Approach to Equilibrium in Markovian Open Quantum Systems

by

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Abstract

Markovian open quantum systems are of great importance in quantum statistical mechanics and quantum information theory, whose dynamics are governed by the von Neumann-Lindblad equation (or the quantum master equation), obtained by integrating over the environmental degrees of freedom. In this thesis, we study the evolution of such systems and, in particular, their asymptotic behaviors. We will show the return-to-equilibrium for such systems under the quantum detailed balance condition. The major techniques employed here are spectral theory and the theory of C^*-algebras.

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Contents

Chapter 1

Introduction

1.1 Von Neumann-Lindblad equation

Consider a quantum system described by a quantum Hamiltonian H on a Hilbert space $\mathcal{H}$. Assume this system interacts with a (quantum) environment and we trace/integrate out the environment's degrees of freedom. The resulting physical system is called the *open quantum system*. Its states are given by density operators ρ (i.e. positive, trace-class operators, $\rho = \rho^* \geq 0$, $\mathrm{Tr}(\rho) < \infty$) on $\mathcal{H}$, and its dynamic, $\beta_t : \rho_0 \mapsto \rho_t$, results from a unitary dynamics of the total system by tracing out the environment. Under the assumption that β_t is Markovian and uniformly continuous, $\beta_t(\rho_0)$ satisfies the *von Neumann-Lindblad (vNL) equation* (or the *quantum master equation*)

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \sum_{j \geq 1} \left(W_j \rho W_j^* - \frac{1}{2}\{W_j^* W_j, \rho\} \right), \tag{1.1.1}$$

with the initial condition $\rho|_{t=0} = \rho_0$, where H is, as above, the quantum Hamiltonian of a proper quantum system, which is bounded in this case, and W_j are bounded operators on $\mathcal{H}$ such that $\sum_j W_j^* W_j$ converges weakly. (Here and in what follows we use the units in which the Planck constant is set to 2π and speed of light, to one: $\hbar = 1$ and $c = 1$.) The converse statement is also true. Moreover, it is shown in [28] that for H self-adjoint and W_j that is H-bounded for each j, the operator on the r.h.s. of (1.1.1) generates a one-parameter, trace-preserving, positive contraction semigroup β_t. It is also shown in [27, 57] and in Appendix A below the semigroup β_t is completely positive, which, according to [51, 52], must be of the form

$$\beta_t(\rho) = \sum_{k \geq 1} V_k(t) \rho V_k(t)^*, \quad \text{with} \sum_{k \geq 1} V_k^*(t) V_k(t) = \mathbf{1}, \tag{1.1.2}$$

for some family of bounded operators $\{V_k(t)\}$ on $\mathcal{H}$. Such semigroup defines a weak solution to (1.1.1) (in the sense explained below) on the Schatten space $\mathcal{S}_1$ of trace-class operators on $\mathcal{H}$.

The set of operators W_j, called *jump* or *Kraus-Lindblad operators*, is what is left over from the interaction with the environment. If we set them to zero, then the second term on the r.h.s. of (1.1.1) drops out and (1.1.1) reduces to the von Neumann equation

$$\frac{\partial \rho}{\partial t} = -i[H, \rho], \qquad \rho_{t=0} = \rho_0, \tag{1.1.3}$$

for the resulting closed system. The latter extends the Schrödinger equation to quantum statistics (see [43] for some definitions). Put differently, Markovian open quantum systems present an extension of quantum mechanics incorporating terms resulting from interaction with the environment.

Due to the presence of jump operators, the evolution described by (1.1.1) is generally dissipative, i.e., the energy of the system will decrease ("dissipating into environment"). Hence, in contrast to evolutions generated by the von Neumann equation (1.1.3), the evolutions generated by (1.1.1) are in general irreversible. The irreversiblity of solution generated by (1.1.1) can be demonstrated explicitly as the decreasing of relative entropy (or positive entropy production), which is strictly decreasing in general unless the Lindblad part disappears, i.e., the relative entropy is constant in time if the evolution is generated by (1.1.3), see [43, 55, 56, 65].

We will call the operators due to the two terms on the r.h.s. of (1.1.1) as the *von Neumann operator* and *Lindblad operator*, respectively.

It is assumed that the Markovian open quantum dynamics approximate open quantum dynamics coming from quantum systems weakly coupled to an environment. So far this is proven, under some technical conditions, for the simplest environment given by the free massless fermion quantum field and simplest interaction (linear in field) in the van Hove limit, see [24, 48, 49] for finite-dimensional systems and [25, 26], for infinite-dimensional ones.

In quantum computations, the vNL equation is also used for preparation of the Gibbs and ground states (a quantum version of Monte-Carlo method), see [18, 19, 22, 32, 49, 63, 69, 71]. For a given state ρ_*, it is shown that, under suitable conditions, we can construct jump operators W_j's such that, for any initial state ρ, the solution $\beta_t(\rho)$ converges to ρ_*. The study of converging rate of these convergence remains an active area of reseaching.

The vNL equation is closely related to the non-commutative analogue of the linear dissipative stochastic master equations in a separable Hilbert space

$\mathcal{H}$ (see [7, 17, 33, 47, 59, 67])

$$d\psi(t) = \sum_j W_j \psi(t) dw_j(t) - K\psi(t)dt. \tag{1.1.4}$$

Here $w_j(t)$, $j = 1, 2, ...$, are independent standard Wiener processes, $K = -iH - \frac{1}{2}\sum_{j\geq 1} W_j^* W_j$, with H and W_j's as given above. Let $\psi(t)$ be a solution to (1.1.4) with an initial condition $\psi_0 \in \mathcal{H}$ and $\mathbf{E}$ stands for expectation w.r.t. the Wiener measure. Under some technical conditions, the equation

$$\langle \psi_0, \beta_t'(A)\psi_0\rangle_{\mathcal{H}} = \mathbf{E}\langle \psi(t), A\psi(t)\rangle_{\mathcal{H}}, \quad t \geq 0, \tag{1.1.5}$$

defines the quantum dynamical semigroup β_t' on $\mathcal{B}$, which is dual to β_t (see [47], Section 3).

The principle of detailed balance (also known as the *micro-reversilibity condition*) is originated in the kinetic gas theory of the (classical) statistical mechanics, which essentially states, at the equilibrium, each elementary process (such as collision, elementary reaction, etc) and its time-reverse process are equally probable. The validity of detailed balance condition is essentially attributed to the invariance of the microscopic equation of motion under time-reversal transformation. In symbols, by denoting $p_i^{(\mathrm{eq})}$ as the equilibirum probability of the ith state and $w_{i\to j}$ is the transition rate (or transition probability) from the ith to the jth state, the detailed balance condition is formulated as

$$p_i^{(\mathrm{eq})} w_{i\to j} = p_j^{(\mathrm{eq})} w_{j\to i}. \tag{1.1.6}$$

Here $w_{j\to i}$ can be intepreted as the time-reversal of the process $w_{i\to j}$. For many system of physical and chemical kinetics, the detailed balance condition is sufficient (however, not necessary in general) for positive entropy production, such as Boltzman's H-theorem.

The detailed balance condition is naturally formulated for Markov processes in classical probability theory. Markov processes that obey detailed balance condition is called the reversible Markovian chain, which is due to the fact that, under detailed balance condition, around any closed cycle of state transition, the net flux of probability is zero. For example, if we have a closed cycle of transition of three states $i \to j \to k \to i$, then the transition probabilities satisfy

$$w_{i\to j} w_{j\to k} w_{k\to i} = w_{i\to k} w_{k\to j} w_{j\to i}. \tag{1.1.7}$$

For quantum Markovian systems (either closed or open), various extensions of the classical detailed balance condition were given. In this thesis, we adapt the general approach according to [2, 15, 16, 40, 58, 70] in terms of the generator of (1.1.1). The relation between our formulation of quantum detailed balance condition and time-reversal was considered in [3, 60]. A specific form of the generators of quantum dynamic semigroups under the quantum detailed balance condition was derived in [2, 15, 16, 24, 58] for a finite-dimensional $\mathcal{H}$. For the Hilbert spaces $\mathcal{H}$ with $\dim \mathcal{H} = \infty$, a formula for bounded Lindblad operator satisfying the quantum detailed balance condition was obtained in [40]. We also remark the quantum detailed balance condition is closely related to the theory of symmetric and KMS-symmetric Markovian semigroups on von Neumann algebras (see [1, 41] for definitions).

In this thesis, we are interested in the long time behaviour of solutions of the vNL equation on an infinite-dimensional Hilbert space $\mathcal{H}$, under the assumptions

(H) H is a self-adjoint operator on a Hilbert space $\mathcal{H}$;

(W) W_j are bounded operators on $\mathcal{H}$ such that the sum $\sum_{j\geq 1} W_j^* W_j$ converges weakly,

and, under the *quantum detailed balance condition* (QDB) (see Subsection 2.3 for the definition). We derive the results for β_t (vNL equation (1.1.1)) from our results for the dual dynamics β_t' on $\mathcal{B}$ defined by

$$\mathrm{Tr}(\beta_t'(A)\rho) = \mathrm{Tr}(A\beta_t(\rho)) \qquad \forall A \in \mathcal{B}, \quad \rho \in \mathcal{S}_1 \tag{1.1.8}$$

(see Subsections 2.1 and 2.2 below). One of our main results, Theorem 3.1.3, formulated in Section 3 (and below), establishes the ergodic convergence of β_t' to its static solutions.

To state this theorem here, we define, for a given density operator $\rho_* > 0$, the Hilbert space $\mathcal{B}_*$ as the completion of the space $\mathcal{B}$ of bounded operators on $\mathcal{H}$ in the norm corresponding to the inner product

$$\langle A, B\rangle_{\mathrm{obs},*} := \mathrm{Tr}(A^* B\rho_*). \tag{1.1.9}$$

Theorem 1.1.1. *Assume Conditions* (H), (W) *and* (QDB), *with* $\rho_* = \rho_\beta$, *the Gibbs state at temperature* $1/\beta$ *(see* (2.3.4)*), hold.*

Then, the dual quantum evolution $\beta_t'(A)$ *converges, in the ergodic sense, to the subspace* $\mathcal{B}_{\mathrm{stat}} := \mathrm{Null}(G')$ *in* $\mathcal{B}_*$*:*

$$s\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T \beta_t' dt = P', \tag{1.1.10}$$

strongly in $\mathcal{B}_*$*, where* P' *is the orthogonal projection onto* $\mathcal{B}_{\mathrm{stat}}$.

This result shows the approach to stationary states for systems of infinite number of degrees of freedom. We expect it could be extended to unbounded operators W_j's.

As followed from (1.1.8) and the fact that β_t is trace-preserving, $\mathbf{1}$ is a stationary state β_t', i.e., $\beta_t'(\mathbf{1}) = \mathbf{1}$. It is shown in Proposition 4.5.1 below that the Spohn's conditions on the W_j's garanteeing the uniqueness of $\mathbf{1}$ also holds for the present context.

Theorem 1.1.1 can be also used to justify the application of the vNL equation for a state preparation. Indeed, given a quantum Hamiltonian H and the Gibbs state ρ_β, one constructs jump operators W_j satisfying (QDB) with $\rho_* = \rho_\beta$ and a uniqueness condition. Then, any solution of the vNL equation with these W_j's and H converges to ρ_β.

1.2 Remarks on related works

To our knowledge, the vNL equation (1.1.1) was first studied by Davies in [23, 24, 25, 26, 27] for bounded H and W_j's and in [28], for unbounded operators H and W_j's bounded relatively to H.

The existence theory for the related non-commutative stochastic equation (1.1.4) was developed in [47].

The fact that the generators of the norm continuous quantum dynamic semigroups are given by the r.h.s. of (1.1.1) was first derived in [57] for infinite dimensional $\mathcal{H}$, and in [42] for the finite dimensional $\mathcal{H}$.

A finite dimensional version of Theorem 1.1.1 was proven by Spohn in [64], who also provided an algebraic condition on the W_j's such that, under (QDB), the vNL equation (1.1.1) has the unique stationary solution ρ_{st}. Later, the same condition was obtained in [65] using the concept of entropy production. Under the same condition, it was shown by Frigerio in [38] that the solutions to (1.1.1) converge to ρ_{st} strongly as $t \to \infty$ for infinite dimensional $\mathcal{H}$ and bounded H and W_j's, provided that the corresponding dynamic semigroup β_t has a faithful normal stationary state and satisfies a "weak coupling" condition[1]. The latter says that β_t describes the reduced dynamics of a system coupled weakly with a reservoir in a KMS state.

1.3 Organization of the thesis

This thesis is organized as follows. In Section 2, we give some basic definitions, including that of the dual, Heisenberg-Lindblad (HL) equation, and formulate

[1]This condition says essentially that β_t is the reduced dynamics of a system coupled weakly to a reservoir initially in a KMS state.

the standard existence and uniqueness results, proved in Appendix A for the readers' convenience.

In Section 3, we state the result on existence and convergence to stationary states for the vNL and HL dynamics. These results are proved in Section 4 and Appendix A. The uniqueness of stationary solution under vNL dynamics is considered in Subsection 4.5 and a sufficient condition for which the von Neumann part and the Lindblad part commute is formulated in Subsection 4.2.1.

In Appendices B and C, we prove some technical results used in the main text.

Notation. Throught out this thesis, we fix the Hilbert space $\mathcal{H}$ and denote $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}$ whose norm $\|\cdot\|$ is given by the operator norm. The identity element in $\mathcal{B}$ is denoted by $\mathbf{1}$. The norm and inner product on $\mathcal{H}$ are denoted by $\|\cdot\|_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

We denote the p-th Schatten spaces on the Hilbert space $\mathcal{H}$ by $\mathcal{S}_p$, which is a complex Banach space w.r.t. the norm $\|\cdot\|_{\mathcal{S}_p}$ given by

$$\|\kappa\|_{\mathcal{S}_p} := (\operatorname{Tr}|\kappa|^p)^{1/p}, \qquad \text{where } |\kappa| := (\kappa^*\kappa)^{1/2}. \tag{1.3.1}$$

In particular, $\mathcal{S}_1, \mathcal{S}_2$ are spaces of trace-class operators and Hilbert-Schmidt (HS) operators on $\mathcal{H}$, respectively. The latter is a Hilbert space, equipped with the HS-inner product:

$$\langle\kappa,\sigma\rangle_{\mathcal{S}_2} := \operatorname{Tr}(\kappa^*\sigma). \tag{1.3.2}$$

Chapter 2

Problem and set-up

2.1 States and observables

We denote $\mathcal{S}_{1,+} \subset \mathcal{S}_1$ be the subset of positive elements in $\mathcal{S}_1$. Those elements in $\mathcal{S}_{1,+}$ with unit trace-norm are called *density operators*.

We will call elements in $\mathcal{B}$ as *observables* and positive, normalized, continuous linear functionals on $\mathcal{B}$ as *states*. Any density operator ρ defines a state on $\mathcal{B}$ by $A \mapsto \mathrm{Tr}(A\rho)$. Such states are known as *normal states* on $\mathcal{B}$.[1]

In addition to the von Neumann-Lindblad evolution, we also consider its dual evolution

$$\partial_t A_t = i[H, A_t] + \sum_{j\geq 1}(W_j^*[A_t, W_j] + [W_j^*, A_t]W_j) \tag{2.1.1}$$

on the space $\mathcal{B}$ of observables. We call (2.1.1) the *Heisenberg-Lindblad (HL) equation.*

2.2 von Neumann-Lindblad and Heisenberg-Lindblad operators

We write the vNL equation (1.1.1) as

$$\partial_t \rho_t = L\rho_t, \tag{2.2.1}$$

[1] One may regard $\mathcal{S}_1$ as the non-commutative analogue of $L^1(\Omega)$ or the space of complex measures on Ω, where Ω is some measure space. In this way, states can be regarded as probability measures. Similarly, the Schatten spaces $\mathcal{S}_p$ can be regarded as the non-commutative analogue of $L^p(\Omega)$. Following this argument, we may also regard $\mathcal{B}$ as $L^\infty(\Omega)$.

where the operator L, called the *vNL operator*, is defined as

$$L = L_0 + G, \qquad L_0\rho := -i[H, \rho], \tag{2.2.2}$$

$$\begin{aligned} G(\rho) &= \frac{1}{2}\sum_{j\in J}\left([W_j, \rho W_j^*] + [W_j\rho, W_j^*]\right) \\ &= \sum_{j\geq 1}\left(W_j\rho W_j^* - \frac{1}{2}\{W_j^*W_j, \rho\}\right), \end{aligned} \tag{2.2.3}$$

where $\{A, B\} := AB + BA$. Since, by Proposition A.2.2, G is bounded on $\mathcal{S}_1$, the domain of L is given by (see [27])

$$\begin{aligned} \mathscr{D}(L) = \mathscr{D}(L_0) := \{\rho \in \mathcal{S}_1 \mid \rho(\mathscr{D}(H)) \subseteq \mathscr{D}(H) \text{ and } \\ H\rho - \rho H \text{ defined on } \mathscr{D}(H) \text{ extends to} \\ \text{an element in } \mathcal{S}_1\}. \end{aligned} \tag{2.2.4}$$

We will call the operator L_0 the *von Neumann (vN) operator* and G the *Lindblad operator*.

Let L' be the dual operator of L w.r.t. the coupling $(A, \rho) := \mathrm{Tr}(A\rho)$, i.e.

$$\mathrm{Tr}(AL\rho) = \mathrm{Tr}((L'A)\rho) \tag{2.2.5}$$

for all $\rho \in \mathscr{D}(L)$ and $A \in \mathscr{D}(L')$.

Similarly to (2.2.2), we have $L' = L_0' + G'$, where L_0' and G' are the dual operators to L_0 and G, w.r.t. the coupling $(A, \rho) = \mathrm{Tr}(A\rho)$. We have

$$L_0'(A) = i[H, A], \qquad (\text{abusing notation}: \; L_0'(A) = -L_0(A)), \tag{2.2.6}$$

and G' is given by

$$\begin{aligned} G'(A) &= \frac{1}{2}\sum_{j}\left(W_j^*[A, W_j] + [W_j^*, A]W_j\right) \\ &= \sum_{j\geq 1}\left(W_j^*AW_j - \frac{1}{2}\{W_j^*W_j, A\}\right). \end{aligned} \tag{2.2.7}$$

In terms of the dual operator L', the HL equation (2.1.1) can be written as

$$\partial_t A_t = L'A_t. \tag{2.2.8}$$

We say that ρ_t is a *weak solution*[2] of (1.1.1) in $\mathcal{S}_1$ if, for any observable A in $\mathscr{D}(L')$, ρ_t satisfies

$$\partial_t \operatorname{Tr}(A\rho_t) = \operatorname{Tr}((L'A)\rho_t). \tag{2.2.10}$$

If L generates a weakly continuous semigroup $\beta_t = e^{Lt}$ on the space $\mathcal{S}_1$, then, for every initial condition $\rho_0 \in \mathcal{S}_1$, Eq. (2.2.1) has a weak solution $\rho_t = \beta_t(\rho_0)$ and, for any $\rho_0 \in \mathscr{D}(L)$, a strong one. The same results hold also for the operator L'.

Theorem 2.2.1. *([23, 27, 28])*

(a) The HL operator L' generates a weakly continuous, completely positive bounded semigroup and therefore (2.1.1) *has a unique weak solution for any initial condition in $\mathcal{B}$ and a unique strong solution for any initial condition in $\mathscr{D}(L')$.*

(b) The vNL operator L generates a weakly continuous, completely positive, bounded semigroup, β_t, on $\mathcal{S}_1$ and therefore (1.1.1) *has a unique weak solution for any initial condition in $\mathcal{S}_1$ and a unique strong solution for any initial condition in $\mathscr{D}(L)$.*

For completeness, we prove this theorem in Appendix A.

As apparent from expressions (2.2.6) and (2.2.7), L_0' and G', and therefore L', has the eigenvalue 0 (with the eigenvector $\mathbf{1}$),

$$L'\mathbf{1} = 0, \quad L_0'\mathbf{1} = 0, \quad \text{and} \quad G'\mathbf{1} = 0. \tag{2.2.11}$$

2.3 Quantum detailed balance condition

We say that the vNL operator $L = L_0 + G$, or what is the same, the HL operator $L' = L_0' + G'$, satisfies the *quantum detailed balance condition* (*QDB*) w.r.t. a strictly positive, density operator ρ_* if

[2]Another possible definition of the weak solution is the one satisfying

$$\int \operatorname{Tr}((\partial_t A_t)\rho_t)dt = \int \operatorname{Tr}((L'A_t)\rho_t)dt \tag{2.2.9}$$

for any differentiable family $A_t \in \mathscr{D}(L')$. Using the facts proven below that L_0 is anti-self-adjoint and G is bounded, one can also defined the mild solution of (2.1.1) with the initial condition $\rho|_{t=0} = \rho_0 \in \mathcal{S}_1$ as a solution to the integral equation

$$\rho_t = e^{L_0 t}\rho + \int_0^t e^{L_0(t-s)} G\rho_s ds.$$

(QDB) (a) $L_0\rho_* = 0$, and (b) the Lindblad operators G and G' satisfy

$$G(A\rho_*) = (G'A)\rho_* \qquad \text{for all } A \in \mathcal{B}. \tag{2.3.1}$$

By the explicit formula (2.2.7), $G'\mathbf{1} = 0$ (see also (2.2.11)). Then, (2.3.1) implies that

$$G\rho_* = 0. \tag{2.3.2}$$

Since $L = L_0 + G$, the relations $L_0\rho_* = 0$ and Eq. (2.3.2) yield

$$L\rho_* = 0, \tag{2.3.3}$$

i.e. ρ_* is a stationary state and 0 is an eigenvalue of L.

