On Global Solutions of the Parabolic Anderson Model and Directed Polymers in Random Environment

by

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Abstract

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and

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This thesis studies global solutions to the semidiscrete stochastic heat equation and the associated Cauchy problem known as Parabolic Anderson Model. Via a Feynman-Kac formula, it is linked with the analysis of directed polymers in random environment, and this thesis establishes a number of results for the corresponding partition function.

We consider a continuous-time simple symmetric random walk on the integer lattice $\mathbb{Z}^d$ in dimension $d \geq 3$, subject to a random potential given by two-sided Wiener processes. In the high-temperature regime, we prove the existence of the L^2- and almost sure limit of the partition function as time $t \to \pm\infty$. We show that the L^2-convergence rate is at least polynomial and that the limiting partition function is positive almost surely. Furthermore, we show that this limiting partition function defines a global stationary solution to the semidiscrete stochastic heat equation which is unique up to a rescaling, and which in some sense attracts solutions to the Parabolic Anderson Model for any subexponentially growing initial data. One of the primary tools in the proof of this uniqueness and attraction result is a factorization formula for the point-to-point partition function, which is related to the ones obtained by Sinai (1995) and Kifer (1997) for other polymer models, but valid not only on the diffusive scale but up to any sub-ballistic scale. This factorization formula allows us to obtain a uniqueness result for physical invariant probability measures of a certain skew product that can be naturally associated with the semidiscrete stochastic heat equation, which in turns gives uniqueness of global stationary solutions.

Synopsis

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Introduction

This thesis addresses the problem of global solutions to the ***semidiscrete stochastic heat equation***:

$$\partial_t u = \Delta u + \beta F^{\omega} u, \tag{sSHE}$$

where $u = u(x,t)$ is a scalar function on the semi-discrete spacetime $\mathbb{Z}^d \times \mathbb{R}$, where Δ is the discrete Laplacian, F^{ω} is a random potential, and $\beta > 0$ is the coupling constant. The Cauchy problem for (sSHE) is known as the ***parabolic Anderson model***. This stochastic partial differential equation is naturally linked via a Feynman-Kac formula with the ***Anderson polymer model***, namely continuous-time directed polymers in the random environment given by the potential F^{ω}.

The main goal of this thesis is to prove that global stationary solutions to the semi-discrete stochastic heat equation (sSHE) are unique up to rescaling at the origin.

This introduction begins with a motivation and history of the two main actors in this thesis, directed polymers in random environment and the sSHE. We also describe the recent work and problems and limitations, setting up the stage for a brief outline of the main results of this thesis.

Contents

1.1. Background.

1.1.1. Directed Polymers in Random Environment

The central role in our analysis of (sSHE) is played by directed polymers in random environment, which are models in statistical mechanics where a stochastic process interacts with its spacetime environment through a random potential. One studies the path of a given stochastic process under a random Gibbs measure that depends on a parameter which is thought of as representing the temperature in the system: as the temperature increases, the influence of the random environment decreases. Many concrete physical systems may be modelled as directed polymers in this way, from elastic strings to bacterial colonies. They were first introduced in the 1980s in physics in the context of ferromagnetism by Huse and Henley [HH85] in order to study the phase boundary of the Ising model subject to random impurities. They then found additional applications in physics in a variety of contexts from tearing sheets of paper [KHW93], understanding topological-defect turbulence in electrically driven liquid crystals [TS10, TS12], to modelling kinetic roughening of growing surfaces [KS91]. The first mathematical treatment was undertaken in the late 1980s by Imbrie and Spencer [IS88] and Bolthausen [Bol89]. A modern treatment of directed polymers in random environments can be found in the recent textbook by Comets [Com17].

The spacetime is parametrized by (x,t) where the space and the time are most typically either continuous (i.e., $x \in \mathbb{R}^d$ for some dimension $d \geq 1$ and $t \in \mathbb{R}$) or discrete (i.e., $x \in \mathbb{Z}^d$ and $t \in \mathbb{Z}$). We will usually say ***continuous spacetime*** to mean $\mathbb{R}^d \times \mathbb{R}$, ***discrete spacetime*** to mean $\mathbb{Z}^d \times \mathbb{Z}$, and ***semidiscrete spacetime*** to mean $\mathbb{Z}^d \times \mathbb{R}$. A point of the lattice $\mathbb{Z}^d$ is often called a ***site***. Sometimes spaces with more interesting topology are considered, most typically a d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ which is used to model periodic systems.

The two most typical choices for the underlying stochastic process are a *Brownian motion* in continuous space $\mathbb{R}^d$ and a *random walk* in discrete space $\mathbb{Z}^d$. In this thesis, we shall only deal with the latter: a ***random walk*** η on an integer lattice $\mathbb{Z}^d$ (sometimes called a *hypercubic lattice*) describes the motion of a walker on $\mathbb{Z}^d$ that jumps at discrete time steps $t = 1, 2, 3, \ldots$ to a new, randomly chosen, site. The most well-studied example is the ***simple symmetric random walk***, in which the "walker" is only allowed to jump from one site to any of its nearest neighbours with equal probabilities. Then a ***directed polymer*** is the graph $(t, \eta(t))_t$ of the random walk η parametrized by time t.

In discrete space, the random environment is given by a family $F = \{F^x(t) : x \in \mathbb{Z}^d\}$ of independent identically distributed random processes. The energy (often called the *Hamiltonian potential*) that a path η acquires in this environment over a time interval $[s,t]$ is given as the integral

$$H_s^t(\eta) := \int_s^t F^{\eta_\tau}(\tau)d\tau. \tag{1.1}$$

Of course, in case of discrete time, this integral should be interpreted as summation. The statistical properties of such a system are then encoded in the ***point-to-point partition function*** given by the following path-integral formula: for any pair of sites x, y and any pair of moments of time s, t,

$$Z_{x,s}^{y,t} := p_{t-s}^{y-x}\mathbf{E}_{x,s}^{y,t}e^{\beta H_s^t} \tag{1.2}$$

where $\mathbf{E}_{x,s}^{y,t}$ is the expectation obtained by conditioning on random walks starting at x at time s and ending at y at time t, and p_{t-s}^{y-x} is the transition probability of the continuous-time simple symmetric random walk going from x to y in time $t-s$. The parameter $\beta > 0$ is called the ***inverse temperature***, which we will discuss in more detail below.

Noise. There are two standard choices for the distribution of the noise F in discrete space:

- A *Bernoulli environment* is the case where $F^x(t) = \pm 1$, each with probability $1/2$. Such models were studied in discrete time by Imbrie and Spencer [IS88], Bolthausen [Bol89], and Song and Zhou [SZ96]. The continuous-time version of this model was first studied in 1995 by Coyle [Coy95, Coy96].
- A *Gaussian environment* is the case where $F^x(t)$ is a standard normal random variable. Such systems were studied in discrete-time by P. Carmona and Hu [CH02].

Some authors consider more general environments. For example, one may consider F to be given by independent identically distributed random variables with finite exponential moment. Important examples are provided by normal distributions and distributions of bounded support. Such models were investigated in discrete spacetime by Sinai [Sin95] and Song and Zhou [SZ96], and later by Vargas [Var06] in both discrete and continuous spacetimes. Other environments in continuous spacetime have also been considered, for example, by Kifer [Kif97].

Weak and Strong Disorder. The inverse temperature parameter β describes the strength of the disorder or the extent to which the directed polymer interacts with the random environment. When $\beta > 0$ is small (i.e., in the high temperature), the interaction is weak, and when $\beta > 0$ is large (i.e., low temperature regime), the interaction is strong. Of particular interest is to understand the longterm behaviour of the directed polymer system; in other words, we are interested in studying the asymptotic behaviour as $t \to \infty$. A directed polymer can be either ***localized*** (the *strong disorder regime*; i.e., the endpoint distribution has bounded variance with large probability) or ***diffusive*** (the *weak disorder regime*; i.e., the variance grows linearly in time). It was shown in [CH02, CY06] that strong disorder always takes place in the low dimensional cases $d = 1, 2$, while in higher dimensions $d \geq 3$ there is a transition from weak to strong disorder as the inverse temperature β increases.

The weak disorder regime has been studied since the late 1980s. In the case $d \geq 3$ and small β, the diffusive behaviour of directed polymers was established in discrete spacetime by Imbrie and Spencer [IS88], Bolthausen [Bol89], and Sinai [Sin95], and in continuous spacetime by Kifer [Kif97] and by Comets, Shiga, and Yoshida [CSY03]. In semidiscrete spacetime, Coyle [Coy95, Coy96] showed diffusivity in a Bernoulli environment.

1.1.2. Parabolic Anderson Model

The analysis of directed polymers in random environment is also known as the ***Anderson Polymer Model***, a name which goes back to the work of the Nobel-prize winning physicist P. W. Anderson on entrapment of electrons in crystals with impurities [And58]. In the mid 1990s, R. Carmona and Molchanov [CM94] (see also [CMS02, CH06]) built a bridge between the analysis of the Anderson polymer model and the analysis of the Cauchy problem for the stochastic heat equation,

$$\begin{cases} \partial_t u = \Delta u + \beta F^\omega u \\ u\big|_{t=0} = f \end{cases} \tag{PAM}$$

which is often called the ***Parabolic Anderson Model***. Specifically, using a Feynman–Kac formula they proved the following statement:

Proposition 1.1 (Proposition 5 in [CH06])
The point to point partition function $Z_{0,0}^{x,t}$ *satisfies the semidiscrete stochastic heat equation* (sSHE).

1.1.3. Random Hamilton-Jacobi and Burgers Equations.

An important motivation for studying (sSHE) is its strong connection with the randomly forced Hamilton-Jacobi equation and the closely related randomly forced Burgers equation. These equations have been very actively studied in the last 30 years, initially motivated by the physics of turbulence-type behaviour, the so-called "burgulence" [FB01, BK07], and later by the connection with the Kardar-Parisi-Zhang (KPZ) equation for the random growth of interfaces [KPZ86].

The Hamilton-Jacobi equation. In its most general form, the *random Hamilton-Jacobi equation* can be written as follows:

$$\partial_t \phi + H(\nabla \phi) = \nu \Delta \phi + F^\omega, \tag{HJ}$$

where ϕ is a scalar function on spacetime $\mathbb{R}^d \times \mathbb{R}$, H is the Hamiltonian, F^ω is a time-dependent random potential, $\nu \geq 0$ is the parameter called *viscosity*, and ∇, Δ are the spacial gradient and spacial Laplacian, respectively. The Hamilton-Jacobi equation has a very long history, naturally arising as in the Hamiltonian approach to classical mechanics.

The Burgers equation. The most important example of a random Hamilton-Jacobi equation corresponds to the quadratic Hamiltonian $H(p) = p^2/2$, where $p \in \mathbb{R}^d$ are the momenta, because it is the only case when the nonlinear equation (HJ) can be linearized, placing it among the simplest physically interesting nonlinear partial differential equations. Indeed, the Lagrangian $L(v) := \max_p [\langle p, v\rangle - H(p)]$ in this case is also quadratic, $L(v) = v^2/2$, and the map between Legendre conjugate variables p and v is the identity; namely, $v(p) = p$. It follows that the velocity field $v = \nabla \phi$ satisfies the *random vector Burgers equation*:

$$\partial_t v + \langle v, \nabla \rangle v = \nu \Delta v + \nabla F^\omega. \tag{Burg}$$

An unforced version of the Burgers equation (Burg) was first introduced in the late 1930s by J.M. Burgers [Bur39] as a one-dimensional model for the dynamics of pressure-less gas, guided by the observation that this equation is a simple model that has similar invariances, conservation laws, and type of hydrodynamical nonlinearity as the Navier-Stokes equation. However, it is the presence of a random forcing term in (Burg) that can be used in order to describe the physics of turbulence-type behaviour, the so-called "burgulence" [FB01, BK07]. The randomly forced Burgers equation has been extensively studied in the last 30 years in both physics [KS91, CY95, GM96, MM97] and mathematics [Sin91, BCJ94a, BCJ94b, DDT94, HLØ+94, HLØ+95, DG95, BG97, Kif97, DD98] literature.

Finally, rescaling the time variable t by ν and using the *Hopf-Cole transformation* [Hop50, Col51]

$$\phi = -2\nu \log u, \tag{1.3}$$

we obtain the stochastic heat equation:

$$\partial_t u = \Delta u + \beta F^\omega u. \tag{SHE}$$

In this way, all three equations (HJ), (Burg), (SHE) are equivalent when the Hamiltonian is quadratic.

Types of Noise. The interpretation of equations (HJ), (Burg), (SHE) depends on the type of the noise term. Normally, the potential F^ω is stationary in space and time and it is important to assume that its correlations decay fast in both space and time. It is usually also assumed to be white in time; this assumption does not affect the regularity of the equation. However, a white in space potential creates very serious difficulties on small scales: in this case, solutions of (HJ) are forced to have very low regularity, making sense of which is a nontrivial problem addressed by Martin Hairer [Hai13, Hai14]. In fact, in the case of a spacetime white noise, the Burgers equation (Burg) has to be understood in some generalized sense where the product $\langle v, \nabla\rangle$ is interpreted as a Wick product, and the solution is given as a distribution-valued process [HLØ+94, HLØ+95]. But if one is interested in large-scale spacetime behaviour (as is the case in this thesis), it can be assumed that the realizations of the potential F^ω are smooth in space: this avoids regularity issues.

Global Solutions in the Compact Case. The case of space-periodic potentials $F^\omega(x+k,t) = F^\omega(x,t)$ for $k \in \mathbb{Z}^d$, has been extensively studied in the series of papers [Sin91, EKMS00, IK03, GIKP05, KZ17], resulting in an almost complete understanding of global solutions in this case. The average velocity

$$b = \int_{\mathbb{T}^d} v(x)dx,$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, is the first integral for the Burgers equation (Burg). The corresponding time-invariant set for (HJ) and (SHE) consists of functions $\phi(x) = \langle b, x\rangle + \psi(x)$ and $u(x) = \exp(\langle b, x\rangle)w(x)$, where $\psi(x)$ and $w(x)$ are periodic: $\psi(x+k) = \psi(x)$, $w(x+k) = w(x)$, $k \in \mathbb{Z}^d$. One of the main results in this periodic case can be formulated in the following way.

Theorem 1.2 ([Sin91],[IK03])
Almost surely for all $b \in \mathbb{R}^d$ and all $\nu \geq 0$, there exists a unique (up to an additive constant) global solution to the random Hamilton-Jacobi equation (HJ) with space-periodic potential and with average velocity b.

Uniqueness in the periodic viscous case ($\nu > 0$) follows from [Sin91], and [IK03] deals with the inviscid case ($\nu = 0$). Of course this result implies the uniqueness of global solutions to the random Burgers equation (Burg) and thanks to the Hopf-Cole transformation, this result also implies the uniqueness (up to a multiplicative constant) of global solutions to the random stochastic heat equation (SHE).

Global Solutions in the Noncompact Case. In the non-periodic case the situation is much more difficult. Although it was conjectured in [BK18] that the almost sure uniqueness of global solutions still holds, at present there are very few mathematical results supporting this conjecture.

Conjecture 1.1 ([BK18]). *For any $d \geq 1$ and any $F^\omega(t,x)$ with exponential decay of space-time correlations, the one force-one solution principle holds; in other words, for every $b \in \mathbb{R}^d$, there is a unique time-stationary (modulo time-dependent additive constants) global solution $\phi_{b,\omega}(t,x) = b \cdot x + \psi_{b,\omega}(t,x)$ where $\psi_{b,\omega}$ has sublinear growth.*

Uniqueness has been established in the case $d = 1$ for the quadratic Hamiltonian in the viscous and inviscid situations under certain assumptions on the forcing [RASY13, BCK14, BL16, BL17]. Some uniqueness results have also been established for the quadratic Hamiltonian in the case $d \geq 3$ with weak forcing. For example, Kifer [Kif97] was able to deduce uniqueness of global solutions to the Burgers equation, but only in a rather weak sense: namely, the solutions to the Cauchy problem converge to the unique global solution if the initial condition $v(\cdot, 0)$ is L^2-stationary. Since different global solutions to the random Hamilton-Jacobi equation depend on a parameter $b \in \mathbb{R}^d$ corresponding to a class of functions of the form $\phi(x,t) = \langle b, x \rangle + \psi(x,t)$ where ψ has sublinear growth in $||x||$ [BK18, IK03], one really has to establish convergence for all $\phi(\cdot, 0)$ with sublinear growth in case $b = 0$. Via the Hopf-Cole transformation, this means that in terms of the stochastic heat equation, the convergence to the global solution has to be established for all initial conditions with subexponential growth; i.e., of the type $u(\cdot, 0) = \exp w(\cdot)$ where w is a function of sublinear growth.

This thesis was mainly inspired by our efforts to address these difficulties. In the next section, we explain what we have achieved.

1.2. Main Results

The principal goal in this thesis is to prove that global solutions (with subexponential growth in space) to the semi-discrete stochastic heat equation (sSHE) are unique up to a normalization at the origin in space. Since our motivation comes from the question of uniqueness for the stochastic partial differential equations (HJ)-(SHE), we work in a continuous-time setting. At the same time, we work in discrete space because it allows for a more transparent presentation. In this section, we describe in detail our setup and main results. Note that the main body of the thesis (i.e., beginning with Chapter 2) can be read completely independently from this section.

The results in this thesis were obtained in collaboration with Dr. Tobias Hurth and Prof. Konstantin Khanin. Their support and encouragement during the work on the thesis are highly appreciated.

1.2.1. The Setup

Noise realizations. For $d \geq 3$, let Ω be the set of functions $\omega : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{Z}^d$, the function $t \mapsto \omega(x,t)$ is continuous and satisfies $\omega(x,0) = 0$. Each $\omega \in \Omega$ represents a realization of the noise in our stochastic model. Let $\mathcal{F}$ denote the canonical σ-field on Ω, and let Q be the probability measure on $(\Omega, \mathcal{F})$ under which $(W^x)_{x\in\mathbb{Z}^d}$, defined by $W_t^x(\omega) := \omega(x,t)$, are independent two-sided Wiener processes. Expectation corresponding to Q will be denoted by $\langle\cdot\rangle$. For any time $s \in \mathbb{R}$, let $\theta_s : \Omega \to \Omega$ be the ***Wiener shift*** defined by

$$\theta_s\big(\omega(x,t)\big) := \omega(x,t+s) - \omega(x,s),$$

for all $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$; i.e., every path $\omega(x,\cdot)$ is shifted by s to the left along the time axis and normalized to equal 0 at time $t = 0$. The probability measure Q is invariant with respect to $(\theta_s)_{s\in\mathbb{R}}$, in the sense that for every $s \in \mathbb{R}$ and for every $A \in \mathcal{F}$, one has $Q(\theta_s(A)) = Q(A)$.

The Parabolic Anderson Model. Let $f : \mathbb{Z}^d \to (0,\infty)$ be any function of subexponential growth and decay in space; namely, such that for a sufficiently small $\epsilon \in (0,1)$, it satisfies

$$\lim_{\|x\|\to\infty} \frac{|\ln(f(x))|}{\|x\|^{1-\epsilon}} = 0. \tag{1.4}$$

We fix $s \in \mathbb{R}$ and $\beta > 0$, and consider the following Cauchy problem for the semidiscrete stochastic heat equation (sSHE), also known as the ***Parabolic Anderson Model*** (PAM):

$$\begin{cases} \partial_t u(y,t) = \Delta_y u(y,t) + \beta u(y,t)\dot{W}_t^y, \quad y \in \mathbb{Z}^d,\ t > s, \\ u(y,s) = f(y), \quad y \in \mathbb{Z}^d. \end{cases} \tag{PAM}$$

Here, Δ_y is the discrete Laplacian given by

$$\Delta_y u(y,t) := \frac{1}{2d} \sum_{z \in \mathbb{Z}^d : \|y-z\|_1 = 1} \big(u(y,t) - u(z,t)\big),$$

and $\dot{W}_t^y$ is the white noise associated with W_t^y. We emphasise that most studies of the PAM consider bounded or even localised initial data, whereas the initial data considered in this thesis is in a much more general class, namely those having subexponential growth as in (1.4).

Directed polymers in a random potential. For any $(x,s) \in \mathbb{Z}^d \times \mathbb{R}$, let $\eta = (\eta_t)_{t \geq s}$ be a continuous-time simple symmetric random walk (SSRW) on $\mathbb{Z}^d$ starting at $\eta_s = x$. The corresponding probability measure is denoted by $\mathbf{P}_{x,s}$ and the corresponding expectation by $\mathbf{E}_{x,s}$. We assume that the jumps of η occur at random times given by independent exponential clocks; i.e., the times between consecutive jumps form an i.i.d. sequence of exponential random variables with rate 1. Note that η is transient because $d \geq 3$. If observed over a time interval $[s,t)$, a sample path of η (which we shall also denote by η) is characterized by

(1) the number $n_{s,t}$ of jumps that occur within the time interval (s,t),

(2) a discrete-time path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{n_{s,t}})$ on $\mathbb{Z}^d$ such that $\gamma_0 = x$ and $\|\gamma_j - \gamma_{j-1}\|_1 = 1$ for $1 \leq j \leq n_{s,t}$, and

(3) the jump times $s < s_1 < \ldots < s_{n_{s,t}} < t$.

It is convenient to introduce the notation $s_0 := s$ and $s_{n_{s,t}+1} := t$, although we do not assume that s and t are jump times. If $s = 0$, we will typically write n_t instead of $n_{0,t}$. To a sample path η and a realization of the noise $\omega \in \Omega$, we assign the action defined by

$$\mathcal{A}_s^t(\eta,\omega) := \sum_{j=0}^{n_{s,t}} \big(\omega(\gamma_j, s_{j+1}) - \omega(\gamma_j, s_j)\big). \tag{1.5}$$

For any time $t > s$ and any site $y \in \mathbb{Z}^d$, denote the probability measure obtained from $\mathbf{P}_{x,s}$ by conditioning on the event $\{\eta_t = y\}$ by $\mathbf{P}^{y,t}_{x,s}$. The corresponding expectation is denoted by $\mathbf{E}^{y,t}_{x,s}$. Also set

$$p^y_t := \mathbf{P}_{0,0}(\eta_t = y).$$

Partition functions. For every $\omega \in \Omega$, we define the random normalized ***point-to-point partition function*** by

$$Z^{y,t}_{x,s}(\omega) := e^{-\frac{\beta^2}{2}(t-s)} p^{y-x}_{t-s}\, \mathbf{E}^{y,t}_{x,s} e^{\beta \mathcal{A}^t_s(\cdot,\omega)}. \tag{1.6}$$

We also define

$$Z^t_{x,s}(\omega) := \sum_{y \in \mathbb{Z}^d} Z^{y,t}_{x,s}(\omega), \qquad \text{and} \qquad Z^{y,t}_s(\omega) := \sum_{x \in \mathbb{Z}^d} Z^{y,t}_{x,s}(\omega). \tag{1.7}$$

Since $e^{-\frac{\beta^2}{2}(t-s)} \langle e^{\beta \mathcal{A}^t_s(\eta,\cdot)} \rangle = 1$ for every η, these partition functions are normalized in the sense that

$$\langle Z^t_{x,s} \rangle = \langle Z^{y,t}_s \rangle = 1.$$

Notice that the law of the stochastic process $(Z^{s+t}_{x,s})_{t\geq 0}$ with respect to Q does not depend on x or s because the law for the increments of the Wiener processes $(W^x)_{x\in\mathbb{Z}^d}$ is stationary in space and time, and because the SSRW η is homogeneous. Besides, $(Z^{s+t}_{x,s})_{t\geq 0}$ and $(Z^{x,s}_{s-t})_{t\geq 0}$ have the same law because of time-reversibility of η.

It was shown in [CM94] that the solution to the Cauchy problem (PAM), if interpreted as an integral equation in the sense of Itô, is given by

$$u^s_f(y,t) := \sum_{x \in \mathbb{Z}^d} f(x) Z^{y,t}_{x,s}, \quad t \geq s. \tag{1.8}$$

Note that $u^s_f(y,s) = f(y)$. In the special case that $s = 0$, we usually write u_f instead of u^0_f. This result can be viewed as a Feynman–Kac formula for the semidiscrete parabolic Anderson model.

1.2.2. Uniqueness of Global Stationary Solutions to the Semi-Discrete Stochastic Heat Equation

We can think of any solution $u_f^s(y,t)$ to (PAM) given by (1.8) as being local in time in the sense that it is defined only for a finite time interval (s,t). The primary focus in this thesis is instead the analysis of solutions which are global in time in the following sense.

Definition 1.3

Let $\Omega' \in \mathcal{F}$ such that $Q(\Omega') = 1$ and $\theta_t(\Omega') = \Omega'$ for all $t \in \mathbb{R}$. A measurable map $Z : \mathbb{Z}^d \times \mathbb{R} \times \Omega' \to \mathbb{R}$ is called a ***global stationary solution*** to (sSHE) if:

(1) For every $y \in \mathbb{Z}^d$, $s,t \in \mathbb{R}$ with $s < t$, and $\omega \in \Omega'$,

$$Z(y,t,\omega) = \sum_{x \in \mathbb{Z}^d} Z(x,s,\omega) Z_{x,s}^{y,t}(\omega);$$

(2) For every $y \in \mathbb{Z}^d$, $t \in \mathbb{R}$, and $\omega \in \Omega'$, we have $Z(y,t,\omega) = Z(y,0,\theta_t \omega)$.

This is Definition 6.1 in the main body of the thesis. The first main result in this thesis is a construction of a particular global stationary solution to (sSHE) that arises as a limit of the partition functions for the parabolic Anderson model for directed polymers in random potential. Namely, we derive the following convergence result, including rate of convergence, for the partition functions (1.7) in the regime of small β.

Theorem 1.4

If β is sufficiently small, the following statements hold.

(1) *For all $(x,s),(y,t) \in \mathbb{Z}^d \times \mathbb{R}$, the partition functions $Z_{x,s}^t$ and $Z_s^{y,t}$ converge in $L^2(Q)$ as $t \to \infty$ and $s \to -\infty$ respectively to the* ***limiting partition functions***

$$Z_{x,s}^{\infty} := \lim_{t \to \infty} Z_{x,s}^t \qquad \text{and} \qquad Z_{-\infty}^{y,t} := \lim_{s \to -\infty} Z_s^{y,t}.$$

In fact, there is $\theta > 0$, independent of x, y and s, t, such that

$$\lim_{t \to \infty} (t-s)^{\theta} \left\langle \left(Z_{x,s}^t - Z_{x,s}^{\infty}\right)^2 \right\rangle = 0 \quad \text{and} \quad \lim_{s \to -\infty} (t-s)^{\theta} \left\langle \left(Z_s^{y,t} - Z_{-\infty}^{y,t}\right)^2 \right\rangle = 0.$$

(2) *There is a subset $\Omega^+ \subset \Omega$ with $Q(\Omega^+) = 1$, such that for all $(x,s),(y,t) \in \mathbb{Z}^d \times \mathbb{R}$ and all $\omega \in \Omega^+$, the limiting partition functions $Z_{x,s}^{\infty}(\omega)$ and $Z_{-\infty}^{y,t}(\omega)$ exist and are positive.*

This is Theorem 3.1 in the main body of the thesis. The limiting partition function $Z_{-\infty}^{y,t}(\omega)$ is of special interest to us because it defines a particular global solution to the (sSHE).

Proposition 1.5

There is an $\mathcal{F}$-measurable subset $\Omega^{\text{sol}} \subset \Omega^{+}$ with $Q(\Omega^{\text{sol}}) = 1$ such that the function

$$\begin{aligned} \mathbb{Z}^d \times \mathbb{R} \times \Omega^{\text{sol}} &\to \mathbb{R} \\ (y, t, \omega) &\mapsto Z_{-\infty}^{y,t}(\omega) \end{aligned} \tag{1.9}$$

is a global stationary solution to (sSHE).

This is Proposition 6.2 in the main body of the text. The main result of this thesis is that, up to normalisation at the origin $0 \in \mathbb{Z}^d$, this global solution $Z_{-\infty}^{y,t}(\omega)$ is unique, implying in particular that the solutions to (PAM) have a rather weak dependence on the initial data.

Theorem 1.6

Let Z be a global stationary solution to (sSHE) *which, Q-almost surely, has subexponential growth in space and satisfies $Z(0, t, \omega) \neq 0$. Then, for β sufficiently small and Q-almost surely for all $(y, t) \in \mathbb{Z}^d \times \mathbb{R}$,*

$$\frac{Z(y, t, \omega)}{Z(0, t, \omega)} = \frac{Z_{-\infty}^{y,t}(\omega)}{Z_{-\infty}^{0,t}(\omega)}.$$

This is Theorem 6.3 in the main body of the thesis. Explicitly, by Z having subexponential growth in space we mean that, there exists $\epsilon > 0$ such that for all t and almost every $\omega \in \Omega$, we have

$$\lim_{\|x\| \to \infty} \frac{\left|\ln\left(Z(x, t, \omega)\right)\right|}{\|x\|^{1-\epsilon}} = 0.$$

Let us make a few remarks regarding our assumptions.

(1) We cannot formulate a corresponding result for the random Hamilton-Jacobi equation (HJ) or the random Burgers equation (Burg) since the Hopf-Cole transformation is not readily available in the setting of discrete space. However, we certainly believe that our results and methods can be extended to the continuous-space setting.

(2) We also note that we have only considered the case $b = 0$, but we expect that the extension of our result to the case of all $b \in \mathbb{R}^d$ should be relatively straightforward. Indeed, in the case of continuous space, this extension would be a simple consequence of shear-invariance, whereas continuous shearing in discrete space will require minor technical adjustments.

(3) The potential F^ω in our model is the white noise associated with space-independent two-sided standard Wiener processes. We remark that the independence in x and the white behaviour in t are purely technical conditions that simplify our proof, and that they can be weakened by considering weakly dependent potentials. At the same time, the Gaussianity of the potential is more important because some of our methods are inspired by Talagrand's approach which requires Gaussian distributions. Nevertheless, we believe that it is still a technical condition.

1.2.3. Attraction to the Unique Global Solution

The first major step towards Theorem 1.6 is a result which says that the particular global stationary solution $Z^{y,t}_{-\infty}$ from (1.9) attracts solutions to the Cauchy problem (PAM) with any subexponentially growing initial data f. For any $c > 0$ and $\epsilon \in (0,1)$, let $\mathcal{L}_{c,\epsilon}$ be the set of functions $f : \mathbb{Z}^d \to (0,\infty)$ such that

$$\big|\ln(f(x))\big| \le c\|x\|^{1-\epsilon}, \qquad \forall x \in \mathbb{Z}^d. \tag{1.10}$$

Note that this condition implies $f(0) = 1$, and is equivalent to

$$e^{-c\|x\|^{1-\epsilon}} \le f(x) \le e^{c\|x\|^{1-\epsilon}}, \quad \forall x \in \mathbb{Z}^d.$$

However, we emphasize that the condition $f(0) = 1$ is not essential and we could consider

$$\big|\ln(f(x))\big| \le c_0 + c\|x\|^{1-\epsilon}, \qquad \forall x \in \mathbb{Z}^d. \tag{1.11}$$

for some $c_0 \ge 0$ instead of (1.10).

Theorem 1.7

For β sufficiently small, the following holds: for every $y \in \mathbb{Z}^d$ and for every $c > 0$, $\epsilon \in (0,1)$, we have

$$\sup_{f\in\mathcal{L}_{c,\epsilon}} \left|\frac{u_f(y,t)}{u_f(0,t)} - \frac{Z^{y,t}_{-\infty}}{Z^{0,t}_{-\infty}}\right| \xrightarrow[t\to\infty]{} 0 \quad \textit{in probability.} \tag{1.12}$$

This is Theorem 6.4 in the main body of the text.

1.2.4. Factorization Formula

The proof of Theorem 1.7 relies on a factorization formula for the point-to-point partition function. In the high dimension, high temperature regime (i.e., with $d \geq 3$ and small β), Sinai, Kifer, and Vargas [Sin95, Kif97, Var06] established a factorization formula for the point-to-point partition function $Z_{x,s}^{y,t}$ for different polymer models. This formula can be viewed as a Local Limit Theorem for directed polymers in random environment. Their results are related to the behaviour on the diffusive scale $\|x\| = O(t^{1/2})$. In [Kif97], Kifer used this kind of factorization formula to show that solutions to the Cauchy problem for the stochastic heat equation converge to the unique stationary global solution if the initial condition is L^2-stationary. However, since we are dealing with fast growing initial conditions, it is necessary to extend the analysis far beyond the diffusive scale, basically up to the ballistic scale $\|x\| = O(t)$. In order to deal with this issue, we prove the following factorization result for partition functions corresponding to polymers with endpoints far away from each other.

Theorem 1.8

For β sufficiently small, the following holds: For any $\sigma \in (0,1)$ there exists $\theta = \theta(\sigma) > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $s < t$ with $\|x-y\| < (t-s)^\sigma$, the partition function $Z_{x,s}^{y,t}$ has the representation

$$Z_{x,s}^{y,t} = p_{t-s}^{y-x} \left(Z_{x,s}^{\infty} Z_{-\infty}^{y,t} + \delta_{x,s}^{y,t} \right), \tag{1.13}$$

where the error term $\delta_{x,s}^{y,t}$ defined by the formula above satisfies

$$\lim_{(t-s)\to\infty} (t-s)^\theta \sup_{x,y\in\mathbb{Z}^d : \|x-y\| < (t-s)^\sigma} \langle |\delta_{x,s}^{y,t}| \rangle = 0. \tag{1.14}$$

This is Theorem 4.1 in the main body of the thesis. Although this result is similar in spirit to the results obtained by Sinai in [Sin95, Theorem 2] and Kifer in [Kif97, Theorem 6.1], we emphasize that the novelty of Theorem 1.8 consists in extending the uniform smallness of the error term far beyond the diffusive regime $\|x-y\| < (t-s)^{\frac{1}{2}}$. Intuitively, this factorization formula says that, even if x and y are far away, conditionally on the event $(\eta_t = y)$, the polymer only "feels" the environment at times close to s when it stays near x and at times close to t when it stays near y, and that for the majority of time in between, it behaves like a conditioned simple random walk.

1.2.5. A Lower Tail Estimate of the Probability Distribution for the Partition Function: Talagrand's Method.

One of the key steps in the proof of Theorem 1.7 is to obtain a lower tail estimate of the probability distribution for the partition function $Z_0^{y,t}$. Such estimates in the case of discrete spacetime have been obtained by P. Carmona and Hu [CH02] using concentration of measure arguments for discrete directed polymers in Gaussian environments that originated in Talagrand's work on spin glasses [Tal98, Tal11]. In contrast, in this thesis, we work in a semidiscrete spacetime in higher dimensions ($d \geq 3$), and we therefore prove the following continuous-time version of [CH02, Theorem 1.5] (see also [Mor14, Theorem 1(a)]).

Theorem 1.9

For β sufficiently small, there exists a constant $c > 0$ such that

$$Q\left(Z_0^{y,t} < e^{-u}\right) < ce^{-u^2/c}, \quad t, u > 0.$$

This is Theorem 5.1 in the main body of the thesis.

1.2.6. Uniqueness of Physical Invariant Probability Measures

After the convergence result Theorem 1.7, the second major step towards Theorem 1.6 goes through the theory of random dynamical systems; namely, we show uniqueness of physical invariant probability measures of a certain skew product that can be naturally associated with the (sSHE).

Recall the sets $\mathcal{L}_{c,\epsilon}$, defined for any $c > 0$ and $\epsilon \in (0,1)$, consisting of functions satisfying (1.10). Define

$$\mathcal{L} := \bigcup_{\substack{c>0 \\ \epsilon\in(0,1)}} \mathcal{L}_{c,\epsilon}, \tag{1.15}$$

which can be thought of as the set of functions $f : \mathbb{Z}^d \to (0, \infty)$ of subexponential asymptotic growth and decay, normalized by imposing $f(0) = 1$. Notice that $\mathcal{L}$ is exactly the set of functions $f : \mathbb{Z}^d \to (0, \infty)$ that satisfy $f(0) = 1$ as well as the condition in (1.4). Note also that for any global stationary solution Z to sSHE from Theorem 1.6 (namely, with subexponential growth and with $Z(0, t, \omega) \neq 0$) the quotient $Z(y, t, \omega)/Z(0, t, \omega)$ is an element of $\mathcal{L}$.

Although the set of functions of subexponential growth has the structure of a vector space, the set

$\mathcal{L}$ is not a vector space due to the requirement that every $f \in \mathcal{L}$ must satisfy $f(0) = 1$. However, $\mathcal{L}$ can be equipped with the structure of a metric space and the corresponding Borel σ-field $\mathcal{B}(\mathcal{L})$ using the metric

$$d(f, g) := \sum_{x \in \mathbb{Z}^d} e^{-\|x\|^2} |f(x) - g(x)|.$$

For $\omega \in \Omega$, $s, t \in \mathbb{R}$ such that $s < t$, and $f \in \mathcal{L}$, we define

$$L_\omega^{s,t} f(y) := \frac{\sum_{x \in \mathbb{Z}^d} f(x) Z_{x,s}^{y,t}(\omega)}{\sum_{x \in \mathbb{Z}^d} f(x) Z_{x,s}^{0,t}(\omega)}, \quad y \in \mathbb{Z}^d.$$

Lemma 1.10

The set $\mathcal{L}$ is Q-almost surely invariant under the dynamics induced by L, i.e. for Q-almost every $\omega \in \Omega$ the following holds: for every $f \in \mathcal{L}$ and for every $s, t \in \mathbb{R}$ such that $s < t$, we have $L_\omega^{s,t} f \in \mathcal{L}$.

Lemma 1.11

There is a set $\widetilde{\Omega} \subset \Omega$ with $Q(\widetilde{\Omega}) = 1$ that satisfies the following conditions:

(1) *$\widetilde{\Omega}$ is invariant under θ_s for every $s \in \mathbb{R}$, i.e. $\theta_s(\widetilde{\Omega}) = \widetilde{\Omega}$ for every $s \in \mathbb{R}$;*

(2) *For every $\omega \in \widetilde{\Omega}$, $f \in \mathcal{L}$, and $s, t \in \mathbb{R}$ such that $s < t$, we have $L_\omega^{s,t} f \in \mathcal{L}$;*

(3) *For every $\omega \in \widetilde{\Omega}$,*

$$\lim_{s \to -\infty} Z_s^{x,0}(\omega) = Z_{-\infty}^{x,0}(\omega) > 0, \quad \forall x \in \mathbb{Z}^d;$$

(4) *For every $\omega \in \widetilde{\Omega}$, the function $x \mapsto \widetilde{Z}_{-\infty}^{x,0}(\omega) := Z_{-\infty}^{x,0}(\omega)/Z_{-\infty}^{0,0}(\omega)$ is an element of $\mathcal{L}$.*

Lemma 1.12

The map $\Phi : [0, \infty) \times \widetilde{\Omega} \times \mathcal{L} \to \mathcal{L}$, given by

$$(t, \omega, f) \mapsto \Phi_\omega^t f := L_\omega^{0,t} f,$$

defines a cocycle; i.e., for all $s, t \geq 0$ and all $\omega \in \widetilde{\Omega}$,

$$\Phi_\omega^{s+t} = \Phi_{\theta_s \omega}^t \circ \Phi_\omega^s.$$

These are Lemmas 6.10, 6.11, and 6.12 in the main body of the thesis. It is not hard to see that Φ is a $(\mathcal{B}([0, \infty)) \otimes \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L}), \mathcal{B}(\mathcal{L}))$-measurable map, where $\mathcal{B}([0, \infty))$ is the Borel σ-field on $[0, \infty)$

and $\widetilde{\mathcal{F}}$ is the restriction of $\mathcal{F}$ to $\widetilde{\Omega}$. On $\widetilde{\Omega} \times \mathcal{L}$, we define the skew product

$$\Theta^t(\omega, f) := (\theta_t(\omega), \Phi^t_\omega f), \quad t \geq 0.$$

Definition 1.13

An ***invariant probability measure*** for the skew product $(\Theta^t)_{t\geq 0}$ is a probability measure μ on $(\widetilde{\Omega} \times \mathcal{L}, \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L}))$ such that

(1) μ has marginal $\widetilde{Q}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$, where $\widetilde{Q}$ is the restriction of Q to $\widetilde{\Omega}$;

(2) $\mu((\Theta^t)^{-1}(\cdot)) = \mu(\cdot), \quad \forall t \geq 0.$

This is Definition 6.13 in the main body of the thesis. If μ is an invariant probability measure for $(\Theta^t)_{t\geq 0}$, then there exists a family $(\mu^\omega)_{\omega \in \widetilde{\Omega}}$ of probability measures on $(\mathcal{L}, \mathcal{B}(\mathcal{L}))$, so-called sample measures, such that for every $A \in \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L})$,

$$\mu(A) = \int_{\widetilde{\Omega}} \mu^\omega(A_\omega)\, Q(d\omega),$$

where $A_\omega := \{f \in \mathcal{L} : (\omega, f) \in A\}$. Notice that sample measures are just conditional distributions under the condition of fixed ω.

Theorem 1.14

The skew product $(\Theta^t)_{t\geq 0}$ admits a unique invariant probability measure whose sample measures are given by

$$\mu^\omega(\cdot) = \delta_{y \mapsto \widetilde{Z}^{y,0}_{-\infty}(\omega)}(\cdot), \quad \omega \in \widetilde{\Omega}.$$

This is Theorem 6.14 in the main body of the thesis.

Remark 1.15

One can define a Markov semigroup $(\mathbf{P}^t)_{t\geq 0}$ on $\mathcal{L}$ by setting

$$\mathbf{P}^t(f, F) := \widetilde{Q}\left(\left\{\omega \in \widetilde{\Omega} : \Phi^t_\omega f \in F\right\}\right), \quad f \in \mathcal{L},\ F \in \mathcal{B}(\mathcal{L}).$$

By Ledrappier-Young [LY88], there is a one-to-one correspondence between invariant probability measures for $(\mathbf{P}^t)_{t\geq 0}$ and so-called physical invariant probability measures for the skew product $(\Theta^t)_{t\geq 0}$. The latter are invariant probability measures μ for $(\Theta^t)_{t\geq 0}$ with sample measures (μ^ω)

such that $\omega \mapsto \mu^\omega$ is measurable with respect to $\sigma(W_u^y : u \leq 0, y \in \mathbb{Z}^d)$. It is easy to see that the unique invariant probability measure from Theorem 1.14 is physical. Therefore, $(\mathbf{P}^t)_{t\geq 0}$ admits a unique invariant probability measure given by

$$\int_{\widetilde{\Omega}} \delta_{x \mapsto \widetilde{Z}^{x,0}_{-\infty}(\omega)}(\cdot)\widetilde{Q}(d\omega).$$

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chapter 2

Transition Probabilities for the Simple Symmetric Random Walk

Contents

In this chapter obtain several estimates on transition probabilities for the simple symmetric random walk on $\mathbb{Z}^d$, both in discrete and in continuous time.

2.1. Discrete-Time

2.1.1. Fundamentals

A random walk is a stochastic process formed by successive summation of independent, identically distributed random variables and it is one of the most basic and well-studied topics in probability theory. The prototypical example is the discrete-time simple symmetric random walk on $\mathbb{Z}^d$.

Definition 2.1 (Simple Symmetric Random Walk on $\mathbb{Z}^d$)
The Simple Symmetric Random Walk on $\mathbb{Z}^d$ $(S)_{n\in\mathbb{N}_0}$ starting at $x \in \mathbb{Z}^d$ is given by

$$S_n := x + X_1 + \quad + X_n$$

where the X_i's are i.i.d. random variables and $\mathbf{P}\{X_j = e_k\} = \mathbf{P}\{X_j = -e_k\} = 1/(2d)$, $k = 1, \dots, d$ and where e_k denotes the unit vector in the k^{th} direction.

Notice that a Simple Symmetric Random Walk on $\mathbb{Z}^d$ can also be considered as a Markov chain with state space $\mathbb{Z}^d$ and transition probabilities $P\{S_{n+1} = z | S_n = y\} = \frac{1}{2d}, \quad z - y \in \{\pm e_1, \dots, \pm e_d\}$. We define the n-step transition probability for the random walk starting at x as

$$q_n(x,y) := \mathbf{P}\{S_n = y \mid S_0 = x\}$$

If the random walks starts at the origin, we denote its transition probability by q_n^y.

