The Hausdorff Dimension of the Level Sets of the Directed Landscape

Lemonte Alie-Lamarche

August 29, 2024

The Hausdorff Dimension of the Level Sets of the Directed Landscape

Lemonte Alie-Lamarche
Doctor of Philosophy

Department of Mathematics
University of Toronto
2024

Abstract

The directed landscape $\mathcal{L}$ introduced by Dauvergne, Ortmann, and Virág in 2018 in their groundbreaking paper [9] has rapidly become a central object of study in modern probability theory. It is believed by many that this random directed metric is possibly *the* universal scaling limit of the random growth models in the KPZ universality class. It was proven by Dauvergne, Nica, and Virág in 2021 in [11] that the directed landscape is, among other things, the scaling limit of at least six different models of last passage percolation in the uniform on compact topology. This universality of $\mathcal{L}$ as a scaling limit, as well as its ties to other random growth models, makes understanding anything about its fractal structure and geometry of significant interest in the wider long-term endeavour to fully understand the structure of the KPZ universality class.

In this thesis, we prove several results about the fractal structure of the level sets of $\mathcal{L}(0,0;\cdot,\cdot)$ as a function on $\mathbb{R}\times\mathbb{R}_{>0}$, which translate quite easily into very similar statements about the corresponding level sets of $\mathcal{L}$ on its domain. We first prove that the $h-$level sets of rescaled Exponential last passage percolation starting at (0,0) converge in the Hausdorff metric induced by the Euclidean norm to the $h-$level set of $\mathcal{L}(0,0;\cdot,\cdot)$ on any convex compact set $K\subseteq\mathbb{R}\times\mathbb{R}_{>0.}$ We then prove that the Hausdorff dimension of the $h-$level set of $\mathcal{L}(0,0;\cdot,\cdot)$ is at most $\frac{5}{3}$ almost surely for all $h\in\mathbb{R}$. We conclude this thesis by developing a strategy to systematically find lower bounds on the Hausdorff dimension of random $h-$level sets of stochastic processes indexed by $\mathbb{R}^2$ that hold with a positive $h-$dependent probability p_h. We apply this strategy to $\mathcal{L}(0,0;\cdot,\cdot)$ to establish that the $h-$ level set of $\mathcal{L}(0,0;\cdot,\cdot)$ has Hausdorff dimension at least $\frac{3}{2}$ with a positive $h-$dependent probability. In the process of doing so, we also construct a partial-two point bound for $\mathcal{L}(0,0;\cdot,\cdot)$.

This thesis is based on several projects of joint work conducted with Virginia Pedreira under the supervision of Bálint Virág.

Acknowledgements

First and foremost, I would thank my family for an entire lifetime of encouragement and support. The faith that you have always had in me, and the amazing role models that you were in teaching me the importance of hard work and the power of education, made me who I am today. The fact that I am even able to be do something like this is only possible because of the generations of hard work before I was even born that made this opportunity possible. My grandparents always told me about how proud my great-grandfather was that his great-grandson had the opportunity to do something he couldn't have even dreamed of having the chance to do when he was my age, and what this represents is something I will never take for granted. He never had the opportunity to go to school himself, and knowing how hard he worked to elevate his family out of poverty, the sacrifices and struggles he experienced in the process, and how much it meant to him to see someone only three generations after him be in the position that I am now in today means the world to me.

I also owe a deep debt of gratitude to Bálint Virág. Right from the very beginning when I first stopped by your office hours to ask you questions about class, you have always been incredibly generous with your time, exceptionally patient and encouraging, and even when I was most frustrated with the research process, the enthusiasm and joy you take in your work has always been infectious and reminded me of why I enjoy doing math in the first place. To this day, I am still amazed by the depth of your passion for what you do. I had heard nothing but the highest of praise about you as a mathematician and an advisor from everyone that I spoke to when I first came to UofT, and I can say with confidence that all the superlatives that I heard still didn't do you justice. I can't thank you enough for being such a fantastic advisor and mentor.

There are also dozens of other people that I met at UofT who played very important roles in shaping my time in grad school. I would like to thank Peter Angelinos, Dinushi Munasinghe, Caleb Jonker, James Munday, and Andy Ramirez-Coté for being great housemates and great friends throughout these years at UofT. I would also like to thank Mario Apetroaie, Christina Issari, Vincent Girard, Jake Zimmerman, Valia Liontou, Adriano Pacifico, Ramon Eduardo Ronzon Lavie, Sina Zabanfahm, Jeffrey Im, Cindy Fang, Reila Zheng, Mario Palasciano, Esmé Tremblay, Eva Politou, Ethan Ross, Lucas Ashbury-Wood, Gaurav Patil, and the countless other friends that I have made over the years in the department. Grad school can easily turn into an incredibly frustrating and isolating time in your life, but it was the years of friendship and wonderful memories with all of you that made the experience as positive as it was for me. I like to think that the influence that being friends with all of you has had on me has made me a better mathematician and a better person.

I would also like to especially thank Jemima Merisca. Since before I even began attending the university, you have always been one of the kindest, most supportive, and caring people that I have ever met in my life. No matter what was going on with work, or exams, or my life in general, every time I stopped by your office to catch up, I would always wind up leaving with the feeling that everything would be alright and work out the way that it was meant to. To this day, every time we talk, I always feel like I'm right back at home. Even though we all tell you how grateful we are for all that you do for us, I genuinely don't think that we can thank you enough for being as wonderful of a person as you are. You truly are the heart and the soul of the grad student community.

Although I have now spent more time at UofT than I did at Queen's, I could never overstate just how much of an impact that my time at Queen's and the people that I met in the math and stats department had on the direction that my life has taken. Undergrad was four of the happiest years of my life, and the friends I made while there are to this day some of the closest friends that I have ever had. I would like to thank Garth Meeks, Helen Yoseph, Mariam Anwar, Susanna Zhang, Jeffrey Diamond, Connor Blandford, Connor Kapahi, AJ Schmidt-Shoukri, Michael Didow, Gabby Bermudez, Trinity Chinner, Lamarr Lundy, Nicole Dozois, Jason Kattan, Nour Fahmy, Eric Han, Prince Zhao, Stanko Banic, Chester Banic, Akshay Gupta, Dan Cloutier, Katie Pennock, Chelsea Crocker, and the many other friends who made my time at Queen's as special as it was. The amount of confidence and trust that all of you have always had and continue to have in me, especially when grad school made me start to doubt myself, even means the world to me. I can't thank all of you enough for being the amazing friends that you are. It's crazy that I've been friends with many of you for almost an entire decade of my life now, and I look to forward to many more decades to come.

There are also several faculty members in the Queen's math and stats department to whom I am particularly grateful. I would like to thank Devon Lin for all the work she put into connecting with the undergraduate student body. The sense of community in the department was one of my most cherished memories of my time at Queen's, and the efforts you make are a huge part of fostering that atmosphere. I would like to thank Francesco Cellarosi and Fady Alajaji for being the first people to really show me how fascinating probability theory could be and igniting my passion for this subject. You were two of the best professors that I have ever had the pleasure of learning from, and I would have never wound up on the academic path that I took if it wasn't for the two of you. Finally, I would like to thank Mike Roth and Andrew Lewis. The department was full of incredibly kind and passionate educators, but the two of you were exceptional even by those standards. Talking to the two of you over the years about the courses I took with you, your experiences in math and academia, my own future plans, and even just my naive questions about math in general was pivotal in convincing me that I could also follow in your footsteps and maybe, just maybe, pursue a PhD myself one day. I cannot thank the two of you enough for instilling in me that level of self-belief in my own capability.

Last but certainly not least, I have to give a particularly special thank you to Virginia Pedreira. Almost immediately, you were one of the warmest and most welcoming people that I met at UofT. You planted the seed that made me want to work with Bálint in the first place, and played a bigger role than you probably realize in fully convincing me that I wanted to spend half a decade of my life working on probability theory. You've been there as a housemate, as a colleague, and most importantly as an amazing friend at almost every step of my time in grad school and more so than possibly anyone else, my experience here would not have been the same without you. Whether it was having very lively debates about sports and current events, hitting academic rock bottom when we thought that we were at a dead end, slowly dying halfway through the hike up Sulfur Mountain, or the elation from realizing that our research *finally* worked out in the end, you were there for all the highest highs and the lowest lows that I experienced in grad school and I couldn't have asked for a better friend to have with me through all of that. In the words of Charles Barkley, "it is an absolute honour and a privilege to call you my friend."

Contents

Chapter 1

Preliminary Material

1.1 Introduction and Motivation

The Central Limit Theorem is arguably the single most important and impactful theorem in the history of mathematics. Its ability to describe mass amounts of physical phenomena in the natural world using the normal distribution has played an irreplaceable role in shaping centuries of scientific and statistical research, and continues to do so to this day. It would not be hyperbole to state the world would not be the same today without the immense utility that this theorem has provided in every single domain with a notion of empirical evidence. The universality of this theorem, i.e. that under light assumptions it does not depend on the specific underlying probability distributions being studied, is essential in explaining its reach and its power. However, despite its vast reaches, there are still many physical and mathematical phenomena that do not fall into the domain of the Central Limit Theorem. Due to the profound impact that this theorem has had, finding other examples of this sort of universal limiting behaviour in such contexts has been a longstanding topic of interest within probability theory.

Several other examples of this sort of universal behaviour for different classes of random objects and phenomena have been found throughout history, but one that has been of particular interest over the past 40 years has been the so-called **KPZ universality class.** In their groundbreaking paper [17] in 1986, Kardar, Parisi, and Zhang introduced the now famous eponymous **KPZ equation**

$$\partial_t \mathcal{H} = \nu \partial_x^2 \mathcal{H} - \lambda \left(\partial_x \mathcal{H}\right)^2 + \sqrt{D}\xi.$$

In this stochastic partial differential equation, ν, λ, and D are physical constants, $\xi = \xi(x,t)$ is spacetime white noise, and $\mathcal{H}(x,t)$ is interpreted as the height of a randomly growing interface $\mathcal{H}$ at the spatial coordinate x at a point t in time. Corwin provides an excellent survey on the rich history of this stochastic partial differential equation in both physics and mathematics in [6]. A similarly comprehensive survey is provided by Quastel in [19]. Much can be and has been said about the KPZ equation, but in the context of this thesis, two aspects of this equation stand out in particular.

The first was that under modest hypotheses, the random growth models which solve the KPZ equation should be stable under changes to parameters of these models, such as the underlying probability distributions or local rules and behaviours. The second and arguably most recognizable aspect is the so-called **KPZ scaling**, often also known as the **1-2-3 scaling**. Roughly speaking, on a window of spacetime with space proportional to $\sigma^{\frac{2}{3}}$ and time proportional to σ that the fluctuations of this random height function about its mean will be proportional to $\sigma^{\frac{1}{3}}$. Despite proofs of most claims about random growth models that fall into this universe remaining elusive, it is the widespread occurrence of this scaling in randomly growing interfaces within physics and mathematics that leads to the belief in this universality. It is precisely the elusiveness of the global structure of the KPZ universality class which had made the **directed landscape**, introduced by Dauvergne, Ortmann, and Virág in [9], of such immediate and profound interest within the probability community upon its discovery. The relationship between the directed will be expanded upon in section 1.4.

The potentially foundational role that the directed landscape will play in this unravelling the long-standing mysteries of the random growth models in this universe makes understanding its structure and geometric properties of significant interest within the probability community at large. Given the high degrees of self-similarity and symmetries of the directed landscape, investigating its fractal structure is a very natural avenue of research. Similar questions have been asked by Ganguly and Zhang in [13], (though their techniques unfortunately do not extend to this context in particular), Basu and Bhatia in [2], Bhatia independently in [4], and Bates, Ganguly, and Hammond in [3].

We will specifically be investigating the level sets of the directed landscape, sometimes restricted to convex and compact sets, in this thesis. A summary of our findings is presented at the end of this chapter in section 1.7. Will now introduce some preliminary background information needed to understand these results. This thesis is relatively self-contained and should be accessible to most readers who have taken a full year of graduate level measure-theoretic probability theory.

1.2 The Parabolic Airy Line Ensemble

We begin our preliminary material by providing an expedited introduction to a particular line ensemble of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ known as the **parabolic Airy line ensemble**. Just as understanding the normal distribution is key to understanding Brownian motion, a baseline level familiarity with the parabolic Airy line ensemble is key to understanding the random objects which will be working with in this thesis. In much the same way that one may intuitively think of a Brownian motion as being an infinite number of normal random variables glued together in a particularly nice way, one may also in a sense think of the **directed landscape,** the central random object upon which we will focus, as being composed of an infinite number of copies of the top line of the parabolic Airy line ensemble glued together in a particularly nice way. We borrow our definition in this thesis from the exposition in [10].

Definition 1.2.1. The **parabolic Airy line ensemble** is a collection of functions $\mathfrak{A} = (A_i)_{i=1}^{\infty}$ in $C(\mathbb{R}, \mathbb{R})$ such that for all $i \in \mathbb{Z}$, $\mathfrak{A}_i > \mathfrak{A}_{i+1}$, and the process

$$\left(\mathfrak{A}_i(r) + r^2\right)_{r\in\mathbb{R}}$$

is the unique determinantal process with kernel K given by

$$K\Big((x,s);(y,t)\Big) = \begin{cases} \int_0^\infty e^{-\lambda(s-t)} Ai(x+\lambda)Ai(y+\lambda)d\lambda, & \text{if } s \geq t \\ -\int_{-\infty}^0 e^{-\lambda(s-t)} Ai(x+\lambda)Ai(y+\lambda)d\lambda, & \text{if } s < t \end{cases}$$

where Ai is the Airy function.

The process $\Big(\mathfrak{A}_i(r)+r^2\Big)_{r\in\mathbb{R}}$ is stationary and is referred to as the **Airy line ensemble**. For the sake of brevity, we will not go into further detail in this thesis, but a more thorough introduction to determinantal processes can be found in [14]. In the universe that we are working in, the full line ensemble $\mathfrak{A}$, or at least large subsets of $\mathfrak{A}$, are often all used at once. However, in the work contained in this thesis, we will only ever be working with $\mathfrak{A}_1$, the top line of the parabolic Airy line ensemble.

It is also important to mention that the distribution of $\mathfrak{A}_i(r)+r^2$ for any fixed $r \in \mathbb{R}$ is well-understood. In particular, the distribution of $\mathfrak{A}_i(r)+r^2$ is what is known as the **Tracy-Widom**$_2$ distribution, denoted TW_2. Much can be said about this distribution and its importance in the context of the KPZ universality class, but we will include only the information strictly necessary to understand this thesis. To that end, the most important facts that we will use about this distribution is that it has a bounded density with respect to the Lebesgue measure on $\mathbb{R}$ and that it has strong upper and lower tail bounds. However, as we will never directly use the precise formula for this density or the tail bounds of this distribution, we do not include them here for brevity. A more thorough discussion about these ideas can be found in [15].

In 2013, Corwin and Hammond developed a technique called the **Brownian Gibbs property** for working with the law of $\mathfrak{A}$ and comparing it to the law of a collection of non-intersecting Brownian bridges. A full exposition can be found in [7], but by thinking of the line ensemble $\mathfrak{A}$ as a function

$$\mathfrak{A} : \mathbb{Z}_{>0} \times \mathbb{R} \to \mathbb{R}$$
$$(i,r) \mapsto \mathfrak{A}_i(r)$$

the Brownian Gibbs property can essentially be thought of as saying that for any compact set

$$K = \{i_1, i_2, \ldots, i_n\} \times [a,b] \subseteq \mathbb{Z}_{>0} \times \mathbb{R},$$

if we condition on the values of $\mathfrak{A}$ on K^C, then the law of $\mathfrak{A}$ restricted to K is absolutely continuous with respect to the law of n non-intersecting Brownian bridges

$$B_{i_1} > B_{i_2} > \cdots > B_{i_n}$$

with diffusion parameter 2 from $(a, \mathfrak{A}_{i_k}(a))$ to $(b, \mathfrak{A}_{i_k}(b))$ for each $k \in \{1,2,\ldots,n\}$. Although the Brownian Gibbs property was originally introduced by Corwin and Hammond, in this thesis we opt to use a version of this absolute continuity statement proven by Dauvergne in 2023 in [8] which has stronger bounds on the Radon-Nikodym derivative than in the original version of the result.

The following absolute continuity result from Theorem 1.8 of [8] will be at the core of our own absolute continuity lemmas that we prove in sections 5.1 and 6.1. We do not need the full power of Dauvergne's theorem, however, so we will only state a weaker version of it here, which is outlined in Example 1.7 of the same paper. We also note that although Dauvergne states the original version of this result for an interval of the form $[0, T]$, the same result holds on a general interval $[a_0, a_0 + T]$ for any $a_0 \in \mathbb{R}$ due to the shift invariance of the parabolic Airy line ensemble.

Theorem 1.2.2 (Dauvergne, Example 1.7 in [8])**.** *For any $a_0 \in \mathbb{R}$ and $T_0 \geq 1$, there exists an absolute constant $c > 0$ such that*

$$\text{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in[a_0,a_0+T_0]}\right) \leq e^{cT_0^3}\,\text{Law}\left(\Big(\mathfrak{L}_1(r)\Big)_{r\in[a_0,a_0+T_0]}\right)$$

where we may decompose the stochastic process $\Big(\mathfrak{L}_1(r)\Big)_{r\in[a_0,a+T_0]}$ as

$$\Big(\mathfrak{L}_1(r)\Big)_{r\in[a_0,a_0+T_0]} \stackrel{d}{=} \Big(L(r) + B(r)\Big)_{r\in[a_0,a_0+T_0]}$$

with B a diffusion parameter two Brownian bridge on $[a_0, a_0+T_0]$ from 0 to 0 and L an affine linear function which is independent of B satisfying

$$L(a_0) \stackrel{d}{=} \mathfrak{L}_1(a_0) \qquad \text{and} \qquad L(a_0+T_0) \stackrel{d}{=} \mathfrak{L}_1(a_0+T_0).$$

Moreover, there exist T_0-dependent (and a_0-independent) constants $d_1, d_2 > 0$ such that for all $m > 0$, we have that

$$\mathbb{P}\Big(L(a_0) \vee L(a_0+T_0) > m\Big) \leq \exp\left(-\frac{4}{3}m^{\frac{3}{2}} + d_1 m^{\frac{5}{4}}\right) \tag{1.2.1}$$

$$\mathbb{P}\Big(L(a_0) \wedge L(a_0+T_0) < -m\Big) \leq 2\exp\left(-d_2 m^3\right) \tag{1.2.2}$$

$$\mathbb{P}\Big(|L(a_0) - L(a_0+T_0)| > m\Big) \leq \exp\left(-\frac{1}{4T_0}m^2 - \frac{2}{3}m^{\frac{3}{2}} + d_1 m^{\frac{5}{4}}\right). \tag{1.2.3}$$

Due to the fact that the bound on the Radon-Nikodym derivative above grows exponentially as $T_0 \to \infty$, Theorem 1.2.2 is not well suited for dealing with $\mathfrak{A}_1$ on arbitrarily long intervals $[a_0, a_0+T_0]$. However, it is often that case that when working with $\mathfrak{A}_1$ on extremely long intervals, we are actually only interested in understanding $\mathfrak{A}_1$ on a very specific and finite collection of subintervals within that singular extremely large interval $[a_0, a_0T_0]$. In this situation where the values of $\mathfrak{A}_1(r)$ are of no concern for the vast majority of the points $r \in [a_0, a_0+T_0]$, Dauvergne extends Theorem 1.8 and provides a stronger version which is capable of handling the distribution of $\mathfrak{A}_1$ on a finite disjoint union of closed intervals. As before, we will not need the full power of Dauvergne's work, so we only present a simpler version of that theorem here.

Theorem 1.2.3 (Dauvergne, Theorem 3.8 in [8])**.** *Fix $T_0 \geq 1$ and $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$ such that $a_1 + T_0 < a_2$. Then there exists an absolute constant $c > 0$ and a random process $(\mathfrak{L}_1^{\boldsymbol{a}}(r))_{r\in\mathbb{R}}$ such* *that*

$$\text{Law}\left((\mathfrak{A}_1(r))_{r\in[a_1,a_1+T_0]}, (\mathfrak{A}_1(r))_{r\in[a_2,a_2+T_0]}\right) \le e^{cT_0^3}\,\text{Law}\left((\mathfrak{L}_1^{\boldsymbol{a}}(r))_{r\in[a_1,a_1+T_0]}, (\mathfrak{L}_1^{\boldsymbol{a}}(r))_{r\in[a_2,a_2+T_0]}\right).$$

Moreover, for each $j \in \{1,2\}$ *we can write*

$$\left(\mathfrak{L}_j^{\boldsymbol{a}}(r)\right)_{r\in[a_j,a_j+T_0]} \stackrel{d}{=} \Big(B_j(r) + L_j(r)\Big)_{r\in[a_j,a_j+T_0]}$$

where B_j *is a diffusion parameter 2 Brownian bridge from 0 to 0 on* $[a_j, a_j + T_0]$*,* B_1 *is independent of* B_2*,* (B_1, B_2) *is independent of* (L_1, L_2)*, we have the decomposition in law*

$$\Big(L_j(r)\Big)_{r\in[a_j,a_j+T_0]} \stackrel{d}{=} \left(\frac{(a_j+T_0)-r}{T_0}\mathfrak{L}_1^{\boldsymbol{a}}(a_j) + \frac{r-a_j}{T_0}\mathfrak{L}_1^{\boldsymbol{a}}(a_j+T_0)\right)_{r\in[a_j,a_j+T_0]} \tag{1.2.4}$$

and for some T_0-*dependent constants* $c_1, c_2 > 0$ *we have that for all* $m > 0$*,*

$$\begin{aligned} \mathbb{P}\Big(|L_j(a_j) + a_j^2| > m\Big) = \mathbb{P}\Big(|\mathfrak{L}(a_j) + a_j^2| > m\Big) \le c_1 e^{-c_2 m^{3/2}} \\ \mathbb{P}\Big(|L_i(a_j+T_0) + (a_j+T_0)^2| > m\Big) = \mathbb{P}\Big(|\mathfrak{L}(a_j+T_0) + (a_j+T_0)^2| > m\Big) \le c_1 e^{-c_2 m^{3/2}}. \end{aligned} \tag{1.2.5}$$

Though this is at its core only a minor adjustment to the exact phrasing used by Dauvergne, this does vary slightly from his original presnetation of this result. For the sake of completeness, we now take a moment to elaborate a bit more on how exactly this follows from the original statement of Dauvergne's Theorem 3.8. In Dauvergne's original, fully generalized version of this theorem, what we first have is that under these same hypotheses,

$$\text{Law}\left((\mathfrak{A}_1(r))_{r\in[a_1,a_2+T_0]}\right) \le\ e^{cT_0^3}\,\text{Law}\left((\mathfrak{L}_1^{\boldsymbol{a}}(r))_{r\in[a_1,a_2+T_0]}\right)$$

where the stochastic process $\mathfrak{L}_1^{\boldsymbol{a}}$ has the property that

$$\text{Law}\left((\mathfrak{L}_1^{\boldsymbol{a}}(r))_{r\in[a_1,a_2+T_0]}\right) = \text{Law}\left((B(r)+L(r))_{r\in[a_1,a_2+T_0]}\right).$$

The stochastic process $\Big(L(r)\Big)_{r\in[a_1,a_2+T_0]}$ is the random line segment on $[a_1, a_2 + T_0]$ such that

$$L(a_1) = \mathfrak{L}_1^{\boldsymbol{a}}(a_1) \qquad \text{and} \qquad L(a_2+T_0) = \mathfrak{L}_1^{\boldsymbol{a}}(a_2+T_0)$$

which characterizes it uniquely almost surely. The stochastic process $\Big(B(r)\Big)_{r\in[a_1,a_2+T_0]}$ is defined by the property (among others) that

$$\Big(B(r)\Big)_{r\in[a_1,a_1+T_0]}$$

is a diffusion parameter 2 Brownian bridge from $\Big(a_1, 0\Big)$ to $\Big(a_1+T_0,\ L(a_1+T_0)\Big)$, and that

$$\Big(B(r)\Big)_{r\in[a_2,a_2+T_0]}$$

is a diffusion parameter 2 Brownian bridge from $\Big(a_2,\ L(a_2)\Big)$ to $\Big(a_2+T_0,\ 0\Big)$. Based on this, we are able to then write that

$$\Big(B(r)\Big)_{r\in[a_1-T_0,a_1+T_0]} \overset{d}{=} \Big(B_1(r)+\ell_1(r)\Big)_{r\in[a_1,a_1+T_0]}$$

where B_1 is a diffusion parameter 2 Brownian bridge from $(a_1, 0)$ to $(a_1+T_0, 0)$ and ℓ_1 is the random affine function such that

$$\ell_1(a_1) = 0 \qquad \text{and} \qquad \ell_1(a_1+T_0) = L(a_1+T_0).$$

Similarly, we can also write that

$$\Big(B(r)\Big)_{r\in[a_2-T_0,a_2+T_0]} \overset{d}{=} \Big(B_2(r)+\ell_2(r)\Big)_{r\in[a_2,a_2+T_0]}$$

where B_2 is a diffusion parameter 2 Brownian bridge from $(a_2, 0)$ to $(a_2+T_0, 0)$ and ℓ_2 is the random affine function such that

$$\ell_2(a_2) = L(a_2) \qquad \text{and} \qquad \ell_2(a_2+T_0) = 0.$$

In this setup, B_1 and B_2 are both independent of all other terms that we have extracted from $\mathfrak{L}_1^{\boldsymbol{a}}$. Putting all of these observations together, we then obtain that

$$\begin{aligned}
&\operatorname{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(\mathfrak{A}_1(r)\Big)_{r\in[a_2,a_2+T_0]}\right)\\
\leq{}& e^{cT_0^3}\operatorname{Law}\left(\Big(\mathfrak{L}_1^{\boldsymbol{a}}(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(\mathfrak{L}_1^{\boldsymbol{a}}(r)\Big)_{r\in[a_2,a_2+T_0]}\right)\\
={}& e^{cT_0^3}\operatorname{Law}\left(\Big(B_1(r)+L(r)+\ell_1(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(B_2(r)+L(r)+\ell_2(r)\Big)_{r\in[a_2,a_2+T_0]}\right)\\
=:{}& e^{cT_0^3}\operatorname{Law}\left(\Big(B_1(r)+L_1(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(B_2(r)+L_2(r)\Big)_{r\in[a_2,a_2+T_0]}\right).
\end{aligned}$$

We also note that it is the equality of laws

$$\begin{aligned}
&\operatorname{Law}\left(\Big(\mathfrak{L}_1^{\boldsymbol{a}}(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(\mathfrak{L}_1^{\boldsymbol{a}}(r)\Big)_{r\in[a_2,a_2+T_0]}\right)\\
={}&\operatorname{Law}\left(\Big(B_1(r)+L(r)+\ell_1(r)\Big)_{r\in[a_1,a_1+T_0]}, \Big(B_2(r)+L(r)+\ell_2(r)\Big)_{r\in[a_2,a_2+T_0]}\right)
\end{aligned}$$

and the fact that $B_1(a_1+T_0) = B_2(a_2) = 0$, which implies that

$$L_1(a_1+T_0) \overset{d}{=} \mathfrak{L}_1^{\boldsymbol{a}}(a_1+T_0) \qquad \text{and} \qquad L_2(a_2) \overset{d}{=} \mathfrak{L}_1^{\boldsymbol{a}}(a_2)$$

in Theorem 1.2.3. The tail bounds in equation (1.2.5) follow from applying the same general techniques used by Dauvergne in [8] to find tail bounds for the process introduced in Theorem 1.8.

1.3 Last Passage Percolation

The first and most fundamental definition that we will introduce in this section is the notion of a **directed metric**. This will essentially be a metric where we relax the requirement that it be non-negative, and allow it to take on at most one of ∞ and $-\infty$ in addition to real values.

Definition 1.3.1. A **directed metric of positive sign** on a set S is a function $d : S^2 \to \mathbb{R} \cup \{\infty\}$ such that

$$\begin{aligned} d(p,p) &= 0 \qquad \text{for all } p \in S \\ d(p,q) + d(q,r) &\geq d(p,r) \qquad \text{for all } p,q,r \in S. \end{aligned}$$

A function $d : S^2 \to \mathbb{R} \cup \{-\infty\}$ is called a **directed metric of negative sign** on S if $-d$ is a directed metric of positive sign on S.

While there is no shortage of examples of directed metrics of negative or positive sign (including all true metrics for instance), the directed metrics of negative sign that we will be most interested in will arise in the random growth model called **last passage percolation.**

Definition 1.3.2. Let $G = (V, E)$ be the directed graph defined by $V := \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ and

$$E := \Big\{ \Big((x_1, y_1), (x_2, y_2) \Big) : (x_2, y_2) = (x_1 + 1, y_1) \text{ or } (x_2, y_2) = (x_1, y_1 + 1) \Big\}.$$

For each $v \in V$ place an i.i.d. random weight W_v on the vertex v. Letting P be the set of paths in G, i.e. the up-right paths on the lattice V, define the random directed metric of negative sign d_{LPP} on the directed graph G by

$$d_{LPP}(p,q) := \sup \left\{ \sum_{v \in \pi} W_v : \pi = \pi_1 \pi_2 \dots \pi_n \in P, \pi_1 = p, \text{ and } \pi_n = q \right\} \tag{1.3.1}$$

for all $p, q \in V$. The **last passage value from** p **to** q is defined as the value $d_{LPP}(p,q)$.

Several other variations of this definition also exist, but for our purposes we will only consider Exponential last passage values (i.e. where the weights are i.i.d. Exponential random variables), with the knowledge that switching between different models has no impact whatsoever on any result in this thesis. Exponential last passage values can also be extended to all points in $\mathbb{R}^2$ via standard quantization methods. The primary reason for introducing last passage percolation, aside from having a concrete example of a random growth model in hand, is that it belongs to the KPZ universality class. An explanation of this membership can be found in [10].

In fact, in this same paper, Dauvergne, Nica, and Virág proved that this model of last passage percolation and several others have an extremely important scaling limit in the uniform on compact topology on $C(\mathbb{R}^4_\uparrow, \mathbb{R})$, where the set $\mathbb{R}^4_\uparrow$ is defined as

$$\mathbb{R}^4_\uparrow := \Big\{ (x,s;y,t) \in \mathbb{R}^4 \ : \ s < t \Big\}.$$

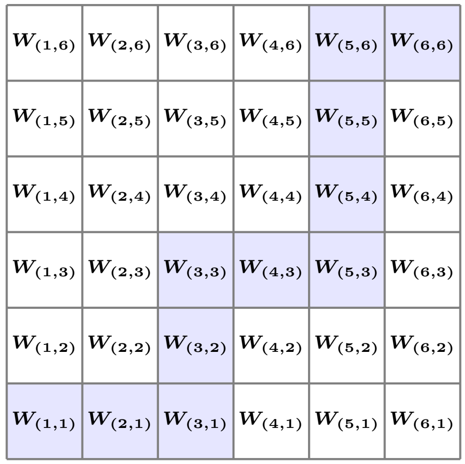

Figure 1.1: A visual representation of a path in a last passage percolation model. Here we are identifying squares in the grid with points in the lattice $V = \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. Paths in this graph are just up-right paths on the grid, and the last passage value gives us the maximal weight of a path between two points. We are indicating the path π of maximal weight from $(1, 1)$ to $(6, 6)$ with blue squares. This optimal path π will be an up-right path-valued random variable in general.

This scaling limit of these various last passage percolation models is a random directed metric of negative sign, which appears to be intimately related to the KPZ universality class as a whole. This random directed metric of negative sign is called the **directed landscape**, denoted $\mathcal{L}$, and was originally introduced by Dauvergne, Ortmann, and Virág in their groundbreaking paper [9] in 2018. The original definition of the directed landscape is independent of its characterization as the uniform on compact limit of lass passage percolation, but requires a modest amount of overhead and terminology to state. In the interest of maximizing readability, we will use Dauvergne, Nica, and Virág's characterization of $\mathcal{L}$ as a limit as our definition in this thesis, and introduce its defining properties afterwards as propositions.

We will introduce this limit theorem in the following section. We also note however that due to the immense generality of the full version of this result, we will again limit our scope to a single demonstrative example in the case of Exponential last passage percolation. We will also remove a degree of choice in the original system by fixing a value of one of the original theorem's parameters ρ. As before, this choice is done simply to maximize readability and has no influence at all on the results which build upon this theorem.

1.4 The Directed Landscape

In this section, we will briefly introduce the definition of the **directed landscape**, as well as the most relevant propositions and theorems related to it for the results in this thesis. Much, much more than this can be said, however, due to the importance of this random object within the context of trying to understand the KPZ universality class. It is the belief of Virág and numerous others working in this domain that the directed landscape is either *the* central limiting random object in this class, or at the very least, a very large component of this elusive limiting random object. A much fuller and richer discussion about the directed landscape and its origins can be read in [9], as well as in [11] where Dauvergne, Nica, and Virág explicitly prove the universality of the directed landscape for several well-known random growth models in the KPZ class. We will provide a significantly simplified and lightly rephrased version of that result in the case of last passage percolation, and take it to be our definition of the directed landscape. After this, we will introduce its most relevant properties for this thesis.

Theorem 1.4.1 (Dauvergne, Nica, and Virág, Theorem 1.7 in [11])**.** *For any* $(p, q) \in \mathbb{R}^4_\uparrow$*, let* $d_{LPP}(p, q)$ *be the Exponential last passage value from* p *to* q*, and define four positive constants* $\alpha, \beta, \chi,$ *and* τ *by*

$$\alpha := 8, \qquad \beta := 2, \qquad \chi := 2^{\frac{4}{3}}, \qquad \text{and} \qquad \tau := 2^{\frac{5}{4}}. \tag{1.4.1}$$

There exists a random directed metric of negative sign $\mathcal{L} \in C(\mathbb{R}^4_\uparrow, \mathbb{R})$*, independent of the choice of* $\alpha, \beta, \chi,$ *and* τ *with the following property: For any sequence* $\sigma \to \infty$*, there is a coupling of* $\mathcal{L}$ *and identically distributed copies* d_σ *of* d_{LPP} *such that*

$$d_\sigma\left(0, 0;\ y\sigma^2\tau + t\sigma^3, -t\sigma^3\right) \to \mathcal{L}(0, 0; y, t)$$

uniformly in (y, t) *as* $\sigma \to \infty$ *on any compact* $K \subset \mathbb{R} \times \mathbb{R}_{>0}$*, as functions in* $C(K, \mathbb{R})$*, almost surely.*

We have kept the notation used in Theorem 1.4.1 unchanged to ease the transition between our formulation and the original theorem in [10], but going forward we will make several notational changes to remove ambiguity in the context of this thesis. Namely, we will use the sequence $(n)_{n=1}^\infty$ instead of an arbitrary sequence σ which goes to ∞, and will refer to the identically distributed copies of d_{LPP} by

$$\left\{d_{LPP}^{(n)}\right\}_{n=1}^{\infty}.$$

This notation and this theorem will only be relevant in Chapter 2, but similar-looking notation will be used throughout our other work, so we choose to remove any ambiguity here as a matter of prudence. With this theorem now clearly stated, we can introduce the following definition.

Definition 1.4.2. The **directed landscape** is defined to be the random directed metric of negative sign $\mathcal{L} \in C(\mathbb{R}^4_\uparrow, \mathbb{R})$ in Theorem 1.4.1.

It was proven via an equivalent definition in [9] that $\mathcal{L}$ exists and is unique almost surely. However, because this is not the original definition of the directed landscape, we will state the properties that uniquely characterize it in its original definition as a proposition here.

Proposition 1.4.3. *The directed landscape* $\mathcal{L}$ *satisfies the following three properties.*

- *(**Metric composition law**): Almost surely, for any* $(x, s; y, t) \in \mathbb{R}^4_\uparrow$ *and any* $s < r < t$ *we always have that*
$$\mathcal{L}(x, s; y, t) = \max_{z \in \mathbb{R}} \Big(\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)\Big).$$
- *(**Independent temporal increments**): For any collection of disjoint time intervals* $\{(s_i, t_i)\}_{i=1}^n$*, the random functions in the collection*
$$\Big\{\mathcal{L}(\cdot, s_i; \cdot, t_i)\Big\}_{i=1}^{n}$$

are all mutually independent.

- ***(Parabolic Airy marginals)**: For any fixed time interval $[s,t]$ with $s < t$, the random function in $C(\mathbb{R}^2, \mathbb{R})$ given by*

$$(x,y) \mapsto \mathcal{L}(x,s;y,t)$$

is equal in distribution to the function

$$(x,y) \mapsto (t-s)^{\frac{1}{3}} \mathfrak{A}_1 \left((t-s)^{-\frac{2}{3}}(x-y) \right).$$

We note here that this third property is a slightly weaker version of the actual statement in [9]. We choose to omit that stronger property for the sake of brevity. The weaker version that we present here will be sufficient for our purposes throughout this thesis.

The second proposition that we will make extensive use of is the fact that the directed landscape has a considerable number of symmetries, giving it its robust fractal structure.

Proposition 1.4.4. *As functions in $C\left(\mathbb{R}^4_{\uparrow}, \mathbb{R}\right)$ we have the following equalities in distribution for any $(x,t;y,t+s) \in \mathbb{R}^4_{\uparrow}$, $r, c \in \mathbb{R}$ and any $q > 0$:*

- ***(Spatial stationarity)**: $\mathcal{L}(x,t;\ y,t+s) \overset{d}{=} \mathcal{L}(x,t+r;\ y,t+s+r)$*
- ***(Temporal stationarity)**: $\mathcal{L}(x,t;\ y,t+s) \overset{d}{=} \mathcal{L}(x+c,t;\ y+c,t+s)$*
- ***(Flip symmetry)**: $\mathcal{L}(x,t;\ y,t+s) \overset{d}{=} \mathcal{L}(-y,-s-t;\ -x,-t)$*
- ***(Skew stationarity)**: $\mathcal{L}(x,t;\ y,t+s) \overset{d}{=} \mathcal{L}(x+ct,t;\ y+ct+sc,t+s) + \frac{\left((x-y-sc)^2-(x-y)^2\right)}{s}$*
- ***(KPZ rescaling)**: $\mathcal{L}(x,t;y,t+s) \ \overset{d}{=}\ \mathcal{L}\left(q^{-2}x, q^{-3}t;\ q^{-2}y, q^{-3}(t+s)\right)$*

Another extremely useful property of the directed landscape is that it satisfies a very strong modulus of continuity on compact subsets of $\mathbb{R}^4_{\uparrow}$. The existence of this modulus of continuity, as well as the very strong tail bound on the random constant appearing within it, will be instrumental in establishing an upper bound on the Hausdorff dimension of the level sets of the directed landscape (intersected with any compact set). However, as we will only use this modulus of continuity Chapter 3 and its statement is quite technical, we defer stating it until it is about to be used, in order to once again maximize readability later on. For the sake of completeness, the modulus of continuity is given as Theorem 3.0.2.

Noting that the convergence in Theorem 1.4.1 is limited to the uniform on compact topology, this suggests that it is quite natural to restrict our attention to the behaviour of the directed landscape on compact subsets $K \subseteq \mathbb{R}^4_{\uparrow}$. Moreover, as a consequence of the temporal and spatial stationarity of $\mathcal{L}$ we also see that

$$\mathcal{L}(x,s;y,t) \overset{d}{=} \mathcal{L}(0,0;y-x,t-s)$$

for any $(x,s;y,t) \in \mathbb{R}^4_{\uparrow}$. In this sense, it is actually more natural to think of $\mathcal{L}$ as really being a two-parameter function of the two increments $y-x$ and $t-s$, where $y-x$ can be arbitrary and $t-s$

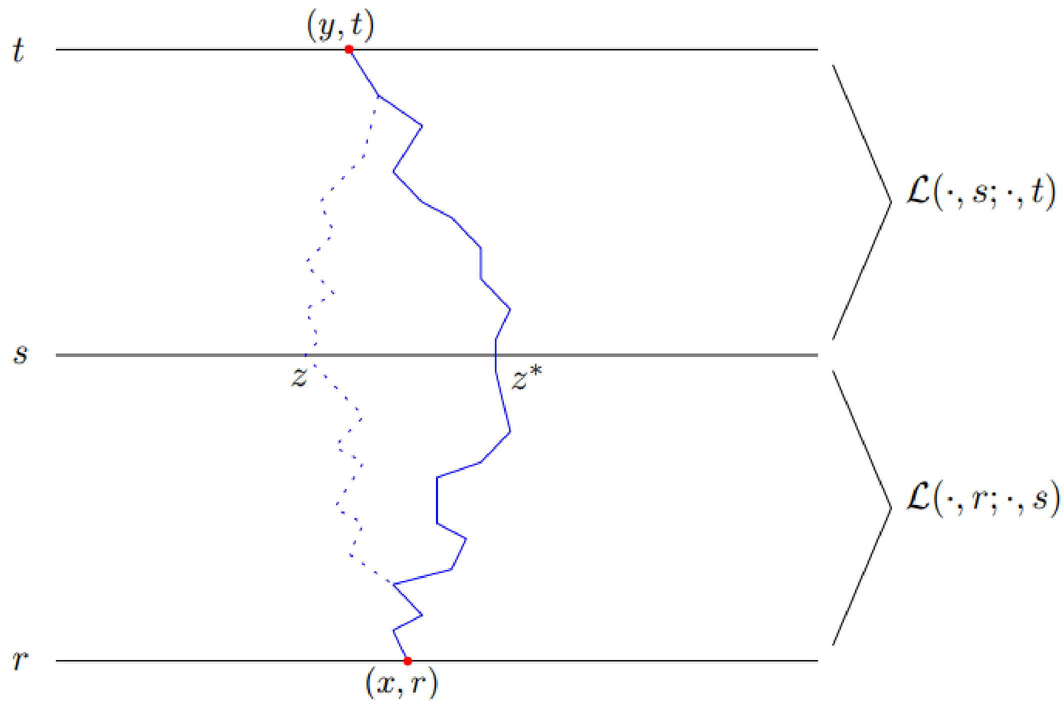

Figure 1.2: A visual representation of the metric composition law of $\mathcal{L}$, borrowed from [9]. $\mathcal{L}$ induces a notion of **directed geodesics** on $\mathbb{R}^4_\uparrow$, and the value of $\mathcal{L}(x, r; y, t)$ is the length of a directed geodesic under $\mathcal{L}$ from (x, r) to (y, t). Given a fixed intermediate point (z, s), concatenating the geodesics under $\mathcal{L}$ from (x, r) to (z, s) and from (z, s) to (y, t) yields a candidate path for the directed geodesic from (x, r) to (y, t). Maximizing the values of $\mathcal{L}$ on these two concatenated paths over all choices of z for a fixed s yields the optimal path, which passes through (z^*, s). It is worth mentioning that this supremal value is always achieved at some $z^* \in \mathbb{R}$, making it a true maximum.

must be strictly positive. As such, in order to understand the behaviour of $\mathcal{L}$ on the whole of $\mathbb{R}^4_\uparrow$, it is equivalent to understand the behaviour of the function

$$\begin{aligned} \mathcal{L}(0, 0, \cdot, \cdot) \ : \ \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R} \\ (x, s) \mapsto \mathcal{L}(0, 0; x, s). \end{aligned}$$

Similarly, to understand the behaviour of $\mathcal{L}$ on a given compact subset $K \subseteq \mathbb{R}^4_\uparrow$ it is equivalent to understand the behaviour of the function

$$\begin{aligned} \mathcal{L}(0, 0, \cdot, \cdot) \ : \ [-n, n] \times \left[\frac{1}{n}, n\right] \to \mathbb{R} \\ (x, s) \mapsto \mathcal{L}(0, 0; x, s). \end{aligned}$$

for a sufficiently large choice of $n \in \mathbb{Z}_{>0}$. For the sake of simplicity, we choose to work with $\mathcal{L}(0, 0; \cdot, \cdot)$ instead of with $\mathcal{L}$ throughout this thesis, with the knowledge that every statement we prove about $\mathcal{L}(0, 0; \cdot, \cdot)$ translates into a statement about $\mathcal{L}$ as a whole by replacing (x, s) with a tuple of increments $(y - x, t - s) \in \mathbb{R} \times \mathbb{R}_{>0}$.

1.5 The Hausdorff Metric

In Chapter 2, we will be investigating whether the level sets of the rescaled last passage values in Theorem 1.4.1 converge to the corresponding level sets of $\mathcal{L}(0,0;\cdot,\cdot)$ on compact and convex set subsets of $\mathbb{R}\times\mathbb{R}_{>0}$. This means that we must necessarily decide on what exactly it means for a sequence of subsets of $\mathbb{R}\times\mathbb{R}_{>0}$ to converge to another limiting subset of the same space, and whether that precise mathematical definition agrees with what we intuitively imagine this sort of convergence would look like. There is by no means a unique answer to this problem, but one such possible answer is to use the topology generated by the **Hausdorff metric.**

Definition 1.5.1. Let (M,ρ) be an arbitrary metric space. For any $x\in M$ and $A\subseteq M$, let

$$d(x,A)=d(a,X):=\inf_{a\in A}\rho(x,a).$$

The **Hausdorff metric**, induced by ρ, is the metric d_H on 2^M, the power set of M, defined for any two subsets $A_1,A_2\subseteq M$ by

$$d_H(A_1,A_2)=d_H(A_2,A_1):=\max\left\{\sup_{a_1\in A_1}d(a_1,A_2),\ \sup_{a_2\in A_2}d(A_1,a_2)\right\}.$$

Equivalently, if for each $A\subseteq\mathbb{R}^m$ and each $\delta>0$ we define

$$A_\delta:=\bigcup_{a\in A}\left\{x\in\mathbb{R}^m\ :\ \rho(x,a)\le\delta\right\}$$

then we may also define $d_H(A_1,A_2)$ by

$$d_H(A_1,A_2)=d_H(A_2,A_1):=\inf\left\{\delta>0\ :\ A_1\subseteq\Big(A_2\Big)_\delta\text{ and }A_2\subseteq\Big(A_1\Big)_\delta\right\}.$$

In simpler terms, the Hausdorff distance between two sets A_1 and A_2 is the infimal $\delta>0$ such that every point $a_1\in A_1$ is within δ of a point in A_2, and every point $a_2\in A_2$ is within δ of a point in A_1. This second characterization of the Hausdorff metric also yields a very nice visual interpretation of what convergence in the topology generated by d_H means. In particular, it essentially says that if for some sequence of sets $(A_n)_{n=1}^\infty$ and A in $\mathbb{R}^m$ we have that

$$\lim_{n\to\infty}d_H(A_n,A)=0$$

then intuitively speaking, A_n and A must have extremely similar shapes in the naive visual sense for sufficiently large $n\in\mathbb{Z}_{>0}$. More specifically, for every $\delta>0$, there exists an $N=N(\delta)\in\mathbb{Z}_{>0}$ such that for $n\ge N$, A_n looks exactly the same as A up to perturbations along its boundary of size no more than δ. Thus, thinking of subsets of $\mathbb{R}^m$ as being $m-$dimensional shapes, the topology generated by this metric aligns extremely well with a naive guess of what it should mean for a sequence of shapes to converge to a limit shape. We do note however that this is by no means the only possible choice here. It is entirely possible that in the context of our work in Chapter 2 that using a different underlying metric for the Hausdorff metric, or even a different metric than the Hausdorff metric altogether, could be of equal or greater interest.