Remark 2.3.1. $\rho_* = f(H)$ for any (reasonable) functions satisfies Condition (a). Under some conditions on H, the converse is also true.

An example of ρ_* is provided by the *Gibbs state* at a temperature $T = \beta^{-1}$:

$$\rho_\beta = e^{-\beta H}/Z(\beta), \quad \beta > 0, \quad \text{provided that } \operatorname{Tr} e^{-\beta H} < \infty, \tag{2.3.4}$$

where $Z(\beta) := \operatorname{Tr} e^{-\beta H}$ (the partition function).

On the other hand, any $\rho_* > 0$ can be written in the form of (2.3.4). Indeed, since $\rho_* > 0$, it can be written as $\rho_* = e^{-\beta H_*}$ for the self-adjoint operator $H_* := -\beta^{-1} \ln \rho_*$. Hence, ρ_* is of the form (2.3.4) with $H = H_* + \mu$ and $Z = \operatorname{Tr}\big(e^{-\beta(H-\mu)}\big) = e^{\beta\mu}$ for any $\mu \in \mathbb{R}$.

2.4 Spaces

Given a density operator $\rho_* > 0$, define the inner product on $\mathcal{B}$

$$\langle A, B\rangle_{\text{obs},*} := \operatorname{Tr}(A^* B \rho_*), \qquad (A, B \in \mathcal{B}). \tag{2.4.1}$$

We define the Hilbert space $\mathcal{B}_*$ as the completion of $\mathcal{B}$ w.r.t. the norm $\|A\|_{\text{obs},*} := \sqrt{\langle A, A\rangle_{\text{obs},*}}$. We study the HL equation on this space.

To study the vNL equation, we introduce, for a given density operator $\rho_* > 0$, the Hilbert space $\mathcal{S}_*$ of density operators as

$$\mathcal{S}_* := \overline{\tilde{\mathcal{S}}_*}, \qquad \tilde{\mathcal{S}}_* := \{\lambda \in \mathcal{S}_1 \mid \lambda\rho_*^{-1/2} \in \mathcal{S}_2\}, \tag{2.4.2}$$

with the completion is taken in the norm induced by the inner product

$$\langle \lambda, \mu\rangle_{\text{st},*} := \operatorname{Tr}\big(\lambda^* \mu \rho_*^{-1}\big). \tag{2.4.3}$$

Remark 2.4.1. (i) We show below (see Theorem 3.1.2) that $L' = L'_0 + G'$ is the decomposition of L' into anti-self-adjoint and self-adjoint parts w.r.t. the inner product (2.4.1).

(ii) The space $\mathcal{B}_*$ is the GNS representation space for a finite number of degrees of freedom, i.e., $\mathrm{Tr}(\rho_*) < \infty$. Indeed, we define the state

$$\omega_*(A) := \mathrm{Tr}(A\rho_*) \tag{2.4.4}$$

on the C^*-algebra $\mathcal{B}$. Then, $\langle A, B\rangle_{\mathrm{obs},*} = \omega_*(A^*B)$ and $\mathcal{B}_*$ is (isomorphic to) the GNS Hilbert space for the C^*-algebra $\mathcal{B}$ and the state ω_* on it (see Appendix C for details).

(iii) The norm associated with the inner product (2.4.3) satisfies

$$\|\lambda\|_{\mathrm{st},*} \geq \|\lambda\|_{\mathcal{S}_1} (\mathrm{Tr}\,\rho_*)^{-1/2}. \tag{2.4.5}$$

Indeed, using the non-commutative Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\lambda\|_{\mathcal{S}_1} &= \mathrm{Tr}\,|\lambda| = \mathrm{Tr}\big(|\lambda|\rho_*^{-1/2}\rho_*^{1/2}\big) \leq \||\lambda|\rho_*^{-1/2}\|_{\mathcal{S}_2}\|\rho_*^{1/2}\|_{\mathcal{S}_2} \\ &= (\mathrm{Tr}\big(\rho_*^{-1/2}\lambda^*\lambda\rho_*^{-1/2}\big))^{1/2}(\mathrm{Tr}\,\rho_*)^{1/2} \\ &= \|\lambda\|_{\mathrm{st},*}(\mathrm{Tr}\,\rho_*)^{1/2}. \end{aligned} \tag{2.4.6}$$

In the next two sections, we formulate and prove our main results on existence and long-time behavior (convergence to equilibrium) for HL and vNL equations, respectively. Along the way, we prove some spectral properties of operators L' and L and establish their dissipative nature on the respective spaces.

Chapter 3

Main results

3.1 The Heisenberg-Lindblad (HL) equation

In this subsection, we formulate our results on the HL equation.

Theorem 3.1.1. *Suppose L' satisfies Condition (H), (W) and (QDB). Then,* (2.1.1) *has a unique solution on $\mathcal{B}_*$ for any initial condition in $\mathscr{D}(L')$.*

This theorem is proven in Appendix A.

Theorem 3.1.2. *Suppose Conditions (H), (W) and (QDB) hold. Then, on $\mathcal{B}_*$, (a) the Lindblad operator G' is self-adjoint, (b) $G' \leq 0$, (c) L'_0 is anti-self-adjoint, and (d) 0 is an eigenvalue of G'.*

Let P' is the orthogonal projector in $\mathcal{B}_*$ onto $\mathrm{Null}(G')$ and $P'^{\perp} = \mathbf{1} - P'$.

Theorem 3.1.3. *Assume Conditions (H), (W) and (QDB) with $\rho_* = \rho_\beta$ (see* (2.3.4)*) hold. Then, the dual quantum evolution $\beta'_t(A)$ converges to $\mathrm{Null}(G')$ in the ergodic sense:*

$$s\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T \beta'_t dt = P', \tag{3.1.1}$$

strongly in $\mathcal{B}_*$.

Theorem 3.1.4. *Assume Conditions (H), (W) and (QDB) with $\rho_* = \rho_\beta$. Suppose further that*

$(Null)$ $\mathrm{Null}(G') \subseteq \mathrm{Null}(L'_0)$.

(a) If, in addition, the following condition is satisfied

$(Spec)$ *G' has no singular continuous spectrum in a neighborhood of 0,* *then the dual quantum evolution* $\beta'_t(A)$ *converges to* $\mathrm{Null}(G')$ *in the sense that, for all* $A \in \mathscr{D}(L')$,

$$\|\beta'_t(A) - P'A\|_{\mathrm{obs},*} \to 0 \qquad \text{as } t \to \infty. \tag{3.1.2}$$

(b) If, instead of Condition $(Spec)$, *we have*

(Gap) *the eigenvalue* 0 *of* G' *is isolated,*

then the dual quantum evolution $\beta'_t(A)$ *converges to the subspace* $\mathrm{Null}(G')$ *exponentially fast: for all* $A \in \mathscr{D}(L')$, *we have*

$$\|\beta'_t(A) - P'A\|_{\mathrm{obs},*} \le e^{-\theta t}\|A\|_{\mathrm{obs},*}, \tag{3.1.3}$$

where $\theta := \mathrm{dist}(0, \sigma(G') \setminus \{0\})$.

(c) If, in addition to Condition (Gap), *the eigenvalue* 0 *of* G' *is simple, then*

$$\|\beta'_t(A) - c^A\mathbf{1}\|_{\mathrm{obs},*} \le e^{-\theta t}\|A\|_{\mathrm{obs},*}, \tag{3.1.4}$$

where $c^A = \langle \mathbf{1}, A\rangle_{\mathrm{obs},*} = \mathrm{Tr}(A\rho_*)$.

Theorem 3.1.5. *Assume the Conditions* (H), (W), (QDC) *with* $\rho_* = \rho_\beta$, *and*

$(Compl)$ $[W_j, A] = [W_j^*, A] = 0$ *for all* $j \ge 1 \implies A \in \mathbb{C}\cdot\mathbf{1}$.

Then, for all $A \in \mathcal{B}_*$, *we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \beta'_t(A)dt = \mathrm{Tr}(A\rho_*)\cdot\mathbf{1}. \tag{3.1.5}$$

Proofs of Theorems 3.1.2, 3.1.3, 3.1.4 and 3.1.5 are given in Subsection 4.1, 4.2, 4.4, and 4.6 respectively.

Remark 3.1.6. By considering $\mathcal{B}$ as a C^*-algebra, Condition $(Compl)$ is satisfied if the collection $\{W_j, W_j^*\}_{j\ge1}$ of operators generates the whole algebra $\mathcal{B}$.

Remark 3.1.7. For $\mathcal{H} = L^2(\mathbb{R}^d)$, the collection $\{W_j\}_{j=1}^{2d}$ of operators with $W_{2k-1} = x_k/\langle x\rangle$ and $W_{2k} = p_k/\langle p\rangle$, $k = 1, ..., d$, on $L^2(\mathbb{R}^d)$ satisfies Condition $(Compl)$, where $p_j := i\hbar\partial_{x_j}$ are the quantum momentum operators, $\langle x\rangle \equiv \sqrt{1+|x|^2}$ and similarly for $\langle p\rangle$.

Remark 3.1.8. Spectra of operators G and G' (regarding them as operators on $\mathcal{S}_2$) are described in Appendix B.

3.2 The vNL equation

We begin with the existence result for the vNL equation.

Theorem 3.2.1. *Assume Conditions* (H), (W) *and* (QDB) *hold. Then, the vNL operator generates a bounded semigroup,* β_t*, on* $\mathcal{S}_*$*. Consequently, the vNL equation* (1.1.1) *has a unique weak solution for any initial condition in* $\mathcal{S}_*$ *and unique strong solution for any initial condition in* $\mathscr{D}(L)$ *(the domain of* L *in* $\mathcal{S}_*$*).*

This theorem is proven in Appendix A.3. Recall that, by (2.3.2), 0 is an eigenvalue of G with the eigenvector ρ_*. The relation between the resolvents of G' and G shows that:

If 0 is an isolated eigenvalue of G', then the operator G also has an isolated eigenvalue 0 (see the proof of Proposition 4.7.2 (c)).

Let P be the orthogonal projector onto $\mathrm{Null}(G)$ in $\mathcal{S}_*$. We have

Theorem 3.2.2. *Suppose the Conditions* (H), (W), (QDB), $(Null)$ *and* (Gap) *hold. Then,*

(a) the quantum evolution $\beta_t(\rho)$ *converges exponentially fast to the subspace* $\mathrm{Null}(G)$*:*

$$\|\beta_t(\rho) - P(\rho)\|_{\mathrm{st},*} \leq e^{-\theta t}\|\rho\|_{\mathrm{st},*}, \tag{3.2.1}$$

for all $\rho \in \mathcal{S}_*$*, where, recall,* $\theta = \mathrm{dist}(0, \sigma(G') \setminus \{0\})$*.*

(b) if, in addition, 0 *is a simple eigenvalue of* G'*, then, for all density operator* $\rho \in \mathscr{D}(L)$*,*

$$\|\beta_t(\rho) - \rho_*\|_{\mathrm{st},*} \leq e^{-\theta t}\|\rho\|_{\mathrm{st},*}, \tag{3.2.2}$$

where ρ_* *is the eigenvector of* G *corresponding to the eigenvalue* 0*.*

Theorem 3.2.2 is proven in Subsection 4.7.

Remark 3.2.3. In the case where ρ_* is the unique stationary solution of β_t, the asymptotic convergence of the dynamic $\beta_t(\rho)$ w.r.t. the trace-norm immediately follows from Theorem 3.2.2 and the inequality:

$$\|\rho - \rho_*\|_{\mathcal{S}_1} \leq \|\rho - \rho_*\|_{\mathrm{st},*} \tag{3.2.3}$$

for $\rho \in \mathcal{S}_*$, see (2.4.5).

Chapter 4

Proofs of main results

4.1 Lindblad operator G': Proof of Theorem 3.1.2

(a) We begin with

Lemma 4.1.1. *The dual Lindblad operator G' satisfies Condition* (QDB) (2.3.1) *w.r.t. ρ_* if and only if G' is symmetric w.r.t. the inner product* (2.4.1)*:*

$$\langle A, G'(B)\rangle_{\mathrm{obs},*} = \langle G'(A), B\rangle_{\mathrm{obs},*} \tag{4.1.1}$$

for all $A, B \in \mathcal{B}$.

Proof. Suppose G' satisfies Condition (QDB) (2.3.1). Since G' is a $*$-map and is dual operator of G, relation (2.3.1) yields

$$\begin{aligned}\langle G'(A), B\rangle_{\mathrm{obs},*} &= \mathrm{Tr}((G'(A))^* B\rho_*) = \mathrm{Tr}(G'(A^*)B\rho_*) \\ &= \mathrm{Tr}(A^* G(B\rho_*)) = \mathrm{Tr}(A^* G'(B)\rho_*) = \langle A, G'(B)\rangle_{\mathrm{obs},*}\end{aligned} \tag{4.1.2}$$

for all $A, B \in \mathcal{B}$, which proves (4.1.1). Conversely, suppose (4.1.1) holds for all $A, B \in \mathcal{B}$. By (4.1.2), we have

$$\begin{aligned}\mathrm{Tr}(A^* G'(B)\rho_*) &= \langle A, G'(B)\rangle_{\mathrm{obs},*} = \langle G'(A), G\rangle_{\mathrm{obs},*} \\ &= \mathrm{Tr}(G'(A)^* B\rho_*) = \mathrm{Tr}(G'(A^*)B\rho_*)\end{aligned} \tag{4.1.3}$$

Since G is dual w.r.t. G', (4.1.3) implies that, for all $A, B \in \mathcal{B}$,

$$\mathrm{Tr}(A^* G'(B)\rho_*) = \mathrm{Tr}(A^* G(B\rho_*)). \tag{4.1.4}$$

Since this is true for all $A, B \in \mathcal{B}$, we have $G'(B)\rho_* = G(B\rho_*)$ for all $B \in \mathcal{B}$. Therefore, G satisfies (QDB) (2.3.1) w.r.t. ρ_*. □

By Proposition A.1.1 and Lemma 4.1.1, G' is bounded and symmetric w.r.t. (2.4.1) on $\mathcal{B}$ so that G' is symmetric on $\mathcal{B}_*$ using a density argument of $\mathcal{B}$ in $\mathcal{B}_*$. Therefore, G' is self-adjoint on $\mathcal{B}_*$.

(b) First, we introduce the *dissipation function* on $\mathcal{B}$ (c.f. [57])

$$D_{G'}(A,B) := G'(A^*B) - G'(A)^*B - A^*G'(B). \tag{4.1.5}$$

Lemma 4.1.2. *For any $A, B \in \mathcal{B}$, we have*

$$D_{G'}(A,B) = \sum_{j\geq 1} [W_j, A]^*[W_j, B]. \tag{4.1.6}$$

Proof. By a straightforwad calculation, we have

$$\begin{aligned} D_{G'}(A,B) = &\sum_{j\geq 1}(W_j^* A^* B W_j - \frac{1}{2}\{W_j^* W_j, A^* B\}) \\ &- \sum_{j\geq 1}(W_j^* A^* W_j B - \frac{1}{2}\{W_j^* W_j, A^*\}B) \\ &- \sum_{j\geq 1}(A^* W_j^* B W_j - \frac{1}{2}A^*\{W_j^* W_j, B\}) \\ = &\sum_{j\geq 1}(W_j^* A^* B W_j - W_j^* A^* W_j B - A^* W_j^* B W_j + A^* W_j^* W_j B) \\ = &\sum_{j\geq 1}[A^*, W_j^*][W_j, B]. \end{aligned} \tag{4.1.7}$$

Since $[A^*, W_j^*] = [W_j, A]^*$ for each j, this gives (4.1.6). □

Lemma 4.1.3. *For all $A, B \in \mathcal{B}$,*

$$\begin{aligned} \langle A, G'(B)\rangle_{\mathrm{obs},*} &= -\frac{1}{2}\operatorname{Tr}(D_{G'}(A,B)\rho_*) \\ &= -\frac{1}{2}\sum_{j\geq 1}\operatorname{Tr}([W_j, A]^*[W_j, B]\rho_*). \end{aligned} \tag{4.1.8}$$

Proof. Since G' is self-adjoint on $\mathcal{B}_*$, $G(\rho_*) = 0$ and $\operatorname{Tr}(AG(\rho)) = \operatorname{Tr}(G'(A)\rho)$, we have, by (4.1.5), for all $A, B \in \mathcal{B}$,

$$\begin{aligned}\langle A, G'(B)\rangle_{\text{obs},*} &= \frac{1}{2}(\langle A, G'(B)\rangle_{\text{obs},*} + \langle G'(A), B\rangle_{\text{obs},*}) \\ &= \frac{1}{2}(\mathrm{Tr}(A^*G'(B)\rho_*) + \mathrm{Tr}(G'(A)^*B\rho_*) - \mathrm{Tr}(A^*BG(\rho_*))) \\ &= \frac{1}{2}\,\mathrm{Tr}((A^*G'(B) + G'(A)^*B - G'(A^*B))\rho_*) \\ &= -\frac{1}{2}\,\mathrm{Tr}(D_{G'}(A, B)\rho_*). \end{aligned} \tag{4.1.9}$$

By Lemma 4.1.2, we obtain (4.1.8). □

By Lemma 4.1.3, we have, for any $A \in \mathcal{B}$,

$$\langle A, G'(A)\rangle_{\text{obs},*} = -\frac{1}{2}\sum_{j\geq 1}\mathrm{Tr}([W_j, A]^*[W_j, A]\rho_*). \tag{4.1.10}$$

Since $[W_j, A]^*[W_j, A] \geq 0$ for all $A \in \mathcal{B}$ and $j \geq 1$, we have that $\langle A, G'(A)\rangle_{\text{obs},*} \leq 0$ for all $A \in \mathcal{B}$. By a density argument, since G' is bounded on $\mathcal{B}_*$, we conclude that $G' \leq 0$ on $\mathcal{B}_*$. This proves Theorem 3.1.2 (b).

(c) We show that L_0' is anti-symmetric on $\mathcal{B}_*$ under (QDB). Indeed, since $L_0\rho_* = 0$ under (QDB), then, for all $A, B \in \mathscr{D}(L_0')$, we have

$$\begin{aligned}\langle L_0'A, B\rangle_{\text{obs},*} &= \mathrm{Tr}((i[H, A])^*B\rho_*) = \mathrm{Tr}(i[H, A^*]B\rho_*) \\ &= \mathrm{Tr}(i[H, A^*B]\rho_*) - \mathrm{Tr}(A^*(i[H, B])\rho_*) \\ &= \mathrm{Tr}(A^*B(L_0\rho_*)) - \langle A, L_0'B\rangle_{\text{obs},*} \\ &= -\langle A, L_0'B\rangle_{\text{obs},*}. \end{aligned} \tag{4.1.11}$$

(d) The fact that 0 is an eigenvalue of G' on $\mathcal{B}_*$ follows from the computation:

$$G'(\mathbf{1}) = \sum_{j\geq 1}(W_j^*W_j - \frac{1}{2}\{W_j^*W_j, \mathbf{1}\}) = 0. \tag{4.1.12}$$

This completes the proof of Theorem 3.1.2. □

Remark 4.1.4. The r.h.s. of (4.1.10) is a special example of non-commutative Dirichlet quadratic forms. See [1, 21, 31] for non-commutative Dirichlet quadratic form on general C^*- and von Neumann algebras and relations to non-commutative Markovian semigroups.

Define $D_jA := [W_j, A]$ and $-\Delta := \sum_{j\geq 1} D_j^\dagger D_j/2$, where $D_j^\dagger$ is the adjoint operator of D_j w.r.t. the inner product (2.4.1). Then, (4.1.10) becomes

$$\langle A, G'(A)\rangle_{\text{obs},*} = -\frac{1}{2}\sum_{j\geq 1}\langle D_j A, D_j A\rangle_{\text{obs},*} = \langle A, \Delta A\rangle_{\text{obs},*}. \tag{4.1.13}$$

Thus, if we ignore L_0', then the HL equation (2.1.1) on $\mathcal{B}_*$ can be considered as a non-commutative analogue of the heat equation.

Since $D_j(\mathbf{1}) = 0$ for all $j \geq 1$, the operator Δ has the eigenvalue 0 with eigenvector $\mathbf{1}$. An estimate of the second highest eigenvalue of Δ would give an estimate of the relaxation/mixing time.

4.2 Proof of Theorem 3.1.3

Throughout Subsections 4.2–4.4, we omit the subindex "obs, $*$" in the inner product and norm in $\mathcal{B}_*$. One should not confuse this norm with the operator norm on $\mathcal{B}$.

4.2.1 Commutativity of L_0' and G'

First, we show the commutativity of L_0' and G', and then we use it to prove Theorem 3.1.3.

Theorem 4.2.1. *Assume Conditions* (*H*), (*W*) *and* (*QDB*) *w.r.t.* $\rho_* = \rho_\beta$ *(defined in* (2.3.4)*). Then the semigroups* $e^{L_0' t}$ *and* $e^{G' t}$ *on* $\mathcal{B}_*$ *commute, and, consequently,* $\beta_t' = e^{L_0' t} e^{G' t}$.

