R}$ with sufficiently nice tail bounds. To obtain these bounds we will use the fact, proved in [14], that the parabolic Airy process and in general the first k lines of the Airy line ensemble in a compact $[0,t]$ are absolutely continuous with respect to k independent Brownian bridges and that the endpoints have comparable tail bounds to that of parabolic Airy process at a particular point (see Theorem 8).

Lemma 24. *Let $a \in \mathbb{R}$ and $T > \frac{1}{6}$. Let ℓ_a be the function on $\mathbb{R}$ defined by*

$$\ell_a(r) = (r-a)^2 - r^2 = -a(2r-a) = a^2 - 2ra$$

and let I_a denote the interval

$$I_a := [a-T, a+T]$$

Then there exists an absolute constant $c > 0$, two $T-$dependent constants $c_1, c_2 > 0$, and a random function $(\mathcal{F}(r))_{r \in I_a}$ such that

$$\text{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r \in I_a}\right) \leqslant e^{cT^3}\,\text{Law}\left(\left(\sqrt{2T}N + \left(\mathcal{F}(r) + \ell_a(r)\right)_{r \in I_a}\right)\right) \tag{6.10}$$

where N is a standard Gaussian independent of $\left(\mathcal{F}(r)\right)_{r \in I_a}$ and for all $m > 0$,

$$\mathbb{P}\left(\sup_{r \in I_a} |\mathcal{F}(r)| \geqslant m\right) \leqslant c_1 e^{-c_2 m^{\frac{3}{2}}}. \tag{6.11}$$

More generally, for any constant $\lambda > 0$, let $\mathfrak{A}_1^{(\lambda)}$ be as in (6.6) and denote by $I_a^{(\lambda)}$ the interval

$$I_a^{(\lambda)} := \left[a\lambda^{\frac{2}{3}} - T\lambda^{\frac{2}{3}}, a\lambda^{\frac{2}{3}} + T\lambda^{\frac{2}{3}}\right].$$

Then as a consequence of (6.10), there exists a random $\lambda-$dependent function $(F^{(\lambda)}(r))_{r \in I_a^{(\lambda)}}$ and an independent standard Gaussian N such that

$$\text{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r \in I_a^{(\lambda)}}\right) \leqslant e^{cT^3}\,\text{Law}\left(\left(\lambda^{\frac{1}{3}}\sqrt{2T}N + \left(\mathcal{F}^{(\lambda)}(r) + \lambda^{\frac{1}{3}} l_a(r\lambda^{-\frac{2}{3}})\right)\right)_{r \in I_a^{(\lambda)}}\right), \tag{6.12}$$

and that for the same constants c_1 and c_2 and all $m > 0$,

$$\mathbb{P}\left(\sup_{r\in I_a^{(\lambda)}}\left|\lambda^{-\frac{1}{3}}\mathcal{F}^{(\lambda)}(r)\right| \geqslant m\right) \leqslant c_1 e^{-c_2 m^{\frac{3}{2}}}. \tag{6.13}$$

In particular, there exist random constants A and C such that we may write

$$\left(\mathcal{F}^{(\lambda)}(a\lambda^{\frac{2}{3}}+\delta)\right)_{\delta\in I_0^{(\lambda)}} \stackrel{d}{=} \left(\mathcal{W}(2\delta+6T\lambda^{\frac{2}{3}})+\lambda^{-\frac{1}{3}}A\delta+\lambda^{\frac{1}{3}}C\right)_{\delta\in I_0^{(\lambda)}} \tag{6.14}$$

where $\mathcal{W}$ is a standard two-sided Brownian motion, and for all $m > 0$,

$$\mathbb{P}\Big(|A| \geqslant m\Big) + \mathbb{P}\Big(|C| \geqslant m\Big) \leqslant 2c_1 e^{-c_2 m^{\frac{3}{2}}}. \tag{6.15}$$

No claims are made about the independence or lack thereof amongst $A, \mathcal{W}$ and C.

Remark.

- The bound on the Radon-Nikodym derivative, e^{cT^3} is crucially not dependent on anything but the length of the interval where we are resampling.
- The proof of this Lemma can be found in Section 6.3. We give now an idea about it. Essentially, from Theorem 8, we know that the first line of the parabolic Airy line ensemble is absolutely continuous with respect to a Brownian bridge that only depends on the boundary conditions; this Brownian bridge can be decomposed into a diffusion 2 Brownian bridge from 0 to 0 and a linear function L. From B, we can 'take' a Gaussian random variable N independent of $B-N$ from Proposition 17. The linear function L is the linear shift to the boundary conditions. These boundary conditions are not equally distributed as the Airy process but preserve the same tails as the Tracy-Widom distribution. Therefore, we get that $\mathcal{F}$ is the sum of $B-N$ and L and those random variables have either Gaussian or Tracy-Widom type of tails so we will be able to work with them.
- The constants A and C are random variables that are linear combinations of N and the endpoints of the linear function L which are the boundary conditions.

Recall that the notation $\mathfrak{A}^{(\sigma)}$ and $\mathfrak{A}^{(s)}$ is introduced in (6.6). For the sake of readability we defer the proof of Lemma 24 for Section 6.3.

Let $I_{-i}^{(\sigma)} = \sigma^{2/3}[-i-\frac{1}{2}, -i+\frac{1}{2}]$ and $I_0^{(s)} = s^{2/3}[-8s^{-2/3}, 8s^{-2/3}]$. In the second case, the length of the interval depends on s; in fact, $T = 8s^{-2/3}$. However, since $s \geqslant 1$, $T \leqslant 8$ so the constants c_1 and c_2 from the theorem above are absolute contants. Notice that if $z \in [\sigma^{2/3}i - \frac{1}{2}, \sigma^{2/3}i + \frac{1}{2}]$, then $-z \in I_{-i}^{(\sigma)}$ and, since $|i| \leqslant \frac{7}{\sigma^{2/3}}$ and $\sigma \leqslant 1$ we know that

$$|z| \leqslant |z + i\sigma^{2/3}| + |i\sigma^{2/3}| \leqslant \frac{\sigma^{2/3}}{2} + 7 \leqslant \frac{1}{2} + 7 \leqslant 8$$

so $z \in I_0^{(s)}$. Now by invoking Lemma 24 twice, once on $\tilde{\mathfrak{A}}_1^{(\sigma)}|_{I_{-i}^{(\sigma)}}$ and again on $\mathfrak{A}_1^{(s)}|_{I_0^{(s)}}$, we can upper bound the probability (6.9) by

$$\mathbb{P}\Bigg(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon,\ \Bigg|\sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \tilde{\mathfrak{A}}_1^{(\sigma)}(-z)\Bigg| \leqslant 2\varepsilon\Bigg)$$
$$\leqslant e^{c(\frac{1}{2})^3} e^{c(8s^{-2/3})^3} \mathbb{P}\Bigg(|4N + \mathcal{F}^{(s)}(0) - h| \leqslant \varepsilon,\ \Big|\sigma^{1/3}\tilde{N} + \sigma^{1/3}G_i - \sigma^{1/3}i^2\Big| \leqslant 2\varepsilon\Bigg), \quad (6.16)$$

where $\ell_0 \equiv 0$,

$$\sigma^{1/3}\tilde{\ell}_{-i}(-z\sigma^{-2/3}) = -i^2\sigma^{1/3} + 2i\Big(-\frac{z}{\sigma^{1/3}} + i\sigma^{1/3}\Big)$$

and

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} + 2i(i - \frac{z}{\sigma^{2/3}}). \quad (6.17)$$

Notice that the bound on the Radon-Nikodym derivative obtained from the absolute continuity Lemma 24 is bounded

$$e^{c(\frac{1}{2})^3} e^{c(8s^{-2/3})^3} \leqslant e^{c\frac{1}{2^3}} e^{c8^3} =: c_0$$

since $s \geqslant 1$ and c is an absolute constant. We can then write the probability in the right hand side of (6.16) as

$$\mathbb{P}\Big(|4N + G - h| \leqslant \varepsilon,\ \Big|\tilde{N} + G_i - i^2\Big| \leqslant 2\varepsilon/\sigma^{1/3}\Big) \quad (6.18)$$

where $G = \mathcal{F}^{(s)}(0)$. It is important to note that the random vector $(G, \tilde{G}_i)$ is independent of the Gaussian random vector $(4N, \tilde{N})$.

We now introduce a lemma that will prove that the vector $(4N, \tilde{N}) + (G, G_i)$ has a continuous density and give bounds to the Radon-Nikodym derivative of the random vector with respect to the Lebesgue measure. The lemma is a version of Young's inequality.

Lemma 25. *Let μ and ν be independent finite measures on $\mathbb{R}^n$. Let the random vector μ be absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative f_μ. Then, the measure $\mu * \nu$ is also absolutely continuous with respect to the Lebesgue measure and*

$$\|f_{\mu*\nu}\|_\infty \leqslant \|f_\mu\|_\infty \nu(\mathbb{R}^n),$$

where $f_{\mu*\nu}$ is the Radon–Nikodym derivative of the measure $\mu * \nu$ with respect to the

Lebesgue measure.

Proof. We begin by proving that $\mu * \nu$ is absolutely continuous with respect to the Lebesgue measure. We denote the Lebesgue measure as $|\cdot|$. Take A a Lebesgue measurable set such that $|A| = 0$. Then,

$$\mu * \nu(A) = \iint 1_A(x+y)d\mu(x)d\nu(y) = \iint 1_{A-y}(x)f_\mu(x)dxd\nu(y).$$

The Lebesgue measure is invariant under translation so $|A - y| = |A| = 0$. This implies that the inner integral in the right hand side expression above is 0.

We have proved that $\mu * \nu \ll |\cdot|$ so there exists an integrable function $f_{\mu*\nu}$ on $\mathbb{R}^n$ such that

$$\mu * \nu(A) = \int_A f_{\mu*\nu}(x)dx$$

for all Borel sets A in $\mathbb{R}^n$. To prove that the Radon-Nikodym derivative $f_{\mu*\nu}$ is bounded it suffices to show that

$$\mu * \nu(A) \leqslant \|f_\mu\|_\infty \nu(\mathbb{R}^n)|A| \tag{6.19}$$

for all Borel sets A in $\mathbb{R}^n$. Let A be a Borel set in $\mathbb{R}^n$. Then, as before,

$$\mu * \nu(A) = \iint 1_{A-y}(x)f_\mu(x)dxd\nu(y).$$

We bound the function f_μ with its norm and get an upper bound on the right hand side as follows:

$$\int\int_{A-y} \|f_\mu\|_\infty dxd\nu(y) = \|f_\mu\|_\infty \int |A|d\nu(y) = \|f_\mu\|_\infty \nu(\mathbb{R}^n)|A|$$

where again we have used the fact that the Lebesgue measure is invariant under translation. □

To estimate probability (6.18), we will first bound the density $\rho_i(x, y)$ of the random variable $(G, G_i) + (4N, \tilde{N})$ at the value (x, y). Notice that by Lemma 25, the density exists. We take a moment here to also observe that we can write (6.18) as

$$\int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3},2\varepsilon/\sigma^{1/3}]} \rho_i(x+h, y+i^2)dxdy. \tag{6.20}$$

The strategy here is to find a sequence of upper bounds a_i such that

$$\rho_i(x+h, y+i^2) \leqslant a_i(y+i^2)$$

for $|i| \leqslant 7\sigma^{-\frac{2}{3}}$ and

$$\sum_{|i| \leqslant 7\sigma^{-\frac{2}{3}}} a_i(y+i^2) < c''$$

for an absolute constant c'' which is independent of h and the bounds of summation for i. This in conjunction with (6.8), the bound (6.16), and its reformulation (6.18) would give us that

$$\begin{aligned}
\mathbb{P}(\mathcal{L}(0,0;0,s) \in & (h-\varepsilon, h+\varepsilon); \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \\
\leqslant & \sum_{i \in \mathbb{Z}} p_{i,\varepsilon} \\
= & c_0 \sum_{|i| \leqslant 7\sigma^{-\frac{2}{3}}} \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} \rho_i(x+h, y+i^2)\,dxdy + \sum_{|i| > 7\sigma^{-\frac{2}{3}}} p_{i,\varepsilon} \\
\leqslant & c_0 \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} c''\,dxdy + \sum_{|i| > 7\sigma^{-\frac{2}{3}}} p_{i,\varepsilon} \\
= & 4c_0 c'' \frac{\varepsilon^2}{\sigma^{1/3}} + \sum_{|i| > 7\sigma^{-\frac{2}{3}}} p_{i,\varepsilon} \qquad (6.21)
\end{aligned}$$

where

$$p_{i,\varepsilon} = \mathbb{P}\left(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, \left| \sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \tilde{\mathfrak{A}}_1^{(t-s)}(-z) \right| \leqslant 2\varepsilon \right).$$

This would thereby reduce our goal in (6.1) to understanding the remaining tail sum in (6.21). Let X be a random variable absolutely continuous with respect to the Lebesgue measure. We denote

$$\mathbb{P}(X \in dx) = \lim_{\varepsilon \searrow 0} \frac{\mathbb{P}\Big(X \in [x, x+\varepsilon)\Big)}{\varepsilon}$$

With the preceding strategy in mind, let $y > 0$ be arbitrary. By conditioning on the value of the second coordinate $\tilde{N} + G_i$, we obtain the following description:

$$\begin{aligned}\rho_i(x,y) =& \mathbb{P}\Big(4N + G \in dx, \widetilde{N} + \tilde{G}_i \in dy\Big) \qquad (6.22)\\ =& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{P}\Big(4N + G \in [x, x+\varepsilon), \widetilde{N} + G_i \in [y, y+\varepsilon)\Big)\\ =& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{P}\Big(4N + G \in [x, x+\varepsilon), \widetilde{N} + G_i \in [y, y+\varepsilon)\Big|G_i \geqslant y/2\Big)\mathbb{P}(G_i \geqslant y/2)\\ &+ \frac{1}{\varepsilon^2}\mathbb{P}\Big(4N + G \in [x, x+\varepsilon), \widetilde{N} + G_i \in [y, y+\varepsilon)\Big|\widetilde{N} \geqslant y/2\Big)\mathbb{P}(\widetilde{N} \geqslant y/2).\end{aligned} \qquad (6.23)$$

Notice that the inequality above makes use of the fact that for any $r > 0$, the event $\{\widetilde{N} + G_i \in [y, y+r)\}$ occurring implies that either the event $\{\widetilde{N} \geqslant y/2\}$ or the event $\{G_i \geqslant y/2\}$ must occur and the definition of $\mathbb{P}(X \in dx)$ as a limit.

We will call the distribution above

$$\mathbb{P}\Big(4N + G \in du, \widetilde{N} + G_i \in dv\Big|G_i \geqslant y/2\Big) =: \hat{f}(u,v)$$

and in general, we will denote the distribution function of any continuous random vector (W, Z) at the point (u, v) as $f_{(W,Z)}(u, v)$. Then,

$$\mathbb{P}\Big(4N+G \in [x, x+\varepsilon), \widetilde{N}+G_i \in [y, y+\varepsilon)\Big|G_i \geqslant y/2\Big) = \int_x^{x+\varepsilon}\int_y^{y+\varepsilon}\hat{f}(u,v)dudv. \qquad (6.24)$$

To bound the density $\hat{f}$, we use Lemma 25.