1.6 The Hausdorff Dimension and Related Ideas

When working with geometric objects that display a high degree of self-similarity, i.e. a fractal-like structure, one of the most useful and intrinsic properties of such objects is their **fractal dimension**. These definitions in this section are all standard, but for the sake of transparency, we borrow them from [5]. There are several different notions of what the dimension of a fractal is, all of which extend the usual notion of dimension for regular geometric objects such as hyperplanes and polygons, each having their own strengths and weaknesses. We will use two such notions of fractal dimension in this thesis, the **Minkowski dimension** and the **Hausdorff dimension**, and will then provide several extremely useful lemmas which enable us to find them systematically. We begin by recalling the definition of a totally bounded set in a metric space.

Definition 1.6.1. Let (M,ρ) be a metric space, and let $K \subseteq M$. K is **totally bounded** if for any $\varepsilon > 0$, there exists a finite collection of points $m_1, m_2, \ldots, m_n \in M$ such that

$$K \subseteq \bigcup_{j=1}^{n} \Big\{ x \in M \ : \ \rho(x, m_1) \leq \varepsilon \Big\}.$$

In a Euclidean space $\mathbb{R}^m$, this property is equivalent to simply being a bounded set by the Heine-Borel theorem. Thus, as we will only be looking at subsets of $\mathbb{R}^2$ in this thesis, we will never need to verify that this property holds before using the subsequent definitions.

In order to motivate the definition of a fractal dimension below, we first observe the following pattern in $\mathbb{R}^d$ for any $d \in \mathbb{Z}_{>0}$. Suppose that for each $\varepsilon > 0$ we wish to cover the unit $d-$cube $D \subseteq \mathbb{R}^d$ using a finite number of sets of diameter at most ε. Upon a moment of reflection, we would see that the most efficient such covering would be to cover D by $d-$cubes of side length ε and that the total number of such cubes needed to cover D is ε^{-d}. More generally, if we replace D by a $d-$cube of arbitrary side length r, the number of such cubes that will be needed is $C\varepsilon^{-d}$ for some $\varepsilon-$independent but $r-$dependent constant C. Given that this holds for any $\varepsilon > 0$, this means that we can write for any fixed $d-$cube D that

$$\lim_{\varepsilon \to 0^+} \frac{\log\left(N(D,\varepsilon)\right)}{\log\left(\frac{1}{\varepsilon}\right)} = \lim_{\varepsilon \to 0^+} \frac{\log\left(C\varepsilon^{-d}\right)}{\log\left(\frac{1}{\varepsilon}\right)} = \lim_{\varepsilon \to 0^+} \frac{\log\left(\varepsilon^{-d}\right)}{\log\left(\frac{1}{\varepsilon}\right)} - \lim_{\varepsilon \to 0^+} \frac{\log\left(C\right)}{\log\left(\frac{1}{\varepsilon}\right)} = d$$

where $N(D,\varepsilon)$ is the infimal number of sets of diameter at most ε needed to cover D.

Thus, the limit above successfully recovers the dimension of any $d-$cube D in $\mathbb{R}^d$, for any $d \in \mathbb{Z}_{>0}$. The same phenomenon will also hold for balls in $\mathbb{R}^d$ as well by an analogous argument. Since any sensible notion of a fractal dimension should agree with the usual notion of dimension for any nice sets such as these, it is this illustrative example which motivates our first definition of the fractal dimension of a general set. Note that it for these reasons that the **Minkowski dimension** is often referred to as the **box-counting dimension.**

Definition 1.6.2. Let (M,ρ) be a metric space and let $K \subseteq M$ be totally bound. For each $\varepsilon > 0$, let $N(K,\varepsilon)$ be the minimal number of open balls of diameter ε needed to cover K. The **upper**

Minkowski dimension of K, denoted $\overline{\dim}_{\mathscr{M}}K)$, is defined as

$$\overline{\dim}_{\mathscr{M}}(K) := \limsup_{\varepsilon\to 0^+} \frac{\log\left(N(K,\varepsilon)\right)}{\log\left(\frac{1}{\varepsilon}\right)}.$$

The **lower Minkowski dimension of** K, denoted $\underline{\dim}_M$(K), is defined as

$$\underline{\dim}_{\mathscr{M}}(K) := \liminf_{\varepsilon\to 0^+} \frac{\log\left(N(K,\varepsilon)\right)}{\log\left(\frac{1}{\varepsilon}\right)}.$$

If it is the case that $\underline{\dim}_{\mathscr{M}}(K) = \overline{\dim}_{\mathscr{M}}(K)$ then this value, denoted by $\dim_{\mathscr{M}}(K)$, is called **the Minkowski dimension of** K.

It is worth mentioning that rather than just using balls of radius $\varepsilon > 0$, $N(K,\varepsilon)$ can equivalently be defined using sets of diameter at most ε. Additional intuition motivating this definition can be found in [5] and in [18]. Though the upper and lower Minkowski dimensions are relatively easy to describe, they do not match in general, so many sets will have different upper and lower Minkowski dimensions. Another important pitfall of the upper and lower Minkowski dimensions is that even when they do agree, the Minkowski dimension is finitely stable but not countably stable. This is a major limitation in many situations. To obtain these nicer properties, a stronger notion of fractal dimension is needed. For us, this stronger notion will be the **Hausdorff dimension**, which requires us to first define the $\alpha-$ **Hausdorff content** of a set.

Definition 1.6.3. Let (M,ρ) be a metric space, and let $K \subseteq M$. For any $\alpha > 0$, the $\alpha-$**Hausdorff content** of K, denoted $\mathscr{H}^{\alpha}_{\infty}(K)$ is defined as

$$\mathscr{H}^{\alpha}_{\infty}(K) := \inf\left\{\sum_{i\in I}(\mathrm{diam}(U_i))^{\alpha} \ : \ K \subseteq \bigcup_{i\in I} U_i \text{ and } \{U_i\}_{i\in I} \text{ is a countable set of subsets of } M\right\}.$$

This preliminary definition now allows us to state our first definition of the **Hausdorff dimension.** Note that the $\alpha-$Hausdorff content is a direction generalization of the quantity $N(K,\varepsilon)$ that we had in the definition of the Minkowski dimension, lending credence to the idea that the definition below does indeed generalize the Minkowski dimension.

Definition 1.6.4. Let (M,ρ) be a metric space, and let $K \subseteq M$. The **Hausdorff dimension of** K, denoted $\dim_H(K)$, is defined as

$$\dim_H(K) := \inf\left\{\alpha \in \mathbb{R}_{>0} \ : \ \mathscr{H}^{\alpha}_{\infty}(K) = 0\right\}.$$

Though the physical intuition for the Hausdorff dimension is not as immediately obvious as that of the Minkowski dimension, the reasonability of this definition can again be understood by observing how it applies to $d-$cubes and $d-$balls in $\mathbb{R}^d$. Readers who are not already familiar with this definition are encouraged to prove that if K is any unit cube in $\mathbb{R}^d$, then

$$\mathscr{H}^{\alpha}_{\infty}(K) = 0$$

for any $\alpha < d$. This in turn means that $\dim_H(K) \geq d$, and since we clearly have that

$$\dim_H(K) \leq \dim_H\left(\mathbb{R}^d\right) \leq d$$

based on this definition of $\dim_H$, we do indeed have that the Hausdorff dimension of any $d-$cube or $d-$ball is again always d. More generally, we also observe in these simpler tangible cases that $\mathscr{H}^\alpha_\infty(K) = \infty$ for any $\alpha > d$, in the exact same way that the total length of any covering of a plane by line segments is infinite, or the total area of any covering of a $d-$ball by finite rectangles is infinite. This observation that the only possible answers when measuring the dimensionality of a set in $\mathbb{R}^d$ using the "wrong dimension" are 0 and ∞ will appear frequently in the exposition to come.

Note that because the Hausdorff dimension is defined as an infimum, it always exists for any set, unlike the Minkowski dimension. Moreover, for any totally bounded set K in a metric space (M, ρ),

$$\dim_H(K) \leq \underline{\dim}_{\mathscr{M}}(K) \leq \overline{\dim}_{\mathscr{M}}(K).$$

Since the upper Minkowski dimension is always an upper bound on the Hausdorff dimension, upper bounding $\overline{\dim}_{\mathscr{M}}(K)$ is often a convenient way to indirectly bound $\dim_H(K)$ from above. This technique is extremely common in the literature and is what we will employ later on in chapter 3. It simply requires showing that a lim sup goes to zero for one specific type of finite covering of the set K in question. Another very important advantage that the Hausdorff dimension has over the Minkowski dimension is that the Hausdorff dimension is countably stable. In other words, for any collection of sets $(K_j)_{j=1}^\infty$ it is always true that

$$\dim_H\left(\bigcup_{j=1}^\infty K_j\right) = \sup_{j \in \mathbb{Z}_{>0}} \dim_H(K_j),$$

The countable stability of $\dim_H$ will also be of the utmost utility in our work in Chapter 3.

Unfortunately, this definition of the Hausdorff measure $\dim_H$ can be a bit challenging to work with directly. To help combat this, a different but closely related quantity called the **α-dimensional Hausdorff measure** is typically used to recharacterize $\dim_H$. This measure also serves the dual purpose of providing a more intuitive understanding of what exactly it means to say that a set has Hausdorff dimension α.

Definition 1.6.5. Let (M, ρ) be a metric space, and let $K \subseteq M$. For any $\alpha, \delta > 0$, define the quantity $\mathscr{H}^\alpha_\delta(K)$ by

$$\mathscr{H}^\alpha_\delta(K) := \inf\left\{\sum_{i \in I} (\operatorname{diam}(U_i))^\alpha \ : \ K \subseteq \bigcup_{i \in I} U_i \ , \ I \text{ is at most countable, and } \sup_{i \in I} \operatorname{diam}(U_i) < \delta\right\}.$$

The $\alpha-$**dimensional Hausdorff measure of** K, denoted $\mathscr{H}^\alpha(K)$, is defined as

$$\mathscr{H}^\alpha(K) := \lim_{\delta \to 0^+} \mathscr{H}^\alpha_\delta(K).$$

Note that for any fixed set $K \subseteq M$ and fixed $\alpha \geq 0$, the function $\delta \mapsto \mathscr{H}^\alpha_\delta(K)$ is monotonically decreasing in δ, as any covering eligble for a smaller δ remains eligible for a larger one. This means that the limit as $\delta \to 0+$ is well-defined. Moreover, under these hypotheses, the function $K \mapsto \mathscr{H}^\alpha_\delta(K)$ is monotonic and countably subadditive, so it defines an outer measure on (M, ρ). In particular, the $\alpha-$dimensional Hausdorff measure $\mathscr{H}^\alpha_\delta$ is even a Borel measure on Euclidean spaces $\mathbb{R}^m$.

By observing that for any $\delta, \alpha, \beta > 0$, with $\beta > \alpha$ and any subset K of a metric space (M, ρ) that

$$\mathscr{H}^\beta_\delta(K) \leq \delta^{\beta-\alpha}\mathscr{H}^\alpha_\delta(K)$$

it immediately follows by letting $\delta \to 0^+$ that

$$\mathscr{H}^\alpha(K) < \infty \implies \mathscr{H}^\beta(K) = 0$$

for all $\beta > \alpha$. It can be shown that this elementary observation then implies the following proposition about the relationship between the $\alpha-$dimensional Hausdorff dimension of a set, and the Hausdorff dimension of that same set.

Proposition 1.6.6. *Let (M, ρ) be a metric space and let $\alpha > 0$. Then $\mathscr{H}^\alpha(M) = 0$ if and only if $\mathscr{H}^\alpha_\infty(M) = 0$. Furthermore, this means that*

$$\begin{aligned}
\dim_H(M) &= \inf\Big\{\alpha \in \mathbb{R}_{>0} \ : \ \mathscr{H}^\alpha(M) = 0\Big\} \\
&= \inf\Big\{\alpha \in \mathbb{R}_{>0} \ : \ \mathscr{H}^\alpha(M) < \infty\Big\} \\
&= \sup\Big\{\alpha \in \mathbb{R}_{>0} \ : \ \mathscr{H}^\alpha(M) > 0\Big\} \\
&= \sup\Big\{\alpha \in \mathbb{R}_{>0} \ : \ \mathscr{H}^\alpha(M) = \infty\Big\}.
\end{aligned}$$

This alternate characterization of $\dim_H$ is particularly useful when trying to bound the Hausdorff dimension of a set from below. Finding a lower bound on the Hausdorff dimension is always significantly harder than finding an upper bound, as this requires making a statement about every possible covering of the set at once. This process generally varies significantly from setting to setting, and typically depends heavily on the nature and structure of the set in question. Fortunately, some general techniques for finding a lower bound on the Hausdorff dimension of a set more systematically exist. The first of these such results is the **Mass Distribution Principle** below. We introduce one final preliminary definition before stating this theorem.

Definition 1.6.7. Let (M, ρ) be a metric space, and let μ be a Borel measure on (M, ρ). If

$$0 < \mu(M) < \infty$$

then μ is a **mass distribution** on (M, ρ).

With this in mind, we now state the Mass Distribution Principle. Note that we once again state a slightly weaker but simpler version that is sufficient for our purposes. The full version of this theorem can be found in [5].

Theorem 1.6.8 (Mass Distribution Principle)**.** *Let (M, ρ) be a metric space and suppose that $\alpha \geq 0$. If there exists a mass distribution μ on (M, ρ) and an absolute constant $C > 0$ such that for any $\delta > 0$ and $x \in M$*

$$\mu\left(\left\{m \in M \ : \ \rho(x, m) \leq \delta\right\}\right) \leq C\delta^{\alpha},$$

then we have that

$$\mathscr{H}_{\alpha}(M) \geq \mathscr{H}_{\alpha}^{\infty}(M) \geq \frac{\mu(M)}{C} > 0$$

and hence that $\dim_H(M) \geq \alpha$.

However in many situations, including those which will appear in this thesis, using the following consequence of the Mass Distribution Principle is much more directly useful. We refer the reader to [18] or [5] for proofs of these two results, as well as the additional discussion surrounding them. This **energy method** was originally proven by Frostman in 1935.

Theorem 1.6.9 (Energy Method)**.** *Let (M, ρ) be a metric space, and let $K \subseteq M$. If μ is a mass distribution on (K, ρ) and*

$$\int_K \int_K \frac{1}{\rho(x, y)^{\alpha}} d\mu(x) d\mu(y) < \infty$$

for some $\alpha \geq 0$, then $\mathscr{H}_{\infty}^{\alpha}(K) = \infty$ and hence $\dim_H(K) \geq \alpha$.

Constructing a mass distribution μ_h on the $h-$level set of the $\mathcal{L}(0, 0; \cdot, \cdot)$, which we will again restrict to a compact subset of $\mathbb{R} \times \mathbb{R}_{>0}$, in order to use Theorem 1.6.9 will be the focus of Chapter 4.

1.7 Outline of Thesis and Synopses of Main Results

There are four main ideas that are explored in this thesis. The first topic that we address is proving that when restricted to compact sets, the level sets of the sequence of rescaled last passage values in Theorem 1.4.1 converge to the corresponding level sets of $\mathcal{L}(0, 0; \cdot, \cdot)$ restricted to that same compact set. We first prove a sufficient pathwise condition for level set convergence under the (Euclidean) Hausdorff metric for deterministic functions, and prove that this condition is satisfied pathwise almost surely in the setting of Theorem 1.4.1. The key idea is establishing that the minimum and the maximum of $\mathcal{L}(0, 0; \cdot, \cdot)$ have no atoms in any convex and compact subset of $\mathbb{R} \times \mathbb{R}_{>0}$. This property, along with several small technical lemmas, will then yield our first result. Our main tools for proving that this criterion is satisfied pathwise almost surely will be the Brownian Gibbs property of $\mathfrak{A}$, and the numerous symmetries and properties of $\mathcal{L}$. Our final result is stated as Theorem 2.0.1. Chapter 2 is independent of the other chapters that follow.

Our next major topic is establishing an upper bound on the upper Minkowski dimension of the entire $h-$level set of $\mathcal{L}(0, 0; \cdot, \cdot)$, Our approach is to mimic the standard proof used to prove that the zero set of Brownian motion has upper Minkowski dimension at most $\frac{1}{2}$. However, the classic proof relies quite heavily on the strong Markov property, which $\mathcal{L}(0,0;\cdot,\cdot)$ has no analogue for. The strong Markov property is used to obtain an upper bound on

$$\mathbb{P}\Big(0 \in \mathcal{W}([a, a+\varepsilon)\Big) \tag{1.7.1}$$

where $\mathcal{W}$ is a standard Brownian motion, $a \in \mathbb{R}$, and $\varepsilon > 0$. This upper bound on the probability of having a zero in any given interval of length $\varepsilon > 0$ is then used to obtain an upper estimate on $\mathbb{E}[N_m]$, the expected number of intervals of length 2^{-m} needed to cover the zero set of $\mathcal{W}$, for each $m \in \mathbb{Z}_{>0}$. By bounding (1.7.1) by an adequately sharp function of ε, it can then be shown that

$$\lim \sup_{m\to\infty} \frac{\mathbb{E}\left[N_m\right]}{2^{\frac{m}{2}+\gamma}} < \infty$$

for any $\gamma > 0$, which proves that the upper Minkowski dimension of the zero set of $\mathcal{W}$ is at most $\frac{1}{2}$ with probability 1. Since we do not have the luxury of the strong Markov property to bound the analogue of (1.7.1) in the case of $\mathcal{L}(0,0;\cdot,\cdot)$, we instead rely on some careful manipulations of the modulus of continuity of $\mathcal{L}$ on a specific class of compact subsets of $\mathbb{R}^4_\uparrow$ to achieve a different but similar result in the end. This final result for the upper bound is stated in Theorem 3.0.1.

The third major topic is systematically constructing a mass distribution on the $h-$level set of stochastic process $(X(x,s))_{(x,s)\in\mathbb{R}^2}$. This mass distribution will then be used with Theorem 1.6.9 to obtain a lower bound on the Hausdorff dimension of the $h-$level set of X that holds with a positive $h-$dependent probability. The inputs for this technique will be upper and lower bounds on the one point distribution of X, and an upper bound on its two-point distribution. Both bounds need only hold on a compact subset of $\mathbb{R}^2$ for our purposes.

This procedure was originally inspired by a similar result for Brownian motion outlined in [1]. However, that result relied on the existence of characteristic functions for the marginal distributions of Brownian motion, and the fact that Brownian motion has the strong Markov property. These tools were once again not available to us, so quite a few modifications to that general argument needed to be made in order for it to generalize to the types of stochastic processes we are interested in here. The final version of this argument for stochastic processes indexed by subsets of $\mathbb{R}^2$ is given in Theorem 4.1.1.

Chapters 5 and 6 are dedicated to establishing a partial upper bound on the two-point distribution of the directed landscape. In particular, we will be developing an upper bound on probabilities of the form

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{L}(0,0;x,s) \in (h-\varepsilon, h+\varepsilon),\ \mathcal{L}(0,0;y,t) \in (h-\varepsilon, h+\varepsilon)\Big) \\
&\leq \mathbb{P}\Big(\mathcal{L}(0,0;x,s) \in (h-\varepsilon, h+\varepsilon),\ |\mathcal{L}(0,0;y,t) - \mathcal{L}(0,0;x,s)| \leq 2\varepsilon\Big)
\end{aligned} \tag{1.7.2}$$

where $(x,s),(y,t) \in [1,2] \times \left[1, \frac{11}{10}\right]$. This task is the most challenging part of this project and comprises the majority of the work within it. To set up the construction of this two-point bound, we first heavily exploit the symmetries of the directed landscape in Proposition 1.4.4 and its defining properties in Proposition 1.4.3 to reduce this problem to understanding the behaviour of a certain measurable function $s_i(\mathfrak{A}_1, \tilde{\mathfrak{A}}_1)$ (where $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are independent copies of the parabolic Airy line ensemble) on a partition of $\mathbb{R}$ into small closed intervals indexed by $i \in \mathbb{Z}$. This partition will be of the form

$$\mathbb{R} = \bigcup_{i=-\infty}^{\infty} \left[|t-s|^{\frac{2}{3}} i - \frac{1}{2}|t-s|^{\frac{2}{3}} \ , \ |t-s|^{\frac{2}{3}} i + \frac{1}{2}|t-s|^{\frac{2}{3}} \right].$$

We will be assuming that $t > s$ without loss of generality. We use this particular partition because upon using the metric composition law, the temporal and spatial stationarity of $\mathcal{L}$, its independent increments property, and its relationship to $\mathfrak{A}_1$, the dominant term that emerges in the difference $|\mathcal{L}(0,0;y,t) - \mathcal{L}(0,0;x,s)|$ will be equal in distribution to the process

$$\left((t-s)^{\frac{1}{3}} \tilde{\mathfrak{A}}_1 \left(\frac{z + (x-y)}{(t-s)^{\frac{2}{3}}} \right) \right)_{z \in \left[i - \frac{1}{2}, i + \frac{1}{2}\right]}$$

where $i \in \mathbb{Z}$. Working on intervals of length $(t-s)^{\frac{2}{3}}$ will therefore neutralize the rescaling in the argument of the dominant term and set everything else in motion afterwards.

Next, we use Dauvergne's specific formulations of the Brownian Gibbs property of $\mathfrak{A}_1$ in Theorem 1.2.2 and Theorem 1.2.3 to construct a family of random vectors $\left\{(X_i, Y_i)\right\}_{i \in \mathbb{Z}}$ such that Law $\Big((X_i, Y_i)\Big)$ dominates Law $\Big(s_i(\mathfrak{A}_1, \tilde{\mathfrak{A}}_1)\Big)$ for each $i \in \mathbb{Z}$. The construction of these random vectors will be split into two cases depending on how the index i compares to $10|t-s|^{-\frac{2}{3}}$, for reasons that will be explained later on. We then establish that each Law $\Big((X_i, Y_i)\Big)$ has a density ρ_i with respect to the Lebesgue measure on $\mathbb{R}^2$, that each ρ_i has a strong uniform bound, and that the two sums

$$\sum_{|i| \le 10|t-s|^{-\frac{2}{3}}} \rho_i \qquad \text{and} \qquad \sum_{|i| > 10|t-s|^{-\frac{2}{3}}} \rho_i$$

each have a relatively strong uniform upper bound on their domain in $\mathbb{R}^2$. The construction of (X_i, Y_i), and hence of ρ_i, will again be somewhat different when we pass from the first regime to the second, but will involve very similar ideas. These sufficiently good uniform upper bounds on the sums above are then used to prove the existence of a sufficiently good two-point bound for probabilities of the form (1.7.2). The two-point bound is stated explicitly in Theorem 4.3.1 before being proven. The work done across these chapters to understand the Hausdorff dimension of the level sets of $\mathcal{L}(0,0;\cdot,\cdot)$ is compiled into one final main result in Theorem 4.3.2 before we prove the existence of our partial two-point bound.

We will conclude this thesis by providing closing thoughts on the work that we have done, and will provide several ideas for possibly improving our main result in Theorem 4.3.2. We do not expect this result to be optimal by any stretch of the imagination, and view it primarily as an initial attempt to use this general technique that we have developed. This can be found in Chapter 7.

1.8 Explanation of Joint Work

In this section, we provide an explanation of how the joint work during these projects was distributed. The **one-parameter case** will refer to the endeavour of understanding the Hausdorff dimension of the level sets of the function $t \mapsto \mathcal{L}(0,0;0,t)$, in which the goal was to find upper and lower bounds of $\frac{2}{3}$. The **two-parameter case** refers to the content of this thesis in particular, i.e. understanding the Hausdorff dimension of the level sets of the function $(x,s) \mapsto \mathcal{L}(0,0;x,s)$.

The original work underlying the results in Chapter 2 and Chapter 3 was done almost entirely collaboratively. This was originally done several years ago and mistakes were recently found, so new proofs with more mature approaches to correct these mistakes were written independently by each of us in our theses. Note that for Chapter 3, our original work was in the two-parameter case. Simplifying our problems to the one-parameter case only became initially necessary when working on the lower bound. A noticeably more clear-cut division of labour emerged during the course of our third and fourth projects.

In the third project, the definition of the measures $\mu_{h,\varepsilon}$ suggested by our advisor. After conducting literature review to find a proof in [1] which used the energy method to find a lower bound on the Hausdorff dimension of the zero set of Brownian motion, Virginia adapted the argument for analogues of subsections 4.2.2, 4.2.3, and 4.2.5 in the one-parameter case from Adler's proof. The original version of the argument in subsection 4.2.2 and 4.2.5 specifically required a considerable amount of care, as much of Adler's proof did not generalize without the numerous powerful properties possessed by Brownian motion on $\mathbb{R}$. I worked on subsection 4.2.4 independently during this time. I later adapted the work in those subsections independently to the two-parameter case, and made several modest generalizations to the level of generality of the argument. Lemma 4.2.2 was the result of a one-on-one discussion with my advisor and I.

The fourth project began with the one-parameter case. Numerous intermediate problems not appearing in this thesis were first worked on to build intuition and a rough framework for the one-parameter result. Virginia spent a considerable amount of time and energy at the beginning of this trying to set up an initial deconstruction of

$$\Big\{\mathcal{L}(0,0;0,s) \in (h-\varepsilon, h+\varepsilon)\Big\} \cap \left\{\left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;0,s) + \mathcal{L}(z,s;0,t) - \mathcal{L}(0,0;0,s)\right| \leq 2\varepsilon\right\}$$

into more manageable sub-events that we could work with, before we arrived at the right notation to crystallize these ideas. During this phase of the process, I moulded Lemma 5.1.1 into a concrete and precise statement mostly independently. This was based on heuristics from our advisor and several joint discussions with the three of us about the intuition underlying this lemma. At this point, our work for the first regime in the one-parameter case was quite messy as it contained work from three different people with three different voices, so after completing Lemma 5.1.1, I focused on a first attempt at reshaping our work at the time into something more cohesive. During that phase of our work, Virginia focused on utilizing Lemma 5.1.1 while thinking about what exactly should follow our extant work at the time. She also worked out Lemma 5.2.1 during this same general timeframe.

After completing my first round of revisions to our old work, I worked independently on generalizing Lemma 5.1.1 to Lemma 6.1.1. Upon completing this new lemma, Virginia and I split off and worked mostly independently for the remainder of our work. Virginia completed the remaining work in the one-parameter case after building intuition for how the one-parameter version of the second regime should work. The generalization of our old work to the full two-parameter case was done almost entirely independently by me. The process of further restructuring our still somewhat hazy original work in the one-parameter case and working out how to generalize it to the two-parameter case led to several insights about mistakes and optimizations that could be used back in the one-parameter case as well. The most valuable of these new insights was using the approach in section 5.3 to thoroughly understand the structure of G_i as it is, which led to key realizations about how to successfully set up density bounds for ρ_i.

The initial idea for navigating the presence of the random fluctuation ξ_i in our tail bounds was first worked out by Virginia in the one-parameter case. As the parabola appearing in the two-parameter is significantly more complex due to its coefficients' additional heavy dependence on the values of $x, (x-y)$, and $(t-s)$, quite a bit of additional work was still required in order to generalize that approach to the two-parameter case. Upon completing this step, the last remaining piece in both the one and two-parameter cases was working out Lemma 5.4.1 and Lemma 5.4.2. I wrote these lemmas independently based on a one-on-one discussion with our advisor following a close but incorrect earlier attempt at lemmas of this form.

The results and joint work from the one-parameter case have been omitted from this thesis for length-based considerations, but can be found in Virginia's thesis.

Chapter 2

Level Set Convergence in the Hausdorff Metric

Our goal in this chapter is to establish level set convergence in the context of the uniform convergence on compact sets in Theorem 1.4.1. More specifically, we want to show that the following theorem is true. Note that in this theorem, we will be viewing the last passage metrics and $L(0,0;\cdot,\cdot)$ as being random functions in $C(K,\mathbb{R})$ for each $K \subseteq \mathbb{R}\times\mathbb{R}_{>0}$. We will also be using the metric space $(M,\rho) = (K, ||\cdot||_2)$, meaning that the topology with respect to which the convergence below takes place will change as the choice of K varies.

Theorem 2.0.1. *Let $K \subseteq \mathbb{R}\times\mathbb{R}_{>0}$ be compact and convex, and for each $h \in \mathbb{R}$ define the random set $Z_h^{(K)} \subseteq \mathbb{R}\times\mathbb{R}_{>0}$ by*

$$Z_h^{(K)} := \Big\{(x,s)\in K \ : \ \mathcal{L}(0,0;x,s) = h\Big\}.$$

For any point $(p;\ q) \in \mathbb{R}^4_{\uparrow}$, let $d_{LPP}(p,q)$ be the Exponential last passage value from p to q. Define the constants $\alpha,\beta,\tau,$ and χ according to equation (1.4.1), and for each $n \in \mathbb{Z}_{>0}$ define the random function $f_n^{(K)} \in C(K,\mathbb{R})$ by

$$f_n^{(K)}(y,t) := \frac{d_{LPP}^{(n)}\left(0,0\ ;\ yn^2\tau + tn^3, -tn^3\right) - (\beta-2)\,\tau n^2 y - n^3\alpha t}{n\chi}$$

where we have used the coupling of $\mathcal{L}$ with an infinite set of identically distributed copies

$$\Big\{d_{LPP}^{(n)}\Big\}_{n=1}^{\infty}$$

of the last passage directed metric d_{LPP} from Theorem 1.4.1. For each $h \in \mathbb{R}$ we define the set

$$Z_{h,n}^{(K)} := \Big\{(x,s)\in K \ : \ f_n^{(K)}(x,s) = h\Big\}.$$

If d_H is the Hausdorff metric induced by the Euclidean norm on K, then for all $h \in \mathbb{R}$,

$$\lim_{n\to\infty} d_H\Big(Z_{h,n}^{(K)},\ Z_h^{(K)}\Big) = 0.$$

Note that although we will only explicitly prove this result for Exponential last passage percolation specifically, the proof is completely agnostic of this choice provided that the constants α, β, τ, and χ are adjusted accordingly as the last passage model is changed. We only restrict our focus to this case in particular for concreteness, and to remove the additional overhead necessary to explain the other models of last passage percolation that exist. Due to the fact that the convergence in Theorem 1.4.1 is in the uniform on compact topology, if any other random growth models are found to fall into the scope of this theorem, then our argument should also immediately extend to those new random growth models as well.

Our strategy to prove that Theorem 2.0.1 is true will be to find a sufficient condition for level set convergence under the Euclidean Hausdorff metric on convex and compact sets that holds for a sequence of deterministic functions $(g_n)_{n=1}^{\infty}$ with uniform limit g. We will then show that with probability 1, this sufficient condition for deterministic functions holds pathwise for each realization of $\mathcal{L}$ and the last passage percolation model in the setting of Theorem 2.0.1. Our sufficient condition will only place impositions on the limit function g, which is why our theorem here is independent of the choice of the model of last passage percolation, provided we use the coupling stated in Theorem 1.4.1.

Lemma 2.0.2. *Let $K \subseteq \mathbb{R}^2$ be a convex and compact set, and let $(g_n)_{n=1}^{\infty}$ be a sequence in $C(K, \mathbb{R})$ with a uniform limit g. Suppose that g has no local maximum or minimum with value 0. Then the sequence of level sets $\left(g_n^{-1}(0)\right)_{n=1}^{\infty}$ converges to $g^{-1}(0)$ with respect to the Hausdorff metric d_H on K induced by the Euclidean norm.*

Proof. For any subset $A \subseteq K$ and any $\delta > 0$ we will denote by $A_\delta \subseteq K$ the set

$$A_\delta := \left\{(x, y) \in K : \sqrt{(x - a_1)^2 + (y - a_2)^2} < \delta \text{ for some } (a_1, a_2) \in A\right\}.$$

For each $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$, we will denote by $B(x, r)$ the set

$$\overline{B(x, r)} := \left\{(y_1, y_2) \in K : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq r\right\}$$

We will first show that for any arbitrarily small $\delta > 0$, $g^{-1}(0) \subseteq (f_n^{-1}(0))_\delta$ for n sufficiently large.

First, suppose that $x \in g^{-1}(0)$. Since x cannot be a local maximum or minimum of g, there must be a sequence of values $(\delta_m)_{m=1}^{\infty}$ decreasing monotonically to 0 such that for each $m \in \mathbb{Z}_{>0}$, there are two points $x_m^+, x_m^- \in \overline{B(x, \delta_m)}$ such that

$$f(x_m^-) < f(x) = 0 < f(x_m^+).$$

Since we are assuming that $g_n \to g$ uniformly, we know that there must exist a positive integer $N_1 = N_1(x_m^-, \delta_m)$ such that for all $n \geq N_1$, $g_n(x_m^-) < 0$. Similarly, there must also exist another positive integer $N_2 = N_2(x_m^+, \delta_m)$ such that for all $n \geq N_2$, $0 < g_n(x_m^+)$. Thus if we take $N = N_1 \vee N_2$, then for all $n \geq N$ we will have that

$$g_n(x_m^-) < 0 < g_n(x_m^+).$$

By the assumption that each g_n is a continuous function on the convex and compact set K (and hence on the convex and compact subset $\overline{B(x,\delta_m)}$ as well), the two-dimensional version of the Intermediate Value Theorem applies and there must exist at least one point $x_n \in \overline{B(x,\delta_m)} \cap g_n^{-1}(0)$ for all $n \geq N = N(x,\delta_m)$. Note that $\overline{B(x,\delta_m)}$ being convex and compact is what ensures that each zero x_n does indeed belong to $\overline{B(x,\delta_m)}$. Moreover, because $g_n \to g$ uniformly, this threshold N will be independent of the choice of x and only depends on the choice of $\delta_m > 0$. As such, we thus see that by let $\delta_m \to 0$, $g^{-1}(0)$ is always contained in $\left(g_n^{-1}(0)\right)_\delta$ for any $\delta > 0$ if we take $n \in \mathbb{Z}_{>0}$ sufficiently large.

We now prove that for any $\delta > 0$, the reverse inclusion $g_n^{-1}(0) \subseteq (g^{-1}(0))_\delta$ holds for $n \in \mathbb{Z}_{>0}$ sufficiently large. Suppose for contradiction that there exists some $\delta > 0$ such that there is no threshold $n \geq N$ after which the inclusion $g_n^{-1}(0) \subseteq (g^{-1}(0))_\delta$ always holds. This implies that there must exist a sequence of zeros $(y_{n_k})_{k=1}^\infty$ with $y_{n_k} \in g_{n_k}^{-1}(0)$ such that for any $x \in g^{-1}(0)$, we always have that $||x - y_{n_k}||_2 > \delta$.

Fix an arbitrary $x \in g^{-1}(0)$. By the compactness of K, we know that $(y_{n_k})_{k=1}^\infty$ must have a convergent subsequence, so without loss of generality we will assume that $(y_{n_k})_{k=1}^\infty$ itself converges to some $y \in K$. We claim that $y \in g^{-1}(0)$. To see that this is true, the triangle inequality gives us that

$$\begin{aligned} |g(y)| &= \ |g(y) - g_{n_k}(y) + g_{n_k}(y) - g_{n_k}(y_{n_k}) + g_{n_k}(y_{n_k})| \\ &\leq \ |g(y) - g_{n_k}(y)| + |g_{n_k}(y) - g_{n_k}(y_{n_k})| + |g_{n_k}(y_{n_k})| \,. \end{aligned}$$

By definition of y_{n_k}, $g(y_{n_k}) = 0$. Secondly, by the uniform convergence of $g_n \to g$, we see immediately that $\lim_{k\to\infty} |g(y) - g_{n_k}(y)| = 0$. Thirdly, by the continuity of each function g_{n_k} we also have that $\lim_{k\to\infty} |g_{n_k}(y) - g_{n_k}(y_{n_k})| = 0$. This means that by taking $k \to \infty$, we can upper bound $|g(y)|$ by an arbitrarily small positive number. Thus, the only remaining possibility is that $g(y) = 0$. However, since $y_{n_k} \to y \in g^{-1}(0)$, this is a contradiction, which proves that the reverse inclusion does indeed hold.

Based on these two inclusions, we have therefore proven that for any $\delta > 0$,

$$d_H\left(g_n^{-1}(0), g^{-1}(0)\right) = \inf\left\{\delta \in (0,\infty) : \ g_n^{-1}(0) \subseteq \left(g^{-1}(0)\right)_\delta \text{ and } g^{-1}(0) \subseteq \left(g_n^{-1}(0)\right)_\delta \right\} < \delta$$

for all $n \in \mathbb{Z}_{>0}$ sufficiently large. By definition of convergence under the Hausdorff metric d_H, this means that we have thus proven that

$$\lim_{n\to\infty} d_H\left(g_n^{-1}(0), g^{-1}(0)\right) = 0.$$

□

As an immediate consequence of this lemma holding for the zero set, we have the following corollary for any arbitrary $h-$level set.

Corollary 2.0.3. *Let $K \subseteq \mathbb{R}^2$ be a convex and compact set, and let $(g_n)_{n=1}^\infty$ be a sequence in $C(K,\mathbb{R})$ with a uniform limit g. Let $h \in \mathbb{R}$ be arbitrary and suppose that g has no local maximum* *or minimum with value h. Then the sequence of level sets $\left(g_n^{-1}(h)\right)_{n=1}^{\infty}$ converges to $g^{-1}(h)$ with respect to the Hausdorff metric d_H on K induced by the Euclidean distance.*

Proof. Replace the sequence of functions $(g_n)_{n=1}^{\infty}$ with the sequence $(g_n - h)_{n=1}^{\infty}$ and the limit function g with $g - h$ in the proof of Lemma 2.0.2. □

With Corollary 2.0.3 established, we will prove one additional preliminary but fairly elementary technical lemma about the law of a sum of independent random variables.

Lemma 2.0.4. *Let X and Y be independent $\mathbb{R}-$valued random variables and assume that* $\text{Law}(X)$ *has a density f_X with respect to the Lebesgue measure on $\mathbb{R}$. Then* $\text{Law}(X+Y)$ *has no atoms.*

Proof. Let F_Y be the cumulative distribution function of Y. Since X and Y are independent by hypothesis, this means that

$$\text{Law}(X+Y) = \text{Law}(X) * \text{Law}(Y).$$

Since $\text{Law}(X)$ has a density f_X, this in turn means that $\text{Law}(X+Y)$ also has a density f_{X+Y} with respect to the Lebesgue measure. Explicitly, letting F_{X+Y} be the cumulative distribution function of $X+Y$, we may write that for any $z \in \mathbb{R}$,

$$F_{X+Y}(z) = \int_{\mathbb{R}} F_Y(x) f_X(z-x)dx.$$

To see that F_{X+Y} is continuous, we observe that for any $z, \delta_0 \in \mathbb{R}$,

$$\begin{aligned}
|F_{X+Y}(z) - F_{X+Y}(z+\delta_0)| &= \left| \int_{\mathbb{R}} F_Y(x) f_X(z-x)dx - \int_{\mathbb{R}} F_Y(x) f_X(z+\delta_0-x)dx \right| \\
&= \int_{\mathbb{R}} |F_Y(x)||f_X(z-x) - f_X(z+\delta_0-x)|dx \\
&\leq \int_{\mathbb{R}} |f_X(z-x) - f_X(z+\delta_0-x)|dx \\
&= \int_{\mathbb{R}} |f_X(u) - f_X(u-\delta_0)|du \\
&= ||f_X(\cdot) - f_X(\cdot - \delta_0)||_{L^1(\mathbb{R})}. \qquad (2.0.1)
\end{aligned}$$

Note that the norm in the last line is the $1-$norm in the function space $L^1(\mathbb{R})$ and that every probability density function belongs to $L^1(\mathbb{R})$ by definition. It is a standard fact from functional analysis that for any function $f \in L^1(\mathbb{R})$, it is always true that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\delta_0| < \delta \implies ||f_X(\cdot) - f_X(\cdot - \delta_0)||_{L_1(\mathbb{R})} < \varepsilon.$$

Using this standard fact in conjunction with equation (2.0.1) immediately implies that F_{X+Y} is a continuous function on all of $\mathbb{R}$, and so it cannot have any jump discontinuities. Thus, since the atoms of $\text{Law}(X+Y)$ are precisely the jump discontinuities of F_{X+Y}, this means that $\text{Law}(X+Y)$ has no atoms. □

We now turn our attention towards our specific problem. To prove Theorem 2.0.1, we will need two more supporting results. We begin by proving the existence of an elementary but extremely useful way to decompose a Brownian bridge on an arbitrary domain into a Gaussian and independent stochastic process. Though elementary, this lemma will be used extensively throughout this thesis.

Lemma 2.0.5. *Let $a \in \mathbb{R}$, $T > 0$, and let $(B(r))_{r\in[a,a+T]}$ be a Brownian bridge with diffusion parameter $k > 0$ with arbitrary start and end values. Then for any $\delta \in \left(0, \frac{1}{2}\right)$, we may write*

$$\Big(B(r)\Big)_{r\in[a+\delta T, a+(1-\delta)T]} = N + \Big(B(r) - N\Big)_{r\in[a+\delta T, a+(1-\delta)T]}$$

where $N \sim \mathcal{N}\left(0, \frac{k\delta T}{2}\right)$ and is independent of the process $\Big(B(r) - N\Big)_{r\in[a+\delta T, a+(1-\delta)T]}$.

Proof. By subtracting a deterministic linear function, we may assume that $B(a) = B(a+T) = 0$ without loss of generality. Moreover, by using the Brownian scaling of Brownian bridges and multiplying every value of B in the computations below by $\sqrt{k}$, we may also assume without loss of generality that $k = 1$. With these simplifications in mind, define

$$N := \frac{B(a+\delta T) + B(a+(1-\delta)T)}{2}$$

and let $r \in [a+\delta T, a+(1-\delta)T]$. We then compute that

$$\begin{aligned}
\mathrm{Cov}\left(N, B(r)\right) &= \mathrm{Cov}\left(\frac{B(a+\delta T)}{2}, B(r)\right) + \mathrm{Cov}\left(\frac{B(a+(1-\delta)T)}{2}, B(r)\right) \\
&= \frac{1}{2}\frac{((a+T)-r)((a+\delta T)-a)}{T} + \frac{1}{2}\frac{((a+T)-(a+(1-\delta)T))(r-a)}{T} \\
&= \frac{1}{2}\frac{((a+T)-r)\delta T}{T} + \frac{1}{2}\frac{\delta T(r-a)}{T} \\
&= \frac{1}{2}\delta T.
\end{aligned}$$

We also see immediately that

$$\begin{aligned}
\mathrm{Var}(N) &= \frac{1}{4}\mathrm{Var}\left(B(a+\delta T)\right) + \frac{1}{4}\mathrm{Var}\left(B(a+(1-\delta)T\right) + \frac{1}{2}\mathrm{Cov}\left(B(a+\delta T), B(a+(1-\delta)T)\right) \\
&= \frac{1}{4}\frac{(1-\delta)T(\delta T)}{T} + \frac{1}{4}\frac{\delta T(1-\delta)T}{T} + \frac{1}{2}\frac{((a+T)-(a+(1-\delta)T))((a+\delta T)-a)}{T} \\
&= \frac{1}{4}(\delta T - \delta^2 T) + \frac{1}{4}(\delta T - \delta^2 T) + \frac{1}{2}\delta^2 T \\
&= \frac{1}{2}\delta T.
\end{aligned}$$

Therefore, for each $r \in [a+\delta T, a+(1-\delta)T]$ we have that

$$\mathrm{Cov}\left(N, B(r) - N)\right) = \mathrm{Cov}\left(N, B(r)\right) - \mathrm{Var}(N) = 0.$$

Thus the process $(B(r) - N)_{r\in[a+\delta T, a+(1-\delta)T]}$ is uncorrelated with N, and hence the two are independent as claimed. ☐

Next we prove the following Proposition about the extrema of $\mathcal{L}(0,0;\cdot,\cdot)$ on compact sets.

Proposition 2.0.6. *Let $K \subseteq \mathbb{R} \times \mathbb{R}_{>0}$ be a convex and compact set. Then for any $h \in \mathbb{R}$,*

$$\mathbb{P}\left(\max_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\right) = \mathbb{P}\left(\min_{(y,t)\in K} \mathcal{L}(0,0;y,t) = h\right) = 0. \tag{2.0.2}$$

Proof. Fix an arbitrary $h \in \mathbb{R}$. Since every compact and convex set $K \subseteq \mathbb{R} \times \mathbb{R}_{>0}$ is contained in a finite square, it suffices to prove that

$$\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) = h\right) = \mathbb{P}\left(\min_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) = h\right) = 0 \tag{2.0.3}$$

for any $(a,b) \in \mathbb{R} \times \mathbb{R}_{>0}$ with $T \geq 1$. In other words, we need only show that the hypotheses of Corollary 2.0.3 are satisfied pathwise almost surely on every square $K = [a, a+T] \times [2b, 2b+T]$. This will be a consequence of the Brownian-Gibbs property of the parabolic Airy line ensemble, the metric composition law of $\mathcal{L}$, and the independent increments property of $\mathcal{L}$.