Similarly, $e^{L_0 t}$ *and* e^{Gt} *commute on* $\mathcal{S}_*$ *and* $\beta_t = e^{L_0 t} e^{Gt}$.

Theorem 4.2.1 was proven in [15] for the finite-dimensional case, and was sketched for the infinite-dimensional case in [40]. Below, we give a detailed proof of this theorem.

We begin with some generalities. We use the following definition (see [12]).

Definition 4.2.2. We say two unbounded, self-adjoint operators H_1 and H_2 *commute strongly* if the spectral projectors of H_1 and H_2 commute, or, equivalently, if $e^{iH_1 s}$ and $e^{iH_2 t}$ commute for all $s, t \in \mathbb{R}$.

Lemma 4.2.3. *For a bounded operator* B *and unbounded self-adjoint operator* A*, the following conditions are equivalent:*

(i) B *commutes strongly with* A.

(ii) There is some core $\mathscr{D}$ *for* A *such that* $B\mathscr{D} \subseteq \mathscr{D}$ *and* $BA\psi = AB\psi$ *for all* $\psi \in \mathscr{D}$.

(iii) The domain $\mathscr{D}(A)$ is invariant under B in the sense of $B\mathscr{D}(A) \subseteq \mathscr{D}(A)$ and, for all $\psi \in \mathscr{D}(A)$, $BA\psi = AB\psi$.

(iv) AB is an extension of BA, written as $BA \subset AB$ (see [50], Chapter 3, Section 5.6).

For a proof, see [12], Lemma 1.

Remark 4.2.4. The statements (ii)–(iv) for strong commutativity can be extended for general closed (not necessarily self-adjoint) operators A.

Remark 4.2.5. Note that $T \subset S$ implies that $S^* \subset T^*$ for general operators T and S on a Hilbert space. If a bounded operator B commutes strongly with some operator A, we have $B^*A^* \subset A^*B^*$, i.e., B^* commutes strongly with A^*.

We present the proof of Theorem 4.2.1 in the subsubsections 4.2.2–4.2.3.

4.2.2 Consequence of G' being a $*$-map

We have shown in Appendix C that the triple $(\mathcal{S}_2, \pi_*, \Omega_*)$, where $\mathcal{S}_2$ is the space of Hilbert-Schmidt operators on $\mathcal{H}$, $\Omega_* := \rho_*^{1/2}$ and π_* is the linear representation of $\mathcal{B}$ on $\mathcal{S}_2$ given by

$$\pi_*(A)\kappa = A\kappa \qquad \text{for all } A \in \mathcal{B}, \tag{4.2.1}$$

yields the GNS representation associated with the pair $(\mathcal{B}, \omega_*)$. Note that Ω_* is cyclic for $\pi_*(\mathcal{B})$ in $\mathcal{S}_2$ (see Appendix C for details).

The Lindblad operator G' induces a linear operator $\widehat{G'}$, defined by

$$\widehat{G'}(\pi_*(A)\Omega_*) := \pi_*(G'(A))\Omega_*, \qquad \text{for all } A \in \mathcal{B}, \tag{4.2.2}$$

on the dense set in $\mathcal{S}_2$, given by

$$\mathscr{F} := \pi_*(A)\Omega_*. \tag{4.2.3}$$

Since G' is bounded on $\mathcal{B}$, $\widehat{G'}$ extends to a bounded operator on $\mathcal{S}_2$. We retain the same notation $\widehat{G'}$ also for this extension.

Define the anti-unitary operator J on $\mathcal{S}_2$ by

$$J\kappa = \kappa^*, \quad \text{for all } \kappa \in \mathcal{S}_2. \tag{4.2.4}$$

Note that $J = J^* = J^{-1}$ (see Proposition C.0.1 (d) for the proof).

A key to the proof of Theorem 4.2.1 is the following theorem, proven in Appendix C:

Theorem 4.2.6. *([44], Section V.1.4, Theorem 1.4.2) Let* $\alpha_t(A) := e^{iH_* t} A e^{-iH_* t}$*, where* $H_* = -\ln \rho_*$*. Then, there is a (unbounded) self-adjoint operator* L_* *such that*

$$\pi_*(\alpha_t(A)) = e^{iL_* t}\pi_*(A)e^{-iL_* t}, \quad L_*\Omega_* = 0. \tag{4.2.5}$$

Moreover, we have the following relation for operators J *and* L_**: For any* $A \in \mathcal{B}$*,* $\pi_*(A)\Omega_* \in \mathscr{D}(e^{-L_*/2})$ *and*

$$Je^{-L_*/2}(\pi_*(A)\Omega_*) = \pi_*(A^*)\Omega_*, \quad J\Omega_* = \Omega_*. \tag{4.2.6}$$

We define the anti-linear operator S on $\mathcal{S}_2$ by

$$S = Je^{-L_*/2}. \tag{4.2.7}$$

By relation (4.2.6), we have

$$S(\pi_*(A)\Omega_*) = \pi_*(A^*)\Omega_*. \tag{4.2.8}$$

Lemma 4.2.7. *The set* $\mathscr{F}$ *is a core for* S*.*

Proof. Since J is bounded and invertible, we have $\mathscr{D}(S) = \mathscr{D}(e^{-L_*/2})$. Thus, to show that $\mathscr{F}$ is a core for S, it suffices to show that it is a core for $e^{-L_*/2}$. Since L_* is self-adjoint on $\mathcal{S}_2$, it generates a one-parameter group $e^{iL_* t}$, $t \in \mathbb{R}$, of unitary operators on $\mathcal{S}_2$. By (4.2.5), for each $t \in \mathbb{R}$ and $A \in \mathcal{B}$, we have

$$e^{iL_* t}(\pi_*(A)\Omega_*) = (e^{iL_* t}\pi_*(A)e^{-iL_* t})\Omega_* = \pi_*(\alpha_t(A))\Omega_* \in \mathscr{F}. \tag{4.2.9}$$

Thus, $e^{iL_* t}\mathscr{F} \subseteq \mathscr{F}$ for each $t \in \mathbb{R}$ so that, by the density of $\mathscr{F}$ in $\mathcal{S}_2$, $\mathscr{F}$ is a core for L_* and therefore for S. □

Recall that an operator K on $\mathcal{B}$ is called a $*$-map if it commutes with taking the adjoint: $K(A^*) = (KA)^*$ for every $A \in \mathcal{B}$.

In the next proposition, we relate G' and S:

Proposition 4.2.8. *The Lindblad operator* G' *is a* $*$*-map on* $\mathcal{B}$ *if and only if its induced operator* $\widehat{G'}$ *on* $\mathcal{S}_2$ *commutes strongly with* S*.*

Proof. Suppose G' is a $*$-map. Then, by Lemma 4.2.7, since $\mathscr{F}$ is a core for S and $\widehat{G'}\mathscr{F} \subseteq \mathscr{F}$, for each $A \in \mathcal{B}$, we have, by (4.2.8),

$$\begin{aligned} S\widehat{G'}(\pi_*(A)\Omega_*) &= S(\pi_*(G'A)\Omega_*) = \pi_*((G'A)^*)\Omega_* \\ &= \pi_*(G'(A^*))\Omega_* = \widehat{G'}(\pi_*(A^*)\Omega_*) = \widehat{G'}S(\pi_*(A)\Omega_*). \end{aligned} \tag{4.2.10}$$

Thus, $\widehat{G'}$ commutes strongly with S.

Conversely, by the same computations above, we obtain, for each $A \in \mathcal{B}$,

$$(G'(A^*))\rho_*^{1/2} = \pi_*(G'(A^*))\Omega_* = \pi_*((G'A)^*)\Omega_* = (G'A)^*\rho_*^{1/2}. \qquad (4.2.11)$$

Since $\rho_* > 0$, we have $G'(A^*) = (G'A)^*$ for all $A \in \mathcal{B}$. Hence, G' is a $*$-map. □

4.2.3 Consequence of Condition (QDB)

Now, we impose Condition (QDB) on G'. An immediate consequence of Condition (QDB) is that the induced operator $\widehat{G'}$ is self-adjoint on $\mathcal{S}_2$:

Lemma 4.2.9. *The Lindblad operator G' satisfies Condition (QDB) w.r.t. ρ_* if and only if the induced operator $\widehat{G'}$ is self-adjoint on $\mathcal{S}_2$.*

Proof. Suppose G' satisfies Condition (QDB). For any $A, B \in \mathcal{B}$, we have

$$\begin{aligned}\langle \widehat{G'}(\pi_*(A)\Omega_*), \pi_*(B)\Omega_*\rangle_{\mathcal{S}_2} &= \mathrm{Tr}((G'A)^*B\rho_*)\\ &= \mathrm{Tr}(A^*G(B\rho_*))\\ &= \mathrm{Tr}(A^*G'(B)\rho_*)\\ &= \langle \pi_*(A)\Omega_*, \widehat{G'}(\pi_*(B)\Omega_*)\rangle_{\mathcal{S}_2}. \end{aligned} \qquad (4.2.12)$$

Since $\mathscr{F}$ is dense in $\mathcal{S}_2$ and $\widehat{G'}$ is bounded, we can extend this equality to the entire space $\mathcal{S}_2$. Thus, $\widehat{G'}$ is self-adjont on $\mathcal{S}_2$.

The converse follows from the same computation so that $\widehat{G'}$ is self-adjoint on $\mathcal{S}_2$ implies that G' satisfies Condition (QDB). □

Using Proposition 4.2.8 and Lemma 4.2.9, we establish the strong commutativity between $\widehat{G'}$ and e^{-L_*}:

Theorem 4.2.10. *Suppose the Lindblad operator G' satisfies Condition (QDB) w.r.t. ρ_*. Then, the induced operator $\widehat{G'}$ commutes strongly with e^{-L_*}.*

Proof. First, by Proposition 4.2.8, $\widehat{G'}$ commutes strongly with $S = Je^{-L_*/2}$ so that $\widehat{G'}S \subset S\widehat{G'}$ (see Remark 4.2.4). Then, by taking adjoint and by Lemma 4.2.9 (see also Remark 4.2.5), we obtain

$$(S\widehat{G'})^* = \widehat{G'}S^* \subset (\widehat{G'}S)^* = S^*\widehat{G'}. \qquad (4.2.13)$$

Thus, $\widehat{G'}$ commutes strongly with $S^* = e^{-L_*/2}J$.

Now, using $J^2 = \mathbf{1}$, we note that $e^{-L_*} = (e^{-L_*/2}J)(Je^{-L_*/2}) = S^*S$. Since $\widehat{G'}$ commutes strongly with both S and S^*, it also commutes strongly with e^{-L_*}. □

Proof of Theorem 4.2.1. Suppose $\rho_* = e^{-\beta H}/Z(\beta)$. In this case, we have $H_* = \beta H + \ln Z(\beta)$ so that $\alpha_t = e^{\beta L_0' t}$. Then, by Theorem 4.2.10, $\widehat{G'}$ commutes with $e^{iL_* t}$ so that, for all $A \in \mathcal{B}$,

$$\begin{aligned}\pi_*(e^{L_0' t} G' A)\Omega_* &= \pi_*(\alpha_{t/\beta}(G'A))\Omega_* = e^{iL_* t/\beta}(\pi_*(G'A))\Omega_* \\ &= e^{iL_* t/\beta}\widehat{G'}(\pi_*(A)\Omega_*) = \widehat{G'} e^{iL_* t/\beta}(\pi_*(A)\Omega_*) \\ &= \widehat{G'}(\pi_*(e^{L_0' t} A)\Omega_*) = \pi_*(G' e^{L_0' t} A)\Omega_*. \qquad (4.2.14)\end{aligned}$$

Since $\mathscr{F}$ is dense in $\mathcal{S}_2$, we have $e^{L_0' t}$ and G' commute on $\mathcal{B}$. Since $e^{L_0' t}$ and G' are bounded and $\mathcal{B}$ is dense in $\mathcal{B}_*$, the commutativity between $e^{L_0' t}$ and G' can be extended to the entire space $\mathcal{B}_*$. Also, by the duality relation, we obtain the strong commutativity between L_0 and G. This proves Theorem 4.2.1. □

4.3 Proof of Theorem 3.1.3

Theorem 3.1.3 follows from Theorem 4.2.1 and the following result.

Theorem 4.3.1. *Assume Conditions* (H), (W) *and* (QDB) *with* $\rho_* = \rho_\beta$ *(see* (2.3.4)*) hold and assume*

(Com) *G' commutes with $e^{L_0' t}$ for all $t \in \mathbb{R}$.*

Then, the dual quantum evolution $\beta_t'(A)$ converges to $\mathrm{Null}(G')$ *in the ergodic sense:*

$$s\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T \beta_t' dt = P', \qquad (4.3.1)$$

strongly in $\mathcal{B}_*$.

Remark 4.3.2. Under Condition (Com), L' is normal, i.e., $[\mathrm{Re}(L'), \mathrm{Im}(L')] = [G', L_0'] = 0$.

Recall that P' is the orthogonal projector onto the subspace $\mathrm{Null}(G')$ in $\mathcal{B}_*$ w.r.t. its inner product and let $P'^{\perp} = \mathbf{1} - P'$.

Let $G'^{\perp} := G' P'^{\perp}$. First, we show that $\mathrm{Ran}(G'^{\perp}) = \mathscr{D}((G'^{\perp})^{-1})$ is dense in $\mathrm{Ran}\, P^{\perp} =: \mathcal{B}_*^{\perp}$. Let $E(\lambda)$ be the spectral resolution of the self-adjoint operator $G'^{\perp}$. The set

$$\mathscr{D} := \{\int f(\lambda) dE(\lambda) A \mid A \in \mathcal{B}_*^{\perp}, f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), \text{ and } f = 0 \text{ in a vicinity of } 0.\} \qquad (4.3.2)$$

is dense in $\operatorname{Ran} P'^{\perp}$ and $(G'^{\perp})^{-1}$ is defined on $\mathscr{D}$ and is given by

$$(G'^{\perp})^{-1}\int f(\lambda)dE(\lambda)A = \int \lambda^{-1}f(\lambda)dE(\lambda)A. \tag{4.3.3}$$

(By the condition of f, the operator on the r.h.s. is bounded in $\mathcal{B}_*$ by $\sup_\lambda |\lambda^{-1}f(\lambda)|\|A\|$.)

Next, by Theorem 4.2.1, we have $\beta'_t = e^{L'_0 t}e^{G' t}$ for all $t \geq 0$. Then, the Cauchy-Schwarz and the Hölder inequalities yield, since L'_0 is anti-self-adjoint, for any $A \in \mathscr{D}(L')$ such that $A^\perp := P'^\perp A \in \mathscr{D}((G'^\perp)^{-1})$ and any $B \in \mathcal{B}_*$,

$$\begin{aligned}|\langle B, \frac{1}{T}\int_0^T \beta'_t(P'^\perp A)dt\rangle_*| &\leq \frac{1}{T}\int_0^T |\langle B, e^{L'_0 t}e^{G' t}A^\perp\rangle_*|dt \\ &= \frac{1}{T}\int_0^T |\langle e^{-L'_0 t}B, e^{G'^\perp t}A^\perp\rangle_*|dt \leq \frac{1}{T}\int_0^T \|e^{-L'_0 t}B\|\|e^{G'^\perp t}A^\perp\|dt \\ &= \frac{\|B\|}{T}\int_0^T \|e^{G'^\perp t}A^\perp\|dt \leq \frac{\|B\|}{\sqrt{T}}\left(\int_0^T \|e^{G'^\perp t}A^\perp\|^2 dt\right)^{1/2}.\end{aligned} \tag{4.3.4}$$

Now, since G' is self-adjoint on $\mathcal{B}_*$, we have

$$\begin{aligned}\int_0^T \|e^{G'^\perp t}A^\perp\|^2 dt &= \int_0^T \langle A^\perp, e^{2G'^\perp t}A^\perp\rangle dt \\ &= \langle A^\perp, (e^{2G'^\perp T} - \mathbf{1})(2G'^\perp)^{-1}A^\perp\rangle.\end{aligned} \tag{4.3.5}$$

Since $G' \leq 0$, we have $\|e^{2G'^\perp T}\| \leq 1$ and therefore, by (4.3.5),

$$\int_0^T \|e^{G'^\perp t}A^\perp\|^2 dt \leq \|(G'^\perp)^{-1/2}A^\perp\|^2. \tag{4.3.6}$$

By (4.3.4) and (4.3.6), we have, for such $A, B \in \mathcal{B}_*$,

$$|\langle B, \frac{1}{T}\int_0^T \beta'_t(P^\perp A)dt\rangle_*| \leq \frac{1}{\sqrt{T}}\|B\|\|(G'^\perp)^{-1/2}A^\perp\|, \tag{4.3.7}$$

which implies that, for all $A \in \mathcal{B}_*$ such that $P'^\perp A \in \mathscr{D}((G'^\perp)^{-1/2})$,

$$\begin{aligned}\|\frac{1}{T}\int_0^T \beta'_t(A)dt - P'A\| &= \|\frac{1}{T}\int_0^T \beta'_t(P^\perp A)dt\| \\ &\leq \frac{1}{\sqrt{T}}\|(G'^\perp)^{-1/2}P'^\perp A\| \to 0\end{aligned} \tag{4.3.8}$$

as $T \to \infty$. Since $\mathscr{D}((G'^\perp)^{-1})$ is dense in $\mathcal{B}_*^\perp$ and $\frac{1}{T}\int_0^T \beta'_t dt - P'$ is bounded uniformly in T,

$$\|\frac{1}{T}\int_0^T \beta'_t(A)dt - P'A\| \leq 2\|A\|, \tag{4.3.9}$$

for all $A \in \mathcal{B}_*$, (4.3.8) implies that, strongly in $\mathcal{B}_*$,

$$s\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T \beta'_t dt = P'. \tag{4.3.10}$$

This proves Theorem 4.3.1. □

Proof of Theorem 3.1.3. Theorem 3.1.3 follows from Theorems 4.2.1 and 4.3.1. □

4.4 Proof of Theorem 3.1.4

(a) Before we proceed to the proof, we note that, by Theorem 4.2.1, since $[L'_0, G'] = 0$, L'_0 also commutes with all the spectral projections of G'. Hence, for $\tilde{P}$ any spectral projection of G' and any $A \in \operatorname{Ran}\tilde{P} \cap \mathscr{D}(L')$, by Leibniz rule and Theorem 3.1.2 (a) and (c), we have

$$\begin{aligned}\partial_t\|A_t\|^2 &= \partial_t\langle A_t, A_t\rangle = \langle \partial_t A_t, A_t\rangle + \langle A_t, \partial_t A_t\rangle \\ &= \langle L'A_t, A_t\rangle + \langle A_t, L'A_t\rangle \\ &= \langle A_t, (L'^\dagger + L')A_t\rangle \\ &= 2\langle A_t, G'A_t\rangle, \end{aligned} \tag{4.4.1}$$

where we denote, throughout this subsection, $A_t := \beta'_t(A)$.

Recall that $P'^\perp := \mathbf{1} - P'$ and let U_0 be an open neighborhood of 0 such that G' has no singular continuous spectrum on U_0. We denote P_1 and P_2 as the orthogonal projections onto the subspaces of point and absolutely continuous spectra in U_0, respectively. Furthermore, we denote P_3 the spectral projection of G' onto $\sigma(G') \setminus U_0$. It follows from Condition (*Spec*) that $P'^\perp = \sum_{i=1}^3 P_i$. We note that, by Theorem 4.2.1, L'_0 and G' commute strongly, which then implies that L'_0 also commutes with all the spectral projections of G', which implies that $[L'_0, P_i] = 0$ for each $i = 1, 2, 3$.

We denote $A_{t,i} := P_i A_t$ for $i = 1, 2, 3$ and consider the Lyapunov functional $\|P'^\perp A_t\|^2$. By triangle inequality, we have

$$\|P'^\perp A_t\|^2 \leq \sum_{i=1}^3 \|A_{t,i}\|^2. \tag{4.4.2}$$

We will show separately that $\|A_{t,i}\|^2 \to 0$ for $i = 1, 2, 3$ as $t \to \infty$.

We begin with $\|A_{t,1}\|^2$. We observe that we can write P_1 and $G'|_{\operatorname{Ran} P_1}$ as

$$P_1 = \sum_n P_{e_n}, \qquad G'|_{\operatorname{Ran} P_1} = \sum_n \lambda_n P_{e_n}, \tag{4.4.3}$$

where λ_n are eigenvalues of G' in U_0 and e_n are the corresponding orthonormal eigenvectors of G'. Since $G' < 0$ on $\operatorname{Ran} P_1$, we have $\lambda_n < 0$ for all n.

Let $A_t^{(n)} := P_{e_n} A_{t,1}$ for each n. By Theorem 4.2.1 and (4.4.1), we have

$$\partial_t \|A_t^{(n)}\|^2 = 2\langle A_t^{(n)}, G' A_t^{(n)}\rangle. \tag{4.4.4}$$

With this notation, we have

$$\langle A_t^{(n)}, G' A_t^{(n)}\rangle = \lambda_n |\langle e_n, A_{t,1}\rangle|^2. \tag{4.4.5}$$

Using the Plancherel's theorem, we have

$$\langle A_t^{(n)}, G' A_t^{(n)}\rangle = \lambda_n |\langle e_n, A_t\rangle|^2 \tag{4.4.6}$$

and introducing the notation $a_n(t) := |\langle e_n, A_t\rangle|^2$, we find from (4.4.4) that

$$\partial_t \|A_t^{(n)}\|^2 = \partial_t a_n(t) = 2\lambda_n a_n(t) \tag{4.4.7}$$

with the initial condition

$$\|A_t^{(n)}\|^2|_{t=0} = \|A_0^{(n)}\|^2 = a_n(0), \tag{4.4.8}$$

which is solved to be

$$a_n(t) = e^{2\lambda_n t} a_n(0). \tag{4.4.9}$$

Since

$$\begin{aligned} \|A_{t,1}\|^2 = \langle A_{t,1}, A_{t,1}\rangle &= \sum_n \langle A_t^{(n)}, A_t^{(n)}\rangle \\ &= \sum_n a_n(t) = \sum_n e^{2\lambda_n t} a_n(0) \end{aligned} \tag{4.4.10}$$

and $\lambda_n < 0$ for each n, we have $\|A_{t,1}\|^2 \to 0$ as $t \to \infty$.