We say that $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$ have the same parity and write $n \leftrightarrow x$ if $n + \sum_{k=1}^{\infty} x_k$ is even. Notice that if n and x have different parity, then $q_n^x = 0$.

We remind the classical local limit theorem for the simple random walk (see for example, Theorem 2.1.1 in [LL10]):

Theorem 2.2 (Local Central Limit Theorem)

For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, define $\overline{q_n^x}$ to be the Gaussian approximation of q_n^x:

$$\overline{q_n^x} := 2\left(\frac{d}{2\pi n}\right)^{d/2} e^{-d\frac{\|x\|^2}{2n}}.$$

Then,

$$\sup_{x:n\leftrightarrow x} |q_n^x - \overline{q_n^x}| = O\left(\frac{1}{n^{\frac{d}{2}+1}}\right). \tag{2.1}$$

In particular,

$$\sup_{x:n\leftrightarrow x} q_n^x = O\left(\frac{1}{n^{d/2}}\right) \tag{2.2}$$

and for every $A > 0$, there exists a constant $c > 0$ such that

$$\inf_{\substack{x:n\leftrightarrow x \\ \|x\|\leq A\sqrt{n}}} q_n^x \geq c\frac{1}{n^{d/2}}. \tag{2.3}$$

This estimate is good for typical x (i.e. for x such that $\|x\| \leq A\sqrt{n}$), but is not very sharp for atypically large x.

The following result, which was also proved in [LL10] (see Theorem 2.3.11), is an improvement of the lower bound for the transition probability obtained in (2.3) since it allows us to consider far away points. We include the proof here for the sake of completeness.

Lemma 2.3

There are constants $c_1, c_2 > 0$ such that the following holds: For every $\sigma \in (\frac{3}{4}, 1)$ and $\tilde\sigma \in (\sigma, 1)$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and $y \in \mathbb{Z}^d$ with $q_n^y > 0$ and $\|y\| \leq n^\sigma$,

$$q_n^y \geq c_1 \left(\tfrac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\tfrac{d}{2n}\|y\|^2\right) \exp\Big(-c_2 n^{4\tilde\sigma-3}\Big). \tag{2.4}$$

Proof. The argument is standard, so we will be brief (see for instance [LL10]). Let $(\gamma_n)_{n\in\mathbb{N}_0}$ be a simple symmetric random walk on $\mathbb{Z}^d$. For $n \in \mathbb{N}_0$, we set $\gamma_n^* := \gamma_{2n}$. Then γ^* is a random walk on the lattice $(\mathbb{Z}^d)_{\text{ev}}$ in $\mathbb{R}^d$ consisting of points whose coordinate sum is even. If $\{e_j\}_{1\leq j\leq d}$ is the standard basis for $\mathbb{R}^d$, then $\{e_1 + e_j : 1 \leq j \leq d\}$ is a basis for $(\mathbb{Z}^d)_{\text{ev}}$. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation mapping $e_1 + e_j$ to e_j for $1 \leq j \leq d$, and define $\tilde\gamma_n := T\gamma_n^*$. Then, $\tilde\gamma$ is an aperiodic, irreducible, symmetric random walk on $\mathbb{Z}^d$ with bounded increments, so it satisfies the conditions of Theorem 2.3.11 in [LL10]. Thus, there is $\rho > 0$ such that for any $i \in \mathbb{N}$ and for any $z \in \mathbb{Z}^d$ satisfying $\|z\| < \rho i$, we have

$$\tilde q_i^z := \mathbf{P}(\tilde\gamma_i = z) = \frac{1}{(2\pi i)^{\frac{d}{2}}\sqrt{\det A}} \exp\left(-\frac{\langle z, A^{-1}z\rangle}{2i}\right) \exp\left(O\left(\frac{1}{i} + \frac{\|z\|^4}{i^3}\right)\right),$$

where A is the covariance matrix of the increment distribution for $\tilde\gamma$. Now, we fix $\sigma \in (\frac{3}{4}, 1)$, $\tilde\sigma \in (\sigma, 1)$, and let $n \in \mathbb{N}$ be so large that $1 + n^\sigma < (n-1)^{\tilde\sigma}$ and $n^{\tilde\sigma} < \frac{\rho}{2\|T\|}n$, where $\|T\|$ is the operator norm of T. We distinguish between two cases: n is either even or odd.

Even case. If $n = 2m$ for some $m \in \mathbb{N}$ then we can prove a stronger statement:

Claim 2.1

If $n \in \mathbb{N}$ is even, then there are constants $c_1, c_2 > 0$, independent of n, $\tilde\sigma$, and σ, such that (2.4) holds for every $y \in \mathbb{Z}^d$ with $q_n^y > 0$ and $\|y\| \leq n^{\tilde\sigma}$.

To prove this claim, we fix $y \in \mathbb{Z}^d$ such that $q_{2m}^y > 0$ and $\|y\| \leq (2m)^{\tilde\sigma}$. Then $q_{2m}^y = \tilde q_m^{Ty}$. Moreover, since $\|Ty\| \leq \|T\|\|y\| \leq \|T\| n^{\tilde\sigma} < \rho m$, we have

$$\begin{aligned}\tilde q_m^{Ty} &= \frac{1}{(2\pi m)^{\frac{d}{2}}\sqrt{\det A}} \exp\left(-\frac{\langle Ty, A^{-1}Ty\rangle}{2m}\right) \exp\left(O\left(\frac{1}{m} + \frac{\|Ty\|^4}{m^3}\right)\right) \\ &= 2\left(\tfrac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\tfrac{d}{2n}\|y\|^2\right) \exp\left(O\left(\frac{1}{m} + \frac{\|Ty\|^4}{m^3}\right)\right) \\ &\geq 2\left(\tfrac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\tfrac{d}{2n}\|y\|^2\right) c_1' \exp\left(-c_2' n^{4\tilde\sigma-3}\right)\end{aligned}$$

for some universal constants $c_1', c_2' > 0$. This proves the claim if we take $c_1 := 2c_1'$ and $c_2 := c_2'$.

ODD CASE. Now, suppose $n = 2m+1$ for some $m \in \mathbb{N}$. Fix $y \in \mathbb{Z}^d$ such that $q^y_{2m+1} > 0$ and $\|y\| \leq (2m+1)^\sigma$. Let E be the set of standard unit vectors in $\mathbb{R}^d$ and their additive inverses. Then

$$q^y_{2m+1} = \sum_{z \in \mathbb{Z}^d} q^{y-z}_{2m} q^z_1 = \frac{1}{2d} \sum_{z \in E} q^{y-z}_{2m}.$$

Since $\|y - z\| < 1 + n^\sigma < (n-1)^{\tilde{\sigma}} = (2m)^{\tilde{\sigma}}$ and $q^{y-z}_{2m} > 0$ for all $z \in E$, then using Claim 2.1, we can bound q^y_{2m+1} from below as follows: there are $c_1', c_2' > 0$ such that

$$\begin{aligned} q^{y-z}_{2m+1} &\geq \frac{1}{2d} \sum_{z \in E} c_1' \left(\tfrac{d}{2\pi(2m)}\right)^{\frac{d}{2}} \exp\left(-\tfrac{d}{2(2m)}\|y-z\|^2\right) \exp\left(-c_2'(2m)^{4\tilde{\sigma}-3}\right) \\ &= \frac{c_1'}{2d} \left(\tfrac{d}{4\pi m}\right)^{\frac{d}{2}} \exp\left(-c_2'(2m)^{4\tilde{\sigma}-3}\right) \sum_{z \in E} \exp\left(-\tfrac{d}{4m}\|y-z\|^2\right) \\ &\geq \frac{c_1'}{2d} \left(\tfrac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-c_2' n^{4\tilde{\sigma}-3}\right) \exp\left(-\tfrac{d}{4m}\|y-e_1\|^2\right). \end{aligned} \tag{2.5}$$

Assume that n is so large that

$$\exp\left(-\frac{d(1+2n^\sigma)}{2n}\left(1 + \frac{1}{n-1}\right)\right) \exp\left(-\frac{dn^{2\sigma}}{2n(n-1)}\right) > \frac{1}{2}.$$

Since $\|y - e_1\|^2 = \|y\|^2 + 1 - 2\langle y, e_1 \rangle \leq \|y\|^2 + 1 + 2\|y\|$, it follows that

$$\exp\left(-\tfrac{d}{2n}\|y-e_1\|^2\right) \geq \exp\left(-\tfrac{d}{2n}\left(\|y\|^2 + 1 + 2\|y\|\right)\left(1 + \tfrac{1}{2m}\right)\right) \geq \tfrac{1}{2} \exp\left(-\tfrac{d}{4m}\|y\|^2\right).$$

Plugging this into (2.5), we obtain the desired estimate. □

2.1.2. Further Estimates

In this section we obtain further estimates on the transition probabilities of the discrete-time simple symmetric random walk. Of special importance is Lemma 2.6 which provides an upper bound estimate for the ratio of transition probabilities $q^{z'}_{n'}/q^z_n$ even for z large ($\|z\| \leq An$ for some $A > 0$). We express special gratitude to Fedor Nazarov, who kindly shared with us his unpublished notes that allowed us to prove Lemma 2.6.

We first show a result for linear functionals that will help us obtain estimates related to the discrete-time transition probabilities.

Fix a linear functional φ on $\mathbb{R}^d$ such that $|\varphi(x)| \le \|x\|$ for all $x \in \mathbb{R}^d$. To simplify notation, we set $\varphi_j := \varphi(e_j)$ for $1 \le j \le d$, where $\{e_j\}$ is the standard basis in $\mathbb{R}^d$. Define, for all $\theta = (\theta^1, \dots, \theta^d) \in \mathbb{R}^d$,

$$\Phi(\theta) := \mathbf{E}\left[e^{i\langle\theta,\gamma_1\rangle} e^{\varphi(\gamma_1)}\right] = \frac{1}{2d}\sum_{j=1}^{d}\left(e^{i\theta^j}e^{\varphi_j} + e^{-i\theta^j}e^{-\varphi_j}\right). \tag{2.6}$$

Notice that for all θ,

$$\left|\Phi(\theta)\right| \le \Phi(0) = \sum_{z\in\mathbb{Z}^d} q_1^z e^{\varphi(z)}. \tag{2.7}$$

Furthermore,

$$\Phi(0)^n = \sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)}. \tag{2.8}$$

Notice also that Φ is 2π-periodic in every argument, so it will be convenient to work with the cube $\mathcal{C} := (-\frac{\pi}{2}, \frac{3\pi}{2}]^d$. It is not hard to see that the inequality (2.7) is strict for all $\theta \in \mathcal{C}$ except for $\theta^0 = (0, \dots, 0)$ and $\theta^1 := (\pi, \dots, \pi)$. For any $\varepsilon > 0$ and $j \in \{0, 1\}$, let $\mathcal{D}_j^\varepsilon := \{\theta \in \mathbb{R}^d \ : \ \|\theta - \theta^j\| < \varepsilon\}$.

Claim 2.2

There exist $\varepsilon, \delta > 0$ such that, for $j \in \{0, 1\}$,

$$\left|\frac{\Phi(\theta)}{\Phi(\theta^j)}\right| \le e^{-\delta\|\theta-\theta^j\|^2} \qquad \textit{for all } \theta \in \mathcal{C} \setminus \mathcal{D}_{1-j}^\varepsilon, \tag{2.9}$$

Proof of Claim. For each $j \in \{0, 1\}$, we define scaled versions of the gradient vector and the Hessian matrix of Φ at θ^j:

$$G_j := -i\frac{\nabla\Phi(\theta^j)}{\Phi(\theta^j)} \qquad \text{and} \qquad H_j := -\frac{1}{2}\frac{\nabla^2\Phi(\theta^j)}{\Phi(\theta^j)}.$$

A simple computation shows that the matrix H_j is diagonal, and that for every $l \in \{1, \dots, d\}$, the l-th component of G_j and the (l, l)-entry of H_j are, respectively,

$$G_j^l = \frac{\sinh(\varphi_l)}{d\Phi(0)} \qquad \text{and} \qquad H_j^l = \frac{\cosh(\varphi_l)}{2d\Phi(0)}. \tag{2.10}$$

If we Taylor expand Φ around θ^j, we get

$$\begin{aligned}\left|\frac{\Phi(\theta)}{\Phi(\theta^j)}\right| &= \left|1 + i\langle G_j, \theta - \theta^j\rangle - \big\langle \theta - \theta^j, H_j(\theta - \theta^j)\big\rangle + O\big(\|\theta - \theta^j\|^3\big)\right| \\ &= \Big(1 - 2\big\langle \theta - \theta^j, H_j(\theta - \theta^j)\big\rangle + \langle G_j, \theta - \theta^j\rangle^2 + O\big(\|\theta - \theta^j\|^3\big)\Big)^{1/2}.\end{aligned}$$

Here and anywhere this notation appears, $g(\theta) = O(f(\theta))$ means there is a universal constant $c > 0$, possibly depending on the dimension d, but independent of φ, n, z, etc., such that $|g(\theta)| \le cf(\theta)$. In the Taylor expansion above, the constant c corresponding to the error term $O\big(\|\theta - \theta^j\|^3\big)$ may be chosen independently of φ because of the assumption that $\|\varphi\| \le 1$. Notice from (2.10) that $G_0 = G_1$ and $H_0 = H_1$, so in order to prove Claim 2.2, it is enough to consider the case $j = 0$. If we write $\theta = (\theta^1, \ldots, \theta^d)$, then using Jensen's inequality for sums,

$$\big\langle G_0, \theta \big\rangle^2 \le \frac{1}{d\Phi(0)} \sum_{l=1}^{d} \sinh|\varphi_l| (\theta^l)^2.$$

Thus, using the expression for H_0 in (2.10) as well as the fact that $\|\varphi\| \le 1$, we get

$$2\big\langle \theta, H_0 \theta \big\rangle - \big\langle G_0, \theta \big\rangle^2 \ge \frac{1}{d\Phi(0)} \sum_{l=1}^{d} e^{-|\varphi_l|} (\theta^l)^2 \ge \frac{1}{de\Phi(0)} \|\theta\|^2.$$

Thus, there is an $\varepsilon > 0$ and a universal constant $c > 0$ such that for all $\|\theta\| \le \varepsilon$,

$$\left| \frac{\Phi(\theta)}{\Phi(0)} \right| \le \left(1 - \frac{1}{de\Phi(0)} \|\theta\|^2 + O\big(\|\theta\|^3\big) \right)^{1/2} \le \big(1 - c\|\theta\|^2\big)^{1/2}.$$

Since the map $\theta \mapsto \big|\Phi(\theta)/\Phi(0)\big|$ is continuous and strictly less than 1 for all $\theta \in \mathcal{C}$ except θ^0, θ^1, it follows that if σ is the supremum of $\big|\Phi(\theta)/\Phi(0)\big|$ over $\theta \in \mathcal{C}$ satisfying $\|\theta - \theta^0\| \ge \varepsilon$ and $\|\theta - \theta^1\| \ge \varepsilon$, then $\sigma < 1$. Thus, if we choose $\tilde{c} \in (0, c)$ so small that $\big(1 - \tilde{c}\|\theta\|^2\big) \ge \sigma^2$ for all $\theta \in \mathcal{C}$, then the claim follows if we take $\delta := \tilde{c}/2$. □

Lemma 2.4

There is $c_1 > 0$ such that for every $y \in \mathbb{Z}^d$ and for every linear functional φ on $\mathbb{R}^d$ with $|\varphi(x)| \le \|x\|$, $x \in \mathbb{R}^d$, we have

$$q_n^y e^{\varphi(y)} \le c_1 n^{-\frac{d}{2}} \sum_{z \in \mathbb{Z}^d} q_n^z e^{\varphi(z)}, \qquad n \in \mathbb{N}.$$

In particular,

$$q_n^y \lesssim n^{-\frac{d}{2}}, \quad n \in \mathbb{N}, y \in \mathbb{Z}^d. \tag{2.11}$$

Proof. For $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, let $\widehat{\Phi^n}$ be the Fourier transform of Φ^n; i.e.,

$$\widehat{\Phi^n}(z) := \frac{1}{(2\pi)^d} \int_{\mathcal{C}} \Phi(\theta)^n e^{-i\langle \theta, z \rangle} \, d\theta.$$

Since $\Phi(\theta)^n = \mathbf{E}\left[e^{i\langle\theta,\gamma_n\rangle}e^{\varphi(\gamma_n)}\right]$, we get

$$\widehat{\Phi^n}(z) = \sum_{y\in\mathbb{Z}^d} \mathbf{P}(\gamma_n = y)e^{\varphi(y)}\frac{1}{(2\pi)^d}\int_{\mathcal{C}} e^{i\langle\theta,y-z\rangle}\,d\theta = q_n^z e^{\varphi(z)}. \tag{2.12}$$

Now, using Claim 2.2, we estimate:

$$\begin{aligned} q_n^z e^{\varphi(z)} &\le \frac{1}{(2\pi)^d}\int_{\mathcal{C}}\left|\Phi(\theta)\right|^n d\theta = \frac{1}{(2\pi)^d}\int_{\mathcal{C}}\left|\frac{\Phi(\theta)}{\Phi(0)}\right|^n d\theta \sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)} \\ &\le \frac{1}{(2\pi)^d}\left(\int_{\mathcal{C}\setminus\mathcal{D}_1^\varepsilon} e^{-\delta n\|\theta\|^2}\,d\theta + \int_{\mathcal{C}\setminus\mathcal{D}_0^\varepsilon} e^{-\delta n\|\theta-\theta^1\|^2}\,d\theta\right)\sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)} \\ &\le \frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-\delta n\|\theta\|^2}\,d\theta \sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)}. \end{aligned}$$

The last integral can be estimate from above by

$$C\int_0^\infty r^{d-1}e^{-\delta n r^2}\,dr = Cn^{-\frac{d}{2}}\int_0^\infty \rho^{d-1}e^{-\delta\rho^2}\,d\rho,$$

where $\rho := n^{\frac{1}{2}}r$ and C is some constant depending on d. □

Lemma 2.5

There are $\rho_1, \rho_2 > 0$ such that the following holds: For any $n \in \mathbb{N}$ and for any $z \in \mathbb{Z}^d$ such that $\|z\| \le \rho_1 n$ and $\|z\|_1 \equiv n$, there is a linear functional φ on $\mathbb{R}^d$ of norm $\|\varphi\| \le \rho_2\frac{\|z\|}{n}$ which satisfies

$$\frac{1}{(2\pi)^d}\int_{\mathcal{C}}\left|\Phi(\theta)\right|^n d\theta \le \left(1 + O(n^{-\frac{2}{5}})\right) q_n^z e^{\varphi(z)}.$$

Moreover,

$$q_n^z e^{\varphi(z)} \gtrsim n^{-\frac{d}{2}}\sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)}.$$

Proof. For each $j \in \{0,1\}$, let $\mathcal{B}_j := \{\theta \;:\; \|\theta - \theta^j\| \le n^{-2/5}\}$. Recall from (2.12) that for all $z \in \mathbb{Z}^d$, all $n \in \mathbb{N}$, and any linear functional φ on $\mathbb{R}^d$ satisfying $\|\varphi\| \le 1$, we have the following equality:

$$q_n^z e^{\varphi(z)} = \frac{1}{(2\pi)^d}\int_{\mathcal{C}} \Phi(\theta)^n e^{-i\langle\theta,z\rangle}\,d\theta = I_0 + I_1 + I, \tag{2.13}$$

where

$$I_j := \frac{1}{(2\pi)^d}\int_{\mathcal{B}_j} \Phi(\theta)^n e^{-i\langle\theta,z\rangle}\,d\theta \quad \text{and} \quad I := \frac{1}{(2\pi)^d}\int_{\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)} \Phi(\theta)^n e^{-i\langle\theta,z\rangle}\,d\theta.$$

Then we find

$$\frac{1}{(2\pi)^d}\int_{\mathcal{C}}\left|\Phi(\theta)\right|^n d\theta - q_n^z e^{\varphi(z)} \leq \sum_{j=0}^{1}\left(\frac{1}{(2\pi)^d}\int_{\mathcal{B}_j}\left|\Phi(\theta)\right|^n d\theta - \mathrm{Re}(I_j)\right) + \frac{2}{(2\pi)^d}\int_{\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)}\left|\Phi(\theta)\right|^n d\theta. \tag{2.14}$$

By Claim 2.2, there are $\varepsilon, \delta > 0$ such that the integral over $\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)$ in (2.14) can be bounded from above by

$$\frac{2\Phi(0)^n}{(2\pi)^d}\left(\int_{\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{D}_1^\varepsilon)} e^{-\delta n\|\theta\|^2} d\theta + \int_{\mathcal{C}\setminus(\mathcal{B}_1\cup\mathcal{D}_0^\varepsilon)} e^{-\delta n\|\theta-\theta^1\|^2} d\theta\right) \leq 4\Phi(0)^n e^{-\delta n^{1/5}}, \tag{2.15}$$

where $\mathcal{D}_j := \{\theta \in \mathbb{R}^d \,:\, \|\theta - \theta^j\| < \varepsilon\}$. Note that we used that $\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{B}_1) \subseteq \mathcal{C}\setminus\big[(\mathcal{B}_0\cap\mathcal{D}_0)\cup(\mathcal{B}_1\cap\mathcal{D}_1)\big] \subseteq \big[\mathcal{C}\setminus(\mathcal{B}_0\cup\mathcal{D}_1)\big]\cup\big[\mathcal{C}\setminus(\mathcal{B}_1\cup\mathcal{D}_0)\big]$.

To find a suitable upper bound on the integrals over $\mathcal{B}_j$ in (2.14), we will choose φ, n and z in such a way that for $j \in \{0,1\}$, the linear term in the Taylor expansion of $\Phi(\theta)e^{-\frac{i}{n}\langle z,\theta\rangle}$ around θ^j vanishes.

To find a suitable upper bound on the expression in the second line of (2.14), we will choose φ, n and z in such a way that for $j \in \{0,1\}$, the linear term

$$\Phi(\theta^j)\nabla(e^{-\frac{1}{n}\langle z,\theta^j\rangle}) + e^{-\frac{1}{n}\langle z,\theta^j\rangle}\nabla\Phi(\theta^j)$$

in the Taylor expansion of $\Phi e^{-\frac{1}{n}\langle z,\theta^j\rangle}$ around $\theta^{(j)}$ vanishes. If we denote the kth component of z by $z^{(k)}$, this is equivalent to

$$F_k(\varphi_1,\ldots,\varphi_d) := \frac{\sinh(\varphi_k)}{\sum_{l=1}^d \cosh(\varphi_l)} = \frac{z^{(k)}}{n}, \quad 1 \leq k \leq d,$$

where $\varphi_k = \varphi(e_k)$. Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be given by

$$F(x_1,\ldots,x_d) := \sum_{k=1}^d \frac{\sinh(x_k)}{\sum_{l=1}^d \cosh(x_l)} e_k.$$

For $r > 0$ and $x \in \mathbb{R}^d$, let $B_r(x)$ denote the open Euclidean ball of radius r centered at x. Since $F(0) = 0$ and

$$\det DF(0) = \frac{1}{d^d} \neq 0,$$

the inverse function theorem yields existence of $\rho_1 > 0$ and an open neighbourhood U of 0 such that $F : U \to B_{\rho_1}(0)$ is a diffeomorphism. Therefore, for any $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ with $\|z\| < \rho_1 n$ there is $\varphi \in U \subseteq \mathbb{R}^d$ such that $F(\varphi) = \frac{z}{n}$. Since F^{-1} is differentiable and $F^{-1}(0) = 0$, there exists $\rho_2 > 0$ such that

$$\|\varphi\| = \|F^{-1}(\tfrac{z}{n})\| \le \rho_2 \frac{\|z\|}{n}.$$

Without loss of generality, we may assume that $\rho_1\rho_2 \le 1$ so that $\|\varphi\| \le 1$. Fix $n \in \mathbb{N}$, $z \in \mathbb{Z}^d$ such that $\|z\| \le \rho_1 n$ and $\|z\|_1 \equiv n$, and the corresponding $\varphi \in \mathbb{R}^d$ (which we identify with the linear functional mapping e_k to φ_k for $1 \le k \le d$). For $j \in \{0, 1\}$ and for $\theta \in \mathcal{B}_j$ such that $\|\theta - \theta^j\| \le n^{-\frac{2}{5}}$, we have

$$\begin{aligned}
\Phi(\theta)e^{-\frac{i}{n}\langle z,\theta\rangle} &= \Phi(\theta^j)e^{-\frac{i}{n}\langle z,\theta^j\rangle} + \Big\langle \theta - \theta^j, A_j(\theta - \theta^j)\Big\rangle + O\big(\|\theta - \theta^j\|^3\big) \\
&= \Phi(\theta^j)e^{-\frac{i}{n}\langle z,\theta^j\rangle}\left(1 + \frac{\big\langle \theta - \theta^j, A_j(\theta - \theta^j)\big\rangle}{\Phi(\theta^j)e^{-\frac{i}{n}\langle z,\theta^j\rangle}} + O(n^{-6/5})\right),
\end{aligned} \tag{2.16}$$

where A_j is the quadratic form in the Taylor expansion of $\Phi(\theta)e^{-\frac{i}{n}\langle z,\theta^j\rangle}$. Note that the error term $O(n^{-\frac{6}{5}})$ is complex-valued and can be bounded by $cn^{-\frac{6}{5}}$ for some constant c that depends neither on φ nor on n or z (this is because $\|\varphi\| \le 1$ and $\frac{\|z\|}{n} \le \rho_1$). Also, using the expressions (2.10) for the gradient and the Hessian of $\Phi(\theta)$, it is easy to see that the entries of $A_j/\Phi(\theta^j)e^{-\frac{i}{n}\langle z,\theta^j\rangle}$ are real. Let $x_j(\theta)$ and $y_j(\theta)$ denote respectively the real and imaginary parts of the expression in parentheses in (2.16). Then each of the summands on the righthand side of the first line in (2.14) can be written as follows:

$$\sum_{j=0}^{1} \frac{\Phi(0)^n}{(2\pi)^d} \int_{\mathcal{B}_j} \left(\big|x_j(\theta) + iy_j(\theta)\big|^n - \mathrm{Re}\Big(\big(x_j(\theta) + iy_j(\theta)\big)^n\Big)\right) d\theta. \tag{2.17}$$

Here, we used the assumption that n and $\|z\|_1$ have the same parity: as $n \equiv \|z\|_1$, we have $\Phi(\theta^1)e^{i\langle z,\theta^1\rangle} = \Phi(0)^n(-1)^n e^{-i\pi\|z\|_1} = \Phi(0)^n$. If we represent $x_j(\theta) + iy_j(\theta)$ in polar form, then the modulus is $\big|x_j(\theta) + iy_j(\theta)\big| = \big|\Phi(\theta)/\Phi(0)\big|$ and the argument is of order $O(n^{-6/5})$. As a result, the integrand in (2.17) equals

$$\frac{\big|\Phi(\theta)\big|^n}{\Phi(0)^n}\Big(1 - \cos\big(O(n^{-1/5})\big)\Big) = \frac{\big|\Phi(\theta)\big|^n}{\Phi(0)^n} O(n^{-2/5}).$$

We can therefore write (2.17) as $\Phi(0)^n O(n^{-2/5}) J_n$ where

$$J_n := \int_{\mathcal{C}} \left(\frac{\left|\Phi(\theta)\right|}{\Phi(0)} \right)^n \, d\theta. \tag{2.18}$$

Now we claim that $J_n \gtrsim n^{-d/2}$. Our choice of φ implies

$$\begin{aligned} J_n &\geq \frac{1}{\Phi(0)^n} \int_{\mathcal{B}_0} \left|\Phi(\theta) e^{-\frac{i}{n}\langle z,\theta\rangle}\right|^n \, d\theta = \int_{\mathcal{B}_0} \left|x_0(\theta) + i y_0(\theta)\right|^n \, d\theta \\ &\geq \cos\left(O(n^{-1/5})\right) \int_{\mathcal{B}_0} \left|x_0(\theta)\right|^n \, d\theta = \left(1 + O(n^{-2/5})\right) \int_{\mathcal{B}_0} \left|x_0(\theta)\right|^n \, d\theta. \end{aligned}$$

For $\theta \in \mathcal{C} \cap \mathcal{B}_0$, we have $x_0(\theta) = \exp\left(\langle \theta, A_0 \theta \rangle / \Phi(0)\right)\left(1 + O(n^{-6/5})\right)$, so we can continue the above chain of inequalities as follows:

$$\begin{aligned} &\geq \left(1 + O(n^{-2/5})\right)\left(1 + O(n^{-6/5})\right)^n \int_{\mathcal{B}_0} \exp\left(n \frac{\langle \theta, A_0 \theta \rangle}{\Phi(0)} \right) d\theta \\ &\geq \left(1 + O(n^{-1/5})\right) \int_{\mathcal{B}_0} \exp\left(n \frac{\langle \theta, A_0 \theta \rangle}{\Phi(0)} \right) d\theta \\ &\gtrsim \int_{\mathcal{B}_0} e^{-cn\|\theta\|^2} \, d\theta \gtrsim n^{-d/2} \end{aligned}$$

for some universal constant $c > 0$. The estimate $J_n \gtrsim n^{-\frac{d}{2}}$ implies that $O(n^{-2/5}) J_n$ decays polynomially in n, whereas $e^{-\delta n^{1/5}}$ decays stretch-exponentially. Therefore, (2.14) implies

$$\frac{1}{(2\pi)^d} \int_{\mathcal{C}} \left|\Phi(\theta)\right|^n \, d\theta - q_n^z e^{\varphi(z)} \leq O(n^{-2/5}) \frac{1}{(2\pi)^d} \int_{\mathcal{C}} |\Phi(\theta)|^n \, d\theta,$$

which gives proves the first claim.

Using (2.13), we write

$$q_n^z e^{\varphi(z)} \geq \mathrm{Re}(I_0) + \mathrm{Re}(I_1) - 2\Phi(0)^n e^{-\delta n^{1/5}} = \mathrm{Re}(I_0) + \mathrm{Re}(I_1) - 2e^{-\delta n^{1/5}} \sum_{y \in \mathbb{Z}^d} q_n^y e^{\varphi(y)},$$

$$\mathrm{Re}(I_0) + \mathrm{Re}(I_1) = \sum_{j=0}^{1} \frac{\Phi(0)^n}{(2\pi)^d} \int_{\mathcal{B}_j} \mathrm{Re}\Big(\left(x_j(\theta) + i y_j(\theta)\right)^n \Big) \, d\theta. \tag{2.19}$$

Moreover, for $j \in \{0, 1\}$ and $\theta \in \mathcal{C} \cap \mathcal{B}_j$,

$$x_j(\theta) = \exp\left(\frac{\left\langle \theta - \theta^j, A_j(\theta - \theta^j) \right\rangle}{\Phi(\theta^j) e^{-\frac{i}{n}\langle z, \theta^j \rangle}} \right) \left(1 + O(n^{-6/5})\right).$$

As the argument in the polar form of $x_j(\theta)+iy_j(\theta)$ is of order $O(n^{-6/5})$, we can bound Re$\Big(\big(x_j(\theta)+iy_j(\theta)\big)^n\Big)$ from below by

$$x_j(\theta)^n \cos\big(O(n^{-1/5})\big) = x_j(\theta)^n\big(1+O(n^{-2/5})\big).$$

For the right side of (2.19), this yields the lower bound

$$\big(1+O(n^{-1/5})\big)\frac{\Phi(0)^n}{(2\pi)^d}\sum_{j=0}^{1}\int_{\mathcal{B}_j}\exp\left(n\frac{\big\langle \theta-\theta^j, A_j(\theta-\theta^j)\big\rangle}{\Phi(\theta^j)e^{-\frac{i}{n}\langle z,\theta^j\rangle}}\right)\,d\theta,$$

which is greater than a constant times

$$n^{-d/2}\Phi(0)^n = n^{-d/2}\sum_{y\in\mathbb{Z}^d} q_n^y e^{\varphi(y)}.$$

As $e^{-\delta n^{\frac{1}{5}}}$ decays faster than $n^{-\frac{d}{2}}$, we obtain the desired estimate. □

Lemma 2.6

There are constants $\rho, c>0$ such that for any $n,n'\in\mathbb{N}$ and for any $z,z'\in\mathbb{Z}^d$ with $\|z\|\le\rho n$ and $\|z\|_1\equiv n$, we have

$$\frac{q_{n'}^{z'}}{q_n^z}\le\Big(1+O(n^{-\frac{2}{5}})\Big)\exp\left(c\left(\frac{\|z\|}{n}\big(\|z-z'\|+|n'-n|\big)+\ln(n)\frac{|n-n'|}{n}\right)\right).$$

Proof. Let ρ_1,ρ_2 be as in Lemma 2.5, relabel ρ_1 as ρ, and let $n,n'\in\mathbb{N}$, $z,z'\in\mathbb{Z}^d$ be such that $\|z\|\le\rho n$ and $\|z\|_1\equiv n$. Let φ be the linear functional from Lemma 2.5 that corresponds to n and z, and for which

$$\frac{1}{(2\pi)^d}\int_{\mathcal{C}}|\Phi(\theta)|^n\,d\theta\le\Big(1+O(n^{-\frac{2}{5}})\Big)\,q_n^z e^{\varphi(z)}. \tag{2.20}$$

We consider two cases: $n'>n$ and $n'\le n$.

Case "$n'>n$". If $n'>n$, we have

$$q_{n'}^{z'}e^{\varphi(z')}\le\frac{1}{(2\pi)^d}\int_{\mathcal{C}}|\Phi(\theta)|^{n'}\,d\theta\le\Phi(0)^{n'-n}\frac{1}{(2\pi)^d}\int_{\mathcal{C}}|\Phi(\theta)|^n\,d\theta.$$

Furthermore,

$$\Phi(0)^{n'-n}\le e^{\|\varphi\|(n'-n)}\le e^{\rho_2\frac{\|z\|}{n}(n'-n)}.$$

The estimate in (2.20) then implies

$$\frac{q_{n'}^{z'}}{q_n^z} \le \left(1 + O(n^{-\frac{2}{5}})\right) e^{\varphi(z-z')} e^{\rho_2 \frac{\|z\|}{n}(n'-n)}$$
$$\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(\rho_2 \frac{\|z\|}{n}\left(\|z - z'\| + |n' - n|\right)\right).$$

CASE "$n' \le n$". If $n' \le n$, the function $x \mapsto x^{\frac{n}{n'}}$ is convex, and Jensen's inequality implies

$$q_{n'}^{z'} e^{\varphi(z')} \le \left(\frac{1}{(2\pi)^d} \int_{\mathcal{C}} |\Phi(\theta)|^n \, d\theta\right)^{\frac{n'}{n}} = \Phi(0)^{n'} J_n^{\frac{n'}{n}} \le \Phi(0)^n J_n^{\frac{n'}{n}}, \tag{2.21}$$

where J_n was defined in (2.18). Since $J_n \gtrsim n^{-d/2}$,

$$J_n^{\frac{n'}{n}} \le \left(c_2 n^{\frac{d}{2}}\right)^{\frac{n-n'}{n}} J_n \le \exp\left(c_3 \ln(n) \frac{n - n'}{n}\right) J_n \tag{2.22}$$

for some constants $c_2, c_3 > 0$. Combining (2.21) and (2.22), we obtain

$$q_{n'}^{z'} \le e^{-\varphi(z')} \frac{1}{(2\pi)^d} \int_{\mathcal{C}} |\Phi(\theta)|^n \, d\theta \ \exp\left(c_3 \ln(n) \frac{n - n'}{n}\right).$$

Together with (2.20), this yields

$$\frac{q_{n'}^{z'}}{q_n^z} \le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(\varphi(z - z') + c_3 \ln(n) \frac{n - n'}{n}\right)$$
$$\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(\frac{\|z\|}{n}\|z - z'\| + \ln(n) \frac{n - n'}{n}\right)\right)$$

for some universal constant $c > 0$. □

For $y \in \mathbb{Z}^d$ and $n \in \mathbb{N}_0$, set

$$\iota(y, n) = \begin{cases} n, & \|y\|_1 \equiv n, \\ n + 1, & \|y\|_1 \not\equiv n. \end{cases}$$

Let $\nu \in (\frac{1}{2}, 1)$ and set

$$J(t) := \left\{n \in \mathbb{N} : \left|\frac{n}{t} - 1\right| < 1 - \nu\right\} = \{n \in \mathbb{N} : \nu t < n < (2 - \nu)t\}. \tag{2.23}$$

Lemma 2.7

Let $\sigma \in (\frac{3}{4}, 1)$ and $\xi_1 \in (0, 1-\sigma)$. There are constants $T, c > 0$ such that the following holds: For any $t \geq T$, $y \in \mathbb{Z}^d$ such that $\|y\| \leq t^\sigma$, $m \in J(2t^{\xi_1})$, and $l \in J(t - 2t^{\xi_1})$, we have

$$q^y_{m+l} \leq c q^y_{\iota(y,l)}.$$

Proof. Since $\iota(y,l) > \|y\|_1$ for $\|y\| \leq t^\sigma$, $l \in J(t - 2t^{\xi_1})$, and t sufficiently large, and since $\iota(y,l) \equiv \|y\|_1$, we have $q^y_{\iota(y,l)} > 0$. For $m \in J(2t^{\xi_1})$, Lemma 2.6 implies the estimate

$$\begin{aligned} q^y_{m+l} &= \frac{q^y_{m+l}}{q^y_{\iota(y,l)}} q^y_{\iota(y,l)} \\ &\leq \left(1 + O(l^{-\frac{2}{5}})\right) \exp\left(c\left(\frac{\|y\|}{\iota(y,l)} m + \ln(\iota(y,l)) \frac{m}{\iota(y,l)}\right)\right) q^y_{\iota(y,l)} \lesssim q^y_{\iota(y,l)}. \end{aligned}$$

For the last estimate, we used the assumption that $\xi_1 < 1 - \sigma$. □

Lemma 2.8

Let $\|y\| < t^\sigma$, $l_\bullet \in J(t - 2t^{\xi_1})$, $l_-, l_+ \in \mathbb{N}_0$, then there exists a constant $C > 0$ such that

$$q^y_{l_- + l_\bullet + l_+} \lesssim \prod_{s \in \{-,+\}} \exp\left(C t^{\sigma - 1} l_s\right) q^y_{\iota(y, l_\bullet)}.$$

This lemma is an easy consequence of Lemma 2.6 and it can be proved in the same way as Lemma 2.7.

2.2. Continuous-Time

Now, we turn to the continuous-time simple symmetric randoms walk on $\mathbb{Z}^d$.

Definition 2.9 (Continuous-time Simple Symmetric Random Walk)

We define ***continuous-time simple symmetric random walk*** $\eta = (\eta_t)_{t\in\mathbb{R}}$ on $\mathbb{Z}^d$ starting at point x by

$$\eta_t = x + X_1 + \cdots + X_{n_t}$$

where n_t is Poisson distributed (i.e. $\mathbf{P}(n_t = n) = \frac{e^{-t}t^n}{n!}$) and the X_i's are i.i.d. random variables and $\mathbf{P}\{X_j = e_k\} = \mathbf{P}\{X_j = -e_k\} = 1/(2d)$, $k = 1, \ldots, d$ and where e_k denotes the unit vector in the k^{th} direction.

That is, a continuous-time simple symmetric random walk starts at some point $x \in \mathbb{Z}^d$ at time 0 and after a waiting time (exponentially distributed with rate 1) it jumps to one of its $2d$ nearest neighbours with equal probability.

Given a continuous-time simple symmetric random walk $\eta = (\eta_t)_{t\in\mathbb{R}}$ on $\mathbb{Z}^d$ starting at x, if observed over a time interval $[s, t)$, a sample path of η (which we shall also denote by η) is characterized by

(1) the number $n_{s,t}$ of jumps that occur within the time interval (s, t),

(2) a discrete-time path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{n_{s,t}})$ on $\mathbb{Z}^d$ such that $\gamma_0 = x$ and $\|\gamma_j - \gamma_{j-1}\|_1 = 1$ whenever $1 \leq j \leq n_{s,t}$, and

(3) the jump times $s < s_1 < \ldots < s_{n_{s,t}} < t$.

Let

$$p_t^z := \mathbf{P}_{0,0}(\eta_t = z)$$

be the transition probability. for the continuous-time simple symmetric random walk starting at the origin. Since the number of jumps up to time t n_t in the continuous-time random walk is Poisson distributed, we have that $p_t^z = \mathbf{E}q_{n(t)}^z$, where $n(t)$ is a Poisson random variable with intensity t, namely

$$p_t^z = \sum_{n=0}^{\infty} \frac{t^n e^{-t}}{n!} q_n^z,$$

where q_n^z is the transition probability for the associated discrete-time random walk $\gamma_n = x + X_1 + \cdots + X_n$.

Lemma 2.10

For any $y \in \mathbb{Z}^d$ *such that* $\|y\| \leq \frac{t}{2\sqrt{d}}$, *we have*

$$p_t^y = \left(\tfrac{d}{2\pi t}\right)^{\frac{d}{2}} \exp\left(-\tfrac{d}{2t}\|y\|^2\right) \exp\left(O\left(\frac{1}{\sqrt{t}} + \frac{\|y\|^3}{t^2}\right)\right). \tag{2.24}$$

Therefore, for any $\sigma \in (0,1)$ *and* t *sufficiently large,*

$$\frac{1}{p_t^y} \leq e^{t^\sigma}, \quad y : \|y\| \leq t^\sigma. \tag{2.25}$$

Proof. Denote the coordinate components of the continuous-time Simple Symmetric Random Walk η on $\mathbb{Z}^d$ starting at 0 by $\eta^{(1)}, \ldots, \eta^{(d)}$ and set $\hat{\eta}_t^{(i)} := \eta_{dt}^{(i)}$ for $1 \leq i \leq d$. Then Proposition 1.2.2 in [LL10] implies that $\hat{\eta}^{(1)}, \ldots, \hat{\eta}^{(d)}$ are independent continuous-time Simple Symmetric Random Walk in dimension 1 starting at 0. According to Theorem 2.5.6 in [LL10], we have

$$\mathbf{P}(\hat{\eta}_t^{(i)} = z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} \exp\left(O\left(\frac{1}{\sqrt{t}} + \frac{|z|^3}{t^2}\right)\right)$$

for $1 \leq i \leq d$, and for all $t > 0$ and $z \in \mathbb{Z}$ such that $|z| \leq \frac{t}{2}$. Now, if $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ such that $\|\mathbf{z}\| \leq \frac{t}{2\sqrt{d}}$, the above equality implies (2.24) and the estimate (2.25) because

$$p_t^{\mathbf{z}} = \prod_{i=1}^{d} \mathbf{P}\big(\hat{\eta}_{\frac{t}{d}}^{(i)} = z_i\big).$$

□

Recall that for $\nu \in (\frac{1}{2}, 1)$

$$J(t) = \left\{ n \in \mathbb{N} : \left|\frac{n}{t} - 1\right| < 1 - \nu \right\} = \{ n \in \mathbb{N} : \nu t < n < (2-\nu)t \}. \tag{2.26}$$

For $t > 0$, $y \in \mathbb{Z}^d$, and $x \in \mathbb{Z}^d$ such that $\|x\| \leq t^\sigma$, let

$$\mathcal{D}_t(y, x) := \frac{\sum_{n \notin J(t)} e^{-t} \frac{t^n}{n!} q_n^{y-x}}{\sum_{n \in J(t)} e^{-t} \frac{t^n}{n!} q_n^{y-x}}.$$

Lemma 2.11

Let $y \in \mathbb{Z}^d$. For any $\sigma \in (\frac{3}{4}, 1)$, we have

$$\lim_{t\to\infty} \sup_{\|x\|\le t^\sigma} \mathcal{D}_t(y,x) = 0.$$

Proof. First observe that for $0 \le n \le \nu t - 1$, we have

$$\frac{t^n}{n!} \le \frac{t^n}{n!}\frac{t}{n+1} = \frac{t^{n+1}}{(n+1)!}.$$

As a result,

$$\frac{t^n}{n!} \le \frac{t^{\lfloor \nu t\rfloor}}{\lfloor \nu t\rfloor!}, \quad 0 \le n \le \nu t.$$

This implies

$$\sum_{0\le n\le \nu t} e^{-t}\frac{t^n}{n!}q_n^{y-x} \le e^{-t}\sum_{0\le n\le \nu t}\frac{t^n}{n!} \le e^{-t}(\nu t+1)\frac{t^{\lfloor \nu t\rfloor}}{\lfloor \nu t\rfloor!}.$$

Since $x > e^{1-\frac{1}{x}}$ for all $x \in (0,\infty)\setminus\{1\}$, we may select $\kappa \in (e^{1-\frac{1}{\nu}}, \nu)$. As $\kappa < \nu$, we have $\lfloor \nu t\rfloor > \kappa t$ for t sufficiently large. By Stirling's formula,

$$\frac{t^{\lfloor \nu t\rfloor}}{\lfloor \nu t\rfloor!} \le e^{\lfloor \nu t\rfloor}\frac{t^{\lfloor \nu t\rfloor}}{\sqrt{2\pi\lfloor \nu t\rfloor}\lfloor \nu t\rfloor^{\lfloor \nu t\rfloor}} \le \frac{1}{\sqrt{2\pi\kappa t}}\left(\frac{e}{\kappa}\right)^{\nu t}.$$

Let

$$\delta = \frac{1}{2}\left(e + \left(\frac{e}{\kappa}\right)^{\nu}\right).$$

As $\kappa > e^{1-\frac{1}{\nu}}$, we have

$$\left(\frac{e}{\kappa}\right)^{\nu} < e$$

and hence

$$\left(\frac{e}{\kappa}\right)^{\nu} < \delta < e.$$

It follows that

$$\sum_{0\le n\le \nu t} e^{-t}\frac{t^n}{n!}q_n^{y-x} \le \left(\frac{\delta}{e}\right)^t \frac{\nu t+1}{\sqrt{2\pi\kappa t}}.$$

For any $k > t$ we have the following tail estimate

$$\sum_{n\ge k}\frac{t^n}{n!} \le \frac{t^k}{k!}\sum_{n=k}^{\infty}\left(\frac{t}{k}\right)^{n-k} = \frac{t^k}{k!}\frac{1}{1-\frac{t}{k}}. \tag{2.27}$$

Because $2-\nu > 1$, we can combine the estimate in (2.27) and Stirling's formula to obtain

$$\sum_{n\geq(2-\nu)t} e^{-t}\frac{t^n}{n!}q_n^{y-x} \leq \frac{1}{1-\frac{1}{2-\nu}}\frac{1}{\sqrt{2(2-\nu)\pi t}}\left(\frac{e^{1-\nu}}{(2-\nu)^{2-\nu}}\right)^t.$$

Notice that $\frac{e^{1-\nu}}{(2-\nu)^{2-\nu}} < 1$ because $\frac{e^{x-1}}{x^x} < 1$ for all $x \in (0,\infty)\setminus\{1\}$.