We begin by fixing an arbitrary $(a,b) \in \mathbb{R} \times \mathbb{R}_{>0}$ and $T \geq 1$. Without loss of generality we will only prove that

$$\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) = h\right) = 0, \tag{2.0.4}$$

with the knowledge that by replacing each instance of

$$\max_{(y,t)\in[a,a+T]\times[2b,2b+T]}$$

in the subsequent argument with an instance of

$$\min_{(y,t)\in[a,a+T]\times[2b,2b+T]}$$

we are left with a complete and virtually identical proof that

$$\mathbb{P}\left(\min_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) = h\right) = 0$$

as well, with no other changes required. With this in mind, we recall that by the metric composition law of the directed landscape, we may write that

$$\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) \stackrel{d}{=} \max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\mathbb{R}} \mathcal{L}(0,0;x,b) + \mathcal{L}(x,b;y,t)\right),$$

with $\mathcal{L}(0,0;x,b)$ independent of $\mathcal{L}(x,b;y,t)$ for all $x \in \mathbb{R}$. Moreover, note that the only variable remaining in the arguments of $\mathcal{L}(x,b;y,t)$ is the spatial variable x, so its distribution is completely unaffected by the choice of $(y,t) \in [a,a+T] \times [2b, 2b+T]$. This independence of the value of (y,t) will be very important momentarily.

By these observations and Proposition 1.4.3, we may then say that

$$\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) \stackrel{d}{=} \max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\mathbb{R}} b^{\frac{1}{3}}\mathfrak{A}_1\left(b^{-\frac{2}{3}}x\right) + \mathcal{L}(x,b;y,t)\right).$$

Now by taking a union bound, we may use this equality in distribution to write that

$$\begin{aligned}
&\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t)=h\right)\\
\leq& \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in[-n,n]} b^{\frac{1}{3}}\mathfrak{A}_1\left(b^{-\frac{2}{3}}x\right)+\mathcal{L}(x,b;y,t)\right)=h\right)\\
=& \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}}\mathfrak{A}_1(x)+\mathcal{L}\left(b^{\frac{2}{3}}x,b;y,t\right)\right)=h\right)
\end{aligned}$$

so to prove that equation (2.0.4) holds, it suffices to prove that

$$\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}}\mathfrak{A}_1(x)+\mathcal{L}\left(b^{\frac{2}{3}}x,b;y,t\right)\right)=h\right)=0$$

for each $n\in\mathbb{Z}_{>0}$. We will do this by using the Brownian-Gibbs property of $\mathfrak{A}$.

In particular, we will apply Theorem 1.2.2 on the interval $[a_0,a_0+T_0]=\left[-1-nb^{-\frac{2}{3}},1+nb^{-\frac{2}{3}}\right]$ to obtain that for some absolute constant $c>0$,

$$\text{Law}\left(\left(\mathfrak{A}_1(x)\right)_{x\in\left[-1-nb^{-\frac{2}{3}},1+nb^{-\frac{2}{3}}\right]}\right)\leq e^{c\left(1+nb^{-\frac{2}{3}}\right)^3}\,\text{Law}\left(\left(B(x)+L(x)\right)_{x\in\left[-1-nb^{-\frac{2}{3}},1+nb^{-\frac{2}{3}}\right]}\right)$$

where B is a diffusion parameter two Brownian bridge on $\left[-1-nb^{-\frac{2}{3}},1+nb^{-\frac{2}{3}}\right]$ from 0 to 0, and L is a random affine function on the same interval independent of B such that

$$L\left(-1-nb^{-\frac{2}{3}}\right)\overset{d}{=}\mathfrak{A}_1\left(-1-nb^{-\frac{2}{3}}\right)\qquad\text{and}\qquad L\left(1+nb^{-\frac{2}{3}}\right)\overset{d}{=}\mathfrak{A}_1\left(1+nb^{-\frac{2}{3}}\right).$$

For convenience, we will denote

$$C_{n,b}:=e^{c\left(1+nb^{-\frac{2}{3}}\right)^3}.$$

Now by elementary measure theory, this Brownian-Gibbs absolute continuity statement implies that

$$\begin{aligned}
&\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}}\mathfrak{A}_1(x)+\mathcal{L}\left(b^{\frac{2}{3}}x,b;y,t\right)\right)=h\right)\\
\leq\; & C_{n,b}\,\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}}B(x)+b^{\frac{1}{3}}L(x)+\mathcal{L}\left(b^{\frac{2}{3}}x,b;y,t\right)\right)=h\right).
\end{aligned}$$

Then, by invoking Lemma 2.0.5 with the parameters $k=2, a=-1-nb^{-\frac{2}{3}}$, $T=2+2nb^{-\frac{2}{3}}>2$, and $\delta=\frac{1}{T}<\frac{1}{2}$, we can write that

$$\text{Law}\left(\left(B(x)\right)_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]}\right)=\text{Law}\left(N+\left(B(x)-N\right)_{x\in\left[-nb^{-\frac{2}{3}},nb^{-\frac{2}{3}}\right]}\right)$$

where $N \sim \mathcal{N}(0,1)$ is independent of the process $\Big(B(x) - N\Big)_{x \in \left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]}$.

At this stage, because $\delta = \left(2 + 2nb^{-\frac{2}{3}}\right)^{-1}$ is completely independent of x, y, and t, the Gaussian random variable N is completely independent of everything else in these two suprema. As such, we may now write that

$$\begin{aligned}
&\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}} B(x) + b^{\frac{1}{3}} L(x) + \mathcal{L}\left(b^{\frac{2}{3}}x, b; y, t\right)\right) \\
\overset{d}{=}\ & b^{\frac{1}{3}} N + \max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}} (B(x) - N) + b^{\frac{1}{3}} L(x) + \mathcal{L}\left(b^{\frac{2}{3}}x, b; y, t\right)\right) \\
&=: b^{\frac{1}{3}} N + S(a,b,n)
\end{aligned}$$

with the random variables $b^{\frac{1}{3}} N$ and $S(a,b,n)$ independent. Moreover, because $b^{\frac{1}{3}} N$ is a Gaussian, it has a density with respect to the Lebesgue measure, and so by Lemma 2.0.4, we see that

$$\text{Law}\left(b^{\frac{1}{3}} N + S(a,b,n)\right) = \text{Law}\left(b^{\frac{1}{3}} N\right) * \text{Law}\left(S(a,b,n)\right)$$

has no atoms. As such, we have now established that

$$\begin{aligned}
&\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}} \mathfrak{A}_1(x) + \mathcal{L}\left(b^{\frac{2}{3}}x, b; y, t\right)\right) = h\right) \\
\leq\ & C_{n,b}\, \mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}} B(x) + b^{\frac{1}{3}} L(x) + \mathcal{L}\left(b^{\frac{2}{3}}x, b; y, t\right)\right) = h\right) \\
&= C_{n,b}\, \mathbb{P}\left(b^{\frac{1}{3}} N + S(a,b,n) = h\right) \\
&= 0
\end{aligned}$$

for all $n \in \mathbb{Z}_{>0}$. This therefore proves that

$$\begin{aligned}
&\mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \mathcal{L}(0,0;y,t) = h\right) \\
\leq\ & \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in[-n,n]} b^{\frac{1}{3}} \mathfrak{A}_1\left(b^{-\frac{2}{3}}x\right) + \mathcal{L}(x,b;y,t)\right) = h\right) \\
=\ & \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{(y,t)\in[a,a+T]\times[2b,2b+T]} \left(\max_{x\in\left[-nb^{-\frac{2}{3}}, nb^{-\frac{2}{3}}\right]} b^{\frac{1}{3}} \mathfrak{A}_1(x) + \mathcal{L}\left(b^{\frac{2}{3}}x, b; y, t\right)\right) = h\right) \\
&= 0
\end{aligned}$$

for any $(a,b) \in \mathbb{R} \times \mathbb{R}_{>0}$, any $T \geq 1$, and any $h \in \mathbb{R}$, which completes our proof. □

With Proposition 2.0.6 now proven, there is nothing left to do in order to prove that Theorem 2.0.1 holds. Proposition 2.0.6 confirms that for any $h \in \mathbb{R}$ and any convex and compact subset

$K \subseteq \mathbb{R} \times \mathbb{R}_{>0}$, the limit function $\mathcal{L}(0, 0; \cdot, \cdot)$ has no local maximum or local minimum with value h on K with probability 1. This means that $\mathbb{P}-$almost surely, we do indeed have that

$$\lim_{n\to\infty} d_H\Big(Z_{h,n}^{(K)},\ Z_h^{(K)}\Big) = 0.$$

This concludes our proof of Theorem 2.0.1 and we now proceed to our next project in the following chapter. The remaining work in this thesis is completely independent of the work done in this chapter.

Chapter 3

An Upper Bound on the Hausdorff Dimension of the $h-$Level Set

Our goal in this chapter will be to establish the following theorem:

Theorem 3.0.1. *For each $h \in \mathbb{R}$, let Z_h be the random set*

$$Z_h := \Big\{(x,s) \in \mathbb{R} \times \mathbb{R}_{>0} : \ \mathcal{L}(0,0;x,s) = h\Big\}.$$

Then for any $h \in \mathbb{R}$,

$$\mathbb{P}\left(\dim_H(Z_h) \leq \frac{5}{3}\right) = 1.$$

We will do so by adapting the standard proof of a similar result for the level sets of Brownian motion found in [5]. In particular, we will derive this almost-sure upper bound on $\dim_H(Z_h)$ by proving that the upper Minkowski dimension of $Z_h \cap \left([-n,n] \times \left[\frac{1}{n}, n\right]\right)$ for any $h \in \mathbb{R}$ and any $n \in \mathbb{Z}_{>0}$ is at most $\frac{5}{3} + \eta + \gamma$ for any $\eta > 0$ and $0 < \gamma < \frac{1}{3}$, and exploiting the countable stability of the Hausdorff dimension. To do this, we begin by recalling that we have the following modulus of continuity for (the stationary version $\mathcal{K}$ of) the directed landscape $\mathcal{L}$ from [9]:

Theorem 3.0.2 (Proposition 10.5 in [9])**.** *Let $\mathcal{K}(x,t,y,t+s) := \mathcal{L}(x,t,y,t+s) + \frac{(x-y)^2}{s}$ for each $(x,t,y,t+s) \in \mathbb{R}^4_\uparrow$ with $s > 0$. For each $n \geq 2$ and each $0 < \delta \leq 1$ define the set*

$$K_n^\delta := [-n,n]^4 \cap \Big\{(x,t;y,t+s) \in \mathbb{R}^4_\uparrow : \ s \geq \delta\Big\}.$$

Let $u_1, u_2 = (x_1,t_1,y_1,t_1+s_1), (x_2,t_2,y_2,t_2+s_2) \in K_n^\delta$ and define the positive constants ξ and τ by

$$\xi = \xi(u_1,u_2) = ||(x_1,y_1) - (x_2,y_2)|| \qquad \text{and} \qquad \tau = \tau(u_1,u_2) = ||(t_1,t_1+s_1) - (t_2,t_2+s_2)||.$$

If $\tau \leq \frac{\delta^3}{n^3}$ then there exists a random constant $C\left(K_n^\delta\right)$ depending only on the choice of the compact set K_n^δ such that

$$|\mathcal{K}(u_1) - \mathcal{K}(u_2)| \leq C\left(K_n^\delta\right)\left(\tau^{\frac{1}{3}} \log^{\frac{2}{3}}\left(\tau^{-1}\right) + \xi^{\frac{1}{2}} \log^{\frac{1}{2}}\left(4n\xi^{-1}\right)\right).$$

Moreover, there exist absolute constants $c, d > 0$ such that for all $M > 0$,

$$\mathbb{P}\left(C\left(K_n^{\delta}\right) > M\right) \leq cn^{10}\delta^{-6}e^{-dM^{\frac{3}{2}}}.$$

This modulus of continuity will be at the foundation of our proof of Theorem 3.0.1. We also take a moment to note that, as previously mentioned in section 1.7, the fact that this modulus of continuity involves a random constant as opposed to the deterministic constant found in Levy's modulus of continuity for Brownian motion (as well as the fact that $\mathcal{L}(0,0;\cdot,\cdot)$ is not Markov) necessitates several adjustments to the classical proof for Brownian motion. While having a random constant in a modulus of continuity could theoretically be quite challenging to navigate in a problem like this, this case in particular will fortunately not pose any significant issues due to the strong upper tail bounds on these random constants.

Proof. The first step in our proof will be to derive a suitable upper bound on

$$\mathbb{P}\Big(Z_h \cap ([a, a+\varepsilon] \times [b, b+\varepsilon]) \neq \emptyset\Big)$$

for an arbitrary $a, b \in \mathbb{R}$ and $\varepsilon > 0$ which decays to 0 as $\varepsilon \to 0^+$. This bound will then be used in conjunction with the Monotone Convergence Theorem to find an upper estimate on the expected number of sets of diameter at most 2^{-m} needed to cover Z_h for an arbitrary (sufficiently large) $m \in \mathbb{Z}_{>0}$. This upper estimate will be precisely what we use to obtain our upper bound on

$$\overline{\dim}_{\mathscr{M}}\left(Z_h \cap \left([-n, n] \times \left[\frac{1}{n}, n\right]\right)\right),$$

and hence an upper bound on

$$\dim_H\left(Z_h \cap \left([-n, n] \times \left[\frac{1}{n}, n\right]\right)\right).$$

By the countable stability of the Hausdorff dimension, showing that

$$\overline{\dim}_{\mathscr{M}}\left(Z_h \cap \left([-n, n] \times \left[\frac{1}{n}, n\right]\right)\right) \leq \frac{5}{3}$$

almost surely for each $n \in \mathbb{Z}_{\geq 2}$ is sufficient. This extra step allows us to limit our focus to compact sets, which then allows us to more easily leverage the modulus of continuity of $\mathcal{K}$ in our argument.

Given this observation, we begin this process by considering the behaviour of the function

$$\begin{aligned}\mathcal{L}(0,0,\cdot,\cdot) : [-n, n] \times \left[\frac{1}{n}, n\right] &\to \mathbb{R}\\ (x, s) &\mapsto \mathcal{L}(0,0;x,s).\end{aligned}$$

where $n \in \mathbb{Z}_{>0}$ is arbitrary. By applying Theorem 3.0.2 with $\delta = \frac{1}{n}$, and $\xi = \tau = \varepsilon \in \left(0, n^{-6}\right)$, there exists an $n-$dependent random constant $C\left(K_n^{\frac{1}{n}}\right)$ such that for any $(x,s),(y,t)\in[-n,n]\times\left[\frac{1}{n},n\right]$,

$$\left|\mathcal{L}(0,0;x,s)+\frac{x^2}{s}-\mathcal{L}(0,0;y,t)-\frac{y^2}{t}\right|\leq C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)\right), \tag{3.0.1}$$

where for any $M>0$ we have that

$$\mathbb{P}\left(C\left(K_n^{\frac{1}{n}}\right)>M\right)\leq cn^{16}e^{-dM^{\frac{3}{2}}}.$$

With this setup in mind, fix $a,b\in\mathbb{R}$ and $0<\varepsilon<n^{-6}$ such that

$$[a,a+\varepsilon]\times[b,b+\varepsilon]\subseteq[-n,n]\times\left[\frac{1}{n},n\right]$$

and suppose that for some $(x,s)\in[a,a+\varepsilon]\times[b,b+\varepsilon]$ that $\mathcal{L}(0,0;x,s)=h$. If this is true, then based on our choice of $[a,a+\varepsilon]\times[b,b+\varepsilon]$ we have by equation (3.0.1) and the reverse triangle inequality that for all $(y,t)\in[a,a+\varepsilon]\times[b,b+\varepsilon]$,

$$\begin{aligned}|\mathcal{L}(0,0;y,t)-h|\leq\ & C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)\right)+\left|\frac{x^2}{s}-\frac{y^2}{t}\right|\\ \leq\ & C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)\right)+\frac{(|a|+\varepsilon)^2}{b}-\frac{a^2}{b+\varepsilon}\\ =\ & C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)\right)+\frac{(|a|+\varepsilon)^2(b+\varepsilon)-a^2b}{b(b+\varepsilon)}\\ =:\ & C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)\right)+K_{a,b,\varepsilon}.\end{aligned} \tag{3.0.2}$$

Because this bound holds uniformly for all choices of $(y,t)\in[a,a+\varepsilon]\times[b,b+\varepsilon]$, this means that we then obtain a bound

$$\begin{aligned}&\mathbb{P}\Big(Z_h\cap([a,a+\varepsilon]\times[b,b+\varepsilon])\neq\emptyset\Big)\\ &\leq\mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h|\leq C\left(K_n^{\delta}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(1+\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(1+\varepsilon^{-1}\right)\right)+K_{a,b,\varepsilon}\right)\\ &\leq\mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h|\leq C\left(K_n^{\delta}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(1+\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(1+\varepsilon^{-1}\right)+K_{a,b,\varepsilon}\right)\right).\end{aligned} \tag{3.0.3}$$

We will now develop an upper bound on the probability above. We first recall that for any $0<\gamma<\frac{1}{3}$ we have that

$$\lim_{\varepsilon\to0^+}\frac{\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}(\varepsilon^{-1})}{\varepsilon^{\frac{1}{3}-\gamma}}=\lim_{\varepsilon\to0^+}\frac{\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}(4n\varepsilon^{-1})}{\varepsilon^{\frac{1}{2}-\gamma}}=0$$

for any fixed value of $n\in\mathbb{Z}_{>0}$, which in turn gives us the pair of upper bounds

$$\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}(\varepsilon^{-1})\leq\varepsilon^{\frac{1}{3}-\gamma}$$

and

$$\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}(4n\varepsilon^{-1})\leq\varepsilon^{\frac{1}{2}-\gamma}$$

for these same sufficiently small positive values of γ. Combining these bounds with equation (3.0.3), we get a further upper bound

$$\begin{aligned}
&\mathbb{P}\Big(Z_h \cap ([a,a+\varepsilon]\times[b,b+\varepsilon]) \neq \emptyset\Big) \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq C\left(K_n^{\delta}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)+K_{a,b,\varepsilon}\right)\right) \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq C\left(K_n^{\delta}\right)\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right)\right). \qquad (3.0.4)
\end{aligned}$$

We will now invoke the relationship between the directed landscape at a fixed point in $\mathbb{R}^4_{\uparrow}$ and the top line of the parabolic Airy process $\mathfrak{A}_1$ to bound this probability in terms of a, b, and ε. In particular, we will be using the fact

$$\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon) \overset{d}{=} (b+\varepsilon)^{\frac{1}{3}}\mathfrak{A}_1\left(\frac{a+\varepsilon}{(b+\varepsilon)^{\frac{2}{3}}}\right)$$

in conjunction with the fact that for any fixed $x \in \mathbb{R}$, $\mathfrak{A}_1(x)+x^2 \sim TW_2$ has a bounded density with respect to the Lebesgue measure on $\mathbb{R}$, with the bound independent of the choice of x. We also take a moment to recall that a translation by a constant and a dilation of the argument of $\mathfrak{A}_1(x)+x^2$ does not change the uniform bound on its density. We also observe that if X, Y are random variables such that $|X| \leq Y$ almost surely and $x, y \in \mathbb{R}_{>0}$ then we always have that

$$\begin{aligned}
\mathbb{P}\Big(|X| \leq Yx\Big) &= \mathbb{P}\Big(\{|X| \leq Yx\} \cap \{Y \leq y\}\Big) + \mathbb{P}\Big(\{|X| \leq Yx\} \cap \{Y > y\}\Big) \\
&\leq \mathbb{P}\Big(|X| \leq yx\Big) + \mathbb{P}\Big(Y > y\Big)
\end{aligned}$$

When these observations are applied to equation (3.0.4), our upper bound can be extended to

$$\begin{aligned}
&\mathbb{P}\Big(Z_h \cap ([a,a+\varepsilon]\times[b,b+\varepsilon]) \neq \emptyset\Big) \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}}\log^{\frac{2}{3}}\left(\varepsilon^{-1}\right)+\varepsilon^{\frac{1}{2}}\log^{\frac{1}{2}}\left(4n\varepsilon^{-1}\right)+K_{a,b,\varepsilon}\right)\right) \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq C\left(K_n^{\frac{1}{n}}\right)\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right)\right) \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq M\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right)\right) + \mathbb{P}\left(C\left(K_n^{\frac{1}{n}}\right) > M\right). \\
&\leq \mathbb{P}\left(|\mathcal{L}(0,0,a+\varepsilon,b+\varepsilon)-h| \leq M\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right)\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\
&\leq \mathbb{P}\left(\left|(b+\varepsilon)^{\frac{1}{3}}\mathfrak{A}_1\left(\frac{a+\varepsilon}{(b+\varepsilon)^{\frac{2}{3}}}\right)-h\right| \leq M\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right)\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\
&\leq 2M\kappa(b+\varepsilon)^{-1}\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\
&\leq 2Mn\kappa\left(\varepsilon^{\frac{1}{3}-\gamma}+\varepsilon^{\frac{1}{2}-\gamma}+K_{a,b,\varepsilon}\right) + cn^{16}e^{-dM^{\frac{3}{2}}}
\end{aligned}$$

where $M > 0$ and $\kappa > 0$ is the uniform upper bound on the density of $\mathfrak{A}_1\left(\frac{a+\varepsilon}{(b+\varepsilon)^{\frac{2}{3}}}\right)$ i.e. the density bound of the Tracy-Widom$_2$ distribution. We will specify the precise value of $M > 0$ that we will be using later on when the motivation for that particular choice is more readily apparent. Moreover, by using the elementary facts that

$$0 < \varepsilon < n^{-6} \quad \text{and} \quad [a,a+\varepsilon]\times[b,b+\varepsilon] \subseteq [-n,n]\times\left[\frac{1}{n},n\right] \quad \text{and} \quad \lim_{\varepsilon\to 0}(a+\varepsilon)^2(b+\varepsilon)-a^2b = 0,$$

we see that for any $\varepsilon > 0$ sufficiently small, we can bound the constant $K_{a,b,\varepsilon}$ for any fixed pair of values $a, b \in [-n, n] \times \left[\frac{1}{n}, n\right]$ by

$$\begin{aligned} K_{a,b,\varepsilon} &= \frac{(|a|+\varepsilon)^2(b+\varepsilon) - a^2 b}{b(b+\varepsilon)} \\ &= \frac{\varepsilon\Big(b(2|a|+\varepsilon) + (|a|+\varepsilon)^2\Big)}{b(b+\varepsilon)} \\ &\leq \frac{\varepsilon\Big(b(3|a|) + (2|a|)^2\Big)}{b^2} \\ &\leq 7n^2 \frac{\varepsilon}{b^2}. \end{aligned}$$

Given this and the fact that $\frac{1}{n} \leq (b+\varepsilon)^{-1} < b^{-1} \leq n$, we can simplify our new upper bound even further and write that

$$\begin{aligned} \mathbb{P}\Big(Z_h \cap ([a, a+\varepsilon] \times [b, b+\varepsilon]) \neq \emptyset\Big) &\leq 2Mn\kappa\left(\varepsilon^{\frac{1}{3}-\gamma} + \varepsilon^{\frac{1}{2}-\gamma} + K_{a,b,\varepsilon}\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\ &\leq 2Mn\kappa\left(\varepsilon^{\frac{1}{3}-\gamma} + \varepsilon^{\frac{1}{2}-\gamma} + 7n^2\frac{\varepsilon}{b^2}\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\ &\leq 2Mn\kappa\left(\varepsilon^{\frac{1}{3}-\gamma} + \varepsilon^{\frac{1}{2}-\gamma} + 7n^4\varepsilon\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\ &\leq 14Mn^5\kappa\left(\varepsilon^{\frac{1}{3}-\gamma} + \varepsilon^{\frac{1}{2}-\gamma} + \varepsilon\right) + cn^{16}e^{-dM^{\frac{3}{2}}}. \end{aligned}$$

With this bound on the probability of hitting h somewhere a given square of area ε^2 in $[-n,n]\times\left[\frac{1}{n}, n\right]$, we are now ready to prove that the upper Minkowski dimension of $Z_h \cap \left([-n, n] \times \left[\frac{1}{n}, n\right]\right)$ is at most $\frac{5}{3}$. To do this, we will set $\varepsilon = 2^{-m}$ with $m \geq 6\log_2(n)$, so that $2^{-m} < n^{-6}$, which will yield that

$$\begin{aligned} \mathbb{P}\Big(Z_h \cap \left(\left[a, a+2^{-m}\right] \times \left[b, b+2^{-m}\right]\right) \neq \emptyset\Big) &\leq 14Mn^5\kappa\left(2^{-\frac{m}{3}+m\gamma} + 2^{-\frac{m}{2}+m\gamma} + 2^{-m}\right) + cn^{16}e^{-dM^{\frac{3}{2}}} \\ &\leq \left(42Mn^5\kappa\right)2^{-\frac{m}{3}+m\gamma} + cn^{16}e^{-dM^{\frac{3}{2}}}. \end{aligned} \tag{3.0.5}$$

Noting that for any $m \in \mathbb{Z}_{>0}$ we have the covering

$$[-n,n]\times\left[\frac{1}{n}, n\right] \subseteq \bigsqcup_{j=1-2^m n}^{2^m n} \bigsqcup_{k=1}^{\lceil (n-\frac{1}{n})2^m \rceil} \left(\left[2^{-m}(j-1), 2^{-m}j\right] \times \left[2^{-m}\left(\frac{1}{n}+k-1\right),\ 2^{-m}\left(\frac{1}{n}+k\right)\right]\right),$$

if for each $m \in \mathbb{Z}_{>0}$ we define the random variable N_m by

$$N_m := \sum_{j=1-2^m n}^{2^m n} \sum_{k=1}^{\lceil (n-\frac{1}{n})2^m \rceil} \mathbb{1}_{\left\{Z_h \cap \left(\left[2^{-m}(j-1), 2^{-m}j\right] \times \left[2^{-m}\left(\frac{1}{n}+k-1\right),\ 2^{-m}\left(\frac{1}{n}+k\right)\right]\right) \neq \emptyset\right\}},$$

then $\mathbb{E}\left[N_m\right]$ will give an upper bound on the expected number of sets of diameter no more than 2^{-m} needed to cover $Z_h \cap \left([-n, n] \times \left[\frac{1}{n}, n\right]\right)$. By applying the bound we obtained in equation (3.0.5) and using the linearity of expectation, we obtain the bound

$$\mathbb{E}\left[N_m\right] = \mathbb{E}\left[\sum_{j=1-2^m n}^{2^m n} \sum_{k=1}^{\lceil (n-\frac{1}{n})2^m \rceil} \mathbb{1}_{\left\{Z_h \cap \left([2^{-m}(j-1), 2^{-m}j] \times \left[2^{-m}\left(\frac{1}{n}+k-1\right),\, 2^{-m}\left(\frac{1}{n}+k\right)\right]\right) \neq \emptyset\right\}}\right]$$

$$= \sum_{j=1-2^m n}^{2^m n} \sum_{k=1}^{\lceil (n-\frac{1}{n})2^m \rceil} \mathbb{P}\left(Z_h \cap \left(\left[2^{-m}(j-1), 2^{-m}j\right] \times \left[2^{-m}\left(\frac{1}{n}+k-1\right),\, 2^{-m}\left(\frac{1}{n}+k\right)\right]\right) \neq \emptyset\right)$$

$$\leq \sum_{j=1-2^m n}^{2^m n} \sum_{k=1}^{\lceil (n-\frac{1}{n})2^m \rceil} \left(\left(42Mn^5\kappa\right) 2^{-\frac{m}{3}+m\gamma} + \, cn^{16} e^{-dM^{\frac{3}{2}}}\right)$$

$$\leq \, 2n\left(\left\lceil n - \frac{1}{n}\right\rceil\right) 2^{2m}\left(\left(42Mn^5\kappa\right) 2^{-\frac{m}{3}+m\gamma} + \, cn^{16} e^{-dM^{\frac{3}{2}}}\right)$$

$$\leq \, 2n^2 2^{2m}\left(\left(42Mn^5\kappa\right) 2^{-\frac{m}{3}+m\gamma} + \, cn^{16} e^{-dM^{\frac{3}{2}}}\right)$$

$$\leq \, 84Mn^7\kappa\left(2^{m\left(\frac{5}{3}+\gamma\right)}\right) + 2cn^{18} e^{-dM^{\frac{3}{2}}}\left(2^{2m}\right).$$

At this stage, we will now set $M = \left(\frac{m}{3d}\right)^{\frac{2}{3}}$ so that our final version of this upper bound on $\mathbb{E}\left[N_m\right]$ becomes the far more useful bound

$$\mathbb{E}\left[N_m\right] \leq \, 84n^7(3d)^{-\frac{2}{3}}\left(m^{\frac{2}{3}} 2^{m\left(\frac{5}{3}+\gamma\right)}\right) + 2cn^{18}\left(2^{2m} e^{-\frac{m}{3}}\right). \tag{3.0.6}$$

Now given this bound on $\mathbb{E}\left[N_m\right]$ for a given $m \geq 6\log_2(n)$, consider the expectation of the infinite series

$$\mathbb{E}\left[\sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} \frac{N_m}{2^{m(\frac{5}{3}+\gamma+\eta)}}\right]$$

for an arbitrary $\eta > 0$. By using the Monotone Convergence Theorem and equation (3.0.6), we can then see that the expected value of this series is no more than

$$\mathbb{E}\left[\sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} \frac{N_m}{2^{m(\frac{5}{3}+\gamma+\eta)}}\right] \leq \sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} \frac{\mathbb{E}\left[N_m\right]}{2^{m(\frac{5}{3}+\gamma+\eta)}}$$

$$\leq \, 84n^7(3d)^{-\frac{2}{3}} \sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} \frac{m^{\frac{2}{3}} 2^{m\left(\frac{5}{3}+\gamma\right)}}{2^{m(\frac{5}{3}+\gamma+\eta)}} + 2cn^{18} \sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} \frac{2^{2m} e^{-\frac{m}{3}}}{2^{m(\frac{5}{3}+\gamma+\eta)}}$$

$$= \, 84n^7(3d)^{-\frac{2}{3}} \sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} m^{\frac{2}{3}} 2^{-m\eta} + 2cn^{18} \sum_{m=\lceil 6\log_2(n)\rceil}^{\infty} 2^{-m(\gamma+\eta)}\left(\frac{2}{e}\right)^{\frac{m}{3}}$$

which is finite for all $\eta > 0$ and $\gamma \in \left(0, \frac{1}{3}\right)$. Thus, since its expected value is finite, this implies that the random series

$$\sum_{m=1}^{\infty} \frac{N_m}{2^{m(\frac{5}{3}+\gamma+\eta)}}$$

is also finite almost surely for any $\eta > 0$ and $\gamma \in (0, 3)$. Furthermore, because this series converges almost surely, we must also have that for any fixed choice of these constants γ and η,

$$\limsup_{m\to\infty} \frac{N_m}{2^{m(\frac{5}{3}+\gamma+\eta)}} = 0$$

almost surely. Note that this is also true independently of the choice of $n \in \mathbb{Z}_{>0}$.

Now by recalling the definition of the upper Minkowski dimension in Definition 1.6.2, this shows that

$$\overline{\dim}_{\mathscr{M}}\left(Z_h \cap [-n,n] \times \left[\frac{1}{n}, n\right]\right) \leq \frac{5}{3} + \gamma + \eta$$

for all $n \in \mathbb{Z}_{>0}, \eta > 0$, and $\gamma \in \left(0, \frac{1}{3}\right)$. Letting η and γ go to 0 then gives the sharper upper bound

$$\overline{\dim}_{\mathscr{M}}\left(Z_h \cap [-n,n] \times \left[\frac{1}{n}, n\right]\right) \leq \frac{5}{3}$$

for each $n \in \mathbb{Z}_{>0}$. Finally, using the countable stability of the Hausdorff dimension, the fact that the upper Minkowski dimension is always greater than the Hausdorff dimension, and the fact

$$\mathbb{R} \times (0,\infty) = \bigcup_{n=1}^{\infty}\left([-n,n] \times \left[\frac{1}{n}, n\right]\right),$$

we conclude our proof of this theorem by observing that

$$\begin{aligned}
\dim_H(Z_h) &= \dim_H\left(\bigcup_{n=1}^{\infty}\left([-n,n] \times \left[\frac{1}{n}, n\right]\right) \cap Z_h\right) \\
&= \sup_{n\in\mathbb{Z}_{>0}} \dim_H\left(\left([-n,n] \times \left[\frac{1}{n}, n\right]\right) \cap Z_h\right) \\
&\leq \sup_{n\in\mathbb{Z}_{>0}} \overline{\dim}_{\mathscr{M}}\left(\left([-n,n] \times \left[\frac{1}{n}, n\right]\right) \cap Z_h\right) \\
&\leq \frac{5}{3}.
\end{aligned}$$

□

As is typical when finding the Hausdorff dimension of any set, this upper bound is relatively quick and painless to obtain as we only needed to find one covering of dimension at most $\frac{5}{3}$. However, the task of finding a lower bound on the Hausdorff dimension of *any* possible covering of Z_h is considerably more difficult and complex. We present our strategy for finding these sorts of bounds systematically in the next chapter. We will establish a general strategy that works for any sufficiently nice stochastic process before verifying that the directed landscape meets the hypotheses required for this technique.

Chapter 4

Lower Bounding the Hausdorff Dimension of Random Level Sets

4.1 Statement of General Strategy

In this chapter we provide a general argument for systematically establishing a lower bound on the Hausdorff dimension $\dim_H$ of the level sets of a stochastic process X indexed by $\mathbb{R}^2$. that holds with an $h-$dependent positive probability $p_h > 0$. This procedure was assembled with the $h-$level sets Z_h of $\mathcal{L}(0,0,\cdot,\cdot)$ and results we are able to obtain for the directed landscape in particular in mind, so many of our choices and assumptions below are purely for convenience in that setting. The dimension that our processes take values in, the dimension of their index sets, the subsets with which we intersect their level sets, and even the nature of the bound (4.1.3) are all highly customizable in general.

Theorem 4.1.1. *Let $\Big(X(x_1,t_1)\Big)_{(x_1,t_1)\in\mathbb{R}^2}$ be a stochastic process, let $h, x_0, y_0 \in \mathbb{R}$ be arbitrary, and let $\delta_x, \delta_t \in (0,1]$. Assume that for some $m \in \mathbb{Z}_{>0}$ there exist constants $a_1,\dots,a_m,b_1,\dots,b_m \in (-1,1)$, some real number $\varepsilon_0 > 0$, and two positive h (and in general x_0, t_0, δ_x, and δ_t)-dependent constants c_h and c_h' such that for all tuples $(x_1,t_1),(x_2,t_2) \in [x_0, x_0+\delta_x]\times[t_0,t_0+\delta_t]$ with $t_1 \neq t_2$,*

$$\mathbb{P}\Big(X(x_1,t_1) \in (h-\varepsilon, h+\varepsilon)\Big) \le 2c_h\varepsilon \tag{4.1.1}$$

$$\mathbb{P}\Big(X(x_1,t_1) \in (h-\varepsilon, h+\varepsilon)\Big) \ge 2c_h'\varepsilon \tag{4.1.2}$$

$$\mathbb{P}\Big(\,|X(x_1,t_1)-h| < \varepsilon, |X(x_2,t_2)-h| < \varepsilon\Big) \le 4c_h\varepsilon^2\left(\sum_{i=1}^{m}|x_1-x_2|^{a_i}|t_1-t_2|^{b_i}\right) \tag{4.1.3}$$

for all $0 < \varepsilon \le \varepsilon_0$. If $\beta := 2 \wedge \left(\min_{i\in\{1,\dots m\}}(a_i+b_i+2)\right)$, then

$$\mathbb{P}\Big(\dim_H\left(X^{-1}(h)\right) \ge \beta\Big) \ge \mathbb{P}\left(\dim_H\left(X^{-1}(h)\cap\Big([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\Big)\right) \ge \beta\right) \ge p_h,$$

where the positive constant $p_h > 0$ is defined as

$$p_h := \frac{(c_h')^2}{64c_h} \left(\sum_{i=1}^{m} \frac{\delta_x^{a_i} \delta_t^{b_i}}{(a_i+1)(b_i+1)} \right)^{-1}.$$

Note. Because $\mathbb{P}$ is a probability measure, we must necessarily have that the positive constants p_h tend to 0 as $|h| \to \infty$. In particular, this will happen because c_h' must shrink to 0 as $|h| \to \infty$. Moreover, the probability p_h of this lower bound on the Hausdorff dimension holding will achieve its largest values on the regions of $\mathbb{R}$ in which

$$\text{Law}\left(\left(X(x_1, t_1) \right)_{(x_1,t_1) \in [x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]} \right)$$

concentrates its mass the most, since these will be the regions which lead to the largest possible constants c_h'. In the event that this law has a continuous, strictly positive density with respect to the Lebesgue measure whose support contains $[h-\varepsilon, h+\varepsilon]$, the existence of these constants $c_h' > 0$ and $c_h > 0$ will always be an immediate consequence of the Extreme Value Theorem.

4.2 Proof of General Strategy

Proof. We will break down the proof of this theorem into 5 steps as follows.

4.2.1 Build a Sequence of Random Measures Supported on $h-$Level Set

Fix an arbitrary $h \in \mathbb{R}$. For each $\varepsilon \in (0, \varepsilon_0)$, we define a random measure $\mu_{h,\varepsilon}$ on $[x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]$ by

$$\mu_{h,\varepsilon}(A) := \frac{1}{2\varepsilon} \lambda \left(\left\{ (x_1, t_1) \in [x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t] :\ (x, s, X(x_1, t_1)) \in A \times (h-\varepsilon, h+\varepsilon) \right\} \right)$$

where λ is the Lebesgue measure on $\mathbb{R}^2$. We will rewrite the measure of each such event as the integral

$$\mu_{h,\varepsilon}(A) = \frac{1}{2\varepsilon} \int_A \mathbb{1}_{\{X(x_1,t_1) \in (h-\varepsilon, h+\varepsilon\}} d\lambda(x_1, t_1). \tag{4.2.1}$$

The intention will be to use this collection of measures to build a mass distribution of the $h-$level set of X. Though this choice of approximating measures for our eventual mass distribution feels reasonable intuitively, we do note that this is by no means the only possible choice for this sequence. It is quite possible in general, and possibly even here in the case of the directed landscape, that a different type of measure may yield stronger results than this.

4.2.2 Sequence of Random Measures has a Subsequential Limit in Law

A natural way to prove the precompactness of a sequence of random measures is to prove that the sequence of random measures is tight. We will use the following form of Prokhorov's Theorem:

Theorem 4.2.1 (Kallenberg, Lemma 14.15 in [16]). *Let $\mu_1, \mu_2, \ldots$ be random measures on a locally compact simply connected Hausdorff space S. Then the sequence (μ_n) is relatively compact in distribution if and only if $(\mu_n(A))$ is tight in $\mathbb{R}_{\geq 0}$ for every measurable $A \in S$.*

Let $(\varepsilon_n)_{n=1}^{\infty}$ be some monotone sequence of positive real numbers in $(0, \varepsilon_0)$ such that $\varepsilon_n \to 0$. This time, we will take $S = [x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]$, $\mu_n = \mu_{h,\varepsilon_n}$, and will let A be an arbitrary measurable subset of $[x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]$. To prove that our sequence $(\mu_{h,\varepsilon_n}(A))_{n=1}^{\infty}$ is tight in $\mathbb{R}_+$, it is enough to prove that the sequence $(\mathbb{E}[\mu_{h,\varepsilon_n}(A)])_{n=1}^{\infty}$ is uniformly bounded in n. Since $(\mu_{h,\varepsilon}(A))$ is non-zero for all $h \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0)$, this means that we only need to find a uniform upper bound in n on $\mathbb{E}[\mu_{h,\varepsilon_n}(A)]$.

Using (4.2.1), we have that by definition,

$$\mathbb{E}[\mu_{h,\varepsilon}(A)] = \frac{1}{2\varepsilon}\mathbb{E}\left[\int_A \mathbb{1}_{\{X(x_1,t_1)\in(h-\varepsilon,h+\varepsilon\}} d\lambda(x_1,t_1)\right].$$

for all $\varepsilon > 0$. By Fubini's theorem, the expectation above is equal to

$$\frac{1}{2\varepsilon}\int_A \mathbb{P}\Big(X(x_1,t_1) \in (h-\varepsilon, h+\varepsilon)\Big) d\lambda(x_1,t_1). \tag{4.2.2}$$

By hypothesis, the integrand above can be bounded using (4.1.1) by $2c_h\varepsilon$ leaving us with

$$\mathbb{E}[\mu_{h,\varepsilon}(A)] \leq c_h \int_A \mathbb{1} d\lambda(x_1,t_1) \leq c_h,$$

where we use that $\lambda(A) \leq \lambda([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]) \leq 1$. This proves that $(\mu_{h,\varepsilon_n}(A))_{n=1}^{\infty}$ is tight for all measurable sets A and thus there exists a convergent subsequence of measures $(\mu_{h,\varepsilon_{n_k}})_{k=1}^{\infty}$ that converges in distribution to a measure μ_h. Without loss of generality, we will simply take $(\mu_{h,\varepsilon_n})_{n=1}^{\infty}$ to be $(\mu_{h,\varepsilon_{n_k}})_{k=1}^{\infty}$. We will now prove that this limiting measure μ_h is mass distribution with finite $\alpha-$energy for all $\alpha < \beta$ on $X^{-1}(h)$ over the course of the next three sections.

4.2.3 Limit is a Positive Measure with Positive Probability

If we can establish bounds of the form

$$\mathbb{E}\Big[\mu_{h,\varepsilon}([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])\Big] \geq K_h > 0$$
$$\mathbb{E}\Big[\mu_{h,\varepsilon}([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])\Big] \leq C_h < \infty$$

where K and C are strictly positive constants independent of the choice of ε, we can prove that $\mu_{h,\varepsilon}([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]) > 0$ with positive probability. If we can find such bounds and choose

$\theta_h \in [0, 1 \wedge K_h]$, then the Payley-Zigmund inequality implies that for each $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\mathbb{P}\Big(\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right) > \theta_h\Big) \\
&\geq \left(1 - \frac{\theta_h}{\mathbb{E}\Big[\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\Big]}\right)^2 \frac{\mathbb{E}\Big[\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\Big]^2}{\mathbb{E}\Big[\mu_{h,\varepsilon}\Big(\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)^2\Big)\Big]} \\
&\geq \left(1 - \frac{\theta_h}{K_h}\right)^2 \frac{K_h^2}{C_h}.
\end{aligned}$$

In particular, this will mean that

$$\mathbb{P}\Big(\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right) > 0\Big) \geq \frac{K_h^2}{C_h} =: p_h > 0.$$

We will now prove that these bounds on the moments of $\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)$ exist.

We first need to find a lower bound for $\mathbb{E}\Big[\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\Big]$. By combining equation (4.2.2) with the assumed bound (4.1.2), we see immediately that

$$\begin{aligned}
\mathbb{E}\Big[\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\Big] &= \frac{1}{2\varepsilon}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_o+\delta_x} \mathbb{P}(X(x_1,t_1) \in (h-\varepsilon, h+\varepsilon))dxdt \\
&\geq \int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x} c_h' dxdt \\
&= \delta_t\delta_x c_h' =: K_h.
\end{aligned}$$

To find an upper bound C_h on the second moment, we start by observing that we may write

$$\begin{aligned}
&\mathbb{E}\Big[\left(\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\right)^2\Big] \\
&= \frac{1}{4\varepsilon^2}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x} \mathbb{P}\Big(|X(x_1,t_1) - h| \leq \varepsilon, |X(x_2,t_2) - h| \leq \varepsilon\Big)dx_1dt_1dx_2dt_2.
\end{aligned}$$

By invoking the assumed two-point distribution bound in (4.1.3), this can be bounded further as

$$\begin{aligned}
&\mathbb{E}\Big[\left(\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)\right)^2\Big] \\
&\leq c_h \int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x} \sum_{i=1}^{m} |x_1 - x_2|^{a_i}|t_1 - t_2|^{b_i} dxdsdydt.
\end{aligned}$$

Next, by performing the change of variables

$$(u_1, v_1, u_2, v_2) = (x_1 - x_2, t_1 - t_2, x_1 + x_2, t_1 + t_2) \tag{4.2.3}$$

this then becomes

$$\begin{aligned}
&\mathbb{E}\left[\Big(\mu_{h,\varepsilon}\left([x_0, x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)\Big)^2\right] \\
&\leq c_h \int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x}\sum_{i=0}^{m}|x_1-x_2|^{a_i}|t_1-t_2|^{b_i}dx_1dt_1dx_2dt_2 \\
&= 4c_h\sum_{i=1}^{m}\int_{2t_0}^{2t_0+2\delta_t}\int_{2x_0}^{2x_0+2\delta_x}\int_{-\delta_t}^{\delta_t}\int_{-\delta_x}^{\delta_x}|u_1|^{a_i}|v_1|^{b_i}du_1dv_1du_2dv_2 \\
&= 64\delta_x\delta_t c_h\sum_{i=1}^{m}\int_0^{\delta_t}\int_0^{\delta_x}u_1^{a_i}v_1^{b_i}du_1dv_1 \\
&= 64\delta_x\delta_t c_h\sum_{i=1}^{m}\frac{\delta_x^{a_i+1}\delta_t^{b_i+1}}{(a_i+1)(b_i+1)} =:\; C_h,
\end{aligned}$$

thus giving us our desired upper bound C_h, which is independent of the choice of ε. Moreover, because the bounds (4.1.2) and (4.1.3) hold for all $\varepsilon \in (0, \varepsilon_0)$, we have that these bounds hold for the same choice of C_h and K_h for each random measure in $(\mu_{h,\varepsilon_n})_{n=1}^{\infty}$. Hence, for each measure μ_{h,ε_n} we have just shown that

$$\mathbb{P}\Big(\mu_{h,\varepsilon_n}\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)>0\Big)\geq p_h>0$$

for all $\theta_h \in (0, 1\wedge K_h)$. All that remains now is to show that this lower bound is also inherited by the limiting measure μ_h. Because μ_h is the limit in distribution of $(\mu_{h,\varepsilon_n})_{n=1}^{\infty}$ we indeed have that

$$\begin{aligned}
\mathbb{P}\Big(\mu_h\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)\geq 0\Big) &\geq \limsup_{n\to\infty}\mathbb{P}\Big(\mu_{h,\varepsilon_n}\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)\geq 0\Big) \\
&\geq \limsup_{n\to\infty}\mathbb{P}\Big(\mu_{h,\varepsilon_n}\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)> 0\Big) \\
&\geq p_h,
\end{aligned}$$

and thus μ_h is a positive measure with non-zero probability.