Now, we consider $\|A_{t,2}\|^2$. Recall P_2 is the orthogonal projection of G' onto the subspace of absolutely continuous spectrum in U_0. Let $dE_{G'}(\lambda)$ be the spectral measure of G' on $\operatorname{Ran} P_2$. Then, for each $A \in \operatorname{Ran} P_2$, we have

$$\langle A, A\rangle = \int_{U_0} d\mu_A(\lambda), \qquad \langle A, G'A\rangle = \int_{U_0} \lambda d\mu_A(\lambda), \tag{4.4.11}$$

where $d\mu_A(\lambda) := \langle A, dE_{G'}(\lambda)A\rangle_*$. Since $d\mu_A(\lambda)$ is absolutely continuous w.r.t. the Lebesgue measure, there exists some positive, Lebesgue measurable function $f_A(\lambda)$ such that $d\mu_A(\lambda) = f_A(\lambda)d\lambda$.

Let $f_t(\lambda) := f_{A_t}(\lambda)$. Then, by (4.4.1) and (4.4.11), we have

$$\partial_t \int_{U_0} f_t(\lambda)d\lambda = \int_{U_0} 2\lambda f_t(\lambda)d\lambda \tag{4.4.12}$$

with the initial condition

$$\left(\int_{U_0} f_t(\lambda)d\lambda\right)|_{t=0} = \int_{U_0} f_0(\lambda)d\lambda. \tag{4.4.13}$$

Next, we observe that, for any subset $V \subseteq U_0$ and P_V corresponding spectral projection onto the absolutely continuous spectrum in V, by Condition (W') and (4.4.1), we have

$$\partial_t \int_{V} f_t(\lambda)d\lambda = \int_{V} 2\lambda f_t(\lambda)d\lambda \tag{4.4.14}$$

with the initial condition

$$\left(\int_{V} f_t(\lambda)d\lambda\right)|_{t=0} = \int_{V} f_0(\lambda)d\lambda. \tag{4.4.15}$$

Now, for each $\lambda \in U_0$, we take a sequence of open neighborhoods $V_n \subseteq U_0$ of λ such that $V_n \supseteq V_{n+1}$ and $V_n \to \{\lambda\}$ as $n \to \infty$. By the Lebesgue differentiation theorem, we have, for almost all $\lambda \in U_0$,

$$\begin{aligned}\partial_t f_t(\lambda) &= \lim_{n\to\infty} \frac{1}{m(V_n)} \int_{V_n} f_t(\lambda)d\lambda \\ &= \lim_{n\to\infty} \frac{1}{m(V_n)} \int_{V_n} 2\lambda f_t(\lambda)d\lambda \\ &= 2\lambda f_t(\lambda)d\lambda\end{aligned} \tag{4.4.16}$$

and

$$f_0(\lambda) = \lim_{n\to\infty} \frac{1}{m(V_n)} \int_{V_n} f_0(\lambda)d\lambda, \tag{4.4.17}$$

where $m(V)$ is the Lebesgue measure of V. Then, for almost all $\lambda \in U_0$, we have

$$f_t(\lambda) = e^{2\lambda t} f_0(\lambda) \tag{4.4.18}$$

so that

$$\|A_{t,2}\|^2 = \int_{U_0} f_t(\lambda) d\lambda = \int_{U_0} e^{2\lambda t} f_0(\lambda) d\lambda. \tag{4.4.19}$$

Since $U_0 \subseteq (-\infty, 0)$, we have $\|A_{t,2}\|^2 \to 0$ as $t \to \infty$.

Finally, we consider $\|A_{t,3}\|^2$. Since U_0 is an open neighborhood of 0, there exists some $\theta > 0$ such that $[-\theta, \theta] \in U_0$ so that

$$G'|_{\operatorname{Ran} P_3} \leq -\theta. \tag{4.4.20}$$

Then, by (4.4.1) and (4.4.20), we have

$$\partial_t \|A_{t,3}\|^2 = 2\langle A_{t,3}, G' A_{t,3}\rangle_* \leq -2\theta \|A_{t,3}\|^2 \tag{4.4.21}$$

with the initial condition

$$\|A_{t,3}\|^2|_{t=0} = \|A_{0,3}\|^2. \tag{4.4.22}$$

Inequality (4.4.21) is solved as

$$\|A_{t,3}\| \leq e^{-\theta t} \|A_{0,3}\|, \tag{4.4.23}$$

which converges to 0 as $t \to \infty$. Therefore, (4.4.2) gives, as $t \to \infty$,

$$\|P'^{\perp} A_t\| \leq \sum_{i=1}^{3} \|P_i A_t\| \to 0. \tag{4.4.24}$$

To conclude this proof, by Condition $(Null)$, we have $[L_0, P'] = 0$ and $\operatorname{Ran} P = \operatorname{Null}(G') \subseteq \operatorname{Null}(L_0')$. It follows that $[L', P'] = [L_0' + G', P'] = 0$ and $\operatorname{Ran} P \subseteq \operatorname{Null}(L')$. Thus, for all $A \in \mathcal{B}_*$, we have

$$P' A_t = P' \beta_t'(A) = P' e^{L' t} A = e^{L' t} P' A = P' A. \tag{4.4.25}$$

Therefore, we have, as $t \to \infty$,

$$\|A_t - P' A\| = \|A_t - P' A_t\| = \|P^{\perp} A_t\| \leq \sum_{i=1}^{3} \|P_i A_t\| \to 0, \tag{4.4.26}$$

which proves (3.1.2).

(b) By Condition (Gap), we have $P_1 = P_2 = 0$ so that $P'^{\perp} = P_3$. It follows from proof for part (a) that there exists some $\theta > 0$ such that

$$\|A_t - P'A_t\| = \|P'^{\perp}A_t\| \le e^{-\theta t}\|P'^{\perp}A_0\| \le e^{-\theta t}\|A\|. \tag{4.4.27}$$

It then follows from the same reason as in the last paragraph of proof for part (a) that $P'A_t = P'A$, which, together with (4.4.27), implies (3.1.3).

(c) Since 0 is a simple eigenvalue of G' and $\mathbf{1} \in \mathrm{Null}(G')$, we have $\mathrm{Null}(G') = \mathbb{C}\cdot\mathbf{1} \subseteq \mathrm{Null}(L'_0)$ so that $\mathrm{Null}(G') \subseteq \mathrm{Null}(L')$, $[L', P'] = 0$, and

$$P'A = \langle \mathbf{1}, A\rangle_* \mathbf{1} = \mathrm{Tr}(A\rho_*)\mathbf{1}. \tag{4.4.28}$$

This and Theorem 3.1.4 (b) yield (3.1.4). □

4.5 Uniqueness of stationary state

The next result gives a sufficient condition for an eigenvalue of G' at 0 to be simple (see [34, 38] for different proofs):

Proposition 4.5.1. *Suppose Condition* (W)*,* (QDB) *and* ($Compl$)*. Then* 0 *is a simple eigenvalue of* G' *on* $\mathcal{B}_*$*.*

Proof. First, let $A \in \mathrm{Null}(G') \cap \mathcal{B}$. By (4.1.8), we have

$$0 = \langle A, G'(A)\rangle_{\mathrm{obs},*} = -\frac{1}{2}\sum_{j\ge 1} \|[W_j, A]\|^2_{\mathrm{obs},*}. \tag{4.5.1}$$

Since $\|[W_j, A]\| \ge 0$ for all $j \ge 1$, (4.5.1) implies that $[W_j, A] = 0$ for all $j \ge 1$.

Next, since G' is a $*$-map, we have $G'(A^*) = 0$ if $A \in \mathrm{Null}(G') \cap \mathcal{B}$ and, by (4.1.8) again,

$$\begin{aligned} 0 = \langle A^*, G'(A^*)\rangle_{\mathrm{obs},*} &= -\frac{1}{2}\sum_{j\ge 1} \|[W_j, A^*]\|^2_{\mathrm{obs},*} \\ &= -\frac{1}{2}\sum_{j\ge 1} \|[W_j^*, A]\|^2_{\mathrm{obs},*} \end{aligned} \tag{4.5.2}$$

so that $[W_j^*, A] = 0$ for all $j \ge 1$. By Condition ($Compl$), we must have $A \in \mathbb{C}\cdot\mathbf{1}$.

Since $G'(\mathbf{1}) = 0$, we have $\mathrm{Null}(G') \cap \mathcal{B} = \mathbb{C}\cdot\mathbf{1}$. Since $\mathcal{B}$ is dense in $\mathcal{B}_*$ and G' is bounded, we conclude that $\mathrm{Null}(G') = \mathbb{C}\cdot\mathbf{1}$. □

By Proposition 4.5.1, under Condition (*Compl*), the orthogonal projection P' onto $\mathrm{Null}(G')$ in $\mathcal{B}_*$ is given by

$$P'A = \mathrm{Tr}(A\rho_*) \cdot \mathbf{1}, \qquad \forall A \in \mathcal{B}_*. \tag{4.5.3}$$

4.6 Proof of Theorem 3.1.5

Recall P' denotes the orthogonal projection onto the subspace $\mathrm{Null}(G')$ in $\mathcal{B}_*$ w.r.t. the inner product (2.4.1). By Theorem 3.1.3, the integral $\frac{1}{T}\int_0^T \beta_t' dt$ converges strongly to P' as $T \to \infty$. By Proposition 4.5.1, Condition (*Compl*) implies (4.5.3). This completes the proof of Theorem 3.1.5. □

4.7 Proof of Theorem 3.2.2

We define the convenient subset of the Schatten space $\mathcal{S}_1$:

$$\widehat{\mathcal{S}}_* = \{\lambda \in \mathcal{S}_1 \mid \lambda = A\rho_*, A \in \mathcal{B}\}. \tag{4.7.1}$$

Lemma 4.7.1. *(i) The set $\widehat{\mathcal{S}}_*$ is dense in $\mathcal{S}_*$.*

(ii) The map $\varphi : A \in \mathcal{B} \mapsto A\rho_* \in \widehat{\mathcal{S}}_*$ can be extended to a unitary map from $\mathcal{B}_*$ to $\mathcal{S}_*$ (for which we keep the symbol φ),

$$\langle \varphi(A), \varphi(B)\rangle_{\mathrm{st},*} = \langle A, B\rangle_{\mathrm{obs},*} \quad \textit{for all } A, B \in \mathcal{B}_*. \tag{4.7.2}$$

Proof. (i) Since $\widehat{\mathcal{S}}_* \subseteq \mathcal{S}_*$, it suffices to show that $\lambda = 0$ if $\lambda \in \mathcal{S}_*$ and $\lambda \perp \widehat{\mathcal{S}}_*$. Suppose $\lambda \in \mathcal{S}_*$ satisfies $\lambda \perp \widehat{\mathcal{S}}_*$. Since, for any $A \in \mathcal{B}$, we have

$$0 = \langle \lambda, A\rho_*\rangle_{\mathrm{st},*} = \mathrm{Tr}\big(\lambda^*(A\rho_*)\rho_*^{-1}\big) = \mathrm{Tr}(\lambda^* A). \tag{4.7.3}$$

Since $\mathcal{B}$ is the dual of $\mathcal{S}_1$, this implies that $\lambda = 0$. Therefore, $\widehat{\mathcal{S}}_*$ is dense in $\mathcal{S}_*$.

(ii) It is easy to show (4.7.2) holds for any $A, B \in \mathcal{B}$. Hence, φ is an isometry. Since φ is invertible with the inverse $\varphi^{-1}(\lambda) = \lambda\rho_*^{-1}$ and since $\mathcal{B}, \widehat{\mathcal{S}}_*$ are dense in $\mathcal{B}_*, \mathcal{S}_*$, respectively, we can extend this map to a unitary map $\varphi : \mathcal{B}_* \to \mathcal{S}_*$ (using the same symbol). □

The significance of the map φ and therefore the space $\mathcal{S}_*$ lie in the fact that (QDB) (2.3.1) can be written as

$$L_0 \circ \varphi = -\varphi \circ L_0', \qquad G \circ \varphi = \varphi \circ G'. \tag{4.7.4}$$

This will be used to translate our results for the HL equation to ones on the vNL equation.

Equations (4.7.2) and (4.7.4) imply that, for all $A, B \in \mathcal{B}_*$,

$$\langle G(\varphi(A)), \varphi(B)\rangle_{\mathrm{st},*} = \langle G'(A), B\rangle_{\mathrm{obs},*}. \tag{4.7.5}$$

Proposition 4.7.2. *Suppose G satisfies (QDB). Then,*

(a) G is self-adjoint on $\mathcal{S}_*$,

(b) $G \le 0$ on $\mathcal{S}_*$,

(c) $\sigma(G) = \sigma(G')$, $\sigma_d(G) = \sigma_d(G')$,

(d) $\dim \mathrm{Null}(G) = \dim \mathrm{Null}(G')$.

(e) If Condition $(Compl)$ holds, then $\mathrm{Null}(G) = \mathbb{C} \cdot \rho_*$.

(f) L_0' and G' commute if and only if L_0 and G commute,

(g) $\mathrm{Null}(G') \subseteq \mathrm{Null}(L_0')$ if and only if $\mathrm{Null}(G) \subseteq \mathrm{Null}(L_0)$.

In the rest of this section, we drop the subscript "st, $*$" in the inner product $\langle\cdot,\cdot\rangle_{\mathrm{st},*}$ and norm $\|\cdot\|_{\mathrm{st},*}$ and denote the adjoint of operators L_0 and G on $\mathcal{S}_*$ by $L_0^\dagger$ and $G^\dagger$, respectively.

Proof of Proposition 4.7.2. We compute

$$\begin{aligned}\langle G(\varphi(A)), \varphi(B)\rangle &= \langle \varphi(G'(A)), \varphi(B)\rangle = \langle G'(A), B\rangle_{\mathrm{obs},*} \\ &= \langle A, G'(B)\rangle_{\mathrm{obs},*} = \langle \varphi(A), \varphi(G'(B))\rangle \\ &= \langle \varphi(A), G(\varphi(B))\rangle.\end{aligned} \tag{4.7.6}$$

Thus, G is self-adjoint on $\mathcal{S}_*$, which proves (a). For (b), for all $A \in \mathcal{B}_*$, using (4.7.5) and Theorem 3.1.2 (or (4.1.8)), we obtain

$$\langle \varphi(A), G(\varphi(A))\rangle = \langle A, G'(A)\rangle_{\mathrm{obs},*} \le 0. \tag{4.7.7}$$

Therefore $G \le 0$ on $\mathcal{S}_*$.

For (c), (4.7.4) implies $(G - z)^{-1} \circ \varphi = \varphi \circ (G' - z)^{-1}$, which, together with (4.7.2), yields $\sigma(G) = \sigma(G')$ and $\sigma_d(G) = \sigma_d(G')$.

For (d), we note that, by (4.7.4), we have $\varphi(\mathrm{Null}(G')) = \mathrm{Null}(G)$. Since $\varphi : \mathcal{B}_* \to \mathcal{S}_*$ is unitary, this implies (d).

If, in addition, Condition $(Compl)$ holds, then, by (4.5.3), we have

$$\begin{aligned}\langle \mu, P\lambda\rangle_{\mathrm{st},*} &= \mathrm{Tr}\big(\mu^*(P\lambda)\rho_*^{-1}\big) = \mathrm{Tr}\big((\rho_*^{-1}\mu^*)P\lambda\big) = \mathrm{Tr}\big(P'(\rho_*^{-1}\mu^*)\lambda\big) \\ &= \mathrm{Tr}\big(\rho_*^{-1}\mu^*\rho_*\big)\,\mathrm{Tr}(\mathbf{1}\cdot\lambda) = \langle \mu, \rho_*\rangle_{\mathrm{st},*}\,\mathrm{Tr}(\lambda)\end{aligned} \tag{4.7.8}$$

for any $\lambda, \mu \in \mathcal{S}_1$ so that

$$P\lambda = \mathrm{Tr}(\lambda)\rho_*, \tag{4.7.9}$$

which implies that $\mathrm{Null}(G) = \mathbb{C} \cdot \rho_*$, i.e., ρ_* is the unique density operator in $\mathrm{Null}(G)$. This proves (e).

For (f), by (4.7.4), since φ is unitary, we have

$$e^{L_0' t} = e^{(-\varphi^{-1} \circ L_0 \circ \varphi)t} = \varphi^{-1} \circ e^{-L_0 t} \circ \varphi, \tag{4.7.10}$$
$$e^{G' t} = \varphi^{-1} \circ e^{G' t} \circ \varphi. \tag{4.7.11}$$

Thus, $e^{L_0' t} e^{G' t} = e^{G' t} e^{L_0' t}$ if and only if $\varphi^{-1} \circ (e^{-L_0 t} e^{G t}) \circ \varphi = \varphi^{-1} \circ (e^{G t} e^{-L_0 t}) \circ \varphi$, which is true if and only if $e^{L_0 t} e^{G t} = e^{G t} e^{L_0 t}$. This proves (f).

Finally, since φ is unitary, we have $\mathrm{Null}(L_0) = \varphi(\mathrm{Null}(L_0'))$ and $\mathrm{Null}(G) = \varphi(\mathrm{Null}(G'))$, which proves (g). □

Proof of Theorem 3.2.2. (a) Recall that P denotes the orthogonal projector onto the subspace $\mathrm{Null}(G) \subseteq \mathcal{S}_*$ w.r.t. the inner product $\langle \cdot, \cdot \rangle$ in the space of observables and define $P^\perp = \mathbf{1} - P$. In the following, we denote $\rho_t = \beta_t(\rho)$ and $\bar{\rho}_t = P^\perp \rho_t$. Similar to the proof of Theorem 3.1.2 in Subsection 4.4, we consider the Lyapunov functional $\|\bar{\rho}_t\|^2/2$.

First, we note that, by Condition (*Null*), $[L_0, P] = 0$. Since $L_0^\dagger = -L_0$, $G^\dagger = G$ on $\mathcal{S}_*$ and $[L_0, P^\perp] = [L_0, \mathbf{1} - P] = 0$, by Leibniz rule, we have

$$\partial_t \|\bar{\rho}_t\|^2/2 = \partial_t \langle \bar{\rho}_t, \bar{\rho}_t \rangle/2 = \langle \bar{\rho}_t, (L^\dagger + L)\bar{\rho}_t \rangle/2 = \langle \bar{\rho}_t, G\bar{\rho}_t \rangle. \tag{4.7.12}$$

By the Condition (*Gap*) and Proposition 4.7.2 (c), using spectral theory, there exists some $\theta > 0$ such that $(-\theta, \theta) \cap (\sigma(G) \setminus \{0\}) = \varnothing$ so that

$$G|_{\mathrm{Ran}\, P^\perp} \leq -\theta < 0 \tag{4.7.13}$$

so that, by substituting (4.7.13) into (4.7.12), we obtain the following inequality

$$\partial_t \|\bar{\rho}_t\|^2/2 = \langle \bar{\rho}_t, G\bar{\rho}_t \rangle \leq -\theta \|\bar{\rho}_t\|^2 \tag{4.7.14}$$

with the initial condition

$$\|\bar{\rho}_t\|^2|_{t=0} = \|\bar{\rho}_0\|^2. \tag{4.7.15}$$

We solve this inequality to obtain

$$\|\bar{\rho}_t\| \leq e^{-\theta t} \|\bar{\rho}_0\|, \tag{4.7.16}$$

giving

$$\|\rho_t - P\rho_t\| \leq e^{-\theta t}\|\rho_0\|. \tag{4.7.17}$$

To complete the proof, we compute $P\rho_t$. Since, by Condition $(Null)$ and Proposition 4.7.2 (g), $\operatorname{Ran} P = \operatorname{Null}(G) \subseteq \operatorname{Null}(L_0)$ and $[L_0, P] = 0$, we have $\operatorname{Ran} P \subseteq \operatorname{Null}(L)$ and $[L, P] = [L_0 + G, P] = 0$. It follows that

$$P\rho_t = P\beta_t(\rho) = Pe^{Lt}\rho = e^{Lt}P\rho = P\rho, \tag{4.7.18}$$

which, together with (4.7.17), implies (3.2.1).

(b) Now, we suppose 0 is a simple eigenvalue of G. Then, we have $\operatorname{Null}(G) = \mathbb{C} \cdot \rho_*$ because $G(\rho_*) = 0$ by (QDB). Let $\rho \in \mathscr{D}(L)$ with $\operatorname{Tr}\rho = 1$. Then, by cyclicity of trace, we have

$$P\rho = \langle \rho_*, \rho\rangle \rho_* = \operatorname{Tr}\big(\rho_*\rho\rho_*^{-1}\big) \cdot \rho_* = \operatorname{Tr}(\rho) \cdot \rho_* = \rho_*. \tag{4.7.19}$$

By substituting this into (3.2.1), we obtain (3.2.2). □

Chapter 5

Remarks and extensions

In this subsection, we present remarks, extensions and variants of the results from Subsections 2.4, 3.1 and 3.2.