By Lemma 25, we get that

$$\begin{aligned}\hat{f}(u,v) &\leqslant \Big\|\hat{f}(u,v)\Big\|_{\infty,\mathbb{R}\times[0,\infty)} = \Big\|f_{(4N,\tilde{N})+(G,G_i)|\{G_i\geqslant y/2\}}(u,v)\Big\|_{\infty,\mathbb{R}\times[0,\infty)}\\ &\leqslant \Big\|f_{(4N,\tilde{N})|\{G_i\geqslant y/2\}}(u,v)\Big\|_{\infty,\mathbb{R}\times[0,\infty)}\mathbb{P}((G,G_i) \in \mathbb{R}\times[0,\infty)|G_i \geqslant y/2)\\ &\leqslant \Big\|f_{(4N,\tilde{N})}(u,v)\Big\|_{\infty,\mathbb{R}\times[0,\infty)} = \frac{1}{8\pi},\end{aligned}$$

where in the last inequality we used that $(N, \tilde{N})$ and $\tilde{G}_i$ are independent. Then, we can use the bound of the density that we just obtained on the integral (6.24) to get

$$\mathbb{P}\Big(4N + G \in [x, x+\varepsilon), \widetilde{N} + G_i \in [y, y+\varepsilon)\Big|G_i \geqslant y/2\Big) \leqslant \frac{\varepsilon^2}{4\pi}. \qquad (6.25)$$

Similarly, by Lemma 25,

$$f_{(4N,\tilde{N})+(G,G_i)|\{\tilde{N}\geqslant y/2\}}(u,v) \leqslant \left\| f_{(4N,\tilde{N})|\tilde{N}\geqslant y/2\}}(u,v) \right\|_{\infty,\mathbb{R}\times[0,\infty)} \mathbb{P}((G,G_i)\in\mathbb{R}\times[0,\infty)|\tilde{N}\geqslant y/2).$$

By a trivial bound to the probability on the right hand side of the equation above, we get that

$$f_{(4N,\tilde{N})+(G,G_i)|\{\tilde{N}\geqslant y/2\}}(u,v) \leqslant \left\| f_{(4N,\tilde{N})|\{\tilde{N}\geqslant y/2\}}(u,v) \right\|_{\infty,\mathbb{R}\times[0,\infty)}.$$

Notice that

$$\begin{aligned} f_{(4N,\tilde{N})|\{\tilde{N}\geqslant y/2\}}(u,v) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{P}\Big(4N\in[u,u+\varepsilon),\tilde{N}\in[v,v+\varepsilon)\Big|\tilde{N}\geqslant y/2\Big) \\ &= \lim_{\varepsilon\to 0}\frac{\mathbb{P}\Big(4N\in[u,u+\varepsilon),\tilde{N}\in[v,v+\varepsilon),\tilde{N}\geqslant y/2\Big)}{\varepsilon^2\mathbb{P}(\tilde{N}\geqslant y/2)} \\ &= \lim_{\varepsilon\to 0}\frac{\mathbb{P}\Big(4N\in[u,u+\varepsilon)\Big)\mathbb{P}\Big(\tilde{N}\in[v,v+\varepsilon),\tilde{N}\geqslant y/2\Big)}{\varepsilon^2\mathbb{P}(\tilde{N}\geqslant y/2)}, \end{aligned} \tag{6.26}$$

where in the last line we have used the independence of N and $\tilde{N}$. Notice that

$$\mathbb{P}\Big(\tilde{N}\in[v,v+\varepsilon),\tilde{N}\geqslant y/2\Big) = \begin{cases} 0 & \text{if } y/2 > v+\varepsilon \\ \mathbb{P}\Big(\tilde{N}\in[y/2,v+\varepsilon)\Big) & \text{if } v\leqslant y/2\leqslant v+\varepsilon \\ \mathbb{P}\Big(\tilde{N}\in[v,v+\varepsilon)\Big) & \text{if } y/2 < v. \end{cases}$$

Notice that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{P}\Big(\tilde{N}\in[v,v+\varepsilon),\tilde{N}\geqslant y/2\Big) = \mathbb{P}\Big(\tilde{N}\in dv\Big)\mathbb{1}\{y\leqslant 2v\} = \frac{\sqrt{\pi}}{e^{-v^2}}\mathbb{1}\{y\leqslant 2v\}.$$

Then, by taking the limit $\varepsilon\to 0$ on (6.26), we obtain that

$$f_{(4N,\tilde{N})|\{\tilde{N}\geqslant y/2\}}(u,v) = \frac{1}{4\sqrt{2\pi}}e^{-\frac{u^2}{32}}\frac{1}{\sqrt{2\pi}\mathbb{P}(\tilde{N}\geqslant y/2)}e^{-v^2}\mathbb{1}\{y\leqslant 2v\}.$$

Taking, supremum over (u,v) we get the desired bound on the density:

$$\left\|f_{(4N,\tilde{N})|\{\tilde{N}\geqslant y/2\}}(u,v)\right\|_{\infty,\mathbb{R}\times[0,\infty)} \leqslant \left\|\frac{e^{-\frac{u^2}{32}}}{8\pi}\frac{e^{-v^2}}{\mathbb{P}(\tilde{N}\geqslant y/2)}\mathbb{1}\{y\leqslant 2v\}\right\|_{\infty,\mathbb{R}\times[0,\infty)}$$
$$\leqslant \frac{1}{8\pi}\frac{e^{-\frac{y^2}{4}}}{\mathbb{P}(\tilde{N}\geqslant y/2)}.$$

Then, we can bound the integral

$$\mathbb{P}\Big(4N+G\in[x,x+\varepsilon),\tilde{N}+G_i\in[y,y+\varepsilon)\Big|\tilde{N}\geqslant y/2\Big)$$
$$=\int_x^{x+\varepsilon}\int_y^{y+\varepsilon} f_{(4N,\tilde{N})+(G,G_i)|\{\tilde{N}\geqslant y/2\}}(u,v)dudv$$
$$\leqslant \int_x^{x+\varepsilon}\int_y^{y+\varepsilon}\frac{1}{8\pi}\frac{e^{-\frac{y^2}{4}}}{\mathbb{P}(\tilde{N}\geqslant y/2)}dudv$$
$$\leqslant \frac{\varepsilon^2}{8\pi}\frac{e^{-\frac{y^2}{4}}}{\mathbb{P}(\tilde{N}\geqslant y/2)}.$$

Adding this last bound to the one found in (6.25), we obtain that

$$\rho_i(x,y)\leqslant \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\frac{\varepsilon^2}{8\pi}\left(\mathbb{P}(G_i\geqslant y/2)+\frac{e^{-\frac{y^2}{4}}}{\mathbb{P}(\tilde{N}\geqslant y/2)}\mathbb{P}(\tilde{N}\geqslant y/2)\right).$$

Therefore, we have bounded the density of the random vector $(4N,\tilde{N})+(G,G_i)$;

$$\rho_i(x,y)\leqslant c\mathbb{P}(G_i\geqslant y/2)+ce^{-\frac{y^2}{4}}$$

for some constants $c>0$.

A similar, symmetric argument shows that

$$\rho_i(x,y)\leqslant c\mathbb{P}(G_i\leqslant y/2)+ce^{-\frac{y^2}{4}}$$

for $y<0$. Notice that there is no bound for $y=0$ but densities can be redefined at single points so for completeness we are going to say that

$$\rho_i(x,0)\leqslant c\mathbb{P}(\tilde{G}_i\geqslant 0)+c.$$

Recall that we need to bound:

$$\int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3},2\varepsilon/\sigma^{1/3}]}\rho_i(x+h,y+i^2)dxdy$$

where ρ_i is the density of the vector $(4N+G, \tilde{N}+G_i)$. Through the Young's inequality argument above, we have proved that we can bound the density above by

$$\rho_i(x+h, y+i^2) \leqslant \begin{cases} c\mathbb{P}(G_i \geqslant (y+i^2)/2) + ce^{-\frac{(y+i^2)^2}{8}} & \text{if } y+i^2 \geqslant 0 \\ c\mathbb{P}(G_i \leqslant (y+i^2)/2) + ce^{-\frac{(y+i^2)^2}{8}} & \text{if } y+i^2 < 0. \end{cases} \tag{6.27}$$

In the next section we will provide suitable summable bounds for the probability of the tail bounds of G_i written above.

6.2.3 Bound for the two-point density of the directed landscape if a geodesic from $(0,0)$ to $(0,t)$ is close to the origin at time s

We need to find bounds for 6.27; in fact our bounds need to be summable on i and those series need to be uniformly bounded on y. Recall that as seen in (6.17),

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} + 2i(i - \frac{z}{\sigma^{2/3}}).$$

For the rest of the argument, I will assume that $i \neq 0$. From now on, $c, c' > 0$ are positive constants whose value might change.

<u>**If** $y + i^2 < 0$.</u>

We need to get a bound for

$$\mathbb{P}\Big(G_i \leqslant \frac{y+i^2}{2}\Big),$$

where $y + i^2 < 0$ and

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} + 2i(i - \frac{z}{\sigma^{2/3}}).$$

We can bound the supremum for the function evaluated at the center of the interval and we obtain

$$G_i \geqslant \frac{(\mathcal{F}^{(s)}(\sigma^{2/3} i) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-\sigma^{2/3} i)}{\sigma^{1/3}}.$$

We have obtained a decomposition of $\mathcal{F}^{(s)}$ as the sum of a double sided Brownian motion and a random linear function in Lemma 24. We will use that decomposition (and the well know tail bounds of the double sided Brownian motion and this particular random linear function, again from Lemma 24), to get the tail bounds that we need for the terms related to $\mathcal{F}^{(s)}$ in G_i. By Lemma 24,

$$\frac{(\mathcal{F}^{(s)}(\sigma^{2/3}i) - \mathcal{F}^{(s)}(0))}{\sigma^{1/3}} \overset{d}{=} \frac{\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)}{\sigma^{1/3}} + A\left(\frac{\sigma}{s}\right)^{1/3} i,$$

where $\mathcal{W}$ is a standard double sided Brownian motion and A is a random constant, whose tail bounds are made explicit in Lemma 24.

Using a union bound argument , we get that

$$\mathbb{P}\bigg(\tilde{G}_i \leqslant \frac{y+i^2}{2}\bigg) \leqslant \mathbb{P}\bigg(\frac{\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)}{\sigma^{1/3}} \leqslant \frac{y+i^2}{6}\bigg) \tag{6.28}$$

$$+\mathbb{P}\bigg(A\left(\frac{\sigma}{s}\right)^{1/3} i \leqslant \frac{y+i^2}{6}\bigg) \tag{6.29}$$

$$+\mathbb{P}\bigg(\frac{\tilde{\mathcal{F}}^{(\sigma)}(-\sigma^{2/3}i)}{\sigma^{1/3}} \leqslant \frac{y+i^2}{6}\bigg). \tag{6.30}$$

In the first case, we need to bound probability (6.28). Recall that $|\sigma^{2/3}i| \leqslant 7$ so $\mathcal{W}$ $2\sigma^{2/3}i + 48$ is positive so we can treat $\mathcal{W}$ as a (one-sided) Brownian motion. Then, $\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)$ is distributed according to a normal random variable Z. In fact, for every z such that $\frac{z}{\sigma^{2/3}} \in [i - \frac{1}{2}, i + \frac{1}{2}]$,

$$\mathcal{W}(2z + 48) - \mathcal{W}(48) \overset{d}{=} \begin{cases} Z(0, 2z) & \text{if } z > 0 \\ Z(0, -2z) & \text{if } z < 0. \end{cases} \tag{6.31}$$

We deduce that $|\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)| \overset{d}{=} |Z(0, 2\sigma^{2/3}|i|)|$. We are now ready to bound probability (6.28).

$$\begin{aligned} \mathbb{P}\bigg(\frac{\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)}{\sigma^{1/3}} \leqslant \frac{y+i^2}{6}\bigg) &\leqslant \mathbb{P}\bigg(\frac{|\mathcal{W}(2\sigma^{2/3}i + 48) - \mathcal{W}(48)|}{\sigma^{1/3}} \geqslant -\frac{y+i^2}{6}\bigg) \\ &= \mathbb{P}\bigg(\frac{|Z(0, 2\sigma^{2/3}|i|)|}{\sigma^{1/3}} \geqslant -\frac{y+i^2}{6}\bigg) \\ &\leqslant c e^{c' \frac{-(y+i^2)^2}{|i|}}, \end{aligned} \tag{6.32}$$

where in the last inequality we have used the Brownian scaling and the tail bounds of a Gaussian random variable of mean 0 and variance $2|i|$.

For probability (6.29), first we use that $A \geqslant -|A|$:

$$\mathbb{P}\bigg(A\left(\frac{\sigma}{s}\right)^{1/3} i \leqslant \frac{y+i^2}{6}\bigg) \leqslant \mathbb{P}\bigg(|A|\left(\frac{\sigma}{s}\right)^{1/3} |i| \geqslant -\frac{y+i^2}{6}\bigg).$$

Using that $s \geqslant 1$ and that $\sigma \leqslant 1$, we can further bound the above probability by

$$\mathbb{P}\left(|A| \geqslant -\frac{y+i^2}{6|i|}\right) = \mathbb{P}\left(|A| \geqslant \frac{|y+i^2|}{6|i|}\right) \leqslant ce^{-c'\frac{|y+i^2|^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}}, \tag{6.33}$$

where we have used that $y + i^2 < 0$ and the bound in tail bounds on A found on Lemma 24 again.

Lastly, for the probability (6.30), and again based on Lemma 24 we get that

$$\mathbb{P}\left(\frac{\tilde{\mathcal{F}}^{(\sigma)}(-\sigma^{2/3}i)}{\sigma^{1/3}} \leqslant \frac{y+i^2}{6}\right) \leqslant \mathbb{P}\left(\frac{|\tilde{\mathcal{F}}^{(\sigma)}(-\sigma^{2/3}i)|}{\sigma^{1/3}} \geqslant \frac{|y+i^2|}{6}\right) \leqslant ce^{-c'|y+i^2|^{\frac{3}{2}}}. \tag{6.34}$$

Thus, for each y and each i such that $y + i^2 < 0$, following (6.27) and the bounds (6.32), (6.33) and (6.34), we get that

$$\sum_{|i| \leqslant \lfloor \frac{7}{\sigma^{2/3}} \rfloor} \rho_i(x+h, y+i^2) 1_{\{y+i^2<0\}} \leqslant$$

$$c\left(1 + \sum_{|i| \geqslant 1}^{\lfloor \frac{7}{\sigma^{2/3}} \rfloor} \left(e^{-c'\frac{(y+i^2)^2}{|i|}} + e^{-c'\frac{|y+i^2|^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}} + e^{-c'|y+i^2|^{\frac{3}{2}}} + e^{-\frac{(y+i^2)^2}{8}}\right) 1_{\{y+i^2<0\}}\right) \tag{6.35}$$

The term 1 comes from the trivial bound to the term corresponding to $i = 0$. Notice that $|i| \geqslant 1$ so $|y+i^2|^{\frac{3}{2}} \geqslant \frac{|y+i^2|^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}$, $(y+i^2)^2 \geqslant \frac{(y+i^2)^2}{|i|}$ and $\frac{(y+i^2)^2}{|i|} \geqslant \frac{(y+i^2)^2}{|i|^2}$. Therefore, we can bound the series above with the simplified series

$$c\left(1 + 2\sum_{|i| \geqslant 1}^{\lfloor \frac{7}{\sigma^{2/3}} \rfloor} \left(e^{-c'\frac{(y+i^2)^2}{|i|^2}} + e^{-c'\frac{|y+i^2|^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}}\right) 1_{\{y+i^2<0\}}\right).$$

Using that for all $u \geqslant 0$ and $p \geqslant 1$, $u^p \geqslant u - 1$, we can bound

$$\begin{aligned}\sum_{|i| \leqslant \lfloor \frac{7}{\sigma^{2/3}} \rfloor} \rho_i(x+h, y+i^2) 1_{\{y+i^2<0\}} &\leqslant \sum_{|i| \geqslant 1}^{\lfloor \frac{7}{\sigma^{2/3}} \rfloor} \left(e^{-c'\frac{(y+i^2)^2}{|i|}} + e^{-c'\frac{|y+i^2|^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}}\right) 1_{\{y+i^2<0\}} \\ &\leqslant \sum_{|i| \geqslant 1}^{\infty} 2e^{-c'\left|\frac{y}{|i|}+|i|\right|} 1_{\{y+i^2<0\}} < \frac{4}{1-e^{-c'}}.\end{aligned} \tag{6.36}$$

The last bound (uniform on $y \in [-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]$) was obtained, by the fact that $y < 0$, through the following lemma:

Lemma 26. *Let $\beta_1, \gamma, r \in \mathbb{R}$ with $\beta_1 \geqslant 0$, $\gamma > 0$, and $r \geqslant 1$. For each $i \in \mathbb{Z}_{\neq 0}$ define*

$$f(i) := |i| - \frac{\beta_1}{|i|}.$$

Then we have that

$$\sum_{i \in \mathbb{Z}} e^{-\gamma |f(i)|^r} 1_{\{f(i)<0\}} < \frac{2}{1-e^{-\gamma}}.$$