4.2.4 Limit is a Mass Distribution on the $h-$Level Set of X

We will next prove that the limiting measure μ_h is supported on (a subset of) the $h-$level set of the random process $(X(x_1,t_1))_{(x_1,t_1)\in[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]}$. This step is necessary since we only have a limit in law, and not a pointwise or uniform limit. To begin, take $\Big(\Omega,\mathcal{F},\mathbb{P}\Big)$ to be the underlying probability space on which X and our sequence of random measures $(\mu_{h,\varepsilon_n})_{n=1}^{\infty}$ are defined, i.e.

$$\begin{aligned}
X:\Big(\Omega,\mathcal{F},\mathbb{P}\Big)&\to\Big(C\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right),\sigma(\tau_{\text{unif}})\Big) \\
\mu_{h,\varepsilon_n}:\Big(\Omega,\mathcal{F},\mathbb{P}\Big)&\to\Big(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]},\sigma(\tau_{\text{vague}})\Big)
\end{aligned}$$

where $M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]}$ is the space of finite (positive) measures on $[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]$, $\sigma(\tau_{\text{unif}})$ is the sigma algebra generated by the topology of uniform convergence, and $\sigma(\tau_{\text{vague}})$ is the sigma algebra generated by the vague topology on $M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]}$. Note that $M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]}$ is a Polish space and that $C\left([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]\right)$ is a complete, separable metric space under the sup-norm by the Stone-Weierstrass theorem.

Next, let d_{vague} be the metric generating the vague topology on $[x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]$ and define a metric d_{prod} on the product of these two spaces by

$$d_{\text{prod}} : \left(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C\left([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]\right)\right)^2 \to \mathbb{R}_+$$

$$\Big((f,\mu),(g,\nu)\Big) \mapsto \sqrt{d_{\text{vague}}(\mu,\nu)^2 + \left(\sup_{(x_1,t_1)\in[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} |f(x_1,t_1) - g(x_1,t_1)|\right)^2}. \quad (4.2.4)$$

Under this metric then see immediately that the product space

$$\Big(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C\left([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]\right), d_{\text{prod}}\Big)$$

is again a complete, separable metric space. Let τ_{prod} denote the topology generated by the metric d_{prod} and $\sigma(\tau_{\text{prod}})$ be the Borel sigma algebra generated by this topology. Under these conventions, any finite measure on the product space endowed with $\sigma(\tau_{\text{prod}})$ will automatically have compact support.

We will now shift our attention back to the sequence of random measures $(\mu_{h,\varepsilon_n})_{n=1}^{\infty}$ and its limit in distribution μ_h. With the conventions in this subsection thus far, we can view the pairs $(\mu_{h,\varepsilon_n}, X)$ as random elements

$$\Big(\mu_{h,\varepsilon_n}, X\Big) : \Big(\Omega, \mathcal{F}, \mathbb{P}\Big) \to \Big(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C\left([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]\right), \sigma(\tau_{\text{prod}})\Big).$$

Given this, if we define for each $n \in \mathbb{Z}_{>0}$ the probability measure

$$Q_n = \text{Law}\Big((\mu_{h,\varepsilon_n}, X)\Big)$$

on $\big(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C\left([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]\right), \sigma(\tau_{\text{prod}})\big)$, then the sequence of probability measures $(Q_n)_{n=1}^{\infty}$ will have a weak limit Q_∞. Moreover, each probability measure Q_n (including $n = \infty$) will also have separable support. Thus, we may use the Skorokhod Representation Theorem to construct a new probability space $\Big(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\Big)$ and random elements

$$Y_n : \Big(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\Big) \to \Big(M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C\left([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t]\right), \sigma(\tau_{\text{prod}})\Big)$$

for each $n > 0$ (including $n = \infty$) such that $Y_n \to Y_\infty$ $\tilde{\mathbb{P}}-$almost surely, and such that

$$\text{Law}\Big(Y_n\Big) = Q_n$$

for each n. For the sake of convenience, for each $n \in \mathbb{Z}_{>0}$ we will write

$$Y_n =: \Big(\tilde{\mu}_{h,\varepsilon_n}, X^{(n)}\Big)$$

and for $n = \infty$, we will similarly write

$$Y_\infty =: \left(\mu_h, X^{(\infty)}\right).$$

Now because the almost-sure convergence $Y_n \to Y_\infty$ is with respect to the metric d_{prod}, we have by (4.2.4) that as $n \to \infty$,

$$\tilde{\mu}_{h,\varepsilon_n} \to \mu_h \tag{4.2.5}$$

$\tilde{\mathbb{P}}-$almost surely in the vague topology on $M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]}$. The same reasoning gives us that

$$X^{(n)} \to X^{(\infty)}$$

with respect to the sup-norm on $C\left([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]\right)$ $\tilde{\mathbb{P}}-$almost surely.

We now take a moment to observe that each $X^{(n)}$ (including $n = \infty$) is a copy of the random process X created on $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, though they are not necessarily the same realization of X. We will show that this potential ambiguity with different realizations of the same process is not an issue. To that end, we will make several elementary observations before resolving this. First, we establish that for each finite n,

$$\operatorname{supp} \tilde{\mu}_{h,\varepsilon_n} \subseteq \left(X^{(n)}\right)^{-1}\left([h-\varepsilon_n, h+\varepsilon_n]\right) \tag{4.2.6}$$

$\tilde{\mathbb{P}}-$almost surely. Letting π_1 and π_2 be the usual projection maps on the product space

$$M_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} \times C([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t]),$$

this follows immediately from the fact that

$$\begin{aligned}
&\tilde{\mathbb{P}}\left(\tilde{\mu}_{h,\varepsilon_n}\left(\left(X^{(n)}\right)^{-1}\left([h-\varepsilon_n, h+\varepsilon_n]^C\right)\right) > 0\right) \\
&= \tilde{\mathbb{P}}\left(\pi_1(Y_n)\left(\pi_2\left(Y_n\right)^{-1}\left([h-\varepsilon_n, h+\varepsilon_n]^C\right)\right) > 0\right) \\
&= \mathbb{P}\Big(\mu_{h,\varepsilon_n}\left(X^{-1}\left([h-\varepsilon_n, h+\varepsilon_n]^C\right)\right) > 0\Big) \\
&= 0
\end{aligned}$$

based on our original definition of μ_{h,ε_n} in terms of X.

Next we fix some small $\delta > 0$ and recall that $\tilde{\mathbb{P}}-$almost surely, there exists $N = N(\delta, \omega)$ such that for all $n > N$ and $\varepsilon_n < \delta/2$,

$$\sup_{(x_1,t_1)\in[1,2]\times\left[1,\frac{11}{10}\right]} \left|X^{(n)}(\omega)(x_1,t_1) - X^{(\infty)}(\omega)(x_1,t_1)\right| < \delta/2. \tag{4.2.7}$$

As an immediate consequence of (4.2.7) and the fact that $\varepsilon_n + \delta/2 < \delta$, we have that for all $n > N$,

$$\left(X^{(n)}\right)^{-1}\left([h-\varepsilon_n, h+\varepsilon_n]\right) \subseteq \left(X^{(\infty)}\right)^{-1}\left([h-\delta, h+\delta]\right) \tag{4.2.8}$$

$\tilde{\mathbb{P}}-$almost surely. In conjunction with (4.2.6), this means that

$$\operatorname{supp} \tilde{\mu}_{h,\varepsilon_n} \subseteq \left(X^{(\infty)}\right)^{-1}([h-\delta, h+\delta])$$

$\tilde{\mathbb{P}}-$almost surely. With this we are now able to establish that $\tilde{\mathbb{P}}-$almost surely,

$$\operatorname{supp} \mu_h \subseteq (X^{(\infty)})^{-1}([h-\delta, h+\delta]) \tag{4.2.9}$$

for any $\delta > 0$, i.e. that

$$\operatorname{supp} \mu_h \subseteq \left(X^{(\infty)}\right)^{-1}(h) \tag{4.2.10}$$

as desired. To prove that (4.2.9) is true, it suffices to show that for any continuous

$$f : [x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t] \to \mathbb{R}$$

vanishing on an arbitrary open neighbourhood of $\left(X^{(\infty)}\right)^{-1}([h-\delta, h+\delta])$ that

$$\iint_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} f(x_1,t_1) d\mu_h(\omega)(x_1,t_1) = 0.$$

By (4.2.5), we know that for any such function f,

$$\iint_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} f(x_1,t_1) d\mu_h(\omega)(x_1,t_1) = \lim_{n\to\infty} \iint_{[x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t]} f(x_1,t_1) d\mu_{h,\varepsilon_n}(\omega)(x_1,t_1).$$

By (4.2.8) and the hypothesis about the support of f, we know that for $n > N = N(\delta, \omega)$, each integral on the right-hand side above will be exactly 0. This in turn establishes (4.2.9) for any fixed $\delta > 0$ and by letting $\delta \to 0$, this finally proves that (4.2.10) is true. This completes our goal for this subsection and confirms that μ_h is a valid mass distribution for the $h-$level set of X. For the sake of convenience going forward, we will take the random measures and stochastic processes that we have on the original probability space $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ to be the realizations that we have just constructed on $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$.

4.2.5 Mass Distribution has Finite Energy for $\alpha < \beta$

In order to invoke Theorem 1.6.9 with our mass distribution μ_h, we need to determine for which $\alpha \geq 0$ the integral

$$\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^\alpha} d\mu_h(x_1,t_1) d\mu_h(x_2,t_2) < \infty \tag{4.2.11}$$

converges almost surely. Notice that since this is a random integral, to prove that this integral is finite, it is enough to show that its expectation is finite. First, we will prove that the mean of the energy integral for each $\mu_{h,\varepsilon}$ is uniformly bounded. By the definition of $\mu_{h,\varepsilon}$,

$$\mathbb{E}\left[\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^\alpha} d\mu_{h,\varepsilon}(x_1,t_1)d\mu_{h,\varepsilon}(x_2,t_2)\right]$$
$$= \mathbb{E}\left[\frac{1}{4\varepsilon^2}\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{\mathbb{1}_{\{(X(x_1,t_1)\in[h-\varepsilon,h+\varepsilon]\}}\mathbb{1}_{\{(X(x_2,t_2)\in[h-\varepsilon,h+\varepsilon]\}}}{||(x_1-x_2,s-t)||^\alpha} d\lambda(x_1,t_1)d\lambda(x_2,t_2)\right].$$

By Fubini's Theorem, the expected value on the right-hand side is equal to

$$\frac{1}{4\varepsilon^2}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x}\int_{t_0}^{t_0+\delta_t}\int_{x_0}^{x_0+\delta_x} \frac{\mathbb{P}\Big(|X(x_1,t_1)-h|<\varepsilon, |X(x_2,t_2)-h|<\varepsilon\Big)}{||(x_1-x_2,s-t)||^\alpha} dx_1dt_1dx_2dt_2.$$

As before, we can use the two-point distribution bound (4.1.3) to obtain the subsequent bound

$$\mathbb{E}\left[\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^\alpha} d\mu_{h,\varepsilon}(x_1,t_1)d\mu_{h,\varepsilon}(x_2,t_2)\right]$$
$$\leq c_h\sum_{i=1}^m\int_{t_0}^{t_0+1}\int_{x_0}^{x_0+1}\int_{t_0}^{t_0+1}\int_{x_0}^{x_0+1} \frac{|x_1-x_2|^{a_i}|t_1-t_2|^{b_i}}{||(x_1-x_2,s-t)||^\alpha} dx_1dt_1dx_2dt_2.$$

By once again using the change of variables (4.2.3), the right-hand side of the expression above can be rewritten as

$$\begin{aligned} & c_h\sum_{i=1}^m\int_{t_0}^{t_0+1}\int_{x_0}^{x_0+1}\int_{t_0}^{t_0+1}\int_{x_0}^{x_0+1} \frac{|x_1-x_2|^{a_i}|t_1-t_2|^{b_i}}{||(x_1-x_2,s-t)||^\alpha} dx_1dt_1dx_2dt_2 \\ = {} & 4c_h\sum_{i=1}^m\int_{2t_0}^{2t_0+2}\int_{2x_0}^{2x_0+2}\int_{-1}^{1}\int_{-1}^{1}\frac{|u_1|^{a_i}|v_1|^{b_i}}{||(u_1,v_1)||^\alpha}du_1dv_1du_2dv_2 \\ = {} & 64c_h\sum_{i=1}^m\int_0^1\int_0^1\frac{u_1^{a_i}v_1^{b_i}}{(u_1^2+v_1^2)^{\alpha/2}}du_1dv_1. \end{aligned} \tag{4.2.12}$$

The convergence of the integral above in the right-hand side of (4.2.12) is best understood by using the following proposition about the calculus of the $\Gamma-$distribution.

Lemma 4.2.2. *For any real numbers $a,b>0$, the integral*

$$\int_0^1\int_0^1\frac{u^{a-1}v^{b-1}}{(u^2+v^2)^{\frac{\alpha}{2}}}dudv = \int_0^1\int_0^1\frac{u^{a-1}v^{b-1}}{||(u,v)||_2^\alpha}dudv$$

converges for all $0\leq\alpha<a+b$.

Proof. We first recall that since all norms on $\mathbb{R}^2$ are equivalent to the infinity norm, there exist positive constants $c,C>0$ such that

$$c||(u,v)||_1^\alpha \leq ||(u,v)||_2^\alpha \leq C||(u,v)||_1^\alpha$$

so we immediately have the upper bound

$$\int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u^2+v^2)^{\frac{\alpha}{2}}} dudv \leq c \int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u+v)^{\alpha}} dudv.$$

Now let U and V be independent random variables with $U \sim \Gamma(a,1)$ and $V \sim \Gamma(b,1)$ and consider

$$\mathbb{E}\left[\frac{1}{(U+V)^{\alpha}} 1_{\{U \leq 1\}} 1_{\{V \leq 1\}}\right].$$

By definition, this expectation is equal to

$$\int_0^1 \int_0^1 \frac{1}{(u+v)^{\alpha}} \left(\frac{u^{a-1} e^{-u}}{\Gamma(a)}\right) \left(\frac{v^{b-1} e^{-v}}{\Gamma(b)}\right) dudv \geq \frac{e^{-2}}{\Gamma(a)\Gamma(b)} \int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u+v)^{\alpha}} dudv$$

which then allows us to say that

$$\int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u^2+v^2)^{\frac{\alpha}{2}}} dudv \leq ce^2 \Gamma(a)\Gamma(b) \mathbb{E}\left[\frac{1}{(U+V)^{\alpha}} 1_{\{U \leq 1\}} 1_{\{V \leq 1\}}\right].$$

Moreover, by the independence of U and V, we also know that $W := U + V \sim \Gamma(a+b, 1)$. With this observation in mind, we can then say that

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u^2+v^2)^{\frac{\alpha}{2}}} dudv &\leq ce^2 \Gamma(a)\Gamma(b) \mathbb{E}\left[\frac{1}{(U+V)^{\alpha}} 1_{\{U \leq 1\}} 1_{\{V \leq 1\}}\right] \\
&\leq ce^2 \Gamma(a)\Gamma(b) \mathbb{E}\left[\frac{1}{W^{\alpha}} 1_{\{W \leq 2\}}\right] \\
&= ce^2 \Gamma(a)\Gamma(b) \int_0^2 \frac{1}{w^{\alpha}} \frac{w^{a+b-1} e^{-w}}{\Gamma(a+b)} dw \\
&\leq \frac{ce^2 \Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^2 w^{a+b-\alpha-1} dw.
\end{aligned}$$

This upper bound is finite if and only

$$\int_0^1 w^{a+b-\alpha-1} dw < \infty \iff a+b-1-\alpha > -1 \iff \alpha < a+b.$$

Therefore, for all $0 \leq \alpha < a+b$, we have that

$$\int_0^1 \int_0^1 \frac{u^{a-1} v^{b-1}}{(u^2+v^2)^{\frac{\alpha}{2}}} dudv < \infty.$$

□

Using Lemma 4.2.2, we see immediately that the improper integral

$$\int_0^1 \int_0^1 \frac{u_1^{a_i} v_1^{b_i}}{(u_1^2+v_1^2)^{\frac{\alpha}{2}}} dudv$$

converges for all $0 \leq \alpha < a_i + b_i + 2$. Moreover, because the integrand above is a non-negative measureable function for each $i \in \{1, \ldots, m\}$, Tonelli's theorem tells us that

$$\sum_{i=1}^{m} \int_0^1 \int_0^1 \frac{u_1^{a_i} v_1^{b_i}}{(u_1^2 + v_1^2)^{\alpha/2}} du_1 dv_1 < \infty \iff \int_0^1 \int_0^1 \frac{u_1^{a_i} v_1^{b_i}}{(u_1^2 + v_1^2)^{\frac{\alpha}{2}}} du_1 dv_1 < \infty \text{ for all } i \in \{1, \ldots, m\}.$$

As such, by Tonelli's Theorem and Lemma 4.2.2 we have that

$$\sum_{i=1}^{m} \int_0^1 \int_0^1 \frac{u_1^{a_i} v_1^{b_i}}{(u_1^2 + v_1^2)^{\alpha/2}} du_1 dv_1 < \infty$$

for all $0 \leq \alpha < \beta = 2 \wedge \left(\min_{i \in \{1, \ldots m\}} (a_i + b_i + 2) \right)$. From these observations, we then get that for all choices of $0 \leq \alpha < \beta$,

$$\mathbb{E}\left[\iiiint_{([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])^2} \frac{1}{||(x_1, t_1) - (x_2, t_2)||^\alpha} d\mu_{h,\varepsilon}(x_1, t_1) d\mu_{h,\varepsilon}(x_2, t_2) \right] \leq 64 c_h R_\alpha < \infty. \tag{4.2.13}$$

for all $\varepsilon \in (0, \varepsilon_0)$, for some $\alpha-$dependent but $h-$independent constant $R_\alpha \in \mathbb{R}_{\geq 0}$.

We now return to our original goal of verifying that equation (4.2.11) holds almost surely. We will do this by showing that the expected value of the energy integral in (4.2.11) is finite almost surely, and relating the expected value of that energy integral to the uniform bound we have found in (4.2.13).

Let the probability space $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ be as in the preceding section. As previously discussed, we know that as $n \to \infty$,

$$\mu_{h,\varepsilon_n}(\omega) \longrightarrow \mu_h(\omega) \tag{4.2.14}$$

for $\mathbb{P}-$almost every $\omega \in \Omega$, in the sense described in (4.2.5). For any such $\omega \in \Omega$, $(\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n})(\omega)$ is a product measure on $([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t])^2$. Since we have previously established that the convergence in (4.2.14) is true, we have immediately that for any such $\omega \in \Omega$,

$$\left(\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n}\right)(\omega) \longrightarrow \left(\mu_h \times \mu_h\right)(\omega)$$

in distribution in $\left(M_{([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])^2}, \sigma(\tau_{\text{vague}})\right)$, which we recall is the space of finite positive measures on $([x_0, x_0 + \delta_x] \times [t_0, t_0 + \delta_t])^2$ equipped with the topology of vague convergence.

To relate the energy integral of the sequence to the energy integral of the limit, we will make use of Fatou's Lemma. In particular, we will use Fatou's Lemma for weakly convergent measures. By invoking Theorem 2.4 in [12], we know that

$$\begin{aligned} &\iiiint_{([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])^2} \liminf_{(x', s', y', t') \to (x, s, y, t)} \frac{1}{||(x', s') - (y', t')||^\alpha} d\mu_h(x_1, t_1) d\mu_h(x_2, t_2) \\ \leq\ &\liminf_{n \to \infty} \iiiint_{([x_0, x_0+\delta_x] \times [t_0, t_0+\delta_t])^2} \frac{1}{||(x_1, t_1) - (x_2, t_2)||^\alpha} d\mu_{h,\varepsilon_n}(x_1, t_1) d\mu_{h,\varepsilon_n}(x_2, t_2). \end{aligned}$$

since $(\mu_{h,\varepsilon_n} \times \mu_{h,\varepsilon_n})$ converges weakly to $(\mu_h \times \mu_h)$ and

$$f(x,s,y,t) := ||(x_1 - x_2, s-t)||^{-\alpha}$$

is a measurable function taking values in $\mathbb{R} \cup \{\infty\}$. Moreover, because we have that

$$\liminf_{(x',s',y',t')\to(x,s,y,t)} \frac{1}{||(x',s')-(y',t')||^{\alpha}} = \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}}$$

this new integral bound can be simplified to

$$\begin{aligned}&\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_h(x_1,t_1)d\mu_h(x_2,t_2)\\ \leq \liminf_{n\to\infty} &\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_{h,\varepsilon_n}(x_1,t_1)d\mu_{h,\varepsilon_n}(x_2,t_2).\end{aligned}$$

By taking the expectation on both sides of the inequality above, we then get that

$$\begin{aligned}&\mathbb{E}\left[\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_h(x_1,t_1)d\mu_h(x_2,t_2)\right]\\ \leq\ &\mathbb{E}\left[\liminf_{n\to\infty}\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_{h,\varepsilon_n}(x_1,t_1)d\mu_{h,\varepsilon_n}(x_2,t_2)\right].\end{aligned}$$

Finally, by using the regular version of Fatou's Lemma on the upper bound above in conjunction with our uniform bound (4.2.13), we finally obtain that

$$\begin{aligned}&\mathbb{E}\left[\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_h(x_1,t_1)d\mu_h(x_2,t_2)\right]\\ \leq \ &\mathbb{E}\left[\liminf_{n\to\infty}\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_{h,\varepsilon_n}(x_1,t_1)d\mu_{h,\varepsilon_n}(x_2,t_2)\right] < \infty,\end{aligned}$$

for all $0 \leq \alpha < \beta$. Thus we have now proven that

$$\iiiint_{([x_0,x_0+\delta_x]\times[t_0,t_0+\delta_t])^2} \frac{1}{||(x_1,t_1)-(x_2,t_2)||^{\alpha}} d\mu_h(x_1,t_1)d\mu_h(x_2,t_2) < \infty$$

for all $0 \leq \alpha < \beta$, completing our proof of Theorem 4.1.1 via Theorem 1.6.9. □

4.3 The General Strategy Applied to $\mathcal{L}(0,0;\cdot,\cdot)$

In this section, we will outline how we intend to use Theorem 4.1.1 to develop a lower bound on the level sets of the Hausdorff dimension of the directed landscape restricted to a compact set. This task will comprise almost all the remaining work in this thesis. As mentioned previously in section 1.2, the bounds in (4.1.2) and (4.1.1) are already known to exist in the case where the stochastic process $X = \mathcal{L}(0,0;\cdot,\cdot)$, via its relationship with $\mathfrak{A}_1$. Thus, the only hypothesis that needs to be verified is the existence of a two-point bound of the form (4.1.3) for $\mathcal{L}(0,0;\cdot,\cdot)$.

Given this information, our goal over the course of the rest of this paper will be to establish a partial two-point bound for the directed landscape of the following form.

Theorem 4.3.1. *Let $h \in \mathbb{R}$ be arbitrary and define a positive constant $\varepsilon_0 > 0$ by*

$$\varepsilon_0 := \min_{(x,s)\in[1,2]\times\left[1,\frac{11}{10}\right]} \frac{x^2}{2s} = \frac{5}{11}.$$

Then there exists an absolute, $h-$independent constant $\kappa > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$\mathbb{P}\Big(|\mathcal{L}(0,0;x,s) - h| \le \varepsilon, |\mathcal{L}(0,0;y,t) - h| \le \varepsilon \Big) \le \kappa\varepsilon^2 \left(1 + |t-s|^{-\frac{1}{2}} + |x-y|^{\frac{1}{2}}|t-s|^{-\frac{2}{3}}|\right) \quad (4.3.1)$$

for all $(x,s),(y,t) \in [1,2] \times \left[1, \frac{11}{10}\right]$.

Upon establishing Theorem 4.3.1, we will have the final ingredient needed to obtain a lower bound of $\frac{3}{2}$ on the Hausdorff dimension of the $h-$level set of $\mathcal{L}(0,0;\cdot,\cdot)$ which holds with ($h-$dependent) probability at least $p_h > 0$. We also note that the two-point bound (4.3.1), while sufficient for our purposes, is almost certainly not optimal. It is highly likely and expected that a true optimal two-point using these techniques would be defined piecewise and will depend heavily on the relationship between $x-y$ and $t-s$, as well as their actual values and their signs. Because of this, there is likely quite a bit of room for further optimization, even within the context of this specific result. Some suggestions for possible improvements will be discussed later in Chapter 7.

Before beginning our proof of the partial two-point bound in Theorem 4.3.1, we will use its conclusion to state and prove the cumulative result of the work in this thesis, via Theorem 4.1.1.

Theorem 4.3.2. *Let f_{TW_2} be the density of $\mathfrak{A}_1(0)$ with respect to the Lebesgue measure, fix an arbitrary $h \in \mathbb{R}$, and let the absolute constant $\kappa > 0$ be as in Theorem 4.3.1. If $h \ge 0$, define two positive constants $c'_h, c_h > 0$ by*

$$c'_h := \left(\frac{10}{11}\right)^{\frac{1}{3}} \min\left\{ f_{TW_2}(u) \ : \ u \in \left[\left(\frac{10}{11}\right)^{\frac{5}{3}} + \left(\frac{10}{11}\right)^{\frac{1}{3}} h - \frac{5}{11},\ 4 + h + \frac{5}{11} \right] \right\}$$

and

$$c_h := \kappa \vee \max\left\{ f_{TW_2}(u) \ : \ u \in \left[\left(\frac{10}{11}\right)^{\frac{5}{3}} + \left(\frac{10}{11}\right)^{\frac{1}{3}} h - \frac{5}{11},\ 4 + h + \frac{5}{11} \right] \right\}.$$

If $h < 0$ then define these positive constants $c_h, c'_h > 0$ instead by

$$c'_h := \left(\frac{10}{11}\right)^{\frac{1}{3}} \min\left\{ f_{TW_2}(u) \ : \ u \in \left[\left(\frac{10}{11}\right)^{\frac{5}{3}} + h - \frac{5}{11},\ 4 + \left(\frac{10}{11}\right)^{\frac{1}{3}} h + \frac{5}{11} \right] \right\}$$

and

$$c_h := \frac{\kappa}{4} \vee \max\left\{ f_{TW_2}(u) \ : \ u \in \left[\left(\frac{10}{11}\right)^{\frac{5}{3}} + h - \frac{5}{11},\ 4 + \left(\frac{10}{11}\right)^{\frac{1}{3}} h + \frac{5}{11} \right] \right\}.$$

If the random set $Z_h \subseteq \mathbb{R} \times \mathbb{R}_{>0}$ *is defined as* $Z_h := \Big\{(x,s) \in \mathbb{R} \times \mathbb{R}_{>0} \ : \ \mathcal{L}(0,0;x,s) = h\Big\}$, *then*

$$\mathbb{P}\left(\dim_H(Z_h) \leq \frac{5}{3}\right) = 1 \qquad \text{and} \qquad \mathbb{P}\left(\dim_H(Z_h) \geq \frac{3}{2}\right) \geq \frac{(c_h')^2}{64c_h\left(1+2\sqrt{10}+2\sqrt[3]{100}\right)}.$$

Proof. Theorem 3.0.1 directly gives us that

$$\mathbb{P}\left(\dim_H(Z_h) \leq \frac{5}{3}\right) = 1$$

for all $h \in \mathbb{R}$, so we only need to verify that

$$\mathbb{P}\left(\dim_H(Z_h) \geq \frac{3}{2}\right) \geq \frac{(c_h')^2}{64c_h\left(1+2\sqrt{10}+2\sqrt[3]{100}\right)}.$$

Now since we know that, by definition of the directed landscape,

$$\mathcal{L}(0,0;x,s) \stackrel{d}{=} \mathfrak{A}_1^{(s)}(x) = s^{\frac{1}{3}}\mathfrak{A}_1\left(s^{-\frac{2}{3}x}\right) \stackrel{d}{=} s^{\frac{1}{3}}\mathfrak{A}_1(0) - s^{-\frac{4}{3}}x^2$$

for any $(x,s) \in [1,2] \times \left[1, \frac{11}{10}\right]$. As such, for fixed (x,s) we see that

$$\begin{aligned}
\mathbb{P}\Big(\mathcal{L}(0,0;x,s) \in (h-\varepsilon, h+\varepsilon)\Big) &= \mathbb{P}\Big(s^{\frac{1}{3}}\mathfrak{A}_1(0) - s^{-\frac{4}{3}}x^2 \in (h-\varepsilon,h+\varepsilon)\Big) \\
&= \mathbb{P}\Big(\mathfrak{A}_1(0) \in \Big(s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h-\varepsilon), s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h+\varepsilon)\Big)\Big) \\
&= \mathbb{P}\Big(TW_2 \in \Big(s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h-\varepsilon), s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h+\varepsilon)\Big)\Big) \\
&= \int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)} f_{TW_2}(u)du. \qquad (4.3.2)
\end{aligned}$$

This means that, in accordance with equations (4.1.2) and (4.1.1), we need to find two $h-$dependent constants $c_h, c_h' > 0$ such that

$$2c_h'\varepsilon \leq \int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)} f_{TW_2}(u)du \leq 2c_h\varepsilon$$

holds for each $0 < \varepsilon < \frac{5}{11}$ and simultaneously for all choices of $(x,s) \in [1,2] \times \left[1, \frac{11}{10}\right]$.

As the subsequent argument changes only superficially as sign (h) changes, without loss of generality, assume that $h \geq 0$. To find these constants $c_h, c_h' > 0$ we start by observing that for all choices of (x,s) and all $0 < \varepsilon < \frac{5}{11}$ we have the inequalities

$$\left(\frac{10}{11}\right)^{\frac{5}{3}} + \left(\frac{10}{11}\right)^{\frac{1}{3}} h - \frac{5}{11} \leq \ s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h-\varepsilon) < \ s^{-\frac{5}{3}}x^2 + s^{-\frac{1}{3}}(h+\varepsilon) \leq \ 4 + h + \frac{5}{11}.$$

These inequalities in turn lead the chain of inequalities

$$\begin{aligned}
&\min\left\{f_{TW_2}(u)\ :\ u\in\left[\left(\frac{10}{11}\right)^{\frac{5}{3}}+\left(\frac{10}{11}\right)^{\frac{1}{3}}h-\frac{5}{11},\ 4+h+\frac{5}{11}\right]\right\}\\
&\leq \min\left\{f_{TW_2}(u)\ :\ u\in\left[s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon),\ s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)\right]\right\}\\
&\leq \max\left\{f_{TW_2}(u)\ :\ u\in\left[s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon),\ s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)\right]\right\}\\
&\leq \max\left\{f_{TW_2}(u)\ :\ u\in\left[\left(\frac{10}{11}\right)^{\frac{5}{3}}+\left(\frac{10}{11}\right)^{\frac{1}{3}}h-\frac{5}{11},\ 4+h+\frac{5}{11}\right]\right\}
\end{aligned}$$

for any $(x,s)\in[1,2]\times\left[1,\frac{11}{10}\right]$ and $0<\varepsilon<\frac{5}{11}$. Given this, we will define

$$c_h' := \left(\frac{10}{11}\right)^{\frac{1}{3}}\min\left\{f_{TW_2}(u)\ :\ u\in\left[\left(\frac{10}{11}\right)^{\frac{5}{3}}+\left(\frac{10}{11}\right)^{\frac{1}{3}}\left(h-\frac{5}{11}\right),\ 4+\left(h+\frac{5}{11}\right)\right]\right\} \tag{4.3.3}$$

and letting $\kappa>0$ be as in Theorem 4.3.1,

$$c_h := \frac{\kappa}{4}\vee\left(\max\left\{f_{TW_2}(u)\ :\ u\in\left[\left(\frac{10}{11}\right)^{\frac{5}{3}}+\left(\frac{10}{11}\right)^{\frac{1}{3}}\left(h-\frac{5}{11}\right),\ 4+\left(h+\frac{5}{11}\right)\right]\right\}\right). \tag{4.3.4}$$

With these definitions in place, we see that for all choices of $h\in\mathbb{R}$, $\varepsilon\in\left(0,\frac{5}{11}\right)$, and (x,s) we have

$$\int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)} f_{TW_2}(u)du\ \geq\ \int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)}\left(\frac{11}{10}\right)^{\frac{1}{3}}c_h'du\ \geq\ 2s^{-\frac{1}{3}}\left(\frac{11}{10}\right)^{\frac{1}{3}}(c_h'\varepsilon)\ \geq\ 2c_h'\varepsilon,$$

and similarly that

$$\int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)} f_{TW_2}(u)du\ \leq\ \int_{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h-\varepsilon)}^{s^{-\frac{5}{3}}x^2+s^{-\frac{1}{3}}(h+\varepsilon)} c_h du\ =\ (2c_h)s^{-\frac{1}{3}}\varepsilon\ \leq\ 2c_h\varepsilon.$$

This proves that the definitions of $c_h, c_h'>0$ in equations (4.3.3) and (4.3.4) satisfy the requirements in (4.1.2) and (4.1.1) of Theorem 4.1.1. In the language of our general strategy, the parameters $a_1,\dots,a_m, b_1,\dots b_m$, x_0, t_0, δ_x, and δ_t that appear in our two-point bound (4.3.1) are

$$(a_1,a_2,a_3)=\left(0,0,\frac{1}{2}\right)\ ,\ (b_1,b_2,b_3)=\left(0,-\frac{1}{2},-\frac{2}{3}\right)\ ,\ (x_0,t_0)=(1,1)\ ,\ \text{and}\ (\delta_x,\delta_t)=\left(1,\frac{1}{10}\right).$$

We also note that this means that the threshold $\beta>0$ becomes

$$\beta = 2\wedge\left(\min_{i\in\{1,2,3\}} a_i+b_i+2\right)=\frac{3}{2}.$$

Thus, in conjunction with Theorem 4.3.1, the conclusion of Theorem 4.1.1 yields that

$$\mathbb{P}\left(\dim_H(Z_h)\geq\frac{3}{2}\right)\geq p_h=\frac{(c_h')^2}{64c_h}\left(\sum_{i=1}^{3}\frac{\delta_x^{a_i}\delta_t^{b_i}}{(a_i+1)(b_i+1)}\right)^{-1}=\frac{(c_h')^2}{64c_h\left(1+2\sqrt{10}+2\sqrt[3]{100}\right)}>0.$$

□

Chapter 5

A Partial Two-Point Bound for $\mathcal{L}(0,0;\cdot,\cdot)$: The First Regime

With our ultimate goal now clearly in mind, we proceed to the long task of proving the existence of our partial two-point bound for $\mathcal{L}(0,0;\cdot,\cdot)$. Throughout the course of these arguments, we will work with the rectangle $[1,2]\times[1,\frac{11}{10}]$ for simplicity/convenience, but the subsequent arguments will also generalize to any box of the form $[u,u+\delta_u]\times[v,v+\delta_v]$ provided that $u,v>0$ and that numerous constants, bounds, and inequalities are adjusted accordingly along the way. Note that because the directed landscape $\mathcal{L}$ is only real-valued on $\mathbb{R}^4_\uparrow$, $\mathbb{R}\times\mathbb{R}_{>0}$ is the only domain on which bounding

$$\mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),\ \mathcal{L}(0,0;y,t)\in(h-\varepsilon,h+\varepsilon)\Big)$$

would be meaningful and non-trivial. For technical reasons later on in the course of our arguments, the assumption that $0<\varepsilon<\varepsilon_0$ will be crucial for obtaining the specific polynomial density bound that we have.

We begin by remarking that if both $\mathcal{L}(0,0;x,s)$ and $\mathcal{L}(0,0;y,t)$ are close to h then intuitively they must also be quite close to one another. Recall that we are assuming without loss of generality that $s<t$. As such, we immediately obtain the elementary inequality

$$\begin{aligned}&\mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),\mathcal{L}(0,0;y,t)\in(h-\varepsilon,h+\varepsilon)\Big)\\ &\le\mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),|\mathcal{L}(0,0;y,t)-\mathcal{L}(0,0;x,s)|\le 2\varepsilon\Big).\end{aligned}\tag{5.0.1}$$

By using the metric composition law of $\mathcal{L}$, we also have that

$$\mathcal{L}(0,0;y,t)=\sup_{z\in\mathbb{R}}\mathcal{L}(0,0;z,s)+\mathcal{L}(z,s;y,t).$$

Moreover, as a consequence of the skew stationarity property of the directed landscape, we have for all fixed $x,s\in[1,2]$ and for all $z\in\mathbb{R}$ the equality in distribution

$$\mathcal{L}(0,0;z,s)\overset{d}{=}\mathcal{L}(0,0;z-x,s)+s^{-1}(2xz+x^2).$$

As such, we can rewrite the right-hand side of the inequality (5.0.1) as

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{L}(0,0;x,s) \in (h-\varepsilon, h+\varepsilon), \left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z,s;y,t) - \mathcal{L}(0,0;x,s)\right| \leq 2\varepsilon\right) \\
&= \mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z-x,s) + \mathcal{L}(z,s;y,t) - \mathcal{L}(0,0;0,s) + \frac{2xz-x^2}{s}\right| \leq 2\varepsilon\right\}\right) \\
&= \mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \mathcal{L}(z+x,s;y,t) - \mathcal{L}(0,0;0,s) + \frac{2xz+x^2}{s}\right| \leq 2\varepsilon\right\}\right). \qquad (5.0.2)
\end{aligned}$$

where we have defined the $(x,s), h, \varepsilon-$dependent event $A_{x,s,h,\varepsilon}$ as

$$A_{x,s,h,\varepsilon} := \left\{\left|\mathcal{L}(0,0;0,s) + \frac{3x^2}{s} - h\right| \leq \varepsilon\right\}.$$

The independent increment property of the directed landscape yields that $\mathcal{L}(0,0;z,s)$ and $\mathcal{L}(z,s;0,t)$ are independent, which is what allows us to legitimately use this sheer on every term except for $\mathcal{L}(z,s;y,t)$ above. In addition to this independence, by the temporal stationarity, spatial stationarity, and flip symmetry of $\mathcal{L}$ we also have that for all $z \in \mathbb{R}$,

$$\mathcal{L}(z,s;y,t) \stackrel{d}{=} \widetilde{\mathcal{L}}(0,0;z-y,t-s)$$

where $\widetilde{\mathcal{L}}$ an independent copy of the directed landscape. This allows us to rewrite the probability (5.0.2) once more as

$$\mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{z\in\mathbb{R}} \mathcal{L}(0,0;z,s) + \widetilde{\mathcal{L}}(0,0;z+x-y,t-s) - \mathcal{L}(0,0;0,s) + \frac{2xz+x^2}{s}\right| \leq 2\varepsilon\right\}\right). \qquad (5.0.3)$$

Now by the definition of the directed landscape, we have that for all $z,s,y,t \in \mathbb{R}$ the equality in distribution

$$\widetilde{\mathcal{L}}(0,0;z+(x-y),t-s) \stackrel{d}{=} (t-s)^{1/3}\mathfrak{A}_1\left(\frac{z+(x-y)}{(t-s)^{2/3}}\right) \qquad (5.0.4)$$

where $\mathfrak{A}_1$ is the top line of the parabolic Airy line ensemble. In order to obtain our (partial) two-point bound (4.3.1), we will condition on the location of the argmax of the sup in the second coordinate of (5.0.3) and then use a union bound to bound the probability of these events by an infinite sum of the probabilities of somewhat simpler events. In particular, we will partition $\mathbb{R}$ into the intervals of length $\sigma^{\frac{2}{3}} := (t-s)^{\frac{2}{3}}$

$$\left[i\sigma^{\frac{2}{3}} \pm \frac{\sigma^{\frac{2}{3}}}{2}\right] = \left[i\sigma^{\frac{2}{3}} - \frac{\sigma^{\frac{2}{3}}}{2}, i\sigma^{\frac{2}{3}} + \frac{\sigma^{\frac{2}{3}}}{2}\right]$$

where we allow i to range over all of $\mathbb{Z}$ and adopt the notational convention that $[a \pm b] := [a-b, a+b]$ for convenience. With this in mind, we are able to obtain the upper bound

$$\begin{aligned} &\mathbb{P}\Big(\mathcal{L}(0,0;x,s) \in (h-\varepsilon, h+\varepsilon), \mathcal{L}(0,0;y,t) \in (h-\varepsilon, h+\varepsilon)\Big) \\ &\leq \sum_{i \in \mathbb{Z}} \mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \mathcal{L}(0,0;z,s) + \widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma) - \mathcal{L}(0,0;0,s) + \frac{2xz+x^2}{s}\right| \leq 2\varepsilon\right\}\right) \end{aligned} \tag{5.0.5}$$

where, for ease of readability, we have also defined $\Delta := (x-y)$.

Based on our choice of the partition for $\mathbb{R}$, we will observe two distinct behaviours in the supremum in $A_{(x,s),(y,t),i,\varepsilon}$ as i ranges over $\mathbb{Z}$. One such reason for this distinction is that these behaviours partly correspond to whether or not the deterministic parabola "hidden" inside the terms from $\mathcal{L}$ appearing in the supremum above is positive or negative. This will be expounded upon in more detail in section 5.3.

At a high level, we will essentially have a parabola in the variable i with $x, s, \sigma^{-1}, \Delta-$dependent coefficients. Moreover, this parabola will open upwards and will always have a positive root and a negative root based on our domain for x and our choice of ε_0. Thus, in our analysis of the probabilities

$$\mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \mathcal{L}(0,0;z,s) + \widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma) - \mathcal{L}(0,0;0,s) + \frac{2xz+x^2}{s}\right| \leq 2\varepsilon\right\}\right)$$

the behaviour of the second event will be heavily influenced by where each $i \in \mathbb{Z}$ falls in relation to the zeros of this parabola. By analyzing the discriminant of this polynomial, finding an upper bound on its magnitude, and noting that σ^{-1} is the only unbounded parameter in the parabola's coefficients, we can therefore find a partition of $\mathbb{Z}$ of the form

$$\mathbb{Z} = \Big([-n(\sigma), n(\sigma)] \cap \mathbb{Z}\Big) \bigsqcup \Big([-n(\sigma), n(\sigma)]^C \cap \mathbb{Z}\Big)$$

for some function n of σ with $[-n(\sigma), n(\sigma)] \cap \mathbb{Z}$ containing the integers in between the zeros of this parabola. As a result, we will have that the parabola is strictly positive on the second subset of $\mathbb{Z}$ and will assume all of its negative values on the first set. Because of this, splitting our analysis into two distinct cases already feels quite natural.

The other, and much more significant, reason for splitting this task in two is that when we work on

$$[-n(\sigma), n(\sigma)] \cap \mathbb{Z},$$

we will only ever need to understand the behaviour of $\mathcal{L}(0,0;\cdot,\cdot)$ on a compact set, which greatly simplifies many of our computations and the types of tools that we need to develop. In particular, this is because there will always be a finite maximal distance between the arguments of

$\mathcal{L}(0,0;z,s)$, $\mathcal{L}(0,0;0,s)$, and $\mathcal{L}(0,0;z+\delta,\sigma)$. Conversely, when we work on the complementary region

$$[-n(\sigma), n(\sigma)]^C \cap \mathbb{Z},$$

z will belong to an interval that can be arbitrarily far away from 0, meaning that we will need to understand the behaviour of $\mathcal{L}(0,0;\cdot,\cdot)$, and hence the behaviour of $\mathfrak{A}_1$ on arbitrarily large intervals. This is a fundamentally very different task than the former, and so a different approach will be needed to elegantly manage the presence of these arbitrarily large gaps between the intervals that we actually care about.

It is important to note however that in a vacuum, this problem of large gaps is agnostic to how we define our cutoff point $n(\sigma)$. Given this information, it is therefore quite natural to choose the cutoff point $n(\sigma)$ in a way such that the analysis of the second regime of these probabilities, i.e. the regime where the intervals can be arbitrarily far apart, is simplified as much as possible. Thus, given the dependence of the event

$$\left\{ \left| \sup_{\frac{z}{\sigma^{2/3}} \in [i \pm 1/2]} \mathcal{L}(0,0;z,s) + \widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma) - \mathcal{L}(0,0;0,s) + \frac{2xz+x^2}{s} \right| \leq 2\varepsilon \right\}$$

on the behaviour of the hidden deterministic parabola, we will split $\mathbb{Z}$ in a way such that we are guaranteed that this parabola is strictly positive in the second regime.

Although there are an infinite number of valid choices for the cutoff point $N(\sigma)$, we will choose to use $n(\sigma) = 10\sigma^{-\frac{2}{3}}$. The relevance of having a factor of $\sigma^{-\frac{2}{3}}$ will become apparent throughout the course of our work, and the factor is simply chosen for convenience. A sharper constant factor would likely exists, but there is no additional benefit gained from optimizing that particular factor. This means that we will define our two distinct regimes to be when $|i| \leq 10\sigma^{-\frac{2}{3}}$ and when $|i| > 10\sigma^{-\frac{2}{3}}$. We also note that because we have defined $\sigma := (t-s)$, $\sigma \in \left[0, \frac{1}{10}\right]$. Importantly, we are only interested in deriving a bound when $\sigma > 0$, since $t \neq s$ by hypothesis in Theorem 4.1.1.

For the sake of readability we introduce the notation that for any $\lambda \neq 0$

$$\mathfrak{A}_1^{(\lambda)}(x) := \lambda^{1/3}\mathfrak{A}_1\left(\frac{x}{\lambda^{2/3}}\right). \tag{5.0.6}$$

With this convention in mind, we now have that by the definition of the directed landscape the equality in distribution

$$\mathcal{L}(0,0;x,s) \overset{d}{=} s^{1/3}\mathfrak{A}_1\left(\frac{x}{s^{2/3}}\right) = \mathfrak{A}_1^{(s)}(x).$$

As such, we may now rewrite each summand in (5.0.5) as

$$\mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \mathcal{L}(0,0;z,s)+\widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma)-\mathcal{L}(0,0;0,s)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right\}\right)$$
$$= \mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0)-h+\frac{3x^2}{s}\right|\le\varepsilon,\ \left|\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \mathfrak{A}_1^{(s)}(z)+\widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta)-\mathfrak{A}_1^{(s)}(0)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right). \tag{5.0.7}$$

Our goal will be to dominate these probabilities for each i by the probability of a certain random vector with a bounded density being in a box of area $8\varepsilon^2\sigma^{-\frac{1}{3}}$. The precise construction of this random vector will change as we switch from the first regime to the second regime, but will be quite similar in both cases. In order to construct these random vectors, however, we will need to develop a pair of absolute continuity statements building off of the work of Dauvergne in [8]. We introduce the first of these absolute continuity statements in the following section.