On the other hand, on account of (2.24), for $y \in \mathbb{Z}^d$ there is a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}q_n^{y-x} = p_t^{y-x} \geq \exp\left(-Ct^{2\sigma-1}\right), \quad \|x\| \leq t^\sigma.$$

Since $2\sigma - 1 < 1$, we obtain the desired convergence. □

Lemma 2.12

Let $\xi \in (0,1)$ and $\sigma \in (\frac{3}{4},1)$. There are constants $T, c > 0$ such that for any $t \geq T$ and $y \in \mathbb{Z}^d$ such that $\|y\| \leq t^\sigma$, we have

$$\frac{p^y_{t-2t^\xi}}{p^y_t} \leq ce^{\beta^2 t^\xi}.$$

Proof. Choose the parameter $\nu \in (\frac{1}{2},1)$ in the definition of $J(t)$ so close to 1 that $4(1-\nu) < \beta^2$, and let $\tilde{\sigma} \in (\sigma,1)$. We have

$$\limsup_{t\to\infty}\sup_{\|y\|\leq t^\sigma}\mathcal{D}_{t-2t^\xi}(y,0) \leq \lim_{t\to\infty}\sup_{\|y\|\leq(t-2t^\xi)^{\tilde{\sigma}}}\mathcal{D}_{t-2t^\xi}(y,0),$$

and the righthand side tends to 0 as $t\to\infty$ by virtue of Lemma 2.11. Thus,

$$\begin{aligned}\frac{p^y_{t-2t^\xi}}{p^y_t} =& \frac{\sum_{n\in J(t-2t^\xi)} e^{2t^\xi - t}\frac{(t-2t^\xi)^n}{n!}q_n^y(1+\mathcal{D}_{t-2t^\xi}(y,0))}{\sum_{n=0}^\infty e^{-t}\frac{t^n}{n!}q_n^y} \\ \leq&(1+\mathcal{D}_{t-2t^\xi}(y,0))e^{2t^\xi}\frac{\sum_{n\in J(t-2t^\xi)}\frac{(t-2t^\xi)^n}{n!}q_n^y}{\sum_{n\in J(t-2t^\xi)}\frac{t^n}{n!}q_n^y} \lesssim e^{2t^\xi}\frac{\sum_{n\in J(t-2t^\xi)}\frac{(t-2t^\xi)^n}{n!}q_n^y}{\sum_{n\in J(t-2t^\xi)}\frac{t^n}{n!}q_n^y}.\end{aligned}$$

For $n \in J(t-2t^\xi)$,

$$\left(\frac{t-2t^\xi}{t}\right)^n = (1-2t^{\xi-1})^n \leq (1-2t^{\xi-1})^{\nu(t-2t^\xi)} \leq (1-2t^{\xi-1})^{(2\nu-1)t} \leq e^{-2(2\nu-1)t^\xi}.$$

Accordingly,

$$\frac{p^y_{t-2t^\xi}}{p^y_t} \lesssim e^{4(1-\nu)t^\xi} \lesssim e^{\beta^2 t^\xi}.$$

□

Lemma 2.13

Let $\sigma \in (0,1)$ *and* $y_1, y_2 \in \mathbb{Z}^d$. *Then,*

$$\lim_{t\to\infty} \inf_{\|x\|\le t^\sigma} \frac{p_t^{y_1-x}}{p_t^{y_2-x}} = \lim_{t\to\infty} \sup_{\|x\|\le t^\sigma} \frac{p_t^{y_1-x}}{p_t^{y_2-x}} = 1.$$

Proof. Thanks to Lemma 2.6, there are $\rho, c > 0$ such that for any $n, n' \in \mathbb{N}$ with $n' \le n$ and for any $z, z' \in \mathbb{Z}^d$ with $\|z\| \le \rho n$ and $\|z\|_1 \equiv n \mod 2$,

$$\frac{q_{n'}^{z'}}{q_n^z} \le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(\frac{\|z\|}{n}\|z - z'\| + \ln(n)\frac{n-n'}{n}\right)\right). \tag{2.28}$$

Fix $\delta > 0$ and choose the parameter $\nu \in (\frac{1}{2}, 1)$ in the definition of $J(t)$ so close to 1 that

$$\nu^{-1} < (1+\delta)^{\frac{1}{3}}.$$

With the help of Lemma 2.11, we can choose $\tau > 0$ so large that

$$\nu^{-1} + \tau^{-1} < (1+\delta)^{\frac{1}{3}},$$
$$1 + \sup_{\|x\|\le t^\sigma} \mathcal{D}_t(y_1, x) < (1+\delta)^{\frac{1}{3}}, \quad \text{for all } t \ge \tau,$$

and such that for all $n > \nu\tau$, we have $\|y_2\| + (\nu^{-1}n)^\sigma < \rho n$ and

$$\left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(\frac{\|y_2\| + (\nu^{-1}n)^\sigma}{n}\|y_1 - y_2\| + \frac{\ln(n)}{n}\right)\right) < (1+\delta)^{\frac{1}{3}}. \tag{2.29}$$

For $y \in \mathbb{Z}^d$ and $n \in \mathbb{N}_0$, let

$$\iota(y,n) := \begin{cases} n, & \|y\|_1 \equiv n \mod 2, \\ n+1, & \|y\|_1 \not\equiv n \mod 2. \end{cases}$$

Let $t \ge \tau$ and let $x \in \mathbb{Z}^d$ such that $\|x\| \le t^\sigma$. For $n > \nu t$,

$$\|y_2 - x\| \le \|y_2\| + \|x\| \le \|y_2\| + (\nu^{-1}n)^\sigma < \rho n \le \rho\iota(y_2 - x, n).$$

By definition of ι, we also have $\|y_2 - x\|_1 \equiv \iota(y_2 - x, n) \mod 2$. Hence, using (2.28) and (2.29),

$$\frac{q_n^{y_1-x}}{q_{\iota(y_2-x,n)}^{y_2-x}} \le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(\frac{\|y_2 - x\|}{n}\|y_1 - y_2\| + \frac{\ln(n)}{n}\right)\right) < (1+\delta)^{\frac{1}{3}}. \tag{2.30}$$

Notice that for $n < \nu^{-1}t$, we can write

$$\frac{t^n}{n!} \le (\nu^{-1} + t^{-1})\frac{t^{\iota(y_2-x,n)}}{\iota(y_2-x,n)!} < (1+\delta)^{\frac{1}{3}}\frac{t^{\iota(y_2-x,n)}}{\iota(y_2-x,n)!}.$$

Then

$$\begin{aligned} p_t^{y_1-x} =& (1+\mathcal{D}_t(y_1,x)) \sum_{n\in J(t)} e^{-t}\frac{t^n}{n!}q_n^{y_1-x} \qquad (2.31)\\ \le& (1 + \sup_{\|x\|\le t^\sigma} \mathcal{D}_t(y_1,x))e^{-t} \sum_{n\in J(t)} \frac{t^n}{n!}q_n^{y_1-x} \\ \le& (1+\delta)^{\frac{1}{3}} e^{-t} \sum_{n\in J(t): q_n^{y_1-x}>0} \frac{t^n}{n!}q_n^{y_1-x} \\ \le& (1+\delta)^{\frac{2}{3}} e^{-t} \sum_{n\in J(t): q_n^{y_1-x}>0} \frac{t^{\iota(y_2-x,n)}}{\iota(y_2-x,n)!}q_{\iota(y_2-x,n)}^{y_2-x}\frac{q_n^{y_1-x}}{q_{\iota(y_2-x,n)}^{y_2-x}} \\ \le& (1+\delta)e^{-t} \sum_{n\in J(t): q_n^{y_1-x}>0} \frac{t^{\iota(y_2-x,n)}}{\iota(y_2-x,n)!}q_{\iota(y_2-x,n)}^{y_2-x} \end{aligned}$$

where in the last line we used the estimate in (2.30). Furthermore,

$$p_t^{y_2-x} = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}q_n^{y_2-x} \ge e^{-t} \sum_{n\in J(t): q_n^{y_1-x}>0} \frac{t^{\iota(y_2-x,n)}}{\iota(y_2-x,n)!}q_{\iota(y_2-x,n)}^{y_2-x}. \tag{2.32}$$

Combining the estimates (2.31) and (2.32) we have

$$\frac{p_t^{y_1-x}}{p_t^{y_2-x}} < 1 + \delta.$$

Since our choice of τ did not depend on x and since $\delta > 0$ is arbitrary, this shows that

$$\limsup_{t\to\infty} \sup_{\|x\|\le t^\sigma} \frac{p_t^{y_1-x}}{p_t^{y_2-x}} \le 1.$$

Interchanging the roles of y_1 and y_2, we obtain

$$\limsup_{t\to\infty} \sup_{\|x\|\le t^\sigma} \frac{p_t^{y_2-x}}{p_t^{y_1-x}} \le 1$$

and thus the desired result. □

Lemma 2.14

There is a constant $C > 0$ (depending on the dimension) such that the continuous-time transition probability of going from 0 to any point $x \in \mathbb{Z}^d$ in time $t = 1$ satisfies the following bound:

$$p_1^x \le \frac{C}{\|x\|^2}.$$

Proof. From Theorem 2.1.1 in [LL10], there is a constant $C > 0$ (depending on the dimension) such that for any $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that

$$q_n^x \le \frac{C}{n^{d/2}\|x\|^2}.$$

Therefore,

$$p_1^x = \sum_{n=0}^{\infty} e^{-1}\frac{1}{n!}q_n^x \le \sum_{n=0}^{\infty} e^{-1}\frac{1}{n!}\frac{C}{n^{d/2}\|x\|^2} \le \frac{C}{\|x\|^2}.$$

□

chapter 3

Directed Polymers in Random Environment

Contents

For $d \geq 3$, let Ω be the set of functions $\omega : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{Z}^d$, the function $t \mapsto \omega(x,t)$ is continuous and satisfies $\omega(x,0) = 0$. Each $\omega \in \Omega$ represents a realization of the noise in our stochastic model. Let $\mathcal{F}$ denote the canonical σ-field on Ω, and let Q be the probability measure on $(\Omega, \mathcal{F})$ under which $(W^x)_{x \in \mathbb{Z}^d}$, defined by $W^x_t(\omega) := \omega(x,t)$, are independent two-sided Wiener processes. Expectation corresponding to Q will be denoted by $\langle \cdot \rangle$. These constitute the random potential in our setting.

For any $(x,s) \in \mathbb{Z}^d \times \mathbb{R}$, let $\eta = (\eta_t)_{t \geq s}$ be a continuous-time simple symmetric random walk (SSRW) on $\mathbb{Z}^d$ starting at x at time s. The corresponding probability measure is denoted by $\mathbf{P}_{x,s}$ and the corresponding expectation by $\mathbf{E}_{x,s}$. We assume that the jumps of η occur at random times given by independent exponential clocks; i.e., the times between consecutive jumps form an i.i.d. sequence of exponential random variables with rate 1. Note that η is transient because $d \geq 3$. If observed over a time interval $[s,t)$, a sample path of η (which we shall also denote by η) is characterized by

(1) the number $n_{s,t}$ of jumps that occur within the time interval (s,t),

(2) a discrete-time path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n_{s,t}})$ on $\mathbb{Z}^d$ such that $\gamma_0 = x$ and $\|\gamma_j - \gamma_{j-1}\|_1 = 1$ whenever $1 \le j \le n_{s,t}$, and

(3) the jump times $s < s_1 < \dots < s_{n_{s,t}} < t$.

It is convenient to introduce the notation $s_0 := s$ and $s_{n_{s,t}+1} := t$, although we do not assume that s and t are jump times. If $s = 0$, we will typically write n_t instead of $n_{0,t}$. To a sample path η and a realisation of the noise $\omega \in \Omega$, we assign the action defined by

$$\mathcal{A}_s^t(\eta, \omega) := \sum_{j=0}^{n_{s,t}} \big(\omega(\gamma_j, s_{j+1}) - \omega(\gamma_j, s_j)\big). \tag{3.1}$$

For any time $t > s$ and any site $y \in \mathbb{Z}^d$, denote the probability measure obtained from $\mathbf{P}_{x,s}$ by conditioning on the event $\{\eta_t = y\}$ by $\mathbf{P}_{x,s}^{y,t}$. The corresponding expectation is denoted by $\mathbf{E}_{x,s}^{y,t}$. Also set

$$p_t^y := \mathbf{P}_{0,0}(\eta_t = y).$$

1. Partition functions. Fix a parameter $\beta > 0$, called *inverse temperature*, which we will always think of as small; i.e., we study the high-temperature regime. For every $\omega \in \Omega$, we define the random ***point-to-point partition function*** by

$$Z_{x,s}^{y,t}(\omega) := e^{-\frac{\beta^2}{2}(t-s)} p_{t-s}^{y-x}\, \mathbf{E}_{x,s}^{y,t} e^{\beta \mathcal{A}_s^t(\cdot,\omega)}. \tag{3.2}$$

We also define

$$Z_{x,s}^{t}(\omega) := \sum_{y\in\mathbb{Z}^d} Z_{x,s}^{y,t}(\omega), \qquad \text{and} \qquad Z_{s}^{y,t}(\omega) := \sum_{x\in\mathbb{Z}^d} Z_{x,s}^{y,t}(\omega). \tag{3.3}$$

Since $e^{-\frac{\beta^2}{2}(t-s)}\langle e^{\beta\mathcal{A}_s^t(\eta,\cdot)}\rangle = 1$ for every η, these partition functions are normalized in the sense that

$$\langle Z_{x,s}^t \rangle = \langle Z_s^{y,t} \rangle = 1.$$

Notice that the law of the stochastic process $(Z_{x,s}^{s+t})_{t\ge 0}$ with respect to Q does not depend on x or s because the law for the increments of the Wiener processes $(W^x)_{x\in\mathbb{Z}^d}$ is stationary in space and time, and because the SSRW η is homogeneous. Besides, $(Z_{x,s}^{s+t})_{t\ge0}$ and $(Z_{s-t}^{x,s})_{t\ge0}$ have the same law because of time-reversibility of η.

The first major achievement of this thesis is to construct a limit of the partition functions for the Anderson polymer model. Namely, we derive the following convergence result, including the rate of convergence, for the partition functions (3.3) in the regime of small β.

Theorem 3.1

If β is sufficiently small, the following statements hold.

(1) *For all $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$, the partition functions $Z^t_{x,s}$ and $Z^{y,t}_s$ converge in $L^2(Q)$ as $t \to \infty$ and $s \to -\infty$ respectively to the* ***limiting partition functions***

$$Z^{\infty}_{x,s} := \lim_{t\to\infty} Z^t_{x,s} \qquad \textit{and} \qquad Z^{y,t}_{-\infty} := \lim_{s\to-\infty} Z^{y,t}_s.$$

In fact, there is $\theta > 0$, independent of x, y and s, t, such that

$$\lim_{t\to\infty} (t-s)^{\theta} \Big\langle \big(Z^t_{x,s} - Z^{\infty}_{x,s}\big)^2 \Big\rangle = 0 \quad \textit{and} \quad \lim_{s\to-\infty} (t-s)^{\theta} \Big\langle \Big(Z^{y,t}_s - Z^{y,t}_{-\infty}\Big)^2 \Big\rangle = 0$$

(2) *There is a subset $\Omega^+ \subset \Omega$ with $Q(\Omega^+) = 1$, such that for all $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$ and all $\omega \in \Omega^+$, the limiting partition functions $Z^{\infty}_{x,s}(\omega)$ and $Z^{y,t}_{-\infty}(\omega)$ exist and are positive.*

Remark 3.2

Note that, due to symmetry, it is enough to henceforth only study the limit as $t \to \infty$ of $Z^t_{x,s}$.

This chapter is devoted to the proof of this theorem which we structure as follows. To prove Part (1), we first prove the existence of the $L^2(Q)$-limit of the partition functions in Proposition 3.3. Then, in Proposition 3.5, we establish the rate of convergence. Of course, Proposition 3.5 immediately implies Proposition 3.3, but we believe that the proof of Proposition 3.3 is sufficiently simple and illustrative to be included for pedagogical reasons.

For Part (2), the proof is constructed in a sequence of three incremental steps. First, in Corollary 3.4 we argue that for every $(x, s) \in \mathbb{Z}^d \times \mathbb{R}$, there is a full-measure subset $\Omega_{x,s}$ such that $Z^{\infty}_{x,s}(\omega)$ exists for every $\omega \in \Omega_{x,s}$. This follows immediately from the proof of Part (1) (specifically, Proposition 3.3). The second and crucial step is to flip the quantifiers: using the existence of subsets $\Omega_{x,s}$ and by restricting our attention to integer time intervals, we argue in Proposition 3.8 that there is a full-measure subset Ω^{lim}, independent of (x, s), such that $Z^{\infty}_{x,s}(\omega)$ exists for every $\omega \in \Omega_{x,s}$ and every $(x, s) \in \mathbb{Z}^d \times \mathbb{R}$. The third and final step, which is Proposition 3.11, is to prove, this time by first restricting our attention to rational times, that there is a full-measure subset $\Omega^+ \subset \Omega^{\lim}$, again independent of (x,s), such that $Z^\infty_{x,s}(\omega)$ is positive for every $\omega \in \Omega^+$ and every $(x,s) \in \mathbb{Z}^d \times \mathbb{R}$.

3.1. Existence of Limiting Partition Functions

Define

$$\alpha := \sum_{n=1}^{\infty} \sum_{z \in \mathbb{Z}^d} (q_n^z)^2 . \tag{3.4}$$

By Theorem 2.2, there exists $C > 0$ such that $q_n^z \leq C n^{-\frac{d}{2}}$ for all $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. Therefore, because $d \geq 3$, we have

$$\alpha \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\frac{d}{2}}} \sum_{z \in \mathbb{Z}^d} q_n^z = C \sum_{n=1}^{\infty} \frac{1}{n^{\frac{d}{2}}} < \infty. \tag{3.5}$$

Proposition 3.3

For $\beta < \frac{1}{\sqrt{1+\alpha}}$, the following holds: For all $x \in \mathbb{Z}^d$ and $s \in \mathbb{R}$, as $t \to \infty$, $Z^t_{x,s}$ converges in $L^2(Q)$ to a limiting partition function $Z^\infty_{x,s}$.

As pointed out by Bolthausen ([Bol89]), $(Z^t_{x,s})_{t \geq s}$ is a martingale with respect to the filtration $\mathcal{F}_t := \sigma(W^y_u : s \leq u \leq t, y \in \mathbb{Z}^d)$, so the Martingale Convergence Theorem implies the following.

Corollary 3.4

For every $x \in \mathbb{Z}^d$ and $s \in \mathbb{R}$ there exists a set $\Omega_{x,s} \subseteq \Omega$ with $Q(\Omega_{x,s}) = 1$ such that for every $\omega \in \Omega_{x,s}$, $Z^t_{x,s}(\omega)$ converges to a limit $Z^\infty_{x,s}(\omega)$ as $t \to \infty$.

Before we prove Proposition 3.3, we derive an expansion for the partition function $Z^{y,t}_{x,s}$, and state analogous versions for $Z^t_{x,s}$ and $Z^{y,t}_s$. In addition to being a key step in proving Proposition 3.3, they will also help us prove Proposition 3.5 and 4.1.

3.1.1. Partition Function Expansions.

To derive an expansion for $Z^t_{x,s}$, let us first write

$$Z^{y,t}_{x,s} = p^{y-x}_{t-s} e^{-\frac{\beta^2}{2}(t-s)} \mathbf{E}^{y,t}_{x,s} \prod_{j=0}^{n_{s,t}} \exp\left(\beta\left(W^{\gamma_j}_{s_{j+1}} - W^{\gamma_j}_{s_j}\right)\right). \tag{3.6}$$

In the expression above, it is important to note that $s_{n_{s,t}+1} = t$, in line with the notation for random walk paths in continuous time over an interval $[s,t)$ that we introduced earlier. Given $z \in \mathbb{Z}^d$ and $s,t \in \mathbb{R}$ such that $s < t$, define

$$h(z;s,t) := e^{-\frac{\beta^2}{2}(t-s)} e^{\beta(W_t^z - W_s^z)} - 1 = \frac{e^{\beta(W_t^z - W_s^z)} - e^{\frac{\beta^2}{2}(t-s)}}{e^{\frac{\beta^2}{2}(t-s)}}. \tag{3.7}$$

Notice that $\langle h(z;s,t)\rangle = 0$ and $\langle h(z;s,t)^2\rangle = e^{\beta^2(t-s)} - 1$. Moreover, $h(z;s,t)$ and $h(z';s',t')$ are independent if $z \neq z'$ or if $(s,t) \cap (s',t') = \varnothing$. Then we can rewrite (3.6) as follows:

$$\begin{aligned}
Z_{x,s}^{y,t} &= p_{t-s}^{y-x} \mathbf{E}_{x,s}^{y,t} \prod_{j=0}^{n_{s,t}} \big(1 + h(\gamma_j; s_j, s_{j+1})\big) \\
&= p_{t-s}^{y-x} + p_{t-s}^{y-x} \mathbf{E}_{x,s}^{y,t} \Bigg[\sum_{r=1}^{n_{s,t}+1} \sum_{\substack{0 \le i_1 < \ldots < i_r \le n_{s,t}, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} \mathbb{1}_{\gamma_{i_j} = z_j, 1 \le j \le r} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \Bigg] \\
&= p_{t-s}^{y-x} + \mathbf{E} \Bigg[\sum_{r=1}^{n_{s,t}+1} \sum_{\substack{0 \le i_1 < \ldots < i_r \le n_{s,t}, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} q_{i_1}^{z_1 - x} \ldots q_{n-i_r}^{y - z_r} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \Bigg] \\
&= p_{t-s}^{y-x} + \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \sum_{\substack{0 \le i_1 < \ldots < i_r \le n, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} q_{i_1}^{z_1 - x} \ldots q_{n-i_r}^{y - z_r} \mathbf{E} \Bigg[\prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \mathbb{1}_{n_{s,t} = n} \Bigg].
\end{aligned} \tag{3.8}$$

Where $\mathbf{E}$ should be understood as averaging with respect to the the Poisson point process on the real line. Similarly, we have:

$$Z_s^{y,t} = 1 + \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \sum_{\substack{0 \le i_1 < \ldots < i_r \le n, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} q_{i_2 - i_1}^{z_2 - z_1} \ldots q_{n-i_r}^{y - z_r} \mathbf{E} \Bigg[\prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \mathbb{1}_{n_{s,t} = n} \Bigg]$$

and

$$Z_{x,s}^{t} = 1 + \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \sum_{\substack{0 \le i_1 < \ldots < i_r \le n, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} q_{i_1}^{z_1 - x} \ldots q_{i_r - i_{r-1}}^{z_r - z_{r-1}} \mathbf{E} \Bigg[\prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \mathbb{1}_{n_{s,t} = n} \Bigg].$$

3.1.2. Proof of Proposition 3.3 and Corollary 3.4

Assume without loss of generality that $s = 0$ and that x is the zero vector in $\mathbb{R}^d$. First notice that $(Z_{0,0}^t)_{t>0}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t>0} := \sigma\{W_u^y : 0 \le u \le t,\ y \in \mathbb{Z}^d\}$. Since $\langle Z_{0,0}^t \rangle = 1$, we have that

$$\langle Z_{0,0}^t | \mathcal{F}_s \rangle = \left\langle \sum_{y \in \mathbb{Z}^d} Z_{0,0}^{y,s} Z_{y,S}^t \,\middle|\, \mathcal{F}_s \right\rangle = \sum_{y \in \mathbb{Z}^d} Z_{0,0}^{y,s} \langle Z_{y,s}^t \rangle = \sum_{y \in \mathbb{Z}^d} Z_{0,0}^{y,s} = Z_{0,0}^t.$$

Following the approach in [Sin95], by the L^2-bounded Martingale Convergence Theorem, to complete the proof of Proposition 3.3 and Corollary 3.4, it suffices to show that

$$\sup_{t>0} \langle (Z_{0,0}^t)^2 \rangle < \infty.$$

Observe that the series in (3.8) has an orthogonality structure, which we shall now exploit. Since $h(z;s,t)$ and $h(z';s',t')$ are independent if $z \neq z'$ or if $(s,t) \cap (s',t') = \varnothing$, and since $\langle h(z;s,t) \rangle = 0$, we have with Jensen's inequality and Fubini's Theorem that $\langle (Z_{0,0}^t)^2 \rangle$ is bounded from above by

$$2 + 2 \sum_{r=1}^{\infty} \sum_{\substack{0 \le i_1 < \ldots < i_r, \\ z_1, \ldots, z_r \in \mathbb{Z}^d}} \left(q_{i_1}^{z_1}\right)^2 \ldots \left(q_{i_r - i_{r-1}}^{z_r - z_{r-1}}\right)^2 \mathbf{E}\left[\left\langle \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1})^2 \right\rangle\right]. \tag{3.9}$$

Since $\langle h(z;s,t)^2 \rangle = e^{\beta^2(t-s)} - 1$, and since $s_{i_j+1} - s_{i_j}$ is an exponentially distributed random variable with rate 1, we find

$$\mathbf{E}\left[\left\langle \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1})^2 \right\rangle\right] = \mathbf{E}\left[\prod_{j=1}^{r} \left(e^{\beta^2(s_{i_j+1} - s_{i_j})} - 1\right)\right] = \lambda^r. \tag{3.10}$$

where

$$\lambda := \frac{\beta^2}{1 - \beta^2}. \tag{3.11}$$

Hence, the expression in (3.9) is finite provided that

$$\sum_{r=1}^{\infty} (\alpha\lambda)^r < \infty,$$

which holds for $\lambda < \frac{1}{\alpha}$, or equivalently, for $\beta^2 < 1/(1+\alpha)$. This completes the proof of Proposition 3.3 and Corollary 3.4.

3.2. Rate of Convergence

Proposition 3.5

For β sufficiently small, there is $\theta \in (0, \min\{\frac{d}{2}-1, -\ln(\frac{\alpha\beta^2}{1-\beta^2})\})$ such that

$$\lim_{t\to\infty} (t-s)^{\theta} \left\langle \left(Z^t_{x,s} - Z^{\infty}_{x,s}\right)^2 \right\rangle = 0.$$

Before presenting the proof of Proposition 3.5, we introduce some additional notation and establish two lemmas. For every $n \in \mathbb{N}_0$ and $r \in \{1, \dots n+1\}$, let

$$I_{n,r} := \{\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{N}_0^r \; : \; 0 \le i_1 < \dots < i_r \le n\}.$$

For $\mathbf{i} \in I_{n,r}$ and $\mathbf{z} \in (\mathbb{Z}^d)^r$, define $q_r(\mathbf{i}, \mathbf{z}) := q_{i_1}^{z_1} \dots q_{i_r - i_{r-1}}^{z_r - z_{r-1}}$ and a sequence $\mathbf{t}$ of positive numbers, let

$$M_n(\mathbf{t}) := \sum_{r=1}^{n+1} \sum_{\mathbf{i}\in I_{n,r},\mathbf{z}} q_r(\mathbf{i}, \mathbf{z})^2 \prod_{j=1}^{r} \left(e^{\beta^2 t_{i_j+1}} - 1\right),$$

which is monotone increasing in n. Set

$$M(\mathbf{t}) := \lim_{n\to\infty} M_n(\mathbf{t}) \in (0, +\infty].$$

Lemma 3.6

Let $\tau = (\tau_k)_{k\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables with rate 1. Then,

$$\lim_{n\to\infty} n^{\theta} \mathbf{E}\left[M(\tau) - M_{n-1}(\tau)\right] = 0, \quad \theta \in (0, \tfrac{d}{2} - 1).$$

Proof. If τ_1 is an exponentially distributed random variable with rate 1, we have for $\beta < 1$

$$\mathbf{E}\left[e^{\beta^2 \tau_1} - 1\right] = \lambda,$$

where $\lambda = \frac{\beta^2}{1-\beta^2}$. Therefore,

$$\begin{aligned}
&\mathbf{E}\left[M(\tau) - M_{n-1}(\tau)\right] \qquad (3.12)\\
&\leq \sum_{r>\ln(n)} \sum_{\substack{0\leq i_1<\ldots<i_r,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2 \prod_{j=1}^{r} \mathbf{E}\left[e^{\beta^2\tau_{i_j+1}} - 1\right]\\
&\quad + \sum_{1\leq r\leq \ln(n)} \sum_{\substack{0\leq i_1<\ldots<i_r, i_r\geq n,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2 \prod_{j=1}^{r} \mathbf{E}\left[e^{\beta^2\tau_{i_j+1}} - 1\right]\\
&= \sum_{r>\ln(n)} \lambda^r \sum_{\substack{0\leq i_1<\ldots<i_r,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2\\
&\quad + \sum_{1\leq r\leq \ln(n)} \lambda^r \sum_{\substack{0\leq i_1<\ldots i_r, i_r\geq n,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2.
\end{aligned}$$

Notice that for $r \geq 1$,

$$\sum_{\substack{j_1,\ldots,j_r\in\mathbb{N},\\ c_1,\ldots,c_r\in\mathbb{Z}^d}} \left(q_{j_1}^{c_1}\right)^2 \cdots \left(q_{j_r}^{c_r}\right)^2 = \prod_{k=1}^{r} \sum_{j_1,\ldots,j_r\in\mathbb{N}} \sum_{c_k\in\mathbb{Z}^d} \left(q_{j_k}^{c_k}\right)^2 = \alpha^r,$$

where $\alpha = \sum_{n=1}^{\infty}\sum_{z\in\mathbb{Z}^d}(q_n^z)^2$. Suppose λ is so small that $\alpha\lambda < 1$. The term in the fourth line of (3.12) can be written as

$$\sum_{r>\ln(n)} \lambda^r(\alpha^r + \alpha^{r-1}) < 2 \sum_{r>\ln(n)-1} (\lambda\alpha)^r < \frac{2}{\alpha\lambda(1-\alpha\lambda)} n^{\ln(\alpha\lambda)},$$

and

$$\lim_{n\to\infty} n^{\theta} \frac{2}{\alpha\lambda(1-\alpha\lambda)} n^{\ln(\alpha\lambda)} = 0, \quad \theta \in (0, -\ln(\alpha\lambda)).$$

The term in the fifth line of (3.12) is less than

$$2 \sum_{1\leq r\leq\ln(n)} \lambda^r \sum_{\substack{j_1,\ldots,j_r\in\mathbb{N},\\ j_1+\ldots+j_r\geq n}} \sum_{c_1,\ldots,c_r\in\mathbb{Z}^d} \left(q_{j_1}^{c_1}\right)^2 \cdots \left(q_{j_r}^{c_r}\right)^2. \qquad (3.13)$$

For $1 \leq r \leq \ln(n)$, $c_1,\ldots,c_r \in \mathbb{Z}^d$ and $j_1,\ldots,j_r \in \mathbb{N}$ with $j_1+\ldots+j_r \geq n$, there is $l \in \{1,\ldots,r\}$

such that $j_l \geq \frac{n}{\ln(n)}$. Thus,

$$\sum_{\substack{j_1,\ldots,j_r \in \mathbb{N}, \\ j_1+\ldots+j_r \geq n}} \sum_{c_1,\ldots,c_r \in \mathbb{Z}^d} \left(q_{j_1}^{c_1}\right)^2 \ldots \left(q_{j_r}^{c_r}\right)^2 \leq \sum_{l=1}^{r} \sum_{\substack{j_1,\ldots,j_r \in \mathbb{N}, \\ j_l \geq \frac{n}{\ln(n)}}} \prod_{k=1}^{r} \left(\sum_{c_k \in \mathbb{Z}^d} \left(q_{j_k}^{c_k}\right)^2 \right)$$
$$= r\alpha^{r-1} \sum_{j \geq \frac{n}{\ln(n)}} \sum_{c \in \mathbb{Z}^d} \left(q_j^c\right)^2$$
$$\lesssim r\alpha^{r-1} \sum_{j \geq \frac{n}{\ln(n)}} \frac{1}{j^{\frac{d}{2}}}.$$

As a result, the expression in (3.13) is less than a constant times

$$\sum_{r=1}^{\infty} r(\alpha\lambda)^r \sum_{j \geq \frac{n}{\ln(n)}} \frac{1}{j^{\frac{d}{2}}} \lesssim \sum_{r=1}^{\infty} r(\alpha\lambda)^r \left(\frac{n}{\ln(n)} - 1 \right)^{1-\frac{d}{2}},$$

and

$$\lim_{n\to\infty} n^{\theta} \left(\frac{n}{\ln(n)} - 1 \right)^{1-\frac{d}{2}} = 0, \quad \theta \in \left(0, \tfrac{d}{2} - 1\right).$$

□

Lemma 3.7

Let $T > t > 0$. For $\mathbf{P}_{0,0}$-almost every realization of the continuous-time random walk η and for every sequence $\mathbf{t} = (t_1, t_2, \ldots)$ of positive numbers such that $t_i = s_i - s_{i-1}$, $1 \leq i \leq n_T + 1$, we have

$$\left\langle \left(e^{-\frac{\beta^2}{2}T} e^{\beta \mathcal{A}_0^T} - e^{-\frac{\beta^2}{2}t} e^{\beta \mathcal{A}_0^t} \right)^2 \right\rangle \leq 3 \left(M(\mathbf{t}) - M_{n_t - 1}(\mathbf{t}) \right) + 3 \left\langle \left(R^t\right)^2 \right\rangle,$$

where R^t is a random variable depending on η and $(W^x)_{x \in \mathbb{Z}^d}$ that satisfies

$$\lim_{t\to\infty} t^{\theta} \mathbf{E} \left\langle \left(R^t\right)^2 \right\rangle = 0, \quad \theta \in \left(0, \tfrac{d}{2}\right). \tag{3.14}$$

Proof. Since t is $\mathbf{P}_{0,0}$-almost surely not a jump time for η, we have

$$\begin{aligned}
&e^{-\frac{\beta^2}{2}T}e^{\beta\mathcal{A}_0^T} - e^{-\frac{\beta^2}{2}t}e^{\beta\mathcal{A}_0^t}\\
=&\sum_{r=1}^{n_t}\sum_{\substack{0\le i_1<\ldots<i_r\le n_T,\ i_r\ge n_t,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})\prod_{j=1}^{r} h(z_j;s_{i_j},s_{i_j+1})\\
&+\sum_{r=n_t+1}^{n_T+1}\sum_{\substack{0\le i_1<\ldots<i_r\le n_T,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})\prod_{j=1}^{r} h(z_j;s_{i_j},s_{i_j+1}) - R^t,
\end{aligned}$$

where $s_{n_T+1}=T$ and

$$R^t := \sum_{r=1}^{n_t+1}\sum_{\substack{0\le i_1<\ldots<i_r=n_t,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})\prod_{j=1}^{r-1} h(z_j;s_{i_j},s_{i_j+1})h(z_r;s_{n_t},t).$$

If we set $t_i=s_i-s_{i-1}$ for $1\le i\le n_T+1$, we obtain for any positive $(t_i)_{i>n_T+1}$ the estimate

$$\begin{aligned}
&\left\langle\left(e^{-\frac{\beta^2}{2}T}e^{\beta\mathcal{A}_0^T} - e^{-\frac{\beta^2}{2}t}e^{\beta\mathcal{A}_0^t}\right)^2\right\rangle\\
\le&3\sum_{r=1}^{n_t}\sum_{\substack{0\le i_1<\ldots<i_r\le n_T, i_r\ge n_t,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2\prod_{j=1}^{r}\left\langle h(z_j;s_{i_j},s_{i_j+1})^2\right\rangle\\
&+3\sum_{r=n_t+1}^{n_T+1}\sum_{\substack{0\le i_1<\ldots<i_r\le n_T,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2\prod_{j=1}^{r}\left\langle h(z_j;s_{i_j},s_{i_j+1})^2\right\rangle + 3\left\langle\left(R^t\right)^2\right\rangle\\
\le&3\left(M\left(\mathbf{t}\right)-M_{n_t-1}\left(\mathbf{t}\right)\right)+3\left\langle\left(R^t\right)^2\right\rangle.
\end{aligned}$$

The only point left to show is (3.14). The D-sequence associated with R^t is

$$D_n^{\vartheta}=\sum_{r=1}^{n+1}\vartheta^r\sum_{\substack{0\le i_1<\ldots<i_r=n,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2,\quad \vartheta>0.$$

For $n\ge 2$,

$$D_n^{\vartheta}=\vartheta U_{n,1}+\sum_{r=2}^{n}\vartheta^r\left(U_{n,r}+U_{n,r-1}\right)+\vartheta^{n+1}U_{n,n}\le 2\sum_{r=1}^{n}\vartheta^r U_{n,r},$$

where

$$U_{n,r} := \begin{cases} \sum\limits_{\substack{0<i_1<\ldots<i_{r-1}<n, \\ z_1,\ldots,z_r\in\mathbb{Z}^d}} \left(q_{i_1}^{z_1}\right)^2 \ldots \left(q_{n-i_{r-1}}^{z_r-z_{r-1}}\right)^2, & 2\leq r\leq n, \\ \sum\limits_{z\in\mathbb{Z}^d} \left(q_n^z\right)^2, & r=1. \end{cases}$$

Let $C>0$ such that $q_n^y \leq Cn^{-\frac{d}{2}}$ for all $n\in\mathbb{N}$ and $y\in\mathbb{Z}^d$. Then,

$$U_{n,r} \leq \begin{cases} C^r \sum\limits_{0<i_1<\ldots<i_{r-1}<n} i_1^{-\frac{d}{2}} \ldots (n-i_{r-1})^{-\frac{d}{2}}, & 2\leq r\leq n, \\ Cn^{-\frac{d}{2}}, & r=1. \end{cases}$$

By Lemma 7.3, we have that there is a constant $c>0$, depending only on the dimension d, such that for any $r\in\mathbb{N}$,

$$\sum_{0<i_1<\ldots<i_r<n} i_1^{-\frac{d}{2}}(i_2-i_1)^{-\frac{d}{2}} \ldots (i_r-i_{r-1})^{-\frac{d}{2}}(n-i_r)^{-\frac{d}{2}} \leq c^r n^{-\frac{d}{2}}, \quad n\geq r+1. \tag{3.15}$$

Hence,

$$U_{n,r} \leq (Cc)^r n^{-\frac{d}{2}}, \quad 1\leq r\leq n.$$

As a result,

$$D_n^{\vartheta} \leq \frac{2}{n^{\frac{d}{2}}} \sum_{r=1}^{\infty} (\vartheta Cc)^r. \tag{3.16}$$

For $\vartheta < (Cc)^{-1}$ and $\theta < \frac{d}{2}$, we obtain $\lim_{n\to\infty} n^{\theta} D_n^{\vartheta} = 0$. Remark 4.5 yields (3.14) for β sufficiently small. □

3.2.1. Proof of Proposition 3.5

Let $T>t>0$ and let $b\in(0,1)$. With Jensen's inequality and Lemma 3.7, we obtain

$$\begin{aligned} &\left\langle \left(Z^T - Z^t\right)^2 \right\rangle \\ \leq &\mathbf{E}_{0,0}\left[\left\langle \left(e^{-\frac{\beta^2}{2}T} e^{\beta\mathcal{A}_0^T} - e^{-\frac{\beta^2}{2}t} e^{\beta\mathcal{A}_0^t}\right)^2 \right\rangle \left(\mathbb{1}_{s_{\lfloor t^b\rfloor}<t} + \mathbb{1}_{s_{\lfloor t^b\rfloor}\geq t}\right)\right] \\ \leq &\mathbf{E}\left[3\left(M(\mathbf{t}) - M_{\lfloor t^b\rfloor-1}(\mathbf{t})\right)\right] + \mathbf{E}\left[3M(\mathbf{t})\ \mathbb{1}_{s_{\lfloor t^b\rfloor}\geq t}\right] + 3\mathbf{E}\left\langle \left(R^t\right)^2 \right\rangle, \end{aligned}$$

where $\mathbf{t}$ is an i.i.d. sequence of exponentially distributed random variables with rate 1. As a consequence of Lemma 3.6,

$$\lim_{t\to\infty} t^{\theta}\mathbf{E}\left[3\left(M(\mathbf{t})-M_{\lfloor t^b\rfloor-1}(\mathbf{t})\right)\right]=0,\quad \theta\in\left(0,b\left(\tfrac{d}{2}-1\right)\right).$$

Furthermore, (3.14) gives

$$\lim_{t\to\infty} t^{\theta}\mathbf{E}\left\langle\left(R^t\right)^2\right\rangle=0,\quad \theta\in\left(0,\tfrac{d}{2}\right).$$

Finally, the Cauchy–Schwarz inequality implies

$$\begin{aligned}
&\mathbf{E}\left[M(\mathbf{t})\mathbb{1}_{s_{\lfloor t^b\rfloor}\geq t}\right]\\
=&\sum_{r=1}^{\infty}\sum_{\substack{0\leq i_1<\ldots<i_r,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2\mathbf{E}\left[\prod_{j=1}^{r}\left(e^{\beta^2 t_{i_j+1}}-1\right)\mathbb{1}_{s_{\lfloor t^b\rfloor}\geq t}\right]\\
\leq&\mathbf{P}\left(s_{\lfloor t^b\rfloor}\geq t\right)^{\frac12}\sum_{r=1}^{\infty}\rho^r\sum_{0\leq i_1<\ldots<i_r}\sum_{z_1,\ldots,z_r\in\mathbb{Z}^d}q_r(\mathbf{i},\mathbf{z})^2,
\end{aligned}$$

where

$$\rho=\mathbf{E}\left[\left(e^{\beta^2 t_1}-1\right)^2\right]^{\frac12}<\alpha^{-1}$$

for β sufficiently small (recall that α was defined in (3.4)). We have

$$\mathbf{P}\left(s_{\lfloor t^b\rfloor}\geq t\right)=e^{-t}\sum_{k=0}^{\lfloor t^b\rfloor-1}\frac{t^k}{k!}.$$

By Stirling's formula,

$$t^{\theta}e^{-t}\sum_{k=0}^{\lfloor t^b\rfloor-1}\frac{t^k}{k!}\leq\frac{t^{\theta+1}e^{-t}e^{\lfloor t^b\rfloor-1}t^{\lfloor t^b\rfloor-1}}{\sqrt{2\pi(\lfloor t^b\rfloor-1)}(\lfloor t^b\rfloor-1)^{\lfloor t^b\rfloor-1}}\lesssim t^{\theta+1-\frac{b}{2}}e^{-t}\left(\frac{et}{\lfloor t^b\rfloor-1}\right)^{\lfloor t^b\rfloor-1},$$

and the right side tends to 0 as $t\to\infty$ for every $\theta>0$. Since b was arbitrarily chosen from $(0,1)$, we obtain

$$\lim_{t\to\infty} t^{\theta}\left\langle\left(Z^t-Z^{\infty}_{0,0}\right)^2\right\rangle=0,\quad \theta\in(0,\tfrac{d}{2}-1).$$

This completes the proof of Proposition 3.5, and, with it, the proof Part (1) of Theorem 3.1.