Proof. Firstly, notice that $f(i)$ is symmetric with respect to i so it is enough to prove that

$$\sum_{i \in \mathbb{Z}_{>0}} e^{-\gamma |f(i)|^r} 1_{\{f(i)<0\}} < \frac{1}{1-e^{-\gamma}}.$$

We start by assuming that the set $\{i \in \mathbb{Z}_{>0} : f(i) < 0\}$ is not empty, otherwise the bounds would be trivially true. Notice that for any $u \geqslant 0$ and $r \geqslant 1$, $u^r \geqslant u - 1$, then

$$\sum_{i=1}^{\infty} e^{\gamma |f(i)|^r} 1_{\{f(i)<0\}} \leqslant e^{\gamma} \sum_{i=1}^{\infty} e^{-\gamma |f(i)|} 1_{\{f(i)<0\}}.$$

So it is enough to prove that

$$\sum_{i \in \mathbb{Z}} e^{-\gamma |f(i)|} 1_{\{f(i)<0\}} < \frac{e^{-\gamma}}{1-e^{-\gamma}}.$$

When $i > 0$, since $\beta_1 \geqslant 0$ we see that

$$f'(i) = \left(i - \frac{\beta_1}{i}\right)' = 1 + \frac{\beta_2}{i^2} \geqslant 1.$$

This means that for any $i > 0$, $f(i+1) \geqslant f(i) + 1$ or equivalently, $f(i) \leqslant f(i+1) - 1$. Set

$$i_{\max} := \max \{i \in \mathbb{Z}_{>0} : f(i) \leqslant 0\}.$$

By iterating this property, we see then that $f(i) \leqslant f(i_{\max}) - (i_{\max} - i) \leqslant 0$ for any

$0 < i \leqslant i_{\max}$. Then this means that

$$\begin{aligned}
\sum_{i=1}^{\infty} e^{-\gamma|f(i)|} 1_{\{f(i)<0\}} &= \sum_{i=1}^{i_{\max}} e^{\gamma f(i)} \\
&\leqslant \sum_{i=1}^{i_{\max}} e^{\gamma(f(i_{\max})-(i_{\max}-i))} \\
&\leqslant \sum_{i=1}^{i_{\max}} e^{\gamma(i-i_{\max})} e^{\gamma f(i_{\max})} \\
&\leqslant \sum_{i=1}^{i_{\max}} e^{\gamma(i-i_{\max})} \\
&\leqslant \sum_{k=1}^{\infty} e^{-\gamma k} \\
&< \frac{e^{-\gamma}}{1-e^{-\gamma}}
\end{aligned}$$

since $e^{\gamma f(i_{\max})} \leqslant 1$ by definition of $i_{\max}$. □

If $y + i^2 \geqslant 0$. Again, we need to bound

$$\mathbb{P}\left(G_i \geqslant \frac{y+i^2}{2} \right).$$

Recall the definition of G_i in (6.17):

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} + 2i(i - \frac{z}{\sigma^{2/3}})$$

Therefore, for each $z \in [\sigma^{2/3} i \pm \sigma^{2/3}/2]$, we can bound

$$2i\left(i - \frac{z}{\sigma^{2/3}} \right) \leqslant 2|i| \left| \frac{z}{\sigma^{2/3}} - i \right| \leqslant |i|.$$

In turn, this means that

$$G_i \leqslant \sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} + |i|$$

and so

$$\mathbb{P}\left(G_i \geqslant \frac{y+i^2}{2} \right) \leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}} \in [i-\frac{1}{2}, i+\frac{1}{2}]} \frac{(\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)) + \tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} \geqslant \frac{y}{2} + \frac{i^2}{2} - |i| \right). \tag{6.37}$$

Here there is a big problem: notice that it is possible that $y+i^2 \geqslant 0$ but $y+i^2-2|i| \leqslant 0$. In that case, we will never get a good bound of the probability above and our best bet is to bound the probability by 1. We need to understand for how many terms of the series, this is true. In short we need to know for each y, how many integers i satisfy that $y + i^2 \geqslant 0$ and $y + i^2 - 2|i| \leqslant 0$.

Before moving on with the proof, we will digress to give some intuition on this. We are trying to understand the density of

$$\sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \frac{\mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(-z)}{\sigma^{1/3}}.$$

The stochastic process $\mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0)$ is absolutely continuous with respect to a Brownian bridge. However, the stochastic process $\widetilde{\mathfrak{A}}_1^{(\sigma)}(-z)$ is a parabolic Airy process so it fluctuates around the parabola $-z^2$. Therefore, in certain intervals close to the origin and when $y + i^2$ is positive but very small, then probability (6.37) might be large. Is this correct? Should I remove it?

If $y > 0$, then $y+i^2 \geqslant 0$ trivially and we need to count how many is make $y+i^2-2|i|$ negative. By completing squares:

$$y + i^2 - 2|i| = y + (|i| - 1)^2 - 1$$

so

$$y + i^2 - 2|i| \leqslant 0 \qquad \text{iff} \qquad y + (|i| - 1)^2 - 1 \leqslant 0 \qquad \text{iff} \qquad (|i| - 1)^2 \leqslant 1 - y.$$

Since y is positive, then $1 - y \leqslant 1$. Then,

$$\{i \in \mathbb{Z} : y + i^2 - 2|i| \leqslant 0\} \subseteq \{i \in \mathbb{Z} : (|i| - 1)^2 \leqslant 1\} = [-2, 2] \cap \mathbb{Z}.$$

This means that there are 5 terms of the series that we need to be bounded by 1.

If $y < 0$, then we have that

$$y + i^2 \geqslant 0 \qquad \text{iff} \qquad |i| \geqslant \sqrt{-y} = \sqrt{|y|}$$

and

$$y + i^2 - 2|i| \leqslant 0 \quad \text{iff} \quad y + (|i| - 1)^2 - 1 \leqslant 0 \quad \text{iff} \quad |i| \leqslant \sqrt{1 - y} + 1 = \sqrt{1 + |y|} + 1.$$

Now, we need to determine the number of integers i that fit in the interval

$[\sqrt{|y|}, \sqrt{1+|y|}+1]$. But

$$\sqrt{1+|y|}+1-\sqrt{|y|} \leqslant \sqrt{(1+2\sqrt{|y|}+|y|)}+1-\sqrt{|y|} = 1+\sqrt{|y|}+1-\sqrt{|y|} = 2.$$

Since the interval $[\sqrt{|y|}, \sqrt{1+|y|}+1]$ has length 2, we have at most 4 terms (accounting for both signs, positive and negative) for which the best bound possible for $\mathbb{P}\big(G_i \geqslant \frac{y+i^2}{2}\big)$ is 1. If we include the term corresponding to $i=0$ which we had singled out before, we conclude that there are 5 terms of the series that need to be bounded by 1.

The rest of the bound is similar as before. Once again,

$$\frac{(\mathcal{F}^{(s)}(z)-\mathcal{F}^{(s)}(0))}{\sigma^{1/3}} \stackrel{d}{=} \frac{\mathcal{W}(2z+48)-\mathcal{W}(48)}{\sigma^{1/3}} + A\frac{z}{(s\sigma)^{1/3}}.$$

Assuming that $\frac{y}{2}+\frac{i^2}{2}-|i| \geqslant 0$, and using a union bound we get

$$\mathbb{P}\left(G_i \geqslant \frac{y+i^2}{2}\right) \leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]} \frac{\mathcal{W}(2z+48)-\mathcal{W}(48)}{\sigma^{1/3}} \geqslant \frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\right) \tag{6.38}$$

$$+\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]} A\frac{z}{(s\sigma)^{1/3}} \geqslant \frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\right) \tag{6.39}$$

$$+\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]} \frac{\tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}} \geqslant \frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\right). \tag{6.40}$$

We will bound the probabilities above using arguments similar to the ones used earlier in this section and Lemma 24.

We start with (6.38). Notice that if $i \neq 0$, the interval $[i-\frac{1}{2}, i+\frac{1}{2}]$ contains only positive numbers or negative numbers (depending on the sign of i). Since $i \neq 0$ for us, we will assume that $i > 0$ without loss of generality. Making the interval over which we take the supremum larger, we get that

$$\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]} \frac{\mathcal{W}(2z+48)-\mathcal{W}(48)}{\sigma^{1/3}} \geqslant \frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\right)$$
$$\leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[0,i+\frac{1}{2}]} \frac{\mathcal{W}(2z+48)-\mathcal{W}(48)}{\sigma^{1/3}} \geqslant \frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\right).$$

By the Markov property of the Brownian motion, we know that

$$(\mathcal{W}(2z+48)-\mathcal{W}(48))_{z\in[0,\sigma^{2/3}i+\sigma^{2/3}/2]} \stackrel{d}{=} (B(2z))_{z\in[0,\sigma^{2/3}i+\sigma^{2/3}/2]}$$

where B is a Brownian motion. Then,

$$\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[0,i+\frac{1}{2}]}\frac{\mathcal{W}(2z+48)-\mathcal{W}(48)}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)$$
$$\leqslant\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[0,i+\frac{1}{2}]}\frac{B(2z)}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg). \quad (6.41)$$

Using the reflection principle, Brownian scaling and the tail bound for a Gaussian random variable, we get that

$$\begin{aligned}\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[0,i+\frac{1}{2}]}\frac{B(2z)}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\leqslant&2\mathbb{P}\bigg(\frac{B(2\sigma^{2/3}i+\sigma^{2/3})}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\\=&2\mathbb{P}\bigg(B(2i+1)\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\\\leqslant&ce^{-c'\frac{|y+i^2-2|i||^2}{2|i|+1}}. \end{aligned}\quad (6.42)$$

For probability (6.39), we get that,

$$\begin{aligned}\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]}A\frac{z}{(s\sigma)^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\leqslant&\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]}|A|\frac{|z|}{(s\sigma)^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\\\leqslant&\mathbb{P}\bigg(|A|\frac{|i|\sigma^{1/3}+\sigma^{1/3}/2}{s^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\\\leqslant&\mathbb{P}\bigg(2|A||i|\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg),\end{aligned}$$

where in the last inequality we have used the $\sigma\leqslant 1$ and $s\geqslant 1$. The tail bounds of the random constant A are determined in Lemma (24), so from the right hand side of the equation above, we get that

$$\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]}A\frac{z}{(s\sigma)^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\leqslant ce^{-c'\frac{|y+i^2-|i||^{3/2}}{|i|^{3/2}}}. \quad (6.43)$$

Lastly, for equation (6.40), by Lemma 24, we immediately get that

$$\begin{aligned}\mathbb{P}\bigg(\sup_{\frac{z}{\sigma^{2/3}}\in[i-\frac{1}{2},i+\frac{1}{2}]}\frac{\tilde{\mathcal{F}}^{(\sigma)}(-z)}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)=&\mathbb{P}\bigg(\sup_{z\in I_{-i}^{(\sigma)}}\frac{\tilde{\mathcal{F}}^{(\sigma)}(z)}{\sigma^{1/3}}\geqslant\frac{y}{6}+\frac{i^2}{6}-\frac{|i|}{3}\bigg)\\\leqslant&ce^{-c'|y+i^2-|i||^{3/2}}.\end{aligned}\quad (6.44)$$

In this way, compiling (6.27), (6.42), (6.43) and (6.44) we have obtained a bound for the density. For each $|i|\leqslant\frac{7}{\sigma^{2/3}}$ and $y\in[-2\varepsilon/\sigma^{1/3},2\varepsilon/\sigma^{1/3}]$ such that $y+i^2-2|i|\geqslant 0$

we get that

$$\rho_i(x+h, y+i^2) \leqslant e^{-\frac{(y+i^2)^2}{4}} + e^{-c'\frac{|y+i^2-2|i||^2}{2|i|+1}} + e^{-c'\frac{|y+i^2-2|i||^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}} + e^{-c'|y+i^2-2|i||^{\frac{3}{2}}}. \quad (6.45)$$

The first term comes from bounding the Gaussian term on the density as shown in (6.27).

We are left with dealing with the series obtained by summing the right hand side of equation (6.45). Before moving on to the series, let's try to simplify its terms. Recall that in this case, both $y+i^2$ and $y+i^2-2|i|$ are positive so $(y+i^2)^2 \geqslant (y+i^2-2|i|)^2$. Since $|i| \geqslant 1$,

$$\begin{aligned} |y+i^2-2|i||^{\frac{3}{2}} &\geqslant \frac{|y+i^2-2|i||^{\frac{3}{2}}}{|i|^{\frac{3}{2}}} \\ (y+i^2)^2 \geqslant (y+i^2-2|i|)^2 &\geqslant \frac{(y+i^2-2|i|)^2}{|i|^2} \\ \frac{|y+i^2-2|i||^2}{2|i|+1} \geqslant \frac{|y+i^2-2|i||^2}{3|i|} &\geqslant \frac{(y+i^2-2|i|)^2}{9|i|^2}. \end{aligned}$$

Using these inequalities on (6.45) we get that

$$\rho_i(x+h, y+i^2) \leqslant 2e^{-c'\frac{(y+i^2-2|i|)^2}{|i|^2}} + 2e^{-c'\frac{|y+i^2-2|i||^{\frac{3}{2}}}{|i|^{\frac{3}{2}}}}.$$

Furthermore, we know that for all $u \geqslant 0$ and all $p \geqslant 1$, $u^p \geqslant u-1$ so we can simplify the bound on the density further

$$\rho_i(x+h, y+i^2) \leqslant 4e^{c'}e^{-c'\frac{(y+i^2-2|i|)}{|i|}}$$

for all $|i| \leqslant \frac{7}{\sigma^{2/3}}$ and $y \in [-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]$ such that $y+i^2-2|i| \geqslant 0$.

The series turns out to be bounded by

$$\sum_{|i| \leqslant \lfloor \frac{7}{\sigma^{2/3}} \rfloor} \rho_i(x+h, y+i^2)\mathbb{1}_{\{y+i^2 \geqslant 0\}} \leqslant 5 + c \sum_{|i| \leqslant \lfloor \frac{7}{\sigma^{2/3}} \rfloor} e^{-c'\frac{(y+i^2-2|i|)}{|i|}}\mathbb{1}_{\{y+i^2-2|i| \geqslant 0\}}.$$

Recall that the constant term 5 comes from the summands for which $y+i^2 \geqslant 0$ but $y+i^2-2|i| < 0$. We will now split in two cases:

If $y > 0$: In this case,

$$\begin{aligned}\sum_{|i|\leqslant\lfloor\frac{7}{\sigma^{2/3}}\rfloor} e^{-c'\frac{(y+i^2-2|i|)}{|i|}}\mathbb{1}_{\{y+i^2-2|i|\geqslant 0\}} &\leqslant \sum_{|i|\leqslant\lfloor\frac{7}{\sigma^{2/3}}\rfloor} e^{-c'\frac{(y+i^2-2|i|)}{|i|}}\mathbb{1}_{\{i^2-2|i|\geqslant 0\}} \\ &= \sum_{|i|\leqslant\lfloor\frac{7}{\sigma^{2/3}}\rfloor} e^{-c'(|i|-2)} \\ &\leqslant e^{2c'}\sum_{i=-\infty}^{\infty} e^{-c'|i|} = c''\end{aligned}$$

If $y \leqslant 0$: We can rewrite the series as follows:

$$\sum_{|i|\leqslant\lfloor\frac{7}{\sigma^{2/3}}\rfloor} e^{-c'\frac{(y+i^2-2|i|)}{|i|}}\mathbb{1}_{\{y+i^2-2|i|\geqslant 0\}} = \sum_{|i|\leqslant\lfloor\frac{7}{\sigma^{2/3}}\rfloor} e^{-c'(|i|-2+\frac{y}{|i|})}\mathbb{1}_{\{y+i^2-2|i|\geqslant 0\}} \leqslant c'',$$

where the last inequality was obtained through the following lemma:

Lemma 27. *Let $\beta_1 \in \mathbb{R}$, $\beta_2 \leqslant 0$, $\gamma > 0$, and $r \geqslant 1$ and for each $i \in \mathbb{Z}_{\neq 0}$ define the sequence $(\Psi_i)_{i\in\mathbb{Z}_{\neq 0}}$ by*

$$\Psi_i := |i| + \beta_1 + \frac{\beta_2}{|i|}.$$

Then, there exists an absolute constant $b_1(\gamma) > 0$ such that

$$\sum_{i=-\infty}^{-1}\left(e^{-\gamma(\Psi_i)^r}\right)1_{\{\Psi_i\geqslant 0\}} + \sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i)^r}\right)1_{\{\Psi_i\geqslant 0\}} < b_1(\gamma) < \infty.$$

Proof. We start by noticing that Ψ_i is symmetric with respect to i so

$$\sum_{i=-\infty}^{-1}\left(e^{-\gamma(\Psi_i)^r}\right)1_{\{\Psi_i\geqslant 0\}} = \sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i)^r}\right)1_{\{\Psi_i\geqslant 0\}}$$

and we can restrict ourselves to lookings at the case when $i > 0$.