5.1 The First Airy Comparison Lemma

The following lemma builds off of Theorem 1.2.2 to dominate the law of $\mathfrak{A}_1$ on an interval of the form $[a-T,a+T]$ by a mean zero Gaussian random variable and an independent stochastic process with strong $T-$dependent upper and lower tail bounds. Importantly, this lemma also extends to law of $\mathfrak{A}_1^{(\lambda)}$ for any $\lambda>0$. It will be instrumental in developing bounds on probabilities of the form (5.0.7) in both the first and second regimes.

Lemma 5.1.1. *Let $a\in\mathbb{R}$ and let $T>\frac{1}{6}$. Let ℓ_a be the function on $\mathbb{R}$ defined by*

$$\ell_a(r)=(r-a)^2-r^2=-a(2r-a)=a^2-2ra$$

and let I_a denote the interval

$$I_a := [a-T,a+T]$$

Then there exists an absolute constant $c>0$, two $T-$dependent constants $c_1,c_2>0$, and a random function $(\mathcal{F}(r))_{r\in I_a}$ such that

$$\operatorname{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in I_a}\right)\le e^{cT^3}\operatorname{Law}\left(\left(\sqrt{2T}N+\left(\mathcal{F}(r)+l_a(r)\right)_{r\in I_a}\right)\right) \tag{5.1.1}$$

where N is a standard Gaussian independent of $\left(\mathcal{F}(r)\right)_{r\in I_a}$ and for all $m>0$,

$$\mathbb{P}\left(\sup_{r\in I_a}|\mathcal{F}(r)|\ge m\right)\le c_1e^{-c_2m^{\frac{3}{2}}}. \tag{5.1.2}$$

More generally, for any constant $\lambda>0$, let $\mathfrak{A}_1^{(\lambda)}$ be as in (5.0.6) and denote by $I_a^{(\lambda)}$ the interval

$$I_a^{(\lambda)} := \left[a\lambda^{\frac{2}{3}}-T\lambda^{\frac{2}{3}},\, a\lambda^{\frac{2}{3}}+T\lambda^{\frac{2}{3}}\right].$$

Then as a consequence of (5.1.1)*, there exists a random* $\lambda-$*dependent function* $(F^{(\lambda)}(r))_{r\in I_a^{(\lambda)}}$ *and an independent standard Gaussian* N *such that*

$$\text{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}\right) \leq e^{cT^3}\,\text{Law}\left(\left(\lambda^{\frac{1}{3}}\sqrt{2T}N + \left(\mathcal{F}^{(\lambda)}(r) + \lambda^{\frac{1}{3}} l_a(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_a^{(\lambda)}}\right), \tag{5.1.3}$$

and that for the same constants c_1 *and* c_2 *and all* $m > 0$*,*

$$\mathbb{P}\left(\sup_{r\in I_a^{(\lambda)}} \left|\lambda^{-\frac{1}{3}}\mathcal{F}^{(\lambda)}(r)\right| \geq m\right) \leq c_1 e^{-c_2 m^{\frac{3}{2}}}. \tag{5.1.4}$$

In particular, there exist random constants A *and* C *such that we may write*

$$\left(\mathcal{F}^{(\lambda)}(a\lambda^{\frac{2}{3}}+\delta)\right)_{\delta\in I_0^{(\lambda)}} \stackrel{d}{=} \left(\mathcal{W}(2\delta + 6T\lambda^{\frac{2}{3}}) + \lambda^{-\frac{1}{3}}A\delta + \lambda^{\frac{1}{3}}C\right)_{\delta\in I_0^{(\lambda)}} \tag{5.1.5}$$

where $\mathcal{W}$ *is a standard two-sided Brownian motion, and for all* $m > 0$*,*

$$\mathbb{P}\Big(|A| \geq m\Big) + \mathbb{P}\Big(|C| \geq m\Big) \leq 2c_1 e^{-c_2 m^{\frac{3}{2}}}. \tag{5.1.6}$$

No claims are made about the independence or lack thereof amongst $A, \mathcal{W}$ *and* C*.*

Proof. By the stationarity of the stationary Airy process, we have the equality in distribution

$$\Big(\mathfrak{A}_1(r)\Big)_{r\in[a\pm 3T]} \stackrel{d}{=} \left(\Big(\mathfrak{A}_1(r-a) + (r-a)^2\Big) - r^2\right)_{r\in[a\pm 3T]} = \Big(\mathfrak{A}_1(r) + r^2 - (r+a)^2\Big)_{r\in[0\pm 3T]}.$$

which can be thus be condensed into the equality of laws

$$\text{Law}\left((\mathfrak{A}_1(r))_{r\in[a\pm 3T]}\right) = \text{Law}\left((\mathfrak{A}_1(r) + r^2 - (r+a)^2)_{r\in[0\pm 3T]}\right).$$

By Theorem 1.2.2, there exists a diffusion parameter 2 Brownian bridge B on $[-3T, 3T]$ from 0 to 0 and an independent random affine function L on $[-3T, 3T]$ such that

$$\begin{aligned}\text{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in[a\pm 3T]}\right) &= \text{Law}\left(\Big(\mathfrak{A}_1(r) + r^2 - (r+a)^2\Big)_{r\in[0\pm 3T]}\right)\\ &\leq e^{216cT^3}\,\text{Law}\left(\Big(B(r) + L(r) + r^2 - (r+a)^2\Big)_{r\in[0\pm 3T]}\right)\end{aligned}$$

We note here that although we do have the option to apply Theorem 1.2.2 on the original interval $[a\pm 3T]$, it is better for our purposes to apply it on $[0\pm 3T]$. The main benefit to making this choice is that this isolates the dependency on the center of the interval a in a single deterministic parabolic term. In doing so, we see that any and all behaviours of L and B on $[0\pm 3T]$ now have absolutely no relation to the value of a. This will be quite useful later on.

Thus by restricting both $(\mathfrak{A}_1(r))_{r\in[a\pm 3T]}$ and $\big(B(r) + L(r) + r^2 - (r+a)^2\big)_{r\in[0\pm 3T]}$ to the middle thirds of their domains, we may use Lemma 2.0.5 with the parameters $\delta = \frac{1}{3}$ and $k = 2$ to conclude that

$$\begin{aligned}\mathrm{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in I_a}\right) &\le e^{216cT^3}\,\mathrm{Law}\left(\left(B(r)+L(r)+r^2-(r+a)^2\right)_{r\in[0\pm T]}\right)\\ &= e^{216cT^3}\,\mathrm{Law}\left(\left(\sqrt{2T}N+(B(r)-\sqrt{2T}N)+L(r)+r^2-(r+a)^2\right)_{r\in[0\pm T]}\right)\\ &= e^{216cT^3}\,\mathrm{Law}\left(\left(\sqrt{2T}N+(B(r-a)-\sqrt{2T}N)+L(r-a)+(r-a)^2-r^2\right)_{r\in I_a}\right)\\ &= e^{216cT^3}\,\mathrm{Law}\left(\left(\sqrt{2T}N+\mathcal{F}(r)+l_a(r)\right)_{r\in I_a}\right)\end{aligned}$$

with N a standard Gaussian independent of the process $(B-\sqrt{2T}N)_{r\in[0\pm T]}$, and where we have defined

$$\left(\mathcal{F}(r)\right)_{r\in I_a} := \left(\left(B(r-a)-\sqrt{2T}N\right)+L(r-a)\right)_{r\in I_a}. \tag{5.1.7}$$

Now that we have defined our process $(\mathcal{F}(r))_{r\in I_a}$, we will next prove that the tail bound (5.1.2) is true. We first observe that

$$\sup_{r\in I_a}|\mathcal{F}(r)| = \sup_{r\in[a\pm T]}\left|B(r-a)-\sqrt{2T}N+L(r-a)\right| \le \sup_{r\in[-T,T]}|B(r)|+\sqrt{2T}\,|N|+\sup_{r\in[-T,T]}|L(r)|. \tag{5.1.8}$$

Moreover, recalling that L is a straight line segment from $L(-3t)$ to $L(3t)$ we have the elementary bound

$$\sup_{r\in[-T,T]}|L(r)| \le \sup_{r\in[-3T,3T]}|L(r)| \le \max\{|L(-3t)|,|L(3t)|\} \le |L(-3t)|+|L(3t)|. \tag{5.1.9}$$

By invoking the bounds (1.2.1) and (1.2.2) in Theorem 1.2.2, we obtain in our case that

$$\mathbb{P}\Big(L(-3t)\wedge \text{Ł}(3t) < -m\Big) \le 2e^{-dm^3} \qquad \text{and} \qquad \mathbb{P}\Big(L(-3t)\vee L(3t) > m\Big) \le e^{-\frac{4}{3}m^{\frac{3}{2}}+cm^{\frac{5}{4}}} \tag{5.1.10}$$

for some $T-$dependent constants $c,d>0$. This means that by taking union bounds, combining equations (5.1.9) and (5.1.10) gives us a chain of inequalities

$$\begin{aligned}\mathbb{P}\left(\sup_{r\in[-T,T]}|L(r)|\ge 2m\right) &\le \mathbb{P}\Big(|L(-3t)|+|L(3t)|\ge 2m\Big) \le \mathbb{P}\Big(|L(-3t)|\ge m\Big)+\mathbb{P}\Big(|L(3t)|\ge m\Big)\\ &\le 2\mathbb{P}\Big(L(-3t)\wedge L(3t) < -m\Big)+2\mathbb{P}\Big(L(-3t)\vee L(3t)>m\Big)\\ &\le 2\left(e^{-\frac{4}{3}m^{\frac{3}{2}}+cm^{\frac{5}{4}}}+2e^{-dm^3}\right) \le c_1e^{-c_2m^{\frac{3}{2}}}\end{aligned} \tag{5.1.11}$$

for some $T-$dependent constants $c_1,c_2>0$. We now turn our attention to the Brownian bridge B. Since B is a Brownian bridge of diffusion parameter 2, we may write

$$\Big(B(r)\Big)_{r\in[-3T,3T]} \overset{d}{=} \left(\mathcal{W}(2r+6T)-\frac{r+3T}{6T}\mathcal{W}(12T)\right)_{r\in[-3T,3T]} \tag{5.1.12}$$

where $\mathcal{W}$ is a standard two-sided Brownian motion. Based on this decomposition in law we have the upper bound

$$\sup_{r\in[-T,T]} |B(r)| \leq |\mathcal{W}(12T)| + \sup_{r\in[0,6T]} |\mathcal{W}(2r)| = |\mathcal{W}(12T)| + \sup_{r\in[0,12T]} |\mathcal{W}(r)|\,.$$

In turn this, this implies the chain of inequalities

$$\begin{aligned}\mathbb{P}\left(\sup_{r\in[-T,T]} |B(r)| \geq 2m\right) &\leq \mathbb{P}\left(|\mathcal{W}(12T)| + \sup_{r\in[0,12T]} |\mathcal{W}(r)| \geq 2m\right)\\ &\leq \mathbb{P}\Big(|\mathcal{W}(12T)| \geq m\Big) + \mathbb{P}\left(\sup_{r\in[0,12T]} |\mathcal{W}(r)| \geq m\right)\\ &= \mathbb{P}\Big(|\mathcal{W}(12T)| \geq m\Big) + \mathbb{P}\left(\sup_{r\in[0,12T]} \mathcal{W}(r) \geq m\right) + \mathbb{P}\left(\sup_{r\in[0,12T]} -\mathcal{W}(r) \geq m\right)\\ &= \mathbb{P}\Big(|\mathcal{W}(12T)| \geq m\Big) + 2\mathbb{P}\left(\sup_{r\in[0,12T]} \mathcal{W}(r) \geq m\right)\\ &= 3\mathbb{P}\Big(|\mathcal{W}(12T)| \geq m\Big)\\ &\leq 6e^{-m^2/(24t)} \end{aligned} \tag{5.1.13}$$

using the fact that $-\mathcal{W} \overset{d}{=} \mathcal{W}$, the known distribution for the running maximum of a Brownian motion, and the standard Gaussian concentration bound. As such, by combining equations (5.1.8), (5.1.11), and (5.1.13), we see that we can find positive $T-$dependent constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\mathbb{P}\Big(\sup_{r\in I_a} |\mathcal{F}(r)| \geq m\Big) &\leq \mathbb{P}\Big(\sup_{r\in[-T,T]} |B(r)| \geq m/3\Big) + \mathbb{P}\Big(\sqrt{2T}|N| \geq m/3\Big) + \mathbb{P}\Big(\sup_{r\in[-T,T]} |L(r)| \geq m/3\Big)\\ &\leq c_1 e^{-c_2 m^{3/2}}\end{aligned}$$

as claimed in equation (5.1.2). The extension of our result to the rescaled Airy process $\mathcal{A}_1^{(\lambda)}$ in (5.1.3) is an immediate consequence of the base case when $\lambda = 1$. To see this explicitly, we observe that

$$\begin{aligned}\text{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}\right) &= \text{Law}\left(\left(\lambda^{\frac{1}{3}}\mathfrak{A}_1(r)\right)_{r\in I_a}\right)\\ &\leq e^{216cT^3}\,\text{Law}\left(\left(\lambda^{\frac{1}{3}}\sqrt{2T}N + \lambda^{\frac{1}{3}}\mathcal{F}(r) + \lambda^{\frac{1}{3}} l_a(r)\right)_{r\in[a\pm T]}\right)\\ &= e^{216cT^3}\,\text{Law}\left(\left(\lambda^{\frac{1}{3}}\sqrt{2T}N + \lambda^{\frac{1}{3}}\mathcal{F}(r\lambda^{-2/3}) + \lambda^{\frac{1}{3}} l_a(r\lambda^{-2/3})\right)_{r\in[a\lambda^{2/3}\pm T\lambda^{2/3}]}\right)\\ &= e^{216cT^3}\,\text{Law}\left(\left(\lambda^{\frac{1}{3}}\sqrt{2T}N + \mathcal{F}^{(\lambda)}(r) + \lambda^{\frac{1}{3}} l_a(r\lambda^{-2/3})\right)_{r\in I_a^{(\lambda)}}\right)\end{aligned}$$

where we have defined the rescaled random function $\left(\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}$ by

$$\left(\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}} = \left(\lambda^{\frac{1}{3}}\left(B(r\lambda^{-2/3} - a) - \sqrt{2T}N + L(r\lambda^{-2/3} - a)\right)\right)_{r\in I_a^{(\lambda)}} \tag{5.1.14}$$

via our original definition in equation (5.1.7). Equation (5.1.4) then follows from (5.1.2) and the fact that

$$\left(\lambda^{-\frac{1}{3}}\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}} = \left(\mathcal{F}(r)\right)_{r\in I_a}.$$

We now turn our attention to establishing the decomposition in law in equation (5.1.5). Given the definition of $\left(\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}$ above, we first recall that we may write

$$\left(L(r)\right)_{r\in[-3T,3T]} = \left(\frac{1}{6T}\Big(\mathfrak{L}_1(3T)-\mathfrak{L}_1(-3T)\Big)r + \frac{1}{2}\Big(\mathfrak{L}_1(-3T)-\mathfrak{L}_1(3T)\Big)\right)_{r\in[-3T,3T]}.$$

Similarly, equation (5.1.12) can be decomposed and rewritten in law as

$$\left(B(r)\right)_{r\in[-3T,3T]} \overset{d}{=} \left(\mathcal{W}(2r+6T) - \frac{1}{2}\mathcal{W}(12T) - \frac{1}{6T}\mathcal{W}(12T)r\right)_{r\in[-3T,3T]}.$$

As an immediate consequence of the above, we also have that

$$\left(\lambda^{\frac{1}{3}}B\left(\delta\lambda^{-\frac{2}{3}}\right)\right)_{\delta\in I_0^{(\lambda)}} \overset{d}{=} \left(\mathcal{W}\left(2\delta+6T\lambda^{\frac{2}{3}}\right) - \lambda^{\frac{1}{3}}\frac{1}{2}\mathcal{W}(12T)\lambda^{-\frac{1}{3}}\frac{\mathcal{W}(12T)}{6T}\delta\right)_{\delta\in[-3,3]}$$

where the first term was simplified by Brownian scaling. Using these decompositions and the explicit definition of $\left(\mathcal{F}^{(\lambda)}(r)\right)_{r\in I_a^{(\lambda)}}$ in (5.1.14), we have the equalities in distribution

$$\begin{aligned}\left(\mathcal{F}^{(\lambda)}\left(a\lambda^{2/3}+\delta\right)\right)_{\delta\in I_0^{(\lambda)}} &= \left(\lambda^{\frac{1}{3}}B\left(\delta\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}\sqrt{2T}N + \lambda^{\frac{1}{3}}L\left(\delta\lambda^{-\frac{2}{3}}\right)\right)_{\delta\in I_0^{(\lambda)}}\\ &\overset{d}{=} \left(\mathcal{W}\left(2\delta+6T\lambda^{\frac{2}{3}}\right) + \lambda^{-\frac{1}{3}}A\delta + \lambda^{\frac{1}{3}}C\right)_{\delta\in I_0^{(\lambda)}}\end{aligned}$$

where we have defined the random constants A and C as

$$\begin{aligned}A &:= \frac{1}{6T}\Big(\mathcal{W}(12T) + \mathfrak{L}_1(3T) - \mathfrak{L}_1(-3T)\Big)\\ C &:= \frac{1}{2}\Big(-\mathcal{W}(12T) + \mathfrak{L}_1(-3T) - \mathfrak{L}_1(3T)\Big) - \sqrt{2T}N.\end{aligned}$$

Observing that the triangle inequality gives us the two upper bounds

$$\begin{aligned}|A| &\le \frac{1}{6T}|\mathcal{W}(12T)| + \frac{1}{6T}\left|\mathfrak{L}_1(3T)-\mathfrak{L}_1(-3T)\right|\\ |C| &\le \frac{1}{2}|\mathcal{W}(12T)| + \frac{1}{2}\left|\mathfrak{L}_1(3T)-\mathfrak{L}_1(-3T)\right| + \sqrt{2T}|N|,\end{aligned}$$

the tail bounds in equation (5.1.6) follow immediately from the standard Gaussian concentration inequality and equation (1.2.3) of Theorem 1.2.2, after possibly redefining our original choice of the $T-$dependent constants $c_1, c_2 > 0$. This completes our proof. □

Remark. Note that although the constants $c, c_1, c_2 > 0$ are $T-$dependent, if we only ever use values of T that are bounded above and below by absolute constants, we can take c, c_1, c_2 to actually be absolute constants as well without of loss of generality. The values of T that we choose to work with specifically will be continuous univariate functions of s, and since s lives in a finite interval, our choices of T will indeed be bounded by absolute constants. Thus, in the work that follows, we will implicitly optimize our choices of c, c_1, c_2 as functions of s to obtain absolute constants.

5.2 The Big Picture in the First Regime

Recall that the notation $\widetilde{\mathfrak{A}}_1^{(\sigma)}$ and $\mathfrak{A}_1^{(s)}$ is introduced in equation (5.0.6). As both terms appear in (5.0.7), we will use Lemma 5.1.1 individually on both processes. The application of Lemma 5.1.1 to the process $\widetilde{\mathfrak{A}}_1^{(\sigma)}$ appearing in the probability (5.0.7) will be fairly standard. Based on the domain of the supremum that we see in the second coordinate and the fact that the argument has a translation by $\Delta = (x-y)$, our first application of Lemma 5.1.1 will use the parameters $a = i - \Delta\sigma^{-\frac{2}{3}}, \lambda = \sigma$, and $T = \frac{1}{2}$. This corresponds to the absolute continuity statement

$$\begin{aligned}
&\operatorname{Law}\left(\left(\mathfrak{A}_1^{(\sigma)}(z+\Delta)\right)_{z\in[i\sigma^{2/3}\pm\sigma^{2/3}]}\right)\\
&= \operatorname{Law}\left(\left(\mathfrak{A}_1^{(\sigma)}(z)\right)_{z\in[(i\sigma^{2/3}+\Delta)\pm\sigma^{2/3}]}\right)\\
&= \operatorname{Law}\left(\left(\mathfrak{A}_1^{(\sigma)}(z)\right)_{z\in I_a^{(\sigma)}}\right)\\
&\le e^c \operatorname{Law}\left(\left(\sigma^{\frac{1}{3}}\sqrt{2T}N + \left(\mathcal{F}^{(\sigma)}(z) + \sigma^{\frac{1}{3}}\ell_{i+\Delta\sigma^{-\frac{2}{3}}}(z\sigma^{-\frac{2}{3}})\right)\right)_{z\in[(i\sigma^{2/3}+\Delta)\pm\sigma^{2/3}]}\right)\\
&\le e^c \operatorname{Law}\left(\left(\sigma^{\frac{1}{3}}N + \left(\mathcal{F}^{(\sigma)}(z+\Delta) + \sigma^{\frac{1}{3}}\ell_{i+\Delta\sigma^{-\frac{2}{3}}}((z+\Delta)\sigma^{-\frac{2}{3}})\right)\right)_{z\in[i\sigma^{2/3}\pm\sigma^{2/3}]}\right).
\end{aligned} \tag{5.2.1}$$

The application of Lemma 5.1.1 to the process $\mathfrak{A}_1^{(s)}$ is more complicated however since in the second coordinate of (5.0.7), $\mathfrak{A}_1^{(s)}$ does not naturally appear on an interval whose length is of order $s^{\frac{2}{3}}$. Moreover, there is a mismatch between the scale of the parabolic Airy values and the domain of the supremum. We also need the interval that we apply Lemma 5.1.1 on to contain every value of $\mathfrak{A}_1^{(s)}$ that appears in the two coordinates. Because of this, extra care is needed when selecting the parameters a and T during this application of our lemma.

The first coordinate of (5.0.7) necessitates a process dominating $\mathfrak{A}_1^{(s)}$ near 0. This poses no problems. For the second coordinate of (5.0.7), we need to dominate $\mathfrak{A}_1^{(s)}$ over an interval of scale $s^{\frac{2}{3}}$ which contains every value in the interval $[i\sigma^{\frac{2}{3}} \pm \frac{1}{2}\sigma^{\frac{2}{3}}]$. However, σ will eventually take on every value in $(0, \frac{1}{10}]$ so an interval whose length is simply a multiple of $\sigma^{\frac{2}{3}}$ would violate the hypothesis in Lemma 5.1.1 which requires that the parameter $T > \frac{1}{6}$. To avoid this incompatibility, we will apply our lemma to $\mathfrak{A}_1^{(s)}$ on an interval which is a superset of $[i\sigma^{\frac{2}{3}} \pm \frac{1}{2}\sigma^{\frac{2}{3}}]$ and whose length is a multiple of $s^{\frac{2}{3}}$.

Recalling that $\sigma \in (0, \frac{1}{10}]$ and that $|i| \le 10\sigma^{\frac{2}{3}}$ in the first regime, we know that every possible value that could appear in $[i\sigma^{\frac{2}{3}} \pm \frac{1}{2}\sigma^{\frac{2}{3}}]$ will belong to the interval $[-\frac{21}{2}, \frac{21}{2}]$. Moreover, because $s \in [1, \frac{11}{10}]$, $s^{-\frac{2}{3}} \in \left[\sqrt[3]{\frac{100}{121}}, 1\right]$. This means that we always have that $\frac{21}{2}s^{-\frac{2}{3}} > \frac{1}{6}$, making it a legitimate choice for the parameter T. However, we will take a slightly larger choice of T to simplify the value of $\sqrt{2T}$. With these considerations in mind, if we apply Lemma 5.1.1 to the process $\mathfrak{A}_1^{(s)}$ on the interval

$$I_0^{(s)} = [-16, 16] = \left[-s^{\frac{2}{3}}(16s^{-\frac{2}{3}}), s^{\frac{2}{3}}(16s^{-\frac{2}{3}})\right],$$

corresponding to the parameters $a = 0, \lambda = s$, and $T = 8s^{-\frac{2}{3}}$, we will indeed have a process dominating $\mathfrak{A}_1^{(s)}$ at every value in the first and second coordinates of (5.0.7). In particular, this will translate to the absolute continuity statement

$$\begin{aligned}
\text{Law}\left(\left(\mathfrak{A}_1^{(s)}(r)\right)_{r\in I_0^{(s)}}\right) &\leq e^{729cs^{-2}}\,\text{Law}\left(\left(s^{\frac{1}{3}}\sqrt{16s^{-\frac{2}{3}}}N+\left(\mathcal{F}^{(s)}(r)\right)\right)_{r\in[-16,16]}\right)\\
&\leq e^{729c/4}\,\text{Law}\left(\left(4N+\left(\mathcal{F}^{(s)}(r)\right)\right)_{r\in[-16,16]}\right) \qquad (5.2.2)
\end{aligned}$$

where we have also used that $\ell_0 \equiv 0$. Note that in the more general setting where s, t belong to an interval other than $[1, \frac{11}{10}]$, it is quite possible that a different choice for T would be more suitable. There are even other equally valid choices of T even in the simplified case we present here in the first regime. In either case however, this will not have a significant impact on the arguments that follow.

For the sake of readability, we will denote by (G, G_i) the random vector

$$\begin{aligned}
(G,G_i) =&\ (G(x,s), G_i(x,s,\Delta,\sigma))\\
:=&\ \left(\mathcal{F}^{(s)}(0)+\frac{3x^2}{s}\,,\ \sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]}\frac{\left(\mathcal{F}^{(s)}(z)-\mathcal{F}^{(s)}(0)+\mathcal{F}^{(\sigma)}\,(z+\Delta)\right)}{\sigma^{\frac{1}{3}}}+g_i(z)\right) \qquad (5.2.3)
\end{aligned}$$

where the deterministic function $g_i(z)$ is defined as

$$g_i(z) = g_i(z;x,s,\Delta,\sigma) := l_{i+\Delta\sigma^{-\frac{2}{3}}}\left(\frac{z+\Delta}{\sigma^{\frac{2}{3}}}\right)+\frac{2xz+x^2}{s\sigma^{\frac{1}{3}}} \qquad (5.2.4)$$

for each i in the first regime. Recalling that we set $\ell_a(r) = (r-a)^2 - r^2 = a^2 - 2ra$, we note that

$$\begin{aligned}
l_{i+\Delta\sigma^{-\frac{2}{3}}}\left(\frac{z+\Delta}{\sigma^{\frac{2}{3}}}\right) =&\ (i+\Delta\sigma^{-\frac{2}{3}})^2 - 2\sigma^{-\frac{2}{3}}(z+\Delta)(i+\Delta\sigma^{-\frac{2}{3}})\\
=&\ i^2 - 2iz\sigma^{-\frac{2}{3}} - 2\Delta\sigma^{-\frac{4}{3}}(z-\Delta). \qquad (5.2.5)
\end{aligned}$$

for completeness. We will next define the bivariate Gaussian vector $\left(N', \tilde{N}\right)$ by

$$\left(N',\tilde{N}\right) := \left(4N,\tilde{N}\right).$$

With these choices in mind, equation (5.2.1), equation (5.2.2), and basic measure theory allow us to extend the upper bound in probability (5.0.7) and write that

$$\begin{aligned}
&\mathbb{P}\left(A_{x,s,h,\varepsilon} \cap \left\{\left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]} \mathcal{L}(0,0;z,s)+\widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma)-\mathcal{L}(0,0;0,s)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right\}\right)\\
&= \mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0)-h+\frac{3x^2}{s}\right|\le\varepsilon, \left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]} \mathfrak{A}_1^{(s)}(z)+\widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta)-\mathfrak{A}_1^{(s)}(0)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right)\\
&= \mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0)-h+\frac{3x^2}{s}\right|\le\varepsilon, \left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]} \frac{\mathfrak{A}_1^{(s)}(z)+\widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta)-\mathfrak{A}_1^{(s)}(0)+\frac{2xz+x^2}{s}}{\sigma^{\frac{1}{3}}}\right|\le 2\varepsilon\sigma^{-\frac{1}{3}}\right)\\
&\le \kappa_1^2\mathbb{P}\left(\left(N'+G,\tilde{N}+G_i\right)\in\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]\right)
\end{aligned} \tag{5.2.6}$$

where $\kappa_1=\max\{e^{729c/4},e^c\}=e^{729c/4}$. Furthermore, with this setup we also have that $\left(N',\tilde{N}\right)$ and $\left(G,G_i\right)$ are independent random vectors. This independence in equation (5.2.6) will be instrumental in establishing the overall two-point bound that we desire. Before proceeding further, we now recall a simple but extremely useful fact from measure theory, which will guarantee that the sum of these two random vectors has a density with respect to the Lebesgue measure.

Lemma 5.2.1. *Let μ and ν be independent finite measures on $\mathbb{R}^n$. Let the random measure μ be absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative f_μ. Then, the measure $\mu*\nu$ is also absolutely continuous with respect to the Lebesgue measure and*

$$\|f_{\mu*\nu}\|_\infty \le \|f_\mu\|_\infty \nu(\mathbb{R}^n),$$

where $f_{\mu*\nu}$ is the Radon–Nikodym derivative of the measure $\mu*\nu$ with respect to the Lebesgue measure.

Proof. We begin by proving that $\mu*\nu$ is absolutely continuous with respect to the Lebesgue measure. We denote the Lebesgue measure as $|\cdot|$. Take A a Lebesgue measurable set such that $|A|=0$. Then,

$$\mu*\nu(A)=\iint \mathbb{1}_A(x+y)d\mu(x)d\nu(y)=\iint \mathbb{1}_{A-y}(x)f_\mu(x)dxd\nu(y).$$

The Lebesgue measure is invariant under translation so $|A-y|=|A|=0$. This implies that the inner integral in the right hand side expression above is 0.

We have proved that $\mu*\nu \ll |\cdot|$ so there exists an integrable function $f_{\mu*\nu}$ on $\mathbb{R}^n$ such that

$$\mu*\nu(A)=\int_A f_{\mu*\nu}(x)dx$$

for all Borel sets A in $\mathbb{R}^n$. To prove that the Radon-Nikodym derivative $f_{\mu*\nu}$ is bounded it suffices to show that

$$\mu*\nu(A)\le\|f_\mu\|_\infty\nu(\mathbb{R}^n)|A|$$

for all Borel sets A in $\mathbb{R}^n$. Let A be a Borel set in $\mathbb{R}^n$. Then, as before,

$$\mu * \nu(A) = \iint \mathbb{1}_{A-y}(x) f_\mu(x) dx d\nu(y).$$

We bound the function f_μ with its norm and get an upper bound on the right-hand side as follows:

$$\int \int_{A-y} \|f_\mu\|_\infty dx d\nu(y) = \|f_\mu\|_\infty \int |A| d\nu(y) = \|f_\mu\|_\infty \nu(\mathbb{R}^n)|A|$$

where again we have used the fact that the Lebesgue measure is invariant under translation. □

Since $\left(N', \tilde{N}\right)$ is a bivariate Gaussian vector (with independent components), Lemma 5.2.1 guarantees that

$$\text{Law}\left(\left(N', \tilde{N}\right) + \left(G, G_i\right)\right)$$

will have a density ρ_i with respect to the Lebesgue measure for each i in the first regime. Given this, we can take the last line in our expression (5.2.6) one step further and write it as an integral

$$\mathbb{P}\left(\left(N' + G, \tilde{N} + G_i\right) \in \left[h \pm \varepsilon\right] \times \left[0 \pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]\right) = \iint_{\left[h\pm\varepsilon\right] \times \left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]} \rho_i(h_1, h_2) dh_1 dh_2. \quad (5.2.7)$$

We will now explain our general strategy for using these findings to start the construction of the bound in equation (4.3.1). Suppose that for all $(h_1, h_2) \in \left[h \pm \varepsilon\right] \times \left[0 \pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]$ we had a uniform bound of the form

$$\sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \rho_i\left(h_1, h_2\right) < \left(k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}\right),$$

where k' was a non-negative absolute constant independent of the choice of $(x, s), (y, t) \in [1,2] \times [1, \frac{11}{10}]$ (and hence also independent of σ and Δ). In this case, if we define for each i and for each $0 < \varepsilon < \varepsilon_0$ the probabilities

$$P_{i,\varepsilon} = \mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0) + \frac{3x^2}{s} - h\right| \le \varepsilon, \left|\sup_{z\sigma^{-\frac{2}{3}} \in \left[i \pm \frac{1}{2}\right]} \mathfrak{A}_1^{(s)}(z) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(z + \Delta) - \mathfrak{A}_1^{(s)}(0) + \frac{2xz + x^2}{s\sigma^{\frac{1}{3}}}\right| \le 2\varepsilon\right), \quad (5.2.8)$$

then by combining equations (5.0.1), (5.0.5), (5.0.7), (5.2.6), and (5.2.7) we would have that

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon);\mathcal{L}(0,0;y,t)\in(h-\varepsilon,h+\varepsilon)\Big)\\
&\le \mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),|\mathcal{L}(0,0;y,t)-\mathcal{L}(0,0;x,s)|\le 2\varepsilon\Big)\\
&\le \sum_{i\in\mathbb{Z}}\mathbb{P}\left(A_{x,s,h,\varepsilon}\cap\left\{\left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]}\mathcal{L}(0,0;z,s)+\widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma)-\mathcal{L}(0,0;0,s)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right\}\right)\\
&=\sum_{i\in\mathbb{Z}}\mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0)+\frac{3x^2}{s}-h\right|\le\varepsilon,\left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac{1}{2}\right]}\mathfrak{A}_1^{(s)}(z)+\widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta)-\mathfrak{A}_1^{(s)}(0)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right)\\
&\le \kappa_1^2\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\mathbb{P}\left(\Big(N'+G\Big)\in\Big[h\pm\varepsilon\Big],\ \Big(\tilde{N}+G_i\Big)\in\Big[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\Big]\right)+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=\kappa_1^2\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]}\rho_i(h_1,h_2)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=\kappa_1^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]}\left(\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\rho_i(h_1,h_2)\right)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&\le\kappa_1^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]}\left(k'+k'\sigma^{-\frac{1}{6}}+k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}\right)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac{1}{2}}\right)+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}.
\end{aligned}\tag{5.2.9}$$

This would reduce our remaining work to developing a similar upper bound for the tail sum in the second regime above. We will revisit this problem in the following section after verifying that such a bound $(k'+k'\sigma^{-\frac{1}{6}}+k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}})$ does exist.

5.3 The Internal Structure of G_i for Small i

Before we proceed further, we will take some time to better understand the nature of the random variables G_i, which will inform the manner in which we construct our density bounds on ρ_i. Due to the presence of the parabola $g_i(z)$ in the definition (5.2.3), there will be quite a few distinct cases in the first regime in our following strategy. We first note that by combining (5.2.2) with the decomposition in law (5.1.5), we may write that for all $z\in[-16,16]$ and $x\in[1,2]$,

$$\begin{aligned}
\sigma^{-\frac{1}{3}}\left(\mathcal{F}^{(s)}(z)-\mathcal{F}^{(s)}(0)\right)&\stackrel{d}{=}\sigma^{-\frac{1}{3}}\left(\mathcal{W}(2z+96)-\mathcal{W}(96)+s^{\frac{1}{3}}Az\right)\\
&\stackrel{d}{=}\sigma^{-\frac{1}{3}}\mathcal{W}(2|z|)+\left(\frac{s}{\sigma}\right)^{\frac{1}{3}}Az.
\end{aligned}$$

In particular, this means that for any $z\in\left[i\pm\frac{1}{2}\right]$ we will have

$$\sigma^{-\frac{1}{3}}\left(\mathcal{F}^{(s)}\left(\sigma^{\frac{2}{3}}z\right)-\mathcal{F}^{(s)}(0)\right)\stackrel{d}{=}\mathcal{W}(2|z|)+(s\sigma)^{\frac{1}{3}}Az.\tag{5.3.1}$$

We also note that both $\mathcal{W}$ and A have symmetric upper and lower tail bounds about 0, via (5.1.6) and the fact that $\mathcal{W}$ is a standard Brownian motion. Similarly, by recalling the absolute continuity statement (5.2.1) and the tail bound (5.1.4) we see that

$$\begin{aligned}
\mathbb{P}\left(\sigma^{-\frac{1}{3}}\mathcal{F}^{(\sigma)}(z+\Delta)\geq m\right)+\mathbb{P}\left(\sigma^{-\frac{1}{3}}\mathcal{F}^{(\sigma)}(z+\Delta)\leq -m\right) &= \mathbb{P}\left(\sigma^{-\frac{1}{3}}|\mathcal{F}^{(\sigma)}(z+\Delta)|\geq m\right)\\
&\leq \mathbb{P}\left(\sup_{z\sigma^{-\frac{2}{3}}\in[i\pm\frac{1}{2}]}\sigma^{-\frac{1}{3}}|\mathcal{F}^{(\sigma)}(z+\Delta)|\geq m\right)\\
&= \mathbb{P}\left(\sup_{z\in I_a^{(\sigma)}}\sigma^{-\frac{1}{3}}|\mathcal{F}^{(\sigma)}(z)|\geq m\right)\\
&\leq c_1\exp\left(-c_2 m^{\frac{3}{2}}\right) \qquad (5.3.2)
\end{aligned}$$

for all $z\in[i\sigma^{\frac{2}{3}}\pm\frac{1}{2}\sigma^{\frac{2}{3}}]$ and all $m\geq 0$. Now for each i in the first regime, define the $\mathbb{R}-$valued random variable $z_i^*=z_i^*(x,s;\Delta,\sigma)$ by

$$z_i^*=\arg\max_{z\in\left[i\pm\frac{1}{2}\right]}\frac{\left(\mathcal{F}^{(s)}\left(\sigma^{\frac{2}{3}}z\right)-\mathcal{F}^{(s)}(0)+\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}}z+\Delta\right)\right)}{\sigma^{\frac{1}{3}}}+g_i\left(\sigma^{\frac{2}{3}}z\right)$$

which is guaranteed to exist by the Extreme Value Theorem. If multiple maximizers exist, we will take $\sigma^{\frac{2}{3}}z_i^*$ to be the largest amongst them to make the choice unique. With this in mind, we can then write G_i more explicitly as

$$\begin{aligned}
G_i &\overset{d}{=} \frac{\left(\mathcal{F}^{(s)}\left(\sigma^{\frac{2}{3}}z_i^*\right)-\mathcal{F}^{(s)}(0)+\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}}z_i^*+\Delta\right)\right)}{\sigma^{\frac{1}{3}}}+g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)\\
&\overset{d}{=} \mathcal{N}\left(0,|2z_i^*|\right)+(s\sigma)^{\frac{1}{3}}Az_i^*+\frac{\left(\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}}z_i^*+\Delta\right)\right)}{\sigma^{\frac{1}{3}}}+g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)\\
&=: G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+g_i\left(\sigma^{\frac{2}{3}}z_i^*\right). \qquad (5.3.3)
\end{aligned}$$

Given this new more explicit decomposition of G_i we can now state the previously mentioned two distinct cases. In the first case where the parabola $g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)\leq 0$, we will have that G_i has the same distribution as a random variable $G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)$ with symmetric tail bounds about 0 plus a (random and dependent) negative parabola. This makes it harder for G_i to achieve large positive values so in this setting we expect to have strong upper tail bounds for G_i. Similarly, when the parabola $g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)\geq 0$, G_i will be distributed as a random variable with symmetric tail bounds about 0 plus a positive (random and dependent) parabola, making it harder to achieve large negative values. In this second case, we expect to have strong lower tail bounds for G_i.

For our purposes in particular, we will be interested in bounding the sets of values (keeping in mind that $0<\varepsilon<\varepsilon_0=\frac{1}{2}\min_{(x,s)}\frac{x^2}{s}=\frac{5}{11}$ by hypothesis)

$$\left\{\rho_i\left(h_1,h_2\right)\;\middle|\;h_1\in\mathbb{R},|h_2|\leq 2\varepsilon\sigma^{-\frac{1}{3}}\right\}$$

for each $i \in \mathbb{Z}$ with $|i| \leq 10\sigma^{-\frac{2}{3}}$.

In the work that follows, we will show that there exists a collection of random variables

$$\{\xi_i\}_{|i|\leq 10\sigma^{-\frac{2}{3}}}$$

such that the value $\rho_i(h_1, h_2)$ for each pair can be bounded above by either

$$\begin{aligned} &e^{-\frac{1}{8}\left(h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right)^2} + \mathbb{P}\left(G_i \geq \frac{h_2}{2}\right) \\ &= e^{-\frac{1}{8}\left(h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right)^2} + \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right)\end{aligned}$$

whenever $h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0$, or by

$$\begin{aligned} &e^{-\frac{1}{8}\left(h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right)^2} + \mathbb{P}\left(G_i \leq \frac{h_2}{2}\right) \\ &= e^{-\frac{1}{8}\left(h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right)^2} + \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \leq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right)\end{aligned}$$

whenever $h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \leq 0$.

Because of this, rather than being interested in just the parabola $g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$, we are actually more interested in the behaviour of $h_2 - g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$. In doing so, we can more easily leverage the tail bounds of $G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right)$ about 0 in the two probabilities above. For the remainder of this section, we will be operating under the assumption that $|h_2| \leq 2\varepsilon\sigma^{-\frac{1}{3}}$.

We will now analyze the behaviour of the parabola $h_2 - g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$ as a function of $|i|$. We will begin with the simpler problem of understanding the explicit form of $h_2 - g_i\left(i\sigma^{\frac{2}{3}}\right)$ which by (5.2.4) simplifies to the (deterministic) parabola

$$h_2 - g_i\left(i\sigma^{\frac{2}{3}}\right) = i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x}{s}\sigma^{\frac{2}{3}}\right)i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} - h_2\right). \tag{5.3.4}$$

While there are in principle many distinct possible behaviours for this parabola, particularly depending on the relationship between Δ and σ, for reasons that will be discussed later on, all that will matter throughout the course of these arguments is that this parabola opens upwards and that the constant term is guaranteed to be strictly negative. The assumption that $\varepsilon < \varepsilon_0$ in particular is what ensures that the constant term is always strictly negative. We also note that this guarantees that for all i and all choices of $(x, s), (y, t)$ the parabola $h_2 - g_i\left(i\sigma^{\frac{2}{3}}\right)$ will always have a strictly positive root and a strictly negative root.

We now consider the more general form of $g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$ and how it differs from the case when $z_i^* = i$.

Using the fact that we can always express $z_i^* = i + p_i$ for some random $p_i \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ we can write

$$\begin{aligned} &h_2 - g_i\left(\sigma^{\frac{2}{3}} z_i^*\right) \\ &= h_2 - g_i\left(i\sigma^{\frac{2}{3}} + p_i\sigma^{\frac{2}{3}}\right) \\ &= i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} + 2p_i - \frac{2x}{s}\sigma^{\frac{2}{3}}\right) i - \left(2\Delta^2\sigma^{-\frac{4}{3}} - 2\Delta\sigma^{-\frac{2}{3}}p_i + \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{2x}{s}p_i\sigma^{\frac{1}{3}} - h_2\right) \\ &= h_2 - g_i\left(i\sigma^{\frac{2}{3}}\right) + \left(2\Delta\sigma^{-\frac{2}{3}} + 2i - \frac{2x}{s}\sigma^{\frac{1}{3}}\right) p_i \qquad (5.3.5) \\ &=: h_2 - g_i\left(i\sigma^{\frac{2}{3}}\right) - \xi_i. \end{aligned}$$

Thus, $h_2 - g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$ is the same deterministic parabola as before plus a randomly fluctuating degree 1 polynomial arising from the uncertainty in the location of z_i^*. Given this, by carefully manoeuvring around the random fluctuation ξ_i, it will be sufficient to understand the behaviour of the deterministic sequence of parabolas $g_i\left(\sigma^{\frac{2}{3}} i\right)$. With these observations, we can now say that for all i in the first regime, we will have the decomposition in law

$$G_i \stackrel{d}{=} G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + g_i\left(\sigma^{\frac{2}{3}} z_i^*\right) \stackrel{d}{=} G_i^*\left(\sigma^{\frac{2}{3}} i\right) + \xi_i + g_i\left(\sigma^{\frac{2}{3}} z_i^*\right).$$

It is this final reformulation of G_i in particular, where we have extracted and isolated all of its randomness into the random variables $G_i^*\left(\sigma^{\frac{2}{3}} i\right) + \xi_i$, that will enable us to efficiently find upper and lower tail bounds for the random variable G_i. In turn, as we previously claimed, those tail bounds will be essential in proving that the sum of densities ρ_i in the first regime is uniformly bounded. We will now use these newfound insights into G_i to build our density bounds on ρ_i.

5.4 Technical Lemma for Summing Exponential Series

Before proceeding to the construction of our density bounds, we introduce two technical lemmas about certain exponential sums. These lemmas will be crucial in understanding why the numerous sums that appear in the following sections all converge to absolute constants independently of the choice of $(x, s), (y, t)$.