3.3. Uniform Almost Sure Convergence of the Partition Functions

Proposition 3.8

There is a subset $\Omega^{\mathrm{lim}} \subset \Omega$ with $Q(\Omega^{\mathrm{lim}}) = 1$, such that for all $(x, s) \in \mathbb{Z}^d \times \mathbb{R}$ and any $\omega \in \Omega^{\mathrm{lim}}$, we have that $Z^t_{x,s}(\omega)$ converges to a limit $Z^\infty_{x,s}(\omega)$ as $t \to \infty$.

In order to prove this proposition, we first need to establish some properties of the partition functions.

3.3.1. Some Properties of the Partition Functions.

Lemma 3.9

Let $x \in \mathbb{Z}^d$, $\omega \in \Omega$, and $0 < s < t$. Then,

$$Z^t_{x,0}(\omega) = \sum_{y \in \mathbb{Z}^d} Z^{y,s}_{x,0}(\omega) Z^t_{y,s}(\omega)$$

Proof. For any $y \in \mathbb{Z}^d$ and any sequence $\mathrm{s} := (s_n)_{n \in \mathbb{N}_0}$ of jump times, let $\Gamma^{n_s}_y$ be the collection of all paths of length n_s starting at y, and let $\Gamma^{y,n_{t-s}}_x$ be the collection of all paths of length n_{t-s} starting at x and ending at y. If γ is any path of length n_t starting at x with $\gamma_{n_s} = y$, let $\gamma' \in \Gamma^{n_s}_y$ and $\gamma'' \in \Gamma^{y,n_{t-s}}_x$ be the unique paths which concatenate to γ. By a slight abuse of notation, let the sample path for the continuous-time random walk described by γ and s be denoted by γ as well. Note that

$$\mathcal{A}^t_0(\gamma, \omega) = \mathcal{A}^s_0(\gamma', \omega) + \mathcal{A}^t_s(\gamma'', \omega),$$

provided that $s \notin \mathrm{s}$; i.e., that s is not a jump time. Then, using the fact that it is enough to average over the sequences of jump times s not containing s (because the complement has measure zero), we have:

$$
\begin{aligned}
Z^{t}_{x,0}(\omega) =& e^{-\frac{1}{2}\beta^2 t}\mathbf{E}\left[\sum_{\gamma\in\Gamma_x^{n_t}} \frac{1}{(2d)^{n_t}} e^{\beta\mathcal{A}_0^t(\gamma,\omega)}\right] \\
=& e^{-\frac{1}{2}\beta^2 s} e^{-\frac{1}{2}\beta^2 (t-s)}\mathbf{E}\left[\frac{1}{(2d)^{n_s}}\frac{1}{(2d)^{n_{t-s}}}\sum_{y\in\mathbb{Z}^d}\sum_{\gamma'\in\Gamma_x^{y,n_s}}\sum_{\gamma''\in\Gamma_y^{n_{t-s}}} e^{\beta\mathcal{A}_0^s(\gamma',\omega)} e^{\beta\mathcal{A}_s^t(\gamma'',\omega)}\right] \\
=& \sum_{y\in\mathbb{Z}^d} e^{-\frac{1}{2}\beta^2 s}\mathbf{E}\left[\sum_{\gamma'\in\Gamma_x^{y,n_s}} \frac{e^{\beta\mathcal{A}_0^s(\gamma',\omega)}}{(2d)^{n_s}}\right] e^{-\frac{1}{2}\beta^2 (t-s)}\mathbf{E}\left[\sum_{\gamma''\in\Gamma_y^{n_{t-s}}} \frac{e^{\beta\mathcal{A}_s^t(\gamma'',\omega)}}{(2d)^{n_{t-s}}}\right] \\
=& \sum_{y\in\mathbb{Z}^d} Z^{y,s}_{x,0}(\omega) Z^{t}_{y,s}(\omega),
\end{aligned}
$$

where $\mathbf{E}$ should be understood as averaging with respect to the Poisson point process on the real line. □

Lemma 3.10
$\left\langle \sup_{s,t\in(-M,M),s<t} \mathbf{E}^{y,t}_{x,s}\left[e^{\mathcal{A}_s^t(\eta,\omega)}\right]\right\rangle$ *is bounded by a constant that only depends on* M *and* β. *Notice that this immediately implies* $\left\langle \sup_{s,t\in(-M,M),s<t} Z^{y,t}_{x,s}\right\rangle$ *is bounded by a constant that only depends on* M *and* β.

Proof. For fixed $\omega \in \Omega$ and $s,t \in (-M,M)$ such that $s<t$, we can estimate

$$
\begin{aligned}
\mathbf{E}^{y,t}_{x,s}\left[e^{\mathcal{A}_s^t(\eta,\omega)}\right] <& \frac{1}{p^0_{s+M}}\int \mathbb{1}_{\eta_s=x,\eta_t=y} e^{\beta\mathcal{A}_s^t(\eta,\omega)}\, \mathbf{P}_{x,-M}(d\eta) \\
\le& \left(\frac{p^{y-x}_{t-s}}{p^0_{s+M}}\right)^{\frac{1}{2}} \left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\, \mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}},
\end{aligned}
\tag{3.17}
$$

where the integral is taken over possible realizations of η, and where the Cauchy–Schwarz inequality was used. For any $s \in (-M,M)$, we have

$$
p^0_{s+M} \ge e^{-(s+M)} \ge e^{-2M}. \tag{3.18}
$$

Therefore,

$$
\mathbf{E}^{y,t}_{x,s}\left[e^{\mathcal{A}_s^t(\eta,\omega)}\right] < e^{M}\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\, \mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}.
$$

Hence, to complete the proof of the lemma, we just need to show that there is a constant $h(M)$

depending only on M and β, such that

$$\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}} < h(M).$$

Let η be a continuous-time random-walk path starting from x at time $-M$, and observed on the time interval $[-M,M]$, with jump sites $\gamma'_1,\ldots,\gamma'_{n_{-M,M}}$, and jump times $s'_1<\ldots<s'_{n_{-M,M}}$. Assume in addition that $\eta_s=x$ and $\eta_t=y$. Set $\gamma'_0:=x$, $s'_0:=-M$, and $s'_{n_{-M,M}+1}:=M$. We use the following relabelling of the jump times and jump sites: Let $j^*:=\min\{j:s'_j>s\}$ and

$$s_k := s'_{j^*+k-1},\quad \gamma_k := \gamma'_{j^*+k-1},\quad \forall k\in\{1,\ldots,n_{s,t}\}.$$

Also set $\gamma_0:=x$, $s_0:=s$, and $s_{n_{s,t}+1}:=t$. Then

$$\begin{aligned}|\mathcal{A}_s^t(\eta,\omega)| \le& \sum_{j=0}^{n_{s,t}}|\omega(\gamma_j,s_{j+1})-\omega(\gamma_j,s_j)|\\ \le& 4\max_{z\in\{x,y\}}\sup_{r\in[-M,M]}|\omega(z,r)|+\sum_{j=0}^{n_{-M,M}}\left|\omega(\gamma'_j,s'_{j+1})-\omega(\gamma'_j,s'_j)\right|.\end{aligned}$$

Since the expression on the right does not depend on s or t, we obtain

$$\begin{aligned}&\left\langle\sup_{s,t\in(-M,M),s<t}\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\cdot)|}\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}\right\rangle\\ \le&\left\langle\left(\int\exp\left(2\beta\left(4\max_{z\in\{x,y\}}\sup_{r\in[-M,M]}|\omega(z,r)|\right.\right.\right.\right.\\ &\left.\left.\left.\left.+\sum_{j=0}^{n_{-M,M}}\left|\omega(\gamma'_j,s'_{j+1})-\omega(\gamma'_j,s'_j)\right|\right)\right)\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}\right\rangle.\end{aligned}$$

By Jensen's inequality, Fubini, and Cauchy–Schwarz, the expression on the right is bounded from above by

$$\begin{aligned}&\left(\int\left\langle\exp\left(16\beta\max_{z\in\{x,y\}}\sup_{r\in[-M,M]}|\omega(z,r)|\right)\right\rangle^{\frac{1}{2}}\right.\\ &\left.\left\langle\exp\left(4\beta\sum_{j=0}^{n_{-M,M}}\left|\omega(\gamma'_j,s'_{j+1})-\omega(\gamma'_j,s'_j)\right|\right)\right\rangle^{\frac{1}{2}}\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}.\end{aligned} \tag{3.19}$$

For a fixed realization η of the random walk and for $0 \le j \le n_{-M,M}$, we have

$$\begin{aligned}&\left\langle \exp\left(4\beta\left|\omega(\gamma'_j, s'_{j+1}) - \omega(\gamma'_j, s'_j)\right|\right)\right\rangle \\ =&\frac{2e^{8(s'_{j+1}-s'_j)\beta^2}}{\sqrt{2\pi(s'_{j+1}-s'_j)}}\int_0^\infty \exp\left(-\frac{(x-4(s'_{j+1}-s'_j)\beta)^2}{2(s'_{j+1}-s'_j)}\right)dx \le 2e^{8(s'_{j+1}-s'_j)\beta^2}.\end{aligned}$$

Thus,

$$\left\langle \exp\left(4\beta\sum_{j=0}^{n_{-M,M}}\left|\omega(\gamma'_j, s'_{j+1}) - \omega(\gamma'_j, s'_j)\right|\right)\right\rangle \le 2^{n_{-M,M}+1}e^{16M\beta^2}.$$

For $z \in \{x, y\}$, let $\xi_z := \sup_{r\in[-M,M]}|\omega(z,r)|$, and

$$\begin{aligned}\xi_z^+(1) &:= \sup_{r\in[0,M]} \omega(z,r), & \xi_z^+(2) &:= \sup_{r\in[0,M]} (-\omega(z,r)),\\ \xi_z^-(1) &:= \sup_{r\in[-M,0]} \omega(z,r), & \xi_z^-(2) &:= \sup_{r\in[-M,0]} (-\omega(z,r)).\end{aligned}$$

Then

$$\max_{z\in\{x,y\}}\sup_{r\in[-M,M]}|\omega(z,r)| \le \xi_x + \xi_y \le \sum_{z\in\{x,y\}}\sum_{i\in\{1,2\}}(\xi_z^+(i) + \xi_z^-(i)).$$

It follows that

$$\begin{aligned}&\left\langle \exp\left(16\beta\max_{z\in\{x,y\}}\sup_{r\in[-M,M]}|\omega(z,r)|\right)\right\rangle\\ \le&\left\langle \exp\left(16\beta\sum_{z\in\{x,y\}}\sum_{i\in\{1,2\}}(\xi_z^+(i)+\xi_z^-(i))\right)\right\rangle\\ =&\prod_{z\in\{x,y\}}\left\langle e^{16\beta\xi_z^+(1)}e^{16\beta\xi_z^+(2)}\right\rangle\left\langle e^{16\beta\xi_z^-(1)}e^{16\beta\xi_z^-(2)}\right\rangle\\ \le&\prod_{z\in\{x,y\}}\prod_{i\in\{1,2\}}\left\langle e^{32\beta\xi_z^+(i)}\right\rangle^{\frac{1}{2}}\left\langle e^{32\beta\xi_z^-(i)}\right\rangle^{\frac{1}{2}} = \left\langle e^{32\beta\xi_0^+(1)}\right\rangle^4.\end{aligned}$$

The random variable $\xi_0^+(1)$ has the same distribution as absolute value of a Gaussian random variable with mean 0 and variance M. Therefore,

$$\left\langle e^{32\beta\xi_0^+(1)}\right\rangle \le 2e^{512M\beta^2}.$$

We have thus shown the estimate

$$\left\langle \exp\left(16\beta \max_{z\in\{x,y\}} \sup_{r\in[-M,M]} |\omega(z,r)|\right)\right\rangle^{\frac{1}{2}} \left\langle \exp\left(4\beta \sum_{j=0}^{n_{-M,M}} \left|\omega(\gamma'_j, s'_{j+1}) - \omega(\gamma'_j, s'_j)\right|\right)\right\rangle^{\frac{1}{2}}$$
$$\leq 4e^{1032M\beta^2} 2^{\frac{n_{-M,M}+1}{2}}.$$

As a result, the expression in (3.19) is bounded from above by

$$\begin{aligned}
&\left(\int 4e^{1032M\beta^2} 2^{\frac{n_{-M,M}+1}{2}}\, \mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}} \\
=&2e^{516M\beta^2}\left(\sum_{n=0}^{\infty} e^{-2M}\frac{(2M)^n}{n!} 2^{\frac{n+1}{2}}\right)^{\frac{1}{2}} \\
=&2\cdot\sqrt{2}e^{516M\beta^2}\exp(2M(\sqrt{2}-1)).
\end{aligned}$$

Hence,

$$\left(\int e^{2\beta|\mathcal{A}^t_s(\eta,\omega)|}\, \mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}} < h(M), \tag{3.20}$$

where $h(M) = 2\cdot\sqrt{2}e^{516M\beta^2}\exp(2M(\sqrt{2}-1))$.

□

3.3.2. Proof of Proposition 3.8

The subset $\Omega^{\lim}$ is defined as follows:

$$\begin{aligned}
\Omega^{\lim} &:= \left\{\omega\in\Omega \;:\; \lim_{t\to\infty} Z^t_{x,s}(\omega) \text{ exists } \forall (x,s)\in\mathbb{Z}^d\times\mathbb{R}\right\} \\
&= \left\{\omega\in\Omega \;:\; (Z^t_{x,s})_{(t\geq s)} \text{ is Cauchy } \forall(x,s)\right\} \\
&= \left\{\omega\in\Omega \;:\; \forall\epsilon>0,\ \exists M>s \text{ s.t. } |Z^T_{x,s}(\omega) - Z^t_{x,s}(\omega)| < \epsilon \quad \forall(x,s)\right\} \\
&= \left\{\omega\in\Omega \;:\; \forall\epsilon>0,\ \exists M>s \text{ s.t. } \sup_{T,t>M} |Z^T_{x,s}(\omega) - Z^t_{x,s}(\omega)| < \epsilon \quad \forall(x,s)\right\}.
\end{aligned}$$

For $x\in\mathbb{Z}^d$, $s\in\mathbb{R}$, $\omega\in\Omega$, let

$$f^\omega_{x,s}(M) := \sup_{T,t\geq M} \left|Z^T_{x,s}(\omega) - Z^t_{x,s}(\omega)\right|.$$

Note that $f^{\omega}_{x,s}$ is non-increasing and non-negative. With this notation, we have

$$\begin{aligned}\Omega^{\lim} &= \{\omega \in \Omega \ : \ \forall \epsilon > 0,\ \exists M > s \text{ s.t. } f^{\omega}_{x,s}(m) < \epsilon \quad \forall m \geq M,\ \forall (x,s)\} \\ &= \left\{\omega \in \Omega \ : \ \lim_{M\to\infty} f^{\omega}_{x,s}(M) = 0 \quad \forall (x,s)\right\}\end{aligned}$$

For every $x \in \mathbb{Z}^d$ and $m \in \mathbb{Z}$, consider the set

$$\Omega^{\lim}_{x,m} := \left\{\omega \in \Omega \ : \ \lim_{M\to\infty} \sup_{s\in[m,m+1]} f^{\omega}_{x,s}(M) = 0\right\},$$

and notice that $\Omega^{\lim}$ equals the total intersection of all sets $\Omega^{\lim}_{x,m}$. Thus, in order to prove this theorem, it is sufficient to show that $Q(\Omega^{\lim}_{x,m}) = 1$ for every $x \in \mathbb{Z}^d$ and every $m \in \mathbb{Z}$. From the translation-invariance of $Z^t_{x,s}(\omega)$ it follows that $Q(\Omega^{\lim}_{x,m}) = Q(\Omega^{\lim}_{0,0})$. To prove that $Q(\Omega^{\lim}_{0,0}) = 1$, we show that $\Omega_{0,0} \subset \Omega^{\lim}_{0,0}$, where $\Omega_{0,0}$ is the Q-full measure set from Proposition 3.3.

Choose any $\omega \in \Omega_{0,0}$. Since $f^{\omega}_{0,s}$ is non-increasing and non-negative for any $s \in \mathbb{R}$, the sequence $\big(\sup f^{\omega}_{0,s}(M)\big)_{M\in\mathbb{Z}}$, where the supremum is taken over $s \in [0,1]$, is a non-increasing sequence of non-negative terms. This sequence therefore has a limit as $M \to \infty$ which is likewise non-negative. Now, Fatou's Lemma gives

$$\left\langle \lim_{M\to\infty} \sup_{s\in[0,1]} f^{\omega}_{0,s}(M) \right\rangle \leq \liminf_{M\to\infty} \left\langle \sup_{s\in[0,1]} f^{\omega}_{0,s}(M) \right\rangle,$$

so to prove that $\omega \in \Omega^{\lim}_{0,0}$, it is enough to show that the righthand side equals 0. Using Lemma 3.9

and Lemma 2.14, we estimate the expected value on the righthand side as follows:

$$\begin{aligned}
\left\langle \sup_{s\in[0,1]} f^{\omega}_{0,s}(M)\right\rangle &= \left\langle \sup_{s\in[0,1]} \sup_{T,t\geq M} |Z^T_{0,s}(\omega) - Z^t_{0,s}(\omega)|\right\rangle \\
&\leq \sum_{x\in\mathbb{Z}^d} \left\langle \sup_{s\in[0,1]} Z^{x,1}_{0,s}\right\rangle \left\langle \sup_{T,t\geq M} |Z^T_{x,1}(\omega) - Z^t_{x,1}(\omega)|\right\rangle \\
&\leq \sum_{x\in\mathbb{Z}^d} \left\langle \sup_{s\in[0,1]} Z^{x,1}_{0,s}\right\rangle \left\langle \sup_{T,t\geq M-1} |Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)|\right\rangle \\
&= \sum_{x\in\mathbb{Z}^d} \left\langle \sup_{s\in[0,1]} e^{-\frac{\beta^2}{2}(1-s)} p^x_{1-s} \mathbf{E}^{x,1}_{0,s}[e^{\beta\mathcal{A}^1_s}]\right\rangle \left\langle \sup_{T,t\geq M-1} |Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)|\right\rangle \\
&= \sum_{x\in\mathbb{Z}^d} \left\langle \sup_{s\in[0,1]} p^x_1 \mathbf{E}^{x,1}_{0,s}[e^{\beta\mathcal{A}^1_s}]\right\rangle \left\langle \sup_{T,t\geq M-1} |Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)|\right\rangle \\
&= \sum_{x\in\mathbb{Z}^d} \frac{C}{\|x\|^2} \left\langle \sup_{s\in[0,1]} \mathbf{E}^{x,1}_{0,s}\left[e^{\beta\mathcal{A}^1_s}\right]\right\rangle \left\langle \sup_{T,t\geq M-1} |Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)|\right\rangle .
\end{aligned}$$

Observe that this is a convergent series because, thanks to Lemma 3.10, the first expected value factor is bounded by $eh(1)$ and the second expected value factor is independent of x. At the same time, only the second expected value factor depends on M, and we claim that it vanishes as $M \to \infty$. Indeed, since $\omega \in \Omega_{0,0}$, the limit of $Z^t_{0,0}(\omega)$ as $t \to \infty$ exists, so the sequence

$$\left(\sup_{T,t\geq M-1} \left|Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)\right|\right)_{M\geq 1}$$

has a limit as $M \to \infty$. But this sequence is non-increasing, so by using the Monotone Convergence Theorem,

$$\lim_{M\to\infty} \left\langle \sup_{T,t\geq M-1} |Z^T_{0,0}(\omega) - Z^t_{0,0}(\omega)|\right\rangle = 0.$$

This completes the proof of Proposition 3.8.

3.4. Positivity of the Limiting Partition Functions

From the definition of the partition functions $Z^{y,t}_{x,s}(\omega)$, it is clear that the limiting partition function $Z^{\infty}_{x,s}(\omega)$ is nonnegative Q-almost surely. In this section, we show that in fact it is positive Q-almost surely; namely, we prove the following proposition.

Proposition 3.11

Let

$$\Omega^+ := \{\omega \in \Omega \ : \ Z^{\infty}_{x,s}(\omega) > 0 \ \forall x \in \mathbb{Z}^d, \forall s \in \mathbb{R}\}.$$

Then, $Q(\Omega^+) = 1$*, provided that* $\beta < \frac{1}{\sqrt{1+\alpha}}$.

The proof of this proposition relies on further properties of the limiting partition functions $Z^{\infty}_{x,S}$ and $Z^{\infty}_{y,S+1}$, which we establish in the next subsection.

3.4.1. Properties of the Limiting Partition Functions

Lemma 3.12

For every $x \in \mathbb{Z}^d$*,* $s \in \mathbb{R}$ *and* $\omega \in \Omega^{\text{lim}}$

$$Z^{\infty}_{x,s}(\omega) = \sum_{y \in \mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^{\infty}_{y,r}(\omega). \tag{3.21}$$

Proof. Let $0 < s < t$. Then, by Lemma 3.9, for $s < r < t$,

$$Z^{t}_{x,s}(\omega) = \sum_{y \in \mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^{t}_{y,r}(\omega)$$

For every $\omega \in \Omega^{\text{lim}}$, the limit as $t \to \infty$ exists and

$$Z^{\infty}_{x,s}(\omega) = \lim_{t \to \infty} \sum_{y \in \mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^{t}_{y,r}(\omega). \tag{3.22}$$

Therefore, to complete the proof of the lemma is enough to show that the limit on the righthand side of (3.22) converges to $\sum_{y \in \mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^{\infty}_{y,s}(\omega)$ in L^1.

$$\begin{aligned}\Big\langle \Big| \sum_{y\in\mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^t_{y,r}(\omega) - \sum_{y\in\mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) Z^\infty_{y,s}(\omega) \Big| \Big\rangle &\le \Big\langle \sum_{y\in\mathbb{Z}^d} Z^{y,r}_{x,s}(\omega) \left| Z^t_{y,r}(\omega) - Z^\infty_{y,r}(\omega) \right| \Big\rangle \\ &= \sum_{y\in\mathbb{Z}^d} \left\langle Z^{y,r}_{x,s}(\omega) \left| Z^t_{y,r}(\omega) - Z^\infty_{y,r}(\omega) \right| \right\rangle \\ &= \sum_{y\in\mathbb{Z}^d} \left\langle Z^{y,r}_{x,s}(\omega) \right\rangle \left\langle \left| Z^t_{y,r}(\omega) - Z^\infty_{y,r}(\omega) \right| \right\rangle \\ &= \sum_{y\in\mathbb{Z}^d} \left\langle Z^{y,r}_{x,s}(\omega) \right\rangle \left\langle \left| Z^t_{0,r}(\omega) - Z^\infty_{0,r}(\omega) \right| \right\rangle \\ &= \left\langle Z^{r}_{x,s}(\omega) \right\rangle \left\langle \left| Z^t_{0,r}(\omega) - Z^\infty_{0,r}(\omega) \right| \right\rangle \\ &= \left\langle \left| Z^t_{0,r}(\omega) - Z^\infty_{0,r}(\omega) \right| \right\rangle,\end{aligned}$$

which converges to 0 as $t \to \infty$.

Notice that in the last line we used the fact that $\left\langle Z^r_{x,s}(\omega) \right\rangle = 1$. □

Lemma 3.13

Let $x, y \in \mathbb{Z}^d$ and $s \in \mathbb{R}$. Then, for every $\omega \in \Omega^{\lim}$, there is $c > 0$ such that

$$Z^\infty_{x,s}(\omega) \ge c Z^\infty_{y,s+1}(\omega).$$

Proof. Let $s \in \mathbb{R}$ and $x, y \in \mathbb{Z}^d$. For any $N \in \mathbb{N}$ and $\tilde{y} \in \mathbb{Z}^d$, let $\Gamma^{\tilde{y},N}_x$ denote the collection of all paths in $\mathbb{Z}^d$ of length N starting at x and ending at $\tilde{y}$ and let $\gamma' := (x, \gamma'_1, \dots, \gamma'_{N-1}, \tilde{y}) \in \Gamma^{\tilde{y},N}_x$. Now, we consider a sequence of jump times $\mathrm{s} = (s_n)_{n\in\mathbb{N}_0}$ such that

$$s+1 \notin \{s_n : \; n \in \mathbb{N}_0\} \quad \text{and} \quad n_{s,s+1} = N.$$

Fix $t > s+1$, then to simplify notation, we set $\hat{N} := n_{s+1,t}$. For any $\tilde{y} \in \mathbb{Z}^d$, let $\Gamma^{\hat{N}}_{\tilde{y}}$ denote the collection of all paths in $\mathbb{Z}^d$ of length $\hat{N}$ starting at $\tilde{y}$ and let $\gamma'' := (\tilde{y}, \gamma''_1, \dots, \gamma''_{\hat{N}}) \in \Gamma^{\hat{N}}_{\tilde{y}}$. Consider the concatenation of γ' and γ'', which is given by $\gamma = (x, \gamma'_1, \dots, \gamma'_{N-1}, \tilde{y}, \gamma''_1, \dots, \gamma''_{\hat{N}})$. By a slight abuse of notation, let the sample path for the continuous-time random walk described by γ and s be denoted by γ as well. Note that

$$\mathcal{A}^t_s(\gamma, \omega) = \mathcal{A}^{s+1}_s(\gamma', \omega) + \mathcal{A}^t_{s+1}(\gamma'', \omega),$$

where the action $\mathcal{A}_s^t(\gamma,\omega)$ was defined in (3.1). Therefore, for $\omega \in \Omega^{\text{lim}}$, we have

$$
\begin{aligned}
Z_{x,s}^t(\omega) \geq\ & e^{-\frac{\beta^2}{2}(t-s)}\mathbf{E}_{x,s}\left[e^{\beta\mathcal{A}_s^t(\gamma,\omega)}\mathbb{1}_{n_{s,s+1}=N}\right] \\
=& e^{-\frac{\beta^2}{2}}e^{-\frac{\beta^2}{2}(t-s-1)}\mathbf{E}_{x,s}\left[\sum_{\tilde{y}\in\mathbb{Z}^d} e^{\beta\mathcal{A}_s^{s+1}(\gamma',\omega)}e^{\beta\mathcal{A}_{s+1}^t(\gamma'',\omega)}\mathbb{1}_{n_{s,s+1}=N}\right] \\
=& \sum_{\tilde{y}\in\mathbb{Z}^d} e^{-\frac{\beta^2}{2}}\mathbf{E}\left[\frac{1}{(2d)^N}\sum_{\gamma'\in\Gamma_x^{\tilde{y},N}} e^{\beta\mathcal{A}_s^{s+1}(\gamma',\omega)}\mathbb{1}_{n_{s,s+1}=N}\right]e^{-\frac{1}{2}\beta^2(t-s-1)}\mathbf{E}\left[\frac{1}{(2d)^{\hat{N}}}\sum_{\gamma'\in\Gamma_{\tilde{y}}^{\hat{N}}} e^{\beta\mathcal{A}_{s+1}^T(\gamma'',\omega)}\right] \\
\geq& e^{-\frac{\beta^2}{2}}\mathbf{E}\left[\frac{1}{(2d)^N}\sum_{\gamma'\in\Gamma_x^{y,N}} e^{\beta\mathcal{A}_s^{s+1}(\gamma',\omega)}\mathbb{1}_{n_{s,s+1}=N}\right]e^{-\frac{1}{2}\beta^2(t-s-1)}\mathbf{E}\left[\frac{1}{(2d)^{\hat{N}}}\sum_{\gamma'\in\Gamma_y^{\hat{N}}} e^{\beta\mathcal{A}_{s+1}^t(\gamma'',\omega)}\right] \\
=& e^{-\frac{\beta^2}{2}}\mathbf{E}\left[\frac{1}{(2d)^N}\sum_{\gamma'\in\Gamma_x^{y,N}} e^{\beta\mathcal{A}_s^{s+1}(\gamma',\omega)}\mathbb{1}_{n_{s,s+1}=N}\right]Z_{y,s+1}^t(\omega),
\end{aligned}
$$

where $\mathbf{E}$ should be understood as averaging with respect to the Poisson point process on the real line.

Notice that the term $e^{-\frac{\beta^2}{2}}\mathbf{E}\left[\frac{1}{(2d)^N}\sum_{\gamma'\in\Gamma_x^{y,N}} e^{\beta\mathcal{A}_s^{s+1}(\gamma',\omega)}\mathbb{1}_{n_{s,s+1}=N}\right]$ is positive because $\mathbf{P}(n_{s,s1} = N) > 0$. Since this term does not depend on t, the result follows by taking the limit as $t \to \infty$. □

3.4.2. Proof of Proposition 3.11

In order to prove this proposition, it is enough to restrict our attention to rational times; i.e., it is enough to show that $Q(\Omega_{\mathbb{Q}}^+) = 1$ where

$$\Omega_{\mathbb{Q}}^+ := \{\omega \in \Omega \ : \ Z_{x,r}^\infty(\omega) > 0 \ \forall x \in \mathbb{Z}^d, \forall r \in \mathbb{Q}\}.$$

Indeed, given any $\omega \in \Omega_{\mathbb{Q}}^+ \cap \Omega^{\text{lim}}$, $x \in \mathbb{Z}^d$, and $s \in \mathbb{R}$, we choose $r \in \mathbb{Q}$ such that $r > s$, and use Lemma 3.12:

$$Z_{x,s}^\infty(\omega) = \sum_{y\in\mathbb{Z}^d} Z_{x,s}^{y,r}(\omega)Z_{y,r}^\infty(\omega) > 0.$$

This shows that $\omega \in \Omega^+$. But since $Q(\Omega_{\mathbb{Q}}^+ \cap \Omega^{\text{lim}}) = 1$, it follows that $Q(\Omega^+) = 1$.

In order to prove that $Q(\Omega_{\mathbb{Q}}^+) = 1$, we show for every $x \in \mathbb{Z}^d$ and $s \in \mathbb{R}$ there exists a subset $\Omega_{x,s}^+$ of $Q-$full measure such that for every $\omega \in \Omega_{x,s}^+$ the limiting partition function $Z_{x,s}^\infty(\omega)$ is positive. For

$\omega \in \Omega$, we define $\theta(\omega)$ as the element of Ω for which

$$(\theta(\omega))^x = \omega^{x-e_1}, \quad x \in \mathbb{Z}^d,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$. In other words, θ is the spatial shift in the direction of e_1. Notice that θ preserves the measure Q on Ω. Let $x \in \mathbb{Z}^d$ and $S \in \mathbb{R}$, define the random variable

$$I(x) = \inf\{k \in \mathbb{Z} : \; Z^{\infty}_{x,s+k}(\omega) = 0\}.$$

We start by showing that $I(x)$ is infinite with Q-probability 1. To obtain a contradiction, let us assume that $Q(I(x) \in \mathbb{Z}) > 0$. Then, there is $k \in \mathbb{Z}$ such that $Q(I(x) = k) > 0$. Since Q is invariant under shifting the Wiener processes with respect to time and then subtracting the value of the shifted processes at time 0, and since $\mathbf{P}$ is invariant under shifting the Poisson point process, the discrete-time stochastic processes $(Z^{\infty}_{x,s+i}(\omega))_{i\in\mathbb{Z}}$ and $(Z^{\infty}_{x,s+j+i}(\omega))_{i\in\mathbb{Z}}$ have the same distribution under Q for any $j \in \mathbb{Z}$. As

$$\{I(x) = l\} = \left\{\inf\{i \in \mathbb{Z} : \; Z^{\infty}_{x,s+l-j+i}(\omega) = 0\} = j\right\}, \quad j \in \mathbb{Z},$$

it follows that $Q(I(x) = j) = Q(I(x) = l) > 0$ for all $j \in \mathbb{Z}$. This leads to a contradiction because the events $(\{I(x) = j\})_{j\in\mathbb{Z}}$ are disjoint. Next, we show that $Q(I(x) = -\infty) = 0$. By virtue of Lemma 3.13,

$$\begin{aligned} Q(I(x) = -\infty) =& Q\left(\bigcap_{k\in\mathbb{Z}} \bigcup_{j<k} \{Z^{\infty}_{x,s+j}(\omega) = 0\}\right) \\ =& Q\left(\bigcap_{k\in\mathbb{Z}} \bigcap_{y\in\mathbb{Z}^d} \{Z^{\infty}_{y,s+k}(\omega) = 0\}\right) = Q\left(\bigcap_{y\in\mathbb{Z}^d} \{I(y) = -\infty\}\right). \end{aligned}$$

Since $\bigcap_{y\in\mathbb{Z}^d}\{I(y) = -\infty\}$ is invariant under the shift θ introduced earlier and by Lemma 7.5, θ is mixing; so in particular, it is ergodic, we have

$$Q\left(\bigcap_{y\in\mathbb{Z}^d} \{I(y) = -\infty\}\right) \in \{0, 1\}.$$

If the set $\{I(x) = -\infty\}$ had Q-measure 1, we would have in particular

$$Z^{\infty}_{x,S}(\omega) = 0, \quad Q - a.s.$$

This cannot happen because

$$\langle Z^{\infty}_{x,S}(\omega)\rangle = \lim_{T\to\infty} \langle Z^{T}_{x,S}(\omega)\rangle = 1.$$

Hence, $Q(I(x) = -\infty) = 0$. Together with $Q(I(x) \in \mathbb{Z}) = 0$, this yields

$$Q(I(x) = \infty) = 1,$$

so in particular $Q(Z^{\infty}_{x,S}(\omega) > 0) = 1$. This completes the proof of Proposition 3.11 and, with it, the proof Part (2) of Theorem 3.1.

chapter 4

Factorization Formula for the Partition Function of Directed Polymers

Contents

The following factorization formula for the point-to-point partition function $Z_{x,s}^{y;t}$ is the main result of this chapter.

Theorem 4.1

For β sufficiently small, the following holds: For any $\sigma \in (0,1)$ there exists $\theta = \theta(\sigma) > 0$ such that for all $x, y \in \mathbb{Z}^d$ and $s < t$ with $\|x-y\| < (t-s)^\sigma$, the partition function $Z_{x,s}^{y,t}$ has the representation

$$Z_{x,s}^{y,t} = p_{t-s}^{y-x}\left(Z_{x,s}^{\infty} Z_{-\infty}^{y,t} + \delta_{x,s}^{y,t}\right), \tag{4.1}$$

where the error term $\delta_{x,s}^{y,t}$ defined by the formula above satisfies

$$\lim_{(t-s)\to\infty} (t-s)^\theta \sup_{x,y\in\mathbb{Z}^d:\|x-y\|<(t-s)^\sigma} \langle|\delta_{x,s}^{y,t}|\rangle = 0. \tag{4.2}$$

Notice that the formula looks similar to the ones obtained by Sinai in [Sin95, Theorem 2], Kifer in [Kif97, Theorem 6.1], and Vargas in [Var06, Theorems 2.3 and 2.9]. However, we show that the error term is small not only within the diffusive regime $\|x-y\| < O(t-s)^{\frac{1}{2}}$, but also for $\|x-y\| < (t-s)^\sigma$ with σ arbitrarily close to 1. This extension beyond the diffusive regime is nontrivial because the error term in (4.1) is multiplied by the random-walk transition probability p_{t-s}^{y-x}, which is itself extremely small for $\|x-y\| \geq (t-s)^{\frac{1}{2}}$. We believe that the smallness condition for β in Theorems 3.1 and 4.1 is technical and the result of Theorem 4.1 should hold for all $\beta < \frac{1}{\sqrt{1+\alpha}}$.

The main idea behind the factorization formula, which goes back at least to [Sin95], is that there is strong averaging for times neither too close to 0 nor too close to t.

For fixed $i_1, \ldots, i_r$ and $z_1, \ldots, z_r$, the random walk is pinned to the points $z_1, \ldots, z_r$ at the corresponding times $s_{i_1}, \ldots, s_{i_r}$. The proof of Proposition 3.3 suggests that the contribution to $Z_{x,0}^{y,t}$ from r on the order of n is negligible. If r is not on the order of n, at least one of the gaps $i_j - i_{j-1}$ must be in some sense large (as defined below). In Subsection 4.2.2, we show that the contribution to $Z_{x,0}^{y,t}$ coming from two or more large gaps is negligible as well. Thus, the main contribution comes from having exactly one large gap $i_j - i_{j-1}$, which is then on the order of n. In order for $q_{i_j-i_{j-1}}^{z_j-z_{j-1}}$ to be positive, z_{j-1} must be close to x and z_j must be close to y. The transition probability $q_{i_j-i_{j-1}}^{z_j-z_{j-1}}$ is then close to q_n^{y-x}.

Notice that to prove Theorem 4.1, it is enough to show that for β sufficiently small there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{y\in\mathbb{Z}^d:\|y\|<t^\sigma} \langle|\delta_{0,0}^{y,t}|\rangle = 0.$$

This is because for a fixed realization ω of the disorder, $\delta_{x,s}^{y,t}(\omega)$ can be written as $\delta_{0,0}^{y-x,t-s}(\hat{\omega})$, where $\hat{\omega}$ is obtained from ω by making a shift in space and time. The distribution of the disorder is invariant under such shifts.

Let $I_{r,n} := \{(i_1, \ldots, i_r) : 0 \leq i_1 < \cdots < i_r \leq n\}$. For $\mathbf{i} \in I_{r,n}$ and $\mathbf{z} \in (\mathbb{Z}^d)^r$, define

$$q_n^y(\mathbf{i}, \mathbf{z}) := q_{i_1}^{z_1} q_{i_2-i_1}^{z_2-z_1} \ldots q_{n-i_r}^{y-z_r}.$$

Then, from the expansion for the partition function $Z_{0,0}^{y,t}$ from Subsection 3.1.1, we have:

$$Z_{0,0}^{y,t} = p_t^y + \mathbf{E}\Bigg[\sum_{r=1}^{n_t+1} \sum_{\mathbf{i} \in I_{r,n},\mathbf{z}} q_{n_t}^y(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1})\Bigg].$$

The first step is to split the double sum into terms according to the size of the largest gap between indices, as discussed in the next subsection.

1. Large and huge gaps To quantify what it means to have many gaps between indices, we fix constants κ_1, κ_2 such that $\frac{1}{2}(3\sigma - 1) < \kappa_1 < \kappa_2 < 1$. Let $N_{\kappa_2} \in \mathbb{N}$ be so large that $2(n - n^{\kappa_2}) > n$ for all $n \geq N_{\kappa_2}$. Then we define

$$k(n) := \begin{cases} (n - N_{\kappa_2})^{\kappa_1} - 1, & (n - N_{\kappa_2})^{\kappa_1} - 1 \geq 1, \\ 0, & (n - N_{\kappa_2})^{\kappa_1} - 1 < 1. \end{cases}$$

Note that the integer $k(n)$ grows with n like n^{κ_1}. We will say that any collection of indices $0 \leq i_1 < \ldots < i_r \leq n$ with $r > k(n)$ gaps has *many gaps*.

To classify the size of a gap between indices, fix another constant ξ such that $0 < \xi < \min\{1 - \sigma, \kappa_2 - \kappa_1\}$. Note that $\xi + \kappa_1 < 1$. Let $n \in \mathbb{N}$ be such that $k(n) \geq 1$, let r be such that $1 \leq r \leq k(n)$, and consider a sequence of indices $0 = i_0 \leq i_1 < \ldots < i_r \leq i_{r+1} = n$. We say that the gap between two consecutive indices i_{j-1} and i_j is

- *large* if $i_j - i_{j-1} \geq n^{\xi}$;
- *huge* if $i_j - i_{j-1} \geq n - rn^{\xi}$.

Observe that the size of the largest gap is necessarily greater than $n/(r+1) \geq n^{1-\kappa_1} \geq n^{\xi}$, so there is at least one large gap. A huge gap is necessarily large. If there is only one large gap, then all other gaps are of size less than n^{ξ}, so this large gap is necessarily huge. Thus, if there is no huge gap, there are at least two large gaps. Since n must be greater than N_{κ_2} in order for $k(n) \geq 1$ to hold, we have that $2(n - rn^{\xi}) > 2(n - n^{\kappa_1+\xi}) > 2(n - n^{\kappa_2}) > n$, so there can be at most one huge gap. Note, however, that a huge gap is not necessarily the only large gap.

Let us introduce more notation. Fix any $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. For any $m \in \mathbb{N}$ such that $1 \leq m \leq r+1$, define the following set of r-tuples of indices:

$$I_1(n,r,m) := \left\{ (i_1, \ldots, i_r) \ : \ \begin{matrix} 0 \leq i_1 < \ldots < i_r \leq n \\ \text{the gap between } i_{m-1} \text{ and } i_m \text{ is huge} \end{matrix} \right\}.$$

Also, for any $m \in \mathbb{N}$ such that $1 \leq m \leq r$, define

$$I_2(n,r,m) := \left\{ (i_1, \ldots, i_r) \ : \ \begin{matrix} 0 \leq i_1 < \ldots < i_r \leq n \\ \text{there is no huge gap} \\ \text{the first large gap occurs between } i_{m-1} \text{ and } i_m \end{matrix} \right\}.$$

Then we decompose the expansion of $Z_{0,0}^{y,t}$ as follows:

$$Z_{0,0}^{y,t} = p_t^y + \sum_{j=1}^{3} \mathbf{E} B_j^{y,t},$$

where,

$$\begin{aligned} B_1^{y,t} &:= \sum_{k(n_t) < r \leq n_t + 1} \sum_{\mathbf{i} \in I_{r,n},\mathbf{z}} q_{n_t}^y(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}), \\ B_2^{y,t} &:= \sum_{1 \leq r \leq k(n_t)} \sum_{\mathbf{i} \in I_2(n,r,m),\mathbf{z}} q_{n_t}^y(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}), \\ B_3^{y,t} &:= \sum_{1 \leq r \leq k(n_t)} \sum_{m=1}^{r+1} \sum_{\mathbf{i} \in I_1(n,r,m),\mathbf{z}} q_{n_t}^y(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}). \end{aligned}$$

With this decomposition in hand, Theorem 4.1 follows immediately from the following lemma.

Lemma 4.2 (Central Lemma)

For $\beta > 0$ sufficiently small, the following are true:

(1) *For every $\theta > 0$,*

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y}\left\langle \left|\mathbf{E}B_1^{y,t}\right|\right\rangle = 0. \tag{4.3}$$

(2) *There is $\theta > 0$ such that*

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y}\langle |\mathbf{E}B_2^{y,t}|\rangle = 0. \tag{4.4}$$

(3) *There is $\theta > 0$ such that*

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \left\langle \left| 1 + \frac{1}{p_t^y}\mathbf{E}B_3^{y,t} - Z_{0,0}^\infty Z_{-\infty}^{y,t}\right|\right\rangle = 0. \tag{4.5}$$

4.1. Reduction to the Discrete Time

In this section, we formulate a key lemma (Lemma 4.3) that will allow us in many cases to deduce convergence statements in the continuous-time setting from convergence results in discrete time.