Also, assume that the set $\{i > 0 : \Psi_i \geqslant 0\}$ is not empty. Otherwise, the inequality would be trivially true.

Then, observe that for any $u \in [0, \infty)$ and any $r \geqslant 1$ it is always true that $u^r \geqslant u-1$. This means that we will always have that

$$\sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i)^r}\right)1_{\{\Psi_i\geqslant 0\}} \leqslant \sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i-1)}\right)1_{\{\Psi_i\geqslant 0\}} = e^{\gamma}\sum_{i=1}^{\infty}\left(e^{-\gamma\Psi_i}\right)1_{\{\Psi_i\geqslant 0\}}.$$

Since $\beta_2 \leqslant 0$,

$$(\Psi_i)' = \left(i + \beta_1 + \frac{\beta_2}{i}\right)' = 1 - \frac{\beta_2}{i^2} \geqslant 1$$

for all $i > 0$. By expressing the increment $\Psi_{i+1} - \Psi_i$ as in integral, we then see that

$$\Psi_{i+1} = \Psi_i + (\Psi_{i+1} - \Psi_i) \geqslant \Psi_i + 1$$

for all $i > 0$. Now if we set $i_0 \in \mathbb{Z}_{>0}$ to be the minimal positive integer such that $\Psi_{i_0} \geqslant 0$ then we can say that

$$\begin{aligned} e^\gamma \sum_{i=1}^{\infty} \left(e^{-\gamma \Psi_i}\right) 1_{\{\Psi_i \geqslant 0\}} =& e^\gamma \sum_{i=i_0}^{\infty} e^{-\gamma \Psi_i} \\ \leqslant& e^\gamma \sum_{i=i_0}^{\infty} e^{-\gamma\left(\Psi_{i_0} + (i - i_0)\right)} \\ \leqslant& e^\gamma \sum_{i=i_0}^{\infty} e^{-\gamma(i-i_0)} \\ =:& \frac{1}{2} b_1(\gamma) < \infty. \end{aligned}$$

□

6.2.4 Absolute continuity of the Airy process on disjoint compacts

We move on to the last part of the proof. Recall that we had split the real line in intervals, with the hopes that the supremum of the sum of the Airy process $\mathfrak{A}^{(\sigma)}$ and the increment on the Airy process $\mathfrak{A}^{(s)}$ had small probability to occur in each interval and we could obtain a bound for the two-point probability of $\mathcal{L}(0,0;0,s)$. We have already dealt with the intervals that are close to 0; we now move on to bound the intervals that are far from the origin. In fact, we need to bound

$$p_{i,\varepsilon} = \mathbb{P}\Big(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, |\sup_{z/\sigma^{2/3} \in [i \pm 1/2]} \mathfrak{A}_1^{(s)}(z) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(-z) - \mathfrak{A}_1^{(s)}(0)| \leqslant 2\varepsilon\Big), \quad (6.46)$$

for $|\sigma^{2/3} i| > 7$.

The strategy is very similar to the one used on the intervals such that $|i| \leqslant \frac{7}{\sigma^{2/3}}$, only that now, i and 0 will be far apart and so $\mathfrak{A}^{(s)}(z)$ and $\mathfrak{A}^{(s)}(0)$ will not belong to the same interval, as before. However, this is not a big problem, since the same absolute continuity result over the Airy line ensemble with respect to a locally Brownian ensemble is true for a finite number of disjoint intervals of the same length. The original theorem by Dauvergne (in [14]) can be found in Section 6.3 and our convenient version of that result as a lemma is stated beloe. Notice that this lemma is comparable to Lemma 24 but in this case, the absolute continuity is used in two disjoint intervals of the same length over the parabolic Airy process simultaneously.

Lemma 28. *Let $a_1, a_2 \in \mathbb{R}_{\neq 0}$ and $T > \frac{1}{6}$ such that $a_1+3T < a_2-3T$. Let $I_1 = [a_1 \pm T]$ and $I_2 = [a_2 \pm T]$. Let f_{a_j} be the linear function on $[a_j \pm 3T]$ satisfying*

$$f_{a_j}(a_j - 3T) = -(a_j - 3T)^2 \qquad \text{and} \qquad f_{a_j}(a_j + 3T) = -(a_j + 3T)^2.$$

Then there exists a constant $c \in \mathbb{R}_{>0}$ and random functions $\left(\left(\mathcal{F}_1(r)\right)_{r\in I_1}, \left(\mathcal{F}_2(r)\right)_{r\in I_2}\right)$ such that

$$\begin{aligned} &\text{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in I_1}, \left(\mathfrak{A}_1(r)\right)_{r\in I_2}\right) \\ &\leqslant e^{cT^3}\,\text{Law}\left(\left(\sqrt{2T}N_1 + (\mathcal{F}_1(r) + f_{a_1}(r))_{r\in I_1}\right), \left(\sqrt{2T}N_2 + (\mathcal{F}_2(r) + f_{a_2}(r))_{r\in I_2}\right)\right) \end{aligned} \tag{6.47}$$

where N_1 and N_2 are independent standard Gaussian random variables. Moreover, the pairs (N_1, N_2) and $\left((\mathcal{F}_1(r))_{r\in I_1}, (\mathcal{F}_2(r))_{r\in I_2}\right)$ are independent, and there exist $T-$dependent constants $c_1, c_2 > 0$ such that for each $j \in \{1, 2\}$,

$$\mathbb{P}\left(\sup_{r\in I_j} |\mathcal{F}_i(r)| > m\right) \leqslant c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \tag{6.48}$$

for all $m > 0$. More generally, for any $\lambda > 0$, let $\mathfrak{A}_1^{(\lambda)}$ be as in (6.6). Denote by $I_j^{(\lambda)}$ the interval

$$I_j^{(\lambda)} := \lambda^{2/3} I_j = \left[a_j\lambda^{2/3} - T\lambda^{2/3}, a_j\lambda^{2/3} + T\lambda^{2/3}\right].$$

Then as a consequence of (6.47), we have that

$$\begin{aligned} &\text{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_1^{(\lambda)}}, \left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_2^{(\lambda)}}\right) \\ &\qquad \leqslant e^{cT_0^3}\,\text{Law}\left(\left(\left(\mathcal{F}_j^{(\lambda)}(r) + \lambda^{1/3}\left(\sqrt{2T}N_j + f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^{2}\right) \end{aligned} \tag{6.49}$$

with N_1 and N_2 as before. For each $j \in \{1, 2\}$, $\mathcal{F}_j^{(\lambda)}$ is a $\lambda-$dependent random function such that for all $m > 0$,

$$\mathbb{P}\left(\sup_{r\in I_j^{(\lambda)}} \left|\lambda^{-1/3}\mathcal{F}_j^{(\lambda)}(r)\right| > m\right) \leqslant c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \tag{6.50}$$

for the same $T-$dependent constants $c_1, c_2 > 0$.

We may also write for each $j \in \{1, 2\}$ the random function $F_j^{(\lambda)}$ as

$$\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} \stackrel{d}{=} \left(\mathcal{W}_j\left(2r - 2(a_j - 3T)\lambda^{\frac{2}{3}}\right) + A_j\lambda^{-\frac{1}{3}}r + C_j\lambda^{\frac{1}{3}}\right)_{r\in I_j^{(\lambda)}} \tag{6.51}$$

where W_j is a standard Brownian motion, and A_j and C_j are random constants such that,

$$\mathbb{P}\left(|A_j| > m\right) \leqslant c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \qquad \text{and} \qquad \mathbb{P}\left(|C_j| > m\right) \leqslant c_1 \exp\left(-c_2\left(\frac{m}{|a_j|}\right)^{\frac{3}{2}}\right) \tag{6.52}$$

for all $m > 0$. There are no claims made about any independence amongst $\mathcal{W}_j, A_j$, and C_j.

Remark. Note that although the constants $c_1, c_2 > 0$ are $T-$dependent, if we only ever use values of T that are bounded above and below by absolute constants, we can take c_1, c_2 to actually be absolute constants as well without of loss of generality. The values of T that we choose to work with specifically will be continuous univariate functions of s, and since s lives in a finite interval, our choices of T will indeed be bounded by absolute constants. Thus, in the work that follows, we will implicitly optimize our choice of c_1, c_2 as functions of s to obtain absolute constants.

We will use the previous lemma to bound probability (6.46):

$$\mathbb{P}\Big(|\mathfrak{A}_1^{(s)}(0) - h| \leqslant \varepsilon, |\sup_{z/\sigma^{2/3}\in[i\pm 1/2]} \mathfrak{A}_1^{(s)}(z) - \mathfrak{A}_1^{(s)}(0) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(-z)| \leqslant 2\varepsilon\Big).$$

We are now in the situation in which we need to understand the behaviour of $\mathfrak{A}^{(s)}(z)$ and $\mathfrak{A}_1^{(s)}(0)$ when z is close to $i\sigma^{2/3}$ and far from 0. Since $|i|\sigma^{2/3} > 7$, the points z and 0 are far enough that we can use the absolute continuity lemma above in two different intervals. For simplicity, we will assume that $i > 0$ so the interval around $\sigma^{2/3}i$ will be on the right of the interval that contains 0. The case when $i < 0$ is done in the same way except that the intervals change their order. The intervals are: $I_1 = s^{2/3}[\frac{\sigma^{1/3}}{2s^{2/3}} \pm \frac{1}{2s^{2/3}}]$ and $I_2 = s^{2/3}[\frac{\sigma^{2/3}}{s^{2/3}}i \pm \frac{1}{2s^{2/3}}]$. In the language of the previous theorem this means that $a_1 = \frac{\sigma^{1/3}}{2s^{2/3}}$, $a_2 = \frac{\sigma^{2/3}}{s^{2/3}}i$ and $T = \frac{1}{2s^{2/3}}$. We need to check three things:

1. The point $0 \in I_1$: It is clear that $0 < \frac{\sigma^{1/3}}{2} + \frac{1}{2}$. We need to check that $\frac{\sigma^{1/3}}{2} - \frac{1}{2} < 0$. Since $\sigma \leqslant 1$, this statement is true.
2. The interval $[\sigma^{2/3}i \pm \frac{\sigma^{2/3}}{2}]$, the interval over which we take the supremum, needs

to be a subset of I_2 so that we can compare the supremum of the parabolic Airy $\mathfrak{A}_1^{(s)}(z)$ in (6.46) with the supremum of the function $\mathcal{F}$ as defined on Lemma 28. If $z \in [\sigma^{2/3} i \pm \frac{\sigma^{2/3}}{2}]$,

$$|z - \sigma^{2/3} i| \leqslant \frac{\sigma^{2/3}}{2} \leqslant \frac{1}{2},$$

where in the last inequality we have used that $\sigma \leqslant 1$. We have proved that $[\sigma^{2/3} i \pm \frac{\sigma^{2/3}}{2}] \subseteq I_2$.

3. We need to make sure that I_1 and I_2 are far enough. As per Lemma 28, we need to guarantee that $a_1 + 3T \leqslant a_2 - 3T$:

$$a_1 + 3T \leqslant a_2 - 3T \quad \text{iff} \quad \frac{\sigma^{1/3}}{2s^{2/3}} + \frac{3}{2s^{2/3}} \leqslant \frac{\sigma^{2/3}}{s^{2/3}} i - \frac{3}{2s^{2/3}} \quad \text{iff} \quad 3 \leqslant \sigma^{2/3} i - \frac{\sigma^{1/3}}{2}.$$

Notice that since $\sigma^{2/3} i \geqslant 4$ and $\sigma \leqslant 1$, we know that $\sigma^{2/3} i - \frac{\sigma^{1/3}}{2} > 7 - \frac{1}{2} \geqslant 3$. This implies that the intervals are far enough to use the absolute continuity statement in Lemma 28 in these two intervals.

For $\widetilde{\mathfrak{A}}_1^{(\sigma)}$ we will use Lemma 24 on one interval as before.

Then, by Lemma 24 and Lemma 28, there exist a constant c_0, and independent standard Gaussians N_1, N_2 and $\widetilde{N}$ corresponding to $\mathfrak{A}_1^{(s)}|_{I_1}(z)$, $\mathfrak{A}_1^{(s)}|_{I_2}(z)$ and $\widetilde{\mathfrak{A}}_1^{(\sigma)}|_{[-\sigma^{2/3} i \pm \sigma^{2/3}/2]}$ such that we can bound probability (6.46) with

$$c_0 \mathbb{P}(|N_1 + G - h| \leqslant \varepsilon, |\sigma^{1/3} \widetilde{N} + N_2 - N_1 + \sigma^{1/3} G_i - \sigma^{1/3} i^2| \leqslant 2\varepsilon), \tag{6.53}$$

where $G = F_2^{(s)}(0) - \ell_{a_1,T}(0)$ and

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z)}{\sigma^{1/3}} + \frac{F_2^{(s)}(z)}{\sigma^{1/3}} - \frac{F_1^{(s)}(0)}{\sigma^{1/3}} + g_i(z)$$

where

$$g_i(z) = \frac{s^{1/3}}{\sigma^{1/3}} \ell_{a_2,T}\left(\frac{z}{s^{2/3}}\right) - \frac{s^{1/3}}{\sigma^{1/3}} \ell_{a_1,t}(0) + 2i\left(i - \frac{z}{\sigma^{2/3}}\right).$$

Recall the definition of the linear shifts:

$$\ell_{a_1,T}(r) = -2a_1 r + a_1^2 - 9T^2 = -\frac{\sigma^{1/3}}{s^{2/3}} r + \frac{\sigma^{2/3}}{4s^{4/3}} - \frac{9}{4s^{4/3}}$$
$$\ell_{a_2,T}(r) = -2a_2 r + a_2^2 - 9T^2 = -\frac{2\sigma^{2/3}}{s^{2/3}} ir + \frac{\sigma^{4/3}}{s^{2/3}} i^2 - \frac{9}{4s^{4/3}}.$$

Then,

$$\begin{aligned}\frac{s^{1/3}}{\sigma^{1/3}}\ell_{a_1,T}(0) &= \frac{\sigma^{1/3}}{4s} - \frac{9\sigma^{1/3}}{4s}\\ \frac{s^{1/3}}{\sigma^{1/3}}\ell_{a_2,T}\big(\frac{z}{s^{2/3}}\big) &= -\frac{2\sigma^{1/3}}{s}iz + \frac{\sigma}{s^{1/3}}i^2 - \frac{9\sigma^{1/3}}{4s}.\end{aligned}$$

We can rewrite g_i as follows:

$$g_i(z) = -\frac{2\sigma^{1/3}}{s}iz + \frac{\sigma}{s^{1/3}}i^2 - \frac{\sigma^{1/3}}{4s} + 2i\big(i - \frac{z}{\sigma^{2/3}}\big) \tag{6.54}$$

We can rearrange equation (6.53) as

$$c_0\mathbb{P}(|N_2 + G - h| \leqslant \varepsilon, |\widetilde{N} + \sigma^{-1/3}N_1 - \sigma^{-1/3}N_2 + G_i - i^2| \leqslant 2\varepsilon\sigma^{-1/3}).$$

Notice that $\{\widetilde{N}, N_1, N_2\}$ is independent of $\{G, G_i\}$. Also, since $\widetilde{N}, N_1$ and N_2 are independent of each other and Gaussian,

$$(N_2, \widetilde{N} + \sigma^{-1/3}N_1 - \sigma^{-1/3}N_2) =: (X, Y) \sim \mathcal{N}(0, \Sigma) \tag{6.55}$$

where Σ is the covariance matrix. In fact,

$$\Sigma = \begin{pmatrix} 1 & -\frac{1}{\sigma^{1/3}} \\ -\frac{1}{\sigma^{2/3}} & \frac{2}{\sigma^{1/3}} + 1 \end{pmatrix}. \tag{6.56}$$

We can now rewrite the probability (6.55) as

$$\mathbb{P}(|X+G-h| \leqslant \varepsilon, |Y+G_i-i^2| \leqslant 2\varepsilon\sigma^{-1/3}) = \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} \rho_i(x+h, y+i^2)dxdy, \tag{6.57}$$

where $\rho_i(x + h, y + i^2)$ is the density of the vector $(X, Y) + (G, G_i)$. It is worth mentioning that the density is well defined by the Lemma 25. Notice that for all $y \in [-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]$,

$$y + i^2 \geqslant -\frac{2\varepsilon}{\sigma^{1/3}} + i^2 \geqslant -\frac{2\varepsilon}{\sigma^{1/3}} + \frac{49}{\sigma^{4/3}} = \frac{1}{\sigma^{1/3}}\Big(\frac{49}{\sigma} - 2\varepsilon\Big) \geqslant \frac{1}{\sigma^{1/3}}\Big(49 - 2\Big) = \frac{47}{\sigma^{1/3}} \geqslant 0,$$

where we have used that $|i| \geqslant 7/\sigma^{2/3}$, $\sigma \leqslant 1$ and $\varepsilon \leqslant 1$.