Lemma 5.4.1. *Let $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ with $\beta_2 \le 0$. For each $i \in \mathbb{Z}_{\neq 0}$, define the sequence $(\Psi_i)_{i \in \mathbb{Z}_{\neq 0}}$ by*

$$\Psi_i := |i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3.$$

If $r \ge 1$, and $\gamma > 0$, then

$$\sum_{i=-\infty}^{-1} \left(e^{-\gamma(\Psi_i)^r}\right) \mathbb{1}_{\{\Psi_i \ge 0\}} + \sum_{i=1}^{\infty} \left(e^{-\gamma(\Psi_i)^r}\right) \mathbb{1}_{\{\Psi_i \ge 0\}} < \frac{2}{1 - e^{-\gamma}}.$$

Proof. First observe that for any $u \in [0, \infty)$ and any $r \ge 1$ it is always true that $u^r \ge u - 1$. This means that we will always have that

$$\sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i)^r}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} \leq \sum_{i=1}^{\infty}\left(e^{-\gamma(\Psi_i-1)}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} = e^{\gamma}\sum_{i=1}^{\infty}\left(e^{-\gamma\Psi_i}\right)\mathbb{1}_{\{\Psi_i\geq 0\}}$$

and similarly that

$$\sum_{i=-\infty}^{-1}\left(e^{-\gamma(\Psi_i)^r}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} \leq \sum_{i=-\infty}^{-1}\left(e^{-\gamma(\Psi_i-1)}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} = e^{\gamma}\sum_{i=-\infty}^{-1}\left(e^{-\gamma\Psi_i}\right)\mathbb{1}_{\{\Psi_i\geq 0\}}.$$

We begin with the case when $\beta_2 \leq 0$. If this is true then we have that

$$(\Psi_i)' = \left(i + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{i} + \beta_3\right)' = 1 - \frac{\beta_2}{i^2} \geq 1$$

for all $i > 0$. By expressing the increment $\Psi_{i+1} - \Psi_i$ as in integral, we then see that

$$\Psi_{i+1} = \Psi_i + (\Psi_{i+1} - \Psi_i) \geq \Psi_i + 1$$

for all $i > 0$. Now if we set $i_0 \in \mathbb{Z}_{>0}$ to be the minimal positive integer such that $\Psi_{i_0} \geq 0$ then we can say that

$$e^{\gamma}\sum_{i=1}^{\infty}\left(e^{-\gamma\Psi_i}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} = e^{\gamma}\sum_{i=i_0}^{\infty}e^{-\gamma\Psi_i} \leq e^{\gamma}\sum_{i=i_0}^{\infty}e^{-\gamma\left(\Psi_{i_0}+(i-i_0)\right)} \leq e^{\gamma}\sum_{i=i_0}^{\infty}e^{-\gamma(i-i_0)} = \frac{1}{1-e^{-\gamma}}.$$

Now consider the sum over the negative integers when $\beta \leq 0$. In this case, we see that

$$\Psi_i = -i - \beta_1 + \frac{\beta_2}{-i} + \beta_3$$

so if we make the change of variables $j = -i$ then

$$\Psi_i = j - \beta_1 + \frac{\beta_2}{j} + \beta_3 = \Psi_{-j}$$

with $j \geq 1$. For convenience, let $\zeta_j := \Psi_{-j}$. With this change in mind we see that

$$e^{\gamma}\sum_{i=-\infty}^{-1}\left(e^{-\gamma\Psi_i}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} = e^{\gamma}\sum_{j=1}^{\infty}\left(e^{-\gamma\zeta_j}\right)\mathbb{1}_{\{\zeta_j\geq 0\}}$$

and that as before, for all $j \geq 1$

$$(\zeta_j)' = \left(j - \beta_1 + \frac{\beta_2}{j} + \beta_3\right)' = 1 - \frac{\beta_2}{j^2} \geq 1$$

as was the case when $i \geq 1$. As such, if we set j_0 to the minimal $j \in \mathbb{Z}_{>0}$ such that $\zeta_j \geq 0$ then we have that by the exact same reasoning,

$$e^{\gamma}\sum_{i=-\infty}^{1}\left(e^{-\gamma\Psi_i}\right)\mathbb{1}_{\{\Psi_i\geq 0\}} = e^{\gamma}\sum_{j=1}^{\infty}\left(e^{-\gamma\zeta_j}\right)\mathbb{1}_{\{\zeta_j\geq 0\}} \leq e^{\gamma}\sum_{j=j_0}^{\infty}e^{-\gamma(j-j_0)} = \frac{1}{1-e^{-\gamma}}.$$

Putting everything together, this means that when $\beta_2 \leq 0$,

$$\sum_{i=-\infty}^{-1} \left(e^{-\gamma(\Psi_i)^r}\right) \mathbb{1}_{\{\Psi_i \geq 0\}} + \sum_{i=1}^{\infty} \left(e^{-\gamma(\Psi_i)^r}\right) \mathbb{1}_{\{\Psi_i \geq 0\}} < \frac{2}{1-e^{-\gamma}}.$$

as we originally claimed. □

This lemma will be used to bound terms when the parabola is positive. Note the equivalence of condition with condition on parabola. Mention that in our setup, this is why we introduce the constraint on ε_0 and do not allow x to ever be zero, as this is what ensures that the constants β_2 which actually appear in our work are always non-positive. We also prove an analogous technical lemma for the indices corresponding to the area in between the roots of the parabola.

Lemma 5.4.2. *Let $\beta_1, \beta_2, \beta_3\gamma, r \in \mathbb{R}$ with $\beta_2 \leq 0$. For each $i \in \mathbb{Z}_{\neq 0}$, define*

$$\Psi_i := |i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3.$$

If $\gamma > 0$ and $r \geq 1$, then

$$\sum_{i\in\mathbb{Z}} e^{-\gamma|\Psi_i|^r} \mathbb{1}_{\{\Psi_i < 0\}} < \frac{2}{1-e^{-\gamma}}.$$

Proof. This proof will be largely the same as that of the proof of the preceding lemma. We first observe that if $\beta_3 = 0$ then

$$|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} < 0 \iff i^2 + \beta_1 i + \beta_2 < 0 \iff \left| i + \frac{\beta_1}{2} \right| \leq \frac{\sqrt{\beta_1^2 + 4\beta_2}}{2}$$

meaning that we know exactly which integers are included in this sum. We will be including every integer beginning with the leftmost zero of the parabola $i^2 + \beta_1 i + \beta_2$ to the rightmost zero of the same parabola. Importantly, because the parabola opens upwards and has a negative vertical intercept, it must have both a negative zero and a positive zero. This means that our sum's index set will contain both positive and negative integers. Having a non-zero value of β_3 will translate and dilate this set of integers, but will not change the fact that it is always finite or that it will contain both negative and positive integers. We will now split our work into two cases as before.

When $i > 0$, since $\beta_2 \geq 0$ by hypothesis we see that

$$(\Psi_i)' = \left(i + \beta_1 + \frac{\beta_2}{i} + \beta_3\right)' = 1 - \frac{\beta_2}{i^2} \geq 1.$$

This means that for any $i > 0$, $\Psi_{i+1} \geq \Psi_i + 1$ or equivalently, $\Psi_i \leq \Psi_{i+1} - 1$. Set

$$i_{\max} := \max\{i \in \mathbb{Z}_{>0} : \Psi_i \leq 0\}.$$

By iterating this property until we reach $i_{\max}$, we see then that $\Psi_i \leq \Psi_{i_{\max}} - (i_{\max} - i) < 0$ for all

$0 < i \leq i_{\max}$. Then implies that

$$\begin{aligned}
\sum_{i=1}^{\infty} e^{\gamma|\Psi_i|^r} \mathbb{1}_{\{\Psi_i<0\}} \leq\ e^{\gamma} \sum_{i=1}^{\infty} e^{-\gamma|\Psi_i|} \mathbb{1}_{\{\Psi_i<0\}} &=\ e^{\gamma} \sum_{i=1}^{i_{\max}} e^{\gamma\Psi_i} \\
&\leq\ e^{\gamma} \sum_{i=1}^{i_{\max}} e^{\gamma(\Psi_{i_{\max}}-(i_{\max}-i))} \\
&\leq\ e^{\gamma} \sum_{i=1}^{i_{\max}} e^{\gamma(i-i_{\max})} e^{\gamma\Psi_{i_{\max}}} \\
&\leq\ e^{\gamma} \sum_{i=1}^{i_{\max}} e^{\gamma(i-i_{\max})} \\
&\leq\ e^{\gamma} \sum_{k=1}^{\infty} e^{-\gamma k}
\end{aligned}$$

since $e^{\gamma\Psi_{i_{\max}}} \leq 1$ by definition of $i_{\max}$.

The case when $i < 0$ is similar. If $i < 0$ then

$$(\Psi_i)' = \left(-i - \beta_1 - \frac{\beta_2}{i} + \beta_3\right)' = -1 + \frac{\beta_2}{i^2} \leq -1.$$

This means that for any $i < 0$, $\Psi_i \leq \Psi_{i-1} - 1$. Similarly to before, set

$$i_{\min} := \min\left\{i \in \mathbb{Z}_{<0} : \Psi_i \leq 0\right\}.$$

By iterating property above until we reach $i_{\min}$, we also have that $\Psi_i \leq \Psi_{i_{\min}} - (i - i_{\min}) < 0$ for all $i_{\min} \leq i < 0$. This then implies that

$$\begin{aligned}
\sum_{i=-\infty}^{-1} e^{\gamma|\Psi_i|^r} \mathbb{1}_{\{\Psi_i<0\}} \leq\ e^{\gamma} \sum_{i=-\infty}^{-1} e^{-\gamma|\Psi_i|} \mathbb{1}_{\{\Psi_i<0\}} &=\ e^{\gamma} \sum_{i=i_{\min}}^{-1} e^{\gamma\Psi_i} \\
&\leq\ e^{\gamma} \sum_{i=i_{\min}}^{-1} e^{\gamma\left(\Psi_{i_{\min}}-(i-i_{\min})\right)} \\
&\leq\ e^{\gamma} \sum_{i=i_{\min}}^{-1} e^{-\gamma(i-i_{\min})} e^{\gamma\Psi_{i_{\min}}} \\
&\leq\ e^{\gamma} \sum_{i=i_{\min}}^{-1} e^{\gamma(i_{\min}-i)} \\
&=\ e^{\gamma} \sum_{i=i_{\min}}^{-1} e^{\gamma i} \\
&\leq\ e^{\gamma} \sum_{k=-\infty}^{-1} e^{\gamma k}
\end{aligned}$$

since by definition, $e^{\gamma\Psi_{i_{\min}}} \leq 1$. Thus putting these two sums together we see that

$$\sum_{i\in\mathbb{Z}} e^{-\gamma|\Psi_i|^r}\mathbb{1}_{\{f(i)<0\}} \leq 2e^{\gamma}\sum_{k=1}^{\infty} e^{-\gamma k} = \frac{2}{1-e^{-\gamma}} < \infty.$$

□

With these two results in mind, we now have all the ingredients we will need in order to build good density bounds for each density ρ_i. We begin this process in the next section.

5.5 Density Bounds for ρ_i for Small i

Based on the preceding discussion, since we can now rewrite

$$\mathbb{P}\Big(G_i \geq h_2\Big) = \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + g_i\left(\sigma^{\frac{2}{3}}z_i^*\right) \geq h_2\right)$$

and likewise that

$$\mathbb{P}\Big(G_i \leq h_2\Big) = \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + g_i\left(\sigma^{\frac{2}{3}}z_i^*\right) \leq h_2\right).$$

We now see that, at least intuitively speaking, for any $h_2 \in \mathbb{R}$, G_i will only have good upper tail bounds when

$$h_2 - g_i\left(\sigma^{\frac{2}{3}}z_i^*\right) \geq 0$$

and similarly, will only ever have good lower tail bounds when

$$h_2 - g_i\left(\sigma^{\frac{2}{3}}z_i^*\right) \leq 0.$$

By recalling that the only source of randomness in the parabola $g_i\left(z_i^*\sigma^{\frac{2}{3}}\right)$ is the randomness introduced by p_i, we can isolate all the randomness by again writing the parabola as

$$g_i\left(z_i^*\sigma^{\frac{2}{3}}\right) = g_i\left(i\sigma^{\frac{2}{3}}\right) - \left(2\Delta\sigma^{-\frac{2}{3}} + 2i - \frac{2x}{s}\sigma^{\frac{1}{3}}\right)p_i = g_i\left(i\sigma^{\frac{2}{3}}\right) + \xi_i.$$

From here, if we let φ_i be the density of the random vector

$$\left(N', \tilde{N}\right) + \left(G, G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + \xi_i\right)$$

then we can say that for all i in the first regime,

$$\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]} \rho_i(h_1,h_2)dh_1dh_2 = \iint_{\left[h\pm\varepsilon\right]\times\left[-g_i\left(i\sigma^{\frac{2}{3}}\right)\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]} \varphi_i(h_1,h_2)dh_1dh_2.$$

This observation is elementary but by passing from the original densities ρ_i to the new densities φ_i, the structure and underlying behaviour of our density bounds will become easier to decipher.

We will now split the construction of our upper bounds for the densities φ_i into two distinct cases.

First, for all $h_2 \geq 0$ we will bound $\varphi_i(h_1,h_2)$ by

$$\begin{aligned}
&\varphi_i(h_1,h_2)\\
&= \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \in dh_2\Big)\\
&\leq \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \in dh_2 \,\Big|\, \tilde{N} \geq \frac{h_2}{2}\Big)\mathbb{P}\Big(\tilde{N} \geq \frac{h_2}{2}\Big) +\\
&\quad \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \in dh_2 \,\Big|\, G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \geq \frac{h_2}{2}\Big)\mathbb{P}\Big(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \geq \frac{h_2}{2}\Big).
\end{aligned} \tag{5.5.1}$$

This density bound we eventually obtain from this initial bound will be used to bound the supremum of the set of values

$$\left\{\varphi_i\left(h_1, h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right) \,\middle|\, h_1 \in \mathbb{R}, |h_2| \leq 2\varepsilon\sigma^{-\frac{1}{3}}\right\}$$

for each pair (i, h_2) such that

$$h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0.$$

Similarly, when $h_2 < 0$ we will instead use the density bound

$$\begin{aligned}
&\varphi_i(h_1,h_2)\\
&= \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*(\sigma^{\frac{2}{3}} z_i^*) + \xi_i \in dh_2\Big)\\
&\leq \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*(\sigma^{\frac{2}{3}} z_i^*) + \xi_i \in dh_2 \,\Big|\, \tilde{N} \leq \frac{h_2}{2}\Big)\mathbb{P}\Big(\tilde{N} \leq \frac{h_2}{2}\Big) +\\
&\quad \mathbb{P}\Big(N' + G \in dh_1, \tilde{N} + G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \in dh_2 \,\Big|\, G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \leq \frac{h_2}{2}\Big)\mathbb{P}\Big(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \leq \frac{h_2}{2}\Big).
\end{aligned}$$

This density bound we eventually obtain from this initial bound will be used to bound the supremum of the set of values

$$\left\{\rho_i\left(h_1, h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)\right) \,\middle|\, h_1 \in \mathbb{R}, |h_2| \leq 2\varepsilon\sigma^{-\frac{1}{3}}\right\}$$

for each pair (i, h_2) such that

$$h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0.$$

With this strategy in mind, let $h_2 \geq 0$ be arbitrary. Before proceeding further, we take a moment to recall that for all a, b satisfying $-\infty \leq a < b \leq \infty$ the probability measure

$$\mathrm{Law}\left(\tilde{N} \,\middle|\, \{a \leq \tilde{N} \leq b\}\right)$$

is the law of a truncated standard Gaussian random variable with lower truncation bound a and trivial upper truncation bound b. Crucially, this distribution has an explicit bounded density

$$f_{\tilde{N}|\{a\leq\tilde{N}\leq b\}}(u) = \frac{1}{\sqrt{2\pi}}\frac{\exp\left(-\frac{1}{2}u^2\right)\mathbb{1}_{\{a\leq u\leq b\}}}{\Phi\left(b\right) - \Phi\left(a\right)}$$

where Φ is the cumulative distribution function of a standard Gaussian. Note that if $a=-\infty$ and $b\le 0$ then

$$f_{\tilde N|\{\tilde N\le b\}}(u)\le \frac{e^{-\frac12 b^2}}{\Phi(b)}$$

whereas if $0\le a$ and $b=\infty$ then

$$f_{\tilde N|\{\tilde N\ge a\}}(u)\le \frac{e^{-\frac12 a^2}}{1-\Phi(a)}.$$

Using this, the independence of $(N',\tilde N)$ and $\left(G, G_i^*\left(\sigma^{\frac23} z_i^*\right)+\xi_i\right)$, defining for each $(h_1,h_2)\in\mathbb{R}^2$ and each $\delta>0$ the sets

$$E_{h_1,h_2,\delta}=[h_1,h_1+\delta]\times[h_2,h_2+\delta],$$

and letting $f_{(N',\tilde N)|\{\tilde N\ge \frac{h_2}{2}\}}$ be the density of

$$\mathrm{Law}\left(\left(N',\tilde N\right)\Big|\left\{\tilde N\ge \frac{h_2}{2}\right\}\right)*\mathrm{Law}\left(\left(G, G_i^*\left(\sigma^{\frac23} z_i^*\right)-\xi_i\right)\right)$$

we have that

$$\begin{aligned}
&\mathbb{P}\Big(N'+G\in dh_1, \tilde N+G_i^*\left(\sigma^{\frac23} z_i^*\right)+\xi_i\in dh_2 \;\Big|\; \tilde N\ge \frac{h_2}{2}\Big)\\
&=\lim_{\delta\to0^+}\frac{1}{\delta^2}\mathbb{P}\left(N'+G\in(h_1,h_1+\delta), \tilde N+G_i^*\left(\sigma^{\frac23} z_i^*\right)+\xi_i\in(h_2,h_2+\delta)\;\Big|\;\tilde N\ge\frac{h_2}{2}\right)\\
&=\lim_{\delta\to0+}\frac{1}{\delta^2}\mathrm{Law}\left((N',\tilde N)\;\Big|\;\left\{\tilde N\ge\frac{h_2}{2}\right\}\right)*\mathrm{Law}\left(\left(G, G_i^*(\sigma^{\frac23} z_i^*)+\xi_i\right)\right)(E_{h_1,h_2,\delta})\\
&=\lim_{\delta\to0+}\frac{1}{\delta^2}\mathrm{Law}\left((N', h_2/2\le \tilde N_{\le\infty})\right)*\mathrm{Law}\left(\left(G, G_i^*(\sigma^{\frac23} z_i^*)+\xi_i\right)\right)(E_{h_1,h_2,\delta})\\
&\le \lim_{\delta\to0+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}\left|\left|f_{(N',\tilde N)|\{\tilde N\ge\frac{h_2}{2}\}}\right|\right|_\infty(u,v)\,dudv\right) \qquad (5.5.2)\\
&\le \lim_{\delta\to0+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}||f_{N'}(u)||_\infty\left|\left|f_{\tilde N|\{\tilde N\ge\frac{h_2}{2}\}}(v)\right|\right|_\infty dudv\right)\\
&\le \lim_{\delta\to0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}\frac{\exp\left(-\frac18 h_2^2\right)}{1-\Phi\left(\frac{h_2}{2}\right)}dudv\right)\\
&=\frac{\exp\left(-\frac18 h_2^2\right)}{1-\Phi\left(\frac{h_2}{2}\right)}
\end{aligned}$$

where the inequality (5.5.2) comes from bounding the sup norm of the density of the convolution (which is guaranteed to exist by Lemma 5.2.1) by the sup norm of the joint density of

$$\mathrm{Law}\left(\left(N',\tilde N\right)\Big|\left\{\tilde N\ge\frac{h_2}{2}\right\}\right).$$

This is then bounded above by the product of the marginal densities, since the two components of this truncated bivariate Gaussian vector are still independent even after conditioning. We can now bound the first probability in the second product in the upper bound in (5.5.1) using essentially the same argument. Explicitly, we have that

$$\begin{aligned}
&\mathbb{P}\Big(N'+G\in dh_1, \tilde N + G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\in dh_2 \;\Big|\; G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq \frac{h_2}{2}\Big)\\
&=\lim_{\delta\to 0^+}\frac{1}{\delta^2}\mathbb{P}\Big(|N'+G-h_1|\leq\delta, |\tilde N+G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i-h_2|\leq\delta \;\Big|\; G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}\Big)\\
&=\lim_{\delta\to 0+}\frac{1}{\delta^2}\,\mathrm{Law}\left((N',\tilde N)+\left(G,G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\right)\;\Big|\; G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}\right)(E_{h_1,h_2,\delta})\\
&=\lim_{\delta\to 0+}\frac{1}{\delta^2}\,\mathrm{Law}\left((N',\tilde N)\right)*\mathrm{Law}\left(\left(G,G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\right)\;\Big|\; G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}\right)(E_{h_1,h_2,\delta})\\
&\leq\lim_{\delta\to 0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}1dudv\right)\\
&=1
\end{aligned}$$

where we use the fact that the sup norm of the joint density of the bivariate Gaussian $(N',\tilde N)$ is at most 1. Based on these two computations, we then arrive at an h_2-dependent upper bound

$$\begin{aligned}
\varphi_i(h_1,h_2) = \;&\mathbb{P}\Big(N'+G\in dh_1,\tilde N+G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\in dh_2\Big)\\
\leq\;& e^{-\frac{1}{8}h_2^2}+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}\right)
\end{aligned}$$

which holds at all points $(h_1,h_2)\in\mathbb{R}\times[0,\infty)$. As mentioned previously, by repeating this procedure for all $(h_1,h_2)\in\mathbb{R}\times(-\infty,0]$, we obtain analogously

$$\begin{aligned}
\varphi_i(h_1,h_2) = \;&\mathbb{P}\Big(N'+G\in dh_1,\tilde N+G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\in dh_2\Big)\\
\leq\;& e^{-\frac{1}{8}h_2^2}+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\leq\frac{h_2}{2}\right).
\end{aligned}$$

Overall, we have therefore shown that

$$\varphi_i(h_1,h_2)\leq e^{-\frac{1}{8}h_2^2}+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\leq\frac{h_2}{2}\right)\mathbb{1}_{\{h_2<0\}}+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}\right)\mathbb{1}_{\{h_2\geq 0\}}$$

for all $|i|\leq 10\sigma^{-\frac{3}{2}}$ and all $(h_1,h_2)\in\mathbb{R}^2$. Finally, by putting all the work in this section together, we have thus obtained the density bound

$$\begin{aligned}
&\rho_i(h_1,h_2)\\
&=\varphi_i\left(h_1,h_2-g_i(\sigma^{\frac{2}{3}}i)\right)\\
&\leq e^{-\frac{1}{8}\left(h_2-g_i(\sigma^{\frac{2}{3}}i)\right)^2}+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\leq\frac{h_2}{2}-\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2-g_i(\sigma^{\frac{2}{3}}i)<0\right\}}\\
&\quad+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}-\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2-g_i(\sigma^{\frac{2}{3}}i)\geq 0\right\}}
\end{aligned}\tag{5.5.3}$$

We restate here for convenience that, via equation (5.3.5), we have that

$$\begin{aligned} &\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i<\frac{h_2}{2}-\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2-g_i(\sigma^{\frac{2}{3}}i)<0\right\}} \qquad (5.5.4)\\ =&\,\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)-\left(2\Delta\sigma^{-\frac{2}{3}}+2i-\frac{2x}{s}\sigma^{\frac{1}{3}}\right)p_i<\frac{h_2}{2}-\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2-g_i(\sigma^{\frac{2}{3}}i)<0\right\}}. \end{aligned}$$

and likewise that

$$\begin{aligned} &\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\geq\frac{h_2}{2}+\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2+g_i(\sigma^{\frac{2}{3}}i)\geq0\right\}} \qquad (5.5.5)\\ =&\,\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)-\left(2\Delta\sigma^{-\frac{2}{3}}+2i-\frac{2x}{s}\sigma^{\frac{1}{3}}\right)p_i\geq\frac{h_2}{2}-\frac{g_i(\sigma^{\frac{2}{3}}i)}{2}\right)\mathbb{1}_{\left\{h_2-g_i(\sigma^{\frac{2}{3}}i)\geq0\right\}}. \end{aligned}$$

We will address the summability of the exponential terms for all $h_2\in[0\pm2\varepsilon\sigma^{-\frac{1}{3}}]$ here and deal with the other two summands separately in the following sections. Recall that as stated in the expansion (5.3.4),

$$h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)=i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x}{s}\sigma^{\frac{2}{3}}\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}-h_2\right)$$

so for all $i\neq0$ we will always have that

$$\left|h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\right|\geq\left|\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{|i|}\right|=\left||i|+\beta_1\operatorname{sign}(i)-\frac{\beta_2}{|i|}\right|=:|\Psi_i|,$$

where the constants $\beta_1,\beta_2\in\mathbb{R}$ are defined as

$$(\beta_1,\beta_2):=\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x}{s}\sigma^{\frac{2}{3}},\ -\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}-h_2\right)\right).$$

With this convention, we will always have that $\beta_2<0$ based on our definition of ε_0 and the requirement that $0<\varepsilon<\varepsilon_0$. As such, by invoking Lemma 5.4.1 and Lemma 5.4.2 with $r=2$ and $\gamma=\frac{1}{8}$, we can conclude that

$$\begin{aligned} \sum_{i=-\infty}^{\infty}e^{-\frac{1}{8}\left(h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\right)^2} &\leq 1+\sum_{i\neq0}e^{-\frac{1}{8}\left(h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\right)^2}\\ &\leq 1+\sum_{i\neq0}e^{-\frac{1}{8}(\Psi_i)^2}\mathbb{1}_{\{\Psi_i\geq0\}}+\sum_{i\neq0}e^{-\frac{1}{8}(|\Psi_i|)^2}\mathbb{1}_{\{\Psi_i<0\}}\\ &\leq 1+\frac{4}{1-e^{-\frac{1}{8}}}. \qquad (5.5.6) \end{aligned}$$

Thus we now need only be concerned about finding similar bounds for the series of upper and lower tail bounds of G_i. Doing this will complete our work for the first regime.

5.6 Managing the Fluctuation ξ_i for Small i

Note that while $G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)$ has known upper tail bounds (as mentioned in section 5.3), these bounds are centred at 0 specifically, and we are not able to meaningfully bound probabilities of the form

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) < m\right) \qquad \text{or} \qquad \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) > -m\right)$$

for $m > 0$ with the information available to us about the components inside $G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)$. However, because ξ_i is theoretically capable of having a larger magnitude than the parabola while having the opposite parity of the parabola for a large number of i for some choices of $(x,s),(y,t) \in [1,2]\times\left[1,\frac{11}{10}\right]$, the situation above would become unavoidable during our energy integral computation if we simply try to bound ξ_i from above or below by something deterministic in (5.5.4) and (5.5.5) without any thought. Fortunately, a workaround to this problem does exist.

We begin with the elementary observation that by definition of the argmax z_i^*,

$$G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + \xi_i = G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) - \left(2\Delta\sigma^{-\frac{2}{3}} + 2i - \frac{2x}{s}\sigma^{\frac{1}{3}}\right)(z_i^* - i) \geq G_i^*\left(\sigma^{\frac{2}{3}}i\right)$$

where we are also using the fact that

$$\begin{aligned}
z_i^* &= \underset{z\in[i\pm\frac{1}{2}]}{\arg\max}\left(G_i^*\left(\sigma^{\frac{2}{3}}z\right) + g_i\left(\sigma^{\frac{2}{3}}z\right)\right) \\
&= \underset{z\in[i\pm\frac{1}{2}]}{\arg\max}\left(G_i^*\left(\sigma^{\frac{2}{3}}z\right) + g_i\left(\sigma^{\frac{2}{3}}i\right) - \left(2\Delta\sigma^{-\frac{2}{3}} + 2i - \frac{2x}{s}\sigma^{\frac{1}{3}}\right)(z-i)\right) \\
&= \underset{z\in[i\pm\frac{1}{2}]}{\arg\max}\left(G_i^*\left(\sigma^{\frac{2}{3}}z\right) - \left(2\Delta\sigma^{-\frac{2}{3}} + 2i - \frac{2x}{s}\sigma^{\frac{1}{3}}\right)(z-i)\right) \\
&= \underset{z\in[i\pm\frac{1}{2}]}{\arg\max}\left(G_i^*\left(\sigma^{\frac{2}{3}}z\right) + \xi_i\right).
\end{aligned}$$

With this observation in tow, we can then bound equation (5.5.4) by

$$\begin{aligned}
&\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + \xi_i \leq \frac{h_2}{2} - \frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right)\mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) < 0\right\}} \\
&\leq \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}i\right) \leq \frac{h_2}{2} - \frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right)\mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) < 0\right\}}
\end{aligned}$$

whose tail bounds we will obtain in the following section.

What reminds is determining how to bound equation (5.5.5). We start by observing that

$$\xi_i = -2\left(\Delta\sigma^{-\frac{2}{3}} + i - \frac{x}{s}\sigma^{\frac{1}{3}}\right)p_i \leq |\Delta|\sigma^{-\frac{2}{3}} + |i| + 2$$

for all $|i| \leq 10\sigma^{-\frac{2}{3}}$ and all $(x,s),(y,t) \in [1,2]\times[1,\frac{11}{10}]$. As such, this means that we may bound equation (5.5.5) by

$$\begin{aligned}&\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i \geq \frac{h_2}{2}-\frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right)\mathbb{1}_{\left\{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\geq 0\right\}}\\ &\leq \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\left(\Delta\sigma^{-\frac{2}{3}}+|i|+2\right) \geq \frac{h_2}{2}-\frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right)\mathbb{1}_{\left\{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\geq 0\right\}} \qquad (5.6.1)\end{aligned}$$

which will have exponential tail bounds for all i in the first regime such that

$$h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\geq 0 \qquad \text{and} \qquad h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)-2\left(|\Delta|\sigma^{-\frac{2}{3}}+|i|+2\right)\geq 0.$$

When the second condition above fails to hold, our only recourse is to bound equation (5.6.1) by 1 for all such values of i. Thus we will now shift our attention to finding an upper bound on the number of integers i in the first regime such that

$$h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\geq 0 \qquad \text{and} \qquad h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)-2\left(|\Delta|\sigma^{-\frac{2}{3}}+|i|+2\right)< 0. \qquad (5.6.2)$$

By once again recalling the expansion in equation (5.3.4), we can write

$$\begin{aligned}&h_2-g_i\left(i\sigma^{\frac{2}{3}}\right)\\ &= i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x}{s}\sigma^{\frac{2}{3}}\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}-h_2\right)\geq 0\end{aligned}$$

which as a consequence gives us that

$$\begin{aligned}&h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)-2\left(\Delta\sigma^{-\frac{2}{3}}+|i|+1\right)\\ &= i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x}{s}\sigma^{\frac{2}{3}}\right)i-2|i|-\left(2\Delta^2\sigma^{-\frac{4}{3}}+2|\Delta|\sigma^{-\frac{2}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}-h_2+4\right)<0.\end{aligned}$$

How many i with $|i|\leq 10\sigma^{-\frac{2}{3}}$ satisfy both of the conditions above?

First suppose that $i<0$. In this case, for any choice of $(x,s)\in[1,2]\times[1,\frac{11}{10}]$ and $h_2\in\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]$, the solutions to the system of equations (5.6.2) will all solve the slightly weaker system of inequalities

$$i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-4\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}\right)\geq 0$$

and

$$i^2+\left(2\Delta\sigma^{-\frac{2}{3}}+2\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+2|\Delta|\sigma^{-\frac{2}{3}}+5\sigma^{-\frac{1}{3}}+4\right)<0.$$

Note that we are using the fact that for any functions f_1,f_2,f_3,f_4 from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1\leq f_2$ and $f_3\leq f_4$, we have the implication for all $x\in\mathbb{R}$ that

$$f_1(x)\geq 0 \quad \text{and} \quad f_4(x)<0 \implies f_2(x)\geq 0 \quad \text{and} \quad f_3(x)<0$$

meaning that $\{x \in \mathbb{R} : f_1(x) \geq 0 \text{ and } f_4(x) < 0\} \subseteq \{x \in \mathbb{R} : f_2(x) \geq 0 \text{ and } f_3(x) < 0\}$.

For simplicity, we first solve the analogous system of inequalities

$$i^2 + \left(2ba^2 - 4\right) i - \left(2b^2a^4\right) \geq 0$$
and
$$i^2 + \left(2ba^2 + 2\right) i - \left(2b^2a^4 + 2|b|a^2 + 5a + 4\right) < 0$$

where $2 \leq a$, $-1 \leq b \leq 1$, and $-10a^2 \leq i < 0$. Using Mathematica, there are 10 classes of integer solutions $-10a^2 \leq i < 0$ depending on the relationship between a and b. However, upon repeatedly using the fact that $|\sqrt{u} - \sqrt{v}| \leq \sqrt{|u-v|}$ to determine of the order of each such solution set in terms of powers of a and b, we can see that there exists an absolute constant $k > 0$ such that in all cases, there are no more than

$$k + ka^{\frac{1}{2}} + ka|b|^{\frac{1}{2}}$$

solutions for any choice of a, b. As such, by setting $a = \sigma^{-\frac{1}{3}}$, $b = \Delta$, and using the fact that $\sigma^{-1} \geq 10$, we see that (5.6.2) has no more than

$$k + k\sigma^{-\frac{1}{6}} + k\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}$$

integer solutions i such that $-10\sigma^{-\frac{2}{3}} \leq i < 0$. We also note that although we did not work out an explicit value for k, an explicit value for this absolute constant can be found, albeit with quite a bit of extremely tedious arithmetic.

Now consider the case where $0 \leq i \leq 10\sigma^{-\frac{2}{3}}$. In this setting, all integer solutions to the system of inequalities (5.6.2) will belong to the solution set of the weaker system

$$i^2 + \left(2\Delta\sigma^{-\frac{2}{3}}\right) i - \left(2\Delta^2\sigma^{-\frac{4}{3}}\right) \geq 0$$
and
$$i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} - 6\right) i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + 2|\Delta|\sigma^{-\frac{2}{3}} + 5\sigma^{-\frac{1}{3}} + 4\right) < 0.$$

As before, we consider the analogous system of inequalities

$$i^2 + \left(2ba^2\right) i - \left(2b^2a^4\right) \geq 0$$
and
$$i^2 + \left(2ba^2 - 6\right) i - \left(2b^2a^4 + 2|b|a^2 + 5a + 4\right) < 0$$

with $2 \leq a$, $-1 \leq b \leq 1$, and $0 \leq i \leq 10a^2$. Using Mathematica, there are at most

$$7 + 3a|b|^{\frac{1}{2}} + 3a^{\frac{1}{2}}$$

integer solutions $0 \leq i \leq 10a^2$ for any choice of a, b. As such, by setting $a = \sigma^{-\frac{1}{3}}$, $b = \Delta$, and using the fact that $\sigma^{-1} \geq 10$, we see that (5.6.2) has no more than

$$7 + 3\sigma^{-\frac{1}{6}} + 3\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}$$

integer solutions $0 \leq i \leq 10a^2$ for any choice of a, b.

Moreover, by combining these two cases we see that we can now say that for any pair of points $(x,s),(y,t) \in [1,2] \times [1, \frac{11}{10}]$, the system of inequalities (5.6.2) has no more than

$$k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}$$

integer solutions i with $|i| \leq 10\sigma^{-\frac{2}{3}}$. Hence, there are at most $k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}}$ indices i in the first regime that need to be discarded before we can ensure that the products in (5.6.1) always have exponentially decaying tail bounds.

Note that because we considered the worst case scenario for h_2, this same bound on the number of problematic indices i holds for all other choices of h_2 as well. The specific integer solutions will change as we vary the value of h_2 in general, but this will not impact our overall density bound b'.

Now for each $h_2 \in \left[0 \pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]$, define the subset of $\mathbb{Z}$

$$S(h_2) = S(h_2; x, s, \Delta, \sigma) := \left\{ i \in \mathbb{Z} \cap \left[0 \pm 10\sigma^{-\frac{2}{3}}\right] \ : \text{ the system of inequalities (5.6.2) is consistent} \right\}.$$

Based on the preceeding work in this section, $S(h_2)$ will be empty for all $h_2 \in \left[0 \pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]$ for most choices of $(x,s),(y,t) \in [1,2] \times [1, \frac{11}{10}]$, and for all choices of $(x,s),(y,t)$ that do have solutions, our upper bound on the cardinality of $S(h_2)$ will be independent of the choice of h_2. With this notation established, we can then say that whenever $h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0$,

$$\begin{aligned}
&\sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} \right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0\right\}} \\
&\leq \sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \phi_i(h_2)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0\right\}} \\
&\leq |S(h_2)| + \sum_{i \in (S(h_2))^C} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \phi_i(h_2)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \geq 0\right\}} \\
&\leq |S(h_2)| + \sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \phi_i(h_2)\right) \mathbb{1}_{\{\phi_i(h_2) \geq 0\}} \\
&\leq k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}} + \sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \phi_i(h_2)\right) \mathbb{1}_{\{\phi_i(h_2) \geq 0\}}. \qquad (5.6.3)
\end{aligned}$$

where for the sake of formatting we have temporarily set

$$\phi_i(h_2) = \phi_i\left(h_2; x, s, \Delta, \sigma\right) := \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + |i| + 1\right)$$

We conclude this portion of the argument by briefly summarizing our strategy for using (5.6.3) to build the overall density bound in (5.2.9). The idea will be to use Lemma 5.4.1 to find a positive absolute constant $b_1 > 0$ such that

$$\sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \ge \phi_i(h_2)\right) \mathbb{1}_{\{\phi_i(h_2) \ge 0\}} < \ b_1$$

and similarly, to use Lemma 5.4.2 to find a positive absolute constant $b_2 > 0$ such that

$$\sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} i\right) \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}} < \ b_2$$

Together, these will imply that

$$\begin{aligned}
&\sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}} \\
&+ \sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \ge \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) \ge 0\right\}} \\
&\qquad \le b_2 + \left(k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}} + b_1\right).
\end{aligned}$$

This will in turn mean that in conjunction with equations (5.5.3) and (5.5.6),

$$\begin{aligned}
\sum_{|i| \le 10\sigma^{-\frac{2}{3}}} \rho_i\left(h_1, h_2\right) &\le \ 1 + \frac{2}{1 - e^{-\frac{1}{8}}} + b_2 + \left(k' + k'\sigma^{-\frac{1}{6}} + k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}} + b_1\right) \qquad (5.6.4)\\
&\le \ k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac{1}{2}} + k'\sigma^{-\frac{1}{6}} + k'.
\end{aligned}$$

after redefining the absolute constant $k' > 0$. Once again, the precise value of k' can be computed explicitly, at least in principle, if need be. We will now establish the existence of these absolute constants b_1 and b_2 in the next section.

5.7 Tail Bounds for $G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right)$ and $G_i^*\left(\sigma^{\frac{2}{3}} i\right)$ for Small i

We first recall the decomposition in law in equation (5.3.3)

$$G_i \overset{d}{=} \mathcal{N}(0, 2|z_i^*|) + (s\sigma)^{\frac{1}{3}} A z_i^* + \frac{\left(\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}} z_i^* + \Delta\right)\right)}{\sigma^{\frac{1}{3}}} + g_i\left(\sigma^{\frac{2}{3}} z_i^*\right).$$

We will also define the three random variables $\{G^*_{i,j}(z^*_i)\}_{j=1}^3$ for $i \neq 0$ by

$$G^*_{i,1}(z^*_i) := \mathcal{N}(0, 2|z^*_i|)$$
$$G^*_{i,2}(z^*_i) := (s\sigma)^{\frac{1}{3}} A z^*_i$$
$$G^*_{i,3}(z^*_i) := \frac{\left(\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}} z^*_i + \Delta\right)\right)}{\sigma^{\frac{1}{3}}}.$$

We are excluding the case where $i = 0$ in particular because this will ensure that we are always able to safely divide by $|i|$ and z^*_i in our tail bounds. For the case of $i = 0$ specifically, we will just use the trivial tail bound of 1 for G_i's upper and lower tail bounds. With that said, we now move on to the $i \neq 0$ case.

We begin by deriving an upper bound for the sum

$$\sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G^*_i\left(\sigma^{\frac{2}{3}} z^*_i\right) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}}.$$

Suppose that $|i| \leq 10\sigma^{-\frac{2}{3}}$ and that $h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0$. First, by taking a union bound, we see that

$$\mathbb{P}\left(G^*_i\left(\sigma^{\frac{2}{3}} z^*_i\right) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}}$$
$$= \mathbb{P}\left(\sum_{j=1}^3 G^*_{i,j}(z^*_i) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}}$$
$$\leq \sum_{j=1}^3 \mathbb{P}\left(G^*_{i,j}(z^*_i) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}}.$$

Next, since $z_i^* \in \left[i \pm \frac{1}{2}\right]$ implies that $2|z_i^*| \leq 3|i|$, $G_{i,1}^*(z_i^*)$ has an upper tail bound

$$\begin{aligned}
&\mathbb{P}\left(G_{i,1}^*(z_i^*) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&= \mathbb{P}\left(\mathcal{N}(0, 2|z_i^*|) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&\leq \mathbb{P}\left(3|i||\mathcal{N}(0,1)| \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&\leq 2\exp\left(-\frac{1}{2}\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{18|i|} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{9|i|}\right)^2\right) \\
&= 2\exp\left(-\frac{1}{628}\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{|i|}\right)^2\right) \qquad (5.7.1)
\end{aligned}$$

via the standard Gaussian concentration inequality. For the second summand $G_{i,2}^*(z_i^*)$, we have by equation (5.1.6) of Lemma 5.1.1 the sequence of bounds

$$\begin{aligned}
&\mathbb{P}\left(G_{i,2}^*(z_i^*) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&= \mathbb{P}\left((s\sigma)^{\frac{1}{3}} A z_i^* \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&\leq \mathbb{P}\left(\left|s\sigma|^{\frac{1}{3}} |A z_i^*\right| \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&\leq \mathbb{P}\left(4|i||A| \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
&\leq \mathbb{P}\left(|A| \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{24|i|} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{12|i|}\right) \\
&\leq 2c_1 \exp\left(-c_2\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{24|i|} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{12|i|}\right)^{\frac{3}{2}}\right) \\
&= 2c_1 \exp\left(-\frac{c_2}{24\sqrt{24}}\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{|i|}\right)^{\frac{3}{2}}\right) \qquad (5.7.2)
\end{aligned}$$

where we have used the fact that $s \in [1, \frac{11}{10}]$ and that $\sigma \in (0, \frac{1}{10}]$. Finally, we observe that we have already explained in equation (5.3.2) that

$$\mathbb{P}\left(G^*_{i,3}(z^*_i) \geq m\right) = \mathbb{P}\left(\sigma^{-\frac{1}{3}} \tilde{\mathcal{F}}^{(\sigma)}\left(z^*_i \sigma^{\frac{2}{3}} + \Delta\right) \geq m\right) \leq c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right).$$

for all $m \geq 0$, so as a direct application of that previously stated bound,

$$\begin{aligned}
&\mathbb{P}\left(G^*_{i,3}(z^*_i) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right) \\
\leq & c_1 \exp\left(-c_2 \left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{6} - \frac{\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{3}\right)^{\frac{3}{2}}\right) \\
= & c_1 \exp\left(-\frac{c_2}{6\sqrt{6}}\left(h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right)^{\frac{3}{2}}\right).
\end{aligned} \tag{5.7.3}$$

Using the fact that (5.7.2) decays the slowest amongst equations (5.7.1), (5.7.2), and (5.7.3) as $|i| \to \infty$, there exist absolute constants $c'_1, c'_2 > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left(G^*_i(\sigma^{\frac{2}{3}} z^*_i) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}} \\
\leq\ & c'_1 \exp\left(-c'_2 \left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{|i|}\right)^{\frac{3}{2}}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \geq 0\right\}}
\end{aligned}$$

for all i in the first regime.

Recalling (5.3.4), we see that we may write

$$h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) = |i| + \beta_1 \operatorname{sign}(i) - 2 + \frac{\beta_2}{|i|}$$

where the constants $\beta_1, \beta_2 \in \mathbb{R}$ are defined as

$$(\beta_1, \beta_2) := \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x}{s}\sigma^{\frac{2}{3}},\ -\left(2\Delta^2\sigma^{-\frac{4}{3}} + 2\Delta\sigma^{-\frac{2}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} - h_2 + 2\right)\right).$$

Thus, by invoking Lemma 5.4.1 and observing that we always have that $\beta_2 \le 0$, we obtain that

$$\sum_{|i|\le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}} z_i^*) \ge \frac{h_2}{2} - \frac{g_i(\sigma^{\frac{2}{3}} i)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \ge 0\right\}}$$
$$\le \sum_{i=-\infty}^{\infty} \mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}} z_i^*) \ge \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \ge 0\right\}}$$
$$\le 1 + \sum_{i\ne 0} c_1' \exp\left(-c_2'\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right)}{|i|}\right)^{\frac{3}{2}}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + |i| + 1\right) \ge 0\right\}}$$
$$= 1 + \sum_{i\ne 0} c_1' \exp\left(-c_2'\left(|i| + \beta_1 \operatorname{sign}(i) - 2 + \frac{\beta_2}{|i|}\right)^{\frac{3}{2}}\right) \mathbb{1}_{\left\{|i| + \beta_1 \operatorname{sign}(i) - 2 + \frac{\beta_2}{|i|} \ge 0\right\}}$$
$$\le 1 + \frac{2}{1 - \exp(-c_2')}.$$

We now turn our attention to further bounding the inequality

$$\sum_{|i|\le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}}$$
$$\le \sum_{|i|\le 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} i\right) \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}}.$$

By mimicking the same general sequence of steps used to find the upper bound for the series of upper tail bonds, we arrive at a similar bound

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}}$$
$$\le \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} i\right) \le \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}}$$
$$\le c_1' \exp\left(-c_2'\left|\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{|i|}\right|^{\frac{3}{2}}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}}$$

for all $i \ne 0$ in the first regime, after possibly redefining the original choice of the absolute constants $c_1', c_2' > 0$. Thus, by updating the constants β_1, β_2 to be

$$(\beta_1, \beta_2) := \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x}{s}\sigma^{\frac{2}{3}},\ -\left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} - h_2\right)\right)$$

so that we may write

$$h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) = |i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|}$$

then since we always have that $\beta_2 \leq 0$, Lemma 5.4.2 yields that

$$
\begin{aligned}
&\sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i \leq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}} \\
&\leq \sum_{|i| \leq 10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} i\right) \leq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}} \\
&\leq 1 + \sum_{i \neq 0} c_1' \exp\left(-c_2' \left|\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{|i|}\right|^{\frac{3}{2}}\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) < 0\right\}} \\
&= 1 + \sum_{i \neq 0} c_1' \exp\left(-c_2' \left||i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|}\right|^{\frac{3}{2}}\right) \mathbb{1}_{\left\{|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} < 0\right\}} \\
&\leq 1 + \frac{2}{1 - \exp(-c_2')}.
\end{aligned}
$$

Thus, by setting $b_1 = b_2 = 1 + 2\Big(1 - \exp(-c_2')\Big)^{-1}$, the work above confirms the existence of the absolute constants $b_1, b_2 > 0$ in equation (5.6.4) and completes the argument for the first regime. We now proceed to the second regime in the following chapter.

Chapter 6

A Partial Two-Point Bound for $\mathcal{L}(0,0;\cdot,\cdot)$: The Second Regime

6.1 The Second Airy Comparison Lemma

Recall that in the first regime, because we knew that $|i| \leq 10\sigma^{-\frac{2}{3}}$, this meant that there was a maximal and finite interval $[a-T, a+T] \in \mathbb{R}$ containing both 0 and every possible interval $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$ simultaneously. This meant that we were able to use a single application of Lemma 5.1.1 that applied to every single probability $p_{i,\varepsilon}$ for each i in the first regime. Moreover, because this parameter T was bounded, this meant that the Radon-Nikodym derivative e^{cT^3} appearing in the application of this lemma remained finite.