Remark 5.0.1. Condition (QDB) is a strong restriction implying essentially that the jump operators W_j can be replaced by multiples of the eigenvectors of the modular operator $\widetilde{\Delta}_* A := \rho_* A \rho_*^{-1}$ (see [15] for $\mathcal{H}$ with $\dim \mathcal{H} < \infty$).

Remark 5.0.2. Unlike for the von Neumann dynamics, for the von Neumann-Lindblad one, we cannot pass to the Hilbert space framework by simply writing $\mathrm{Tr}(A\rho) = \mathrm{Tr}\big(\sqrt{\rho}^* A \sqrt{\rho}\big) \equiv \langle \sqrt{\rho}, A\sqrt{\rho}\rangle$. Indeed, since

$$\sqrt{\beta_t \rho} \neq \beta_t(\sqrt{\rho})$$

(unlike for the unitary dynamics), $\sqrt{\beta_t \rho}$, entering in

$$\mathrm{Tr}((\beta_t' A)\rho) = \mathrm{Tr}(A\beta_t\rho) = \mathrm{Tr}\Big((\sqrt{\beta_t\rho})^* A \sqrt{\beta_t\rho}\Big) \equiv \langle \sqrt{\beta_t\rho}, A\sqrt{\beta_t\rho}\rangle,$$

does not satisfy a simple evolution equation.

Remark 5.0.3. Condition (QDB) is intimately related to the KMS-symmetric Markovian semigroup as remarked in Section 1. Let φ be a faithful state on a W^*-algebra $\mathcal{A}$. We call a semigroup P_t on $\mathcal{A}$ is *KMS-symmetric* if, for all $A, B \in \mathcal{A}$,

$$\varphi(P_t A \sigma^\varphi_{-i/2}(B)) = \varphi(\sigma^\varphi_{i/2}(A) P_t B), \tag{5.0.1}$$

where σ_t^φ is the modular automorphism on $\mathcal{A}$ associated with φ. In our case, we have $\mathcal{A} = \mathcal{B}$, $P_t = e^{G't}$, φ is the faithful, normal state ω_* determined by ρ_* and $\sigma_t^\varphi = \alpha_t$, where π_* and α_t are given in Theorem 4.2.6. Indeed, for each $A, B \in \mathcal{B}$, we use the commutativity between G' and $i[H_*, \cdot]$ (i.e., the generator of α_t) due to Condition (QDB), the self-adjointness on $\mathcal{B}_*$ and the relation

$$\omega_*(\alpha_{i/2}(A)B) = \omega_*(A\alpha_{-i/2}(B)) \tag{5.0.2}$$

to obtain

$$\begin{aligned}\omega_*(e^{G't}A\alpha_{-i/2}(B)) &= \omega_*(Ae^{G't}\alpha_{-i/2}(B)) \\ &= \omega_*(A\alpha_{-i/2}(e^{G't}B)) = \omega_*(\alpha_{i/2}(A)e^{G't}B).\end{aligned} \tag{5.0.3}$$

This suggests to a generalization of our results to KMS-symmetric Markovian semigroup on abstract von Neumann algebra.

Inner product (2.4.1) is a member of the family of inner products on $\mathcal{B}$: For each $r \in [0,1]$, we define the inner product

$$\langle A, B\rangle_{\mathrm{obs},r} = \mathrm{Tr}\big(A^*\rho_*^r B\rho_*^{1-r}\big). \tag{5.0.4}$$

Note that $\langle A, B\rangle_{\mathrm{obs},0} = \langle A, B\rangle_{\mathrm{obs},*}$ and $\langle A, B\rangle_{\mathrm{obs},1} = \langle B^*, A^*\rangle_{\mathrm{obs},*}$. G' is symmetric in $\langle \cdot, \cdot\rangle_{\mathrm{obs},1}$. Indeed, using the fact that G' is a $*$-map and is symmetric w.r.t. (2.4.1), we obtain

$$\begin{aligned}\langle G'(A), B\rangle_{\mathrm{obs},1} &= \langle B^*, G'(A^*)\rangle_{\mathrm{obs},*} \\ &= \langle G'(B^*), A^*\rangle_{\mathrm{obs},*} = \langle A, G'(B)\rangle_{\mathrm{obs},1}.\end{aligned} \tag{5.0.5}$$

Similarly, we can define another Hilbert space of states as follows. Consider the set

$$\tilde{\mathcal{S}}_*^{\intercal} := \{\lambda \in \mathcal{S}_1 \mid \rho_*^{-1/2}\lambda \in \mathcal{S}_2\} \tag{5.0.6}$$

We define the Hilbert space $\mathcal{S}_*^{\intercal}$ as the completion of the set in (5.0.6) w.r.t. the norm corresponding to the inner product

$$\langle \lambda, \mu\rangle_{\mathrm{st},*}^{\intercal} := \mathrm{Tr}\big(\lambda^*\rho_*^{-1}\mu\big). \tag{5.0.7}$$

Lemma 5.0.4. *The spaces $\mathcal{S}_*$ and $\mathcal{S}_*^{\intercal}$ are the dual of $\mathcal{B}_*$ w.r.t. the couplings $\mathrm{Tr}\Big(A\rho_*^{1/2}\lambda\rho_*^{-1/2}\Big)$ and $(A,\lambda) := \mathrm{Tr}(A\lambda)$, respectively.*

Proof. We check only $\mathcal{S}_*^{\intercal}$; the space $\mathcal{S}_*$ is done similarly. By the non-Abelian Cauchy-Schwarz inequality, for each $A \in \mathcal{B}_*$ and $\lambda \in \mathcal{S}_*^{\intercal}$, we have

$$\begin{aligned}|\,\mathrm{Tr}(A\lambda)| = |\,\mathrm{Tr}\big(A\rho_*^{1/2}\rho_*^{-1/2}\lambda\big)| &\le \|A\rho_*^{1/2}\|_{\mathcal{S}_2}\|\rho_*^{-1/2}\lambda\|_{\mathcal{S}_2} \\ &= (\mathrm{Tr}(A^*A\rho_*))^{1/2}(\mathrm{Tr}\big(\lambda^*\rho_*^{-1}\lambda\big))^{1/2} \\ &= \|A\|_{\mathrm{obs},*}\|\lambda\|_{\mathrm{st},*}^{\intercal}\end{aligned} \tag{5.0.8}$$

so that

$$\sup_{\|A\|_{\mathrm{obs},*}=1} |\operatorname{Tr}(A\lambda)| \leq \|\lambda\|^{\intercal}_{\mathrm{st},*}. \tag{5.0.9}$$

We now show that this is in fact an equality. By taking $A = \lambda^* \rho_*^{-1}$, we have

$$\|A\|^2_{\mathrm{obs},*} = \operatorname{Tr}(A^* A \rho_*) = \operatorname{Tr}\big(\rho_*^{-1}\lambda\lambda^*\big) = (\|\lambda\|^{\intercal}_{\mathrm{st},*})^2 < \infty \tag{5.0.10}$$

so that $A \in \mathcal{B}_*$ and, by taking $A' = A/\|\lambda\|^{\intercal}_{\mathrm{st},*}$, we have $\|A'\|_{\mathrm{obs},*} = 1$ and

$$\operatorname{Tr}(A'\lambda) = (\|\lambda\|^{\intercal}_{\mathrm{st},*})^{-1} \operatorname{Tr}\big(\lambda^* \rho_*^{-1} \lambda\big) = \|\lambda\|^{\intercal}_{\mathrm{st},*}. \tag{5.0.11}$$

Therefore, $\mathcal{S}^{\intercal}_*$ is the dual space of $\mathcal{B}_*$ w.r.t. the coupling $(\cdot,\cdot)$. □

Similar to (4.7.1), we define a map

$$\varphi^{\intercal}(A) := \rho_*^{1/2} A \rho_*^{1/2}. \tag{5.0.12}$$

By its definition, we have

$$\langle \varphi^{\intercal}(A), \varphi^{\intercal}(B)\rangle^{\intercal}_{\mathrm{st},*} = \langle A, B\rangle_{\mathrm{obs},*}. \tag{5.0.13}$$

Remark 5.0.5. (a) The space $\mathcal{S}_*$ is a point $r = 0$ in the family of Hilbert spaces $\mathcal{S}^{(r)}_*$, $r \in [0,1]$, defined by the completion of the set

$$\tilde{\mathcal{S}}^{(r)} = \{\lambda \in \mathcal{S}_1 \mid \rho_*^{-r/2} \lambda \rho_*^{-(1-r)/2} \in \mathcal{S}_2\}, \tag{5.0.14}$$

under the norm $\|\lambda\|_{\mathrm{st},r} := \sqrt{\langle \lambda, \lambda\rangle_{\mathrm{st},r}}$, corresponding to the inner product

$$\langle \lambda, \sigma\rangle_{\mathrm{st},r} := \operatorname{Tr}\big(\lambda^* \rho_*^{-r} \sigma \rho_*^{-1+r}\big). \tag{5.0.15}$$

For each $r \in [0,1]$, the space $\mathcal{S}^{(r)}_*$ and $\mathcal{S}_2$ are in one-to-one correspondence. Indeed, we define the map

$$\pi^{(r)} : \mathcal{S}_2 \to \mathcal{S}^{(r)}_* \qquad \pi^{(r)}(\kappa) := \rho_*^{r/2} \kappa \rho_*^{(1-r)/2}. \tag{5.0.16}$$

Obviously, the map $\pi^{(r)}$ is linear, bounded and invertible. Moreover, for all $\lambda, \sigma \in \mathcal{S}_2$, by (5.0.15), (5.0.16) and the cyclicity of trace, we have

$$\langle \pi^{(r)}(\lambda), \pi^{(r)}(\sigma)\rangle_{\mathrm{st},r} = \langle \lambda, \sigma\rangle_{\mathcal{S}_2} \tag{5.0.17}$$

so that $\pi^{(r)}$ is unitary. Hence, $\mathcal{S}^{(r)}_*$ is isomorphic to $\mathcal{S}_2$.

(b) We note that the inner products (5.0.15), $r \in [0,1]$, are special examples of the so-called quantum χ^2-divergence (c.f. [68]).

(c) Consider the family of maps $\varphi^{(r)} : \mathcal{B} \to \mathcal{S}_*^{(r)}$, $r \in [0,1]$, as $\varphi^{(r)}(A) = \rho_*^{r/2} A \rho_*^{1-r/2}$. Since $\varphi^{(r)}$ maps $\mathcal{B}$ into $\tilde{\mathcal{S}}_*^{(r)}$ and, for each $A, B \in \mathcal{B}$,

$$\begin{aligned}\langle \varphi^{(r)}(A), \varphi^{(r)}(B)\rangle_{\mathrm{st},r} &= \mathrm{Tr}\big((\rho_*^{1-r/2} A^* \rho_*^{r/2})\rho_*^{-r}(\rho_*^{r/2} B \rho_*^{1-r/2})\rho_*^{-1+r}\big)\\ &= \mathrm{Tr}(A^* B \rho_*) = \langle A, B\rangle_{\mathrm{obs},*}. \end{aligned} \tag{5.0.18}$$

By a density argument, we can extend $\varphi^{(r)}$ to a unitary map from $\mathcal{B}_*$ to $\mathcal{S}_*^{(r)}$, and we denote this extension also by $\varphi^{(r)}$. Thus, $\mathcal{B}_*$ and $\mathcal{S}_*^{(r)}$ are isomorphic.

Maps φ and $\varphi^{\intercal}$ in Lemma 4.7.1 and in (5.0.12) are the $r = 0$ and $r = 1$ cases of tNote that $\varphi^{(r)}$ is not $*$-map, unless $r = 1$.

(d) For any operator K on $\mathcal{S}_1$ and its dual K' on $\mathcal{B}$, we have

$$K' = *\varphi^{-1} K^* \varphi *, \tag{5.0.19}$$

where $* : A \mapsto A^*$ and the adjoint K^* is taken w.r.t. (2.4.3). Indeed, for any $A \in \mathcal{B}$ and $\lambda \in \mathcal{S}_1$, we have

$$\begin{aligned}(*\varphi^{-1}K^*\varphi * A, \lambda) &= \mathrm{Tr}\big((*\varphi^{-1}(K^*(A^*\rho_*)))\lambda\big) = \mathrm{Tr}\big(((K^*(A^*\rho_*))\rho_*^{-1})^*\lambda\big)\\ &= \mathrm{Tr}\big(\rho_*^{-1}(K^*(A^*\rho_*))^*\lambda\big) = \langle K^*(A^*\rho_*), \lambda\rangle_{\mathrm{st},*}\\ &= \langle A^*\rho_*, K\lambda\rangle_{\mathrm{st},*} = \mathrm{Tr}\big(\rho_* A (K\lambda)\rho_*^{-1}\big)\\ &= \mathrm{Tr}(A(K\lambda)) = (A, K\lambda) = (K'A, \lambda), \end{aligned} \tag{5.0.20}$$

which implies (5.0.19).

Remark 5.0.6. The following result presents a variant of Theorem 3.2.2 for the space $\mathcal{S}_*^{\intercal}$.

Theorem 5.0.7. *Assume Conditions (H), (W), (QDB) and that* 0 *is a simple, isolated eigenvalue of* G'*. Then, for all density operator* $\rho \in \mathcal{S}_*^{\intercal}$ *and for* $\theta = \mathrm{dist}(0, \sigma(G') \setminus \{0\})$*, we have*

$$\|\beta_t(\rho) - \rho_*\|_{\mathrm{st},*}^{\intercal} \le e^{-\theta t}\|\rho\|_{\mathrm{st},*}^{\intercal}. \tag{5.0.21}$$

This theorem is proven below.

Remark 5.0.8. (i) The property $G' \le 0$ is related to the contractivity of β_t'. Indeed, for $A(t) := \beta_t'(A)$, using the Leibniz rule and HL equation (2.1.1), we find

$$\begin{aligned}\partial_t \|A(t)\|_{\mathrm{obs},*}^2/2 &= (\langle L'A(t), A\rangle_{\mathrm{obs},*} + \langle A(t), L'A(t)\rangle_{\mathrm{obs},*})/2\\ &= \mathrm{Re}\langle A(t), L'A(t)\rangle_{\mathrm{obs},*}. \end{aligned} \tag{5.0.22}$$

Since L'_0 is anti-self-adjoint on $\mathcal{B}_*$, we have

$$\mathrm{Re}\langle A(t), L'_0 A(t)\rangle_{\mathrm{obs},*} = 0. \tag{5.0.23}$$

This, the relation $L' = L'_0 + G'$ and the self-adjointness of G' imply

$$\mathrm{Re}\langle A(t), L' A(t)\rangle_{\mathrm{obs},*} = 2\langle A(t), G' A(t)\rangle_{\mathrm{obs},*},$$

and therefore,

$$\partial_t \|A(t)\|^2_{\mathrm{obs},*} = 2\langle A(t), G' A(t)\rangle_{\mathrm{obs},*} \leq 0. \tag{5.0.24}$$

(ii) The property $D_{G'}(A, A) \geq 0$ for all $A \in \mathcal{B}_*$ is related to the complete positivity of β'_t. Indeed, we can generalize the operator function (4.1.5) for L' on $\mathscr{D}(L')$ as

$$D_{L'}(A, B) = L'(A^* B) - A^* L'(B) - (L'(A))^* B. \tag{5.0.25}$$

Indeed, since L'_0 is a derivation, the domain $\mathscr{D}(L') = \mathscr{D}(L'_0)$ is a *-subalgebra in $\mathcal{B}$ so that AB and $A^* \in \mathscr{D}(L')$, provided that $A, B \in \mathscr{D}(L')$. By the definition of derivations, we can show immediately that, for all $A, B \in \mathscr{D}(L')$,

$$D_{L'}(A, B) = D_{G'}(A, B). \tag{5.0.26}$$

Since G' is bounded on $\mathcal{B}_*$, we can extend $D_{L'}$ to entire $\mathcal{B}_*$. It then follows from (4.1.8) that $D_{L'}(A, A) \geq 0$ for all $A \in \mathcal{B}_*$.

Now, we show the positivity of $D_{L'}(A, A)$ follows from the property of complete positivity of β'_t. To show this, since β'_t is completely positive and $\beta'_t(\mathbf{1}) = \mathbf{1}$, by Kraus' theorem, we have, for each $t \geq 0$,

$$\beta'_t(A) = \sum_{k\geq 1} V_k^*(t) A V_k(t), \qquad \sum_{k \geq 1} V_k^*(t) V_k(t) = \mathbf{1} \tag{5.0.27}$$

where $V_k(t) \in \mathcal{B}$ for all k. Then, by the operator inequality (A.1.5), we obtain, for each $t \geq 0$,

$$0 \leq \beta'_t(A^* A) - \beta'_t(A)^* \beta'_t(A). \tag{5.0.28}$$

If we take $A \in \mathscr{D}(L')$ and differentiate at $t = 0$, we obtain

$$\begin{aligned} 0 &\leq L'(A^* A) - A^* L'(A) - (L'(A))^* A \\ &= D_{L'}(A, A) = D_{G'}(A, A). \end{aligned} \tag{5.0.29}$$

In fact, it was shown in [57] that, in the case that L' is bounded, the positivity of $D_{L'}(A, A)$ for all $A \in \mathcal{B}_*$ is also sufficient for L' to generate a (norm continuous) completely positive semigroup.

Proof of Theorem 5.0.7. First of all, we note that, since $\mathrm{Tr}(\rho) = 1$,

$$\begin{aligned}\mathrm{Tr}(A(\beta_t(\rho) - \rho_*)) &= \mathrm{Tr}(\beta_t'(A)\rho) - \mathrm{Tr}(A\rho_*) \\ &= \mathrm{Tr}\big((\beta_t'(A) - c^A \mathbf{1})\rho\big),\end{aligned} \tag{5.0.30}$$

where $c^A = \mathrm{Tr}(A\rho_*)$.

Recall that, by Theorem 3.1.2 (b), G' is self-adjoint and $G' \leq 0$ on the space $\mathcal{B}_*$. Since 0 is an isolated eigenvalue of G', $\sigma(G') \cap (-\theta, \theta) = \varnothing$ for $\theta := \mathrm{dist}(0, \sigma(G') \setminus \{0\})$, and so, by (5.0.30) and (5.0.8), we have

$$\begin{aligned}\|\beta_t(\rho) - \rho_*\|_{\mathrm{st},*}^{\intercal} &= \sup_{\|A\|_{\mathrm{obs},*}=1} |\,\mathrm{Tr}(A(\beta_t(\rho) - \rho_*))| \\ &= \sup_{\|A\|_{\mathrm{obs},*}=1} |\,\mathrm{Tr}\big((\beta_t'(A) - c^A\mathbf{1})\rho\big)| \\ &\leq \sup_{\|A\|_{\mathrm{obs},*}=1} \|\beta_t'(A) - c^A\mathbf{1}\|_{\mathrm{obs},*} \|\rho\|_{\mathrm{st},*}^{\intercal} \\ &\leq e^{-\theta t} \|\rho\|_{\mathrm{st},*}^{\intercal} \sup_{\|A\|_{\mathrm{st},*}=1} \|(\mathbf{1} - P')A\|_{\mathrm{obs},*} \\ &\leq e^{-\theta t} \|\rho\|_{\mathrm{st},*}^{\intercal}.\end{aligned} \tag{5.0.31}$$

Now, using (3.1.4) and recalling that P' is the rank-1 projector onto $\mathrm{Null}(G') = \mathbb{C} \cdot \rho_*$, we find $\|\beta_t(\rho) - \rho_*\|_{\mathrm{st},*}^{\intercal} \leq e^{-\theta t}\|\rho\|_{\mathrm{st},*}^{\intercal}$, as desired. □

Appendix A

Proofs of Theorems 2.2.1, 3.1.1 and 3.2.1

A.1 Dual Lindblad operator G'

Throughout this section, we assume Condition (W) holds. Recall from Section 2.2 that the HL operator L', acting on observables, is given by

$$L' = L'_0 + G', \quad L'_0 A = i[H, A], \tag{A.1.1}$$

$$G'(A) := \sum_{j\geq 1}(W_j^* A W_j - \frac{1}{2}\{W_j^* W_j, A\}). \tag{A.1.2}$$

We begin with two propositions stating the boundedness of the operator G' on space $\mathcal{B}$ and $\mathcal{B}_*$.