Mimicking what we have done before, we will first bound the density $\rho_i(x, y)$ of the random vector $(X, Y) + (G, G_i)$ at the value (x, y). The goal is to find a bound a_i independent of h and σ such that if $|i| > 7\sigma^{-2/3}$, then

$$\rho_i(x + h, y + i^2) \leqslant a_i(y + i^2), \qquad \text{for all } x \in \mathbb{R} \tag{6.58}$$

and $\sum_{|i|>7\sigma^{-2/3}} a_i(y+i^2) < d''$ for a constant d'' independent of σ, y and h. Once we obtain that bound, can join this bound with the work done previously in (6.21)

$$\begin{aligned}
&\mathbb{P}(\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon); \mathcal{L}(0,0;0,t) \in (h-\varepsilon, h+\varepsilon)) \\
&\qquad \leqslant \sum_{i\in\mathbb{Z}} p_{i,\varepsilon} = 8c'' \frac{\varepsilon^2}{\sigma^{1/3}} + \sum_{|i|>7\sigma^{-\frac{2}{3}}} p_{i,\varepsilon} \\
&\qquad = 4c'' \frac{\varepsilon^2}{\sigma^{1/3}} + \sum_{|i|>7\sigma^{-\frac{2}{3}}} \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} \rho_i(x+h, y+i^2)dxdy \\
&\qquad \leqslant 4c'' \frac{\varepsilon^2}{\sigma^{1/3}} + \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} \sum_{|i|>7\sigma^{-\frac{2}{3}}} a_i(y+i^2)dxdy \\
&\qquad \leqslant 4c'' \frac{\varepsilon^2}{\sigma^{1/3}} + \int_{[-\varepsilon,\varepsilon]\times[-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]} d''dxdy \\
&\qquad = 4c'' \frac{\varepsilon^2}{\sigma^{1/3}} + 4d'' \frac{\varepsilon^2}{\sigma^{1/3}} \leqslant c'' \frac{\varepsilon^2}{\sigma^{1/3}} = c'' \frac{\varepsilon^2}{|t-s|^{1/3}}.
\end{aligned}$$

In the last inequality we have redefined the value of c''. Notice that obtaining the bound (6.58) completes the proof of this theorem as explained in (6.1).

We now, embark on the task of finding bounds for the density ρ_i. Let $y > 0$ arbitrary. We will use the same strategy as before: condition on the second coordinate to be able to deal with the density bounds of Y and G_i separately. Since (X, Y) and (G, G_i) are independent we will bound the density of the convolution $(X, Y) + (G, G_i)$ conditioned using Lemma 25. Recall that

$$\begin{aligned}
\rho_i(x,y) =& \mathbb{P}(X+G \in dx, Y+G_i \in dy) \\
=& \lim_{\delta\to 0} \frac{1}{\delta^2} \mathbb{P}(X+G \in [x, x+\delta), Y+G_i \in [y, y+\delta)) \\
\leqslant& \lim_{\delta\to 0} \frac{1}{\delta^2} \mathbb{P}(X+G \in [x, x+\delta), Y+G_i \in [y, y+\delta) | Y \geqslant y/2)\mathbb{P}(Y \geqslant y/2) \\
+& \frac{1}{\delta^2} \mathbb{P}(X+G \in [x, x+\delta), Y+G_i \in [y, y+\delta) | G_i \geqslant y/2)\mathbb{P}(G_i \geqslant y/2). \quad (6.59)
\end{aligned}$$

We have used that for every $\delta > 0$, the event $\{Y + G_i \in [y, y+\delta)\}$ implies that either $\{Y \geqslant y/2\}$ or $\{G_i \geqslant y/2\}$ must occur.

We now focus on the last two probabilities above. We start by writing

$$\begin{aligned}
&\mathbb{P}(X+G \in [x, x+\delta), Y+G_i \in [y, y+\delta) | Y \geqslant y/2) \\
&\qquad\qquad = \int_x^{x+\delta} \int_y^{y+\delta} f_{(X,Y)+(G,G_i)|\{Y\geqslant y/2\}}(u,v)dudv. \quad (6.60)
\end{aligned}$$

By Lemma 25, for all $(u, v) \in \mathbb{R} \times [0, \infty)$, we know that

$$\begin{aligned} f_{(X,Y)+(G,G_i)|\{Y \geqslant y/2\}}(u, v) &\leqslant \left\| f_{(X,Y)+(G,G_i)|\{Y \geqslant y/2\}}(u, v) \right\|_\infty \\ &\leqslant \left\| f_{(X,Y)|\{Y \geqslant y/2\}}(u, v) \right\|_\infty \mathbb{P}((G, G_i) \in \mathbb{R} \times [0, \infty) | Y \geqslant y/2) \\ &\leqslant \left\| f_{(X,Y)|\{Y \geqslant y/2\}}(u, v) \right\|_\infty . \end{aligned} \tag{6.61}$$

In a closer inspection to this density we see that

$$\begin{aligned} f_{(X,Y)|\{Y \geqslant y/2\}}(u, v) &= \lim_{\delta \to 0} \frac{1}{\delta^2} \frac{\mathbb{P}(X \in [u, u + \delta), Y \in [v, v + \delta), Y \geqslant y/2)}{\mathbb{P}(Y \geqslant y/2)} \\ &= \frac{f_{(X,Y)}(u, v) \mathbb{1}\{v \geqslant y/2\}}{\mathbb{P}(Y \geqslant y/2)}. \end{aligned} \tag{6.62}$$

The vector (X, Y) follows the distribution of bivariate normal with covariance matrix Σ so its density is

$$f_{(X,Y)}(u, v) = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{1}{2}(u,v)^T \Sigma^{-1}(u,v)} = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{1}{2|\Sigma|}\left(\left(\frac{u}{\sigma^{1/3}} + v\right)^2 + \left(\frac{1}{\sigma^{2/3}} + 1\right)u^2\right)}.$$

Following the definition of Σ on (6.56) we find the determinant:

$$|\Sigma| = \det \Sigma = \frac{1}{\sigma^{2/3}} + 1.$$

By simple inspection we can tell that the maximum of this density is achieved at the origin but since we are maximizing over the half-plane $\{v \geqslant y/2\}$ and $y > 0$, we conclude that the maximum occurs when $v = y/2$. Optimizing the exponential function, we get

$$\left\| f_{(X,Y)}(u, v) \mathbb{1}\{v \geqslant y/2\} \right\|_\infty = \left\| f_{(X,Y)}(u, y/2) \right\|_\infty = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{y^2 \sigma^{2/3}}{4(2+\sigma^{2/3})}}. \tag{6.63}$$

Notice that since $\sigma \leqslant 1$, we can bound the determinant $|\Sigma| \geqslant 2$ and

$$\frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{y^2 \sigma^{2/3}}{4(2+\sigma^{2/3})}} \leqslant \frac{1}{2\sqrt{2}\pi} e^{-\frac{y^2 \sigma^{2/3}}{4(2+\sigma^{2/3})}} \leqslant \frac{1}{2\sqrt{2}\pi} e^{-\frac{y^2 \sigma^{2/3}}{12}}. \tag{6.64}$$

Putting equations (6.60), (6.61), (6.62), (6.63) and (6.64) together, we see that

$$\mathbb{P}(X + G \in [x, x + \delta), Y + G_i \in [y, y + \delta) | Y \geqslant y/2) \leqslant \frac{\delta^2}{2\sqrt{2}\pi} \frac{e^{-\frac{y^2 \sigma^{2/3}}{12}}}{\mathbb{P}(Y \geqslant y/2)}. \tag{6.65}$$

We move onto the bounding the density of the other conditioned distribution:

$$\mathbb{P}(X+G \in [x, x+\delta), Y+G_i \in [y, y+\delta) | G_i \geqslant y/2) = \int_x^{x+\delta} \int_y^{y+\delta} f_{(X,Y)+(G,G_i)|\{G_i \geqslant y/2\}}(u,v) du dv. \quad (6.66)$$

Again, by Lemma 25,

$$\begin{aligned} f_{(X,Y)+(G,G_i)|\{Y \geqslant y/2\}}(u,v) &\leqslant \left\| f_{(X,Y)+(G,G_i)|\{G_i \geqslant y/2\}}(u,v) \right\|_\infty \\ &\leqslant \left\| f_{(X,Y)|\{Y \geqslant y/2\}}(u,v) \right\|_\infty \mathbb{P}((G,G_i) \in \mathbb{R} \times [0,\infty) | G_i \geqslant y/2) \\ &\leqslant \left\| f_{(X,Y)|\{G_i \geqslant y/2\}}(u,v) \right\|_\infty \end{aligned} \quad (6.67)$$

for all $(u,v) \in \mathbb{R} \times [0,\infty)$. Notice that $\{X,Y\}$ are independent of G_i so

$$f_{(X,Y)|\{G_i \geqslant y/2\}}(u,v) = f_{(X,Y)}(u,v)$$

and

$$\left\| f_{(X,Y)|\{G_i \geqslant y/2\}}(u,v) \right\|_\infty \leqslant \frac{1}{2\sqrt{2\pi}}. \quad (6.68)$$

Adding the results obtained in (6.66), (6.67) and (6.68) we get that

$$\mathbb{P}(X+G \in [x,x+\delta), Y+G_i \in [y,y+\delta) | G_i \geqslant y/2) \leqslant \frac{\delta^2}{2\sqrt{2\pi}}. \quad (6.69)$$

Then, following (6.59), (6.65) and (6.69) we obtain that for all $x \in \mathbb{R}$ and $y > 0$

$$\begin{aligned} \rho_i(x,y) \leqslant& \lim_{\delta \to 0} \frac{1}{\delta^2} \mathbb{P}(X+G \in [x,x+\delta), Y+G_i \in [y,y+\delta) | Y \geqslant y/2) \mathbb{P}(Y \geqslant y/2) \\ &+ \frac{1}{\delta^2} \mathbb{P}(X+G \in [x,x+\delta), Y+G_i \in [y,y+\delta) | G_i \geqslant y/2) \mathbb{P}(G_i \geqslant y/2) \\ \leqslant& \frac{1}{2\sqrt{2\pi}} \frac{e^{-\frac{y^2 \sigma^{2/3}}{12}}}{\mathbb{P}(Y \geqslant y/2)} \mathbb{P}(Y \geqslant y/2) + \frac{1}{2\sqrt{2\pi}} \mathbb{P}(G_i \geqslant y/2) \\ =& \frac{1}{2\sqrt{2\pi}} \Big(e^{-\frac{y^2 \sigma^{2/3}}{12}} + \mathbb{P}(G_i \geqslant y/2) \Big). \end{aligned}$$

Then, we can bound the density in (6.57) as follows:

$$\rho_i(x+h, y+i^2) \leqslant \frac{1}{2\sqrt{2\pi}} \Big(e^{-\frac{(y+i^2)^2 \sigma^{2/3}}{12}} + \mathbb{P}\big(G_i \geqslant \frac{y+i^2}{2}\big) \Big). \quad (6.70)$$

In the next section we will concern ourselves with bounding $\mathbb{P}\big(G_i \geqslant \frac{y+i^2}{2}\big)$.

6.2.5 Bound for the two-point density of the directed landscape if a geodesic from $(0,0)$ to $(0,t)$ is far from the origin at time s

Recall the definition of G_i (equation (6.54)):

$$G_i = \sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z) + F_2^{(s)}(z) - F_1^{(s)}(0)}{\sigma^{1/3}} + g_i(z)$$

where

$$g_i(z) = \frac{s^{1/3}}{\sigma^{1/3}} \ell_{a_2,T}\big(\frac{z}{s^{2/3}}\big) - \frac{s^{1/3}}{\sigma^{1/3}} \ell_{a_1,t}(0) + 2i\big(i - \frac{z}{\sigma^{2/3}}\big).$$

We will first focus on the deterministic part of the supremum; we can rewrite g_i as follows:

$$g_i(z) = -\frac{2\sigma^{1/3} i}{s}(z - \sigma^{2/3} i) - \frac{2\sigma i^2}{s} + \frac{\sigma i^2}{s^{1/3}} - \frac{\sigma^{1/3}}{4s} + 2i(i - \frac{z}{\sigma^{2/3}}). \tag{6.71}$$

For any $z \in [\sigma^{2/3} i \pm \sigma^{2/3}/2]$, we can bound the right hand side of the equation above and we get

$$g_i(z) \leqslant \frac{\sigma^{1/3}|i|}{s} - \frac{2\sigma i^2}{s} + \frac{\sigma i^2}{s^{1/3}} - \frac{\sigma^{1/3}}{4s} + |i| = \big(\frac{-2}{s^{2/3}} + 1\big)\frac{\sigma}{s^{1/3}} i^2 + \big(\frac{\sigma^{1/3}}{s} + 1\big)|i| - \frac{\sigma^{1/3}}{4s}.$$

Using the fact that $0 \leqslant \sigma \leqslant 1$ and $1 \leqslant s \leqslant 2$, we can further bound the right hand side of the equation above by

$$(1 - 2^{1/3}) i^2 + 2|i|$$

Notice that the equation above is negative for all $|i| > 7$.

We conclude that for all $z \in [\sigma^{2/3} i \pm \sigma^{2/3}/2]$,

$$g_i(z) \leqslant 0.$$

Then,

$$G_i \leqslant \sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z) + F_2^{(s)}(z) - F_1^{(s)}(0)}{\sigma^{1/3}}.$$

Therefore,

$$\mathbb{P}\left(G_i \geqslant \frac{y + i^2}{2}\right) \leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z) + F_2^{(s)}(z) - F_1^{(s)}(0)}{\sigma^{1/3}} \geqslant \frac{y + i^2}{2}\right).$$

Now, we will try to understand

$$\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z) + F_2^{(s)}(z) - F_1^{(s)}(0)}{\sigma^{1/3}}. \tag{6.72}$$

Recall that by Lemma 28, we know that for all $r \in I_j$

$$\mathcal{F}_j^{(\lambda)}(r) \overset{d}{=} \mathcal{W}_j\left(2r - 2(a_j - 3T)\lambda^{\frac{2}{3}}\right) + A_j\lambda^{-\frac{1}{3}}r + C_j\lambda^{\frac{1}{3}}$$

Then, we can rewrite

$$\begin{aligned}
\frac{F_1^{(s)}(0)}{\sigma^{1/3}} &= \frac{\mathcal{W}_1(-2(\frac{\sigma^{1/3}}{2s^{2/3}} - \frac{3}{2s^{2/3}})s^{2/3})}{\sigma^{1/3}} + \frac{C_1 s^{1/3}}{\sigma^{1/3}} \\
&= \frac{\mathcal{W}_1(-\sigma^{1/3}+3)}{\sigma^{1/3}} + \frac{C_1 s^{1/3}}{\sigma^{1/3}} \\
\frac{F_2^{(s)}(z)}{\sigma^{1/3}} &= \frac{\mathcal{W}_2(2z - 2(\frac{\sigma^{2/3}}{s^{2/3}}i - \frac{3}{2s^{2/3}})s^{2/3})}{\sigma^{1/3}} + \frac{A_2 z}{\sigma^{1/3}s^{1/3}} + \frac{C_2 s^{1/3}}{\sigma^{1/3}} \\
&= \frac{\mathcal{W}_2(2z - 2\sigma^{2/3}i + 3)}{\sigma^{1/3}} + \frac{A_2 z}{\sigma^{1/3}s^{1/3}} + \frac{C_2 s^{1/3}}{\sigma^{1/3}}
\end{aligned}$$

Then, we can rewrite equation (6.72) as

$$\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z)}{\sigma^{1/3}} + \frac{\mathcal{W}_2(2z - 2\sigma^{2/3}i + 3)}{\sigma^{1/3}} + \frac{A_2 z}{\sigma^{1/3}s^{1/3}} + \frac{C_2 s^{1/3}}{\sigma^{1/3}} - \frac{\mathcal{W}_1(-\sigma^{1/3}+3)}{\sigma^{1/3}} - \frac{C_1 s^{1/3}}{\sigma^{1/3}}.$$

Using the fact that the supremum of a sum is larger or equal than the sum of suprema and a union bound, we can see that

$$\mathbb{P}\left(G_i \geqslant \frac{y+i^2}{2}\right) \leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z)}{\sigma^{1/3}} + \frac{F_2^{(s)}(z)}{\sigma^{1/3}} - \frac{F_1^{(s)}(0)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{2}\right)$$

$$\leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right) \tag{6.73}$$

$$+\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\mathcal{W}_2(2z-2\sigma^{2/3}i+3)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right) \tag{6.74}$$

$$+\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{A_2 z}{\sigma^{1/3}s^{1/3}} \geqslant \frac{y+i^2}{12}\right) \tag{6.75}$$

$$+\mathbb{P}\left(\frac{C_2 s^{1/3}}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right) \tag{6.76}$$

$$+\mathbb{P}\left(-\frac{\mathcal{W}_1(-\sigma^{1/3}+3)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right) \tag{6.77}$$

$$+\mathbb{P}\left(-\frac{C_1 s^{1/3}}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right). \tag{6.78}$$

We are going to bound each of the terms above.