Let $(R_n)_{n\in\mathbb{N}_0}$ be any family of sets such that $R_n \subset \{1, \ldots, n+1\}$ for every $n \in \mathbb{N}_0$. Let $(I_{n,r})_{n\in\mathbb{N}_0, r\in R_n}$ be any collection of index sets such that

$$I_{n,r} \subset \{\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{N}_0^r \ : \ 0 \le i_1 < \ldots < i_r \le n\}$$

for every $n \in \mathbb{N}_0$ and $r \in R_n$. Consider any collection of nonnegative real numbers $q_{n,r}^y(\mathbf{i}, \mathbf{z}) \ge 0$ indexed by all $n \in \mathbb{N}_0, r \in R_n, y \in \mathbb{Z}^d, \mathbf{z} \in (\mathbb{Z}^d)^r, \mathbf{i} \in I_{n,r}$, which satisfy the following finiteness condition: for every $y \in \mathbb{Z}^d, n \in \mathbb{N}_0, r \in R_n, \mathbf{i} \in I_{n,r}$,

$$\sum_{\mathbf{z}\in(\mathbb{Z}^d)^r} q_{n,r}^y(\mathbf{i}, \mathbf{z}) \le 2. \tag{4.6}$$

Notice that this condition implies

$$\sum_{\mathbf{z}\in(\mathbb{Z}^d)^r} q_{n,r}^y(\mathbf{i}, \mathbf{z})^2 \le 4, \qquad y \in \mathbb{Z}^d,\ n \in \mathbb{N}_0,\ r \in R_n,\ \mathbf{i} \in I_{n,r}. \tag{4.7}$$

For every $y \in \mathbb{Z}^d$ and all $s, t \in \mathbb{R}$ such that $s < t$, we define

$$\mathcal{T}(y; s, t) := \sum_{r \in R_{n_{s,t}}} \sum_{\substack{\mathbf{i} \in I_{n_{s,t},r} \\ \mathbf{z} \in (\mathbb{Z}^d)^r}} q^y_{n_{s,t},r}(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}).$$

where $n_{s,t}$ is the number of jumps that occur within (s, t). The definition of $\mathcal{T}$ is inspired by the partition function expansions in Subsection 3.1.1.

For any $\vartheta > 0$, we define a sequence of functions $(D^\vartheta_n)_{n \in \mathbb{N}_0}$ on the lattice $\mathbb{Z}^d$ by

$$D^\vartheta_n(y) := \sum_{r \in R_n} \vartheta^r \sum_{\substack{\mathbf{i} \in I_{n,r} \\ \mathbf{z} \in (\mathbb{Z}^d)^r}} q^y_{n,r}(\mathbf{i}, \mathbf{z})^2, \qquad y \in \mathbb{Z}^d.$$

We call $(D^\vartheta_n)_{n \in \mathbb{N}_0}$ the **D-*sequence*** associated with $\mathcal{T}$. As a consequence of Condition (4.6), each term $D^\vartheta_n(y)$ is finite. Recall that n_t is our shorthand for $n_{0,t}$.

Lemma 4.3 (Key Lemma)

Fix any $\tilde{\sigma} \in (\frac{3}{4}, 1)$ and $\vartheta > 0$. For any $\beta > 0$ sufficiently small, the following statement holds: If there is $\theta > 0$ such that

$$\lim_{n \to \infty} n^\theta \sup_{\|y\| \le n^{\tilde{\sigma}}} D^\vartheta_n(y) = 0,$$

then for every $\sigma \in (0, \tilde{\sigma})$,

$$\lim_{t \to \infty} t^{\frac{\theta}{2}} \sup_{\|y\| \le t^\sigma} \frac{1}{p^y_t} \mathbf{E}\left[q^y_{n_t} \langle |\mathcal{T}(y; 0, t)| \rangle \right] = 0.$$

Remark 4.4

We do not attempt to find the supremum of $\beta > 0$ for which the conclusions of Lemma 4.3 hold.

Remark 4.5

If D^ϑ_n does not depend on y and if $\lim_{n \to \infty} n^\theta D^\vartheta_n = 0$ for some $\theta, \vartheta > 0$, then we have for β sufficiently small

$$\lim_{t \to \infty} t^\theta \mathbf{E} \left\langle (\mathcal{T}(0, t))^2 \right\rangle = 0.$$

4.1.1. Some Preliminary Estimates

For any $t > 0$, $l \in \mathbb{N}_0$, and $1 \leq r \leq l+1$, we introduce the following notation:

$$A(t,l,r) := \mathbf{E}\left[\prod_{j=1}^{r-2}\left(e^{\beta^2 t_j} - 1\right) e^{\beta^2(t_{r-1}+t_r)} \middle| n_t = l\right], \tag{4.8}$$

where t_0 should be interpreted as 0, and t_{l+1} as $t - s_l$.

Lemma 4.6

If $\beta > 0$ satisfies $\beta^2 < 1$, then for all t sufficiently large,

$$A(t,l,r) \lesssim (l+1)^2 e^{\beta^2 t}\beta^{2r}\frac{t^r}{(l+1)\dots(l+r)}, \qquad l \in \mathbb{N}_0,\ 1 \leq r \leq l+1. \tag{4.9}$$

Proof. For all $t > 0$, $l \in \mathbb{N}_0$, and $1 \leq r \leq l+1$, define an r-dimensional simplex

$$\Delta(t,r) := \left\{(t_1,\dots,t_r) \in \mathbb{R}_+^r : t_1 + \dots + t_r < t\right\}, \tag{4.10}$$

and, for every $(t_1,\dots,t_r) \in \mathbb{R}_+^r$, a function

$$\rho_{l,r}^t(t_1,\dots,t_r) := \frac{1}{t^r}\prod_{j=0}^{r-1}(l-j)\left(1 - \frac{1}{t}\sum_{j=1}^{r} t_j\right)^{l-r}. \tag{4.11}$$

Then we can write $A(t,l,r)$ as an integral over the simplex $\Delta(t,r)$:

$$A(t,l,r) = \int_{\Delta(t,r)}\prod_{j=1}^{r-2}\left(e^{\beta^2 t_j} - 1\right) e^{\beta^2(t_{r-1}+t_r)}\rho_{l,r}^t(t_1,\dots,t_r)\,dt_1\cdots dt_r. \tag{4.12}$$

Note that if $l = 0$, then $r = 1$ and so $A(t,0,1) = e^{\beta^2 t}$ which obviously satisfies the desired estimate, so we assume that $l \neq 0$. Then for all $t > 0$, $l \in \mathbb{N}$, $1 \leq r \leq l+1$, we define the following integral:

$$\mathcal{I}(t,l,r) := \int_{\Delta(t,l)}\prod_{j=1}^{r-2}\left(e^{\beta^2 t_j} - 1\right) e^{\beta^2(t_{r-1}+t_r)}\,dt_1\cdots dt_l,$$

where $t_0 := 0$ and $t_{l+1} := t - (t_1 + \dots + t_l)$. Then we have the following identity:

$$A(t,l,r) = \frac{l!}{t^l}\mathcal{I}(t,l,r). \tag{4.13}$$

We claim that if $\beta^2 < 1$ and $t \geq \max\{\beta^{-2}, 2\}$, then

$$\mathcal{I}(t,l,r) \leq e^{\beta^2 t}\beta^{2(r-3)}\frac{t^{l+r-2}}{(l+r-2)!}, \qquad l \in \mathbb{N},\ 1 \leq r \leq l+1. \tag{4.14}$$

This estimate implies the estimate (4.9).

It remains to prove the estimate (4.14). Consider three separate cases: $r = l+1$, $r = l$, and $r < l$. Let us first establish it in the case $r = l$. For this, we use the easily verifiable identity

$$\mathcal{I}(t,l+1,l+1) = \int_0^t \left(e^{\beta^2 s} - 1\right)\mathcal{I}(t-s,l,l)\,ds, \qquad l \geq 2, \tag{4.15}$$

and argue by induction on l with bases cases $l = 1, 2$:

$$\mathcal{I}(t,1,1) = e^{\beta^2 t}\beta^{-2} - \beta^{-2} \leq e^{\beta^2 t}\beta^{-4},$$
$$\mathcal{I}(t,2,2) = \beta^{-2}(e^{\beta^2 t}t - \beta^{-2}e^{\beta^2 t} + \beta^{-2}) \leq \beta^{-2}e^{\beta^2 t}t \leq \beta^{-2}e^{\beta^2 t}\frac{t^2}{2}.$$

By the induction hypothesis, the righthand side of (4.15) is bounded by

$$e^{\beta^2 t}\beta^{2(l-3)}\frac{1}{(2(l-1))!}\int_0^t (t-s)^{2(l-1)}\left(1 - e^{-\beta^2 s}\right)\,ds$$
$$\leq e^{\beta^2 t}\beta^{2(l-2)}\frac{1}{(2(l-1))!}\int_0^t (t-s)^{2(l-1)} s\,ds = e^{\beta^2 t}\beta^{2(l-2)}\frac{t^{2l}}{(2l)!},$$

which implies (4.14) for $r = l$. If $r < l$, then we write

$$\mathcal{I}(t,l,r) = \int_{\Delta(t,l-r)} \mathcal{I}\Big(t - \sum_{j=r+1}^{l} t_j, r, r\Big)\,dt_{r+1}\ldots dt_l.$$

The righthand side is less than or equal to

$$\frac{\beta^{2(r-3)}}{(2(r-1))!}\int_{\Delta(t,l-r)}\Big(t - \sum_{j=r+1}^{l} t_j\Big)^{2(r-1)} e^{\beta^2(t-\sum_{j=r+1}^{l} t_j)}\,dt_{r+1}\ldots dt_l. \tag{4.16}$$

If $r = l-1$, the integral above, without the factor in front of it, reads

$$\int_0^t (t-s)^{2(l-2)}\,e^{\beta^2(t-s)}\,ds \leq e^{\beta^2 t}\frac{t^{2l-3}}{2l-3}.$$

Here, we used that for $t > 0$ and $n, k \in \mathbb{N}_0$,

$$\int_0^t (t-s)^k \, s^n \, ds = \frac{n!k!}{(n+k+1)!} t^{n+k+1}. \tag{4.17}$$

If $r < l-1$, we use the change of variables $s = \sum_{j=r+1}^{l} t_j$, $s_1 = t_{r+2}, s_2 = t_{r+3}, \ldots, s_{l-r-1} = t_l$ to convert the integral in (4.16) into

$$\begin{aligned} &\int_0^t \int_{\Delta(s,l-r-1)} (t-s)^{2(r-1)} \, e^{\beta^2(t-s)} \, ds_1 \ldots \, ds_{l-r-1} \, ds \\ =&\frac{1}{(l-r-1)!} \int_0^t (t-s)^{2(r-1)} \, e^{\beta^2(t-s)} \, s^{l-r-1} \, ds. \end{aligned} \tag{4.18}$$

Using again the identity in (4.17), we see that the expression in the second line of (4.18) is less than

$$e^{\beta^2 t} \frac{t^{l+r-2}}{\prod_{j=1}^{l-r}(2r+j-2)}.$$

Similarly, one can show (4.14) in the case $r = l+1$. □

Recall that for $\nu \in (\frac{1}{2}, 1)$, and for all $t > 0$,

$$J(t) := \Big\{ n \in \mathbb{N} \ : \ \nu t < n < (2-\nu)t \Big\}. \tag{4.19}$$

Lemma 4.7

Let $\nu_1 \in (\nu^{-1} - 1, 1)$, and let

$$\psi := \frac{\beta^2}{(1-\nu_1)((2-\nu)(1-\nu_1) - \beta^2)}. \tag{4.20}$$

Then, if β is so small that $\psi > 0$, we have

$$A(t,l,r) \lesssim \psi^r, \quad \nu t < l < (2-\nu)t, \ 1 \le r < \nu_1 l.$$

Proof. We use the setup from the proof of Lemma 4.6. For $\nu t < l < (2-\nu)t$ and $1 \le r < \nu_1 l$, the function $\rho_{l,r}^t$ from (4.11) can be bounded as follows

$$\rho_{l,r}^t(t_1, \ldots, t_r) \le \left(\frac{l}{t}\right)^r \prod_{j=1}^{r} e^{-\frac{l-r}{t} t_j} \le (2-\nu)^r \prod_{j=1}^{r} e^{-\nu(1-\nu_1)t_j}.$$

As a result, the integrand in (4.12) can be bounded by the following expression:

$$(2-\nu)^r \prod_{j=1}^{r-2} \left(e^{(\beta^2-\nu(1-\nu_1))t_j} - e^{-\nu(1-\nu_1)t_j} \right) e^{(\beta^2-\nu(1-\nu_1))t_{r-1}} e^{(\beta^2-\nu(1-\nu_1))t_r}$$

Then if we integrate in (4.12) over the larger domain $\mathbb{R}^r_+$, we can rewrite the integral as a product of integrals over $\mathbb{R}^r_+$. Therefore, we can bound $A(t,l,r)$ by the following expression:

$$(2-\nu)^r \left(\int_0^\infty \left(e^{(\beta^2-\nu(1-\nu_1))s} - e^{-\nu(1-\nu_1)s} \right) ds \right)^{r-2} \left(\int_0^\infty e^{(\beta^2-\nu(1-\nu_1))s}\, ds \right)^2,$$

which is less than a constant times $(2-\nu)^r \left(\frac{\beta^2}{(1-\nu_1)(2-\nu)((2-\nu)(1-\nu_1)-\beta^2)} \right)^r = \psi^r$. □

Fix $\delta \in (0,1)$, $\kappa \in \left(e^{1-\frac{1}{\delta}}, \delta \right)$, $\hat{\kappa} \in \left(e^{1-\frac{1}{\nu}}, \nu \right)$, and $\nu_0 \in (0, \frac{\delta}{2})$. Let $\beta > 0$ be so small that

$$e^{1-\beta^2} > \max\left\{ \left(\frac{e}{\kappa}\right)^\delta, \left(\frac{e}{\hat{\kappa}}\right)^\nu \right\}, \qquad \beta^2 < \nu_0 \alpha^{-1}, \qquad \beta^2 < (2-\nu)\alpha^{-1}.$$

Lemma 4.8

For t sufficiently large, the following estimates hold:

$$\sum_{0 \le l \le \nu_0 t} \frac{t^l}{l!} \sum_{r=1}^{l+1} r\alpha^r A(t,l,r) \lesssim e^{\beta^2 t} \frac{(t+1)^5}{\sqrt{2\pi\kappa t}} \left(\frac{e}{\kappa}\right)^{\delta t}; \tag{4.21}$$

$$\sum_{\nu_0 t < l < \nu t} \frac{t^l}{l!} \sum_{r=1}^{l+1} r\alpha^r A(t,l,r) \lesssim e^{\beta^2 t} \frac{(t+1)^5}{\sqrt{2\pi\hat{\kappa} t}} \left(\frac{e}{\kappa}\right)^{\nu t}; \tag{4.22}$$

$$\sum_{l > (2-\nu)t} \frac{t^l}{l!} \sum_{r=1}^{l+1} r\alpha^r A(t,l,r) \lesssim e^{\beta^2 t} t^4 \sum_{l > (2-\nu)t - 2} \frac{t^l}{l!}. \tag{4.23}$$

Remark 4.9

Notice that all the upper bounds in (4.21) - (4.23) converge to 0 as $t \to \infty$.

Proof. Fix $\delta \in (0,1)$, $\kappa \in \left(e^{1-\frac{1}{\delta}}, \delta \right)$, $\hat{\kappa} \in \left(e^{1-\frac{1}{\nu}}, \nu \right)$, and $\nu_0 \in (0, \frac{\delta}{2})$. Let $\beta > 0$ be so small that

$$e^{1-\beta^2} > \max\left\{ \left(\frac{e}{\kappa}\right)^\delta, \left(\frac{e}{\hat{\kappa}}\right)^\nu \right\}, \qquad \beta^2 < \nu_0 \alpha^{-1}, \qquad \beta^2 < (2-\nu)\alpha^{-1}.$$

First, thanks to Lemma 4.6,

$$\frac{t^l}{l!}\sum_{r=1}^{l+1} r\alpha^r A(t,l,r) \lesssim e^{\beta^2 t}(l+1)^2 \sum_{r=1}^{l+1} r(\alpha\beta^2)^r \frac{t^l}{l!}\frac{t^r}{(l+1)\dots(l+r)}. \tag{4.24}$$

To prove (4.21), let t be so large that $2\nu_0 t+1 < \lfloor \delta t \rfloor$. For $0 \le l \le \nu_0 t$ and $1 \le r \le l+1$, we have that $l+r \le 2l+1 \le 2\nu_0 t+1 < \delta t < t$. Thus,

$$\frac{t^{l+r}}{(l+r)!} \le \frac{t^{2l+1}}{(2l+1)!}.$$

Then, using the fact that $\alpha\beta^2 < \nu_0 < 1$, we bound the righthand side of (4.24) by

$$e^{\beta^2 t}(l+1)^2 \frac{t^{2l+1}}{(2l+1)!}\sum_{r=1}^{l+1} r \le e^{\beta^2 t}(l+1)^4 \frac{t^{2l+1}}{(2l+1)!}.$$

Furthermore, using Stirling's formula, we find

$$\sum_{0\le l\le \nu_0 t} (l+1)^4 \frac{t^{2l+1}}{(2l+1)!} \le \sum_{0\le l\le \delta t}(l+1)^4\frac{t^l}{l!} \le (t+1)^5 \frac{t^{\lfloor \delta t\rfloor}}{\lfloor \delta t\rfloor !} \le \frac{(t+1)^5}{\sqrt{2\pi\kappa t}}\left(\frac{e}{\kappa}\right)^{\delta t}.$$

Therefore, the lefthand side of (4.21) is less than a constant times $e^{\beta^2 t}$ times the righthand side of the above estimate.

To prove (4.22), note that for $l > \nu_0 t$ and $1 \le r \le l+1$,

$$\frac{t^r}{(l+1)\dots(l+r)} \le \left(\frac{t}{l}\right)^r \le \nu_0^{-r}. \tag{4.25}$$

Thus, using the fact that $\alpha\beta^2 < \nu_0$, the righthand side of (4.24) is bounded by

$$e^{\beta^2 t}(l+1)^2\frac{t^l}{l!}\sum_{r=1}^{l+1} r(\alpha\beta^2\nu_0^{-1})^r \le e^{\beta^2 t}(l+1)^4\frac{t^l}{l!}.$$

As a result, the lefthand side of (4.22) is bounded up to a constant by

$$e^{\beta^2 t}\sum_{\nu_0 t<l<\nu t}(l+1)^4\frac{t^l}{l!} \le e^{\beta^2 t} t^5 \frac{t^{\lfloor \nu t\rfloor}}{\lfloor \nu t\rfloor !} \lesssim e^{\beta^2 t} t^5 \frac{t^{\lfloor \nu t\rfloor}}{\sqrt{2\pi(\nu t)}(\nu t)^{\lfloor \nu t\rfloor}} e^{\nu t}$$
$$\lesssim e^{\beta^2 t}\frac{(t+1)^5}{2\pi\hat{\kappa} t}\left(\frac{e}{\kappa}\right)^{\nu t}.$$

Finally, to prove (4.23), note that for $l > (2-\nu)t$ and $1 \le r \le l+1$, the inequality (7.3) holds with

ν_0 replaced by $(2-\nu)$. As a result, thanks to (4.24) and the fact that $\alpha\beta^2 < (2-\nu)$, the lefthand side of (4.23) is bounded by

$$e^{\beta^2 t} \sum_{l>(2-\nu)t} (l+1)^2 \frac{t^l}{l!} \sum_{r=1}^{l+1} r \left(\tfrac{\alpha\beta^2}{2-\nu}\right)^r \lesssim e^{\beta^2 t} t^4 \sum_{l>(2-\nu)t-2} \frac{t^l}{l!},$$

which is what we wanted to show. □

4.1.2. Proof of the Key Lemma (Lemma 4.3)

Let $\tilde{\sigma} \in (\frac{3}{4}, 1)$ and $\vartheta, \theta > 0$ such that

$$\lim_{n\to\infty} n^\theta \sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^\vartheta(y) = 0.$$

Let $\sigma \in (0, \tilde{\sigma})$. By Jensen's inequality,

$$\left(t^{\frac{\theta}{2}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_{n_t}^y \left\langle |\mathcal{T}(y;0,t)| \right\rangle \right] \right)^2 \le t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_{n_t}^y \left\langle \mathcal{T}(y;0,t)^2 \right\rangle \right].$$

Fix $\delta \in (0,1)$, $\kappa \in \left(e^{1-\frac{1}{\delta}}, \delta\right)$, $\hat{\kappa} \in \left(e^{1-\frac{1}{\nu}}, \nu\right)$. Let $\nu_0 \in (0, \frac{\delta}{2})$ be so small that

$$e^{1-\nu_0} > \left(\frac{e}{\kappa}\right)^\delta.$$

Then, choose $\beta > 0$ so small that

$$e^{1-\nu_0-\beta^2} > \left(\frac{e}{\kappa}\right)^\delta, \quad e^{1-\beta^2} > \left(\frac{e}{\hat{\kappa}}\right)^\nu, \quad \beta^2 < \nu_0\vartheta. \tag{4.26}$$

Fix $\nu \in (\frac{1}{2}, 1)$ and let $J(t)$ be as in (2.26). For $t > 0$ and $\|y\| \le t^\sigma$, write $t^\theta (p_t^y)^{-1} \mathbf{E}\left[q_{n_t}^y \left\langle \mathcal{T}(y;0,t)^2 \right\rangle \right]$ as

$$\sum_{0\le l\le \nu_0 t} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t) + \sum_{l>\nu_0 t, l\notin J(t)} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t) + \sum_{l\in J(t)} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t),$$

where

$$Y_l(y,t) = e^{-t} \frac{t^l}{l!} \mathbf{E}\left[\left\langle \mathcal{T}(y;0,t)^2 \right\rangle \middle| n_t = l \right] \le e^{-t} \frac{t^l}{l!} \sum_{r\in R_l} \sum_{\mathbf{i}\in I_{l,r},\mathbf{z}} q_{l,r}^y(\mathbf{i},\mathbf{z})^2 A(t,l,r).$$

For $r \in R_l$ and $\mathbf{i} \in I_{l,r}$, we have from (4.7) the following estimate for $l \leq \nu_0 t$

$$\sum_{\mathbf{i} \in I_{l,r,\mathbf{z}}} q_{l,r}^y(\mathbf{i}, \mathbf{z})^2 \leq 4\binom{l+1}{r} \leq 2^{l+3} \leq 2^3 e^{\nu_0 t}. \tag{4.27}$$

Using Lemma 2.10 and the estimate (4.21) from Lemma 4.8, this implies, for t sufficiently large and $\|y\| \leq t^\sigma$:

$$\begin{aligned}\sum_{0 \leq l \leq \nu_0 t} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t) &\lesssim e^{t^\sigma} e^{(\nu_0 - 1)t} t^\theta \sum_{0 \leq l \leq \nu_0 t} \frac{t^l}{l!} \sum_{1 \leq r \leq l+1} A(t,l,r) \\ &\lesssim e^{t^\sigma + (\beta^2 + \nu_0 - 1)t} t^\theta \frac{(t+1)^5}{\sqrt{2\pi\kappa t}} \left(\frac{e}{\kappa}\right)^{\delta t}.\end{aligned}$$

The righthand side does not depend on y and clearly tends to 0 as $t \to \infty$. Next, we observe that

$$\sum_{l > \nu_0 t, l \notin J(t)} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t) \lesssim e^{t^\sigma} e^{-t} t^\theta \sum_{l > \nu_0 t, l \notin J(t)} \frac{t^l}{l!} \sum_{r \in R_l} \sum_{\mathbf{i} \in I_{l,r,\mathbf{z}}} q_{l,r}^y(\mathbf{i}, \mathbf{z})^2 A(t,l,r).$$

Using Lemma 4.6 and the fact that $\frac{t^r}{(l+1)\cdots(l+r)} \leq \nu_0^{-r}$ for $l > \nu_0 t$, we have that for $t \geq 2$, the right side is less than a constant times

$$\begin{aligned}&e^{t^\sigma} e^{(\beta^2-1)t} t^\theta \sum_{l > \nu_0 t, l \notin J(t)} (l+1)^2 \frac{t^l}{l!} \sum_{r \in R_l} \beta^{2r} \frac{t^r}{(l+1)\ldots(l+r)} \sum_{\mathbf{i} \in I_{l,r,\mathbf{z}}} q_{l,r}^y(\mathbf{i}, \mathbf{z})^2 \\ \leq &e^{t^\sigma} e^{(\beta^2-1)t} \sum_{l > \nu_0 t, l \notin J(t)} (l+1)^2 \frac{t^l}{l!} D_l^{\beta^2 \nu_0^{-1}}(y).\end{aligned}$$

For $\|y\| \leq t^\sigma$ and $l > \nu_0 t$, we have $\|y\| \leq \nu_0^{-\sigma} l^\sigma < l^{\tilde{\sigma}}$ for t large enough. Since $\beta^2 \nu_0^{-1} < \vartheta$,

$$D_l^{\beta^2 \nu_0^{-1}}(y) \leq \sup_{\|z\| \leq l^{\tilde{\sigma}}} D_l^\vartheta(z) \leq \sup_{n \in \mathbb{N}} \sup_{\|z\| \leq n^{\tilde{\sigma}}} D_n^\vartheta(z)$$

for such l and y, so

$$\lim_{t \to \infty} \sup_{\|y\| \leq t^\sigma} \sum_{l > \nu_0 t, l \notin J(t)} t^\theta \frac{q_l^y}{p_t^y} Y_l(y,t) = 0$$

is implied by

$$\lim_{t \to \infty} e^{t^\sigma} e^{(\beta^2-1)t} t^\theta \sum_{l > \nu_0 t, l \notin J(t)} (l+1)^2 \frac{t^l}{l!} = 0,$$

which follows from the estimates (4.22) and (4.23) of Lemma 4.8.

It remains to show that

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \sum_{l\in J(t)} \frac{q_l^y}{p_t^y} Y_l(y,t) = 0.$$

Fix $\nu_1 \in (\nu^{-1}-1, 1)$, and choose $\beta > 0$ so small that, in addition to the constraints in (4.26), it also satisfies $\vartheta > \max\left\{((2-\nu)\psi), \beta^2\right\}$, where ψ was defined in (4.20). For $l \in J(t)$, we set

$$\begin{aligned} Y_l^1(y,t) =& e^{-t}\frac{t^l}{l!} \sum_{r\in R_l, r<\nu_1 l} \sum_{\mathbf{i}\in I_{l,r,\mathbf{z}}} q_{l,r}^y(\mathbf{i},\mathbf{z})^2 A(t,l,r), \\ Y_l^2(y,t) =& e^{-t}\frac{t^l}{l!} \sum_{r\in R_l, r\ge\nu_1 l} \sum_{\mathbf{i}\in I_{l,r,\mathbf{z}}} q_{l,r}^y(\mathbf{i},\mathbf{z})^2 A(t,l,r), \end{aligned}$$

and note that $Y_l(y,t) \le Y_l^1(y,t) + Y_l^2(y,t)$. Let us first show that

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \sum_{l\in J(t)} \frac{q_l^y}{p_t^y} Y_l^1(y,t) = 0.$$

Fix $\epsilon > 0$ and choose $L \in \mathbb{N}$ so large that

$$l^{\theta} \sup_{\|z\|\le l^{\tilde\sigma}} D_l^{\vartheta}(z) < \epsilon\nu^{\theta}, \quad l \ge L.$$

Let t be so large that $\nu t > L$ and $\nu^{-\sigma} l^{\sigma} < l^{\tilde\sigma}$ for all $l > \nu t$. Since $(2-\nu)\psi < \vartheta$, we have for $l > \nu t$ and $y \in \mathbb{Z}^d$ such that $\|y\| \le t^{\sigma}$ the estimate

$$t^{\theta} D_l^{(2-\nu)\psi}(y) \le \nu^{-\theta} l^{\theta} \sup_{\|z\|\le l^{\tilde\sigma}} D_l^{\vartheta}(z) \le \epsilon.$$

By Lemma 4.7,

$$\begin{aligned} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \sum_{l\in J(t)} \frac{q_l^y}{p_t^y} Y_l^1(y,t) \lesssim& t^{\theta} \sup_{\|y\|\le t^{\sigma}} (p_t^y)^{-1} \sum_{l\in J(t)} e^{-t}\frac{t^l}{l!} q_l^y D_l^{(2-\nu)\psi}(y) \\ \le& \epsilon \sup_{\|y\|\le t^{\sigma}} (p_t^y)^{-1} \sum_{l\in J(t)} e^{-t}\frac{t^l}{l!} q_l^y \le \epsilon. \end{aligned}$$

To complete the proof, we show

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \sum_{l\in J(t)} \frac{q_l^y}{p_t^y} Y_l^2(y,t) = 0.$$

By Lemma 4.6, Lemma 2.10, and (4.23), for $t \geq 2$,

$$t^{\theta} \sum_{l \in J(t)} \frac{q_l^y}{p_t^y} Y_l^2(y,t) \lesssim t^{\theta} e^{t^{\sigma}} e^{(\beta^2-1)t} \sum_{l \in J(t)} (l+1)^2 \frac{t^{\lceil (1+\nu_1)l \rceil}}{\lceil (1+\nu_1)l \rceil !} D_l^{\beta^2}(y)$$
$$\lesssim t^{\theta} e^{t^{\sigma}} e^{(\beta^2-1)t} t^2 \sum_{l > \nu(1+\nu_1)t} \frac{t^l}{l!} D_l^{\beta^2}(y). \tag{4.28}$$

To complete the proof of the Key Lemma, it is enough to show convergence to 0 of the term in (4.28), which follows from the exponential tail estimate Lemma 7.4 by taking $f(t) = \lceil \nu(\nu_1+1)t \rceil$, $\rho_1 = \nu(\nu_1+1)$, and $\rho_2 = \rho_1 + \varepsilon$, where ε is so small that $e^{\varepsilon - 1} < \left(\frac{\rho_1}{e}\right)^{\rho_1} \rho_1^{\varepsilon}$. This completes the proof of the Key Lemma (Lemma 4.3).

4.2. Small contributions: Proof of the Central Lemma (Lemma 4.2), Parts 1 and 2

In this subsection, we show that the contributions of the terms $\mathbf{E}B_1^{y,t}$ and $\mathbf{E}B_2^{y,t}$ to $Z_{0,0}^{y,t}$ are negligible. In both cases, the strategy is to show convergence to 0 of the $D-$sequence associate with $B_j^{y,t}/q_{n_t}^y$ for $j = 1, 2$ and where $q_{n_t}^y > 0$, and apply the Key Lemma 4.3. Notice that $B_j^{y,t} = 0$ for $j = 1, 2$ whenever $q_{n_t}^y = 0$. If $q_{n_t}^y > 0$, then for $j = 1, 2$ we get

$$\left(\frac{1}{p_t^y} \left\langle \left| \mathbf{E} B_j^{y,t} \right| \right\rangle \right)^2 \leq \left(\frac{1}{p_t^y} \mathbf{E} \left[q_{n_t}^y \left\langle \left| \frac{B_j^{y,t}}{q_{n_t}^y} \right| \right\rangle \right] \right)^2 \leq \frac{1}{p_t^y} \mathbf{E} \left[q_{n_t}^y \left\langle \left| \frac{B_j^{y,t}}{q_{n_t}^y} \right|^2 \right\rangle \right],$$

where in the second step we applied Jensen's inequality twice.

4.2.1. Many Gaps: Proof of Part 1

For all $y \in \mathbb{Z}^d$, any $\vartheta > 0$, and for all $n \in \mathbb{N}$ so large that $k(n) \geq 1$ the D-sequence associated $B_1^{y,t}/q_{n_t}^y$ is given by

$$D_n^{\vartheta}(y) := \begin{cases} 0 & \text{if } q_n^y = 0, \\ \dfrac{1}{(q_n^y)^2} \displaystyle\sum_{k(n) < r \leq n+1} \vartheta^r \sum_{\mathbf{i} \in I_{r,n,\mathbf{z}}} q_{n_t}^y(\mathbf{i}, \mathbf{z})^2 & \text{if } q_n^y > 0. \end{cases}$$

Fix any $\tilde{\sigma} \in (\sigma, 1)$ such that $\frac{1}{2}(3\sigma - 1) > 2\tilde{\sigma} - 1$. We claim that for any $\theta > 0$ and $\vartheta < \alpha^{-1}$,

$$\lim_{n\to\infty} n^{\theta} \sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^{\vartheta}(y) = 0. \tag{4.29}$$

Thanks to the Key Lemma 4.3, this will imply

$$\lim_{t\to\infty} t^{\theta/2} \sup_{\|y\|\le t^{\tilde{\sigma}}} \frac{1}{p_t^y} \mathbf{E}\left[q_{n_t}^y \left\langle \left| \frac{B_1^{y,t}}{q_{n_t}^y} \right|^2 \right\rangle \right] = 0,$$

and hence the limit (4.3).

To prove (4.29), we argue as follows. Let $n \in \mathbb{N}$ be so large that $k(n) \ge 1$. First we note that we can restrict our attention to those $y \in \mathbb{Z}^d$ such that $\|y\|_1 \equiv n$, for otherwise $q_n^y = 0$, and thus $D_n^{\vartheta}(y) = 0$. Since $\vartheta\alpha < 1$, then using the definition (3.4) of α, we can estimate $D_n^{\vartheta}(y)$ as follows:

$$D_n^{\vartheta}(y) \lesssim \frac{1}{(q_n^y)^2} \sum_{k(n)<r\le n+1} (\vartheta\alpha)^r \le \frac{1}{(q_n^y)^2} \sum_{r>k(n)} (\vartheta\alpha)^r \le \frac{1}{(q_n^y)^2} \frac{(\vartheta\alpha)^{k(n)}}{1-\vartheta\alpha}.$$

To estimate $(q_n^y)^2$ in the denominator, we fix $\hat{\sigma} \in (\tilde{\sigma}, 1)$ such that $4\hat{\sigma} - 3 < 2\tilde{\sigma} - 1$. Then by Lemma 2.3, there are constants $c_1, c_2 > 0$ (independent of $\tilde{\sigma}, \hat{\sigma}$) and $N \in \mathbb{N}$ (depending on $\tilde{\sigma}, \hat{\sigma}$) such that for every $n \ge N$ and $y \in \mathbb{Z}^d$ with $\|y\| \le n^{\tilde{\sigma}}$, we have the following estimate:

$$\begin{aligned} q_n^y &\ge c_1 \left(\tfrac{d}{2\pi n}\right)^{d/2} \exp\left(-\tfrac{d}{2n}\|y\|^2\right) \exp\left(-c_2 n^{4\hat{\sigma}-3}\right) \\ &\gtrsim n^{-d/2} \exp\left(-\tfrac{d}{2} n^{2\tilde{\sigma}-1} - c_2 n^{4\hat{\sigma}-3}\right) \ge n^{-d/2} \exp\left(-c n^{2\tilde{\sigma}-1}\right) \end{aligned}$$

for some universal constant $c > 0$. We therefore obtain the estimate

$$D_n^{\vartheta}(y) \lesssim n^d (\vartheta\alpha)^{k(n)} \exp\left(2cn^{2\tilde{\sigma}-1}\right).$$

Notice that this estimate is independent of y. Since $\kappa_1 > \frac{1}{2}(3\sigma - 1) > 2\tilde{\sigma} - 1$, we have $n^{2\tilde{\sigma}-1}/k(n) \to 0$ as $n \to \infty$, and therefore, for all $\theta > 0$, we obtain

$$n^{\theta} \sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^{\vartheta}(y) \lesssim n^{\theta} n^d (\vartheta\alpha)^{k(n)} \exp\left(2cn^{2\tilde{\sigma}-1}\right) \xrightarrow[t\to\infty]{} 0.$$

4.2.2. No Huge Gaps: Proof of Part 2

For all $y \in \mathbb{Z}^d$, any $\vartheta > 0$, and for all $n \in \mathbb{N}$ so large that $k(n) \geq 1$ the D-sequence associated $B_2^{y,t}/q_{n_t}^y$ is given by

$$D_n^{\vartheta}(y) := \begin{cases} 0 & \text{if } q_n^y = 0, \\ \frac{1}{(q_n^y)^2} \sum_{1 \leq r \leq k(n)} \vartheta^r \sum_{m=1}^{r} \sum_{\mathbf{i} \in I_2(n,r,m), \mathbf{z}} q_{n_t}^y(\mathbf{i}, \mathbf{z})^2 & \text{if } q_n^y > 0. \end{cases}$$

Fix any $\tilde{\sigma} \in (\sigma, 1)$ such that $1 - \tilde{\sigma} > \xi$. We claim that there are $\theta, \vartheta > 0$ such that

$$\lim_{n \to \infty} n^{\theta} \sup_{\|y\| \leq n^{\tilde{\sigma}}} D_n^{\vartheta}(y) = 0. \tag{4.30}$$

Thanks to the Key Lemma 4.3, this will imply the limit (4.4).

To prove (4.30), consider the following double sum for any $n \in \mathbb{N}$ such that $k(n) > 1$, any $y \in \mathbb{Z}^d$ with $\|y\| \leq n^{\tilde{\sigma}}$ and with $q_n^y > 0$, and any $r \in \mathbb{N}$ with $1 \leq r \leq k(n)$:

$$M_{n,r}(y) := \sum_{m=1}^{r} \sum_{\substack{\mathbf{i} \in I_2(n,r,m), \\ i_1 \neq 0, i_r \neq n, \mathbf{z}}} q_{n_t}^y(\mathbf{i}, \mathbf{z})^2$$

Notice that $M_{n,r}(y)$ is almost exactly the expression appearing in the definition of $D_n^{\vartheta}(y)$ but with $i_1 \neq 0, i_r \neq 0$; therefore,

$$D_n^{\vartheta}(y) \lesssim \frac{1}{(q_n^y)^2} \sum_{1 \leq r \leq k(n)} \vartheta^r M_{n,r}(y) \tag{4.31}$$

Now we estimate $M_{n,r}(y)$. Let $n' := n - (n_{l+1} + \ldots + n_r)$ and $z' := y - (z_{l+1} + \ldots + z_r)$. Since no huge gaps implies there are at least two large gaps, we can write

$$M_{n,r}(y) \leq (r+1) \sum_{l=1}^{r} \binom{r}{l} M_{n,r,l}(y), \tag{4.32}$$

where

$$M_{n,r,l}(y) := \sum_{\substack{n_1+\ldots+n_{r+1}=n \\ z_1+\ldots+z_{r+1}=y \\ n_1,\ldots,n_l \geq n^\xi \\ n_{r+1}\geq n_1,\ldots,n_r}} \left(q_{n_1}^{z_1}\right)^2 \cdots \left(q_{n_{r+1}}^{z_{r+1}}\right)^2 = \sum_{n_{l+1},\ldots,n_r<n^\xi} \left(q_{n_{l+1}}^{z_{l+1}}\right)^2 \cdots \left(q_{n_r}^{z_r}\right)^2 M_{n,r,l}^{n'}(z'), \tag{4.33}$$

$$M_{n,r,l}^{n'}(z') := \sum_{\substack{n_1+\ldots+n_l+n_{r+1}=n' \\ z_1+\ldots+z_l+z_{r+1}=z' \\ n_{r+1}\geq n_1,\ldots,n_l\geq n^\xi}} \left(q_{n_1}^{z_1}\right)^2 \cdots \left(q_{n_l}^{z_l}\right)^2 \left(q_{n_{r+1}}^{z_{r+1}}\right)^2. \tag{4.34}$$

In (4.33), the summation condition $n_1, \ldots, n_l \geq n^\xi$ ensures that there is at least one large gap, and the condition $n_{r+1} \geq n_1, \ldots, n_r$ ensures that n_{r+1} is another large gap, thus guaranteeing that we are summing over all collections of indices $0 \leq i_0 < \ldots < r_n \leq n$ involving at least two large gaps. We now search for a bound for each $M_{n,r,l}^{n'}(z')$ when n is sufficiently large.

Claim 4.1

For all n sufficiently large, there are constants $C, C' > 0$ (independent of n) such that for each r, l, n', z' as above,

$$M_{n,r,l}^{n'}(z') \lesssim n^{-\xi/4}(q_n^y)^2 C^l n^{-\xi l(2d-5)/4} \exp\left(C'(r-l)n^{\tilde{\sigma}+\xi-1}\right).$$

We use Claim 4.1 to estimate $M_{n,r,l}(y)$ from (4.33) as follows:

$$\begin{aligned} M_{n,r,l}(y) &\lesssim \alpha^{r-l} n^{-\xi/4}(q_n^y)^2 C^l n^{-\xi l(2d-5)/4} \exp\left(C'(r-l)n^{\tilde{\sigma}+\xi-1}\right) \\ &\leq n^{-\xi/4}(q_n^y)^2 \left(Cn^{-\xi(2d-5)/4}\right)^l \left(\alpha \exp\left(C' n^{\tilde{\sigma}+\xi-1}\right)\right)^{r-l}. \end{aligned}$$

Then we combine this with (4.32) to estimate $M_{n,r}(y)$ as follows:

$$\begin{aligned} M_{n,r}(y) &\lesssim n^{-\xi/4}(q_n^y)^2 (r+1) \sum_{l=1}^{r} \binom{r}{l} \left(Cn^{-\xi(2d-5)/4}\right)^l \left(\alpha \exp\left(C' n^{\tilde{\sigma}+\xi-1}\right)\right)^{r-l} \\ &= n^{-\xi/4}(q_n^y)^2 (r+1) \left(Cn^{-\xi(2d-5)/4} + \alpha \exp\left(C' n^{\tilde{\sigma}+\xi-1}\right)\right)^r. \end{aligned}$$

Finally, combining this estimate with (4.31), we obtain:

$$D_n^\vartheta(y) \lesssim n^{-\xi/4} \sum_{r=1}^{\infty} (r+1)\vartheta^r \left(Cn^{-\xi(2d-5)/4} + \alpha \exp\left(C' n^{\tilde{\sigma}+\xi-1}\right)\right)^r.$$

Since $d \geq 3$ and since we chose $\tilde{\sigma}$ such that $\xi < 1 - \tilde{\sigma}$, then (4.30) is true as long as $\vartheta < \alpha^{-1}$ and $\theta < \xi/4$. To complete the proof, it remains to prove Claim 4.1.