Proposition A.1.1. *The operator G' is bounded on $\mathcal{B}$.*

Proof. Let

$$\Phi(A) := \sum_{j\geq 1} W_j^* A W_j, \qquad Y := \Phi(\mathbf{1}) = \sum_{j\geq 1} W_j^* W_j \tag{A.1.3}$$

so that

$$G'(A) = \Phi(A) - \frac{1}{2}\{Y, A\}. \tag{A.1.4}$$

First, we estimate $\Phi(A)$. We claim that, for all $A \in \mathcal{B}$,

$$\Phi(A)^*\Phi(A) \leq \|\Phi(\mathbf{1})\|\Phi(A^*A). \tag{A.1.5}$$

Indeed, for any $u, v \in \mathcal{H}$ and for any $A \in \mathcal{B}$, we have

$$\begin{aligned}
|\langle u, \Phi(A)v\rangle| &\leq |\sum_{j\geq 1}\langle W_j u, AW_j v\rangle| \\
&\leq \left(\sum_{j\geq 1}\|W_j u\|^2\right)^{1/2}\left(\sum_{j\geq 1}\|AW_j v\|^2\right)^{1/2} \\
&= \left(\sum_{j\geq 1}\langle u, W_j^* W_j u\rangle\right)^{1/2}\left(\sum_{j\geq 1}\langle v, W_j^* A^* AW_j v\rangle\right)^{1/2} \\
&= \langle u, \Phi(\mathbf{1})u\rangle^{1/2}\langle v, \Phi(A^*A)v\rangle^{1/2} \\
&\leq \|\Phi(\mathbf{1})\|^{1/2}\|u\|\langle v, \Phi(A^*A)v\rangle^{1/2}. \qquad \text{(A.1.6)}
\end{aligned}$$

By taking $u = \Phi(A)v$, (A.1.6) implies that

$$\|\Phi(A)v\|^2 \leq \|\Phi(\mathbf{1})\|\langle v, \Phi(A^*A)v\rangle, \qquad \text{(A.1.7)}$$

which, since $\|\Phi(A)v\|^2 = \langle v, \Phi(A)^*\Phi(A)v\rangle$, gives

$$\langle v, \Phi(A)^*\Phi(A)v\rangle \leq \|\Phi(\mathbf{1})\|\langle v, \Phi(A^*A)v\rangle \qquad \text{(A.1.8)}$$

for all $v \in \mathcal{H}$. This is equivalent to (A.1.5).

Next, we estimate, for all $A \in \mathcal{B}$ and $v \in \mathcal{H}$,

$$\begin{aligned}
\langle v, \Phi(A^*A)v\rangle &= \sum_{j\geq 1}\langle AW_j v, AW_j v\rangle = \sum_{j\geq 1}\|AW_j v\|^2 \\
&\leq \|A\|^2\langle v, \sum_{j\geq 1}W_j^* W_j v\rangle \\
&\leq \|Y\|^2\|A\|^2\|v\|^2. \qquad \text{(A.1.9)}
\end{aligned}$$

Using (A.1.5), (A.1.9) and the definition $Y = \Phi(\mathbf{1})$, we conclude that

$$\begin{aligned}
\|\Phi(A)v\|^2 &= \langle v, \Phi(A)^*\Phi(A)v\rangle \\
&\leq \|Y\|\langle v, \Phi(A^*A)v\rangle \leq \|Y\|^2\|A\|^2\|v\|^2. \qquad \text{(A.1.10)}
\end{aligned}$$

Therefore, we have

$$\|\Phi(A)\| \leq \|Y\|\|A\|. \qquad \text{(A.1.11)}$$

Now, for any $A \in \mathcal{B}$, by triangle inequality and (A.1.9), we have

$$\begin{aligned}\|G'(A)\| &\leq \|\Phi(A)\| + \frac{1}{2}\|\{Y, A\}\| \\ &\leq \|\Phi(A)\| + \frac{1}{2}(\|YA\| + \|AY\|) \\ &\leq \|\Phi(A)\| + \|Y\|\|A\| \\ &\leq 2\|Y\|\|A\|. \end{aligned} \tag{A.1.12}$$

Therefore, G' is bounded on $\mathcal{B}$. □

Remark A.1.2. For another proof for (A.1.5), we refer to [20], using the completely positive property of Φ.

For the second proposition, we need the following

Lemma A.1.3. *Suppose Condition* (*QDB*) *holds. Then,* $\sum_{j\geq 1} W_j^* W_j$ *commutes with* ρ_**.*

Proof. Let $Y := \sum_{j\geq 1} W_j^* W_j$. By (2.3.1), for all $A \in \mathcal{B}$, we have

$$\begin{aligned} G'(A)\rho_* &= \sum_{j\geq 1} W_j^* A W_j - \frac{1}{2}\{Y, A\}\rho_* \\ &= \sum_{j\geq 1} W_j A \rho_* W_j^* - \frac{1}{2}\{Y, A\rho_*\} = G(A\rho_*). \end{aligned} \tag{A.1.13}$$

Since this is true for all $A \in \mathcal{B}$, we must have

$$\sum_{j\geq 1} W_j^* A W_j \rho_* = \sum_{j\geq 1} W_j A \rho_* W_j^*, \qquad \{Y, A\}\rho_* = \{Y, A\rho_*\}. \tag{A.1.14}$$

From the second equality in (A.1.14), we must have

$$Y\rho_* = \rho_* Y, \tag{A.1.15}$$

which proves the result. □

Proposition A.1.4. *Suppose Condition* (*QDB*) *holds. Then,* G' *is bounded on* $\mathcal{B}_*$*.*

Proof. Since $\mathcal{B}_*$ is the completion of $\mathcal{B}$ w.r.t. the norm $\|\cdot\|_{\text{obs},*}$, then $\mathcal{B}$ is a dense subspace of $\mathcal{B}_*$. Hence, it suffices to show that G' is bounded on $\mathcal{B}$ with respect the the norm $\|\cdot\|_{\text{obs},*}$.

By (A.1.4) and triangle inequality, we have

$$\|G'(A)\|_{\text{obs},*} \leq \|\Phi(A)\|_{\text{obs},*} + \frac{1}{2}\|\{Y, A\}\|_{\text{obs},*}. \tag{A.1.16}$$

To estimate the first term on the r.h.s. of (A.1.16), we use inequality (A.1.5). Since $\rho_* > 0$, this inequality, together with cyclicity of trace, gives

$$\begin{aligned}\|\Phi(A)\|^2_{\text{obs},*} &= \mathrm{Tr}(\Phi(A)^*\Phi(A)\rho_*) \le \|Y\|\,\mathrm{Tr}(\Phi(A^*A)\rho_*)\\ &= \|Y\|\,\mathrm{Tr}(A^*A\Psi(\rho_*)).\end{aligned} \tag{A.1.17}$$

where $\Psi(\rho) := \sum_{j\ge1} W_j\rho W_j^*$, the dual of Φ. Since $G(\rho_*) = 0$ (see (2.3.1)) and, by Lemma A.1.3, ρ_* and Y commute, (2.2.3) implies that

$$\Psi(\rho_*) = \frac{1}{2}\{Y, \rho_*\} = \rho_* Y = \rho_*^{1/2} Y \rho_*^{1/2}, \tag{A.1.18}$$

where, for the last equality in (A.1.18), we use that $[\rho_*, Y] = 0$ implies $[\rho_*^{1/2}, Y] = 0$. Next, using cyclicity of trace and Lemma A.1.3, we obtain

$$\begin{aligned}\sum_{j\ge1}\mathrm{Tr}\big(A^*AW_j\rho_*W_j^*\big) &= \mathrm{Tr}(A^*A\rho_*Y) = \mathrm{Tr}\big(A^*A\rho_*^{1/2}Y\rho_*^{1/2}\big)\\ &= \mathrm{Tr}\big(A\rho_*^{1/2}Y\rho_*^{1/2}A^*\big) \le \|Y\|\,\mathrm{Tr}(A\rho_*A^*) = \|Y\|\|A\|^2_{\text{obs},*}\end{aligned} \tag{A.1.19}$$

so that, by substituting (A.1.18) into the last term in (A.1.17),

$$\|\Phi(A)\|^2_{\text{obs},*} \le \|Y\|^2\|A\|^2_{\text{obs},*}. \tag{A.1.20}$$

Finally, for the second term on the r.h.s. of (A.1.16), we use the triangle inequality, cyclicty of trace and Lemma A.1.3 to obtain

$$\begin{aligned}\|\{Y, A\}\|^2_{\text{obs},*} &\le \|YA\|^2_{\text{obs},*} + \|AY\|^2_{\text{obs},*}\\ &= \mathrm{Tr}\big(A^*Y^2A\rho_*\big) + \mathrm{Tr}(YA^*AY\rho_*)\\ &= \mathrm{Tr}\big(\rho_*^{1/2}A^*Y^2A\rho_*^{1/2}\big) + \mathrm{Tr}\big(\rho_*^{1/2}YA^*AY\rho_*^{1/2}\big)\\ &= \mathrm{Tr}\big(\rho_*^{1/2}A^*Y^2A\rho_*^{1/2}\big) + \mathrm{Tr}\big(A\rho_*^{1/2}Y^2\rho_*^{1/2}A^*\big).\end{aligned} \tag{A.1.21}$$

Using the operator inequality

$$A^*Y^2A \le \|Y\|^2A^*A, \qquad A\rho_*^{1/2}Y^2\rho_*^{1/2}A^* \le \|Y\|^2A\rho_*A^*, \tag{A.1.22}$$

we find

$$\|\{Y, A\}\|_{\text{obs},*} \le \|Y\|\|A\|_{\text{obs},*}. \tag{A.1.23}$$

By substituting (A.1.17) and (A.1.23) into (A.1.16), we obtain

$$\|G'(A)\|_{\text{obs},*} \le 2\|Y\|\|A\|_{\text{obs},*} \tag{A.1.24}$$

so that G' is bounded on $\mathcal{B}$, hence is bounded on $\mathcal{B}_*$, since $\mathcal{B}$ is dense in $\mathcal{B}_*$. □

A.2 Proof of Theorem 2.2.1

Theorem 2.2.1 (a) follows from the following abstract theorem (c.f. [13] Theorem 3.1.33):

Theorem A.2.1. *Let X be a Banach space and let V_t be a C_0- (resp. C_0^*-) semigroup on X with generator S and let P be some bounded operator on X. Then $S+P$ generates a C_0- (resp. C_0^*-) continuous semigroup V_t^P on X.*

Proof of Theorem 2.2.1 (a). Assume Condition (H) and (W) hold. By Proposition A.1.1, G' is bounded on $\mathcal{B}$, while L_0' generates a one-parameter strongly continuous group $e^{L_0' t}$ of bounded operators on $\mathcal{B}$:

$$e^{L_0' t} A = e^{iHt} A e^{-iHt}. \tag{A.2.1}$$

Hence, Theorem A.2.1 implies that the HL operator $L' = L_0' + G'$ generates a bounded semigroup β_t' on $\mathcal{B}$. Therefore, the HL equation has a unique soluton in $\mathcal{B}$ for any given initial condition in $\mathscr{D}(L')$. □

For part (b), we need the next proposition:

Proposition A.2.2. *The operator G defined in* (2.2.3) *is bounded on the space $\mathcal{S}_1$.*

Proof. Since G is linear, it suffices to show that G is bounded for any $\rho \geq 0$ in $\mathcal{S}_1$. Thus, without loss of generality, we assume $\rho \geq 0$.

Clearly, $\sum_j W_j \rho W_j^* \geq 0$ if $\rho \geq 0$. Hence, let $Y := \sum_{j \geq 1} W_j^* W_j$, by the triangle inequality and Schwartz inequalities and the standing assumption (W),

$$\begin{aligned} \left\| \sum\nolimits_j W_j \rho W_j^* \right\|_{\mathcal{S}_1} &\leq \sum\nolimits_j \| W_j \rho W_j^* \|_{\mathcal{S}_1} = \sum\nolimits_j \operatorname{Tr}\big(W_j \rho W_j^*\big) \\ &= \sum\nolimits_j \operatorname{Tr}\big(W_j^* W_j \rho\big) = \operatorname{Tr}\left(Y \rho\right) \\ &\leq \|Y\| \, \|\rho\|_{\mathcal{S}_1} < \infty. \end{aligned} \tag{A.2.2}$$

On the other hand, again by Condition (W), then

$$\|\{Y, \rho\}\|_{\mathcal{S}_1} \leq \|Y\rho\|_{\mathcal{S}_1} + \|\rho Y\|_{\mathcal{S}_1} \leq 2\|Y\| \|\rho\|_{\mathcal{S}_1} < \infty. \tag{A.2.3}$$

Therefore, the operator G is bounded on $\mathcal{S}_1$. □

Proof of Theorem 2.2.1 (b). Since L_0 generates a one-parameter, strongly continuous group of bounded operators on $\mathcal{S}_1$, given explicitly by $e^{L_0 t}\rho = e^{-iHt}\rho e^{iHt}$, and, by Proposition A.2.2, G is bounded on $\mathcal{S}_1$, it follows from Theorem A.2.1, with $X = S_1$, that $L = L_0 + G$ generates a one-parameter, strongly continuous semigroup on $\mathcal{S}_1$.

For weak uniqueness, for any $A \in \mathscr{D}(L')$ and $\rho_0 \in \mathcal{S}_1$, we write ρ_t for a solution to (2.2.1) with the initial condition ρ_0 and $\langle A, \rho\rangle := \mathrm{Tr}(A\rho)$ so that, by Leibniz rule,

$$\begin{aligned}\partial_s\langle A, \beta_{t-s}(\rho_s)\rangle &= \partial_s\langle \beta'_{t-s}(A), \rho_s\rangle \\ &= \langle -L'\beta'_{t-s}(A), \rho_s\rangle + \langle \beta'_{t-s}(A), L\rho_s\rangle = 0. \end{aligned} \tag{A.2.4}$$

This implies that

$$\langle A, \rho_t\rangle = \langle A, \beta'_{t-s}(\rho_s)\rangle|_{t=s} = \langle A, \beta'_{t-s}(\rho_s)\rangle_{s=0} = \langle A, \beta_t(\rho_0)\rangle. \tag{A.2.5}$$

This proves the weak uniqueness of solution to (2.2.1) with any initial condition $\rho \in \mathcal{S}_1$.

For strong uniqueness, we proceed similarly by taking $\rho_0 \in \mathscr{D}(L)$ and write ρ_t for a solution to (2.2.1) with the initial condition ρ_0. Then, we have

$$\begin{aligned}\partial_s\beta_{t-s}(\rho_s) &= -L\beta_{t-s}(\rho_s) + \beta_{t-s}(L\rho_s) \\ &= \beta_{t-s}(-L\rho_s + L\rho_s) = 0. \end{aligned} \tag{A.2.6}$$

Thus, we have

$$\rho_t = \beta_{t-s}(\rho_s)|_{t=s} = \beta_{t-s}(\rho_s)|_{s=0} = \beta_t(\rho_0). \tag{A.2.7}$$

This completes the proof of Theorem 2.2.1 (b). □

Lemma A.2.3. *The semigroup β_t is completely positive on $\mathcal{S}_1$.*

Proof. The argument follows from Theorem 5.2 in [27]. For this, we rewrite the vNL operator as

$$\begin{aligned}L(\rho) &= -i[H, \rho] + \sum_{j\geq 1}(W_j\rho W_j^* - \frac{1}{2}\{W_j^*W_j, \rho\}) \\ &= [-iH - Y, \rho] + F(\rho), \end{aligned} \tag{A.2.8}$$

where $Y = Y^* = \frac{1}{2}\sum_{j\geq 1} W_j^*W_j$ and $\Psi(\rho) = \sum_{j\geq 1} W_j\rho W_j^*$.

Let $B_t := e^{-iHt - Yt}$, which is well-defined since Y is bounded. Then, the semigroup S_t generated by $-i[H - iY, \cdot]$ is given by

$$T_t(\rho) = B_t\rho B_t^*, \tag{A.2.9}$$

which defines a completely positive semigroup by Kraus' theorem. On the other hand, since, by Lemma A.2.2, the map Ψ is bounded on $\mathcal{S}_1$, Ψ generates a semigroup on $\mathcal{S}_1$

$$e^{\Psi t} := \sum_{k=0}^{\infty} \frac{t^k}{k!} \Psi^k. \tag{A.2.10}$$

We note that, since Ψ is completely positive by Kraus' theorem, so are Ψ^k for any $k = 0, 1, 2, ...$ and any linear combination of them. Therefore, $e^{\Psi t}$ is a completely positive semigroup.

Then, it follows that, for any $n \in \mathbb{N}$, the operator $(T_{t/n} e^{\Psi t/n})^n$ is a completely positive semigroup on $\mathcal{S}_1$.

Finally, by Trotter-Lie formula, we have

$$\beta_t(\rho) = \exp(L_0 + G)(\rho) = \lim_{n\to\infty} (T_{t/n} e^{\Psi t/n})^n(\rho), \tag{A.2.11}$$

where the limit is taken in the trace norm, so that the semigroup β_t is completely positive. □

Note that the proof of Lemma A.2.3 provides a different construction for the semigroup β_t.

A.3 Proof of Theorem 3.1.1

By Proposition A.1.4, G' is bounded on $\mathcal{B}_*$. Also, since L_0' is anti-self-adjoint on $\mathcal{B}_*$, it generates a one-parameter strongly continuous group $e^{L_0' t}$ of unitary operators on $\mathcal{B}_*$ as in (A.2.1). Then, by Theorem A.2.1 again, the HL operator L' generates a bounded semigroup β_t', and therefore the HL equation has a unique soluton in $\mathcal{B}_*$ for any given initial condition in $\mathscr{D}(L')$. □

Remark A.3.1. The unitarity of $e^{L_0' t}$ can be proven directly as follows. For all $A, B \in \mathcal{B}_*$,

$$\begin{aligned} \langle e^{L_0' t} A, e^{L_0' t} B \rangle_{\text{obs},*} &= \text{Tr}\big(e^{iHt} A^* B e^{-iHt} \rho_*\big) \\ &= \text{Tr}\big(A^* B e^{-iHt} \rho_* e^{iHt}\big) \\ &= \text{Tr}(A^* B \rho_*) = \langle A, B \rangle_{\text{obs},*} \end{aligned} \tag{A.3.1}$$

by cyclicity of trace and the fact that $L_0' \rho_* = 0$ by (QDB). Hence, it is unitary.

The semigroup, β_t', generated by L', is dual to β_t. Indeed, the semigroup β_t' has the generator L':

$$\partial_t|_{t=0} \text{Tr}(\beta_t'(A)\rho) = \partial_t|_{t=0} \text{Tr}(A \beta_t(\rho)) = \text{Tr}(A L(\rho)) = \text{Tr}(L'(A)\rho). \tag{A.3.2}$$

Given that G' is bounded on $\mathcal{B}$, the domain of L' is the same as the domain $\mathscr{D}(L'_0)$ of L'_0, where

$$\mathscr{D}(L'_0) := \{A \in \mathcal{B} \mid A(\mathscr{D}(H)) \subset \mathscr{D}(H) \text{ and } HA - AH \\ \text{defined on } \mathscr{D}(H) \text{ extends to an element in } \mathcal{B}\}. \tag{A.3.3}$$

A.4 Proof of Theorem 3.2.1

The proof Theorem 3.2.1 follow the same lines as in the previous subsection.

Proposition A.4.1. *Assume Conditions* (W) *and* (QDB) *hold. Then,* G *is bounded and self-adjoint on* $\mathcal{S}_*$.

Proof. For boundedness, by (2.3.1), we have, for any $\lambda \in \widehat{\mathcal{S}}_*$,

$$\begin{aligned} \|G(\lambda)\|_{\text{st},*} &= \|G'(A)\rho_*\|_{\text{st},*} = \|G'(A)\|_{\text{obs},*} \\ &\le \|G'\|\|A\|_{\text{obs},*} = \|G'\|\|A\rho_*\|_{\text{st},*}, \end{aligned} \tag{A.4.1}$$

where $A = \lambda\rho_*^{-1} \in \mathcal{B}$. Since, by Lemma 4.7.1, $\widehat{\mathcal{S}}_*$ is dense in $\mathcal{S}_*$, then G is bounded on $\mathcal{S}_*$.

Now, we show that G is symmetric on $\mathcal{S}_*$. By Lemma 4.1.1 and cyclicity of trace, we have, for any $\lambda, \mu \in \widehat{\mathcal{S}}_*$, we take $\lambda = A\rho_*$ and $\mu = B\rho_*$ so that

$$\begin{aligned} \langle \lambda, G(\mu)\rangle_{\text{st},*} &= \operatorname{Tr}\big(\lambda^* G(\mu)\rho_*^{-1}\big) = \operatorname{Tr}\big(\rho_* A^* G(B\rho_*)\rho_*^{-1}\big) \\ &= \operatorname{Tr}(A^* G'(B)\rho_*) = \langle A, G'(B)\rangle_{\text{obs},*} = \langle G'(A), B\rangle_{\text{obs},*} \\ &= \operatorname{Tr}((G'(A))^* B\rho_*) = \operatorname{Tr}\big(\rho_*^{-1}(G'(A)\rho_*)^* B\rho_*\big) \\ &= \operatorname{Tr}\big(\rho_*^{-1}(G(A\rho_*))^* B\rho_*\big) = \operatorname{Tr}\big(\rho_*^{-1} G(\lambda)^*\mu\big) \\ &= \langle G(\lambda), \mu\rangle_{\text{st},*}. \end{aligned} \tag{A.4.2}$$

Therefore, G is symmetric on $\widehat{\mathcal{S}}_*$ and hence, by a density argument, G is symmetric on $\mathcal{S}_*$. Since G is bounded on $\mathcal{S}_*$, then G is self-adjoint on $\mathcal{S}_*$. □

Proposition A.4.2. *Assume Conditions* (H) *and* (QDB) *hold. Then,* L_0 *is anti-self-adjoint on* $\mathcal{S}_*$.