We start by bounding the first term of the expression above. The numbers c_i' and c_i'' for $i = 1, \ldots, 6$ are constants that can change from line to line, absorbing the constants into the expressions. By (6.50) in Lemma 24, we bound the first term (equation (6.73)) as follows:

$$\mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{\widetilde{F}^{(\sigma)}(-z)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right) \leqslant \mathbb{P}\left(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \frac{|\widetilde{F}^{(\sigma)}(-z)|}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\right)$$

$$\leqslant c_1' e^{-c_1''(y+i^2)^{3/2}}. \tag{6.79}$$

For the second term (equation (6.74)), recall that $\mathcal{W}_2$ is a standard Brownian motion so

$$\begin{aligned}
\mathbb{P}\Big(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]}\frac{\mathcal{W}_2(2z-2\sigma^{2/3}i+3)}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big) =&\mathbb{P}\Big(\sup_{z\in[-\sigma^{2/3},\sigma^{2/3}]}\frac{\mathcal{W}_2(z+3)}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big)\\
\leqslant&\mathbb{P}\Big(\sup_{z\in[0,\sigma^{2/3}+3]}\frac{\mathcal{W}_2(z)}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big)\\
\leqslant&2\mathbb{P}\Big(\frac{\mathcal{W}_2(\sigma^{2/3}+3)}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big)\\
=&2\mathbb{P}\Big(\mathcal{W}_2\big(1+\frac{3}{\sigma^{2/3}}\big)\geqslant\frac{y+i^2}{12}\Big)\\
\leqslant&2\exp\{-\frac{(y+i^2)^2}{24\big(1+\frac{3}{\sigma^{2/3}}\big)}\}\\
\leqslant&c_2'e^{-c_2''(y+i^2)^2\sigma^{2/3}},
\end{aligned}\tag{6.80}$$

where we have used the reflection principle, the scaling property of the Brownian motion, the Gaussian tail bounds and the fact that $\sigma\leqslant 1$.

For the third term (equation (6.75)), using that $s\leqslant 1$, we get that

$$\begin{aligned}
\mathbb{P}\Big(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]}\frac{A_2z}{\sigma^{1/3}s^{1/3}}\geqslant\frac{y+i^2}{12}\Big) \leqslant&\mathbb{P}\Big(\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]}\frac{|A_2|z}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big)\\
\leqslant&\mathbb{P}\Big(\frac{|A_2|(\sigma^{2/3}i+\sigma^{2/3}/2)}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big)\\
\leqslant&\mathbb{P}\Big(|A_2|\geqslant\frac{y+i^2}{12(\sigma^{1/3}i+\sigma^{1/3}/2)}\Big)\\
\leqslant&\mathbb{P}\Big(|A_2|\geqslant\frac{y+i^2}{12(i+1/2)}\Big)\leqslant c_3'\exp\{-c_3''\frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}\},
\end{aligned}\tag{6.81}$$

where in the last two inequalities we have used that $\sigma\leqslant 1$, $i>0$ and the tail bound of A_2 from (6.52) in Lemma 28 respectively.

For the fourth term (equation (6.76)),

$$\begin{aligned}
\mathbb{P}\Big(\frac{C_2s^{1/3}}{\sigma^{1/3}}\geqslant\frac{y+i^2}{12}\Big) \leqslant&\mathbb{P}\Big(|C_2|\geqslant\frac{(y+i^2)\sigma^{1/3}}{2^{1/3}12}\Big)\\
\leqslant&c_4'\exp\{-c_4''\frac{(y+i^2)^{3/2}\sigma^{1/2}}{(\sigma^{1/3}/2s^{2/3})^{3/2}}\}\\
\leqslant&c_4'e^{-c_4''(y+i^2)^{3/2}},
\end{aligned}\tag{6.82}$$

where we have used that $1\leqslant s\leqslant 2$ and the tail bound of C_2 from (6.52) in Lemma 28.

For the fifth term (equation (6.77)), since $\mathcal{W}_1$ is a Brownian motion,

$$\mathbb{P}\Big(-\frac{\mathcal{W}_1(-\sigma^{1/3}+3)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\Big) = \mathbb{P}\Big(\frac{\mathcal{W}_1(-\sigma^{1/3}+3)}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\Big) \leqslant c_5' e^{-c_5'' \frac{(y+i^2)^2\sigma^{2/3}}{3-\sigma^{1/3}}} \leqslant c_5' e^{-c_5''(y+i^2)^2\sigma^{2/3}}, \tag{6.83}$$

where we have used the symmetry of the standard Gaussian random variable, the tail bounds for a standard Gaussian and the fact that $\sigma \leqslant 1$ in each step of the inequality chain above.

For the sixth term (equation (6.78)),

$$\mathbb{P}\Big(-\frac{C_1 s^{1/3}}{\sigma^{1/3}} \geqslant \frac{y+i^2}{12}\Big) \leqslant c_6' e^{-c_6''(y+i^2)^{3/2}} \tag{6.84}$$

for the same reason as (6.82).

Putting the bounds obtained in (6.79), (6.80), (6.81), (6.82), (6.83) and (6.82) together we get that for all $y \in [-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]$,

$$\mathbb{P}(G_i \geqslant \frac{y+i^2}{2}) \leqslant c_1' e^{-c_1''(y+i^2)^{3/2}} + c_2' e^{-c_2''(y+i^2)^2\sigma^{2/3}} + c_3' e^{-c_3'' \frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}} + c_4' e^{-c_4''(y+i^2)^{3/2}} + c_5' e^{-c_5''(y+i^2)^2\sigma^{2/3}} + c_6' e^{-c_6''(y+i^2)^{3/2}}.$$

Taking $c' = \max\{c_i' = i = 1, \ldots, 6\}$ and $c'' = \min\{c_i'' = i = 1, \ldots, 6\}$, we can bound the sum above by

$$c'\left(3e^{-c''(y+i^2)^{3/2}} + 2e^{-c''(y+i^2)^2\sigma^{2/3}} + e^{-c''\frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}}\right). \tag{6.85}$$

Putting together this last calculation and (6.70), we can bound the density as follows:

$$\begin{aligned}
\rho_i(x+h, y+i^2) &\leqslant \frac{1}{\sqrt{3\pi}}\Big(e^{-\frac{(y+i^2)^2\sigma^{2/3}}{16}} + \mathbb{P}\big(G_i \geqslant \frac{y+i^2}{2}\big)\Big) \\
&\leqslant \frac{1}{\sqrt{3\pi}}\Big(e^{-\frac{(y+i^2)^2\sigma^{2/3}}{16}} + c'\big(3e^{-c''(y+i^2)^{3/2}} + 2e^{-c''(y+i^2)^2\sigma^{2/3}} + e^{-c''\frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}}\big)\Big) \\
&\leqslant c'\Big(e^{-c''(y+i^2)^{3/2}} + e^{-c''(y+i^2)^2\sigma^{2/3}} + e^{-c''\frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}}\Big),
\end{aligned}$$

where in the last inequality we have simply redefined the constants c' and c''. Recall that $i > 0$, $\sigma^{2/3} \geqslant \frac{7}{|i|} \geqslant \frac{7}{i^2}$, and that $1/2 \leqslant |i|$ so

$$e^{-c''(y+i^2)^2\sigma^{2/3}} \leqslant e^{-7c''\left(\frac{y+i^2}{|i|}\right)^2} \quad \text{and} \quad e^{-c''\frac{(y+i^2)^{3/2}}{(i+1/2)^{3/2}}} \leqslant e^{-\frac{c''}{2^{3/2}}\left(\frac{y+i^2}{|i|}\right)^{3/2}}. \tag{6.86}$$

Furthermore, recall that for all $u \geqslant 0$ and all $p \geqslant 1$, $u^p \geqslant u - 1$ so putting together the bounds obtained on (6.85) and (6.86), we get that

$$\begin{aligned}\rho_i(x+h, y+i^2) \leqslant & c'\left(e^{-c''(y+i^2)}e^{c''} + e^{-7c''\left(\frac{y+i^2}{|i|}\right)}e^{7c''} + e^{-\frac{c''}{2^{3/2}}\left(\frac{y+i^2}{|i|}\right)}e^{\frac{c''}{2^{3/2}}}\right) \\ \leqslant & c'\left(e^{-c''(y+i^2)} + e^{-c''\left(\frac{y+i^2}{|i|}\right)}\right)\end{aligned}$$

where in the last inequality we have simply redefined the constants c' and c'' for more convenient ones once again.

As previously explained, all that is there left to do now is finding a bound for the series

$$\sum_{i \geqslant 7/\sigma^{2/3}} \rho_i(x+h, y+i^2) \leqslant \sum_{i \geqslant 7/\sigma^{2/3}} c'\left(e^{-c''(y+i^2)} + e^{-c''\left(\frac{y+i^2}{|i|}\right)}\right)$$

uniform on $y \in [-2\varepsilon/\sigma^{1/3}, 2\varepsilon/\sigma^{1/3}]$. For the first series,

$$\sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''(y+i^2)} \leqslant \sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''|\lfloor y \rfloor + i^2|} \leqslant \sum_{i \in \mathbb{Z}} e^{-c''|\lfloor y \rfloor + i|} = \sum_{j \in \mathbb{Z}} e^{-c''|j|} = d(c''),$$

where we have used in the first inequality the fact that $y + i^2 = |y + i^2|$ and the fact that $y \geqslant \lfloor y \rfloor$, in the second inequality we have added to the series the non quadratic terms and in the third equality, we have performed a change of variables $j = \lfloor y \rfloor + i$.

In the second case, we are going to use Lemma 27.

If $y \leqslant 0$: By Lemma 27, with $\beta_1 = 0$ and $\beta_2 = y$,

$$\sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''\left(\frac{y+i^2}{|i|}\right)} = \sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''\left(|i| + \frac{y}{|i|}\right)} \leqslant \sum_{i \geqslant 1} e^{-c''\left(|i| + \frac{y}{|i|}\right)} 1_{\{|i| + \frac{y}{|i|} \geqslant 0\}} < b_1(c'').$$

If $y > 0$:

$$\sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''\left(\frac{y+i^2}{|i|}\right)} \leqslant \sum_{i \geqslant 7/\sigma^{2/3}} e^{-c''|i|} \leqslant d(c'')$$

We have proved that

$$\sum_{i \geqslant 7/\sigma^{2/3}} \rho_i(x+h, y+i^2) \leqslant 2d(c'') + b_1(c'').$$

This concludes our proof.

□

6.3 Proofs of the absolute continuity lemmas

We conclude this section of our argument by providing the proof of Lemma 24 and Lemma 28 that we deferred earlier. Recall that these lemmas are simple adaptations of Theorem 8 and Theorem 29.

6.3.1 Proof of Lemma 24

We begin with the proof of Lemma 24. We will use Theorem 8 and Proposition 17 (proof available in Subsection 3.4.2).

Proof of Lemma 24. By the stationarity of the stationary Airy process, we have the equality in distribution

$$\begin{aligned}\left(\mathfrak{A}_1(r)\right)_{r\in[a\pm 3T]} &\overset{d}{=} \left(\left(\mathfrak{A}_1(r-a)+(r-a)^2\right)-r^2\right)_{r\in[a\pm 3T]}\\ &\overset{d}{=} \left(\mathfrak{A}_1(r)+r^2-(r+a)^2\right)_{r\in[0\pm 3T]}\end{aligned}$$

which can be thus be condensed into the equality of laws

$$\mathrm{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in[a\pm 3T]}\right) = \mathrm{Law}\left(\left(\mathfrak{A}_1(r)+r^2-(r+a)^2\right)_{r\in[0\pm 3T]}\right).$$

By Lemma 8, there exists a diffusion parameter 2 Brownian bridge B on $[-3T, 3T]$ from 0 to 0 and an independent random affine function L on $[-3T, 3T]$ such that

$$\begin{aligned}\mathrm{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in[a\pm 3T]}\right) &= \mathrm{Law}\left(\left(\mathfrak{A}_1(r)+r^2-(r+a)^2\right)_{r\in[0\pm 3T]}\right)\\ &\leqslant e^{216cT^3}\,\mathrm{Law}\left(\left(B(r)+L(r)+r^2-(r+a)^2\right)_{r\in[0\pm 3T]}\right)\end{aligned}$$

We note here that although we do have the option to apply Lemma 8 on the original interval $[a \pm 3T]$, it is better for our purposes to apply it on $[0 \pm 3T]$. The main benefit to making this choice is that this isolates the dependency on the center of the interval a in a single deterministic parabolic term. In doing so, we see that any and all behaviours of L and B on $[0 \pm 3T]$ now have absolutely no relation to the value of a. This will be quite useful later on.

