Department of Mathematics
Comprehensive Exam 2023
First half: Monday, September 18, 2023 - 6:00 - 9:00 p.m.
BA6183

Last name ....

First name ....

Email ....

NO AIDS ALLOWED. Solve 6 questions out 12 questions. Indicate which questions you want to be graded. Overall passing score is 80 percent from those 6 questions (over 2 days). Each one of the 6 problems chosen has to get a minimum score of 70%. Do not attempt all problems; instead, aim for complete solutions.

1. **(10 pts)** Let $(X, \mathcal{M}, \mu)$ be a **finite** measure space.

(a) (3 points) Let $f \in L^1(\mu)$, show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) < \delta \implies \int_E |f| d\mu < \epsilon.$$

(b) (5 points) Let f_n be a sequence of measurable functions on $(X, \mathcal{M})$, we assume that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) < \delta \implies \sup_n \int_E |f_n| d\mu < \epsilon.$$

Prove that if f_n is a Cauchy sequence in measure, then f_n is a Cauchy sequence in L^1.

(c) (2 points) If f_n is Cauchy in measure and $\sup_n \int |f_n|^2 d\mu < \infty$, then f_n is a Cauchy sequence in L^1.

2. (10 pts) Let u, v in $C(\mathbb{R}^n \times [0,T]) \cap C^2(\mathbb{R}^n \times (0,T))$ be bounded solutions of the following heat equations on the whole space:

$$\begin{cases} u_t - \Delta u = F & \text{in} \quad \mathbb{R}^n \times (0,T), \\ v_t - \Delta v = 0 & \text{in} \quad \mathbb{R}^n \times (0,T), \end{cases} \tag{H}$$

with initial data $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ satisfying

$$\sup_{x\in\mathbb{R}^n} |u_0(x)| + \int_{\mathbb{R}^n} |u_0| \leq A, \qquad \sup_{x\in\mathbb{R}^n} |v_0(x)| + \int_{\mathbb{R}^n} |v_0| \leq B,$$

and given forcing term $F = F(x,t)$.

a) Prove that for some constant $C = C(n) > 0$, v satisfies the estimate

$$\sup_{x\in\mathbb{R}^n} |v(x,t)| \leq C_n \frac{B}{(1+t)^{n/2}}.$$

b) Prove that, for all $t \leq T$, one has

$$\|u(\cdot,t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|F(\cdot,s)\|_{L^2}\, ds,$$

provided the right-hand side is finite.

c) Consider the case when (H) is a system of coupled equations with

$$F(x,t) = v^3(x,t).$$

Determine in which dimension n one has

$$\|u(\cdot,t)\|_{L^2} \leq C = C(A,B).$$

3. (10 pts total) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial in one variable over $\mathbb{Q}$.

a) **(5 pts)** Prove that $F = \mathbb{Q}[x]/(f(x))$ is a field.

b) **(5 pts)** Prove that

$$\{1 \bmod f,\ x \bmod f,\ x^2 \bmod f,\ \ldots,\ x^{(\deg f)-1} \bmod f\}$$

is a basis for F over $\mathbb{Q}$, where $a(x) \bmod f(x)$ denotes the image of $a(x) \in \mathbb{Q}[x]$ under the projection map $\mathbb{Q}[x] \twoheadrightarrow F$.

(You are welcome to utilize the Long Division Algorithm in $\mathbb{Q}[x]$ without proof.)

4. (10 pts)

Let $v \in \Omega^2(S^2)$ satisfy $\int_{S^2} v = 1$, and let $f : S^3 \to S^2$ be a smooth map.

a) **(2 pts)** Prove that there exists $\xi \in \Omega^1(S^3)$ such that $d\xi = f^*v$.

b) **(2 pts)** Prove that if $d\xi' = f^*v$, then $\xi' = \xi + d\lambda$, for $\lambda \in C^\infty(S^3, \mathbb{R})$.

Define the Hopf invariant of f to be the following integral:

$$H(f) = \int_{S^3} \xi \wedge d\xi.$$

c) **(2 pts)** Prove that $H(f)$ does not depend on the choice of ξ such that $d\xi = f^*v$.

d) **(2 pts)**Prove that $H(f)$ is unchanged if v is replaced by $v' \in \Omega^2(S^2)$ which also satisfies $\int_{S^2} v' = 1$.

e) **(2 pts)**Let f_0, f_1 be smooth maps $S^3 \to S^2$. Prove that if f_0, f_1 are smoothly homotopic, then they have the same Hopf invariant.

5. (10 pts) Let X_n, $n \in \mathbb{N}$ be a sequence of independent exponential random variables of mean 1, and let $S_n = X_1 + X_2 + \ldots X_n$. For every positive a find

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{P}(S_n < a).$$

6. (10 pts) An $n \times n$ matrix with complex-valued entries, A, is Hermitian if $A^* = A$ where A^* is the matrix constructed from A via $(A^*)_{ij} = \overline{A_{ji}}$ for all $1 \leq i, j \leq n$. Hermitian matrices can be orthogonally diagonalized and their eigenvalues are real:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

a) Assume A is Hermitian. The Rayleigh quotient $R(A, x)$ is defined for a nonzero vector x via

$$R(A, x) = \frac{x^* A x}{x^* x}$$

Prove one of the two equalities:

$$\lambda_1 = \min_{x \neq 0} R(A, x) \qquad\qquad \lambda_n = \max_{x \neq 0} R(A, x).$$

They're both true but if you can prove one then you can prove the other with similar ideas.

b) From above, you can find λ_1 and λ_n by optimizing over all nonzero vectors. What about other eigenvalues? Find subspaces W_2 and V_2 so that

$$\lambda_2 = \min_{x \neq 0, x \in W_2} \frac{x^* A x}{x^* x} = \max_{x \neq 0, x \in V_2} \frac{x^* A x}{x^* x}$$

and prove that your choices work. What subspaces W_k and V_k would you use to find λ_k? (No proof requested.)

c) The previous part gives a pen-and-paper way of finding intermediate eigenvalues λ_i such that $\lambda_1 < \lambda_i < \lambda_n$. Is it practical? (Do you see any challenges in coding it up on a computer?)

d) Can you think of a promising way to find such intermediate eigenvalues using either a min-max or a max-min optimization? (Note: it will have different computational challenges, if you try to code it up on a computer. But if you come up with the one I'm hoping you come up with, it's an analytically super-powerful one.)