Department of Mathematics
Comprehensive Exam 2023
Second half: Tuesday, September 19, 2023 - 6:00-9:00 p.m.
BA6183

Last name ....

First name ....

Email ....

NO AIDS ALLOWED. Solve 6 questions out 12 questions. Indicate which questions you want to be graded. Overall passing score is 80 percent from those 6 questions. (over 2 days). Each one of the 6 problems chosen has to get a minimum score of 70%. Do not attempt all problems; instead, aim for complete solutions.

1. **(10 pts)**

a) **(1 pt)** Define what it means for a family of functions $\{\phi_t\}_{t>0}$ in $L^1(\mathbb{R})$ to be an approximate identity. Also, given any fixed $L^1(\mathbb{R})$ function ϕ with $\int \phi = 1$, there is a standard way to create an approximate identity from it. Explain what this standard way is and prove that it indeed yields an approximate identity.

b) **(1 pt)** Assume $\{\phi_t\}_{t>0}$ is an approximate identity, and $f \in L^p(\mathbb{R})$. Prove that if $1 \leq p < \infty$, then $\phi_t * f \to f$ in $L^p(\mathbb{R})$, as $t \to 0$.

c) **(2 pts)** Assume that $f \in C_0(\mathbb{R})$ (the space of continuous functions which vanish at infinity, i.e. such that for any $\varepsilon > 0$, there exists a compact $K \subset \mathbb{R}$ such that $|f(x)| < \varepsilon$ for $x \notin K$). Prove that $\phi_t * f \to f$ uniformly, as $t \to 0$.

d) **(3 pts)** Assume that $f \in L^\infty(\mathbb{R})$. Prove that $\phi_t * f \to f$ in the weak-$*$ topology of $L^\infty(\mathbb{R})$, as $t \to 0$.

e) **(3 pts)** Prove that there is no identity for convolution in $L^1(\mathbb{R})$, i.e. there is no function $e \in L^1(\mathbb{R})$ such that, for all $f \in L^1(\mathbb{R})$, $f * e = f$.

2. **(10 pts)**

a) **(2 pts)** Find the Riemann surface X over the Riemann sphere S^2 of the multivalued function

$$w = \sqrt[3]{(z-a)(z-b)(z-c)},$$

where a, b, c are distinct complex numbers.

b) **(1 pt)** How many sheets does X have?

c) **(1 pt)** Where are the branch points of X? (Be sure to check infinity.)

d) **(2 pts)** How are the sheets permuted as we go around simple closed curves in $\mathbb{C}$ which avoid (the images of) the branch points?

e) **(2 pts)** What are the possible values of

$$\int_\delta \frac{dz}{\sqrt[3]{(z-a)(z-b)(z-c)}},$$

where δ is a simple closed curve in X lying over a positively oriented circle γ in $\mathbb{C}$ enclosing the points a, b, c.

f) **(2 pts)** What changes (or remains the same) in (a)–(e) above if two or all of the points a, b, c coincide?

3. **(10 pts)** Consider the initial value problem associated to the linearized Korteweg-de Vries equation

$$\begin{aligned}&\partial_t u + \partial_{xxx} u = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+\\&u(x,0) = u_0(x)\end{aligned}$$

Here $u(x,t)$ is real-valued. Assume $u \in \mathbb{S}(\mathbb{R})$ (i.e. of Schwartz class).

(a) Write the differential equation satisfied by $\hat{u}(k,t)$, the Fourier transform of u (in x), and solve it.

(b) Write u in the form of a convolution of u_0 with a kernel $T(x,t)$.

(c) Write the kernel $T(x,t)$ in terms of the Airy function

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(kx+\frac{k^3}{3})}dk.$$

(d) Consider the Korteweg-de Vries equation

$$\begin{aligned}\partial_t u + \partial_x\Big(u_{xx} + \frac{u^2}{2}\Big) = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+\\u(x,0) = u_0(x)\end{aligned}$$

Here again, $u(x,t)$ is real-valued.
Assume the solution $u \in \mathbb{S}(\mathbb{R})$ (the Schwartz space), prove that

$$I_1(t) = \int_{-\infty}^{\infty} u(x,t)dx$$

$$I_2(t) = \int_{-\infty}^{\infty} u^2(x,t)dx$$

$$I_3(t) = \int_{-\infty}^{\infty} \Big((\partial_x u)^2 - \frac{1}{3}u^3\Big)(x,t)dx$$

are conserved quantities, i.e. they are independent of t.

4. **(10 pts)**

Consider a group G which has the following character table over $\mathbb{C}$, where A, B, C, D, E, F denote the conjugacy classes of G.

	A	B	C	D	E	F
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	$*$	1
χ_3	1	-1	1	-1	1	-1
χ_4	1	1	1	-1	-1	-1
χ_5	2	0	-1	2	0	-1
χ_6	2	0	-1	$*$	0	1

a) **(4 pts)** Determine the unknown entries $*$ and the order of G.

b) **(2 pts)** Find the size of each conjugacy class A, B, C, D, E, F.

c) **(4 pts)** Show that G has a normal subgroup of order 6 and a normal subgroup of order 3. (Hint: it may help to remember that 1-dimensional representations of G are the same as homomorphisms $G \to \mathbb{C}^{\times}$.)

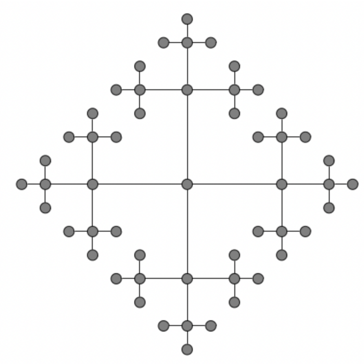
5. **(10 pts)**

a) **(3 pts)** For SY the suspension of Y, show by a Mayer-Vietoris sequence that there are isomorphisms $\tilde{H}_n(SX)$ and $\tilde{H}_n(X)$ for all n.

b) **(2 pts)** For a pair (X, A), let $X \cup CA$ be X with a cone on A attached. Show that if A is contractible in X then X is a retract of $X \cup CA$.

c) **(5 pts)** Use the fact that $(X \cup CA)/X$ is the suspension SA of A to show that if A is contractible in X then $H_n(X, A)$ is isomorphic to $\tilde{H}_n(X) \oplus \tilde{H}_{n-1}(A)$.

6. **(10 pts)** Let G be the infinite 4-regular graph that is connected and has no loops. A picture of part of this graph is:



Let X_n be the random walk on G, starting at a fixed vertex $v_0 \in G$. That is, at every time step, X_n jumps to one of its four adjacent vertices with equal probability.

a) Let Y_n be the distance of X_n from v_0 at time n. Prove that,

$$\lim_{n \to \infty} \frac{Y_n}{n} = \frac{1}{2}$$

b) Show that the random walk X_n is transient (that is, almost surely, it visits any given vertex $v \in G$ only finitely many times).