Thus by restricting both $(\mathfrak{A}_1(r))_{r\in[a\pm 3T]}$ and $(B(r)+L(r)+r^2-(r+a)^2)_{r\in[0\pm 3T]}$ to the middle thirds of their domains, we may use Lemma 17 with the parameters $\delta = \frac{1}{3}$ and $k = 2$ to extract a Gaussian random variable from the Brownian bridge and conclude that

$$\begin{aligned}\text{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in I_a}\right) &\leqslant e^{216cT^3}\,\text{Law}\left(\left(B(r)+L(r)+r^2-(r+a)^2\right)_{r\in[0\pm T]}\right)\\ &= e^{216cT^3}\,\text{Law}\left(\left(\sqrt{2T}N+(B(r)-\sqrt{2T}N)+L(r)+r^2-(r+a)^2\right)_{r\in[0\pm T]}\right)\\ &= e^{216cT^3}\,\text{Law}\left(\left(\sqrt{2T}N+(B(r-a)-\sqrt{2T}N)+L(r-a)+(r-a)^2-r^2\right)_{r\in I_a}\right)\\ &= e^{216cT^3}\,\text{Law}\left(\left(\sqrt{2T}N+\mathcal{F}(r)+\ell_a(r)\right)_{r\in I_a}\right)\end{aligned}$$

with N a standard Gaussian independent of the process $(B-\sqrt{2T}N)_{r\in[0\pm T]}$, and where we have defined

$$\left(\mathcal{F}(r)\right)_{r\in I_a} := \left(\left(B(r-a)-\sqrt{2T}N\right)+L(r-a)\right)_{r\in I_a}. \tag{6.87}$$

Now that we have defined our process $(\mathcal{F}(r))_{r\in I_a}$, we will next prove that the tail bound (6.11) is true. We first observe that

$$\begin{aligned}\sup_{r\in I_a}|\mathcal{F}(r)| &= \sup_{r\in[a\pm T]}\left|B(r-a)-\sqrt{2T}N+L(r-a)\right|\\ &\leqslant \sup_{r\in[-T,T]}|B(r)|+\sqrt{2T}\,|N|+\sup_{r\in[-T,T]}|L(r)|.\end{aligned} \tag{6.88}$$

Moreover, recalling that L is a straight line segment from $L(-3t)$ to $L(3t)$ we have the elementary bound

$$\sup_{r\in[-T,T]}|L(r)| \leqslant \sup_{r\in[-3T,3T]}|L(r)| \leqslant \max\{|L(-3t)|,|L(3t)|\} \leqslant |L(-3t)|+|L(3t)|. \tag{6.89}$$

By invoking the bounds (1.5) and (1.6) in Lemma 8, we obtain in our case that

$$\mathbb{P}\Big(L(-3t)\wedge L(3t) < -m\Big) \leqslant 2e^{-dm^3} \quad \text{and} \quad \mathbb{P}\Big(L(-3t)\vee L(3t) > m\Big) \leqslant e^{-\frac{4}{3}m^{\frac{3}{2}}+cm^{\frac{5}{4}}} \tag{6.90}$$

for some T_0-dependent constants $c,d>0$. This means that by taking union bounds, combining equations (6.89) and (6.90) gives us a chain of inequalities

$$\begin{aligned}\mathbb{P}\left(\sup_{r\in[-T,T]}|L(r)|\geqslant 2m\right)\leqslant&\mathbb{P}\Big(|L(-3t)|+|L(3t)|\geqslant 2m\Big)\\ \leqslant&\mathbb{P}\Big(|L(-3t)|\geqslant m\Big)+\mathbb{P}\Big(|L(3t)|\geqslant m\Big)\\ \leqslant&2\mathbb{P}\Big(L(-3t)\wedge L(3t)<-m\Big)+2\mathbb{P}\Big(L(-3t)\vee L(3t)>m\Big)\\ \leqslant&2\left(e^{-\frac{4}{3}m^{\frac{3}{2}}+cm^{\frac{5}{4}}}+2e^{-dm^3}\right)\leqslant c_1e^{-c_2m^{\frac{3}{2}}}\end{aligned}\tag{6.91}$$

for some $T-$dependent constants $c_1,c_2>0$. We now turn our attention to the Brownian bridge B. Since B is a Brownian bridge of diffusion parameter 2, we may write

$$\Big(B(r)\Big)_{r\in[-3T,3T]}\overset{d}{=}\Big(\mathcal{W}(2r+6T)-\frac{r+3T}{6T}\mathcal{W}(12T)\Big)_{r\in[-3T,3T]}\tag{6.92}$$

where $\mathcal{W}$ is a standard two-sided Brownian motion. Based on this decomposition in law we have the upper bound

$$\sup_{r\in[-T,T]}|B(r)|\leqslant|\mathcal{W}(12T)|+\sup_{r\in[0,6T]}|\mathcal{W}(2r)|=|\mathcal{W}(12T)|+\sup_{r\in[0,12T]}|\mathcal{W}(r)|.$$

In turn, this implies the chain of inequalities

$$\mathbb{P}\Big(\sup_{r\in[-T,T]}|B(r)|\geqslant 2m\Big)\tag{6.93}$$

$$\begin{aligned}&\leqslant\mathbb{P}\left(|\mathcal{W}(12T)|+\sup_{r\in[0,12T]}|\mathcal{W}(r)|\geqslant 2m\right)\\ &\leqslant\mathbb{P}\Big(|\mathcal{W}(12T)|\geqslant m\Big)+\mathbb{P}\left(\sup_{r\in[0,12T]}|\mathcal{W}(r)|\geqslant m\right)\\ &=\mathbb{P}\Big(|\mathcal{W}(12T)|\geqslant m\Big)+\mathbb{P}\left(\sup_{r\in[0,12T]}\mathcal{W}(r)\geqslant m\right)+\mathbb{P}\left(\sup_{r\in[0,12T]}-\mathcal{W}(r)\geqslant m\right)\\ &=\mathbb{P}\Big(|\mathcal{W}(12T)|\geqslant m\Big)+2\mathbb{P}\left(\sup_{r\in[0,12T]}\mathcal{W}(r)\geqslant m\right)\\ &=3\mathbb{P}\Big(|\mathcal{W}(12T)|\geqslant m\Big)\\ &\leqslant 6e^{-m^2/(24t)}\end{aligned}\tag{6.94}$$

using the fact that $-\mathcal{W}\overset{d}{=}\mathcal{W}$, the known distribution for the running maximum of a Brownian motion, and the standard Gaussian concentration bound. As such, by combining equations (6.88), (6.91), and (6.93), we see that we can find positive

T_0-dependent constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{r\in I_a}|\mathcal{F}(r)| \geqslant m\Big) \\
&\quad\leqslant \mathbb{P}\Big(\sup_{r\in[-T,T]}|B(r)| \geqslant m/3\Big) + \mathbb{P}\Big(\sqrt{2T}|N| \geqslant m/3\Big) + \mathbb{P}\Big(\sup_{r\in[-T,T]}|L(r)| \geqslant m/3\Big) \\
&\quad\leqslant c_1 e^{-c_2 m^{3/2}}
\end{aligned}$$

as claimed in equation (6.11). The extension of our result to the rescaled Airy process $\mathcal{A}_1^{(\lambda)}$ in (6.12) is an immediate consequence of the base case when $\lambda = 1$. To see this explicitly, we observe that

$$\begin{aligned}
&\mathrm{Law}\,\Big(\Big(\ \mathfrak{A}_1^{(\lambda)}(r)\Big)_{r\in I_a^{(\lambda)}}\Big) \\
&\quad= \mathrm{Law}\,\Big(\Big(\lambda^{\frac{1}{3}}\mathfrak{A}_1(r)\Big)_{r\in I_a}\Big) \\
&\quad\leqslant e^{216cT^3}\,\mathrm{Law}\,\Big(\Big(\lambda^{\frac{1}{3}}\sqrt{2T}N + \lambda^{\frac{1}{3}}\mathcal{F}(r) + \lambda^{\frac{1}{3}}\ell_a(r)\Big)_{r\in[a\pm T]}\Big) \\
&\quad= e^{216cT^3}\,\mathrm{Law}\,\Big(\Big(\lambda^{\frac{1}{3}}\sqrt{2T}N + \lambda^{\frac{1}{3}}\mathcal{F}(r\lambda^{-2/3}) + \lambda^{\frac{1}{3}}\ell_a(r\lambda^{-2/3})\Big)_{r\in[a\lambda^{2/3}\pm T\lambda^{2/3}]}\Big) \\
&\quad= e^{216cT^3}\,\mathrm{Law}\,\Big(\Big(\lambda^{\frac{1}{3}}\sqrt{2T}N + \mathcal{F}^{(\lambda)}(r) + \lambda^{\frac{1}{3}}\ell_a(r\lambda^{-2/3})\Big)_{r\in I_a^{(\lambda)}}\Big)
\end{aligned}$$

where we have defined the rescaled random function $\Big(\mathcal{F}^{(\lambda)}(r)\Big)_{r\in I_a^{(\lambda)}}$ by

$$\Big(\mathcal{F}^{(\lambda)}(r)\Big)_{r\in I_a^{(\lambda)}} = \Big(\lambda^{\frac{1}{3}}\Big(B(r\lambda^{-2/3}-a) - \sqrt{2T}N + L(r\lambda^{-2/3}-a)\Big)\Big)_{r\in I_a^{(\lambda)}} \tag{6.95}$$

via our original definition in equation (6.87). Equation (6.13) then follows from (6.11) and the fact that

$$\Big(\lambda^{-\frac{1}{3}}\mathcal{F}^{(\lambda)}(r)\Big)_{r\in I_a^{(\lambda)}} = \Big(\mathcal{F}(r)\Big)_{r\in I_a}.$$

We now turn our attention to establishing the decomposition in law in equation (6.14). Given the fact that $L(r)$ is a linear function, we may write

$$\Big(L(r)\Big)_{r\in[-3T,3T]} = \Big(\frac{1}{6T}\Big(L(3T) - L(-3T)\Big)r + \frac{1}{2}\Big(L(-3T) + L(3T)\Big)\Big)_{r\in[-3T,3T]}.$$

Similarly, equation (6.92) can be decomposed and rewritten in law as

$$\Big(B(r)\Big)_{r\in[-3T,3T]} \stackrel{d}{=} \Big(\mathcal{W}(2r+6T) - \frac{1}{2}\mathcal{W}(12T) - \frac{1}{6T}\mathcal{W}(12T)r\Big)_{r\in[-3T,3T]}.$$

As an immediate consequence of the above, we also have that

$$\left(\lambda^{\frac{1}{3}} B\left(\delta\lambda^{-\frac{2}{3}}\right)\right)_{\delta\in I_0^{(\lambda)}} \stackrel{d}{=} \left(\mathcal{W}\left(2\delta + 6T\lambda^{\frac{2}{3}}\right) - \lambda^{\frac{1}{3}}\frac{1}{2}\mathcal{W}(12T)\lambda^{-\frac{1}{3}}\frac{\mathcal{W}(12T)}{6T}\delta\right)_{\delta\in[-3,3]}$$

where the first term was simplified by Brownian scaling. Using these decompositions and the explicit definition of $\left(\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}$ in (6.95), we have the equalities in distribution

$$\begin{aligned}\left(\mathcal{F}^{(\lambda)}\left(a\lambda^{2/3}+\delta\right)\right)_{\delta\in I_0^{(\lambda)}} &= \left(\lambda^{\frac{1}{3}}B\left(\delta\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}\sqrt{T}N + \lambda^{\frac{1}{3}}L\left(\delta\lambda^{-\frac{2}{3}}\right)\right)_{\delta\in I_0^{(\lambda)}} \\ &\stackrel{d}{=} \left(\mathcal{W}\left(2\delta+6T\lambda^{\frac{2}{3}}\right) + \lambda^{-\frac{1}{3}}A\delta + \lambda^{\frac{1}{3}}C\right)_{\delta\in I_0^{(\lambda)}}\end{aligned}$$

where we have defined the random constants A and C as

$$\begin{aligned}A &:= \frac{1}{6T}\Big(\mathcal{W}(12T) + L(3T) - L(-3T)\Big) \\ C &:= \frac{1}{2}\Big(-\mathcal{W}(12T) + L(-3T) + L(3T)\Big) - \sqrt{2T}N.\end{aligned}$$

Observing that the triangle inequality gives us the two upper bounds

$$\begin{aligned}|A| &\leqslant \frac{1}{6T}|\mathcal{W}(12T)| + \frac{1}{6T}\left|L(3T) - L(-3T)\right| \\ |C| &\leqslant \frac{1}{2}|\mathcal{W}(12T)| + \frac{1}{2}\left|L(3T)\right| + \frac{1}{2}\left|L(-3T)\right| + \sqrt{2T}|N|,\end{aligned}$$

the tail bounds in equation (6.15) follow immediately from the standard Gaussian concentration inequality and equation (1.7) of Theorem 8, after possibly redefining our original choice of the $T-$dependent constants $c_1, c_2 > 0$. This completes our proof. □

6.3.2 Proof of Lemma 28

We now move on to the proof of Lemma 28. First, we state the absolute continuity in disjoint intervals of the Airy line ensemble in which we base our arguments.

Theorem 29 (Dauvergne, in [14])**.** *Fix $T_0 \geqslant 1$ and $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$ such that $a_1 + T_0 < a_2$. Then, there exists an absolute constant $c > 0$ and a random process $\left(\mathfrak{L}_1^{\vec{a}}(r)\right)_{r\in\mathbb{R}}$ such that*

$$\begin{aligned}&\text{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in[a_1,a_1+T_0]},\ \left(\mathfrak{A}_1(r)\right)_{r\in[a_2,a_2+T_0]}\right) \\ &\qquad\leqslant e^{cT_0^3}\,\text{Law}\left(\left(\mathfrak{L}_1^{\vec{a}}(r)\right)_{r\in[a_1,a_1+T_0]}, \left(\mathfrak{L}_1^{\vec{a}}(r)\right)_{r\in[a_2,a_2+T_0]}\right).\end{aligned}$$

Moreover, for each $j \in \{1,2\}$ we can write

$$\left(\mathfrak{L}_j^{\vec{a}}(r)\right)_{r\in[a_j,a_j+T_0]} \stackrel{d}{=} \left(B_j(r)+L_j(r)\right)_{r\in[a_j,a_j+T_0]}$$

where B_j is a diffusion parameter 2 Brownian bridge from 0 to 0 on $[a_j, a_j+T_0]$, B_1 is independent of B_2, (B_1, B_2) is independent of (L_1, L_2) and the linear terms L_1 and L_2 can be decomposed in law as

$$\Big(L_j(r)\Big)_{r\in[a_j,a_j+T_0]} \stackrel{d}{=} \left(\frac{(a_j+T_0)-r}{T_0}\mathfrak{L}_1^{\vec{a}}(a_j) + \frac{r-a_j}{T_0}\mathfrak{L}_1^{\vec{a}}(a_j+T_0)\right)_{r\in[a_j,a_j+T_0]}. \quad (6.96)$$

Moreover, for some T_0-dependent constants $c_1, c_2 > 0$ we have that for all $m > 0$,

$$\begin{aligned} \mathbb{P}\Big(|L_j(a_j)+a_j^2| > m\Big) &= \mathbb{P}\Big(|\mathfrak{L}(a_j)+a_j^2| > m\Big) \leqslant c_1 e^{-c_2 m^{3/2}} \\ \mathbb{P}\Big(|L_i(a_j+T_0)+(a_j+T_0)^2| > m\Big) &= \mathbb{P}\Big(|\mathfrak{L}(a_j+T_0)+(a_j+T_0)^2| > m\Big) \leqslant c_1 e^{-c_2 m^{3/2}}. \end{aligned} \quad (6.97)$$

Now we proof the lemma.

Proof of Lemma 28. By invoking Theorem 29 with $T_0 = 6T$ and $\vec{a} = (a_1-3T, a_2-3T)$, we have that

$$\begin{aligned} &\mathrm{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in[a_1\pm 3T]}, \Big(\mathfrak{A}_1(r)\Big)_{r\in[a_2\pm 3T]}\right) \\ &\quad \leqslant e^{216cT^3}\,\mathrm{Law}\left(\Big(B_1(r)+L_1(r)\Big)_{r\in[a_1\pm 3T]}, \Big(B_2(r)+L_2(r)\Big)_{r\in[a_2\pm 3T]}\right). \end{aligned}$$

Since I_j is the middle third of the interval $[a_j \pm 3T]$, we may invoke Lemma 17 with $k = 1$ and $\delta = \frac{1}{3}$ on $[a_j \pm 3T]$ for each $j \in \{1,2\}$ to get the decomposition in law

$$\Big(B_j(r)\Big)_{r\in I_j} \stackrel{d}{=} \sqrt{T}N_j + \Big(B_j(r) - \sqrt{2T}N_j\Big)_{r\in I_j}$$

where N_j is a standard Gaussian, N_j is independent of the process $(B_j(r) - N_j)_{r\in I_j}$, and N_1 is independent of N_2. As such, we can now write that

$$\begin{aligned} &\mathrm{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in I_1}, \Big(\mathfrak{A}_1(r)\Big)_{r\in I_2}\right) \\ &\leqslant\ e^{216cT^3}\,\mathrm{Law}\left(\Big(\sqrt{2T}N_1 + (\mathcal{F}_1(r)+f_{a_1}(r)\Big)_{r\in I_1}, \Big(\sqrt{2T}N_2 + (\mathcal{F}_2(r)+f_{a_2}(r)\Big)_{r\in I_2}\right) \end{aligned}$$

where for each $j \in \{1, 2\}$ we have defined

$$\Big(\mathcal{F}_j(r)\Big)_{r \in I_j} := \Big(L_j(r) - f_{a_j}(r) + B_j(r) - \sqrt{2T}N_j\Big)_{r \in I_j}$$

and for $r \in [a_j \pm 3T]$, we haved defined f_{a_j} by

$$f_{a_j}(r) = -\frac{(a_j + 3T) - r}{6T}(a_j - 3T)^2 - \frac{r - (a_j - 3T)}{6T}(a_j + 3T)^2.$$

This establishes (6.47) so all that remains is to establish (6.48). To that end, we employ the same general argument used in Lemma 24 previously, independently in each coordinate of (6.48).