This trick will no longer work in the second regime, since it is characterized by $|i| > 10\sigma^{-\frac{2}{3}}$. Since the same intervals $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$ can now be arbitrarily far from 0, and we will need to control the behaviour of $\mathfrak{A}_1^{(s)}$ on both these intervals and near 0, this is deeply problematic. This would cause the parameter T in each application of Lemma 5.1.1 to $p_{i,\varepsilon}$ to grow exponentially quickly to ∞ as $|i| \to \infty$. Recalling the definition of $p_{i,\varepsilon}$ in (5.2.8), this would make it *extremely* improbable that any bound we obtained on the tail sum

$$\sum_{|i|>10\sigma^{-\frac{2}{3}}} p_{i,\varepsilon}$$

using our original lemma would converge. In light of this observation, a new lemma is needed to deal with the law of $\mathfrak{A}_1^{(s)}$ for each of these large integers i in the second regime. Fortunately, due to the nature of the events $p_{i,\varepsilon}$, we only actually need to concern ourselves with the behaviour of $\mathfrak{A}_1^{(s)}$ near 0 and on the interval $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$. Because the intermediate values of $\mathfrak{A}_1^{(s)}$ are completely irrelevant to us, this makes Theorem 1.2.3 a perfect foundation upon which to build an analogue of Lemma 5.1.1 for our work in the second regime. We will now use the remainder of this section to prove that generalization of our original Airy comparison lemma.

Lemma 6.1.1. *Let $a_1, a_2 \in \mathbb{R}_{\neq 0}$ and $T > \frac{1}{6}$ such that $a_1 + 3T < a_2 - 3T$. Let $I_1 = [a_1 \pm T]$ and $I_2 = [a_2 \pm T]$. Let f_{a_j} be the linear function on $[a_j \pm 3T]$ satisfying*

$$f_{a_j}(a_j - 3T) = -(a_j - 3T)^2 \qquad \text{and} \qquad f_{a_j}(a_j + 3T) = -(a_j + 3T)^2.$$

Then there exists a constant $c \in \mathbb{R}_{>0}$ and random functions $\left(\left(\mathcal{F}_1(r)\right)_{r\in I_1}, \left(\mathcal{F}_2(r)\right)_{r\in I_2}\right)$ such that

$$\begin{aligned} &\operatorname{Law}\left(\left(\mathfrak{A}_1(r)\right)_{r\in I_1}, \left(\mathfrak{A}_1(r)\right)_{r\in I_2}\right) \\ \le\ & e^{cT^3} \operatorname{Law}\left(\left(\sqrt{2T}N_1 + (\mathcal{F}_1(r) + f_{a_1}(r))_{r\in I_1}\right), \left(\sqrt{2T}N_2 + (\mathcal{F}_2(r) + f_{a_2}(r))_{r\in I_2}\right)\right) \end{aligned} \tag{6.1.1}$$

where N_1 and N_2 are independent standard Gaussian random variables. Moreover, the pairs (N_1, N_2) and $\left((\mathcal{F}_1(r))_{r\in I_1}, (\mathcal{F}_2(r))_{r\in I_2}\right)$ are independent, and there exist $T-$dependent constants $c_1, c_2 > 0$ such that for each $j \in \{1,2\}$,

$$\mathbb{P}\left(\sup_{r\in I_j} |\mathcal{F}_i(r)| > m\right) \le c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \tag{6.1.2}$$

for all $m > 0$. More generally, for any $\lambda > 0$, let $\mathfrak{A}_1^{(\lambda)}$ be as in (5.0.6). Denote by $I_j^{(\lambda)}$ the interval

$$I_j^{(\lambda)} := \lambda^{2/3} I_j = \left[a_j\lambda^{2/3} - T\lambda^{2/3}, a_j\lambda^{2/3} + T\lambda^{2/3}\right].$$

Then as a consequence of (6.1.1), we have that

$$\begin{aligned} &\operatorname{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_1^{(\lambda)}}, \left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_2^{(\lambda)}}\right) \\ \le\ & e^{cT^3} \operatorname{Law}\left(\left(\left(\mathcal{F}_j^{(\lambda)}(r) + \lambda^{1/3}\left(\sqrt{2T}N_j + f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^{2}\right) \end{aligned} \tag{6.1.3}$$

with N_1 and N_2 as before. For each $j \in \{1,2\}$, $\mathcal{F}_j^{(\lambda)}$ is a $\lambda-$dependent random function such that for all $m > 0$,

$$\mathbb{P}\left(\sup_{r\in I_j^{(\lambda)}} \left|\lambda^{-1/3}\mathcal{F}_j^{(\lambda)}(r)\right| > m\right) \le c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \tag{6.1.4}$$

for the same $T-$dependent constants $c_1, c_2 > 0$.

We may also write for each $j \in \{1,2\}$ the random function $F_j^{(\lambda)}$ as

$$\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} \stackrel{d}{=} \left(\mathcal{W}_j\left(2r - 2(a_j - 3T)\lambda^{\frac{2}{3}}\right) + A_j\lambda^{-\frac{1}{3}}r + C_j\lambda^{\frac{1}{3}}\right)_{r\in I_j^{(\lambda)}} \tag{6.1.5}$$

where W_j is a standard Brownian motion, and A_j and C_j are random constants such that,

$$\mathbb{P}\left(|A_j| > m\right) \le c_1 \exp\left(-c_2 m^{\frac{3}{2}}\right) \qquad \text{and} \qquad \mathbb{P}\left(|C_j| > m\right) \le c_1 \exp\left(-c_2 \left(\frac{m}{|a_j|}\right)^{\frac{3}{2}}\right) \tag{6.1.6}$$

for all $m > 0$. There are no claims made about any independence amongst $\mathcal{W}_j, A_j$, and C_j.

Proof. By invoking Theorem 1.2.3 with $T_0 = 6T$ and $\boldsymbol{a} = (a_1 - 3T, a_2 - 3T)$, we have that

$$\begin{aligned}&\operatorname{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in[a_1\pm 3T]}, \Big(\mathfrak{A}_1(r)\Big)_{r\in[a_2\pm 3T]}\right)\\ \le\ & e^{216cT^3}\operatorname{Law}\left(\Big(B_1(r)+L_1(r)\Big)_{r\in[a_1\pm 3T]}, \Big(B_2(r)+L_2(r)\Big)_{r\in[a_2\pm 3T]}\right).\end{aligned}$$

Since I_j is the middle third of the interval $[a_j \pm 3T]$, we may invoke Lemma 2.0.5 with $k = 1$ and $\delta = \frac{1}{3}$ on $[a_j \pm 3T]$ for each $j \in \{1, 2\}$ to get the decomposition in law

$$\Big(B_j(r)\Big)_{r\in I_j} \overset{d}{=} \sqrt{T}N_j + \Big(B_j(r) - \sqrt{2T}N_j\Big)_{r\in I_j}$$

where N_j is a standard Gaussian, N_j is independent of the process $(B_j(r) - N_j)_{r\in I_j}$, and N_1 is independent of N_2. As such, we can now write that

$$\begin{aligned}&\operatorname{Law}\left(\Big(\mathfrak{A}_1(r)\Big)_{r\in I_1}, \Big(\mathfrak{A}_1(r)\Big)_{r\in I_2}\right)\\ \le\ & e^{216cT^3}\operatorname{Law}\left(\Big(\sqrt{2T}N_1 + (\mathcal{F}_1(r) + f_{a_1}(r)\Big)_{r\in I_1}, \Big(\sqrt{2T}N_2 + (\mathcal{F}_2(r) + f_{a_2}(r)\Big)_{r\in I_2}\right)\end{aligned}$$

where for each $j \in \{1, 2\}$ we have defined

$$\Big(\mathcal{F}_j(r)\Big)_{r\in I_j} := \Big(L_j(r) - f_{a_j}(r) + B_j(r) - \sqrt{2T}N_j\Big)_{r\in I_j}$$

and for $r \in [a_j \pm 3T]$, we have defined f_{a_j} by

$$f_{a_j}(r) = -\frac{(a_j+3T)-r}{6T}(a_j-3T)^2 - \frac{r-(a_j-3T)}{6T}(a_j+3T)^2.$$

This establishes (6.1.1) so all that remains is to establish (6.1.2). To that end, we employ the same general argument used in Lemma 5.1.1 previously, independently in each coordinate of (6.1.2).

We begin by observing the chain of inequalities

$$\begin{aligned}\sup_{r\in I_j}|\mathcal{F}_j(r)| &\le \sup_{r\in I_j}|L_j(r)+f_{a_j}(r)| + \sup_{r\in I_j}|B_j(r)| + |\sqrt{2T}N_j|\\ &\le |L_j(a_j-3T) - f_{a_j}(a_j-3T)| \vee |L_j(a_j+3T) - f_{a_j}(a_i+3T)| + \sup_{r\in I_j}|B_j(r)| + |\sqrt{2T}N_j|\\ &= |L_j(a_j-3T) + (a_j-3T)^2| \vee |L_j(a_j+3T) + (a_j+3T)^2| + \sup_{r\in I_j}|B_j(r)| + |\sqrt{2T}N_j|.\end{aligned} \tag{6.1.7}$$

Note that we are using the fact that because $L_j - f_{a_j}$ is a (random) line segment, its maximum absolute value is obtained at one of its two endpoints. We will now adopt the convention that for any $a \in \mathbb{R}$, $B_{a,6T}$ is a diffusion parameter 2 Brownian bridge on $[a, a + 6T]$ from 0 to 0. With this convention, we may write that

$$\Big(B_{0,6T}(r)\Big)_{r\in[0,6T]} \overset{d}{=} \Big(\mathcal{W}(2r) - \frac{r}{6T}\mathcal{W}(12T)\Big)_{r\in[0,6T]}$$

where $\mathcal{W}$ is a standard Brownian motion. Given this, we may then say that

$$\begin{aligned}
\mathbb{P}\left(\sup_{r\in I_j}|B_j(r)| > 2m\right) &= \mathbb{P}\left(\sup_{r\in[2T,4T]}|B_{0,6T}(r)| > 2m\right)\\
&\le \mathbb{P}\left(\sup_{r\in[0,6T]}|B_{0,6T}(r)| > 2m\right)\\
&= \mathbb{P}\left(\sup_{r\in[0,6T]}\left|\mathcal{W}(2r) - \frac{r}{6T}\mathcal{W}(12T)\right| > 2m\right)\\
&\le \mathbb{P}\left(|\mathcal{W}(12T)| + \sup_{r\in[0,6T]}|\mathcal{W}(2r)| > 2m\right)\\
&\le \mathbb{P}\Big(|\mathcal{W}(12T)| > m\Big) + \mathbb{P}\left(\sup_{r\in[0,12T]}|\mathcal{W}(r)| > m\right)\\
&\le \mathbb{P}\Big(|\mathcal{W}(12T)| > m\Big) + 2\mathbb{P}\left(\sup_{r\in[0,12T]}\mathcal{W}(r) > m\right)\\
&= \mathbb{P}\Big(|\mathcal{W}(12T)| > m) + 2\mathbb{P}(|\mathcal{W}(12T)| > m\Big)\\
&= 3\mathbb{P}\Big(|\mathcal{W}(12T)| > m\Big)
\end{aligned}$$

using that W is equal in law to $-W$, and the known distribution of the running maximum of a standard Brownian motion.

We may use this elementary bound in conjunction with (6.1.7) to obtain the union bound

$$\begin{aligned}
&\mathbb{P}\left(\sup_{r\in I_j}|\mathcal{F}_i(r)| > 4m\right)\\
\le\ &\mathbb{P}\Big(|L_j(a_j-3T)+(a_j-3T)^2| \vee |L_j(a_i+3T)+(a_j+3T)^2| > m\Big)\\
&+\mathbb{P}\left(\sup_{r\in I_j}|B_j(r)| > 2m\right) + \mathbb{P}\Big(|\sqrt{2T}N_j| > m\Big)\\
\le\ &\mathbb{P}\Big(|L_j(a_j-3T)+(a_j-3T)^2| \vee |L_j(a_j+3T)+(a_j+3T)^2| > m\Big)\\
&+3\mathbb{P}\Big(|\mathcal{W}(12T)| > m\Big) + P\Big(|\sqrt{2T}N_j| > m\Big).
\end{aligned}$$

Using the standard sub-Gaussian concentration inequalities for the latter two summmands, and the tail bounds in equation (1.2.5) for the first summand above yields

$$\mathbb{P}\left(\sup_{r\in I_j}|\mathcal{F}_j(r)| > 4m\right) \le c_1e^{-c_2m^{\frac{3}{2}}} + 6e^{-\frac{m^2}{2(12T)^2}} + 2e^{-\frac{m^2}{2(2T)^2}} \le c_1e^{-c_2m^{\frac{3}{2}}}$$

by redefining the original choice of c_1 and c_2 as needed, thus establishing (6.1.2) and completing the proof of the base case.

Equations (6.1.3) and (6.1.4) are immediate consequences of (6.1.1) and (6.1.2), respectively. To see this explicitly, we need only observe that (6.1.1) gives us the chain of equalities

$$\begin{aligned}
&\mathrm{Law}\left(\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_1^{(\lambda)}},\left(\mathfrak{A}_1^{(\lambda)}(r)\right)_{r\in I_2^{(\lambda)}}\right)\\
&=\mathrm{Law}\left(\left(\lambda^{1/3}\mathfrak{A}_1(r\lambda^{-2/3})\right)_{r\in\lambda^{2/3}I_1},\left(\lambda^{1/3}\mathfrak{A}_1(r\lambda^{-2/3})\right)_{r\in\lambda^{2/3}I_2}\right)\\
&=\mathrm{Law}\left(\left(\lambda^{1/3}\mathfrak{A}_1(r)\right)_{r\in I_1},\left(\lambda^{1/3}\mathfrak{A}_1(r)\right)_{r\in I_2}\right)\\
&\le e^{216cT^3}\,\mathrm{Law}\left(\left(\lambda^{1/3}\left(\sqrt{2T}N_j+(\mathcal{F}_j(r)+f_{a_j}(r))\right)_{r\in I_j}\right)_{j=1}^2\right)\\
&=e^{216cT^3}\,\mathrm{Law}\left(\left(\lambda^{1/3}\left(\sqrt{2T}N_j+\left(\mathcal{F}_j(r\lambda^{-2/3})+f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^2\right)\\
&=e^{216cT^3}\,\mathrm{Law}\left(\left(\left(\mathcal{F}_j^{(\lambda)}+\lambda^{1/3}\left(\sqrt{2T}N_j+f_{a_j}(r\lambda^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(\lambda)}}\right)_{j=1}^2\right)
\end{aligned}$$

where we have that $\mathcal{F}_j^{(\lambda)}$ is defined for each $j\in\{1,2\}$ by

$$\begin{aligned}
\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} &:= \left(\lambda^{\frac{1}{3}}\mathcal{F}_j(r\lambda^{-\frac{2}{3}})\right)_{r\in I_j^{(\lambda)}} \qquad (6.1.8)\\
&= \left(\lambda^{\frac{1}{3}}L_j(r\lambda^{-\frac{2}{3}})-\lambda^{\frac{1}{3}}f_{a_j}(r\lambda^{-\frac{2}{3}})+\lambda^{\frac{1}{3}}B_j(r\lambda^{-\frac{2}{3}})-\lambda^{\frac{1}{3}}\sqrt{2T}N_j\right)_{r\in\left[\lambda^{\frac{2}{3}}a_j\pm T\lambda^{\frac{2}{3}}\right]}.
\end{aligned}$$

All the claimed independence properties of the decomposition (6.1.8) are inherited from the base case of this proof. Establishing the tail bound (6.1.4) follows immediately from (6.1.2) and the fact that

$$\left(\mathcal{F}_j(r)\right)_{r\in I_j}=\left(\lambda^{-1/3}F_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}}.$$

We now provide a decomposition of the functions $\left(\mathcal{F}_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}}$ which will enable us to establish the tail bounds (6.1.6). By invoking the decomposition in equation (1.2.4), we obtain that

$$\left(L_j(r)\right)_{r\in[a_j\pm 3T]}\overset{d}{=}\left(\frac{(a_j+3T)-r}{6T}\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)+\frac{r-(a_j-3T)}{6T}\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)\right)_{r\in[a_j\pm 3T]}.$$

The right-hand side can then be rewritten as

$$\left(\frac{\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-\mathfrak{L}_1^{a}(a_j-3T)}{6T}\cdot r+\frac{(a_j+3T)\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)\mathfrak{L}_1^{a}(a_j+3T)}{6T}\right)_{r\in[a_j\pm 3T]}. \qquad (6.1.9)$$

Similarly, we may write for each $r \in [a_j \pm 3T]$ that

$$\begin{aligned} f_{a_j}(r) = & -\frac{(a_j+3T)^2-(a_j-3T)^2}{6T}r - \frac{(a_j+3T)(a_j-3T)^2-(a_j-3T)(a_j+3T)^2}{6T} \\ = & -(2a_j)r + \left(a_j^2 - 9T^2\right). \end{aligned} \tag{6.1.10}$$

We begin with the definition in (6.1.8), which gives us for each $r \in I_j^{(\lambda)} = \left[a_j\lambda^{2/3} \pm 3T\lambda^{2/3}\right]$ the decomposition in law

$$\begin{aligned} \mathcal{F}_j^{(\lambda)}(r) &= \lambda^{\frac{1}{3}}\mathcal{F}_j\left(r\lambda^{-2/3}\right) \\ &= \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-2/3}\right) + \lambda^{\frac{1}{3}}B_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}\sqrt{T}N_j \\ &\overset{d}{=} \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-2/3}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-2/3}\right) + \lambda^{\frac{1}{3}}B_{0,6T}\left(r\lambda^{-2/3} - (a_j-3T)\right) - \lambda^{\frac{1}{3}}\sqrt{T}N_j \end{aligned} \tag{6.1.11}$$

where as before, $B_{a,6T}$ is a diffusion parameter 2 Brownian bridge on $[a, a+6T]$ from 0 to 0. Noting that for any $a, k \in \mathbb{R}$ the scaling properties of Brownian bridges give us that

$$\begin{aligned} \left(k^{-1}B_{0,6T}(k^2r - k^2a)\right)_{r\in[a,a+6T]} &\overset{d}{=} \left(k^{-1}B_{k^2a,6k^2T}(r-k^2a)\right)_{r\in[k^2a,k^2(a+6T)]} \\ &\overset{d}{=} \left(k^{-1}B_{0,6k^2T}(r)\right)_{r\in[0,k^2(6T))]} \\ &\overset{d}{=} \left(\mathcal{W}(2r) - \frac{k^{-2}r}{6T}\mathcal{W}(12k^2T)\right)_{r\in[0,k^2(6T))]} \\ &\overset{d}{=} \left(\mathcal{W}(2r-2k^2a) - \frac{k^{-2}r-a}{6T}\mathcal{W}(12k^2T)\right)_{r\in[k^2a,k^2(a+6T)]} \\ &\overset{d}{=} \left(\mathcal{W}(2r-2k^2a) - \frac{k^{-1}r-ka}{6T}\mathcal{W}(12T)\right)_{r\in[k^2a,k^2(a+6T)]} \end{aligned}$$

we may refine (6.1.11) further, for all $r \in I_j^{(\lambda)} = \left[\lambda^{\frac{2}{3}}a_j \pm \lambda^{\frac{2}{3}}T\right]$, as

$$\begin{aligned} &\mathcal{F}_j^{(\lambda)}(r) + \lambda^{\frac{1}{3}}\sqrt{2T}N_j \\ &\overset{d}{=} \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-\frac{2}{3}}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-\frac{2}{3}}\right) + \lambda^{\frac{1}{3}}B_{0,6T}\left(r\lambda^{-\frac{2}{3}} - (a_j-3T)\right) \\ &\overset{d}{=} \lambda^{\frac{1}{3}}L_j\left(r\lambda^{-\frac{2}{3}}\right) - \lambda^{\frac{1}{3}}f_{a_j}\left(r\lambda^{-\frac{2}{3}}\right) + \mathcal{W}_j\left(2r - (2a_j-6T)\lambda^{\frac{2}{3}}\right) - \frac{r\lambda^{-\frac{1}{3}} - (a_j-3T)\lambda^{\frac{1}{3}}}{6T}\mathcal{W}_j(12T). \end{aligned} \tag{6.1.12}$$

To remove any ambiguity in the application of this lemma, we will adopt the convention that $\mathcal{W}_1$ and $\mathcal{W}_2$ are independent standard Brownian motions that we associate with the independent Brownian bridges B_1 and B_2 respectively.

Now by using (6.1.12) in conjunction with (6.1.9) and (6.1.10) we can obtain the decomposition in law

$$\left(F_j^{(\lambda)}(r)\right)_{r\in I_j^{(\lambda)}} \overset{d}{=} \left(\mathcal{W}_j\left(2r - (2a_j-6T)\lambda^{\frac{2}{3}}\right) + A_jr\lambda^{-1/3} + C_j\lambda^{1/3}\right)_{r\in I_j^{(\lambda)}}$$

where the random constants A_j, and C_j are defined as

$$
\begin{aligned}
A_j &:= \left(\frac{(\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2)-(\mathfrak{L}_1^{\boldsymbol{a}}(a_j)-a_j^2)}{3T}-\frac{1}{6T}\mathcal{W}_j(12T)\right)\\
C_j &:= \left(\frac{(a_j+3T)(\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2)-(a_j-3T)(\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2)}{6T}\right)\\
&\quad+\ \frac{a_j-3T}{6T}\mathcal{W}_j(12T)-\sqrt{2T}N_j.
\end{aligned}
$$

We now establish tail bounds for the random constants A_j and C_j. Using the standard Gaussian tail bounds and the tail bounds in (1.2.5), we see that for all $m>0$,

$$
\begin{aligned}
\mathbb{P}\Big(|A_j|>m\Big) &\le \mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2}{6T}\right|+\left|\frac{\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2}{6T}\right|+\left|\frac{1}{6T}\mathcal{W}_j(12T)\right|>m\right)\\
&\le \mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{a}(a_j+3T)-(a_j+3T)^2}{6T}\right|>\frac{m}{3}\right)+\mathbb{P}\left(\left|\frac{\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2}{6T}\right|>\frac{m}{3}\right)\\
&\qquad+\mathbb{P}\Big(\left|\frac{1}{12T}\mathcal{W}_j(6T)\right|>\frac{m}{3}\Big)\\
&\le \mathbb{P}\Big(\left|\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2\right|>2mT\Big)+\mathbb{P}\Big(\left|\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2\right|>2mT\Big)\\
&\qquad+\mathbb{P}\Big(|\mathcal{W}_j(12T)|>2mT\Big)\\
&\le c_1e^{-c_2(2mT)^{\frac{3}{2}}}+c_1e^{-c_2(2mT)^{\frac{3}{2}}}+2e^{-\frac{(2mT)^2}{2(12T)}}\\
&\le c_1e^{-c_2m^{\frac{3}{2}}}
\end{aligned}
$$

where the $T-$dependent constants c_1, c_2 have been redefined in the final inequality as needed. Similarly, for the random constant C_j we may obtain the tail bound

$$
\begin{aligned}
\mathbb{P}\Big(|C_j|>m\Big) &\le \mathbb{P}\left(\left|\frac{(a_j+3T)(\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2)}{6T}\right|>\frac{m}{4}\right)\\
&\quad+\mathbb{P}\left(\left|\frac{(a_j-3T)(\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2)}{6T}\right|>\frac{m}{4}\right)\\
&\quad+\mathbb{P}\left(\left|\frac{(a_j-3T)\mathcal{W}_j(12T)}{6T}\right|>\frac{m}{4}\right)+\mathbb{P}\left(\left|\sqrt{2T}N_j\right|>\frac{m}{4}\right)\\
&= \mathbb{P}\left(\left|\mathfrak{L}_1^{\boldsymbol{a}}(a_j-3T)-(a_j-3T)^2)\right|>\frac{3mT}{2|a_j+3T|}\right)\\
&\quad+\mathbb{P}\left(\left|\mathfrak{L}_1^{\boldsymbol{a}}(a_j+3T)-(a_j+3T)^2)\right|>\frac{3mT}{2|a_j-3T|}\right)\\
&\quad+\mathbb{P}\left(|\mathcal{W}_j(12T)|>\frac{3mT}{2|a_j-3T|}\right)+\mathbb{P}\left(|\sqrt{2T}N_j|>\frac{m}{4}\right)\\
&\le c_1e^{-c_2\left(\frac{3mT}{2|a_j+3T|}\right)^{\frac{3}{2}}}+c_1e^{-c_2\left(\frac{3mT}{2|a_j-3T|}\right)^{\frac{3}{2}}}+2e^{-\frac{1}{24T}\left(\frac{3mT}{2|a_j-3T|}\right)^2}+2e^{-\frac{1}{4T}\left(\frac{m}{4}\right)^2}\\
&\le c_1e^{-c_2\left(\frac{m}{|a_j|}\right)^{\frac{3}{2}}}
\end{aligned}
$$

where we have once again redefined the values of c_1 and c_2 so that the final inequality holds as well. This therefore establishes (6.1.6) and completes the proof of Lemma 6.1.1. □

6.2 The Big Picture in the Second Regime

To complete our proof of the two-point bound (4.3.1), we need to bound $\sum_{|i|>10\sigma^{-\frac{2}{3}}} p_{i,\varepsilon}$, where we remind the reader that we have defined $p_{i,\varepsilon}$ as

$$p_{i,\varepsilon} := \mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0) + \frac{3x^2}{s} - h\right| \leq \varepsilon,\ \left|\sup_{\frac{z}{\sigma^{2/3}}\in[i\pm 1/2]} \mathfrak{A}_1^{(s)}(z) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta) - \mathfrak{A}_1^{(s)}(0) + \frac{2xz+x^2}{s\sigma^{\frac{1}{3}}}\right| \leq 2\varepsilon\right).$$

We will do so by mimicking the same general sequence of steps used in the first regime. To that end, we will use Lemma 5.1.1 and Lemma 6.1.1 to build a set of random vectors

$$\left\{ \left(Z,\tilde{Z}\right) + \left(G, G_i\right)\ :\ |i| > 10\sigma^{-\frac{2}{3}} \right\}$$

with $\left(Z,\tilde{Z}\right)$ a bivariate Gaussian random vector independent of the random vector (G, G_i) such that for some absolute constant $\kappa_2 > 0$

$$p_{i,\varepsilon} \leq \kappa_2^2 \mathbb{P}\left(\left(Z,\tilde{Z}\right) + \left(G, G_i\right) \in \left[h \pm \varepsilon\right] \times \left[0 \pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]\right)$$

for all i in the second regime.

We begin this procedure by first applying Lemma 5.1.1 to the process $\widetilde{\mathfrak{A}}_1^{(\sigma)}$. There will actually be no changes whatsoever to how we use Lemma 5.1.1 on $\widetilde{\mathfrak{A}}_1^{(\sigma)}$ in the second regime, so we will reuse the absolute continuity statement (5.2.1) completely verbatim. However, because the maximal distance between 0 and the interval $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$ is unbounded in the second regime, the absolute continuity statement that we develop for $\mathfrak{A}_1^{(s)}$ will be quite different this time.

In particular, we will need to use Lemma 6.1.1 on $\mathfrak{A}_1^{(s)}$ over an interval containing 0 and on a separate disjoint interval containing the domain of our supremum $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$. In order to do this, the only decision we need to make is the selection of a constant $T > \frac{1}{6}$ and two intervals $I_1 = [a_1 \pm 3T]$ and $[a_2 \pm 3T]$. However, because we require that $I_2 = a_1 + 3T < a_2 - 3T$, there is a natural order to the selection of these intervals which will not by symmetric when $\text{sign}(i)$ changes. If $i > 10\sigma^{-\frac{2}{3}}$ then we will want to have $0 \in I_1^{(s)}$ and $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right] \subseteq I_2^{(s)}$, and if $i < -10\sigma^{-\frac{2}{3}}$, then we will want to have $\left[\sigma^{\frac{2}{3}} i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right] \subseteq I_1^{(s)}$ and $0 \in I_2^{(s)}$. Recall that we have defined $I_j^{(s)} := \left[s^{\frac{2}{3}} a_j \pm T s^{\frac{2}{3}}\right]$.

Fortunately, although this asymmetry will slightly change the nomenclature used to apply Lemma 6.1.1 to $\mathfrak{A}_1^{(s)}$, it will have no impact on the absolute continuity statement generated from its application. So without loss of generality, we will just illustrate the application of this for $i > 10\sigma^{-\frac{2}{3}}$ with the knowledge that reversing the order of the intervals changes nothing important in the end. Given this, since $\frac{1}{2}\sigma^{\frac{2}{3}} \leq 1 \leq 2s^{-\frac{2}{3}}$ for all $s \in \left[1, \frac{11}{10}\right]$, we will choose the parameters $\lambda = s$, $T = 2s^{-\frac{2}{3}}$,

$a_1 = s^{-\frac{2}{3}}\sigma^{\frac{1}{3}}$, and $a_2 = s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}i$ to generate the new absolute continuity statement

$$\begin{aligned}
&\text{Law}\left(\left(\mathfrak{A}_1^{(s)}(r)\right)_{r\in\left[\sigma^{\frac{1}{3}}\pm 4\right]}, \left(\mathfrak{A}_1^{(s)}(r)\right)_{r\in\left[\sigma^{\frac{2}{3}}i\pm 4\right]}\right)\\
=\ &\text{Law}\left(\left(\mathfrak{A}_1^{(s)}(r)\right)_{r\in I_1^{(s)}}, \left(\mathfrak{A}_1^{(s)}(r)\right)_{r\in I_2^{(s)}}\right)\\
\leq\ &e^{64cs^{-2}}\,\text{Law}\left(\left(\left(\mathcal{F}_j^{(s)}(r) + s^{\frac{1}{3}}\left(\sqrt{2T}N_j + f_{a_j}(rs^{-\frac{2}{3}})\right)\right)_{r\in I_j^{(s)}}\right)_{j=1}^{2}\right)\\
\leq\ &e^{64c}\,\text{Law}\left(\left(\left(\mathcal{F}_j^{(s)}(r) + 2N_j + s^{\frac{1}{3}}f_{a_j}(rs^{-\frac{2}{3}})\right)_{r\in I_j^{(s)}}\right)_{j=1}^{2}\right).
\end{aligned}$$

Note that the condition $|i| > 10\sigma^{-\frac{2}{3}}$ is what ensures that $a_1 + 3T < a_2 - 3T$ for all i. For the sake of readability, we will first introduce the definitions of $\left(Z,\tilde{Z}\right)$ and the new random vectors (G, G_i) before using bounding the probabilities $p_{i\varepsilon}$. For all $i > 10\sigma^{-\frac{2}{3}}$ we will define

$$\left(Z,\tilde{Z}\right) := \left(2N_1,\ 2\sigma^{-\frac{1}{3}}N_2 + \tilde{N} - 2\sigma^{-\frac{1}{3}}N_1\right) \tag{6.2.1}$$

and the independent random vector (G, G_i) by

$$\left(G, G_i\right) := \left(\mathcal{F}_1^{(s)}(0) + f_{\frac{1}{10}}(0) + \frac{3x^2}{s},\ \sup_{z\sigma^{-\frac{2}{3}}\in[i\pm 1/2]} \frac{\widetilde{\mathcal{F}}^{(\sigma)}(z+\Delta) + \mathcal{F}_2^{(s)}(z) - \mathcal{F}_1^{(s)}(0)}{\sigma^{\frac{1}{3}}} + g_i(z)\right) \tag{6.2.2}$$

where we define the deterministic function $g_i(z)$ for all such i as

$$\begin{aligned}
g_i(z) =\ & g_i(z; x, s, \Delta, \sigma)\\
:=\ & \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}i}\left(zs^{-\frac{2}{3}}\right) + \ell_{i+\Delta\sigma^{-\frac{2}{3}}}\left((z+\Delta\sigma^{-\frac{2}{3}})\right) - \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\sigma^{\frac{1}{3}}}(0) + \frac{2xz + x^2}{s\sigma^{\frac{1}{3}}}.
\end{aligned} \tag{6.2.3}$$

With these definitions in place, we then have that by elementary measure theory,

$$\begin{aligned}
&\mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0) + \frac{3x^2}{s} - h\right| \leq \varepsilon,\ \left|\sup_{z\sigma^{-\frac{2}{3}}\in[i\pm 1/2]} \mathfrak{A}_1^{(s)}(z) + \widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta) - \mathfrak{A}_1^{(s)}(0) + \frac{2xz+x^2}{s\sigma^{\frac{1}{3}}}\right| \leq 2\varepsilon\right)\\
&\qquad \leq\ \kappa_2^2\mathbb{P}\left(\left(Z+G\right) \in \left[h\pm\varepsilon\right],\ \left(\tilde{Z}+G_i\right) \in \left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]\right)\\
&\qquad =\ \kappa_2^2\mathbb{P}\left(\left(Z,\tilde{Z}\right) + \left(G, G_i\right) \in \left[h\pm\varepsilon\right] \times \left[0\pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]\right)
\end{aligned}$$

where the constant $\kappa_2 := \max\{e^c, e^{64c}\} = e^{64c}$. Before proceeding further, we take a moment to observe that by the independence of the standard Gaussians N, N_1, and N_2, the bivariate Gaussian random vector $(Z,\tilde{Z})$ is distributed as

$$(Z,\tilde{Z}) \sim \mathcal{N}\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right) := \mathcal{N}\left([0,0]^{\mathbf{T}},\ \begin{bmatrix} 4 & -4\sigma^{-\frac{1}{3}} \\ -4\sigma^{-\frac{1}{3}} & 4\sigma^{-\frac{2}{3}}+1 \end{bmatrix}\right).$$

Moreover, by observing that for all $u \in \mathbb{R}$ we always have that

$$\begin{bmatrix} 4 & -4u \\ -4u & 4u^2+1 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 4u^2+1 & 4u \\ 4u & 4 \end{bmatrix} \quad \text{and} \quad \det\left(\begin{bmatrix} 4 & -4u \\ -4u & 4u^2+1 \end{bmatrix}\right) = 4,$$

we can compute explicitly that the joint density $f_{(Z,\tilde{Z})}$ is given for all $(z,\tilde{z}) \in \mathbb{R}^2$ by

$$\begin{aligned} f_{(Z,\tilde{Z})}(z,\tilde{z}) &= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}\begin{bmatrix} z & \tilde{z}_2 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix}\right) \\ &= \frac{1}{4\pi} \exp\left(-\frac{1}{8}\begin{bmatrix} z & \tilde{z} \end{bmatrix} \begin{bmatrix} 4\sigma^{-\frac{2}{3}}+1 & 4\sigma^{-\frac{1}{3}} \\ 4\sigma^{-\frac{1}{3}} & 4 \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix}\right) \\ &= \frac{1}{4\pi} \exp\left(-\frac{1}{8}\left(\left(4\sigma^{-\frac{2}{3}}+1\right)z^2 + 8\sigma^{-\frac{1}{3}} z\tilde{z} + 4\tilde{z}^2\right)\right). \end{aligned} \tag{6.2.4}$$

Note that because $\boldsymbol{\Sigma}$ is positive semi-definite, we have by definition that

$$-\frac{1}{8}\left(\left(4\sigma^{-\frac{2}{3}}+1\right)z^2 + 8\sigma^{-\frac{1}{3}} z\tilde{z} + 4\tilde{z}^2\right) \leq 0$$

for all $(z,\tilde{z}) \in \mathbb{R} \times [0,\infty)$, which ensures that the density is bounded above by a finite absolute constant. In the work to come, this will mean that for any $h_2 \geq 0$, the law of the random vector $(Z,\tilde{Z})$ conditioned on the event $\left\{\tilde{Z} \geq h_2\right\}$ will also have a bounded density. Moreover, this maximal value will be a function of h_2 which decays to $-\infty$ as $h_2 \to \infty$.

As was the case in the first regime, by Lemma 5.2.1 we know that for all $i \in \mathbb{Z}$ with $|i| > 10\sigma^{-\frac{2}{3}}$, there exists a density ρ_i with respect to the Lebesgue measure on $\mathbb{R}^2$, which has $(x,s), \Delta$, and σ as parameters, such that we may write

$$\mathbb{P}\left((Z',Z)+(G,G_i) \in \left[h \pm \varepsilon\right] \times \left[0 \pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]\right) = \iint_{\left[h\pm\varepsilon\right] \times \left[0\pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]} \rho_i(h_1,h_2)\,dh_1 dh_2. \tag{6.2.5}$$

From this point onward, our strategy will be to prove that there exists an absolute constant $b'' > 0$ such that we have a uniform bound of the form

$$\sum_{|i|>10\sigma^{-\frac{2}{3}}} \rho_i\left(h_1,h_2\right) < b''. \tag{6.2.6}$$

Such a uniform bound would allow us to extend the chain of upper bounds (5.2.9) to get

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),\ \mathcal{L}(0,0;y,t)\in(h-\varepsilon,h+\varepsilon)\Big)\\
&\le \mathbb{P}\Big(\mathcal{L}(0,0;x,s)\in(h-\varepsilon,h+\varepsilon),\ |\mathcal{L}(0,0;y,t)-\mathcal{L}(0,0;x,s)|\le 2\varepsilon\Big)\\
&\le \sum_{i\in\mathbb{Z}}\mathbb{P}\left(A_{x,s,h,\varepsilon}\cap\left\{\left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac12\right]}\mathcal{L}(0,0;z,s)+\widetilde{\mathcal{L}}(0,0;z+\Delta,\sigma)-\mathcal{L}(0,0;0,s)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right\}\right)\\
&=\sum_{i\in\mathbb{Z}}\mathbb{P}\left(\left|\mathfrak{A}_1^{(s)}(0)+\frac{3x^2}{s}-h\right|\le\varepsilon,\ \left|\sup_{z\sigma^{-\frac{2}{3}}\in\left[i\pm\frac12\right]}\mathfrak{A}_1^{(s)}(z)+\widetilde{\mathfrak{A}}_1^{(\sigma)}(z+\Delta)-\mathfrak{A}_1^{(s)}(0)+\frac{2xz+x^2}{s}\right|\le 2\varepsilon\right)\\
&\le \kappa_1^2\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\mathbb{P}\left(\left(N',\tilde{N}\right)+\left(G,G_i\right)\in\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]\right)+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=\kappa_1^2\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{1}{3}}\right]}\rho_i(h_1,h_2)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=\kappa_1^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{1}{3}}\right]}\left(\sum_{|i|\le 10\sigma^{-\frac{2}{3}}}\rho_i(h_1,h_2)\right)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&\le\kappa_1^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{1}{3}}\right]}\left(k'+k'\sigma^{-\frac{1}{6}}+k'\sigma^{-\frac{1}{3}}|\Delta|^{\frac12}\right)dh_1dh_2+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&=8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+\sum_{|i|>10\sigma^{-\frac{2}{3}}}p_{i,\varepsilon}\\
&\le 8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+\kappa_2^2\sum_{|i|>10\sigma^{-\frac{2}{3}}}\mathbb{P}\left(\left(Z,\tilde{Z}\right)+\left(G,G_i\right)\in\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{2}{3}}\right]\right)\\
&\le 8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+\kappa_2^2\sum_{|i|>10\sigma^{-\frac{2}{3}}}\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{2}{3}}\right]}\rho_i(h_1,h_2)dh_1dh_2\\
&\le 8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+\kappa_2^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{2}{3}}\right]}\left(\sum_{|i|>10\sigma^{-\frac{2}{3}}}\rho_i(h_1,h_2)\right)dh_1dh_2\\
&\le 8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+\kappa_2^2\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm2\varepsilon\sigma^{-\frac{2}{3}}\right]}b''dh_1dh_2\\
&=8\kappa_1^2\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)+8\kappa_2^2\varepsilon^2b''\\
&\le\kappa\varepsilon^2\left(k'\sigma^{-\frac{1}{3}}+k'\sigma^{-\frac{1}{2}}+k'\sigma^{-\frac{2}{3}}|\Delta|^{\frac12}\right)
\end{aligned}\tag{6.2.7}$$

for some absolute multiplicative constant $\kappa>0$. Upon proving the bound (6.2.6) we will have completed our proof of the two-point bound (4.3.1).

6.3 The Internal Structure of G_i for Large i

The first step towards building the bound (6.2.6) will be understanding the behaviour of the random variable G_i. The decomposition of G_i will be more complex in the second regime than it was in the first regime, due to the fact that we are replacing the old increment $\mathcal{F}^{(s)}(z) - \mathcal{F}^{(s)}(0)$, which had terms cancelling, with a new increment $\mathcal{F}_2^{(s)}(z) - \mathcal{F}_1^{(s)}(0)$, which has no such cancellations. There will be more random variables in our decomposition of G_i which will necessitate moderately more work later on, but the same general ideas overall will still work in this new setting.

We begin by observing that by (6.1.5) we can decompose $\mathcal{F}_1^{(s)}(0)$ in law as

$$\mathcal{F}_1^{(s)}(0) \stackrel{d}{=} W_1\left(-2(a_1 - 3T)s^{\frac{2}{3}}\right) + C_1 s^{\frac{1}{3}} = W_1\left(24 - 2\sigma^{\frac{1}{3}}\right) + C_1 s^{\frac{1}{3}} \tag{6.3.1}$$

and similarly, since $I_2^{(s)} = \left[s^{\frac{2}{3}}a_2 \pm s^{\frac{2}{3}}T\right] = \left[\sigma^{\frac{2}{3}}i \pm 4\right]$, we can write

$$\begin{aligned}\left(\mathcal{F}_2^{(s)}(z)\right)_{z\in\left[\sigma^{\frac{2}{3}}i\pm 4\right]} &\stackrel{d}{=} \left(W_2\left(2z - 2(a_2 - 3T)s^{\frac{2}{3}}\right) + A_2 z s^{-\frac{1}{3}} + C_2 s^{\frac{1}{3}}\right)_{z\in\left[\sigma^{\frac{2}{3}}i\pm 4\right]} \\ &\stackrel{d}{=} \left(W_2\left(2z - 2\sigma^{\frac{2}{3}}i + 24\right) + A_2 z s^{-\frac{1}{3}} + C_2 s^{\frac{1}{3}}\right)_{z\in\left[\sigma^{\frac{2}{3}}i\pm 4\right]}.\end{aligned} \tag{6.3.2}$$

These two decompositions in law mean that we may also write that

$$\begin{aligned}&\sigma^{-\frac{1}{3}}\left(\mathcal{F}_2^{(s)}(z) - \mathcal{F}_1^{(s)}(0)\right) \\ &\stackrel{d}{=} \sigma^{-\frac{1}{3}}\left(W_2\left(2z - 2\sigma^{\frac{2}{3}}i + 24\right) - W_1\left(24 - 2\sigma^{\frac{1}{3}}\right) + A_2 z s^{-\frac{1}{3}} + C_2 s^{\frac{1}{3}} - C_1 s^{\frac{1}{3}}\right) \\ &\stackrel{d}{=} \sigma^{-\frac{1}{3}}\left(\mathcal{N}\left(0, \left|2z - 2\sigma^{\frac{2}{3}}i - 2\sigma^{\frac{1}{3}}\right|\right) + A_2 z s^{-\frac{1}{3}} + C_2 s^{\frac{1}{3}} - C_1 s^{\frac{1}{3}}\right)\end{aligned}$$

using the fact that W_1 and W_2 are independent standard Brownian motions. Thus, by recalling equation (6.2.2) we can decompose H_i more transparently as

$$\begin{aligned}G_i &= \sup_{z\sigma^{-\frac{2}{3}}\in[i\pm 1/2]} \frac{\tilde{\mathcal{F}}^{(\sigma)}(z+\Delta) + \mathcal{F}_2^{(s)}(z) - \mathcal{F}_1^{(s)}(0)}{\sigma^{\frac{1}{3}}} + g_i(z) \\ &\stackrel{d}{=} \sup_{z\sigma^{-\frac{2}{3}}\in[i\pm 1/2]} \frac{\tilde{\mathcal{F}}^{(\sigma)}(z+\Delta) + \mathcal{N}\left(0, \left|2z - 2\sigma^{\frac{2}{3}}i - 2\sigma^{\frac{1}{3}}\right|\right) + A_2 z s^{-\frac{1}{3}} + C_2 s^{\frac{1}{3}} - C_1 s^{\frac{1}{3}}}{\sigma^{\frac{1}{3}}} + g_i(z).\end{aligned} \tag{6.3.3}$$

Note that similarly to our decomposition in law in the first regime, all five random variables in the supremum above have known exponential tail bounds which are symmetric about 0 for all $z \in \left[\sigma^{\frac{2}{3}}i \pm \frac{1}{2}\sigma^{\frac{2}{3}}\right]$.

We now define a family of random variables $\{z_i^*\}_{|i|>10\sigma^{-\frac{2}{3}}}$ by setting for each such i

$$z_i^* := \arg\max_{z\in\left[i\pm\frac{1}{2}\right]} \frac{\left(\mathcal{F}_2^{(s)}(z\sigma^{\frac{2}{3}}) - \mathcal{F}_1^{(s)}(0) + \tilde{\mathcal{F}}^{(\sigma)}(z\sigma^{\frac{2}{3}} + \Delta)\right)}{\sigma^{\frac{1}{3}}} + g_i\left(\sigma^{\frac{2}{3}}z\right).$$

As before, if supremum above has more than one arg max, we will take z_i^* to be the largest amongst them, making the choice unique. Given this new version of z_i^*, if we then define the random variables $\left\{G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)\right\}_{|i|>10\sigma^{-\frac{2}{3}}}$ for all such $i \in \mathbb{Z}$ by

$$G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) := \frac{\tilde{\mathcal{F}}^{(\sigma)}(\sigma^{\frac{2}{3}}z_i^* + \Delta) + \mathcal{N}\left(0, \left|2\sigma^{\frac{2}{3}}z_i^* - 2\sigma^{\frac{2}{3}}i - 2\sigma^{\frac{1}{3}}\right|\right)}{\sigma^{\frac{1}{3}}} + \left(\frac{s}{\sigma}\right)^{-\frac{1}{3}} A_2 z_i^* + \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} (C_2 - C_1) \tag{6.3.4}$$

then we can say that for all i in the second regime that

$$G_i = \sup_{z\sigma^{-\frac{2}{3}} \in [i\pm 1/2]} \frac{\tilde{\mathcal{F}}^{(\sigma)}(z+\Delta) + \mathcal{F}_2^{(s)}(z) - \mathcal{F}_1^{(s)}(0)}{\sigma^{\frac{1}{3}}} + g_i(z) \overset{d}{=} G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)$$

where the random variable $G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)$ has symmetric upper and lower tail bounds about 0, and $g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)$ is a random parabola in the variable i. As we mentioned earlier, because $G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)$ now contains five random variables instead of three, establishing its tail bounds will be slightly more tedious than it was previously but it will be no more complex than that.