Proof. By Stone's theorem, it suffices to show that L_0 generates a strongly continuous group of unitary operators on $\mathcal{S}_*$. By definition of L_0, for any $\lambda \in \mathcal{S}_*$, we have $e^{L_0 t}\lambda = e^{-iHt}\lambda e^{iHt}$. The group property of $e^{L_0 t}$ follows from the group property of $e^{\pm iHt}$. The group $e^{L_0 t}$ is unitary on $\mathcal{S}_*$ since, for any $\lambda, \mu \in \mathcal{S}_*$ and $t \in \mathbb{R}$, by cyclicity of trace and Condition (QDB), we have

$$\begin{aligned} \langle e^{L_0 t}\lambda, e^{L_0 t}\mu\rangle_{\text{st},*} &= \operatorname{Tr}\big((e^{-iHt}\lambda e^{iHt})^*(e^{-iHt}\mu e^{-iHt})\rho_*^{-1}\big) \\ &= \operatorname{Tr}\big(\lambda^*\mu\rho_*^{-1}\big) = \langle \lambda, \mu\rangle_{\text{st},*}. \end{aligned} \tag{A.4.3}$$

Now, we show $e^{L_0 t}$ is strongly continuous on $\mathcal{S}_*$. Recall that $\mathcal{S}_*$ is the completion of $\tilde{\mathcal{S}}_* = \{\lambda \in \mathcal{S}_1 \mid \lambda\rho_*^{-1/2} \in \mathcal{S}_2\}$ w.r.t. to the norm $\|\cdot\|_{\text{st},*}$. Let $\lambda \in \tilde{\mathcal{S}}_*$. Then, there exists some $\kappa \in \mathcal{S}_2$ such that $\kappa = \lambda\rho_*^{-1/2}$ so that, by Condition (QDB), cyclicity of trace and the fact that e^{-iHt} is strongly continuous on $\mathcal{H}$, we have

$$\begin{aligned}\|e^{L_0 t}\lambda - \lambda\|_{\text{st},*}^2 &= \text{Tr}\big((e^{-iHt}\lambda e^{iHt} - \lambda)^*(e^{-iHt}\lambda e^{iHt} - \lambda)\rho_*^{-1}\big)\\ &= \text{Tr}\big((e^{-iHt}\kappa e^{iHt} - \kappa)^*(e^{-iHt}\kappa e^{iHt} - \kappa)\big)\\ &= \|e^{-iHt}\kappa e^{iHt} - \kappa\|_{\mathcal{S}_2}^2 \to 0 \qquad \text{(A.4.4)}\end{aligned}$$

as $t \to 0$. By a density argument and unitary property of $e^{L_0 t}$, we can extend this limit to the whole space $\mathcal{S}_*$ so that $e^{L_0 t}$ is strongly continuous. Therefore, the generator L_0 is anti-self-adjoint on $\mathcal{S}_*$. □

Proof of Theorem 3.2.1. Since the operator L_0 generates a one-parameter strongly continuous group $e^{L_0 t}$ of unitary operators on $\mathcal{S}_*$ (given explicitly by $e^{L_0 t}\rho = e^{-iHt}\rho e^{iHt}$), and, by Proposition A.4.1, G is bounded on $\mathcal{S}_*$. It follows from Theorem A.2.1, with $X = \mathcal{S}_*$, that $L = L_0 + G$ generates a one-parameter, strongly continuous semigroup on $\mathcal{S}_*$, which implies the statement of Theorem 3.2.1. □

Remark A.4.3. One can also prove a version of Proposition A.4.1 with the space $\mathcal{S}^{(1)}$ (see Remark 5.0.5(a)) replacing $\mathcal{S}_*$.

Remark A.4.4. It is *not* clear whether the semigroup β_t is positive on $\mathcal{S}_*$ under (QDB), i.e., $\langle\lambda, \beta_t(\lambda)\rangle_{\text{st},*}$ is not necessarily non-negative. Indeed, β_t is trace-preserving and, by Lemma A.2.3, completely positive on $\mathcal{S}_1$ so, according to [51, 52], there exists some family $\{V_k(t)\}_{k\geq 1}$ of bounded operators on $\mathcal{H}$ such that

$$\beta_t(\rho) = \sum_{k\geq 1} V_k(t)\rho V_k(t)^*, \qquad \text{(A.4.5)}$$

where $\sum_{k\geq 1} V_k(t)^* V_k(t) = \mathbf{1}$. Then, for any $\lambda \in \mathcal{S}_*$, by the cyclicity of trace,

$$\begin{aligned}\langle\lambda, \beta_t(\lambda)\rangle_{\text{st},*} &= \sum_{k\geq 1} \text{Tr}\big(\lambda^* V_k(t)\lambda V_k(t)^*\rho_*^{-1}\big)\\ &= \sum_{k\geq 1}\langle V_k(t)^*\lambda, \lambda V_k(t)^*\rangle_{\text{st},*}. \qquad \text{(A.4.6)}\end{aligned}$$

However, for each $k \geq 1$, $\langle V_k(t)^*\lambda, \lambda V_k(t)^*\rangle_{\text{st},*}$ is not necessarily non-negative, showing β_t is not necessarily positive on $\mathcal{S}_*$.

Appendix B

Spectra of G and G'

In this section, we study the spectra of G and G' by regarding them as operators on $\mathcal{S}_2$.

Proposition B.0.1. *If the jump operators W_j's are normal, mutually commute and have purely discrete spectra ($\sigma(W_j) = \sigma_d(W_j)$ for all $j \geq 1$), then G and G' have purely discrete spectra outside 0.*

Proof. Since W_j's commute, they have a joint orthonormal basis $\{\psi_k\}$. Let $\lambda_{j,k}$ be the corresponding eigenvalues:

$$W_j \psi_k = \lambda_{j,k} \psi_k. \tag{B.0.1}$$

Let $P_{kl} = |\psi_k\rangle \langle\psi_l|$. Since $\{\psi_k\}$ is a complete orthonormal set in $\mathcal{H}$, $\{P_{kl}\}$ forms an orthonormal basis in $\mathcal{S}_2$. Indeed, for each k, l, m, n, we have

$$\begin{aligned} \langle P_{kl}, P_{mn} \rangle_{\mathcal{S}_2} &= \mathrm{Tr}(P_{kl}^* P_{mn}) = \mathrm{Tr}(|\psi_l\rangle \langle \psi_k|\psi_m\rangle \langle\psi_n|) \\ &= \langle \psi_k, \psi_m \rangle_{\mathcal{H}} \langle \psi_n, \psi_l \rangle_{\mathcal{H}} \\ &= \delta_{km}\delta_{ln} \end{aligned} \tag{B.0.2}$$

so that $\{P_{kl}\}$ are orthonormal in $\mathcal{S}_2$. Next, for any $\kappa \in \mathcal{S}_2$ such that $\langle P_{kl}, \kappa\rangle_{\mathcal{S}_2} = 0$ for all k, l, we have

$$0 = \langle P_{kl}, \kappa \rangle_{\mathcal{S}_2} = \mathrm{Tr}(P_{kl}^* \kappa) = \mathrm{Tr}(|\psi_l\rangle \langle\psi_k|\, \kappa) = \langle \psi_k, \kappa\psi_l \rangle_{\mathcal{H}}. \tag{B.0.3}$$

Hence, all matrix components of κ vanish, which implies that $\kappa = 0$. Therefore, $\{P_{kl}\}$ is a basis in $\mathcal{S}_2$.

Now, for each $j \geq 1$,

$$\begin{aligned} W_j P_{kl} W_j^* &= |W_j\psi_k\rangle\langle W_j\psi_l| = \lambda_{j,k}\overline{\lambda_{j,l}}P_{kl}, &\text{(B.0.4)}\\ W_j^* P_{kl} W_j &= \overline{\lambda_{j,k}}\lambda_{j,l}P_{kl}, &\text{(B.0.5)}\\ \{W_j^*W_j, P_{kl}\} &= \left|W_j^*W_j\psi_k\right\rangle\langle\psi_l| + |\psi_k\rangle\left\langle W_j^*W_j\psi_l\right| \\ &= (|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)P_{kl} &\text{(B.0.6)} \end{aligned}$$

so that

$$\begin{aligned} G(P_{kl}) &= \sum_{j\geq 1}\big(W_jP_{kl}W_j^* - \frac{1}{2}\{W_j^*W_j, P_{kl}\}\big) \\ &= \big(\sum_{j\geq 1}\lambda_{j,k}\overline{\lambda_{j,l}} - \frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)\big)P_{kl} &\text{(B.0.7)} \end{aligned}$$

and

$$\begin{aligned} G'(P_{kl}) &= \sum_{j\geq 1}\big(W_j^*P_{kl}W_j - \frac{1}{2}\{W_j^*W_j, P_{kl}\}\big) \\ &= \big(\sum_{j\geq 1}\overline{\lambda_{j,k}}\lambda_{j,l} - \frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)\big)P_{kl}. &\text{(B.0.8)} \end{aligned}$$

Thus, outside 0, G (resp. G') has purely discrete spectrum with eigenvalues $\sum_{j\geq 1}\lambda_{j,k}\overline{\lambda_{j,l}} - \frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)$ (resp. $\sum_{j\geq 1}\overline{\lambda_{j,k}}\lambda_{j,l} - \frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)$), and 0 is an eigenvalue of infinite multiplicity with eigen-operators P_{kk} for all k, proving the result. □

Proposition B.0.2. *Suppose the series $\sum_{j\geq 1} W_j^*W_j$ converges in the operator norm. If the jump operators W_j's are normal, compact and* Null$(W_j) = \{0\}$ *for all $j \geq 1$, then G and G' are compact on $\mathcal{S}_2$.*

Proof. First, we fix j. Let $\Phi_j(\kappa) := W_j\kappa W_j^*$ and $\Psi_j(\kappa) := -\frac{1}{2}\{W_j^*W_j, \kappa\}$. Let $\{\psi_{j,k}\}$ be an orthonormal basis of $\mathcal{H}$ consists of eigenvectors of W_j with corresponding eigenvalues $\lambda_{j,k}$, i.e., $W_j\psi_{j,k} = \lambda_{j,k}\psi_{j,k}$ for each k. Let $P_{kl}^{(j)} = |\psi_{j,k}\rangle\langle\psi_{j,l}|$ so that $\{P_{kl}^{(j)}\}$ forms an orthonormal basis in $\mathcal{S}_2$. Since W_j's are normal, we have $W_j^*\psi_{j,k} = \overline{\lambda_{j,k}}\psi_{j,k}$. By (B.0.4) and (B.0.6), we have, for each k, l,

$$\Phi_j(P_{kl}^{(j)}) = \lambda_{j,k}\overline{\lambda_{j,l}}P_{kl}^{(j)}, \quad \Psi_j(P_{kl}^{(j)}) = -\frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)P_{kl}^{(j)}. \qquad \text{(B.0.9)}$$

We denote the eigenvalues $\mu_{j,kl} := \lambda_{j,k}\overline{\lambda_{j,l}}$ and $\tau_{j,kl} := -\frac{1}{2}(|\lambda_{j,k}|^2 + |\lambda_{j,l}|^2)$ of Φ_j and Ψ_j, respectively. Since Null$(W_j) = \{0\}$, we have $\lambda_{j,k} \neq 0$ so that

$\mathrm{Null}(\Phi_j) = \mathrm{Null}(\Psi_j) = \{0\}$. Furthermore, we note that

$$\mathrm{multi}(\mu_{j,kl}) = \mathrm{multi}(\tau_{j,kl}) = \mathrm{multi}(\lambda_{j,k}) \times \mathrm{multi}(\lambda_{j,l}), \tag{B.0.10}$$

where multi(λ) is the multiplicity of the eigenvalue λ. Since W_j's are compact, the eigenvalues $\mu_{j,kl}$ and $\tau_{j,kl}$ must have finite multiplicities for each k, l and both converges to zero as $k, l \to \infty$. Hence, Φ_j and Ψ_j are compact on $\mathcal{S}_2$.

Next, we define $G_N = \sum_{j=1}^{N}(\Phi_j + \Psi_j)$. Then, the compactness of G_N follows from the compactness of Φ_j and Ψ_j for each j.

Now, we observe that, for each positive $\kappa \in \mathcal{S}_2$, by the cyclicity of trace and the normality of W_j's,

$$\begin{aligned}
\Big\| \sum_{j\geq N+1} \Phi_j(\kappa)\Big\|_{\mathcal{S}_2}^2 &= \sum_{j,\ell\geq N+1} \mathrm{Tr}\left(W_j\kappa^* W_j^* W_\ell \kappa W_\ell^*\right)\\
&= \sum_{j,\ell\geq N+1} \mathrm{Tr}\Big(\kappa^{1/2} W_j^* W_\ell \kappa W_\ell^* W_j \kappa^{1/2}\Big)\\
&\leq \|\kappa\| \sum_{j,\ell\geq N+1} \mathrm{Tr}\left(\kappa^{1/2} W_j^* (W_\ell^* W_\ell) W_j \kappa^{1/2}\right)\\
&\leq \|\kappa\| \Big\| \sum_{\ell\geq N+1} W_\ell^* W_\ell\Big\| \sum_{j\geq N+1} \mathrm{Tr}\Big(\kappa^{1/2} W_j^* W_j \kappa^{1/2}\Big)\\
&\leq \Big\| \sum_{j\geq N+1} W_j^* W_j\Big\|^2 \|\kappa\|_{\mathcal{S}_2}^2,
\end{aligned} \tag{B.0.11}$$

where, in the last line, we used $\|\kappa\| \leq \|\kappa\|_{\mathcal{S}_2}$, and, similarly,

$$\begin{aligned}
\Big\| \sum_{j\geq N+1} \Psi_j(\kappa)\Big\|_{\mathcal{S}_2} &\leq \frac{1}{2}\Big(\Big\| \sum_{j\geq N+1} W_j^* W_j \kappa\Big\|_{\mathcal{S}_2} + \Big\| \sum_{j\geq N+1} \kappa W_j^* W_j\Big\|_{\mathcal{S}_2}\Big)\\
&\leq \Big\| \sum_{j\geq N+1} W_j^* W_j\Big\| \|\kappa\|_{\mathcal{S}_2}.
\end{aligned} \tag{B.0.12}$$

Thus, we have, as $N \to \infty$,

$$\begin{aligned}
\|G - G_N\|_{\mathcal{S}_2\to\mathcal{S}_2} &= \Big\| \sum_{j\geq N+1} \Phi_j\Big\|_{\mathcal{S}_2\to\mathcal{S}_2} + \Big\| \sum_{j\geq N+1} \Psi_j\Big\|_{\mathcal{S}_2\to\mathcal{S}_2}\\
&\leq 2\Big\| \sum_{j\geq N+1} W_j^* W_j\Big\| \to 0
\end{aligned} \tag{B.0.13}$$

since $\sum_{j\geq 1} W_j^* W_j$ converges in norm. Since G_N is compact for each N, (B.0.13) implies that G is compact on $\mathcal{S}_2$. Since G' is the adjoint of G on $\mathcal{S}_2$, G' is also compact on $\mathcal{S}_2$. □

Proposition B.0.3. *Suppose the jump operators $W_j = 0$ for $j \geq 2$ and $W \equiv$*

W_1 is self-adjoint and has a purely absolutely continuous (a.c.) spectrum on $\mathcal{H}$ (i.e., $\sigma(W) = \sigma_{\mathrm{ac}}(W)$). Then, G and G' has a purely a.c. spectrum on $\mathcal{S}_2$.

Proof. First, let $E_W(\lambda)$ be the spectral resolution of the self-adjoint operator W. Then, since W has a purely a.c. spectrum, for each $\psi \in \mathcal{H}$, there exists some unique positive $f_\psi \in L^1(\mathbb{R}, d\lambda)$, whose support is contained in $\sigma(W)$ and $\|f_\psi\|_{L^1} = \|\psi\|^2_{\mathcal{H}}$, such that

$$\langle \psi, W\psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \lambda d\mu_\psi(\lambda) = \int_{\mathbb{R}} \lambda f_\psi(\lambda) d\lambda, \tag{B.0.14}$$

where μ_ψ is the spectral measure of W associated with ψ and $d\lambda$ is the Lebesgue measure on $\mathbb{R}$, i.e., $f_\psi(\lambda) = d\mu_\psi(\lambda)/d\lambda$ is the Randon-Nikodym derivative of μ_ψ w.r.t. λ.

Next, let $\{\psi_k\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H}$ and define $P_{kl} := |\psi_k\rangle \langle \psi_l|$. Then, $\{P_{kl}\}_{k,l\geq 1}$ forms an orthonormal basis for $\mathcal{S}_2$. Furthermore, let $L_A : \kappa \mapsto A\kappa$ and $R_A : \kappa \mapsto \kappa A$ be the left- and the right-multiplication by A on $\mathcal{S}_2$, respectively. First, we note that, since W is bounded and self-adjoint, L_W and R_W are both bounded and self-adjoint on $\mathcal{S}_2$. Indeed, for each $\kappa, \sigma \in \mathcal{S}_2$, we have

$$\|L_W \kappa\|_{\mathcal{S}_2} = \|W\kappa\|_{\mathcal{S}_2} \leq \|W\| \|\kappa\|_{\mathcal{S}_2} \tag{B.0.15}$$

and

$$\begin{aligned} \langle (L_W)^* \sigma, \kappa \rangle_{\mathcal{S}_2} &= \langle \sigma, L_W \kappa \rangle_{\mathcal{S}_2} = \mathrm{Tr}(\sigma^* W \kappa) \\ &= \mathrm{Tr}((W\sigma)^* \kappa) = \langle L_W \sigma, \kappa \rangle_{\mathcal{S}_2}. \end{aligned} \tag{B.0.16}$$

The claims for R_W follows from a similar computations. Since $[L_W, R_W] = 0$, we have

$$G = G' = L_W R_W - \frac{1}{2}(L_W^2 + R_W^2) = -\frac{1}{2}(L_W - R_W)^2. \tag{B.0.17}$$

Hence, the spectral property of G is determined completely by that of $Z_W := L_W - R_W$ through the spectral mapping theorem, i.e., if Z_W has a purely a.c. spectrum on $\mathcal{S}_2$, then so does G. To show our result, since $\{P_{kl}\}_{k,l\geq 1}$ is a basis for $\mathcal{S}_2$, it suffices to show that $\{P_{kl}\}_{k,l\geq 1} \subseteq [\mathcal{S}_2]_{\mathrm{ac}}$, where $[\mathcal{S}_2]_{\mathrm{ac}}$ is the subspace of $\mathcal{S}_2$ consisting of vectors whose spectral measure corresponding to Z_W is absolutely continuous w.r.t. λ.

Now, we fix k, l. Since $[L_W, R_W] = 0$ and L_W, R_W are self-adjoint, by (B.0.14), the cyclicity of trace and the Fubini's theorem, we have, for any

$n, m \in \mathbb{N}$,

$$\begin{aligned}\langle P_{kl}, (L_W)^n (R_W)^m P_{kl} \rangle_{\mathcal{S}_2} &= \mathrm{Tr}(P_{kl}^* W^n P_{kl} W^m) \\ &= \langle \psi_k, W^n \psi_k \rangle_{\mathcal{H}} \langle \psi_l, W^m \psi_l \rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}^2} \lambda_1^n \lambda_2^m \widetilde{F}_{kl}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \end{aligned} \tag{B.0.18}$$

where $F_{kl}(\lambda_1, \lambda_2) := f_{\psi_k}(\lambda_1) f_{\psi_l}(\lambda_2)$. By linearity, this can be extended to arbitrary polynomial $p(x, y)$ on $\mathbb{R}^2$, i.e.,

$$\langle P_{kl}, p(L_W, R_W) P_{kl} \rangle_{\mathcal{S}_2} = \int_{\mathbb{R}^2} p(\lambda_1, \lambda_2) \widetilde{F}_{kl}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \tag{B.0.19}$$

which in turns implies that, for any measurable function χ on $\mathbb{R}$, we have

$$\langle P_{kl}, \chi(Z_W) P_{kl} \rangle_{\mathcal{S}_2} = \int_{\mathbb{R}^2} \chi(\lambda_1 - \lambda_2) \widetilde{F}_{kl}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2. \tag{B.0.20}$$

Let $\widetilde{F}_{kl}(\lambda) := \int_{\mathbb{R}} F_{kl}(\lambda_1, \lambda - \lambda_1) d\lambda_1$, which satisfies $\|F_{kl}\|_{L^1} = 1$ by its definition. Then, (B.0.20) becomes

$$\langle P_{kl}, \chi(Z_W) P_{kl} \rangle_{\mathcal{S}_2} = \int_{\mathbb{R}} \chi(\lambda) \widetilde{F}_{kl}(\lambda) d\lambda. \tag{B.0.21}$$

Hence, the spectral measure of Z_W associated with P_{kl} is $\widetilde{F}_{kl}(\lambda) d\lambda$, which is absolutely continuous w.r.t. λ. Therefore, Z_W has a purely a.c. spectrum on $\mathcal{S}_2$, which implies that, due to the spectral mapping theorem, G has an purely a.c. spectrum on $\mathcal{S}_2$. Since this is true for any k, l, this completes the proof. □

Appendix C

GNS representation of $(\mathcal{B}, \omega_*)$

In this section, we define the GNS representation associated with the pair $(\mathcal{B}, \omega_*)$ (by regarding $\mathcal{B}$ as a von Neumann algebra and ω_* as a normal state on $\mathcal{B}$). This section follows mainly according to [44].