Proof of Claim 4.1. By Lemma 2.5, there are constants $\rho_1, \rho_2 > 0$ such that for any $n \in \mathbb{N}$, there is a linear functional φ on $\mathbb{R}^d$ of norm $\|\varphi\| \leq \rho_2 \|y\|/n$ such that

$$q_n^y e^{\varphi(y)} \gtrsim n^{-d/2} \sum_{z \in \mathbb{Z}^d} q_n^z e^{\varphi(z)}, \tag{4.35}$$

whenever $\|y\| \leq \rho_1 n$ and $\|y\|_1 \equiv n$. Fix $n \in \mathbb{N}$ so large that $k(n) \geq 1$, as well as $n^{\tilde{\sigma}} \leq \rho_1 n$ and $\rho_2 n^{\tilde{\sigma}-1} \leq 1$. Then, for any $n_1, \ldots, n_l, n_{r+1} \in \mathbb{N}$ and any $z_1, \ldots, z_l, z_{r+1} \in \mathbb{Z}^d$ satisfying the conditions in the sum in (4.34), the by Lemma 2.4, there is a constant $c_1 > 0$ such that

$$\begin{aligned} q_{n_1}^{z_1} \cdots q_{n_l}^{z_l} q_{n_{r+1}}^{z_{r+1}} &= e^{\varphi(-z')} \prod_{j \in \{1,\ldots,l,r+1\}} e^{\varphi(z_j)} q_{n_j}^{z_j} \\ &\leq e^{\varphi(-z')} \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \sum_{z \in \mathbb{Z}^d} q_{n_j}^z e^{\varphi(z)} \right) \\ &= e^{\varphi(-z')} \sum_{z \in \mathbb{Z}^d} q_{n'}^z e^{\varphi(z)} \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \right) \\ &= e^{\varphi(-z')} \Phi(0)^{n'} \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \right), \end{aligned} \tag{4.36}$$

where in the third line we used the fact that φ is a linear functional, and in the fourth line we used (2.8). Since $n' < n$ and $\Phi(0) \geq 1$, it follows from (4.35) that $\Phi(0)^{n'} \leq \Phi(0)^n \lesssim n^{d/2} q_n^y e^{\varphi(y)}$. As a result, we obtain

$$\max_{z_1+\ldots+z_l+z_{r+1}=z'} q_{n_1}^{z_1} \cdots q_{n_l}^{z_l} q_{n_{r+1}}^{z_{r+1}} \lesssim n^{d/2} q_n^y e^{\varphi(y-z')} \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \right).$$

Furthermore, the sum $\sum q_{n_1}^{z_1} \cdots q_{n_l}^{z_l} q_{n_{r+1}}^{z_{r+1}}$ over all tuples $(z_1, \ldots, z_l, z_{r+1})$ such that $z_1 + \cdots z_l + z_{r+1} = z'$ equals $q_{n'}^{z'}$, and by Lemma 2.6,

$$q_{n'}^{z'} \leq q_n^y \big(1 + O(n^{-2/5})\big) \exp\left(\frac{c}{n} \Big(\|y\| \cdot \|y - z'\| + \|y\|(n - n') + \ln(n)(n - n') \Big) \right),$$

for some universal constant $c > 0$. Therefore,

$$\begin{aligned} \sum_{z_1+\ldots+z_l+z_{r+1}=z'} \big(q_{n_1}^{z_1}\big)^2 \cdots \big(q_{n_l}^{z_l}\big)^2 \big(q_{n_{r+1}}^{z_{r+1}}\big)^2 &\lesssim n^{d/2} q_n^y q_{n'}^{z'} e^{\varphi(y-z')} \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \right) \\ &\lesssim n^{d/2} (q_n^y)^2 P(n) \prod_{j \in \{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2} \right), \end{aligned}$$

where $P(n) := \exp\left(\frac{c'}{n}\Big(2\|y\|\cdot\|y-z'\| + \|y\|(n-n') + \ln(n)(n-n')\Big)\right)$, for some constant $c' > 0$.

By Lemma 7.2, there exists $c > 0$ such that

$$\sum_{\substack{n_1+\ldots+n_{l+1}=n,\\ n_1,\ldots,n_{l+1}\geq M}} \prod_{j=1}^{l+1} n_j^{-\frac{d}{2}} \leq \frac{c^l}{M^{l(\frac{d}{2}-1)}} n^{-\frac{d}{2}}. \tag{4.37}$$

Therefore, we obtain the following estimate on $M_{n,r,l}^{n'}(z')$:

$$\begin{aligned} M_{n,r,l}^{n'}(z') &\lesssim n^{d/2}(q_n^y)^2 P(n) \sum_{\substack{n_1+\ldots+n_l+n_{r+1}=n'\\ n_{r+1}\geq n_1,\ldots,n_l\geq n^{\xi}}} \prod_{j\in\{1,\ldots,l,r+1\}} \left(c_1 n_j^{-d/2}\right) \\ &\lesssim (q_n^y)^2 C^l n^{-\xi l(d-2)/2} \left(\frac{n}{n'}\right)^{d/2} P(n), \\ &\leq n^{-\xi/4}(q_n^y)^2 C^l n^{-\xi l(2d-5)/4} \left(\frac{n}{n'}\right)^{d/2} P(n), \end{aligned}$$

It remains to bound $(n/n')^{d/2}P(n)$. We estimate the following expressions involved in $(n/n')^{d/2}P(n)$ like so:

$$\left(\frac{n}{n'}\right)^{d/2} \leq \exp\left(\frac{d}{2}\ln(n)\frac{n-n'}{n-1}\right), \qquad n-n' = \sum_{j=l+1}^{r} n_j < (r-l)n^{\xi},$$

$$\|y-z'\| \leq \sum_{j=l+1}^{r} \|z_j\| \leq \sum_{j=l+1}^{r} n_j < (r-l)n^{\xi},$$

provided that $q_{n_1}^{z_1}\ldots q_{n_{r+1}}^{z_{r+1}} > 0$. Then, using $\|y\| \leq n^{\tilde{\sigma}}$, we obtain:

$$\left(\frac{n}{n'}\right)^{d/2} P(n) \leq \exp\left(C'(r-l)n^{\tilde{\sigma}+\xi-1}\right)$$

for some constant $C' > 0$, which completes the proof of Claim 4.1. □

4.3. The Main contribution: Proof of the Central Lemma (Lemma 4.2), Part 3

For $\mathbf{i} \in I_1(n,r,m)$ and $\mathbf{z} \in (\mathbb{Z}^d)^r$, define

$$q_{n,\hat{m}}^y(\mathbf{i},\mathbf{z}) := q_{i_1}^{z_1}\ldots \widehat{q_{i_m-i_{m-1}}^{z_m-z_{m-1}}}\ldots q_{n-i_r}^{y-z_r}, \tag{4.38}$$

where the factor with the hat is absent; in other words, we remove the transition probability corresponding to the huge gap.

Now decompose $B_3^{y,t}$ further, depending on the position of the huge gap 1) at the begining, 2) in the middle, or 3) at the end, as follows:

$$B_3^{y,t} = q_{n_t}^y \sum_{i=1}^{3} \left(F_i^{y,t} + L_i^{y,t} \right),$$

where

$$\begin{aligned}
F_1^{y,t} &:= \sum_{1\le r\le k(n_t)} \sum_{\mathbf{i}\in I_1(n_t,r,1),\mathbf{z}} q^y_{n_t,\hat{1}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),\\
F_2^{y,t} &:= \sum_{2\le r\le k(n_t)} \sum_{m=2}^{r} \sum_{\mathbf{i}\in I_1(n_t,r,m),\mathbf{z}} q^y_{n_t,\hat{m}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),\\
F_3^{y,t} &:= \sum_{1\le r\le k(n_t)} \sum_{\mathbf{i}\in I_1(n_t,r,r+1),\mathbf{z}} q^y_{n_t,\widehat{r+1}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1});
\end{aligned}$$

and the error terms are given by

$$\begin{aligned}
L_1^{y,t} &:= \sum_{1\le r\le k(n_t)} \sum_{\mathbf{i}\in I_1(n_t,r,1),\mathbf{z}} \frac{q_{i_1}^{z_1} - q_{n_t}^y}{q_{n_t}^y} q^y_{n_t,\hat{1}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),\\
L_2^{y,t} &:= \sum_{2\le r\le k(n_t)} \sum_{m=2}^{r} \sum_{\mathbf{i}\in I_1(n_t,r,m),\mathbf{z}} \frac{q_{i_m-i_{m-1}}^{z_m-z_{m-1}} - q_{n_t}^y}{q_{n_t}^y} q^y_{n_t,\hat{m}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),\\
L_3^{y,t} &:= \sum_{1\le r\le k(n_t)} \sum_{\mathbf{i}\in I_1(n_t,r,r+1),\mathbf{z}} \frac{q_{n_t-i_r}^{y-z_r} - q_{n_t}^y}{q_{n_t}^y} q^y_{n_t,\widehat{r+1}} \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),
\end{aligned}$$

whenever $q_{n_t}^y > 0$, and 0 if $q_{n_t}^y = 0$. Central Lemma part (3) now immediately follows from the following lemma.

Lemma 4.10

For sufficiently small $\beta > 0$, there is a $\theta > 0$ such that

(a) $$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q_{n_t}^y \sum_{i=1}^{3} L_i^{y,t} \right] \right| \right\rangle = 0; \tag{4.39}$$

(b) $$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \left\langle \left| p_t^y + \mathbf{E}\left[q_{n_t}^y \sum_{i=1}^{3} F_i^{y,t} \right] - p_t^y Z_{0,0}^{\infty} Z_{-\infty}^{y,t} \right| \right\rangle = 0. \tag{4.40}$$

4.3.1. Contribution from the Error Terms: Proof of Lemma 4.10 (a)

We first show that the contribution from each error term is negligible. By Key Lemma 4.3, it is enough to show that for β small enough there are $\theta, \vartheta > 0$ and $\tilde{\sigma} \in (\sigma, 1)$ such that for $i \in \{1, 2, 3\}$,

$$\lim_{n\to\infty} n^{\theta} \sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^{\vartheta}(i, y) = 0. \tag{4.41}$$

Here, $D^{\vartheta}(i)$ is the D-sequence associated with L_i. We will only show convergence of the D-sequence for $i = 1$ and $i = 2$, because the convergence for $i = 1$ entails convergence for $i = 3$ by symmetry. The D-sequence associated with L_i is:

$$D_n^{\vartheta}(i, y) := \begin{cases} 0 & \text{if } q_n^y = 0, \\ \displaystyle\sum_{1\le r\le k(n)} \vartheta^r a_i(r) \sum_{\mathbf{n}\in\Xi^i_{r,\mathbf{z}}} \left(q_{n_1}^{z_1}\right)^2 \dots \left(q_{n_r}^{z_r}\right)^2 \frac{(q_{n-n_1-\dots-n_r}^{y-z_1-\dots-z_r} - q_n^y)^2}{(q_n^y)^2} & \text{if } q_n^y > 0. \end{cases}$$

where $a_1(r) := 1$, $a_2(r) := r + 1$, and

$$\Xi_r^1 := \left\{ (n_1, \dots, n_r) \in \mathbb{N}^r \ : \ \begin{matrix} n_1 \ge 0, n_2, \dots, n_r > 0 \\ n_1 + \dots + n_r < rn^{\xi} \end{matrix} \right\},$$
$$\Xi_r^2 := \left\{ (n_1, \dots, n_r) \in \mathbb{N}^r \ : \ \begin{matrix} n_1, n_r \ge 0, n_2, \dots, n_{r-1} > 0 \\ n_1 + \dots + n_r < rn^{\xi} \end{matrix} \right\}.$$

The convergence in (4.41) relies on $q_{n-n_1-\dots-n_r}^{y-z_1-\dots-z_r}$ being close to q_n^y in the following sense. Fix $\tilde{\sigma} \in (\sigma, 1)$ so close to σ that $\tilde{\sigma} < 1 - \xi$; and let $n \in \mathbb{N}$ be so large that $k(n) \ge 1$ and $n^{\tilde{\sigma}} \le \rho n$, where ρ is the constant from Lemma 2.6. Let $y \in \mathbb{Z}^d$ such that $\|y\| \le n^{\tilde{\sigma}}$ and $q_n^y > 0$. In addition, let $1 \le r \le k(n)$, $n_1 \ge 0, n_2, \dots, n_r \ge 1$ with $n_1 + \dots + n_r < rn^{\xi}$. Without loss of generality, let $z_1, \dots, z_r \in \mathbb{Z}^d$ such that $q_{n_1}^{z_1}, \dots, q_{n_r}^{z_r} > 0$ as otherwise the contribution to $D_n^{\vartheta}(i, y)$ is zero.

Claim 4.2
$$\left(\frac{q_{n-n_1-\dots-n_r}^{y-z_1-\dots-z_r} - q_n^y}{q_n^y} \right)^2 \le \left(1 + O(n^{-\frac{2}{5}})\right) e^{c_3 \rho n^{\tilde{\sigma}+\xi-1}} - 1.$$

Using this claim we can bound $\sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^{\vartheta}(i, y)$ as follows:

$$\begin{aligned} \sup_{\|y\|\le n^{\tilde{\sigma}}} D_n^{\vartheta}(i, y) &\lesssim \sum_{r=1}^{\infty} \vartheta^r (r+1) \sum_{(n_1,\dots,n_r)\in\mathbb{N}^r, \mathbf{z}} \left(q_{n_1}^{z_1}\right)^2 \dots \left(q_{n_r}^{z_r}\right)^2 \left(\left(1 + O(n^{-\frac{2}{5}})\right) e^{c_3 \rho n^{\tilde{\sigma}+\xi-1}} - 1 \right) \\ &\lesssim \sum_{r=0}^{\infty} (\vartheta\alpha)^r (r+1) \left(\left(1 + O(n^{-\frac{2}{5}})\right) e^{c_3 \rho n^{\tilde{\sigma}+\xi-1}} - 1 \right). \end{aligned}$$

Therefore, for $i = 1, 2$ and $\vartheta < \alpha^{-1}$, $\sup_{\|y\| \le n^{\tilde\sigma}} D_n^{\vartheta}(i, y)$ converges to 0 as $n \to \infty$ faster than $n^{-\theta}$, with $\theta \in (0, 1 - \tilde\sigma - \xi)$. This implies (4.41).

To complete the proof of Lemma (4.10), it only remains to prove Claim 4.2.

Proof of Claim 4.2. Let $z' := z_1 + \cdots + z_r$ and $n' := n_1 + \cdots + n_r$. Observe that $q_{n-n'}^{y-z'} > 0$. Indeed, notice first of all $n - n' \ge n - k(n)n^{\xi}$ since $n' < rn^{\xi}$. Moreover,

$$\|y - z'\| \le n^{\tilde\sigma} + \sum_{j=1}^{r} \|z_j\| \le n^{\tilde\sigma} + \sum_{j=1}^{r} n_j \le n^{\tilde\sigma} + k(n)n^{\xi}.$$

Finally, $n - n'$ and $\|y - z'\|_1$ have the same parity because otherwise, if $n - n'$ and $\|y - z'\|_1$ had distinct parity, then, since $n \equiv |y|_1$, at least one of the probabilities $q_{n_1}^{z_1}, \dots, q_{n_r}^{z_r}$ must be zero in this case, which violates our previously made assumption.

Now, we derive a bound on $|q_{n-n'}^{y-z'} - q_n^y|/q_n^y$. If $q_{n-n'}^{y-z'} \ge q_n^y$, then Lemma 2.6 gives

$$\begin{aligned} \frac{|q_{n-n'}^{y-z'} - q_n^y|}{q_n^y} &\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(2n^{\tilde\sigma - 1}\rho n^{\xi} + \frac{\ln(n)}{n}\rho n^{\xi}\right)\right) - 1 \\ &\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c_1 \rho n^{\tilde\sigma + \xi - 1}\right) - 1 \end{aligned} \tag{4.42}$$

for some constant $c_1 > 0$. Otherwise, if $q_n^y > q_{n-n'}^{y-z'}$, then we argue as follows. For sufficiently large n, we have $n - n' \ge n - k(n)n^{\xi} \ge n/2$. Hence, $\|y - z'\| \le n^{\tilde\sigma} + k(n)n^{\xi} \le \frac{\rho}{2} n \le \rho(n - n')$ for large n. Therefore, again by Lemma 2.6,

$$\frac{|q_{n-n'}^{y-z'} - q_n^y|}{q_n^y} \le \frac{q_n^y}{q_{n-n'}^{y-z'}} - 1 \tag{4.43}$$

$$\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c\left(\left(2n^{\tilde\sigma - 1} + 2\rho n^{\xi - 1}\right) 2\rho n^{\xi} + 2\frac{\ln(n)}{n}\rho n^{\xi}\right)\right) - 1 \tag{4.44}$$

$$\le \left(1 + O(n^{-\frac{2}{5}})\right) \exp\left(c_2 \rho n^{\tilde\sigma + \xi - 1}\right) - 1.$$

Using the general fact that $(a-1)^2 \le a^2 - 1$ for any $a \ge 1$, in either case (4.42) or (4.43), we have the following bound:

$$\left(\frac{q_{n-n'}^{y-z'} - q_n^y}{q_n^y}\right)^2 \le \left(1 + O(n^{-\frac{2}{5}})\right) e^{c_3 \rho n^{\tilde\sigma + \xi - 1}} - 1,$$

where $c_3 > 0$ is a constant. □

4.3.2. Main Contribution: Proof of Lemma 4.10, Part (b)

In order to deal with the F_i's (i.e., Lemma 4.10, Part (b)), the strategy is to first define suitable truncations of the partition functions. Fix ξ_1, ξ_2, ξ_3 satisfying $0 < \xi_1 < \xi_2 < \xi_3 < \xi$, and notice that since $\xi + \sigma < 1$, we have $\xi_1 + \sigma < 1$. For $n \in \mathbb{N}_0$, set

$$v(n) := \left\lceil \frac{n}{2} \right\rceil^{\xi_2}, \quad w(n) := \left\lceil \frac{n}{2} \right\rceil^{\xi_3}.$$

To avoid heavy notation, we write

$$n := n_t, \quad n_- := n_{t^{\xi_1}}, \quad n_\bullet := n_{t^{\xi_1}, t-t^{\xi_1}}, \quad n_+ := n_{t-t^{\xi_1}, t},$$

and introduce the following set of symbols:

$$\oplus := \{-, \bullet, +\}.$$

Now we define

$$T^t_{0,0} := 1 + \sum_{1 \le r \le v(n_-)+1} \sum_{\substack{\mathbf{i} \in I_{r,n}, \\ i_r \le w(n_-), \\ \mathbf{z}}} q^y_{n, \widehat{r+1}}(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}),$$

$$T^{y,t}_0 := 1 + \sum_{1 \le r \le v(n_+)+1} \sum_{\substack{\mathbf{i} \in I_{r,n}, \\ n - w(n_+) \le i_1, \, i_r \le n, \\ \mathbf{z}}} q^y_{n,\hat{1}}(\mathbf{i}, \mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}).$$

Notice that $\mathbf{E}[T^t_{0,0}]$ and $\mathbf{E}[T^{y,t}_0]$ are truncations of the partition functions $Z^t_{0,0}$ and $Z^{y,t}_0$ respectively. The proof of part (b) of Lemma 4.10 boils down to proving the following lemma.

Lemma 4.11

For sufficiently small $\beta > 0$, there is a $\theta > 0$ such that

(a)
$$\lim_{t\to\infty} t^\theta \sup_{\|y\| \le t^\sigma} \frac{1}{p^y_t} \left\langle \left| p^y_t + \mathbf{E}\left[q^y_{n_t} \sum_{i=1}^{3} F^{y,t}_i \right] - \mathbf{E}\left[q^y_{n_t} T^{y,t}_0 T^t_{0,0} \right] \right| \right\rangle = 0; \tag{4.45}$$

(b)
$$\lim_{t\to\infty} t^\theta \sup_{\|y\| \le t^\sigma} \frac{1}{p^y_t} \left\langle \left| \mathbf{E}\left[q^y_{n_t} T^{y,t}_0 T^t_{0,0} \right] - p^y_t Z^\infty_{0,0} Z^{y,t}_{-\infty} \right| \right\rangle = 0. \tag{4.46}$$

We dedicate the whole of next section to proving this lemma.

4.4. Proof of Lemma 4.11

4.4.1. Preliminaries

Recall that for any $t > 0$, $l \in \mathbb{N}_0$, and $1 \le r \le l+1$, we introduced the following notation in Subsection 4.1.1:

$$A(t,l,r) := \mathbf{E}\left[\prod_{j=1}^{r-2}\left(e^{\beta^2 t_j}-1\right)e^{\beta^2(t_{r-1}+t_r)}\middle| n_t = l\right], \tag{4.47}$$

where t_0 should be interpreted as 0, and t_{l+1} as $t - s_l$. Recall that for $\nu \in (\frac{1}{2}, 1)$, and for all $t > 0$, and we also defined the set

$$J(t) := \left\{n \in \mathbb{N} \;:\; \nu t < n < (2-\nu)t\right\}. \tag{4.48}$$

To avoid heavy notation, introduce the following shorthands:

$$\begin{aligned}
\mathbf{P}^i(m) &= \mathbf{P}(n_i = m), \quad i \in \oplus,\\
\mathbf{P}^{i,j}(m,l) &= \mathbf{P}(n_i = m, n_j = l), \quad i,j \in \oplus,\ i \ne j,\\
\mathbf{P}(m,l,k) &= \mathbf{P}(n_- = m, n_\bullet = l, n_+ = k),\\
\mathbf{P}^{(i,j)}(m) &= \mathbf{P}(n_i + n_j = m), \quad i,j \in \oplus,\ i \ne j,\\
\mathbf{P}^{-,(\bullet,+)}(m,l) &= \mathbf{P}(n_- = m, n_\bullet + n_+ = l),\\
\mathbf{P}^{\bullet,(-,+)}(m,l,k) &= \mathbf{P}(n_\bullet = m, n_- + n_+ = l),
\end{aligned}$$

as well as the analogous definitions for $\mathbf{E}^i[\cdot|m]$, $\mathbf{E}^{i,j}[\cdot|m,l]$, etc.

Next, we make the following preliminary simplifications, which will be applied several times troughout the proof of Lemmas 4.13 and 4.11. For any $t \in \mathbb{R}$, and each $s \in \oplus$, let $S_s(t)$ be any nonempty subset of $\mathbb{N}$ and $l_s \in S_s(t)$. Let $l := l_- + l_\bullet + l_+$ and let

$$R \subseteq \{1,\ldots,l\}.$$

For $r \in R$, let

$$M_r \subseteq \{1,\ldots,r\}$$

and for $m \in M_r$, let

$$H_{r,m} \subseteq I_1(l,r,m).$$

Let

$$f := \sum_{r\in R}\sum_{m\in M_r}\sum_{\mathbf{i}\in H_{r,m,\mathbf{z}}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z})^2 \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}). \tag{4.49}$$

Notice that, for example, each $F_i^{y,t}$ has exactly this form. Then,

$$\mathbf{E}\left[q^y_n \left\langle f^2\right\rangle \mathbb{1}_{l_s\in S_s(t),s\in\oplus}\right] = \sum_{\substack{l_s\in S_s\\ s\in\oplus}} q^y_{l_-+l_\bullet+l_+}\mathbf{P}(l_-,l_\bullet,l_+)\mathbf{E}\left[\left\langle f^2\right\rangle \middle| l_-,l_\bullet,l_+\right], \tag{4.50}$$

where

$$\mathbf{E}\left[\left\langle f^2\right\rangle \middle| l_-,l_\bullet,l_+\right] = \sum_{r\in R}\sum_{m\in M_r}\sum_{\mathbf{i}\in H_{r,m,\mathbf{z}}} q^y_{l_-+l_\bullet+l_+,\hat{m}}(\mathbf{i},\mathbf{z})^2 \mathbf{E}\left[\prod_{j=1}^{r}\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| l_-,l_\bullet,l_+\right] \tag{4.51}$$

For $l_-,l_\bullet,l_+\in\mathbb{N}_0$, $r\in R^i$, $m\in M^i_r$, and $\mathbf{i}\in I^i_{r,m}$, let

$$\begin{aligned} r_- &= \left|\{1\le j\le r : i_j < l_-\}\right|, \\ r_\bullet &= \left|\{1\le j\le r : l_- \le i_j < l_- + l_\bullet\}\right|, \\ r_+ &= r - r_- - r_\bullet, \end{aligned}$$

and observe that

$$\mathbf{E}\left[\prod_{j=1}^{r}\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| l_-,l_\bullet,l_+\right] \le A(t^{\xi_1},l_-,r_-+1)A(t-2t^{\xi_1},l_\bullet,r_\bullet+1)A(t^{\xi_1},l_+,r_++1). \tag{4.52}$$

In the proof of Lemma 4.11, we will need the following technical results whose proofs are relegated to the Appendix, Section 7.1. Let $\chi\in(0,\frac{1}{2}(1-\sigma))$ and $\nu\in(\frac{1}{2},1)$. For $t>0$ and $y\in\mathbb{Z}^d$, let $\chi_1(t)$ be the smallest even integer $\ge t(1-t^{-\chi})$, and let $\chi_2(t)$ be the largest odd integer $\le t(1+t^{-\chi})$. Let

$$K(t) = \{l\in\mathbb{N} : \chi_1(t)\le l\le\chi_2(t)\},\quad t>0.$$

Lemma 4.12 (Building Blocks)
Let $\nu \in (\frac{1}{2}, 1)$ *and* $\nu_1 \in (\nu^{-1} - 1, 1)$. *Then, for any* $\theta, c > 0$ *we have:*

(A0) $\limsup_{t\to\infty} e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) < \infty;$

(A1) $\limsup_{t\to\infty} e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) < \infty;$

(A2) $\limsup_{t\to\infty} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r<\nu_1 l-1} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) < \infty;$

(A3) $\lim_{t\to\infty} t^\theta e^{\beta^2 t^{\xi_1}} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{\nu_1 l-1\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) = 0;$

(A4) $\lim_{t\to\infty} t^\theta e^{\beta^2 t^{\xi_1}} \sum_{l\notin J(t^{\xi_1})} e^{ct^{\sigma-1}l} \mathbf{P}^-(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) = 0;$

(A5) $\lim_{t\to\infty} t^\theta e^{t^\sigma} \sum_{l\notin J(t-2t^{\xi_1})} \mathbf{P}^\bullet(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) = 0;$

(A6) $\lim_{t\to\infty} \sum_{l\notin J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) = 0;$

(A7) $\limsup_{t\to\infty} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r<l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) < \infty;$

(A8) $\lim_{t\to\infty} t^\theta e^{t^\sigma} \sum_{l\notin K(t-2t^{\xi_1})} \mathbf{P}^\bullet(l) = 0, \quad \theta > 0.$

4.4.2. Proof of Lemma 4.11: Part (a)

To prove part (a), we further split it into the following three limits.

Lemma 4.13

For sufficiently small $\beta > 0$, *there is* $\theta > 0$ *such that*

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q^y_{n_t} \left(F_2^{y,t} - (T^t_{0,0} - 1)(T_0^{y,t} - 1) \right) \right] \right| \right\rangle = 0, \tag{4.53}$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q^y_{n_t} \left(F_1^{y,t} - (T_0^{y,t} - 1) \right) \right] \right| \right\rangle = 0, \tag{4.54}$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q^y_{n_t} \left(F_3^{y,t} - (T^t_{0,0} - 1) \right) \right] \right| \right\rangle = 0, \tag{4.55}$$

4.4.2.1. Proof of Lemma 4.13, Part 1: Convergence of One Huge Gap in the Middle

In this subsection, we prove the convergence statement in (4.53). Let

$$f_2^{y,t} := F_2^{y,t} - (T_{0,0}^t - 1)(T_0^{y,t} - 1).$$

Claim 4.3

It is enough to show that for β sufficiently small there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q_n^y f_2^{y,t} \mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet \in J(t-2t^{\xi_1})} \right] \right| \right\rangle = 0, \tag{4.56}$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[q_n^y f_2^{y,t} \left(1 - \mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet \in J(t-2t^{\xi_1})}\right) \right] \right| \right\rangle = 0. \tag{4.57}$$

Proof of Claim 4.3, Part 1. We first show (4.56). By symmetry considerations, it suffices to show that for small β there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_n^y \left\langle \left(f_2^{y,t}\right)^2 \right\rangle \mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet \in J(t-2t^{\xi_1}),n_-\le n_+} \right] = 0. \tag{4.58}$$

We have

$$\begin{aligned} &(T_{0,0}^t - 1)(T_0^{y,t} - 1) \\ &= \sum_{1\le r\le v(n_-)+1} \sum_{1\le s\le v(n_+)+1} \sum_{\substack{0\le i_1<\ldots<i_r\le w(n_-),\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} \sum_{\substack{n-w(n_+)\le l_1<\ldots<l_s\le n,\\ c_1,\ldots,c_s\in\mathbb{Z}^d}} \\ &\quad q_{i_1}^{z_1}\ldots q_{i_r-i_{r-1}}^{z_r-z_{r-1}} q_{l_2-l_1}^{c_2-c_1}\ldots q_{n-l_s}^{y-c_s} \prod_{j=1}^r h(z_j; s_{i_j}, s_{i_j+1}) \prod_{k=1}^s h(c_k; s_{l_k}, s_{l_k+1}). \end{aligned} \tag{4.59}$$

Define the set

$$V_{r,m}^{n,n_-,n_+} := \left\{ \mathbf{i} = (i_1,\ldots,i_r) \in I_1(n,r,m) \ : \ \begin{array}{c} 0\le i_1<\ldots<i_{m-1}\le w(n_-) \\ n-w(n_+)\le i_m<\ldots<i_r\le n \end{array} \right\}.$$

and its complement in $I_1(n,r,m)$

$$W_{r,m}^{n,n_-,n_+} := \left\{ \mathbf{i} = (i_1,\ldots,i_r) \in I_1(n,r,m) \ : \ i_m - i_{m-1} < n - w(n_-) - w(n_+) \right\}.$$

If n, n_-, n_+ are clear from the context, we will simply write $V_{r,m}, W_{r,m}$.

Suppose now that $n_- \le n_+$, so that $v(n_-) \le v(n_+)$, and recall the notation $q^y_{r,\hat{m}}(\mathbf{i},\mathbf{z})$ from (4.38), which is

$$q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) := q^{z_1}_{i_1} \dots \widehat{q^{z_m - z_{m-1}}_{i_m - i_{m-1}}} \dots q^{y-z_r}_{n-i_r},$$

for $\mathbf{i} \in I_1(n,r,m)$ and $\mathbf{z} \in (\mathbb{Z}^d)^r$. Making the change of summation indices $r := r+s$ and $m := r+1$ in (4.59), the expression in (4.59) can be written as

$$\begin{aligned}(T^t_{0,0} - 1)(T^{y,t}_0 - 1) = &\sum_{2 \le r \le v(n_-)+2} \sum_{m=2}^{r} \sum_{\mathbf{i} \in V_{r,m},\mathbf{z}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \\ &+ \sum_{v(n_-)+2 < r \le v(n_+)+2} \sum_{2 \le m \le v(n_-)+2} \sum_{\mathbf{i} \in V_{r,m},\mathbf{z}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}) \\ &+ \sum_{\substack{v(n_+)+2<r, \\ r \le v(n_-)+v(n_+)+2}} \sum_{\substack{r - v(n_+) \le m, \\ m \le v(n_-)+2}} \sum_{\mathbf{i} \in V_{r,m},\mathbf{z}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}).\end{aligned} \tag{4.60}$$

Conditional on $n_-, n_+ \in J(t^{\xi_1})$, $n_\bullet \in J(t - 2t^{\xi_1})$, and $n_- \le n_+$, and provided that t is sufficiently large, (4.60) allows us to rewrite $f^{y,t}_2$ as $f^{y,t}_{2;1} + f^{y,t}_{2;2} + f^{y,t}_{2;3}$, where

$$\begin{aligned} f^{y,t}_{2;1} &:= \sum_{r \in R^1} \sum_{m \in M^1_r} \sum_{\mathbf{i} \in W_{r,m},\mathbf{z}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}), \\ f^{y,t}_{2;2} &:= \sum_{r \in R^2} \left(\sum_{m=2}^{r} \sum_{\mathbf{i} \in I_1(n_t,r,m),\mathbf{z}} - \sum_{2 \le m \le v(n_-)+2} \sum_{\mathbf{i} \in V_{r,m},\mathbf{z}} \right) q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}), \\ f^{y,t}_{2;3} &:= \sum_{r \in R^3} \sum_{m \in M^3_r} \sum_{\mathbf{i} \in I_1(n,r,m),\mathbf{z}} q^y_{n,\hat{m}}(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}), \end{aligned}$$

where $R^1 := \{2, \dots, v(n_-)+2\}$, $R^2 := \{v(n_-)+3, \dots, v(n_-)+v(n_+)+2\}$, and $R^3 := \{v(n_-) + v(n_+)+3, \dots, k(n)\}$; and $M^1_r := \{2, \dots, r\}$ and $M^3_r := \{2, \dots, r\}$.

In order to prove (4.58), it is then enough to show existence of $\theta > 0$, for β small, such that for $i = 1, 2, 3$,

$$\lim_{t \to \infty} t^\theta \sup_{\|y\| \le t^\sigma} \frac{1}{p^y_t} \mathbf{E}\left[q^y_n \left\langle \left(f^{y,t}_{2;i} \right)^2 \right\rangle \mathbb{1}_{n_-,n_+ \in J(t^{\xi_1}), n_\bullet \in J(t-2t^{\xi_1}), n_- \le n_+} \right] = 0. \tag{4.61}$$

Let $i = 1, 2, 3$. Conditioning on the number of jumps in $[0, t^{\xi_1})$, $[t^{\xi_1}, t - t^{\xi_1})$, and $[t - t^{\xi_1}, t)$, equation

(4.50) allows us to write

$$\begin{aligned}&\mathbf{E}\left[q^y_n\left\langle\left(f^{y,t}_{2;i}\right)^2\right\rangle\mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet\in J(t-2t^{\xi_1}),n_-\le n_+}\right]\\&=\sum_{\substack{l_s\in S_s(t)\\ l_-\le l_+}}q^y_{l_-+l_\bullet+l_+}\mathbf{P}(l_-,l_\bullet,l_+)\mathbf{E}\left[\left\langle\left(f^{y,t}_{2;i}\right)^2\right\rangle\Big|l_-,l_\bullet,l_+\right]'\end{aligned}$$

where $S_-(t)=S_+(t)=J(t^{\xi_1})$ and $S_\bullet(t)=J(t-2t^{\xi_1})$.

In order to show (4.61) for $i=1,2$, we will show that the term $\mathbf{E}[\langle(f^{y,t}_{2;i})^2\rangle|l_-,l_\bullet,l_+]$ can be bounded buy a function that goes to 0 as $t\to\infty$ at least polynomially fast; then the limit in (4.61) is yielded by the fact that

$$\sum_{\substack{l_s\in S_s(t)\\ l_-\le l_+}}q^y_{l_-+l_\bullet+l_+}\mathbf{P}(l_-,l_\bullet,l_+)\le p^y_t. \tag{4.62}$$

Notice that $\mathbf{E}[\langle(f^{y,t}_{2;i})^2\rangle|l_-,l_\bullet,l_+]$ with $i=1,3$, can be expanded using (4.51) like so:

$$\sum_{r\in R}\sum_{m\in M^i_r}\sum_{\mathbf{i}\in H^i_{r,m},\mathbf{z}}q^y_{l_-+l_\bullet+l_+,\hat{m}}(\mathbf{i},\mathbf{z})^2\mathbf{E}\left[\prod_{j=1}^r\left(e^{\beta^2 t_{i_j+1}}-1\right)\Big|l_-,l_\bullet,l_+\right], \tag{4.63}$$

where $H^1_{r,m}=W_{r,m}$ and $H^3_{r,m}=I_1(n,r,m)$. Notice furthermore that $\mathbf{E}[\langle(f^{y,t}_{2;2})^2\rangle|l_-,l_\bullet,l_+]$ is bounded by (4.63) with $i=2$ and $M^2_r=\{2,\dots,r\}$ and $H^2_{r,m}=I_1(n,r,m)$.

Recall that σ is the coefficient associated with the family of sets $J(t),t>0$, and fix $\nu_1\in(\nu^{-1}-1,1)$. For $i=1,2$, for t sufficiently large, and for $l_-\le l_+$ in $J(t^{\xi_1})$, $l_\bullet\in J(t-2t^{\xi_1})$, we have

$$r+1\le v(l_-)+v(l_+)+3=\left\lceil\frac{l_-}{2}\right\rceil^{\xi_2}+\left\lceil\frac{l_+}{2}\right\rceil^{\xi_2}+3<\nu_1 l_-,\quad r\in R^i,$$

so in particular $r_s+1<\nu_1 l_s$ for $s\in\oplus$. Therefore, since $r=r_-+r_\bullet+r_+$, by Lemma 4.7,

$$A(t^{\xi_1},l_-,r_-+1)A(t-2t^{\xi_1},l_\bullet,r_\bullet+1)A(t^{\xi_1},l_+,r_++1)\lesssim\psi^{r_-+r_\bullet+r_+}=\psi^r. \tag{4.64}$$

Now, we take up cases $i=1,2,3$ separately.

CASE $i = 1$. Using the bound in (4.64) and taking $H_{r,m} = W_{r,m}$, the term in (4.51) is less than a constant times

$$\begin{aligned}&\sum_{r\in R^1}\sum_{m\in M_r^1}\sum_{\substack{\mathbf{j}=(j_1,\dots,j_r)\in\mathbb{N}^r,\\ j_m-j_{m-1}<n-w(l_-)-w(l_+)}}\sum_{\mathbf{z}}\left(q_{j_1}^{z_1}\right)^2\dots\left(q_{j_r}^{z_r}\right)^2\psi^r\\ \lesssim&\sum_{r\in R^1} r\sum_{\substack{j_1,\dots,j_{r-1}\in\mathbb{N},\\ j_1+\dots+j_{r-1}>\frac{1}{3}(w(l_-)+w(l_+))}}\sum_{\mathbf{z}}\left(q_{j_1}^{z_1}\right)^2\dots\left(q_{j_r}^{z_r}\right)^2\psi^r\\ \lesssim&\sum_{r=1}^{\infty} r^2(\alpha\psi)^r\sum_{j>\frac{w(l_-)+w(l_+)}{3(v(l_-)+2)}}\frac{1}{j^{\frac{d}{2}}}\lesssim\left(\frac{w(l_-)+w(l_+)}{v(l_-)}\right)^{1-\frac{d}{2}}.\end{aligned}$$

For the last estimate we assumed that β is so small that $\alpha\psi < 1$. Since

$$\frac{w(l_-)+w(l_+)}{v(l_-)} \gtrsim l_-^{\xi_3-\xi_2} \gtrsim t^{\xi_1(\xi_3-\xi_2)},$$

it follows that

$$\sup_{\|y\|\le t^\sigma}\frac{1}{p_t^y}\mathbf{E}\left[q_n^y\left\langle\left(f_{2;1}^{y,t}\right)^2\right\rangle\mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet\in J(t-2t^{\xi_1}),n_-\le n_+}\right]\lesssim t^{\xi_1(\xi_3-\xi_2)(1-\frac{d}{2})}.$$

This implies (4.61) for $i = 1$ and $\theta < \xi_1(\xi_3-\xi_2)(\frac{d}{2}-1)$.

CASE $i = 2$. Again, using (4.64), we bound (4.63) by

$$\begin{aligned}\sum_{r\in R^2} r\sum_{\mathbf{j}=(j_1,\dots,j_r)\in\mathbb{N}^r}\sum_{\mathbf{z}}\left(q_{j_1}^{z_1}\right)^2\dots\left(q_{j_r}^{z_r}\right)^2\psi^r\lesssim&\sum_{r>\frac{1}{3}v(l_-)} r(\alpha\psi)^r\\ \lesssim&(\alpha\psi)^{\frac{1}{3}v(l_-)}\frac{1}{3}v(l_-)\lesssim(\alpha\psi)^{Ct^{\xi_1\xi_2}},\end{aligned}$$

for some constant $C > 0$. In this estimate we assumed $\alpha\psi < 1$ and used that $l_- \in J(t^{\xi_1})$. From this estimate we deduce (4.61) for $i = 2$ and for any $\theta > 0$.

CASE $i = 3$. For t sufficiently large, $l_- \le l_+$ in $J(t^{\xi_1})$, $l_\bullet \in J(t-2t^{\xi_1})$, we have

$$r_\bullet + 1 \le k(l_- + l_\bullet + l_+) < \nu_1 l_\bullet, \quad r \in R^3.$$

However, it is not true in general that $r_- + 1 < \nu_1 l_-$ and $r_+ + 1 < \nu_1 l_+$. Thus, Lemma 4.7 only gives

$$\mathbf{E}\left[\prod_{j=1}^{r}\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| l_-, l_\bullet, l_+\right] \lesssim \psi^{r_\bullet} A(t^{\xi_1}, l_-, r_-+1) A(t^{\xi_1}, l_+, r_++1).$$

Consequently, for $i=3$, the expression in (4.63) is less than a constant times

$$\sum_{\substack{r_-, r_\bullet, r_+ \in \mathbb{N}_0, \\ r_-+r_\bullet+r_+ \in R^3}} (r_-+r_\bullet+r_+)(\alpha\psi)^{r_-+r_\bullet+r_+} \tag{4.65}$$

$$+ \sum_{\substack{r_- \le l_-,\ r_+ \le l_+,\ r_\bullet \in \mathbb{N}_0, \\ r_- \ge \nu_1 l_- \text{ or } r_+ \ge \nu_1 l_+}} (r_-+r_\bullet+r_+)(\alpha\psi)^{r_\bullet}\alpha^{r_-+r_+} A(t^{\xi_1}, l_-, r_-+1) A(t^{\xi_1}, l_+, r_++1). \tag{4.66}$$

Notice that the expression in (4.65) converges to 0 as $t \to \infty$ faster than any polynomial by the same argument as in the case $i = 2$. So the limit statement for this term is also yielded by (4.62). Therefore, it remains only to consider the expression in (4.66); that is, we need to show that for any $\theta > 0$,

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{\substack{l_-, l_+ \in J(t^{\xi_1}),\, l_\bullet \in J(t-2t^{\xi_1}) \\ l_- \le l_+}} \sum q^y_{l_-+l_\bullet+l_+} \mathbf{P}(l_-, l_\bullet, l_+) \cdot \tag{4.67}$$

$$\cdot \sum_{\substack{r_- \le l_-,\ r_+ \le l_+,\ r_\bullet \in \mathbb{N}_0, \\ r_- \ge \nu_1 l_- \text{ or } r_+ \ge \nu_1 l_+}} (r_-+r_\bullet+r_+)(\alpha\psi)^{r_\bullet}\alpha^{r_-+r_+} A(t^{\xi_1}, l_-, r_-+1) A(t^{\xi_1}, l_+, r_++1) = 0.$$

Recall from Chapter 2 that for $y \in \mathbb{Z}^d$ and $n \in \mathbb{N}_0$, we define

$$\iota(y,n) = \begin{cases} n, & \|y\|_1 \equiv n, \\ n+1, & \|y\|_1 \not\equiv n \end{cases}.$$

By Lemma 2.7, for t sufficiently large, $y \in \mathbb{Z}^d$ such that $\|y\| \le t^\sigma$, $l_-, l_+ \in J(t^{\xi_1})$, and $l_\bullet \in J(t-2t^{\xi_1})$,

$$q^y_{l_-+l_\bullet+l_+} \lesssim q^y_{\iota(y,l_\bullet)}.$$

Also, note that in (4.67) at least one of the conditions $r_- \ge \nu_1 l_-$ or $r_+ \ge \nu_1 l_+$ must be satisfied. Without loss of generality, assume $r_- \ge \nu_1 l_-$ (the other case is argued similarly). Therefore, since

$r_- + r_\bullet + r_+ \leq (r_- + 1)(r_\bullet + 1)(r_+ + 1)$, we can bound the expression in (4.67) from above by

$$\sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \, e^{-\beta^2 t^{\xi_1}} \sum_{l_\bullet \in J(t-2t^{\xi_1})} q^y_{\iota(y,l_\bullet)} \mathbf{P}(l_\bullet) \cdot \tag{4.68}$$

$$\sum_{r_\bullet \in \mathbb{N}_0} (r_\bullet + 1)(\alpha\psi)^{r_\bullet} \cdot \tag{4.69}$$

$$e^{\beta^2 t^{\xi_1}} \sum_{l_- \in J(t^{\xi_1})} \sum_{\nu_1 l_- \leq r_- \leq l_-} (r_- + 1)\alpha^{r_-} \mathbf{P}(l_-, l) A(t^{\xi_1}, l_-, r_- + 1) \cdot \tag{4.70}$$

$$\sum_{l_+ \in J(t^{\xi_1})} \sum_{0 \leq r_+ \leq l_+} (r_+ + 1)\alpha^{r_+} \mathbf{P}(l_+, l) A(t^{\xi_1}, l_+, r_+ + 1) \tag{4.71}$$

Here, note that we multiplied and divided by $e^{\beta^2 t^{\xi_1}}$, so that the term in (4.68) is finite in the limit by (A0) in Lemma 4.12; the term in (4.69) is bounded provided that $\alpha\psi < 1$. Finally, the term in (4.70) goes to 0 as $t \to \infty$ by (A3) in Lemma 4.12 and the term in (4.71) goes to 0 as $t \to \infty$ by (A7) in Lemma 4.12. □

Proof of Claim 4.3, Part 2. With regard to the convergence statement in (4.57), we first note that by the triangle inequality it is enough to show that for β sufficiently small and for any $\theta > 0$,

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \left\langle \mathbf{E}\left[q_n^y \left| F_2^{y,t} \right| \left(1 - \mathbb{1}_{n_-, n_+ \in J(t^{\xi_1}), n_\bullet \in J(t-2t^{\xi_1})}\right)\right]\right\rangle = 0, \tag{4.72}$$

and

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \left\langle \mathbf{E}\left[q_n^y \left| (T^t_{0,0} - 1)(T_0^{y,t} - 1) \right| \left(1 - \mathbb{1}_{n_-, n_+ \in J(t^{\xi_1}), n_\bullet \in J(t-2t^{\xi_1})}\right)\right]\right\rangle = 0. \tag{4.73}$$

We first show (4.72), which will follow from

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_n^y \left\langle \left(F_2^{y,t}\right)^2 \right\rangle \left(1 - \mathbb{1}_{n_-, n_+ \in J(t^{\xi_1}), n_\bullet \in J(t-2t^{\xi_1})}\right)\right] = 0 \tag{4.74}$$

for any $\theta > 0$. From (4.50), for given families of sets $S_-(t), S_\bullet(t), S_+(t) \subset \mathbb{N}_0$, we have

$$\mathbf{E}\left[q_n^y \left\langle \left(F_2^{y,t}\right)^2 \right\rangle \mathbb{1}_{n_s \in S_s(t), s \in \oplus}\right] = \sum_{l_s \in S_s(t), s\in\oplus} q^y_{l_- + l_\bullet + l_+} \mathbf{P}(l_-, l_\bullet, l_+) \mathbf{E}\left[\left\langle \left(F_2^{y,t}\right)^2 \right\rangle \middle| l_-, l_\bullet, l_+ \right].$$

Also, (4.51) and (4.52), imply that for $l_-, l_\bullet, l_+ \in \mathbb{N}_0$,

$$\begin{aligned}
&\mathbf{E}\left[\left\langle \left(F_2^{y,t}\right)^2 \right\rangle \Big| l_-, l_\bullet, l_+\right] \\
&\lesssim \sum_{0\le r_s\le l_s, s\in\oplus} (r_- + r_\bullet + r_+)\alpha^{r_- + r_\bullet + r_+} A(t^{\xi_1}, l_-, r_- + 1)A(t-2t^{\xi_1}, l_\bullet, r_\bullet + 1)A(t^{\xi_1}, l_+, r_+ + 1) \\
&\lesssim \sum_{0\le r_\bullet \le l_\bullet} (r_\bullet + 1)\alpha^{r_\bullet} A(t-2t^{\xi_1}, l_\bullet, r_\bullet + 1) \prod_{s\in\{-,+\}} \sum_{0\le r_s\le l_s} (r_s+1)\alpha^{r_s} A(t^{\xi_1}, l_s, r_s + 1).
\end{aligned}$$

By Lemma 2.10, for t sufficiently large $1/p_t^y \lesssim e^{t^\sigma}$ for $\|y\| \le t^\sigma$. Therefore,

$$\begin{aligned}
&\sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_n^y \left\langle \left(F_2^{y,t}\right)^2 \right\rangle \mathbb{1}_{n_s\in S_s(t), s\in\oplus}\right] \\
&\quad \lesssim e^{t^\sigma} \sum_{l\in S_\bullet(t)} \mathbf{P}^\bullet(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) \\
&\qquad \prod_{s\in\{-,+\}} \sum_{l_s\in S_s(t)} \mathbf{P}^s(l_s) \sum_{0\le r_s\le l_s} (r_s+1)\alpha^{r_s} A(t^{\xi_1}, l_s, r_s+1).
\end{aligned}$$

In order to complete the proof of (4.72), we will consider two main cases: (1) $S_\bullet(t)$ is the complement of $J(t-2t^{\xi_1})$ and (2) $S_\bullet(t)$ is $J(t-2t^{\xi_1})$.