We begin by observing the chain of inequalities

$$\sup_{r \in I_j} |\mathcal{F}_j(r)| \leqslant \sup_{r \in I_j} |L_j(r) - f_{a_j}(r)| + \sup_{r \in I_j} |B_j(r)| + |\sqrt{2T}N_j|. \tag{6.98}$$

Notice that because $L_j - f_{a_j}$ is a (random) line segment, its maximum absolute value is obtained at one of its two endpoints. So we have that

$$\begin{aligned} \sup_{r \in I_j} |L_j(r) - f_{a_j}(r)| \leqslant & |L_j(a_j - 3T) - f_{a_j}(a_j - 3T)| \vee |L_j(a_j + 3T) - f_{a_j}(a_i + 3T)| \\ = & |L_j(a_j - 3T) + (a_j - 3T)^2| \vee |L_j(a_j + 3T) + (a_j + 3T)^2|, \end{aligned} \tag{6.99}$$

where in the last equality we have used the definition of f_{a_j}. We will now adopt the convention that for any $a \in \mathbb{R}$, $B_{a,6T}$ is a diffusion parameter 2 Brownian bridge on $[a, a + 6T]$ from 0 to 0. With this convention, we may write that

$$\Big(B_{0,6T}(r)\Big)_{r \in [0,6T]} \stackrel{d}{=} \Big(\mathcal{W}(2r) - \frac{r}{6T}\mathcal{W}(12T)\Big)_{r \in [0,6T]}$$

where $\mathcal{W}$ is a standard Brownian motion. Given this, we may then say that

$$\begin{aligned}
\mathbb{P}\left(\sup_{r\in I_j}|B_j(r)|>2m\right) &= \mathbb{P}\left(\sup_{r\in[2T,4T]}|B_{0,6T}(r)|>2m\right)\\
&\leqslant \mathbb{P}\left(\sup_{r\in[0,6T]}|B_{0,6T}(r)|>2m\right)\\
&= \mathbb{P}\left(\sup_{r\in[0,6T]}\left|\mathcal{W}(2r)-\frac{r}{6T}\mathcal{W}(12T)\right|>2m\right)\\
&\leqslant \mathbb{P}\left(|\mathcal{W}(12T)|+\sup_{r\in[0,6T]}|\mathcal{W}(2r)|>2m\right)\\
&\leqslant \mathbb{P}\Big(|\mathcal{W}(12T)|>m\Big)+\mathbb{P}\left(\sup_{r\in[0,12T]}|\mathcal{W}(r)|>m\right)\\
&\leqslant \mathbb{P}\Big(|\mathcal{W}(12T)|>m\Big)+2\mathbb{P}\left(\sup_{r\in[0,12T]}\mathcal{W}(r)>m\right)\\
&= \mathbb{P}\Big(|\mathcal{W}(12T)|>m)+2\mathbb{P}(|\mathcal{W}(12T)|>m\Big)\\
&= 3\mathbb{P}\Big(|\mathcal{W}(12T)|>m\Big)
\end{aligned}$$

using that W is equal in law to $-W$, and the known distribution of the running maximum of a standard Brownian motion.

We may use this elementary bound in conjunction with (6.98) and (6.99) to obtain the union bound

$$\begin{aligned}
&\mathbb{P}\left(\sup_{r\in I_j}|\mathcal{F}_i(r)|>4m\right)\\
&\quad\leqslant\ \mathbb{P}\Big(|L_j(a_j-3T)+(a_j-3T)^2|\vee|L_j(a_i+3T)+(a_j+3T)^2|>m\Big)\\
&\qquad\qquad+\mathbb{P}\left(\sup_{r\in I_j}|B_j(r)|>2m\right)+\mathbb{P}\Big(|\sqrt{2T}N_j|>m\Big)\\
&\quad\leqslant\ \mathbb{P}\Big(|L_j(a_j-3T)+(a_j-3T)^2|\vee|L_j(a_j+3T)+(a_j+3T)^2|>m\Big)\\
&\qquad\qquad+3\mathbb{P}\Big(|\mathcal{W}(12T)|>m\Big)+P\Big(|\sqrt{2T}N_j|>m\Big).
\end{aligned}$$

Using the standard sub-Gaussian concentration inequalities for the latter two summmands, and the tail bounds in equation (6.97) for the first summand above yields

$$\mathbb{P}\left(\sup_{r\in I_j}|\mathcal{F}_j(r)|>4m\right)\leqslant c_1e^{-c_2m^{\frac{3}{2}}}+6e^{-\frac{m^2}{2(12T)^2}}+2e^{-\frac{m^2}{2(2T)^2}}\leqslant c_1e^{-c_2m^{\frac{3}{2}}}$$

by redefining the original choice of c_1 and c_2 as needed, thus establishing (6.48) and completing the proof of the base case.

Equations (6.49) and (6.50) are immediate consequences of (6.47) and (6.48), respectively. To see this explicitly, we need only observe that (6.47) gives us the chain of equalities

$$\begin{aligned}
&\mathrm{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_1^{(\lambda)}}, \left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_2^{(\lambda)}} \right) \\
&\quad = \mathrm{Law}\left(\left(\lambda^{1/3}\mathfrak{A}_1(r\lambda^{-2/3})\right)_{r\in\lambda^{2/3}I_1}, \left(\lambda^{1/3}\mathfrak{A}_1(r\lambda^{-2/3})\right)_{r\in\lambda^{2/3}I_2}\right) \\
&\quad = \mathrm{Law}\left(\left(\lambda^{1/3}\mathfrak{A}_1(r)\right)_{r\in I_1}, \left(\lambda^{1/3}\mathfrak{A}_1(r)\right)_{r\in I_2}\right) \\
&\quad \leqslant e^{216cT^3}\,\mathrm{Law}\left(\left(\lambda^{1/3}\left(\sqrt{2T}N_j + (\mathcal{F}_j(r) + f_{a_j}(r))\right)_{r\in I_j}\right)_{j=1}^2\right) \\
&\quad = e^{216cT^3}\,\mathrm{Law}\left(\left(\lambda^{1/3}\left(\sqrt{2T}N_j + \left(\mathcal{F}_j(r\lambda^{-2/3}) + f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^2\right) \\
&\quad = e^{216cT^3}\,\mathrm{Law}\left(\left(\left(\mathcal{F}_j^{(\lambda)} + \lambda^{1/3}\left(\sqrt{2T}N_j + f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^2\right)
\end{aligned}$$

where we have that $\mathcal{F}_j^{(\lambda)}$ is defined for each $j \in \{1,2\}$ by

$$\begin{aligned}
\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} &:= \left(\lambda^{\frac{1}{3}}\mathcal{F}_j(r\lambda^{-\frac{2}{3}})\right)_{r\in I_j^{(\lambda)}} \\
&= \left(\lambda^{\frac{1}{3}}L_j(r\lambda^{-\frac{2}{3}}) - \lambda^{\frac{1}{3}}f_{a_j}(r\lambda^{-\frac{2}{3}}) + \lambda^{\frac{1}{3}}B_j(r\lambda^{-\frac{2}{3}}) - \lambda^{\frac{1}{3}}\sqrt{2T}N_j\right)_{r\in I_j^{(\lambda)}}.
\end{aligned} \tag{6.100}$$

All the claimed independence properties of the decomposition (6.100) are inherited from the base case of this proof. Establishing the tail bound (6.50) follows immediately from (6.48) and the fact that

$$\left(\mathcal{F}_j(r)\right)_{r\in I_j} = \left(\lambda^{-1/3}F_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}}.$$

We now provide a decomposition of the functions $\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}}$ which will enable us to establish the tail bounds (6.52). By invoking the decomposition in equation (6.96), we obtain that

$$\left(L_j(r)\right)_{r\in[a_j\pm 3T]} \overset{d}{=} \left(\frac{(a_j+3T)-r}{6T}\mathfrak{L}_1^{\vec{a}}(a_j-3T) + \frac{r-(a_j-3T)}{6T}\mathfrak{L}_1^{\vec{a}}(a_j+3T)\right)_{r\in[a_j\pm 3T]}. \tag{6.101}$$

We may write for each $r \in [a_j \pm 3T]$ that

$$\begin{aligned} f_{a_j}(r) &= -\frac{(a_j+3T)^2-(a_j-3T)^2}{6T}r - \frac{(a_j+3T)(a_j-3T)^2-(a_j-3T)(a_j+3T)^2}{6T} \\ &= -(2a_j)r + \left(a_j^2 - 9T^2\right). \end{aligned} \tag{6.102}$$

We begin with the definition in (6.100), which gives us for each $r \in I_j^{(\lambda)} = \left[a_j\lambda^{2/3} \pm 3T\lambda^{2/3}\right]$ the decomposition in law

$$\begin{aligned} \mathcal{F}_j^{(\lambda)}(r) &= \lambda^{\frac{1}{3}}\mathcal{F}_j\left(r\lambda^{-2/3}\right) \\ &= \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-2/3}\right) + \lambda^{\frac{1}{3}}B_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}\sqrt{T}N_j \\ &\overset{d}{=} \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-2/3}\right) + \lambda^{\frac{1}{3}}B_{0,6T}\left(r\lambda^{-2/3} - (a_j-3T)\right) - \lambda^{\frac{1}{3}}\sqrt{T}N_j \end{aligned} \tag{6.103}$$

where as before, $B_{a,6T}$ is a diffusion parameter 2 Brownian bridge on $[a, a+6T]$ from 0 to 0. Noting that for any $a, k \in \mathbb{R}$ the scaling properties of Brownian bridges give us that

$$\begin{aligned} \left(k^{-1}B_{0,6T}(k^2r - k^2a)\right)_{r\in[a,a+6T]} &\overset{d}{=} \left(k^{-1}B_{k^2a,6k^2T}(r-k^2a)\right)_{r\in[k^2a,k^2(a+6T)]} \\ &\overset{d}{=} \left(k^{-1}B_{0,6k^2T}(r)\right)_{r\in[0,k^2(6T))]} \\ &\overset{d}{=} \left(\mathcal{W}(2r) - \frac{k^{-2}r}{6T}\mathcal{W}(12k^2T)\right)_{r\in[0,k^2(6T))]} \\ &\overset{d}{=} \left(\mathcal{W}(2r-2k^2a) - \frac{k^{-2}r-a}{6T}\mathcal{W}(12k^2T)\right)_{r\in[k^2a,k^2(a+6T)]} \\ &\overset{d}{=} \left(\mathcal{W}(2r-2k^2a) - \frac{k^{-1}r-ka}{6T}\mathcal{W}(12T)\right)_{r\in[k^2a,k^2(a+6T)]}. \end{aligned}$$

Therefore, we will adopt the convention that $\mathcal{W}_1$ and $\mathcal{W}_2$ are independent standard Brownian motions associated with the independent Brownian bridges B_1 and B_2 respectively such that for all $r \in I_j^{(\lambda)} = \left[\lambda^{\frac{2}{3}}a_j \pm \lambda^{\frac{2}{3}}T\right]$,

$$\lambda^{\frac{1}{3}}B_j\left(r\lambda^{-2/3}\right) \overset{d}{=} \mathcal{W}_j\left(2r - (2a_j - 6T)\lambda^{\frac{2}{3}}\right) - \frac{r\lambda^{-\frac{1}{3}} - (a_j-3T)\lambda^{\frac{1}{3}}}{6T}\mathcal{W}_j(12T). \tag{6.104}$$

Now by using (6.104) in conjunction with (6.101) and (6.102) we can obtain the decomposition in law

$$\left(F_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} \overset{d}{=} \left(\mathcal{W}_j\left(2r-(2a_j-6T)\lambda^{\frac{2}{3}}\right) + A_jr\lambda^{-1/3} + C_j\lambda^{1/3}\right)_{r\in I_j^{(\lambda)}}$$

where the random constants A_j, and C_j are defined as

$$A_j := \left(\frac{(\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2)-(\mathfrak{L}_1^{\vec{a}}(a_j)-a_j^2)}{3T} - \frac{1}{6T}\mathcal{W}_j(12T)\right)$$

$$C_j := \left(\frac{(a_j+3T)(\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2)-(a_j-3T)(\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2)}{6T}\right) + \frac{a_j-3T}{6T}\mathcal{W}_j(12T) - \sqrt{2T}N_j.$$

We now establish tail bounds for the random constants A_j and C_j. First,

$$\mathbb{P}\Big(|A_j| > m\Big) \leqslant \mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2}{6T}\right| + \left|\frac{\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2}{6T}\right| + \left|\frac{1}{6T}\mathcal{W}_j(12T)\right| > m\right). \tag{6.105}$$

Now, using the tail bounds in (6.97), we see that for all $m > 0$,

$$\mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2}{6T}\right| > \frac{m}{3}\right) \leqslant \mathbb{P}\Big(\left|\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2\right| > 2mT\Big) \leqslant c_1 e^{-c_2(2mT)^{\frac{3}{2}}} \tag{6.106}$$

and similarly,

$$\mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2}{6T}\right| > \frac{m}{3}\right) \leqslant \mathbb{P}\Big(\left|\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2\right| > 2mT\Big) \leqslant c_1 e^{-c_2(2mT)^{\frac{3}{2}}} \tag{6.107}$$

Using the standard Gaussian tail bounds, we get that

$$\mathbb{P}\Big(\left|\frac{1}{6T}\mathcal{W}_j(12T)\right| > \frac{m}{3}\Big) \leqslant \mathbb{P}\Big(|\mathcal{W}_j(12T)| > 2mT\Big) \leqslant 2e^{-\frac{(2mT)^2}{2(12T)}} \tag{6.108}$$

Using equations (6.106), (6.107) and (6.108) and a simple union bound, we get that

$$\mathbb{P}\Big(|A_j| > m\Big) \leqslant c_1 e^{-c_2(2mT)^{\frac{3}{2}}} + 2e^{-\frac{(2mT)^2}{2(12T)}} \leqslant c_1 e^{-c_2 m^{\frac{3}{2}}}$$

where the $T-$dependent constants c_1, c_2 have been redefined as needed.

Similarly, for the random constant C_j we may obtain the tail bound

$$\begin{aligned}
\mathbb{P}\Big(|C_j| > m\Big) \leqslant\ & \mathbb{P}\left(\left|\frac{(a_j+3T)(\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2)}{6T}\right| > \frac{m}{4}\right) \\
& + \mathbb{P}\left(\left|\frac{(a_j-3T)(\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2)}{6T}\right| > \frac{m}{4}\right) \\
& + \mathbb{P}\left(\left|\frac{(a_j-3T)\mathcal{W}_j(12T)}{6T}\right| > \frac{m}{4}\right) + \mathbb{P}\left(\left|\sqrt{2T}N_j\right| > \frac{m}{4}\right) \\
=\ & \mathbb{P}\left(\left|\mathfrak{L}_1^{\vec{a}}(a_j-3T)-(a_j-3T)^2)\right| > \frac{3mT}{2|a_j+3T|}\right) \\
& + \mathbb{P}\left(\left|\mathfrak{L}_1^{\vec{a}}(a_j+3T)-(a_j+3T)^2)\right| > \frac{3mT}{2|a_j-3T|}\right) \\
& + \mathbb{P}\left(|\mathcal{W}_j(12T)| > \frac{3mT}{2|a_j-3T|}\right) + \mathbb{P}\left(|\sqrt{2T}N_j| > \frac{m}{4}\right) \\
\leqslant\ & c_1 e^{-c_2\left(\frac{3mT}{2|a_j+3T|}\right)^{\frac{3}{2}}} + c_1 e^{-c_2\left(\frac{3mT}{2|a_j-3T|}\right)^{\frac{3}{2}}} + 2e^{-\frac{1}{24T}\left(\frac{3mT}{2|a_j-3T|}\right)^2} + 2e^{-\frac{1}{4T}\left(\frac{m}{4}\right)^2} \\
\leqslant\ & c_1 e^{-c_2\left(\frac{m}{|a_j|}\right)^{\frac{3}{2}}}
\end{aligned}$$

where we have once again redefined the values of c_1 and c_2 so that the final inequality holds as well. This therefore establishes (6.52) and completes the proof of Lemma 28. □

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