Recall the state ω_* is defined through a density operator $\rho_* > 0$ and is given, for any $A \in \mathcal{B}$, by

$$\omega_*(A) = \mathrm{Tr}(A\rho_*). \tag{C.0.1}$$

The operator $\Omega_* := \rho_*^{1/2}$ is Hilbert-Schmidt. Note that, since $\rho_* > 0$, the state ω_* is faithful. Indeed, for each $A \in \mathcal{B}$, we have

$$\omega_*(A^*A) = \mathrm{Tr}(A^*A\rho_*) = \|A\Omega_*\|_{\mathcal{S}_2}^2 \tag{C.0.2}$$

so that $\omega_*(A^*A) = 0$ implies $A\Omega_* = 0$. Since $\Omega_* > 0$, we have $A = 0$.

Next, we define the representation π_* of $\mathcal{B}$ acting on $\mathcal{S}_2$ by

$$\pi_*(A)\kappa = A\kappa, \qquad \kappa \in \mathcal{S}_2, \quad A \in \mathcal{B}, \tag{C.0.3}$$

i.e., the representation for left-action by $\mathcal{B}$. Then, we have

$$\omega_*(A) = \langle \Omega_*, \pi_*(A)\Omega_* \rangle_{\mathcal{S}_2}. \tag{C.0.4}$$

Since $\mathcal{S}_2$ is a closed, two-sided ideal of $\mathcal{B}$, there is another natural, anti-linear representation π_*' of $\mathcal{B}$ on $\mathcal{S}_2$, given by

$$\pi_*'(A)\kappa = \kappa A^*, \qquad \kappa \in \mathcal{S}_2, \quad A \in \mathcal{B}, \tag{C.0.5}$$

i.e., the representation for right-action by $\mathcal{B}$. It is immediate from the definition that π_*' satisfies

$$\pi_*'(AB) = \pi_*'(A)\pi_*'(B), \quad \pi_*'(A^*) = (\pi_*'(A))^*. \tag{C.0.6}$$

We remark that the superscript * on the r.h.s. of (C.0.6) is the adjoint operation of the right-multiplication of A on $\mathcal{S}_2$, whereas the superscript * on the l.h.s. is the adjoint operation of operators in $\mathcal{B}$ (abuse the notation).

Recall the definition of the anti-unitary operator J:

$$J\kappa = \kappa^*, \qquad \text{for all } \kappa \in \mathcal{S}_2. \tag{C.0.7}$$

Properties of Ω_*, π_* and π_*' are summarized in the following proposition:

Proposition C.0.1. *([44], Section V.1.4, Theorem 1.4.1)*

(a) The operator norm of $\pi_*(A)$ and $\pi_*'(A)$ on $\mathcal{S}_2$ are given by

$$\|\pi_*(A)\| = \|\pi_*'(A)\| = \|A\|. \tag{C.0.8}$$

(b) Ω_* is cyclic and separating for both $\pi_*(\mathcal{B})$ and $\pi_*'(\mathcal{B})$.

(c) $J^* = J$, $J^2 = \mathbf{1}$ and $J\Omega_* = \Omega_*$.

(d) π_* and π_*' are transformed to each other by the anti-unitary operator J as

$$J\pi_*(A)J = \pi_*'(A), \quad \text{for each } A \in \mathcal{B}. \tag{C.0.9}$$

Proof. (a) By definition, we have

$$\|\pi_*(A)\|^2_{\mathcal{S}_2} = \sup_{\|\kappa\|_{\mathcal{S}_2}=1} \|\pi_*(A)\kappa\|^2_{\mathcal{S}_2} = \sup_{\|\kappa\|_{\mathcal{S}_2}=1} \mathrm{Tr}(\kappa^* A^* A\kappa) = \|A\|^2, \tag{C.0.10}$$

where the last equality follows from by choosing κ as some rank-1 projection operator. The norm for $\pi_*'(A)$ can be found in a similar way using cyclicity of trace and the fact that $\|A^*\| = \|A\|$.

(b) Suppose $\kappa \in \mathcal{S}_2$ is an element such that $\kappa \perp \pi_*(A)\Omega_*$ for all $A \in \mathcal{B}$. Then, by taking $A = \kappa$ (by viewing κ as an element in $\mathcal{B}$), we have

$$0 = \langle \kappa, \pi_*(\kappa)\Omega_* \rangle_{\mathcal{S}_2} = \mathrm{Tr}\big(\kappa^*\kappa\rho_*^{1/2}\big). \tag{C.0.11}$$

Since $\rho_*^{1/2} > 0$, we must have $\kappa^*\kappa = 0$, which implies that $\kappa = 0$. Together with the faithfulness of ω_*, we see that Ω_* is a cyclic and separating vector for the representation π_*.

The cyclicity of Ω_* for $\pi_*'(\mathcal{B})$ follows from the separability of Ω_*. The separatibility of Ω_* for $\pi_*(\mathcal{B})$ and $\pi_*'(\mathcal{B})$ are proven similarly.

(c) Let J be as in (C.0.7). Since $J^2\kappa = (\kappa^*)^* = \kappa$ for all $\kappa \in \mathcal{S}_2$, we have

$$J^2 = \mathbf{1}. \tag{C.0.12}$$

Next, for each $\kappa, \sigma \in \mathcal{S}_2$, by cyclicity of trace and by the fact that J is anti-linear, we have

$$\begin{aligned}\langle J^*\kappa, \sigma\rangle_{\mathcal{S}_2} &= \overline{\langle \kappa, J\sigma\rangle_{\mathcal{S}_2}} = \langle J\sigma, \kappa\rangle_{\mathcal{S}_2} = \mathrm{Tr}((\sigma^*)^*\kappa) \\ &= \mathrm{Tr}(\sigma\kappa) = \mathrm{Tr}((\kappa^*)^*\sigma) = \langle \kappa^*, \sigma\rangle_{\mathcal{S}_2} = \langle J\kappa, \sigma\rangle_{\mathcal{S}_2}. \end{aligned} \tag{C.0.13}$$

Thus, $J^* = J$.

Finally, by the definition (C.0.7), we have $J\Omega_* = (\rho_*^{1/2})^* = \rho_*^{1/2} = \Omega_*$.

(d) For each $A \in \mathcal{B}$, since $J\Omega_* = \Omega_*$ from part (c), we have

$$\pi_*'(A)\Omega_* = \rho_*^{1/2}A^* = (A\rho_*^{1/2})^* = J(\pi_*(A)\Omega_*) = (J\pi_*(A)J)\Omega_*. \tag{C.0.14}$$

Since Ω_* is cyclic for both $\pi_*(\mathcal{B})$ and $\pi_*'(\mathcal{B})$, we conclude that $J\pi_*(A)J = \pi_*'(A)$ for all $A \in \mathcal{B}$. □

Thus, the triple $(\mathcal{S}_2, \pi_*, \Omega_*)$ gives the GNS representation associated with $(\mathcal{B}, \omega_*)$.

Proof of Theorem 4.2.6. Recall the definitions of the unbounded, self-adjoint operator

$$H_* := -\ln \rho_*, \qquad \text{so that } \rho_* = e^{-H_*}, \tag{C.0.15}$$

and the automorphism group α_t on $\mathcal{B}$ by

$$\alpha_t(A) = e^{iH_*t}Ae^{-iH_*t}. \tag{C.0.16}$$

We see that ω_* is an invariant state under α_t. Indeed, since $\alpha_t'(\rho_*) \equiv e^{-iH_*t}\rho_*e^{iH_*t} = \rho_*$, we have, for each $A \in \mathcal{B}$ and $t \in \mathbb{R}$,

$$\omega_*(\alpha_t(A)) = \mathrm{Tr}(A\alpha_t'(\rho_*)) = \mathrm{Tr}(A\rho_*) = \omega_*(A). \tag{C.0.17}$$

We define the family of operators $U(t)$ on $\mathcal{S}_2$, given by

$$U(t)(\pi_*(A)\Omega_*) = \pi_*(\alpha_t(A))\Omega_*, \qquad \text{for all } A \in \mathcal{B}. \tag{C.0.18}$$

By (C.0.8) and $\|\alpha_t(A)\| = \|A\|$, for each t, the operator $U(t)$ is an isometry. Since α_t is a *-automorphism group of $\mathcal{B}$ and ω_* is invariant under α_t, $U(t)$ is a one-parameter family of unitary operators and $U(t)\Omega_* = \Omega_*$ for each $t \in \mathbb{R}$ (see [13], Corollary 2.3.17). Furthermore, since α_t is weakly* continuous on $\mathcal{B}$, we have

Lemma C.0.2. *The one-parameter group $U(t)$ of unitary operators defined in* (C.0.18) *is strongly continuous.*

Proof of Lemma C.0.2. For each $A \in \mathcal{B}$, we have

$$\begin{aligned}&\|U(t)(\pi_*(A)\Omega_*) - \pi_*(A)\Omega_*\|^2_{\mathcal{S}_2}\\ &\quad = 2\omega_*(A^*A) - \omega_*(A^*\alpha_t(A)) - \omega_*((\alpha_t(A))^*A).\end{aligned} \tag{C.0.19}$$

Since α_t is weakly* continuous, for each $\epsilon > 0$, there exists some $\delta > 0$ such that, for any $0 < |t| < \delta$, the r.h.s. of (C.0.19) is less than ϵ^2. Thus, for $0 < |t| < \delta$, we have

$$\|U(t)(\pi_*(A)\Omega_*) - \pi_*(A)\Omega_*\|_{\mathcal{S}_2} < \epsilon. \tag{C.0.20}$$

For general $\kappa \in \mathcal{S}_2$, since $\mathscr{F}$ is dense in $\mathcal{S}_2$, for each $\epsilon > 0$, there exists some $A \in \mathcal{B}$ such that $\|\kappa - \pi_*(A)\Omega_*\|_{\mathcal{S}_2} < \epsilon$. By (C.0.20) and the fact that $U(t)$ is unitary for each $t \in \mathbb{R}$, for any $0 < |t| < \delta$, we have

$$\begin{aligned}\|U(t)\kappa - \kappa\|_{\mathcal{S}_2} &\le \|U(t)(\pi_*(A)\Omega_*) - \pi_*(A)\Omega_*\|_{\mathcal{S}_2}\\ &\quad + 2\|\kappa - \pi_*(A)\Omega_*\|_{\mathcal{S}_2} < \epsilon + 2\epsilon = 3\epsilon.\end{aligned} \tag{C.0.21}$$

Thus, $U(t)$ is strongly continuous on $\mathcal{S}_2$. □

Now, by the cyclicity of Ω_* and the fact that $U(t)\Omega_* = \Omega_*$, we have, for all $A \in \mathcal{B}$,

$$U(t)\pi_*(A)U(t)^* = \pi_*(\alpha_t(A)). \tag{C.0.22}$$

Let L_* be the generator of $U(t)$. Then, we have

$$\pi_*(\alpha_t(A)) = e^{iL_*t}\pi_*(A)e^{-iL_*t}, \qquad L_*\Omega_* = 0, \tag{C.0.23}$$

which proves (4.2.5).

To prove relation (4.2.6), we introduce entire analytic elements for α_t (see [13], Section 2.5.3).

Definition C.0.3. We say an operator $A \in \mathcal{B}$ is *entire analytic* for α_t if there exists a function $f : \mathbb{C} \to \mathcal{B}$ such that

(a) $f(t) = \alpha_t(A)$ for $t \in \mathbb{R}$.

(b) The function $z \mapsto \omega(f(z))$ is analytic on the entire plane $\mathbb{C}$ for all $\omega \in \mathcal{B}'$, where $\mathcal{B}'$ is the dual space of $\mathcal{B}$.

We denote the set of entire analytic operators for α_t by $\mathcal{B}_{\text{ana}}$.

By Lemma C.0.2, $U(t)$ is strongly continuous, which leads to the following lemma:

Lemma C.0.4. *For each $A \in \mathcal{B}$ and any $\epsilon > 0$, there exists some $B \in \mathcal{B}_{\text{ana}}$ such that*

$$\|(\pi_*(A) - \pi_*(B))\Omega_*\|_{\mathcal{S}_2} < \epsilon. \tag{C.0.24}$$

Proof. We follow the argument in [13], Proposition 2.5.22. Let $A \in \mathcal{B}$ be fixed. For each integer $n \geq 1$, we define

$$A_n = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(A) e^{-nt^2} dt. \tag{C.0.25}$$

Each A_n is entire analytic for α_t. Indeed, for each $z \in \mathbb{C}$, the function

$$f_n(z) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(A) e^{-n(t-z)^2} dt, \tag{C.0.26}$$

is a well-defined function of $z \in \mathbb{C}$ since the function $t \mapsto e^{-(t-z)^2}$ is integrable for each z and, when $z = s \in \mathbb{R}$, we have

$$\begin{aligned} f_n(s) &= \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(A) e^{-n(t-s)^2} dt = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_{t+s}(A) e^{-nt^2} dt \\ &= \alpha_s \left[\sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(A) e^{-nt^2} dt \right] = \alpha_s(A_n). \end{aligned} \tag{C.0.27}$$

Also, for each $\omega \in \mathcal{B}'$, we have

$$\omega(f_n(z)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \omega(\alpha_t(A)) e^{-n(t-z)^2} dt \tag{C.0.28}$$

so that

$$\begin{aligned} |\omega(f_n(z))| &\leq \|\omega\| \|A\| \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} |e^{-n(t-z)^2}| dt \\ &\leq \|\omega\| \|A\| \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(t-x)^2 + ny^2} dt \leq e^{ny^2} \|\omega\| \|A\|, \end{aligned} \tag{C.0.29}$$

where $z = x + iy$. It then follows from the Lebesgue dominated convergence theorem that the function $z \mapsto \omega(f_n(z))$ is entire analytic. Thus, A_n is analytic.

Next, we show that $\|(\pi_*(A_n) - \pi_*(A))\Omega_*\|_{\mathcal{S}_2} \to 0$ as $n \to \infty$. For notational simplicity, we drop the subscript in the norm $\|\cdot\|_{\mathcal{S}_2}$ in the rest of this proof.

Let $\xi_A = \pi_*(A)\Omega_*$. Since $\sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} e^{-nt^2}dt = 1$, by (C.0.18), we have

$$\begin{aligned}
&\|(\pi_*(A_n) - \pi_*(A))\Omega_*\| \\
&\qquad \le \sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} e^{-nt^2}\|\pi_*(\alpha_t(A)) - \pi_*(A))\Omega_*\|dt \\
&\qquad = \sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} e^{-nt^2}\|U(t)\xi_A - \xi_A\|dt
\end{aligned} \tag{C.0.30}$$

By Lemma C.0.2, $U(t)$ is strongly continuous so that, for any $\epsilon > 0$, there exists some δ such that, for all $|t| < \delta$, $\|U(t)\xi_A - \xi_A\| < \epsilon$. Note that the choice of δ is independent of n. Thus, we choose n large enough so that

$$\sqrt{\frac{n}{\pi}}\int_{|t|\ge\delta} e^{-nt^2}dt < \frac{\epsilon}{2\|A\|}. \tag{C.0.31}$$

It follows that, since $\|U(t)\xi_A\| = \|\xi_A\|$,

$$\begin{aligned}
&\sqrt{\frac{n}{\pi}}\int_{\mathbb{R}} e^{-nt^2}\|U(t)\xi_A - \xi_A\|dt \\
&\quad = \sqrt{\frac{n}{\pi}}\int_{|t|<\delta} e^{-nt^2}\|U(t)\xi_A - \xi_A\|dt \\
&\qquad + \sqrt{\frac{n}{\pi}}\int_{|t|\ge\delta} e^{-nt^2}\|U(t)\xi_A - \xi_A\|dt \\
&\quad \le \epsilon\sqrt{\frac{n}{\pi}}\int_{|t|<\delta} e^{-nt^2}dt + \sqrt{\frac{n}{\pi}}\int_{|t|\ge\delta} e^{-nt^2}(\|U(t)\xi_A\| + \|\xi_A\|)dt \\
&\quad \le \epsilon + 2\|\xi_A\|\sqrt{\frac{n}{\pi}}\int_{|t|\ge\delta} e^{-nt^2}dt \\
&\quad < \epsilon + \epsilon = 2\epsilon.
\end{aligned} \tag{C.0.32}$$

This completes the proof. □

Lemma C.0.5. *Every element in* $\mathscr{F}_{\text{ana}} := \pi_*(\mathcal{B}_{\text{ana}})\Omega_*$ *is entire analytic for* $U(t)$. *Consequently,* $\mathscr{F}_{\text{ana}} \subseteq \mathscr{D}(e^{zL_*})$ *for all* $z \in \mathbb{C}$.

Proof of Lemma C.0.5. For all $A \in \mathcal{B}_{\text{ana}}$ and $\sigma \in \mathcal{S}_2$, the function

$$\begin{aligned}
z \mapsto \langle\sigma, U(z)(\pi_*(A)\Omega_*)\rangle_{\mathcal{S}_2} &= \langle\sigma, \pi_*(\alpha_z(A))\Omega_*\rangle_{\mathcal{S}_2} \\
&= \mathrm{Tr}(\sigma^*\alpha_z(A)\Omega_*)
\end{aligned} \tag{C.0.33}$$

is analytic on $\mathbb{C}$. Thus, every elements in $\mathscr{F}_{\text{ana}}$ is entire analytic for $U(t)$. □

Now, by Lemma C.0.5, for each $A \in \mathcal{B}_{\text{ana}}$, $\pi_*(A)\Omega_* \in \mathscr{D}(e^{-L_*/2})$ and

$$\begin{aligned} e^{-L_*/2}(\pi_*(A)\Omega_*) &= \pi_*(\alpha_{i/2}(A))\Omega_* = \alpha_{i/2}(A)\rho_*^{1/2} \\ &= (\rho_*^{1/2} A \rho_*^{-1/2})\rho_*^{1/2} = \rho_*^{1/2} A \\ &= (A^* \rho_*^{1/2})^* = J(\pi_*(A^*)\Omega_*). \end{aligned} \tag{C.0.34}$$

Since $J^2 = \mathbf{1}$, this yields, for all $A \in \mathcal{B}_{\text{ana}}$,

$$\pi_*(A^*)\Omega_* = J e^{-L_*/2}(\pi_*(A)\Omega_*). \tag{C.0.35}$$

Next, for any $A \in \mathcal{B}$, we construct a sequence $\{A_n\}$ in $\mathcal{B}_{\text{ana}}$ as in the proof of Lemma C.0.4 so that, as $n \to \infty$,

$$\|\pi_*(A_n)\Omega_* - \pi_*(A)\Omega_*\|_{\mathcal{S}_2} \to 0, \tag{C.0.36}$$

$$\begin{aligned} &\|J e^{-L_*/2}(\pi_*(A_n)\Omega_*) - \pi_*(A^*)\Omega_*\|_{\mathcal{S}_2} \\ &\qquad = \|\pi_*(A_n^*)\Omega_* - \pi_*(A^*)\Omega_*\|_{\mathcal{S}_2} \to 0. \end{aligned} \tag{C.0.37}$$

By the closedness of the operator $Je^{-L_*/2}$, we have $\pi_*(A)\Omega_* \in \mathscr{D}(e^{-L_*/2})$ and

$$J e^{-L_*/2}(\pi_*(A)\Omega_*) = \pi_*(A^*)\Omega_*, \tag{C.0.38}$$

which proves (4.2.6). □

Remark C.0.6. In our case, we can also define entire analytic elements for $U(t)$ in $\mathcal{S}_2$ in a similar way: An element $\kappa \in \mathcal{S}_2$ is *entire analytic* for $U(t)$ if, for all $\sigma \in \mathcal{S}_2$, the function $z \mapsto \langle \sigma, U(z)\kappa \rangle_{\mathcal{S}_2}$ is analytic on $\mathbb{C}$.

Furthermore, we can define entire analytic elements for $U(t)$ equivalently using its generator L_*: An element $\kappa \in \mathcal{S}_2$ is *entire analytic* if $\kappa \in \mathscr{D}(L_*^n)$ for all $n \in \mathbb{N}$ and, for all $t > 0$, the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|L_*^n \kappa\|_{\mathcal{S}_2} < \infty. \tag{C.0.39}$$

Hence, $\kappa \in \mathscr{D}(e^{zL_*})$ for all $z \in \mathbb{C}$ if $\kappa \in \mathcal{S}_2$ is entire analytic for $U(t)$. For a proof of the equivalence of the above two definitions for entire analytic elements, see [13], p.178–179.

Remark C.0.7. In fact, the state ω_* is a KMS-state w.r.t. the automorphism group α_t, i.e., ω_* is invariant under α_t and, for each $A, B \in \mathcal{B}$, the function $F_{A,B}(z) := \omega_*(\alpha_z(A)B)$ is analytic on the strip $\mathcal{I} := \{z \in \mathbb{C} \mid 0 < \text{Im}(z) < 1\}$ and continuous on $\overline{\mathcal{I}}$ such that

$$F_{A,B}(t) = \omega_*(\alpha_t(A)B), \qquad F_{A,B}(t+i) = \omega_*(B\alpha_t(A)). \tag{C.0.40}$$

In other words, ω_* is an equilibrium state of α_t.

Remark C.0.8. Formally, by regarding e^{iH_*t} as an element in $\mathcal{B}$, we can write

$$U(t) = \pi_*(e^{iH_*t})\pi'_*(e^{iH_*t}) \tag{C.0.41}$$

or, equivalently,

$$L_* = \pi_*(H_*) - \pi'_*(H_*) \tag{C.0.42}$$

by extending the representations π_* and π'_* to those unbounded operators affiliated to $\mathcal{B}$.

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