Case 1. If $S_\bullet(t)$ is the complement of $J(t-2t^{\xi_1})$, then (A5) in Lemma 4.12 yields

$$\lim_{t\to\infty} t^\theta e^{t^\sigma} \sum_{l\in S_\bullet(t)} \mathbf{P}^\bullet(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) = 0, \quad \theta > 0.$$

Then either of two possibilities occur:

If either S_- or S_+ is the complement of $J(t^{\xi_1})$, then (A6) in Lemma 4.12 implies

$$\lim_{t\to\infty} \sum_{l\notin J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) = 0.$$

If either S_- or S_+ is of $J(t^{\xi_1})$, (A7) implies

$$\lim_{t\to\infty} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) < \infty.$$

Therefore,

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \mathbf{E}\left[q_n^y \left\langle \left(F_2^{y,t}\right)^2 \right\rangle \mathbb{1}_{n_\bullet \notin J(t-2t^{\xi_1})}\right] = 0, \quad \theta > 0.$$

Case 2. If $S_\bullet = J(t-2t^{\xi_1})$, then by Lemma 2.8, for $\|y\| \le t^\sigma$, $l_\bullet \in S_\bullet(t)$, and $l_-, l_+ \in \mathbb{N}_0$, we have

$$q^y_{l_-+l_\bullet+l_+} \lesssim \prod_{s\in\{-,+\}} \exp\left(Ct^{\sigma-1}l_s\right) q^y_{\iota(y,l_\bullet)}. \tag{4.75}$$

Therefore,

$$\begin{aligned}
&\sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \sum_{l_s\in S_s(t), s\in\{-,+\}} \sum_{l_\bullet\in J(t-2t^{\xi_1})} q^y_{l_-+l_\bullet+l_+} \mathbf{P}(l_-, l_\bullet, l_+) \\
&\sum_{0\le r\le l_\bullet} (r+1)\alpha^r A(t-2t^{\xi_1}, l_\bullet, r+1) \prod_{s\in\{-,+\}} \sum_{0\le r_s\le l_s} (r_s+1)\alpha^{r_s} A(t^{\xi_1}, l_s, r_s+1) \\
\lesssim & e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) \qquad (4.76)\\
&e^{\beta^2 t^{\xi_1}} \prod_{s\in\{-,+\}} \sum_{l_s\in S_s(t)} \exp\left(Ct^{\sigma-1}l_s\right) \mathbf{P}^s(l_s) \sum_{0\le r_s\le l_s} (r_s+1)\alpha^{r_s} A(t^{\xi_1}, l_s, r_s+1). \qquad (4.77)
\end{aligned}$$

Notice that the limit as $t \to \infty$ of the term in (4.76) if finite by (A1) in Lemma 4.12. To deal with the term in (4.77), first note that at least one of S_- or S_+ must be the complement of $J(t^{\xi_1})$; without loss of generality, assume it is $S_-(t)$, in which case $S_+(t)$ is allowed to be $J(t^{\xi_1})$ or its complement. Then, (A4) in Lemma 4.12 implies that the factor in (4.77) corresponding to $s = -$ satisfies

$$\lim_{t\to\infty} t^\theta e^{\beta^2 t^{\xi_1}} \sum_{l\notin J(t^{\xi_1})} e^{ct^{\sigma-1}l} \mathbf{P}^-(l) \sum_{0\le r\le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) = 0.$$

If $S_+(t)$ is the complement of $J(t^{\xi_1})$, then the factor in (4.77) corresponding to $s = +$ also satisfies the above. However, if $S_+(t) = J(t^{\xi_1})$, then $\exp(Ct^{\sigma-1}l_+) \le \exp(C't^{\sigma-1+\xi_1})$ which remain bounded because $\sigma - 1 + \xi_1 < 0$. Finally, (A7) implies

$$\limsup_{t\to\infty} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r<l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) < \infty.$$

Therefore, we also have

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \mathbf{E}\left[q^y_n \left\langle \left(F^{y,t}_2\right)^2 \right\rangle \mathbb{1}_{n_\bullet\in J(t-2t^{\xi_1})} \left(1 - \mathbb{1}_{n_-,n_+\in J(t^{\xi_1})}\right)\right] = 0, \quad \theta > 0.$$

This completes the proof of (4.72).

For the convergence statement in (4.73), we only need to note that by the Cauchy–Schwarz inequality, Fubini Theorem, and symmetry,

$$\begin{aligned}&\left\langle \mathbf{E}\left[q_n^y \left|(T_{0,0}^t-1)(T_0^{y,t}-1)\right| \left(1-\mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet\in J(t-2t^{\xi_1})}\right)\right]\right\rangle\\ \leq&\mathbf{E}\left[q_n^y \left\langle (T_{0,0}^t-1)^2\right\rangle \left(1-\mathbb{1}_{n_-,n_+\in J(t^{\xi_1}),n_\bullet\in J(t-2t^{\xi_1})}\right)\right].\end{aligned}$$

The rest of the proof can be carried out in full analogy to the proof of (4.74). □

4.4.2.2. Proof of Lemma 4.13 Parts 2 and 3: Convergence for one huge gap at the start or the end

We only show the convergence statement in (4.55) as the proof of (4.54) is analogous. We write

$$F_3^{y,t}-T_{0,0}^t+1=f_{3;1}^t+f_{3;2}^t$$

where for $i=1,2$,

$$f_{3;i}^t:=\sum_{r\in R^i}\sum_{\mathbf{i}\in H_{r,n}^i,\mathbf{z}} q_r(\mathbf{i},\mathbf{z})\prod_{j=1}^r h(z_j;s_{i_j},s_{i_j+1}) \tag{4.78}$$

where $q_r(\mathbf{i},\mathbf{z}):=q^y_{n,\widehat{r+1}}=q_{i_1}^{z_1}\dots q_{i_r-i_{r-1}}^{z_r-z_{r-1}}$, and

$$R^1:=\{1,\dots,v(n_-)+1\},\qquad R^2:=\{v(n_-)+1,\dots k(n)\},$$

$$H_{r,n}^1:=\left\{\mathbf{i}=(i_1,\dots,i_r)\ :\ \begin{matrix}0\leq i_1<\ldots<i_r\leq rn^\xi\\ i_r>w(n_-)\end{matrix}\right\},\quad\text{and}\quad H_{r,n}^2:=I_1(n,r,r+1).$$

By Jensen's inequality, it is enough to prove the following claim.

Claim 4.4

For $\beta>0$ sufficiently small, there is $\theta>0$ such that for $i\in\{1,2\}$,

$$\lim_{t\to\infty}t^\theta\sup_{\|y\|\leq t^\sigma}\frac{1}{p_t^y}\mathbf{E}\left[q_n^y\left\langle (f_{3;i}^t)^2\right\rangle \mathbb{1}_{n_-\in J(t^{\xi_1}),\ n_\bullet+n_+\in J(t-t^{\xi_1})}\right]=0, \tag{4.79}$$

$$\lim_{t\to\infty}t^\theta\sup_{\|y\|\leq t^\sigma}\frac{1}{p_t^y}\mathbf{E}\left[q_n^y\left\langle (f_{3;i}^t)^2\right\rangle \left(1-\mathbb{1}_{n_-\in J(t^{\xi_1}),n_\bullet+n_+\in J(t-t^{\xi_1})}\right)\right]=0. \tag{4.80}$$

Let $i \in \{1,2\}$. Conditioning on the number of jumps in $[0, t^{\xi_1})$ and $[t^{\xi_1}, t)$, we write

$$\begin{aligned}
&\mathbf{E}\left[q_n^y \left\langle \left(f_{3;i}^t\right)^2 \right\rangle \mathbb{1}_{n_- \in J(t^{\xi_1}), n_\bullet + n_+ \in J(t-t^{\xi_1})}\right] \\
&= \sum_{l_- \in J(t^{\xi_1})} \sum_{l_\oplus \in J(t-t^{\xi_1})} q_{l_- + l_\oplus}^y \, \mathbf{P}^{-,(\bullet,+)}(l_-, l_\oplus) \, \mathbf{E}^{-,(\bullet,+)}\left[\left\langle \left(f_{3;i}^t\right)^2 \right\rangle | l_-, l_\oplus\right].
\end{aligned} \tag{4.81}$$

For $l_-, l_\oplus \in \mathbb{N}_0$,

$$\mathbf{E}^{-,(\bullet,+)}\left[\left\langle \left(f_{3;i}^t\right)^2 \right\rangle | l_-, l_\oplus\right] = \sum_{r \in R^i} \sum_{\mathbf{i} \in I_{r,\mathbf{z}}^i} q_r(\mathbf{i}, \mathbf{z})^2 \mathbf{E}^{-,(\bullet,+)}\left[\prod_{j=1}^r \left(e^{\beta^2 t_{i_j+1}} - 1\right) \Big| l_-, l_\oplus\right], \tag{4.82}$$

and for $r \in R^i$, and $\mathbf{i} \in H_{r,n}^i$, let

$$r_- = |\{1 \le j \le r : i_j < l_-\}|, \quad r_\oplus = r - r_-.$$

Observe that

$$\mathbf{E}^{-,(\bullet,+)}\left[\prod_{j=1}^r \left(e^{\beta^2 t_{i_j+1}} - 1\right) \Big| l_-, l_\oplus\right] \le A(t^{\xi_1}, l_-, r_- + 1) A(t - t^{\xi_1}, l_\oplus, r_\oplus + 1), \tag{4.83}$$

where A was defined in (4.47).

Proof of Claim 4.4, Part 1. We will consider the cases $i = 1$ and $i = 2$ separately.

Case $i = 1$. In order to show (4.79) for $i = 1$, we will show that the term $\mathbf{E}[\langle (f_{3;i}^t)^2 \rangle | l_-, l_\oplus]$ can be bounded but a function that goes to 0 as $t \to \infty$ at least polynomially fast; then the fact that

$$\sum_{\substack{l_s \in S_s(t) \\ l_- \le l_+}} q_{l_- + l_\bullet + l_+}^y \mathbf{P}(l_-, l_\bullet, l_+) \le p_t^y \tag{4.84}$$

yields the limit in (4.79).

For t sufficiently large, $l_- \in J(t^{\xi_1}), l_\oplus \in J(t - t^{\xi_1})$, we have for any $r \in R^1$

$$r + 1 \le v(l_-) + 2 < \nu_1 l_-,$$

so in particular $r_s + 1 < \nu_1 l_s$ for $s \in \{-, \oplus\}$. By Lemma 4.7, the expression on the right side of (4.83) is less than a constant times

$$\psi^{r_- + r_\oplus} = \psi^r.$$

As a result, for $i = 1$, the expression in (4.82) is less than a constant times

$$\begin{aligned}
\sum_{r \in R^1} \sum_{\mathbf{i} \in H^1_{r,m}, \mathbf{z}} \psi^r q_r(\mathbf{i}, \mathbf{z})^2 &\lesssim \sum_{1 \le r \le v(n_-)+1} \psi^r \sum_{\substack{j_1, \dots, j_r \in \mathbb{N}, \\ j_1 + \dots + j_r > w(l_-)}} \sum_{c_1, \dots, c_r \in \mathbb{Z}^d} \left(q^{c_1}_{j_1}\right)^2 \dots \left(q^{c_r}_{j_r}\right)^2 \\
&\lesssim \sum_{1 \le r \le v(n_-)+1} \psi^r \sum_{l=1}^{r} \sum_{\substack{j_1, \dots, j_r \in \mathbb{N}, \\ j_l \ge \frac{1}{2} \frac{w(l_-)}{v(l_-)}}} \sum_{k=1}^{r} \Bigg(\sum_{c_k \in \mathbb{Z}^d} \left(q^{c_k}_{j_k}\right)^2 \Bigg) \\
&\lesssim \sum_{1 \le r \le \infty} r(\alpha\psi)^r \sum_{j > \frac{1}{2} \frac{w(l_-)}{v(l_-)}} \frac{1}{j^{d/2}} \\
&\lesssim \left(\frac{w(l_-)}{v(l_-)} \right)^{1 - \frac{d}{2}}.
\end{aligned}$$

Since

$$\frac{w(l_-)}{v(l_-)} \gtrsim l_-^{\xi_3 - \xi_2} \gtrsim t^{\xi_1(\xi_3 - \xi_2)},$$

this implies (4.79) for $i = 1$ and $\theta < \xi_1(\xi_3 - \xi_2)(\frac{d}{2} - 1)$.

Case $i = 2$. For $l_- \in J(t^{\xi_1})$ and $l_\oplus \in J(t - t^{\xi_1})$,

$$r + 1 \le k(l_- + l_\oplus) + 1 < \nu_1 l_\oplus, \quad r \in R^2,$$

as long as t is sufficiently large. Then,

$$\mathbf{E}^{-,(\bullet,+)} \left[\prod_{j=1}^{r} \left(e^{\beta^2 t_{i_j+1}} - 1 \right) \Big| l_-, l_\oplus \right] \lesssim \psi^{r_\oplus} A(t^{\xi_1}, l_-, r_- + 1).$$

Consequently, for $i = 2$, the expression in (4.82) is less than a constant times

$$\sum_{\substack{0 \le r_- \le l_-, r_\oplus \ge 0, \\ v(l_-)+1 < r_- + r_\oplus}} (\alpha\psi)^{r_\oplus} A(t^{\xi_1}, l_-, r_- + 1)\alpha^{r_-}.$$

If $r_- + 1 < \nu_1 l_-$, we have as before $A(t^{\xi_1}, l_-, r_- + 1) \lesssim (\psi)^{r_-}$, so the term in (4.82) is less than a constant times

$$\sum_{r > (v(l_-)+1)/2} (\alpha\psi)^r \lesssim (\alpha\psi)^{(v(l_-)+1)/2}$$

For $l_- \in J(t^{\xi_1})$, we have that $v(l_-) \gtrsim l_-^{\xi_2} \gtrsim t^{\xi_1 \xi_2}$. Therefore, for $\alpha\psi < 1$, the estimate in (4.84) yields

$$\lim_{t\to\infty} t^\theta \sup_{\|y\| \le t^\sigma} \frac{1}{p_t^y} \sum_{\substack{l_- \in J(t^{\xi_1}) \\ l_\oplus \in J(t - t^{\xi_1})}} q^y_{l_- + l_\oplus} \mathbf{P}^{-,(\bullet,+)}(l_-, l_\oplus) \sum_{\substack{0 \le r_- < \nu_1 l_- \\ r_- + r_\oplus \in R^2}} (\alpha\psi)^{r_- + r_\oplus} = 0.$$

To complete the proof of (4.79) for $i = 2$, it only remains to consider the case $r_- + 1 \ge \nu_1 l_-$.

we now show that for any $\theta > 0$,

$$\lim_{t\to\infty} t^\theta \sup_{\|y\| \le t^\sigma} \frac{1}{p_t^y} \sum_{l_- \in J(t^{\xi_1})} \sum_{l_\oplus \in J(t - t^{\xi_1})} q^y_{l_- + l_\oplus} \mathbf{P}^{-,(\bullet,+)}(l_-, l_\oplus) \sum_{\nu_1 l_- - 1 \le r_- \le l_-, r_\oplus \in \mathbb{N}_0} (\alpha\psi)^{r_\oplus} \alpha^{r_-} A(t^{\xi_1}, l_-, r_- + 1) = 0. \tag{4.85}$$

By Lemma 2.7, for t sufficiently large, $y \in \mathbb{Z}^d$ such that $\|y\| \le t^\sigma$, $l_- \in J(t^{\xi_1})$, and $l_\oplus \in J(t - t^{\xi_1})$, we have that $q^y_{l_- + l_\oplus} \lesssim q^y_{\iota(y, l_\oplus)}$. Therefore,

$$\begin{aligned}
&\sup_{\|y\| \le t^\sigma} \frac{1}{p_t^y} \sum_{l_- \in J(t^{\xi_1})} \sum_{l_\oplus \in J(t - t^{\xi_1})} q^y_{l_- + l_\oplus} \mathbf{P}^{-,(\bullet,+)}(l_-, l_\oplus) \\
&\quad \sum_{r \in \mathbb{N}_0} (\alpha\psi)^r \sum_{\nu_1 l_- - 1 \le r_- \le l_-} (r_- + 1) \alpha^{r_-} A(t^{\xi_1}, l_-, r_- + 1) \\
&\lesssim \sup_{\|y\| \le t^\sigma} \frac{1}{p_t^y} \sum_{l_\oplus \in J(t - t^{\xi_1})} q^y_{\iota(y, l_\oplus)} \mathbf{P}^{(\bullet,+)}(l_\oplus) \\
&\quad \sum_{l_- \in J(t^{\xi_1})} \mathbf{P}^-(l_-) \sum_{\nu_1 l_- - 1 \le r_- \le l_-} \alpha^{r_-} A(t^{\xi_1}, l_-, r_- + 1).
\end{aligned}$$

and convergence to 0 as $t \to \infty$ follows from (A0) and (A3) in Lemma 4.12. □

Proof of Claim 4.4, Part 2. Notice that by (4.82) and (4.83), for $l_-, l_\oplus \in \mathbb{N}_0$, we have

$$\mathbf{E}^{-,(\bullet,+)}\left[\left\langle\left(f^t_{3;i}\right)^2\right\rangle\Big| l_-, l_\oplus\right] \lesssim \sum_{0\le r_-\le l_-} \alpha^{r_-} A(t^{\xi_1}, l_-, r_- + 1) \sum_{0\le r_\oplus \le l_\oplus} \alpha^{r_\oplus} A(t-t^{\xi_1}, l_\oplus, r_\oplus + 1).$$

Therefore, since Lemma 2.10 implies that $1/p^y_t \lesssim e^{t^\sigma}$ for $\|y\| \le t^\sigma$, using the expansion in (4.81), we obtain the following bound

$$\begin{aligned}&\sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t}\mathbf{E}\left[q^y_n \left\langle\left(f^t_{3;i}\right)^2\right\rangle \mathbb{1}_{n_s\in S_s(t), s\in\{-,\oplus\}}\right]\\ \lesssim & e^{t^\sigma} \sum_{l_-\in S_-(t)} \mathbf{P}^-(l_-) \sum_{0\le r_-\le l_-} \alpha^{r_-} A(t^{\xi_1}, l_-, r_- + 1)\\ & \sum_{l_\oplus\in S_\oplus(t)} \mathbf{P}^{(\bullet,+)}(l_\oplus) \sum_{0\le r_\oplus\le l_\oplus} \alpha^{r_\oplus} A(t-t^{\xi_1}, l_\oplus, r_\oplus+1).\end{aligned}$$

We now consider two cases: (1) $S_\oplus(t)$ is the complement of $J(t-t^{\xi_1})$ and (2) $S_\oplus(t)$ is $J(t-t^{\xi_1})$.

Case 1. If $S_\oplus(t)$ is the complement of $J(t-t^{\xi_1})$, (A5) in Lemma 4.12 yields

$$\lim_{t\to\infty} t^\theta e^{t^\sigma} \sum_{l\in S_\oplus(t)} \mathbf{P}^{(\bullet,+)}(l) \sum_{0\le r\le l} \alpha^r A(t-t^{\xi_1}, l, r+1) = 0, \quad \theta > 0,$$

Notice that $S_-(t)$ is allowed to be either $J(t^{\xi_1})$ or its complement. If $S_-(t)$ is the complement of $J(t^{\xi_1})$, (A6) in Lemma 4.12 yields

$$\lim_{t\to\infty} \sum_{l\notin J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r\le l} \alpha^r A(t^{\xi_1}, l, r+1) = 0.$$

If on the contrary, $S_-(t) = J(t^{\xi_1})$, by (A7) in Lemma 4.12 we have

$$\limsup_{t\to\infty} \sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{0\le r\le l} \alpha^r A(t^{\xi_1}, l, r+1) < \infty.$$

Therefore, this establishes for $i \in \{1,2\}$ and any $\theta > 0$:

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t}\mathbf{E}\left[q^y_n \left\langle\left(f^t_{3;i}\right)^2\right\rangle \mathbb{1}_{n_\bullet + n_+ \notin J(t-t^{\xi_1})}\right] = 0.$$

Case 2. If $S_\oplus(t)$ is $J(t-t^{\xi_1})$, then $S_-(t)$ must be the complement of $J(t^{\xi_1})$. Also, notice that by Lemma 2.8, for $\|y\| \le t^\sigma$, $l_\oplus \in J(t-t^{\xi_1})$, and $l_- \in \mathbb{N}_0$, we have

$$q^y_{l_-+l_\oplus} \lesssim \exp\left(Ct^{\sigma-1}l_-\right) q^y_{\iota(y,l_\oplus)}.$$

Hence,

$$\begin{aligned}
&\sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \sum_{l_\oplus \in J(t-t^{\xi_1})} \sum_{l_- \in S_-(t)} q^y_{l_-+l_\oplus} \mathbf{P}^{-,(\bullet,+)}(l_-,l_\oplus) \\
&\sum_{0\le r_\oplus \le l_\oplus} \alpha^{r_\oplus} A(t-t^{\xi_1}, l_\oplus, r_\oplus+1) \sum_{0\le r_-\le l_-} \alpha^{r_-} A(t^{\xi_1}, l_-, r_-+1) \\
\lesssim & e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \sum_{l_\oplus\in J(t-t^{\xi_1})} q^y_{\iota(y,l_\oplus)} \mathbf{P}^\oplus(l_\oplus) \sum_{0\le r_\oplus\le l_\oplus} \alpha^{r_\oplus} A(t-t^{\xi_1}, l_\oplus, r_\oplus+1) \\
& e^{\beta^2 t^{\xi_1}} \sum_{l_-\notin J(t^{\xi_1})} \exp\left(Ct^{\sigma-1}l_-\right) \mathbf{P}^-(l_-) \sum_{0\le r_-\le l_-} \alpha^{r_-} A(t^{\xi_1}, l_-, r_-+1).
\end{aligned}$$

Then, by (A1) and (A4) in Lemma 4.12 , we have for $i \in \{1,2\}$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \mathbf{E}\left[q^y_n \left\langle \left(f^t_{3;i}\right)^2 \right\rangle \mathbb{1}_{n_\bullet+n_+\in J(t-t^{\xi_1}), n_-\notin J(t^{\xi_1})}\right] = 0, \quad \theta > 0.$$

This completes the proof of (4.80). □

4.4.3. Proof of Lemma 4.11 Part (b): Convergence to limiting partition functions

Recall that we write n for n_t, n_- for $n_{t^{\xi_1}}$, $n_\bullet$ for $n_{t^{\xi_1},t-t^{\xi_1}}$, and n_+ for $n_{t-t^{\xi_1},t}$. We also maintain the notational shorthands $\mathbf{P}^-$ $\mathbf{P}^\bullet$, etc., introduced at the beginning of Subsection 4.4.2.2.

Lemma 4.14

For β sufficiently small there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \left\langle \left| \mathbf{E}\left[\left(q^y_n - q^y_{n_\bullet}\right) T^{y,t}_0 T^t_{0,0}\right]\right| \right\rangle = 0.$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \left\langle \left| \mathbf{E}\left[q^y_{n_\bullet} T^{y,t}_0 T^t_{0,0}\right] - p^y_t Z^{y,t}_{-\infty} Z^\infty_{0,0} \right| \right\rangle = 0.$$

4.4.3.1. Proof of Lemma 4.14, Part 1

Let $\chi \in (0, \frac{1}{2}(1-\sigma))$ and $\nu \in (\frac{1}{2}, 1)$. For $t > 0$ and $y \in \mathbb{Z}^d$, let $\chi_1(t)$ be the smallest even integer $\geq t(1-t^{-\chi})$, and let $\chi_2(t)$ be the largest odd integer $\leq t(1+t^{-\chi})$. Let

$$K(t) = \{l \in \mathbb{N} : \chi_1(t) \leq l \leq \chi_2(t)\}, \quad t > 0.$$

We will show that there is $\theta > 0$ such that the following two limits are zero:

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[\left(q_n^y - q_{n_\bullet}^y\right) T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_- + n_+ \in J(2t^{\xi_1}), n_\bullet \in K(t-2t^{\xi_1})} \right] \right| \right\rangle = 0, \tag{4.86}$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[\left(q_n^y - q_{n_\bullet}^y\right) T_0^{y,t} T_{0,0}^t \left(1 - \mathbb{1}_{n_- + n_+ \in J(2t^{\xi_1}), n_\bullet \in K(t-2t^{\xi_1})}\right) \right] \right| \right\rangle = 0. \tag{4.87}$$

Let us first show (4.86). If t is sufficiently large, a point $y \in \mathbb{Z}^d$ such that $\|y\| \leq t^\sigma$ can be connected to the origin by a path of length less than $(t-2t^{\xi_1})(1-(t-2t^{\xi_1})^{-\chi})$. For such t, for $l_\pm \in J(2t^{\xi_1})$, and $l_\bullet \in K(t-2t^{\xi_1})$, it follows that $q_{l_\bullet}^y > 0$ if $l_\bullet \equiv \|y\|_1$, $q_{l_\bullet}^y = 0$ if $l_\bullet \not\equiv \|y\|_1$, $q_{l_\bullet + l_\pm}^y > 0$ if $l_\bullet + l_\pm \equiv \|y\|_1$, and $q_{l_\bullet + l_\pm}^y = 0$ if $l_\bullet + l_\pm \not\equiv \|y\|_1$. Thus,

$$\mathbf{E}\left[(q_n^y - q_{n_\bullet}^y) T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_- + n_+ \in J(2t^{\xi_1}), n_\bullet \in K(t-2t^{\xi_1})} \right] = A(y,t) + B(y,t), \tag{4.88}$$

where

$$A(y,t) := \sum_{\substack{l_\pm \in J(2t^{\xi_1}), \\ l_\pm \equiv 0}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}), \\ l_\bullet \equiv \|y\|_1}} \left(q_{l_\bullet + l_\pm}^y - q_{l_\bullet}^y \right) \mathbf{E}[T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet = l_\bullet, n_- + n_+ = l_\pm}],$$

$$B(y,t) := \sum_{\substack{l_\pm \in J(2t^{\xi_1}), \\ l_\pm \equiv 1}} \sum_{l_\bullet \in K(t-2t^{\xi_1})} \left(\mathbb{1}_{l_\bullet \not\equiv \|y\|_1} q_{l_\bullet + l_\pm}^y - \mathbb{1}_{l_\bullet \equiv \|y\|_1} q_{l_\bullet}^y \right) \mathbf{E}[T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet = l_\bullet, n_- + n_+ = l_\pm}].$$

We will show that there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \langle |A(y,t)| \rangle = 0, \tag{4.89}$$

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\leq t^\sigma} \frac{1}{p_t^y} \langle |B(y,t)| \rangle = 0. \tag{4.90}$$

To do so, we first establish an upper bound for a factor appearing in both $A(y,t)$ and $B(y,t)$:

Claim 4.5

$$\mathbf{E}[T_0^{y,t}T_{0,0}^t \mathbb{1}_{n_\bullet=l_\bullet,n_-+n_+=l_\pm}] \lesssim \sum_{m=0}^{l_\pm} \mathbf{P}(m,l_\bullet,l_\pm-m)\left(1+\sum_{1\le r\le m+1} A(t^{\xi_1},m,r)\alpha^r\right). \tag{4.91}$$

Proof. Since $T_0^{y,t}T_{0,0}^t$ and $n_\bullet$ are independent, we have that

$$\mathbf{E}[T_0^{y,t}T_{0,0}^t \mathbb{1}_{n_\bullet=l_\bullet,n_-+n_+=l_\pm}] = \mathbf{P}^{\bullet,(-,+)}(l_\bullet,l_\pm)\mathbf{E}^{(-,+)}\left[\left\langle\left|T_0^{y,t}T_{0,0}^t\right|\right\rangle |l_\pm\right].$$

By the Cauchy–Schwarz inequality and symmetry,

$$\mathbf{E}^{(-,+)}\left[\left\langle\left|T_0^{y,t}T_{0,0}^t\right|\right\rangle |l_\pm\right] \le \mathbf{E}^{(-,+)}\left[\left\langle(T_{0,0}^t)^2\right\rangle |l_\pm\right].$$

If we also condition on n_-, this implies that the right side of (4.92) is bounded from above by

$$\sum_{m=0}^{l_\pm} \mathbf{P}(m,l_\bullet,l_\pm-m)\mathbf{E}^-\left[\left\langle(T_{0,0}^t)^2\right\rangle |m\right]. \tag{4.92}$$

Now, for a fixed m, we have

$$\mathbf{E}^-[\langle(T_{0,0}^t)^2\rangle|m] \le 2+2\sum_{r\in R}\sum_{\mathbf{i}\in H_r,\mathbf{z}} q_r(\mathbf{i},\mathbf{z})^2\mathbf{E}^-\left[\prod_{j=1}^r\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| m\right], \tag{4.93}$$

where

$$\begin{aligned} R &:= \{1,\dots,v(m)+1\}, \\ H_r &:= \{\mathbf{i}=(i_1,\dots,i_r)\in\mathbb{N}_0^r : 0\le i_1<\dots<i_r\le w(m)\}, \\ q_r(\mathbf{i},\mathbf{z}) &= q_{i_1}^{z_1}\dots q_{i_r-i_{r-1}}^{z_r-z_{r-1}}. \end{aligned}$$

Since

$$\mathbf{E}^-\left[\prod_{j=1}^r\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| m\right] \le A(t^{\xi_1},m,r),$$

we have

$$\sum_{r\in R}\sum_{\mathbf{i}\in I_r,\mathbf{z}} q_r(\mathbf{i},\mathbf{z})^2\mathbf{E}^-\left[\prod_{j=1}^r\left(e^{\beta^2 t_{i_j+1}}-1\right)\middle| m\right] \lesssim \sum_{1\le r\le m+1} A(t^{\xi_1},m,r)\alpha^r,$$

which proves the claim. ∎

Now we show (4.89). By Claim 4.5, we have that $A(y,t)$ is less than a constant times

$$\sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \left(q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet}\right) \sum_{m=0}^{l_\pm} \mathbf{P}(m, l_\bullet, l_\pm - m) \left(1 + \sum_{1\le r\le m+1} A(t^{\xi_1}, m, r)\alpha^r\right). \tag{4.94}$$

We first show that

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p^y_t} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \left|q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet}\right| \mathbf{P}(l_\bullet, l_\pm) = 0. \tag{4.95}$$

Using the fact that $q_{l_\bullet+l_\pm} > 0$, we can write

$$\left|q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet}\right| = q^y_{l_\bullet+l_\pm} \left|\frac{q^y_{l_\bullet}}{q^y_{l_\bullet+l_\pm}} - 1\right|.$$

But as

$$\sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} q^y_{l_\bullet+l_\pm} \mathbf{P}(l_\bullet, l_\pm) \le p^y_t,$$

in order to prove (4.95), it is therefore enough to show that there exists a function g such that $t^\theta g(t)$ converges to 0 as $t \to \infty$ for some $\theta > 0$ and which satisfies

$$\left|\frac{q^y_{l_\bullet}}{q^y_{l_\bullet+l_\pm}} - 1\right| \le g(t). \tag{4.96}$$

By Lemma 2.6, there is $c > 0$ such that for t sufficiently large, for $y \in \mathbb{Z}^d$ with $\|y\| \le t^\sigma$, and for $l_\bullet \in K(t - 2t^{\xi_1})$, $l_\pm \in J(2t^{\xi_1})$ with $q^y_{l_\bullet}, q^y_{l_\bullet+l_\pm} > 0$, we have

$$\frac{q^y_{l_\bullet}}{q^y_{l_\bullet+l_\pm}} - 1 \le \left(1 + O(t^{-\frac{2}{5}})\right) \exp(ct^{\sigma+\xi_1-1}) - 1$$

and

$$1 - \frac{q^y_{l_\bullet}}{q^y_{l_\bullet+l_\pm}} = 1 - \frac{1}{q^y_{l_\bullet+l_\pm}/q^y_{l_\bullet}} \le 1 - \left(1 + O(t^{-\frac{2}{5}})\right) \exp(-ct^{\sigma+\xi_1-1}).$$

Therefore, the inequality (4.96) can be satisfied by

$$g(t) := \left(1 + O(t^{-\frac{2}{5}})\right) \exp(ct^{\sigma+\xi_1-1}) - 1, \tag{4.97}$$

and $\lim_{t\to\infty} t^{\theta} g(t) = 0$ for $\theta \in (0, 1-\sigma-\xi_1)$.

Let us now show that for β sufficiently small, there is $\theta > 0$ such that for $S_-(t) = J(t^{\xi_1})$ and for $S_-(t)$ equal to the complement of $J(t^{\xi_1})$,

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{0\le m\le l_\pm,\\ m\in S_-(t)}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \left| q^y_{l_\bullet + l_\pm} - q^y_{l_\bullet} \right| \\ \mathbf{P}(m, l_\bullet, l_\pm - m) \sum_{1\le r\le m+1} A(t^{\xi_1}, m, r)\alpha^r = 0. \tag{4.98}$$

Notice that if $g(t)$ is as in (4.97), then for $l_\bullet \in K(t-2t^{\xi_1})$ such that $l_\bullet \equiv \|y\|_1$ and for $l_\pm \in J(2t^{\xi_1})$ such that $l_\pm \equiv 0$,

$$\left| q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet} \right| = q^y_{l_\bullet} \left| \frac{q^y_{l_\bullet+l_\pm}}{q^y_{l_\bullet}} - 1 \right| \le q^y_{l_\bullet} g(t).$$

As a result, if S_- is the complement of $J(t^{\xi_1})$, we obtain (4.98) from (A0) and (A4) from Lemma 4.12 as well as the following estimate:

$$\sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{0\le m\le l_\pm,\\ m\notin J(t^{\xi_1})}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \left| q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet} \right| \mathbf{P}(m, l_\bullet, l_\pm - m) \\ \sum_{r\in R} \sum_{\mathbf{i}\in I_r,\mathbf{z}} q_r(\mathbf{i},\mathbf{z})^2 \mathbf{E}^- \left[\prod_{j=1}^{r} \left(e^{\beta^2 t_{i_j+1}} - 1 \right) \middle| m \right] \\ \lesssim g(t) e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^{\bullet}(l) \\ e^{\beta^2 t^{\xi_1}} \sum_{m\notin J(t^{\xi_1})} \mathbf{P}^-(m) \sum_{1\le r\le m+1} \alpha^r A(t^{\xi_1}, m, r).$$

If S_- is $J(t^{\xi_1})$, consider two subcases: (1) $1 \le r < \nu_1 m$ and (2) $\nu_1 m \le r \le m+1$. In the first subcase, Lemma 4.7 gives:

$$\sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{0\le m\le l_\pm,\\ m\in J(t^{\xi_1})}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \\ \left| q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet} \right| \mathbf{P}(m, l_\bullet, l_\pm - m) \sum_{1\le r<\nu_1 m} A(t^{\xi_1}, m, r)\alpha^r \\ \lesssim \sup_{\|y\|\le t^{\sigma}} \frac{1}{p_t^y} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \left| q^y_{l_\bullet+l_\pm} - q^y_{l_\bullet} \right| \mathbf{P}^{\bullet,(-,+)}(l_\bullet, l_\pm).$$

The desired limit then follows from (4.95) for $\theta \in (0, 1-\sigma-\xi_1)$.

In the second subcase, we estimate as follows:

$$
\begin{aligned}
&\sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 0}} \sum_{\substack{0\le m\le l_\pm,\\ m\in J(t^{\xi_1})}} \sum_{\substack{l_\bullet \in K(t-2t^{\xi_1}),\\ l_\bullet \equiv \|y\|_1}} \\
&\left| q^y_{l_\bullet + l_\pm} - q^y_{l_\bullet} \right| \mathbf{P}(m, l_\bullet, l_\pm - m) \sum_{\nu_1 m \le r \le m+1} A(t^{\xi_1}, m, r)\alpha^r \\
\lesssim & g(t) e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \\
& e^{\beta^2 t^{\xi_1}} \sum_{m\in J(t^{\xi_1})} \mathbf{P}^-(m) \sum_{\nu_1 m\le r\le m+1} \alpha^r A(t^{\xi_1}, m, r).
\end{aligned}
$$

By (A0) and (A3), this tends to 0 as $t\to\infty$ faster than $t^{-\theta}$ for any $\theta > 0$. This completes the proof of (4.89).

Let us now show (4.90), which will imply (4.86). For convenience, we assume that $\|y\|_1 \equiv 0$. In the case $\|y\|_1 \equiv 1$, the proof proceeds analogously. We have

$$
B(y,t) = B^{(1)}(y,t) + B^{(2)}(y,t),
$$

where

$$
\begin{aligned}
B^{(1)}(y,t) &:= \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 1}} \sum_{p=\frac{1}{2}\chi_1(t-2t^{\xi_1})}^{\frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)} q^y_{2p+1+l_\pm} \mathbf{E}\left[T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_-+n_+=l_\pm} \left(\mathbb{1}_{n_\bullet = 2p+1} - \mathbb{1}_{n_\bullet = 2p} \right) \right], \\
B^{(2)}(y,t) &:= \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 1}} \sum_{p=\frac{1}{2}\chi_1(t-2t^{\xi_1})}^{\frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)} \left(q^y_{2p+1+l_\pm} - q^y_{2p} \right) \mathbf{E}\left[T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet = 2p, n_-+n_+=l_\pm} \right].
\end{aligned}
$$

Following the proof of (4.89), one shows

$$
\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| B^{(2)}(y,t) \right| \right\rangle = 0
$$

for $\theta \in (0, 1-\sigma-\xi_1)$. To establish (4.90), it remains to show that there is $\theta > 0$ such that

$$
\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| B^{(1)}(y,t) \right| \right\rangle = 0. \tag{4.99}
$$

By Claim 4.5, we have

$$\left\langle \left| B^{(1)}(y,t) \right| \right\rangle \leq \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 1}} \sum_{m=0}^{l_\pm} \sum_{p=\frac{1}{2}\chi_1(t-2t^{\xi_1})}^{\frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)} q^y_{2p+1+l_\pm} \left| \mathbf{P}^\bullet(2p+1) - \mathbf{P}^\bullet(2p) \right| \mathbf{P}^{-,+}(m, l_\pm - m) \left(1 + \sum_{1 \leq r \leq m+1} \alpha^r A(t^{\xi_1}, m, r) \right).$$

Since $n_\bullet$ is Poisson distributed with intensity $t - 2t^{\xi_1}$,

$$\mathbf{P}^\bullet(2p+1) - \mathbf{P}^\bullet(2p) = \mathbf{P}^\bullet(2p+1) \left(1 - \frac{2p+1}{t-2t^{\xi_1}} \right).$$

Therefore, for $\frac{1}{2}\chi_1(t-2t^{\xi_1}) \leq p \leq \frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)$, we have

$$|\mathbf{P}^\bullet(2p+1) - \mathbf{P}^\bullet(2p)| \leq \mathbf{P}^\bullet(2p+1)\hat{g}(t),$$

where $\hat{g}(t) = \left(t - 2t^{\xi_1}\right)^{-\sigma}$. Since

$$\sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 1}} \sum_{p=\frac{1}{2}\chi_1(t-2t^{\xi_1})}^{\frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)} q^y_{2p+1+l_\pm} \mathbf{P}^{\bullet,(-,+)}(2p+1, l_\pm) \leq p^y_t,$$

we have

$$\lim_{t\to\infty} t^\theta \hat{g}(t) \sup_{\|y\| \leq t^\sigma} \frac{1}{p^y_t} \sum_{\substack{l_\pm \in J(2t^{\xi_1}),\\ l_\pm \equiv 1}} \sum_{p=\frac{1}{2}\chi_1(t-2t^{\xi_1})}^{\frac{1}{2}(\chi_2(t-2t^{\xi_1})-1)} q^y_{2p+1+l_\pm} \mathbf{P}^{\bullet,(-,+)}(2p+1, l_\pm) = 0$$

for $\theta \in (0, \sigma)$. Since we also have that

$$q^y_{2p+1+l_\pm} \leq q^y_{2p+1}(1+g(t)) \lesssim q^y_{2p+1} = q_{\iota(y,2p+1)},$$

we can complete the proof of (4.99) by following the reasoning for (4.89).