We will now turn our attention towards understanding the structure of the new random parabola $g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)$. For the sake of simplicity, we will begin by understanding the deterministic parabola $g_i\left(\sigma^{\frac{2}{3}}i\right)$ before moving onto the general case where $z_i^* \neq i$. Recalling equations (5.2.5) and (6.1.10) we have

$$\begin{aligned}
\ell_{i+\Delta\sigma^{-\frac{2}{3}}}\left(\frac{z+\Delta}{\sigma^{\frac{2}{3}}}\right) &= i^2 - 2iz\sigma^{-\frac{2}{3}} - 2\Delta\sigma^{-\frac{4}{3}}(z-\Delta) \\
s^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}i}\left(zs^{-\frac{2}{3}}\right) = \frac{\left(2\sigma^{\frac{2}{3}}iz + \left(\sigma^{\frac{4}{3}}i^2 - 9(16)\right)\right)}{s} &= -\frac{\sigma^{\frac{4}{3}}i^2 + 2\sigma^{\frac{2}{3}}iz - 144}{s} \\
s^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\sigma^{\frac{1}{3}}}(0) &= -\frac{\sigma^{\frac{2}{3}} - 144}{s}.
\end{aligned}$$

Given this and the definition of $g_i(z)$ in (6.2.3), we then have that

$$\begin{aligned}
g_i(z) &:= \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}i}\left(zs^{-\frac{2}{3}}\right) + \ell_{i+\Delta\sigma^{-\frac{2}{3}}}\left((z+\Delta\sigma^{-\frac{2}{3}})\right) - \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} f_{s^{-\frac{2}{3}}\frac{1}{10}}(0) + \frac{2xz+x^2}{s\sigma^{\frac{1}{3}}} \\
&= \left(1-\frac{\sigma}{s}\right)i^2 - \left(2z\sigma^{-\frac{2}{3}} + \frac{2\sigma^{\frac{1}{3}}z}{s}\right)i + \left(2\Delta^2\sigma^{-\frac{4}{3}} - 2\Delta z\sigma^{-\frac{4}{3}} + \frac{2xz+x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}}\right).
\end{aligned}$$

As such, if we set $z = \sigma^{\frac{2}{3}}i$ in particular, this equation becomes

$$\begin{aligned}
g_i\left(\sigma^{\frac{2}{3}}i\right) &= \left(1-\frac{\sigma}{s}\right)i^2 - \left(2i + \frac{2i\sigma}{s}\right)i + \left(2\Delta^2\sigma^{-\frac{4}{3}} - 2i\Delta\sigma^{-\frac{2}{3}} + \frac{2x\sigma^{\frac{2}{3}}i + x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}}\right) \\
&= -\left(1+\frac{3\sigma}{s}\right)i^2 - \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x\sigma^{\frac{1}{3}}}{s}\right)i + \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}}\right).
\end{aligned} \tag{6.3.5}$$

We will again be more interested in the behaviour of $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)$ specifically. For the sake of completeness,

$$h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) = \left(1+\frac{3\sigma}{s}\right)i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x\sigma^{\frac{1}{3}}}{s}\right)i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}} - h_2\right).$$

From this, we see that the general longterm behaviour of $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)$ is more or less the same as that of the parabolas in the first regime, so we will keep our analysis of this new parabola brief. The main point of interest in this case is finding a positive integer $K > 0$ such that $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) \geq 0$ for all $i \in \mathbb{Z}$ with $|i| \geq K\sigma^{-\frac{2}{3}}$. and $|h_2| \leq 2\varepsilon_0\sigma^{-\frac{1}{3}}$. By observing that we can bound the magnitude of the degree ≤ 1 terms by

$$\begin{aligned}
&\left|\left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x\sigma^{\frac{1}{3}}}{s}\right)i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{2x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}} - h_2\right)\right| \\
\leq\ &\left(\left|2\Delta\sigma^{-\frac{2}{3}}\right| + \left|\frac{2x\sigma^{\frac{1}{3}}}{s}\right|\right)|i| + \left|2\Delta^2\sigma^{-\frac{4}{3}}\right| + \left|\frac{x^2}{s}\sigma^{-\frac{1}{3}}\right| + \left|\frac{1}{s}\sigma^{\frac{1}{3}}\right| + |h_2| \\
\leq\ &\left(2\sigma^{-\frac{2}{3}} + 4\right)|i| + 2\sigma^{-\frac{4}{3}} + 1 \\
\leq\ &4\sigma^{-\frac{2}{3}}|i| + 3\sigma^{-\frac{4}{3}} + 5\sigma^{-\frac{1}{3}} + 1
\end{aligned}$$

it suffices to guarantee that the weaker inequality

$$|i|^2 - 4\sigma^{-\frac{2}{3}}|i| - 3\sigma^{-\frac{4}{3}} - 5\sigma^{-\frac{1}{3}} - 1 \geq 0$$

always holds for all $|i| \geq K\sigma^{-\frac{2}{3}}$. This inequality holds whenever i lies outside the region in between the zeros of the parabola. By the quadratic formula, the zeroes of this parabola will have a magnitude of no more than

$$\begin{aligned}
2\sigma^{-\frac{2}{3}} + \frac{1}{2}\sqrt{4\sigma^{-\frac{4}{3}} + 12\sigma^{-\frac{4}{3}} + 20\sigma^{-\frac{1}{3}} + 4} &\leq\ 2\sigma^{-\frac{2}{3}} + \sqrt{4\sigma^{-\frac{4}{3}} + 5\sigma^{-\frac{1}{3}} + 1} \\
&<\ 2\sigma^{-\frac{2}{3}} + \sqrt{10\sigma^{-\frac{4}{3}}} \\
&<\ 6\sigma^{-\frac{2}{3}}.
\end{aligned}$$

Given this and the fact that our parabola opens upwards, we obtain a chain of implications

$$|i| > 10\sigma^{-\frac{2}{3}} \implies |i|^2 - 4\sigma^{-\frac{2}{3}}|i| - 3\sigma^{-\frac{4}{3}} - 5\sigma^{-\frac{1}{3}} - 1 \geq 0 \implies h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) \geq 0.$$

Thus, because the second regime was defined by the requirement that $|i| > 10\sigma^{-\frac{2}{3}}$, we have now verified that $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)$ is always non-negative in the second regime. The guaranteed non-negativity of $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)$ for any h_2 in our domain of integration will also be useful in the work to come. We also take a moment to note that as was the case in the first regime, the constant term of $h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)$ is guaranteed to be negative by definition of ε_0. This will again be extremely useful in the work to come.

We now consider the more general case with $h_2 - g_i\left(\sigma^{\frac{2}{3}}z_i^*\right)$. By once again writing $z_i^* = i + p_i$

where p_i is a random variable taking values in $\left[-\frac{1}{2},\frac{1}{2}\right]$, we see that

$$\begin{aligned} h_2 - g_i\left(\sigma^{\frac{2}{3}} z_i^*\right) &= h_2 - g_i\left(\sigma^{\frac{2}{3}}(i+p_i)\right) \\ &= \left(1+\frac{3\sigma}{s}\right) i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x\sigma^{\frac{1}{3}}}{s}\right) i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}} - h_2\right) - \xi_i \\ &= h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - \xi_i \end{aligned} \tag{6.3.6}$$

where we have defined the family of random fluctuations $\{\xi_i\}_{|i|>10\sigma^{-\frac{2}{3}}}$ for each such i by

$$\xi_i := -2\left(\Delta\sigma^{-\frac{2}{3}} + \frac{(1+\sigma)}{s} i - \frac{x\sigma^{\frac{1}{3}}}{s}\right) p_i. \tag{6.3.7}$$

As was the case in the first regime, the random fluctuation ξ_i represents the uncertainty in the parabola $g_i\left(\sigma^{\frac{2}{3}} z_i^*\right)$ stemming from the fact that the arg max z_i^* is random. Once again, since we will simply be adapting our previous work in the first regime to the natural analogues in the second regime, there is no need at this time for any deeper analysis of ξ_i itself. In conclusion, we have established the decomposition in law

$$G_i \stackrel{d}{=} G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) + \xi_i + g_i\left(\sigma^{\frac{2}{3}} i\right)$$

where $g_i\left(\sigma^{\frac{2}{3}} i\right)$ is deterministic and the remaining two terms are random variables. Our goal in the following work will once again be to leverage the strong tail bounds about 0 of $G_i^*\left(\sigma^{\frac{2}{3}} i\right)$ as much as possible while minimizing the influence of ξ_i on our subsequent computations.

6.4 Density Bounds for ρ_i for Large i

With our newfound understanding of the random variables $\{G_i\}_{|i|>10\sigma^{-\frac{2}{3}}}$ we are now ready to develop density bounds on the family $\{\rho_i\}_{|i|>10\sigma^{-\frac{2}{3}}}$ using Lemma 5.2.1.

Let the collection $\{\varphi_i\}_{|i|>10\sigma^{-\frac{2}{3}}}$ be the densities of the random vectors

$$\left\{\left(Z,\tilde{Z}\right) + \left(G, G_i^*\left(\sigma^{\frac{2}{3}} i\right) + \xi_i\right)\right\}_{|i|>10\sigma^{-\frac{2}{3}}}$$

respectively so that we may write

$$\iint_{\left[h\pm\varepsilon\right]\times\left[0\pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]} \rho_i(h_1,h_2)dh_1dh_2 = \iint_{\left[h\pm\varepsilon\right]\times\left[-g_i\left(\sigma^{\frac{2}{3}} i\right)\pm 2\varepsilon\sigma^{-\frac{2}{3}}\right]} \varphi_i(h_1,h_2)dh_1dh_2.$$

We will bound $\rho_i(h_1,h_2)$ by first bounding $\varphi_i(h_1,h_2)$ and then translating the second coordinate of the latter by $-g_i\left(\sigma^{\frac{2}{3}} i\right)$. Lemma 5.2.1 ensures that φ_i has a bounded density because $\left(Z', Z\right)$ is a bivariate Gaussian. We will now find an explicit bound on each of these densities ρ_i.

By the definition of an absolutely continuous probability density function with respect to the

Lebesgue measure, we can write for each i and all $(h_1,h_2)\in\mathbb{R}^2$ that

$$\begin{aligned}
&\varphi_i(h_1,h_2)\\
&=\mathbb{P}\Big(Z'+G\in dh_1, Z+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2\Big)\\
&\le\mathbb{P}\left(Z'+G\in dh_1, Z+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2\,\middle|\, Z\ge\frac{h_2}{2}\right)\mathbb{P}\left(Z\ge\frac{h_2}{2}\right)+\\
&\quad\mathbb{P}\left(Z'+G\in dh_1, Z+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2\,\middle|\, G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\ge\frac{h_2}{2}\right)\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\ge\frac{h_2}{2}\right).
\end{aligned}\tag{6.4.1}$$

Before proceeding further, we take a moment to review the distribution of a truncated bivariate Gaussian random vector. This is slightly more complicated than it was in the first regime since $\left(Z,\tilde{Z}\right)$ does not have independent components like $\left(N',\tilde{N}\right)$ did, but the same general ideas will still apply here as well.

By the definition of a truncated random vector in $\mathbb{R}^2$, we know that for any $h_2\ge 0$, the conditional law $\mathrm{Law}\left(\left(Z,\tilde{Z}\right)\middle|\{\tilde{Z}\ge h_2\}\right)$ has a density given by

$$f_{(Z,\tilde{Z})|\{\tilde{Z}\ge h_2\}}=\frac{1}{\mathbb{P}\left(\tilde{Z}\ge h_2\right)}f_{(Z,\tilde{Z})}\mathbb{1}_{\{\mathbb{R}\times[h_2,\infty)\}}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$. Using (6.2.4) we have that for all $(z,\tilde{z})\in\mathbb{R}^2$,

$$f_{(Z,\tilde{Z})|\{\tilde{Z}\ge h_2\}}(z,\tilde{z})=\frac{\mathbb{1}_{\{\tilde{z}\ge h_2\}}}{4\pi\mathbb{P}\left(\tilde{Z}\ge h_2\right)}\exp\left(-\frac{1}{8}\left(\left(4\sigma^{-\frac{2}{3}}+1\right)z^2+8\sigma^{-\frac{1}{3}}z\tilde{z}+4\tilde{z}^2\right)\right)$$

The exponent above as a function of $(z,\tilde{z})$ has a single critical point at the origin and is clearly unbounded below on the region $\mathbb{R}^2$ as $||(z,\tilde{z})||\to\infty$. From this, we can infer that the only extrema of the conditional density above over the region $\mathbb{R}\times[h_2,\infty)$ will be a global maximum along the boundary curve $\tilde{z}=h_2$. Upon making the substitution $\tilde{z}=h_2$ and optimizing the resulting exponent as a function of z, we see that the maximum occurs at the point

$$(z,\tilde{z})=\left(\frac{-4\sigma^{-\frac{1}{3}}h_2}{8\sigma^{-\frac{2}{3}}+h_2},\ h_2\right).$$

By evaluating the conditional density at this maximizer, we obtain the uniform bound on $\mathbb{R}\times[h_2,\infty)$

$$f_{(Z,\tilde{Z})|\{\tilde{Z}\ge h_2\}}(z,\tilde{z})\le\frac{1}{4\pi\mathbb{P}\left(\tilde{Z}\ge h_2\right)}\exp\left(-\frac{h_2^2}{8\sigma^{-\frac{2}{3}}+2}\right).\tag{6.4.2}$$

With this established, we can now develop our upper bound on (6.4.1).

For convenience, we will again define for each $\delta>0$ and $(h_1,h_2)\in\mathbb{R}\times[0,\infty)$, the sets

$$E_{h_1,h_2,\delta}:=[h_1,h_1+\delta]\times[h_1,h_2+\delta].$$

By mimicking our work for the density bounds in the first regime, we can then say that

$$\begin{aligned}
&\mathbb{P}\Big(Z'+G\in dh_1, \tilde{Z}+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2 \;\Big|\; \tilde{Z}\geq \frac{h_2}{2}\Big)\\
&=\lim_{\delta\to 0^+}\frac{1}{\delta^2}\mathbb{P}\left(Z'+G\in(h_1,h_1+\delta), \tilde{Z}+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in(h_2,h_2+\delta) \;\Big|\; \tilde{Z}\geq\frac{h_2}{2}\right)\\
&=\lim_{\delta\to 0+}\frac{1}{\delta^2}\operatorname{Law}\left(\left(Z',\tilde{Z}\right)\;\Big|\;\left\{\tilde{Z}\geq\frac{h_2}{2}\right\}\right)*\operatorname{Law}\left(\left(G,G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\right)\right)\left(E_{h_1,h_2,\delta}\right)\\
&\leq\lim_{\delta\to 0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}\left|\left|f_{(Z,\tilde{Z})|\{\tilde{Z}\geq\frac{h_2}{2}\}}(u,v)\right|\right|_\infty dudv\right) \qquad (6.4.3)\\
&\leq\lim_{\delta\to 0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}\frac{1}{4\pi\mathbb{P}\left(\tilde{Z}\geq h_2\right)}\exp\left(-\frac{\left(\frac{h_2}{2}\right)^2}{8\sigma^{-\frac{2}{3}}+2}\right)dudv\right)\\
&=\frac{1}{4\pi\mathbb{P}\left(\tilde{Z}\geq h_2\right)}\exp\left(-\frac{h_2^2}{32\sigma^{-\frac{2}{3}}+8}\right)
\end{aligned}$$

where the inequality (6.4.3) follows from the fact that

$$\operatorname{Law}\left(\left(Z',\tilde{Z}\right)\;\Big|\;\left\{\tilde{Z}\geq\frac{h_2}{2}\right\}\right)*\operatorname{Law}\left(\left(G,G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\right)\right)$$

has a density uniformly bounded above by $\left|\left|f_{(Z,\tilde{Z})|\{\tilde{Z}\geq h_2\}}(u,v)\right|\right|_\infty$ via Lemma 5.2.1. Moreover, because we are assuming that $|i|>10\sigma^{-\frac{2}{3}}$ and $\sigma^{-1}\geq 10$ we can bound this even further as

$$\mathbb{P}\Big(Z'+G\in dh_1, \tilde{Z}+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2 \;\Big|\; \tilde{Z}\geq h_2/2\Big)\leq\frac{1}{\mathbb{P}\left(\tilde{Z}\geq h_2\right)}\exp\left(-\frac{1}{4}\left(\frac{h_2}{|i|}\right)^2\right). \quad (6.4.4)$$

Similarly, with these same conventions we also obtain that

$$\begin{aligned}
&\mathbb{P}\Big(Z+G\in dh_1, \tilde{Z}+G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\in dh_2 \;\Big|\; G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\geq\frac{h_2}{2}\Big)\\
&=\lim_{\delta\to 0^+}\frac{1}{\delta^2}\mathbb{P}\Big(|Z+G-h_1|\leq\delta, \left|\tilde{Z}+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i-h_2\right|\leq\delta \;\Big|\; \left\{G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\geq\frac{h_2}{2}\right\}\Big)\\
&=\lim_{\delta\to 0^+}\frac{1}{\delta^2}\operatorname{Law}\left(\left(Z,\tilde{Z}\right)+\left(G,G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\right)\;\Big|\;\left\{G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\geq\frac{h_2}{2}\right\}\right)\left(E_{h_1,h_2,\delta}\right)\\
&=\lim_{\delta\to 0^+}\frac{1}{\delta^2}\operatorname{Law}\left(\left(Z,\tilde{Z}\right)\right)*\operatorname{Law}\left(\left(G,G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\right)\;\Big|\;\left\{G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\geq\frac{h_2}{2}\right\}\right)\left(E_{h_1,h_2,\delta}\right)\\
&\leq\lim_{\delta\to 0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}\left|\left|f_{(Z,\tilde{Z})}(u,v)\right|\right|_\infty dudv\right)\\
&\leq\lim_{\delta\to 0^+}\frac{1}{\delta^2}\left(\int_{h_1}^{h_1+\delta}\int_{h_2}^{h_2+\delta}1dudv\right)=1.
\end{aligned}$$

By combining these observations with equation (6.4.1), we get the density bound

$$\begin{aligned}\varphi_i(h_1,h_2) =& \ \mathbb{P}\Big(Z+G\in dh_1, \tilde{Z}+G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\in dh_2\Big)\\ \le& \ \exp\left(-\frac{1}{4}\left(\frac{h_2}{|i|}\right)^2\right)+\mathbb{P}\Big(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\ge\frac{h_2}{2}\Big).\end{aligned}$$

Using our previous observation that $h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)$ is always non-negative in the second regime, and the definition of φ_i in terms of ρ_i, this implies that

$$\begin{aligned}\rho_i(h_1,h_2) =& \ \varphi_i\left(h_1,h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)\right)\\ \le& \ \exp\left(-\frac{1}{4}\left(\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{|i|}\right)^2\right)+\mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}}z_i^*)+\xi_i\ge\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right).\end{aligned}$$

Thus, we now have that for all $(h_1,h_2)\in\mathbb{R}\times\left[0\pm 2\varepsilon\sigma^{-\frac{1}{3}}\right]$,

$$\begin{aligned}&\sum_{|i|>10\sigma^{-\frac{2}{3}}}\rho_i(h_1,h_2)\\ \le& \sum_{|i|>10\sigma^{-\frac{2}{3}}}\exp\left(-\frac{1}{4}\left(\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{|i|}\right)^2\right)+\sum_{|i|>10\sigma^{-\frac{2}{3}}}\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)+\xi_i\ge\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right).\end{aligned}$$

We will address the summability of the second tail series in the following two subsections. Proving that the first series converges and is bounded by an absolute constant will again be a consequence of Lemma 5.4.1. In particular, by recalling (6.3.5) we have that for all i in the second regime,

$$\begin{aligned}\frac{h_2-g_i\left(\sigma^{\frac{2}{3}}i\right)}{|i|} =& \ \left(1+\frac{3\sigma}{s}\right)|i|+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x\sigma^{\frac{1}{3}}}{s}\right)\operatorname{sign}(i)-\frac{\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}+\frac{1}{s}\sigma^{\frac{1}{3}}-h_2\right)}{|i|}\\ \ge& \ |i|+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x\sigma^{\frac{1}{3}}}{s}\right)\operatorname{sign}(i)-\frac{\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}+\frac{1}{s}\sigma^{\frac{1}{3}}-h_2\right)}{|i|}\\ =:& \ |i|+\beta_1\operatorname{sign}(i)+\frac{\beta_2}{|i|}.\end{aligned}$$

where we also recall that $|i|+\beta_1\operatorname{sign}(i)+\frac{\beta_2}{|i|}$ is always nonnegative when $|i|>10\sigma^{-\frac{2}{3}}$. As was the case in the first regime, based on our definition of the threshold ε_0, we will always have that $\beta_2\le 0$.

So by invoking Lemma 5.4.1 we see that

$$\begin{aligned}
&\sum_{|i|>10\sigma^{-\frac{2}{3}}} \exp\left(-\frac{1}{4}\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right)}{|i|}\right)^2\right) \\
&\leq \sum_{|i|>10\sigma^{-\frac{2}{3}}} \exp\left(-\frac{1}{4}\left(|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|}\right)^2\right) \\
&\leq \sum_{i\neq 0} \exp\left(-\frac{1}{4}\left(|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|}\right)^2\right) \mathbb{1}_{\left\{|i|+\beta_1 \operatorname{sign}(i)+\frac{\beta_2}{|i|}\geq 0\right\}} \\
&\leq \frac{2}{1-\exp\left(-\frac{1}{4}\right)}. \qquad (6.4.5)
\end{aligned}$$

We will now show that the remaining tail series has a similar geoemtric bound.

6.5 Managing the Fluctuation ξ_i for Large i

To build our density bound in the second regime, we will have to understand

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right) + \xi_i \geq \frac{h_2}{2} - \frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right). \qquad (6.5.1)$$

As before, the problem will be ξ_i since the righthand side is guaranteed to always be positive in the second regime. The trick is to bound ξ_i from above by something deterministic like last time. In particular, because $(x,s) \in [1,2] \times \left[1, \frac{11}{10}\right]$ and $\sigma \in \left(0, \frac{1}{10}\right)$, we have that

$$\xi_i = -2\left(\Delta\sigma^{-\frac{2}{3}} + \frac{(1+\sigma)}{s}i - \frac{x\sigma^{\frac{1}{3}}}{s}\right)p_i \leq |\Delta|\sigma^{-\frac{2}{3}} + \frac{(1+\sigma)}{s}|i| + \frac{x\sigma^{\frac{1}{3}}}{s} \leq |\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2. \qquad (6.5.2)$$

This means that

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}i\right) + \xi_i \geq \frac{h_2}{2} - \frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2}\right) \leq \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}i\right) \geq \frac{h_2}{2} - \frac{g_i\left(\sigma^{\frac{2}{3}}i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)\right)$$

which we will have to bound trivially by 1 for all $|i| > 10\sigma^{-\frac{2}{3}}$ such that

$$h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) \geq 0 \qquad \text{and} \qquad h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) - 2\left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right) < 0. \qquad (6.5.3)$$

How many such bad integers i exist in the second regime?

As we explained earlier,

$$h_2 - g_i\left(\sigma^{\frac{2}{3}}i\right) = \left(1 + \frac{3\sigma}{s}\right)i^2 + \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x\sigma^{\frac{1}{3}}}{s}\right)i - \left(2\Delta^2\sigma^{-\frac{4}{3}} + \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}} - h_2\right)$$

so our system of inequalities can be written more explicitly as

$$\left(1+\frac{3\sigma}{s}\right)i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x\sigma^{\frac{1}{3}}}{s}\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}+\frac{1}{s}\sigma^{\frac{1}{3}}-h_2\right)\geq 0$$

and

$$\left(1+\frac{3\sigma}{s}\right)i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-\frac{2x\sigma^{\frac{1}{3}}}{s}\right)i-4|i|-\left(2\Delta^2\sigma^{-\frac{4}{3}}+\frac{x^2}{s}\sigma^{-\frac{1}{3}}+\frac{1}{s}\sigma^{\frac{1}{3}}-h_2+2|\Delta|\sigma^{-\frac{2}{3}}+4\right)<0.$$

We will split this task into two cases again depending on sign(i). First consider $i<-10\sigma^{-\frac{2}{3}}$. In this case, all solutions to the system above will solve the weaker system of inequalities

$$(1+3\sigma)\,i^2+\left(2\Delta\sigma^{-\frac{2}{3}}+4\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}\right)\geq 0$$

and

$$\left(1+\frac{30\sigma}{11}\right)i^2+\left(2\Delta\sigma^{-\frac{2}{3}}+4\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+2|\Delta|\sigma^{-\frac{2}{3}}+5\sigma^{-\frac{1}{3}}+5\right)<0.$$

As before, consider the simplified system

$$\left(1+\frac{3}{a}\right)i^2+\left(2ba^2+4\right)i-\left(2b^2a^4\right)\geq 0$$

and

$$\left(1+\frac{30}{11a}\right)i^2+\left(2ba^2+4\right)i-\left(2b^2a^4+2|b|a^2+5a+5\right)<0$$

where $-1\leq b\leq 1$, $a\geq 2$, and $i<-10a^2$. Using Mathematica, this system has no integer solutions $i<-10a^2$. Hence neither does our actual system of inequalities. Intuitively, this is once again because the second parabola opens upwards and both of its roots have magnitudes strictly smaller than $10\sigma^{-\frac{2}{3}}$.

Now suppose that $i>10\sigma^{-\frac{2}{3}}$. In this case, all solutions of the original system of inequalities will belong to the solution set of the weaker system

$$(1+3\sigma)\,i^2+\left(2\Delta\sigma^{-\frac{2}{3}}\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}\right)\geq 0$$

and

$$\left(1+\frac{30\sigma}{11}\right)i^2+\left(2\Delta\sigma^{-\frac{2}{3}}-12\right)i-\left(2\Delta^2\sigma^{-\frac{4}{3}}+2|\Delta|\sigma^{-\frac{2}{3}}+5\sigma^{-\frac{1}{3}}+5\right)<0.$$

We now pass to the analogous simplified system of inequalities

$$\left(1+3a^{-1}\right)i^2+\left(2ba^2\right)i-\left(2b^2a^4\right)\geq 0$$

and

$$\left(1+\frac{30a^{-1}}{11}\right)i^2+\left(2ba^2-12\right)i-\left(2b^2a^4+2|b|a^2+5a+5\right)<0$$

where $-1 \le b \le 1$, $a \ge 2$, and $i > 10a^2$. Once again, using Mathematica we see that this system also has no integer solutions $i > 10a^2$. Hence neither does our actual system of inequalities. Given these observations, we conclude that (6.5.3) never occurs when $|i| > 10\sigma^{-\frac{2}{3}}$, and so it is always true that

$$h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right) \ge 0$$

in the second regime. This will allow us to use our yet-to-be derived upper tail bound of $G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right)$ directly on the probability

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \ge \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)\right)$$

without having to throw away any initial indices i. We now construct this upper tail bound.

6.6 An Upper Tail Bound for $G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right)$ for Large i

The last step of our argument is to find upper tail bound for $G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right)$ in second regime and to prove that that tail bound is summable. We begin by letting $m > 0$ be arbitrary and considering the probability

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \ge 5m\right).$$

Recalling (6.3.4), we have the decomposition in law

$$G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \stackrel{d}{=} \frac{\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}} z_i^* + \Delta\right) + \mathcal{N}\left(0, \left|2\sigma^{\frac{2}{3}} z_i^* - 2\sigma^{\frac{2}{3}} i - 2\sigma^{\frac{1}{3}}\right|\right)}{\sigma^{\frac{1}{3}}} + \left(\frac{s}{\sigma}\right)^{-\frac{1}{3}} A_2 z_i^* + \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} (C_2 - C_1)$$

so we will define the collection of five random variables $\left\{G_{i,j}^*\left(z_i^*\right)\right\}_{j=1}^5$ by

$$G_{i,1}^*\left(z_i^*\right) := \sigma^{-\frac{1}{3}} \tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}} z_i^* + \Delta\right)$$
$$G_{i,2}^*\left(z_i^*\right) := \sigma^{-\frac{1}{3}} \mathcal{N}\left(0, \left|2\sigma^{\frac{2}{3}} z_i^* - 2\sigma^{\frac{2}{3}} i - 2\sigma^{\frac{1}{3}}\right|\right)$$
$$G_{i,3}^*\left(z_i^*\right) := (s\sigma)^{-\frac{1}{3}} A_2 z_i^*$$
$$G_{i,4}^*\left(z_i^*\right) := \left(\frac{s}{\sigma}\right)^{\frac{1}{3}} C_2$$
$$G_{i,5}^*\left(z_i^*\right) := -\left(\frac{s}{\sigma}\right)^{\frac{1}{3}} C_1.$$

With these definitions, we can now write more compactly that

$$G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \stackrel{d}{=} \sum_{j=1}^{5} G_{i,j}^*\left(z_i^*\right).$$

Based on this equality in law, we can say now say that via a union bound,

$$\mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}}z_i^*\right)\geq 5m\right)=\mathbb{P}\left(\sum_{j=1}^{5}G_{i,j}^*\left(z_i^*\right)\geq 5m\right)\leq\sum_{j=1}^{5}\mathbb{P}\Big(G_{i,j}^*\left(z_i^*\right)\geq m\Big). \tag{6.6.1}$$

We will now establish exponential upper tail bounds for all five of the summands above. We start with the simplest bound, which is when $j=1$. Just as we observed in the first regime, by equation (6.1.4), there are two absolute constants $c_1, c_2>0$ such that

$$\mathbb{P}\Big(G_{i,1}^*\left(z_i^*\right)\geq m\Big)=\mathbb{P}\Big(\sigma^{-\frac{1}{3}}\tilde{\mathcal{F}}^{(\sigma)}\left(\sigma^{\frac{2}{3}}z_i^*+\Delta\right)\geq m\Big)\leq c_1e^{-c_2m^{\frac{3}{2}}}.$$

The case when $j=2$ is similarly simple to compute. Using the facts that $|i|>10\sigma^{-\frac{2}{3}}$, $s\in\left[1,\frac{11}{10}\right]$, $\sigma^{-1}\geq 10$, and that $|z_i^*-i|=|p_i|\leq\frac{1}{2}$, we arrive at the chain of upper bounds

$$\begin{aligned}
\mathbb{P}\Big(G_{i,2}^*\left(z_i^*\right)\geq m\Big) &= \mathbb{P}\left(\sigma^{-\frac{1}{3}}\mathcal{N}\left(0,\left|2\sigma^{\frac{2}{3}}z_i^*-2\sigma^{\frac{2}{3}}i-2\sigma^{\frac{1}{3}}\right|\right)\geq m\right)\\
&\leq \mathbb{P}\left(\left|\sigma^{-\frac{1}{3}}\mathcal{N}\left(0,\left|2\sigma^{\frac{2}{3}}z_i^*-2\sigma^{\frac{2}{3}}i-2\sigma^{\frac{1}{3}}\right|\right)\right|\geq m\right)\\
&\leq \mathbb{P}\left(\sigma^{-\frac{1}{3}}\sqrt{\left|2\sigma^{\frac{2}{3}}\left(z_i^*-i\right)-2\sigma^{\frac{1}{3}}\right|}\left|\mathcal{N}\left(0,1\right)\right)\right|\geq m\right)\\
&\leq \mathbb{P}\left(|i|\sqrt{\sigma^{\frac{2}{3}}+2\sigma^{\frac{1}{3}}}\left|\mathcal{N}\left(0,1\right)\right)\right|\geq m\right)\\
&\leq \mathbb{P}\Big(2|i|\left|\mathcal{N}\left(0,1\right)\right)\right|\geq m\Big)\\
&\leq 2\exp\left(-\frac{1}{8}\left(\frac{m}{|i|}\right)^2\right).
\end{aligned}$$

We now consider the case when $j=3$. Based on equation (6.1.6) and the domains of σ and s,

$$\begin{aligned}
\mathbb{P}\left(G_{i,2}^*\left(z_i^*\right)\geq m\right) &= \mathbb{P}\left(\left(\frac{s}{\sigma}\right)^{-\frac{1}{3}}A_2z_i^*\geq m\right)\\
&\leq \mathbb{P}\left(\left(\frac{s}{\sigma}\right)^{-\frac{1}{3}}|A_2|\left|z_i^*\right|\geq m\right)\\
&\leq \mathbb{P}\left(\left(|i|+\frac{1}{2}\right)|A_2|\geq m\right)\\
&\leq \mathbb{P}\left(|A_2|\geq\frac{m}{2|i|}\right)\\
&\leq c_1\exp\left(-c_2\left(\frac{m}{2|i|}\right)^{\frac{3}{2}}\right).
\end{aligned}$$

For $j=4$ and $j=5$ we first recall that we have set $a_1=s^{-\frac{2}{3}}\sigma^{\frac{1}{3}}$ and $a_2=s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}i$. Using these once more in conjunction with (6.1.6) we have that

$$\begin{aligned}
\mathbb{P}\Big(G^*_{i,4}\left(z^*_i\right) \geq m\Big) &= \mathbb{P}\left(\left(\frac{s}{\sigma}\right)^{\frac{1}{3}} C_2 \geq m\right) \\
&\leq \mathbb{P}\left(2\sigma^{-\frac{1}{3}}|C_2| \geq m\right) \\
&\leq c_1 \exp\left(-c_2\left(\frac{m\sigma^{\frac{1}{3}}}{2|a_2|}\right)^{\frac{3}{2}}\right) \\
&= c_1 \exp\left(-c_2\left(\frac{m\sigma^{\frac{1}{3}}}{2s^{-\frac{2}{3}}\sigma^{\frac{2}{3}}|i|}\right)^{\frac{3}{2}}\right) \\
&= c_1 \exp\left(-c_2 s\sigma^{-1}\left(\frac{m}{2|i|}\right)^{\frac{3}{2}}\right) \\
&\leq c_1 \exp\left(-10c_2\left(\frac{m}{2|i|}\right)^{\frac{3}{2}}\right)
\end{aligned}$$

and by the exact same reasoning with $j = 5$, our final upper tail bound

$$\mathbb{P}\Big(G^*_{i,5}\left(z^*_i\right) \geq m\Big) = \mathbb{P}\left(-\left(\frac{s}{\sigma}\right)^{\frac{1}{3}} C_1 \geq m\right) \leq c_1 \exp\left(-c_2\left(\frac{m\sigma^{\frac{1}{3}}}{2|a_1|}\right)^{\frac{3}{2}}\right) \leq c_1 \exp\left(-10c_2\left(\frac{m}{2}\right)^{\frac{3}{2}}\right).$$

By combining these five individual tail bounds with (6.6.1), we see that

$$\begin{aligned}
\mathbb{P}\left(G^*_i\left(\sigma^{\frac{2}{3}} z^*_i\right) \geq 5m\right) &\leq c_1 e^{-c_2 m^{\frac{3}{2}}} + 2e^{-\frac{1}{8}\left(\frac{m}{|i|}\right)^2} + c_1 e^{-\frac{c_2}{4}\left(\frac{m}{|i|}\right)^{\frac{3}{2}}} + c_1 e^{-10c_2\left(\frac{m}{2|i|}\right)^{\frac{3}{2}}} + c_1 e^{-10c_2\left(\frac{m}{2}\right)^{\frac{3}{2}}} \\
&\leq c_1' e^{-c_2'\left(\frac{m}{|i|}\right)^{\frac{3}{2}}} \qquad (6.6.2)
\end{aligned}$$

as $m \to \infty$ for some absolute constants $c_1', c_2' > 0$. By specializing to the case where

$$5m = \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)$$

we therefore have established the upper tail bound

$$\begin{aligned}
&\mathbb{P}\left(G^*_i\left(\sigma^{\frac{2}{3}} z^*_i\right) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)\right) \\
&\leq c_1' \exp\left(-c_2'\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{10|i|} - \frac{|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2}{5|i|}\right)^{\frac{3}{2}}\right) \\
&\leq c_1' \exp\left(-10^{-\frac{3}{2}} c_2'\left(\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)}{|i|}\right)^{\frac{3}{2}}\right).
\end{aligned}$$

Given this, if we now set the constants $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ to be

$$(\beta_1, \beta_2, \beta_3) := \left(2\Delta\sigma^{-\frac{2}{3}} - \frac{2x}{s}\sigma^{\frac{1}{3}},\ -\left(2\Delta^2\sigma^{-\frac{4}{3}} + 2|\Delta|\sigma^{-\frac{2}{3}} - \frac{x^2}{s}\sigma^{-\frac{1}{3}} + \frac{1}{s}\sigma^{\frac{1}{3}} - h_2 + 4\right),\ -4\right)$$

then we may write that

$$\frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)}{|i|} = |i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3$$

with $\beta_2 < 0$. Thus by using the fact that $|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3$ is non-negative whenever $|i| > 10\sigma^{-\frac{2}{3}}$, Lemma 5.4.1 yields that

$$\begin{aligned}
&\sum_{|i|>10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*(\sigma^{\frac{2}{3}} z_i^*) + \xi_i \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2}\right) \\
\leq &\sum_{|i|>10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)\right) \\
= &\sum_{|i|>10\sigma^{-\frac{2}{3}}} \mathbb{P}\left(G_i^*\left(\sigma^{\frac{2}{3}} z_i^*\right) \geq \frac{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right)}{2} - \left(|\Delta|\sigma^{-\frac{2}{3}} + 2|i| + 2\right)\right) \mathbb{1}_{\left\{h_2 - g_i\left(\sigma^{\frac{2}{3}} i\right) - 2\left(\Delta\sigma^{-\frac{2}{3}} + 2|i| + 2\right) \geq 0\right\}} \\
\leq &\sum_{|i|>10\sigma^{-\frac{2}{3}}} c_1' \exp\left(-10^{-\frac{3}{2}} c_2' \left(|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3\right)^{\frac{3}{2}}\right) \mathbb{1}_{\left\{|i| + \beta_1 \operatorname{sign}(i) + \frac{\beta_2}{|i|} + \beta_3 \geq 0\right\}} \\
< &\frac{2c_1'}{1 - \exp\left(10^{-\frac{3}{2}} c_2'\right)}.
\end{aligned}$$

In conjunction with (6.4.5), setting $b'' = \frac{2c_1'}{1-\exp\left(10^{-\frac{3}{2}} c_2'\right)} + \frac{2}{1-\exp\left(-\frac{1}{4}\right)}$ completes (6.2.7) and thus establishes the original bound (4.3.1), completing our proof of the two-point bound on $[1,2] \times \left[1, \frac{11}{10}\right]$.
□

Chapter 7

Further Ideas

Due to the amount of concessions and arbitrary decisions made throughout the course of chapters 5 and 6, it is likely that there is a large amount of room for further optimization. Some possible optimizations and improvements for these chapters include but are not limited to:

- **Change the rectangle** $[1,2] \times \left[1, \frac{11}{10}\right]$: Beyond satisfying the requirement that the coordinate (x_0, t_0) of the bottom-left corner of the rectangle that we use in Theorem 4.3.1 has both $x_0, t_0 > 0$, this choice was made purely for convenience. Even the imposition that the rectangle have side lengths at most 1 is not strictly necessary; it would simply mean that there is possibly more tedious work to sort through when developing the two-point bound. It is quite likely that experimenting with the location and size of this rectangle will improve the lower bound p_h to some extent.

- **Develop a sharper, piecewise two-point bound:** As mentioned before, the choice to build a global two-point bound was made because it was simpler but it also certainly quite far from optimal. Using a global upper bound removes quite a bit of sharpness from the two-point bound, which in turn leads to a larger second moment for the measures $\mu_{h,\varepsilon}$ in Chapter 4. These larger second moments then shrink the probability p_h. Optimizing this two-point bound will almost certainly lead to a noticeably higher value of p_h.

- **Improve the lower bound on** $\dim_H(Z_h)$ **to** $\frac{5}{3}$: Based on the Hölder continuity of the directed landscape, it is expected that $\frac{5}{3}$ is the Hausdorff dimension of all of its level sets. It is highly unlikely he lower bound of $\frac{3}{2}$ that we obtained in this thesis is the true Hausdorff dimension of the $h-$level sets, and improving the sharpness of the two-point bound will likely bridge this gap.

- **Building a different mass distribution on** Z_h: As mentioned before, the measures $\mu_{h,\varepsilon}$ who had a subsequential limit which was a mass distribution on Z_h were chosen because they are easy to understand intuitively. There is no inherent reason to believe that they are the optimal choice for the directed landscape specifically. Replacing them with a different sequence of random measures more finely attuned to the structure of $\mathcal{L}$ may lead to a better mass distribution than the one we built in the general case.

- **Find an alternative to the Paley-Zigmund inequality:** The Paley-Zigmund inequality was our only means of verifying that the random measure μ_h was actually a mass distribution on Z_h with positive probability. The quality of the lower bound that we obtain from this inequality places somewhat of a ceiling on the probability of our lower bound on the Hausdorff dimension holding. Finding a way to upgrade this and alter Theorem 4.1.1 could prove very fruitful, but this feels like somewhat of an unrealistic goal, at least in the general case, based on the existing tools in the literature.

- **Extend the argument to a three-point bound and beyond:** It is my personal belief that though it would likely be very cumbersome, these same techniques can also be used to develop a rudimentary three-point bound for $\mathcal{L}$ and possibly even an $n-$point bound in general. Dauvergne's full version of Theorem 1.2.3 works for any finite collection of intervals $\{[a_j, a_j + T_0]\}_{j=1}^n$ instead of just a pair of intervals, and so Lemma 6.1.1 should extend to the case of n disjoint segments of $\mathfrak{A}_1$ fairly easily. Lemma 5.1.1 could likely still be used in much the same way for a three-point bound argument and possibly an $n-$point bound argument. This would likely require an $n-$fold sum and taking several suprema at once, though the exact form that these details would take is also unclear to me. The key would be to impose the condition that $0 < t_1 < t_2 < \cdots < t_n$ and observe that

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\Big\{\mathcal{L}(0,0;x_k,t_k) \in (h-\varepsilon, h+\varepsilon)\Big\}\right)$$
$$\leq \mathbb{P}\left(\Big\{|\mathcal{L}(0,0;x_1,t_1) - h| < \varepsilon\Big\} \cap \bigcap_{k=1}^{n-1}\Big\{|\mathcal{L}(0,0;x_{k+1},t_{k+1}) - \mathcal{L}(0,0;x_k,t_k)| < (k+1)\varepsilon\Big\}\right).$$

Similar techniques with the metric composition law and the independent temporal increments of $\mathcal{L}$ would likely eventually work to bound this upper bound here too, although likely at the expense of a daunting amount of tedium and regret.

Bibliography

[1] Robert J. Adler. *The geometry of random fields*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Ltd., Chichester, 1981, pp. xi+280. ISBN: 0-471-27844-0.

[2] Riddhipratim Basu and Manan Bhatia. *A Peano curve from mated geodesic trees in the directed landscape*. 2023. arXiv: 2304.03269 [math.PR]. URL: https://arxiv.org/abs/2304.03269.

[3] Erik Bates, Shirshendu Ganguly, and Alan Hammond. *Hausdorff dimensions for shared endpoints of disjoint geodesics in the directed landscape*. 2021. arXiv: 1912.04164 [math.PR]. URL: https://arxiv.org/abs/1912.04164.

[4] Manan Bhatia. *Duality in the directed landscape and its applications to fractal geometry*. 2024. arXiv: 2301.07704 [math.PR]. URL: https://arxiv.org/abs/2301.07704.

[5] Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*. Cambridge University Press, 2017.

[6] Ivan Corwin. *The Kardar-Parisi-Zhang equation and universality class*. 2011. arXiv: 1106.1596 [math.PR]. URL: https://arxiv.org/abs/1106.1596.

[7] Ivan Corwin and Alan Hammond. "Brownian Gibbs property for airy line ensembles". In: *Inventiones mathematicae* 195.2 (Mar. 2013), pp. 441–508. DOI: 10.1007/s00222-013-0462-3.

[8] Duncan Dauvergne. *Wiener densities for the Airy line ensemble*. 2023. arXiv: 2302.00097 [math.PR].

[9] Duncan Dauvergne, Janosch Ortmann, and Bálint Virág. "The directed landscape". In: *Acta Math.* 229.2 (2022), pp. 201–285. ISSN: 0001-5962.

[10] Duncan Dauvergne and Bálint Virág. "Bulk properties of the Airy line ensemble". In: *The Annals of Probability* 49.4 (May 2021). ISSN: 0091-1798. DOI: 10.1214/20-aop1492. URL: http://dx.doi.org/10.1214/20-AOP1492.

[11] Duncan Dauvergne and Bálint Virág. "The scaling limit of the longest increasing subsequence". In: (Apr. 2021). arXiv: 2104.08210 [math.PR].

[12] E. A. Feinberg, P. O. Kasyanov, and Y. Liang. "Fatou's lemma for weakly converging measures under the uniform integrability condition". In: *Teor. Veroyatn. Primen.* 64.4 (2019), pp. 771–790. ISSN: 0040-361X.

[13] Shirshendu Ganguly and Lingfu Zhang. *Fractal geometry of the space-time difference profile in the directed landscape via construction of geodesic local times*. 2022. arXiv: 2204.01674 [math.PR]. URL: https://arxiv.org/abs/2204.01674.

[14] John Ben Hough et al. *Zeros of gaussian analytic functions and Determinantal Point Processes*. Vol. 51. American Mathematical Society, 2012.

[15] Kurt Johansson. "Shape Fluctuations and Random Matrices". In: *Communications in Mathematical Physics* 209.2 (2000), pp. 437–476.

[16] Olav Kallenberg. *Foundations of modern probability*. Vol. 99. Probability Theory and Stochastic Modelling. Third edition [of 1464694]. Springer, Cham, 2021, pp. xii+946. ISBN: 978-3-030-61870-4.

[17] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. "Dynamic scaling of growing interfaces". In: *Physical Review Letters* 56.9 (Mar. 1986), pp. 889–892. DOI: 10.1103/physrevlett.56.889.

[18] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010. ISBN: 9781139486576. URL: https://books.google.ca/books?id=e-TbA-dSrzYC.

[19] Jeremy Quastel. "Introduction to KPZ". In: *Current Developments in Mathematics* 2011.1 (2011). DOI: 10.4310/cdm.2011.v2011.n1.a3.