Next, we show (4.87). Let $\|y\| \leq t^\sigma$, then by Lemma 2.10, we have that $1/p^y_t \leq e^{t^\sigma}$. For any $n, n_\bullet \in \mathbb{N}_0$, we also have that $|q^y_n - q_{n_\bullet}| \leq 2$. Therefore, using Claim 4.5, we have that for any given families of sets $S_\bullet(t), S_\pm(t) \subset \mathbb{N}_0$,

$$\begin{aligned}&\sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[\left(q_n^y - q_{n_\bullet}^y\right) T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet \in S_\bullet(t), n_- + n_+ \in S_\pm(t)} \right] \right| \right\rangle \\ \lesssim & e^{t^\sigma} \sum_{l_\bullet \in S_\bullet(t)} \mathbf{P}^\bullet(l_\bullet) \\ & \sum_{l_\pm \in S_\pm(t)} \sum_{m=0}^{l_\pm} \mathbf{P}^{-,+}(m, l_\pm - m) \left(1 + \sum_{1\le r\le m+1} \alpha^r A(t^{\xi_1}, m, r) \right).\end{aligned} \tag{4.100}$$

If $S_\bullet(t)$ is the complement of $K(t - 2t^{\xi_1})$, then part (A8) in Lemma 4.12, allows us to deal with the term in the second line of (4.100).

In addition, the expression in the third line of (4.100) is bounded from above by

$$1 + \sum_{m\in\mathbb{N}_0} \mathbf{P}^-(m) \sum_{1\le r\le m+1} \alpha^r A(t^{\xi_1}, m, r),$$

and (A7) and (A4) imply

$$\limsup_{t\to\infty} \sum_{m\in\mathbb{N}_0} \mathbf{P}^-(m) \sum_{1\le r\le m+1} \alpha^r A(t^{\xi_1}, m, r) < \infty.$$

From this, we may already infer

$$\lim_{t\to\infty} t^\theta \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[\left(q_n^y - q_{n_\bullet}^y\right) T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet \notin K(t-2t^{\xi_1})} \right] \right| \right\rangle = 0, \quad \theta > 0.$$

If $S_\bullet(t)$ is $K(t-2t^{\xi_1})$, then $S_\pm(t)$ must be the complement of $J(2t^{\xi_1})$. For $\|y\| \le t^\sigma$, $l_\bullet \in K(t-2t^{\xi_1}) \subset J(t - 2t^{\xi_1})$, and $l_\pm \in \mathbb{N}_0$, we have as in (4.75)

$$q_{l_\bullet + l_\pm}^y \lesssim \exp\left(C t^{\sigma-1} l_\pm\right) q_{\iota(y, l_\bullet)}^y.$$

Thus,

$$\sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \left\langle \left| \mathbf{E}\left[\left(q_n^y - q_{n_\bullet}^y\right) T_0^{y,t} T_{0,0}^t \mathbb{1}_{n_\bullet \in K(t-2t^{\xi_1}), n_- + n_+ \notin J(2t^{\xi_1})} \right] \right| \right\rangle \tag{4.101}$$

$$\lesssim e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in K(t-2t^{\xi_1})} q_{\iota(y,l)}^y \mathbf{P}^\bullet(l) \tag{4.102}$$

$$e^{\beta^2 t^{\xi_1}} \sum_{\substack{l_-,l_+\in\mathbb{N}_0,\\ l_-+l_+\notin J(2t^{\xi_1})}} \exp\left(Ct^{\sigma-1}(l_- + l_+)\right) \mathbf{P}^{-,+}(l_-,l_+) \sum_{1\le r\le l_-+1} \alpha^r A(t^{\xi_1}, l_-, r).$$

By (A1) of Lemma 4.12,

$$\limsup_{t\to\infty} e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in K(t-2t^{\xi_1})} q_{\iota(y,l)}^y \mathbf{P}^\bullet(l) < \infty.$$

If $l_-, l_+ \in \mathbb{N}_0$ such that $l_- + l_+ \notin J(2t^{\xi_1})$, we have $l_- \notin J(t^{\xi_1})$ or $l_+ \notin J(t^{\xi_1})$. Consequently, for any $\theta > 0$, the product of t^θ and the expression in the third line of (4.101) is less than a constant times

$$t^\theta e^{\beta^2 t^{\xi_1}} \sum_{l\notin J(t^{\xi_1})} \exp\left(Ct^{\sigma-1} l\right) \mathbf{P}^-(l) \sum_{1\le r\le l+1} \alpha^r A(t^{\xi_1}, l, r) \tag{4.103}$$

$$\left(\sum_{l\in J(t^{\xi_1})} \mathbf{P}^-(l) \sum_{1\le r\le l+1} \alpha^r A(t^{\xi_1}, l, r) \right.$$

$$\left. + \sum_{l\notin J(t^{\xi_1})} \exp\left(Ct^{\sigma-1} l\right) \mathbf{P}^-(l) \sum_{1\le r\le l+1} \alpha^r A(t^{\xi_1}, l, r) \right).$$

By (A4) in Lemma 4.12, as $t \to \infty$, the expression in the first line of (4.103) tends to 0 for any $\theta > 0$ and the expression in the third line of (4.103) tends to 0. The second line of (4.103) is bounded on account of (A7). This completes the proof of (4.87) and thus of Lemma 4.14 Part 1.

4.4.3.2. Proof of Lemma 4.14 Part 2

Since $q_{n_\bullet}^y$, $T_0^{y,t}$ and $T_{0,0}^t$ are independent with respect to $\mathbf{P}$, we have

$$\mathbf{E}\left[q_{n_\bullet}^y T_0^{y,t} T_{0,0}^t \right] = p_{t-2t^{\xi_1}}^y \mathbf{E} T_0^{y,t} \mathbf{E} T_{0,0}^t.$$

It is hence enough to show existence of $\theta > 0$ such that

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \frac{|p^y_{t-2t^{\xi_1}} - p^y_t|}{p^y_t} \left\langle \left| \mathbf{E}T^t_{0,0} \mathbf{E}T^{y,t}_0 \right| \right\rangle =0, \tag{4.104}$$

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \left\langle \left| \mathbf{E}T^t_{0,0} \mathbf{E}T^{y,t}_0 - Z^{\infty}_{0,0} Z^{y,t}_{-\infty} \right| \right\rangle =0. \tag{4.105}$$

Let us point out that there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^{\theta} \sup_{\|y\|\le t^{\sigma}} \frac{|p^y_{t-2t^{\xi_1}} - p^y_t|}{p^y_t} = 0.$$

Since $p^y_{t-2t^{\xi_1}} = \mathbf{E}q^y_{n_\bullet}$ and $p^y_t = \mathbf{E}q^y_n$, this can be shown similarly to (4.95). Hence, in order to prove (4.104), it is enough to show that

$$\limsup_{t\to\infty} \left\langle \left| \mathbf{E}T^t_{0,0} \mathbf{E}T^{y,t}_0 \right| \right\rangle < \infty.$$

By the Cauchy–Schwarz inequality and symmetry,

$$\left\langle \left| \mathbf{E}T^t_{0,0} \mathbf{E}T^{y,t}_0 \right| \right\rangle \le \left\langle \left(\mathbf{E}T^t_{0,0} \right)^2 \right\rangle .$$

Let us now show that the truncated partition function $\mathbf{E}T^t_{0,0}$ converges to the limiting partition function $Z^{\infty}_{0,0}$ in the L^2 sense and obtain a rate of convergence. In order to do this we will first prove that there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^{\theta} \left\langle \left(\mathbf{E}T^t_{0,0} - Z^{t^{\xi_1}}_{0,0} \right)^2 \right\rangle = 0. \tag{4.106}$$

We keep the notation introduced at the beginning of Subsection 4.4.2.1. Let $t > 0$. For $\mathbf{P}_{0,0}$-almost every realization of the continuous-time simple symmetric random walk η on $\mathbb{Z}^d$, we have

$$e^{-\frac{\beta^2}{2} t^{\xi_1}} e^{\beta \mathcal{A}^{t^{\xi_1}}_0} - T^t_{0,0} = N^{t^{\xi_1}}_1 + N^{t^{\xi_1}}_2 ,$$

where

$$N_1^{t^{\xi_1}} = \sum_{1\le r\le v(n_-)+1} \sum_{\substack{0\le i_1<\ldots<i_r\le n_-, i_r>w(n_-),\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1})$$

$$N_2^{t^{\xi_1}} = \sum_{v(n_-)+1<r\le n_-+1} \sum_{\substack{0\le i_1<\ldots<i_r\le n_-,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z}) \prod_{j=1}^{r} h(z_j; s_{i_j}, s_{i_j+1}).$$

By Jensen's inequality,

$$\left\langle \left(\mathbf{E}T^t_{0,0} - Z^{t^{\xi_1}}_{0,0}\right)^2 \right\rangle \le \mathbf{E}\left[\left\langle \left(N_1^{t^{\xi_1}} + N_2^{t^{\xi_1}}\right)^2 \right\rangle\right],$$

so it is enough to show existence of $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \mathbf{E}\left[\left\langle (N_i^t)^2 \right\rangle\right] = 0, \quad i \in \{1,2\},$$

i.e., we need to check convergence of the D-sequences

$$D_n^\vartheta(1) = \sum_{1\le r\le v(n)+1} \vartheta^r \sum_{\substack{0\le i_1<\ldots<i_r\le n, i_r>w(n),\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2,$$

$$D_n^\vartheta(2) = \sum_{v(n)+1<r\le n+1} \vartheta^r \sum_{\substack{0\le i_1<\ldots i_r\le n,\\ z_1,\ldots,z_r\in\mathbb{Z}^d}} q_r(\mathbf{i},\mathbf{z})^2.$$

For $\vartheta < \alpha^{-1}$,

$$D_n^\vartheta(2) \lesssim \sum_{v(n)+1<r\le n+1} (\vartheta\alpha)^r \le \frac{(\vartheta\alpha)^{n/2}}{1-\vartheta\alpha},$$

so

$$\lim_{n\to\infty} n^\theta D_n^\vartheta(2) = 0, \quad \theta > 0,\ \vartheta < \alpha^{-1}.$$

Moreover,

$$D_n^{\vartheta}(1) \lesssim \sum_{1\le r\le v(n)+1} \vartheta^r \sum_{\substack{j_1,\ldots,j_r\in\mathbb{N},\\ j_1+\ldots+j_r>w(n)}} \sum_{c_1,\ldots,c_r\in\mathbb{Z}^d} \left(q_{j_1}^{c_1}\right)^2 \ldots \left(q_{j_r}^{c_r}\right)^2$$

$$\le \sum_{1\le r\le v(n)+1} \vartheta^r \sum_{l=1}^{r} \sum_{\substack{j_1,\ldots,j_r\in\mathbb{N},\\ j_l\ge \frac{1}{2}\frac{w(n)}{v(n)}}} \prod_{k=1}^{r}\left(\sum_{c_k\in\mathbb{Z}^d}\left(q_{j_k}^{c_k}\right)^2\right)$$

$$\lesssim \sum_{1\le r\le v(n)+1} r(\vartheta\alpha)^r \sum_{j\ge \frac{1}{2}\frac{w(n)}{v(n)}} \frac{1}{j^{\frac{d}{2}}} \lesssim \sum_{r=1}^{\infty} r(\vartheta\alpha)^r \left(\frac{1}{2}\frac{w(n)}{v(n)} - 1\right)^{1-\frac{d}{2}},$$

and

$$\lim_{n\to\infty} n^{\theta}\left(\frac{1}{2}\frac{w(n)}{v(n)} - 1\right)^{1-\frac{d}{2}} = 0, \quad \theta\in(0,(\xi_3-\xi_2)(\tfrac{d}{2}-1)),$$

from which we deduce (4.106). If we combine this result with Theorem 3.1, we obtain in particular that there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^{\theta}\left\langle\left(\mathbf{E}T_{0,0}^{t} - Z_{0,0}^{\infty}\right)^2\right\rangle = 0. \tag{4.107}$$

Finally, since $\left\langle\left(Z_{0,0}^{\infty}\right)^2\right\rangle < \infty$, we obtain (4.104).

In order to prove (4.105), first notice that for $y\in\mathbb{Z}^d$ such that $\|y\|\le t^{\sigma}$, we have

$$\left\langle\left|\mathbf{E}T_{0,0}^{t}\mathbf{E}T_{0}^{y,t} - Z_{0,0}^{\infty}Z_{-\infty}^{y,t}\right|\right\rangle \le \left\langle\left|\mathbf{E}T_{0}^{y,t}\left(\mathbf{E}T_{0,0}^{t} - Z_{0,0}^{\infty}\right)\right|\right\rangle + \left\langle\left|Z_{0,0}^{\infty}\left(\mathbf{E}T_{0}^{y,t} - Z_{-\infty}^{y,t}\right)\right|\right\rangle.$$

Therefore, we obtain (4.105) by applying Cauchy–Schwarz to the two summands on the right, and using (4.107) together with

$$\lim_{t\to\infty}\left\langle\left(\mathbf{E}T_{0,0}^{t}\right)^2\right\rangle = \left\langle\left(Z_{0,0}^{\infty}\right)^2\right\rangle < \infty.$$

chapter 5

A Lower Tail of the Partition Function: Talagrand's Method

Contents

In this chapter, we prove the following theorem.

Theorem 5.1

For β sufficiently small, there exists a constant $c > 0$ such that

$$Q\left(Z_0^{y,t} < e^{-u}\right) < ce^{-u^2/c}, \quad t, u > 0.$$

This can be thought of as a continuous-time version of Theorem 1.5 in [CH02]. The Gaussianity of the noise is important here.

Remark 5.2

The result is interesting in its own right because it implies that the limiting partition function $Z_{0,0}^{\infty}$ admits all positive and negative moments.

This section is devoted to the proof of Theorem 5.1. The first step is to prove Theorem 5.3, which is a discrete-time version of Theorem 5.1; we do this in Section 5.1. Then, in Section 5.2, we use Theorem 5.3 to prove Theorem 5.1 by a suitable limiting procedure to go from discrete time to continuous time, showing that the estimate on the partition function carries over. To prove Theorem 5.3, we follow closely the strategy laid out in [CH02, Section 4], which goes back to Talagrand [AST03].

5.1. The Discrete-time Case

Fix a positive integer N, and let $\mathcal{S} = (S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}^d$ that starts at the origin and has transition probabilities

$$\mathbf{P}(S_{n+1} = y \,|S_n = x) := \begin{cases} \frac{N}{N+1} & \text{if} \quad y = x, \\ \frac{1}{2d(N+1)} & \text{if} \quad \|y - x\| = 1. \end{cases} \tag{5.1}$$

In addition, let $(\omega(z,k))_{z\in\mathbb{Z}^d, k\in\mathbb{N}_0}$ be an i.i.d. collection of Gaussian random variables with mean 0 and variance $\frac{1}{N}$ that is independent of $\mathcal{S}$. We denote the probability measure corresponding to $(\omega(z,k))_{z\in\mathbb{Z}^d, k\in\mathbb{N}_0}$ by Q^N and the expected value by $\langle\cdot\rangle_N$. For any $t > 0$ such that $tN \in \mathbb{N}$, we let Γ_{tN} denote the set of possible realizations of $(S_0, \ldots, S_{tN-1})$. Given a path $\gamma = (\gamma_0, \ldots, \gamma_{tN-1}) \in \Gamma_{tN}$, we let $p(\gamma)$ denote the probability that $(S_0, \ldots, S_{tN-1}) = \gamma$. Then, we define the partition function

$$Z_t^N := e^{-\frac{\beta^2}{2}t}\mathbf{E}\left[\exp\left(\beta \sum_{i=0}^{tN-1} \omega(S_i, i)\right)\right],$$

where $\beta > 0$ is a small parameter, and $\mathbf{E}$ denotes the expectation taken with respect to $\mathcal{S}$. Then we have:

$$\begin{aligned} Z_t^N =& e^{-\frac{\beta^2}{2}t} \sum_{\gamma\in\Gamma_{tN}} p(\gamma) \exp\left(\beta \sum_{i=0}^{tN-1} \omega(\gamma_i, i)\right) \\ =& \sum_{\gamma\in\Gamma_{tN}} p(\gamma) \prod_{i=0}^{tN-1} \left[\exp(\beta\omega(\gamma_i, i)) \exp\left(-\frac{\beta^2}{2N}\right)\right]. \end{aligned}$$

We now state and prove a discrete-time version of Theorem 5.1.

Theorem 5.3

For β sufficiently small, there exists a constant $c > 0$, independent of N, such that for all $t > 0$ with $tN \in \mathbb{N}$,

$$Q^N\left(Z_t^N < e^{-u}\right) \leq ce^{-u^2/c}, \qquad u > 0.$$

The proof of this theorem relies on three technical lemmas (Lemmas 5.4–5.8) following the approach in [CH02] and [AST03].

Let $\mathcal{S}^1$ and $\mathcal{S}^2$ be two independent copies of the random walk $\mathcal{S}$. Set

$$L_n := \sum_{i=0}^{n-1} \mathbb{1}_{\{S_i^1 = S_i^2\}}, \quad \text{and} \quad L_\infty := \sum_{i=0}^{\infty} \mathbb{1}_{\{S_i^1 = S_i^2\}}.$$

Since $\mathcal{S}$ is transient on $\mathbb{Z}^d$ for $d \geq 3$, we have $L_\infty < \infty$ almost surely. Also consider the random walk $\mathcal{D} := \mathcal{S}^1 - \mathcal{S}^2 = (D_n)_{n \geq 0}$. It is transient because $\mathcal{S}^1$ and $\mathcal{S}^2$ are. Let τ_1 denote the time of first return of the random walk $\mathcal{D}$ to 0, and set

$$q := P(\tau_1 < \infty).$$

Clearly, $q > 0$, and because of transience we also have $q < 1$. In the following lemma, we give more precise bounds on q.

Lemma 5.4

There are positive constants c_1 and c_2 with $c_1 < c_2$, depending only on the dimension d, such that

$$1 - \frac{c_2}{N} \leq q \leq 1 - \frac{c_1}{N}.$$

Remark 5.5

For the conclusion of Theorems 5.3 and 5.1 to hold it is enough to have $\beta < \inf_N \sqrt{N \ln(1/q)}$ (see also Theorem 1.5 in [CH02]). The infimum is strictly positive because $c_1 \leq \inf_N (N(1-q)) \leq \inf_N (N \ln(1/q))$, by virtue of Lemma 5.4.

Proof. For $\mathbf{t} \in \mathbb{R}$, let $\varphi(\mathbf{t})$ denote the characteristic function of D_1. By properties of characteristic functions (see e.g. [Dur10, p.194]),

$$\mathbf{P}_{\mathcal{S}^1 \otimes \mathcal{S}^2}(D_n = 0) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \varphi(\mathbf{t})^n \, d\mathbf{t}, \quad n \in \mathbb{N}_0.$$

It is not hard to see that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n = \sum_{n=0}^{\infty} \mathbf{P}_{\mathcal{S}^1 \otimes \mathcal{S}^2}(D_n = 0) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \frac{1}{1 - \varphi(\mathbf{t})} \, d\mathbf{t}. \tag{5.2}$$

For any $\mathbf{t} \in (-\pi,\pi)^d$,

$$1-\varphi(\mathbf{t}) = \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[1-e^{i\langle \mathbf{t},D_1\rangle}\right] = \sum_{y\in\mathbb{Z}^d} \mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\left(1-e^{i\langle \mathbf{t},y\rangle}\right)$$
$$= \sum_{y\in\mathbb{Z}^d} \mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\left(1-\cos(\langle \mathbf{t},y\rangle)\right).$$

Let us decompose the integral on the righthand side of (5.2) into

$$\int_{(-\frac{\pi}{3},\frac{\pi}{3})^d} \frac{d\mathbf{t}}{1-\varphi(\mathbf{t})} + \int_{(-\pi,\pi)^d\setminus(-\frac{\pi}{3},\frac{\pi}{3})^d} \frac{d\mathbf{t}}{1-\varphi(\mathbf{t})}. \tag{5.3}$$

Fix $\mathbf{t} \in (-\frac{\pi}{3},\frac{\pi}{3})^d$. Since $1-\cos(x) \geq \frac{x^2}{4}$ for $x \in (-\frac{\pi}{3},\frac{\pi}{3})$, we have

$$1-\varphi(\mathbf{t}) \geq \frac{1}{4} \sum_{y\in\mathbb{Z}^d:\langle \mathbf{t},y\rangle\in(-\frac{\pi}{3},\frac{\pi}{3})} \mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\langle \mathbf{t},y\rangle^2.$$

As long as $\mathbf{t}$ is not the zero vector, the expression on the right can be written as

$$\frac{\|\mathbf{t}\|^2}{4} \sum_{y\in\mathbb{Z}^d:\langle \mathbf{t},y\rangle\in(-\frac{\pi}{3},\frac{\pi}{3})} \mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\left\langle \frac{\mathbf{t}}{\|\mathbf{t}\|}, y\right\rangle^2. \tag{5.4}$$

As $\frac{\mathbf{t}}{\|\mathbf{t}\|}$ lies on the unit sphere in $\mathbb{R}^d$, there is $i \in \{1,\ldots,d\}$ such that

$$\left\langle \frac{\mathbf{t}}{\|\mathbf{t}\|}, e_i\right\rangle^2 \geq \frac{1}{d}.$$

where e_i is the i^{th} unit vector in $\mathbb{R}^d$.

Accordingly, using (5.1), the expression in (5.4) is bounded below by

$$\frac{\|\mathbf{t}\|^2}{4d}\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=e_i) = \frac{\|\mathbf{t}\|^2}{4d^2}\frac{N}{(N+1)^2}$$

and the first integral in (5.3) is bounded above by

$$4d^2\frac{(N+1)^2}{N}\int_{(-\frac{\pi}{3},\frac{\pi}{3})^d} \frac{d\mathbf{t}}{\|\mathbf{t}\|^2} < \hat{c}_1 N,$$

for some constant $\hat{c}_1$ depending on d.

Now, let $\mathbf{t} \in (-\pi,\pi)^d \setminus (-\frac{\pi}{3},\frac{\pi}{3})^d$. In this case, there is $i \in \{1,\ldots,d\}$ such that $|t_i| \geq \frac{\pi}{3}$, and $1-\varphi(\mathbf{t})$

is bounded below by

$$\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=e_i)\,(1-\cos(\langle\mathbf{t},e_i\rangle))=\frac{N}{d(N+1)^2}(1-\cos(t_i))\geq\frac{N}{2d(N+1)^2}.$$

Therefore, the second integral in (5.3) is bounded above by

$$2d\frac{(N+1)^2}{N}\int_{(-\pi,\pi)^d\setminus(-\frac{\pi}{3},\frac{\pi}{3})^d}d\mathbf{t}<\tilde{c}_1N$$

for some constant $\tilde{c}_1$ depending on d. Letting $c_1:=\hat{c}_1+\tilde{c}_1$, from (5.2) we obtain the following bound for q:

$$q\leq 1-\frac{c_1}{N}.$$

To obtain a lower bound for q, we estimate

$$1-\varphi(\mathbf{t})\leq\sum_{y\in\mathbb{Z}^d}\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\langle\mathbf{t},y\rangle^2\leq\sum_{y\in\mathbb{Z}^d\setminus\{0\}}\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\|\mathbf{t}\|^2\|y\|^2.$$

Since

$$\sum_{y\in\mathbb{Z}^d\setminus\{0\}}\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(D_1=y)\|y\|^2=\frac{2N}{(N+1)^2}+\frac{2}{d(N+1)^2}+\frac{2d-2}{d(N+1)^2}\lesssim\frac{1}{N},$$

it follows from (5.2) that

$$\frac{1}{1-q}\gtrsim\frac{N}{(2\pi)^d}\int_{(-\pi,\pi)^d}\frac{d\mathbf{t}}{\|\mathbf{t}\|^2}=\tilde{c}_2N$$

for some $\tilde{c}_2>0$ depending on d. Therefore, there is a constant c_2 such that

$$q\geq 1-\frac{c_2}{N}.$$

□

For $tN\in\mathbb{N}$, it is easy to see that the expected value of the partition function Z_t^N with respect to the disorder $(\omega(z,k))_{z\in\mathbb{Z}^d,k\in\mathbb{N}_0}$ is equal to 1. Now, we show that the second moment of Z_t^N with respect to the disorder is bounded uniformly in t and N.

Lemma 5.6

Fix $\beta<\inf_N\sqrt{N\ln(1/q)}$. There is a constant $c_3>0$, independent of N, such that

$$\left\langle(Z_t^N)^2\right\rangle\leq c_3,\quad\textit{for all}\ \ tN\in\mathbb{N}.$$

Proof. We have

$$\begin{aligned}\left\langle (Z_t^N)^2 \right\rangle =& e^{-\beta^2 t}\left\langle \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[\exp\left(\beta\sum_{j=1}^{2}\sum_{i=0}^{tN-1}\omega(S_i^j,i)\right)\right]\right\rangle_N \\ =& e^{-\beta^2 t}\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[\prod_{i=0}^{tN-1}\left\langle \exp\left(\beta\Big(\omega(S_i^1,i)+\omega(S_i^2,i)\Big)\right)\right\rangle_N\right] \\ =& \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[\prod_{i=0}^{tN-1}\left(e^{\frac{\beta^2}{N}}\mathbb{1}_{\{S_i^1=S_i^2\}}+\mathbb{1}_{\{S_i^1\neq S_i^2\}}\right)\right] \\ =& \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[e^{\frac{\beta^2}{N}L_{tN}}\right] \leq \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[e^{\frac{\beta^2}{N}L_\infty}\right].\end{aligned}$$

Since $\mathbf{P}_{\mathcal{S}^1\otimes\mathcal{S}^2}(L_\infty = k) = q^{k-1}(1-q)$, we have

$$\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[e^{\frac{\beta^2}{N}L_\infty}\right] = e^{\frac{\beta^2}{N}}(1-q)\sum_{k=1}^{\infty}\left(e^{\frac{\beta^2}{N}}q\right)^{k-1}. \tag{5.5}$$

Since $\beta < \inf_N \sqrt{N\ln(1/q)}$ and since $1-q \leq c_2/N$ by virtue of Lemma 5.4, the righthand side of (5.5) is bounded above by

$$e^{\frac{\beta^2}{N}}\frac{c_2}{N}\frac{1}{1-e^{\frac{\beta^2}{N}}q} \leq e^{\frac{\beta^2}{N}}\frac{c_2}{\beta^2-(\inf_N\sqrt{N\ln(1/q)})^2}\frac{\frac{\beta^2}{N}+\ln(q)}{1-e^{\frac{\beta^2}{N}}q},$$

and as $N\to\infty$ the expression on the right converges to the finite limit

$$\frac{c_2}{(\inf_N\sqrt{N\ln(1/q)})^2-\beta^2}.$$

So there exists a constant $c_3 > 0$ such that $\left\langle (Z_t^N)^2\right\rangle \leq c_3$ for all $tN\in\mathbb{N}$. □

Lemma 5.7

Fix $\beta < \inf_N\sqrt{N\ln(1/q)}$. There exists a constant $C>0$, independent of t and N, such that

$$Q^N\left(Z_t^N \geq \tfrac{1}{2};\ \mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}\exp\left(\beta\sum_{i=0}^{tN-1}\sum_{j=1}^{2}\omega(S_i^j,i)\right)\right] \leq CNe^{\beta^2 t}(Z_t^N)^2\right) > \frac{1}{C}.$$

Proof. By the Paley–Zygmund inequality (e.g., see [**?**]) and the fact that $\langle Z_t^N\rangle = 1$, we get

$$Q^N\left(Z_t^N \geq \frac{1}{2}\right) \geq \frac{1}{4\langle (Z_t^N)^2\rangle} \geq \frac{1}{4c_3},$$

where c_3 is the constant from Lemma 5.6. First, let C be any positive number. We will only impose a restriction on C towards the end of the proof. Then the lefthand side of the displayline in Lemma 5.7 is bounded below by

$$\frac{1}{4c_3} - Q^N\left(\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}\exp\left(\beta\sum_{i=0}^{tN-1}\sum_{j=1}^{2}\omega(S_i^j,i)\right)\right] > \frac{C}{4}Ne^{\beta^2 t}\right). \tag{5.6}$$

Markov's inequality implies

$$\begin{aligned}
&Q^N\left(\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}\exp\left(\beta\sum_{i=0}^{tN-1}\sum_{j=1}^{2}\omega(S_i^j,i)\right)\right] > \frac{C}{4}Ne^{\beta^2 t}\right)\\
\leq&\frac{4}{CN}e^{-\beta^2 t}\left\langle\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}\exp\left(\beta\sum_{i=0}^{tN-1}\sum_{j=1}^{2}\omega(S_i^j,i)\right)\right]\right\rangle_N\\
=&\frac{4}{CN}e^{-\beta^2 t}\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}\prod_{i=0}^{tN-1}\left\langle\exp\left(\beta(\omega(S_i^1,i)+\omega(S_i^2,i)\right)\right\rangle_N\right]\\
=&\frac{4}{CN}\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_{tN}e^{\frac{\beta^2}{N}L_{tN}}\right],
\end{aligned}$$

where the last step follows from the proof of Lemma 5.6. As a result, the expression in (5.6) is bounded below by

$$\frac{1}{4c_3} - \frac{4}{CN}\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_\infty e^{\frac{\beta^2}{N}L_\infty}\right]. \tag{5.7}$$

From Lemma 5.4, we have

$$\begin{aligned}
\mathbf{E}_{\mathcal{S}^1\otimes\mathcal{S}^2}\left[L_\infty e^{\frac{\beta^2}{N}L_\infty}\right] =&e^{\frac{\beta^2}{N}}(1-q)\sum_{k=1}^{\infty}k\left(e^{\frac{\beta^2}{N}}q\right)^{k-1}\\
\leq&e^{\frac{\beta^2}{N}}\frac{c_2}{N}\left(\frac{1}{1-e^{\frac{\beta^2}{N}}q}\right)^2\\
\leq&e^{\frac{\beta^2}{N}}\frac{c_2 N}{(\beta^2-(\inf_N\sqrt{N\ln(1/q)})^2)^2}\left(\frac{\frac{\beta^2}{N}+\ln(q)}{1-e^{\frac{\beta^2}{N}}q}\right)^2 \leq \tilde{C}N
\end{aligned}$$

for some $\tilde{C} > 0$. Hence, the expression in (5.7) is greater than

$$\frac{1}{4c_3} - \frac{4\tilde{C}}{C},$$

which is greater than $\frac{1}{C}$ if C is chosen sufficiently large. □

Note that for any fixed $tN \in \mathbb{N}$, the random walk $\mathcal{S}$ cannot leave the box

$$B_{tN} := \Big\{ z \in \mathbb{Z}^d : \ \|z\|_1 \le tN \Big\}$$

before time tN, so Z_t^N only depends on $(\omega(z,k))_{z \in B_{tN}, 0 \le k \le tN}$. Let Ξ_t^N be the collection of all arrays $\xi = (\xi(z,k))_{z \in B_{tN}, 0 \le k \le tN}$ of real numbers indexed by $z \in B_{tN}$ and $k \in \{1, \dots, tN\}$. For $\xi \in \Xi_t^N$ define

$$Z_t^N(\xi) := \ e^{-\frac{\beta^2}{2}t}\mathbf{E}\left[\exp\left(\beta \sum_{i=0}^{tN-1} \xi(S_i, i)\right)\right],$$

and let

$$X_t^N := \left\{ \xi : Z_t^N(\xi) \ge \tfrac{1}{2};\ \mathbf{E}_{\mathcal{S}^1 \otimes \mathcal{S}^2}\left[L_{tN} \exp\bigg(\beta \sum_{i=0}^{tN-1} \sum_{j=1}^{2} \xi(S_i^j, i)\bigg)\right] \le CNe^{\beta^2 t} Z_t^N(\xi)^2 \right\},$$

where C is the constant from Lemma 5.7. Let ω_{tN} be the random vector $(\omega(z,i))_{z \in B_{tN}, i \in \{0, \dots, tN\}}$. Then by Lemma 5.7

$$Q^N(\omega_{tN} \in X_t^N) > \frac{1}{C}.$$

Finally, for $m \in \mathbb{N}$, $g, h \in \mathbb{R}^m$ and a measurable set $A \subset \mathbb{R}^m$, define

$$d(g,h) := \|g - h\|, \quad \text{and} \quad d(g,A) := \inf_{h \in A} \|g - h\|.$$

The following lemma is a consequence of the Gaussian concentration inequality (e.g., see [AST03] and [Tal11, Theorem 1.3.4]).

Lemma 5.8

Fix $\beta < \inf_N \sqrt{N \ln(1/q)}$ and let C be the corresponding constant from Lemma 5.7. For any $v > 0$,

$$Q^N\left(d(\omega_{tN}, X_t^N) > v + \sqrt{\frac{2}{N}\ln(C)} \right) \le 2e^{-\frac{N}{2}v^2}.$$

Proof. As the components of $\sqrt{N}\omega_{tN}$ are i.i.d. standard normal, the Gaussian concentration inequality ([Tal11, Theorem 1.3.4]) yields, for any $v > 0$,

$$Q^N\left(\left| d(\omega_{tN}, X_t^N) - \Big\langle d(\omega_{tN}, X_t^N) \Big\rangle_N \right| > v \right) \le 2e^{-\frac{N}{2}v^2}. \tag{5.8}$$

Suppose that $v < \langle d(\omega_{tN}, X_t^N)\rangle$. Then, by Lemma 5.7

$$\frac{1}{C} < Q^N(\omega_{tN} \in X_t^N) \le Q^N\left(\left|d(\omega_{tN}, X_t^N) - \left\langle d(\omega_{tN}, X_t^N)\right\rangle\right| > v\right) \le 2e^{-\frac{N}{2}v^2}.$$

As this inequality holds for any $v < \langle d(\omega_{tN}, X_t^N)\rangle$, we also have

$$\left\langle d(\omega_{tN}, X_t^N)\right\rangle_N \le \sqrt{\frac{2}{N}\ln(2C)}.$$

Together with (5.8), we obtain the desired result. □

5.1.1. Proof of Theorem 5.3

Let $\beta < \inf_N \sqrt{N\ln(1/q)}$, so Lemma 5.8 can be applied. Now fix $\xi' \in X_t^N$ and $\xi \in \Xi_t^N$. Then we can write

$$Z_t^N(\xi) = e^{-\frac{\beta^2}{2}t}\mathbf{E}\left[\exp\left(\beta\sum_{i=0}^{tN-1}\left(\xi(S_i,i) - \xi'(S_i,i)\right)\right)\exp\left(\beta\sum_{i=0}^{tN-1}\xi'(S_i,i)\right)\right]. \tag{5.9}$$

If we define the Gibbs measure ν on the space of path realizations for $\mathcal{S}$ up to step $(tN-1)$ by

$$\nu(B) = \frac{e^{-\frac{\beta^2}{2}t}}{Z_t^N(\xi')}\mathbf{E}\left[\mathbb{1}_B(S_0,\dots,S_{tN-1})\exp\left(\beta\sum_{i=0}^{tN-1}\xi'(S_i,i)\right)\right],$$

then the righthand side of (5.9) becomes

$$\begin{aligned} Z_t^N(\xi')\int \exp\left(\beta\sum_{i=0}^{tN-1}\left(\xi(S_i,i) - \xi'(S_i,i)\right)\right)\,d\nu \\ \ge \frac{1}{2}\exp\left(-\beta\left|\int\sum_{i=0}^{tN-1}\left(\xi(S_i,i) - \xi'(S_i,i)\right)\,d\nu\right|\right), \end{aligned}$$

where we used that $\xi' \in X_t^N$. We have

$$\left|\int\sum_{i=0}^{tN-1}\left(\xi(S_i,i) - \xi'(S_i,i)\right)\,d\nu\right| = \left|\sum_{i=0}^{tN-1}\sum_{z\in B_{tN}}\left(\xi(z,i) - \xi'(z,i)\right)\int \mathbb{1}_{\{S_i=z\}}\,d\nu\right|.$$

By the Cauchy–Schwarz inequality, the expression on the right is less than

$$d(\xi, \xi') \left(\int \int L_{tN} \, d\nu \, d\nu \right)^{\frac{1}{2}}.$$

Again using the fact that $\xi' \in X_t^N$, we infer from the definition of ν the inequality $\int \int L_{tN} \, d\nu \, d\nu \leq CN$ and thus

$$Z_t^N(\xi) \geq \tfrac{1}{2} \exp \left(-\beta d(\xi, \xi') \left(\int \int L_{tN} \, d\nu \, d\nu \right)^{\frac{1}{2}} \right) \geq \tfrac{1}{2} \exp \left(-\beta d(\xi, \xi') \sqrt{CN} \right).$$

As the inequality above holds for any $\xi' \in X_t^N$,

$$Z_t^N(\xi) \geq \tfrac{1}{2} \exp \left(-\beta \sqrt{C} d(\xi, X_t^N) \sqrt{N} \right).$$

Fix $u > \ln(2)$ and suppose

$$\frac{u - \ln(2)}{\beta \sqrt{CN}} \geq d(\xi, X_t^N).$$

Then

$$Z_t^N(\xi) \geq \tfrac{1}{2} \exp \left(-(u - \ln(2)) \right) = e^{-u}.$$

Accordingly,

$$Q^N(Z_t^N \geq e^{-u}) \geq Q^N \left(d(\omega_{tN}, X_t^N) \leq \frac{u - \ln(2)}{\beta \sqrt{CN}} \right).$$

By Lemma 5.8 with $v = \frac{u - \ln(2)}{\beta \sqrt{CN}} - \sqrt{2 \ln(C)/N}$, the expression on the right is greater than

$$1 - 2 \exp \left(-\frac{1}{2} \left(\frac{u - \ln(2)}{\beta \sqrt{C}} - \sqrt{2 \ln(C)} \right)^2 \right),$$

so there is $c > 0$ such that

$$Q^N(Z_t^N \geq e^{-u}) \geq 1 - e^{-u^2/c}$$

for u sufficiently large. This implies the desired inequality, and therefore completes the proof of Theorem 5.3.

5.2. The Continuous-time Case

In this section we present the proof of Theorem 5.1. We will show that there is a constant $c > 0$ such that

$$Q(Z_{0,0}^t < e^{-u}) < ce^{-u^2/c}, \quad t, u > 0.$$

To simplify notation, we write Z^t in lieu of $Z_{0,0}^t$. We start with a result on continuity in time for the partition function.

Lemma 5.9

Fix $t > 0$. For any $\epsilon > 0$, there exists $s_\epsilon > 0$ such that

$$Q(|Z^{t+s} - Z^t| > \epsilon) < \epsilon \textit{ for all } s \in (0, s_\epsilon).$$

Proof. Recall that $\mathcal{A}_s^t$ is the action defined by (3.1). We have

$$\begin{aligned} \langle (Z^{t+s} - Z^t)^2 \rangle &= e^{-\beta^2 t} \left\langle \left(\mathbf{E}_{0,0} \left[\exp(\beta \mathcal{A}_0^t) \left(e^{-\frac{\beta^2}{2}s} \exp(\beta \mathcal{A}_t^{t+s}) - 1 \right) \right] \right)^2 \right\rangle \\ &\le e^{-\beta^2 t} \mathbf{E}_{0,0} \left[\langle \exp(2\beta \mathcal{A}_0^t) \rangle \left\langle \left(e^{-\frac{\beta^2}{2}s} \exp(\beta \mathcal{A}_t^{t+s}) - 1 \right)^2 \right\rangle \right] \\ &= e^{\beta^2 t}(e^{\beta^2 s} - 1). \end{aligned}$$

Let $s_\epsilon > 0$ be so small that $e^{\beta^2 t}(e^{\beta^2 s} - 1) < \epsilon^3$ for all $s \in (0, s_\epsilon)$. Then, by Markov's inequality,

$$Q(|Z^{t+s} - Z^t| > \epsilon) \le \frac{\langle (Z^{t+s} - Z^t)^2 \rangle}{\epsilon^2} < \epsilon. \qquad \square$$

Let $N \in \mathbb{N}$ and $t > 0$ such that $tN \in \mathbb{N}$. We will now represent the partition function Z_t^N from Section 5.1 in a way that mimicks the definition of Z^t. Let $\mathcal{S}^N = (S_i^N)_{i \in \mathbb{N}_0}$ denote the random walk $\mathcal{S}$ from Section 5.1. The change in notation reflects that we now want to vary N. For $s \ge 0$, we define the continuous-time random walk η^N by

$$\eta_s^N := S_i^N \text{ if } s \in [\tfrac{i}{N}, \tfrac{i+1}{N}).$$

A sample path of η^N over the time interval $[0, t)$ is characterized by the number of actual jumps n_t^N that occur in $(0, t)$, the embedded discrete-time path $\gamma^N = (\gamma_0^N, \gamma_1^N, \dots, \gamma_{n_t^N}^N)$ on $\mathbb{Z}^d$ such that $\|\gamma_j^N - \gamma_{j-1}^N\|_1 = 1$ for $1 \le j \le n_t^N$, and the jump times $0 < s_1^N < \ldots < s_{n_t^N}^N < t$, which are of the form $\frac{i}{N}$. To keep the definition of the action compact, we denote $s_0^N := s$ and $s_{n_t^N+1}^N := t$. To such a sample path of η^N, we assign the action

$$\mathcal{A}_t^N := \sum_{j=0}^{n_t^N} \left(W_{s_{j+1}^N}^{\gamma_j^N} - W_{s_j^N}^{\gamma_j^N} \right).$$

Next, we define the probability measure

$$g_N\left(\left\{\frac{k}{N}\right\}\right) := \frac{1}{N+1}\left(\frac{N}{N+1}\right)^{k-1}, \quad k \in \mathbb{N}.$$

Let $(\tau_j^N)_{j\in\mathbb{N}}$ be an i.i.d. sequence of random variables distributed according to g_N. The jump times of η^N can then be represented as

$$s_k^N = \sum_{j=1}^{k} \tau_j^N, \quad k \in \mathbb{N}_0.$$

The partition function Z_t^N from Section 5.1 has the same distribution under Q^N as

$$e^{-\frac{\beta^2}{2}t}\mathbf{E}_{\tau^N}\mathbf{E}_\gamma e^{\beta\mathcal{A}_t^N}, \tag{5.10}$$

where $\mathbf{E}_{\tau^N}$ denotes expectation with respect to $(\tau_j^N)_{j\in\mathbb{N}}$, and $\mathbf{E}_\gamma$ averages with respect to the sample paths of a discrete-time simple symmetric random walk on $\mathbb{Z}^d$. We will therefore also denote the expression in (5.10) by Z_t^N. As $N \to \infty$, g_N converges weakly to the exponential distribution with intensity 1. As a result, for any $k \in \mathbb{N}$, $(\tau_1^N, \ldots, \tau_k^N)$ converges weakly to $(\tau_1, \ldots, \tau_k)$ as $N \to \infty$, where $(\tau_j)_{j\in\mathbb{N}}$ is a sequence of independent exponentially distributed random variables with intensity 1.

Lemma 5.10

Fix $t \in (0,\infty) \cap \mathbb{Q}$. For any $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that for $N \in \mathbb{N}$ sufficiently large and with $tN \in \mathbb{N}$, we have

$$Q\left(e^{-\frac{\beta^2}{2}t}\mathbf{E}_{\tau^N}\left[\mathbf{E}_\gamma\left[\exp(\beta\mathcal{A}_t^N)\right]\mathbb{1}_{n_t^N > M_\epsilon}\right] > \epsilon\right) < \epsilon.$$

Proof. For any $M \in \mathbb{N}$, we have

$$\left\langle e^{-\frac{\beta^2}{2}t}\mathbf{E}_{\tau^N}\left[\mathbf{E}_\gamma\left[\exp(\beta\mathcal{A}_t^N)\right]\mathbb{1}_{n_t^N > M}\right]\right\rangle = \mathbf{P}_{\tau^N}(n_t^N > M).$$

Recall that n_t is the number of jumps that occur within the finite time interval $(0, t)$ for the continuous-time sample path of η (see page 40). By weak convergence,

$$\lim_{N\to\infty} \mathbf{P}_{\tau^N}\big(n_t^N > M\big) = \mathbf{P}\big(n_t > M\big).$$

Choose $M_\epsilon \in \mathbb{N}$ so large that

$$\mathbf{P}\Big(n_t > M_\epsilon\Big) < \frac{\epsilon^2}{2}.$$

Then, for N sufficiently large,

$$\left\langle e^{-\frac{\beta^2}{2}t}\mathbf{E}_{\tau^N}\left[\mathbf{E}_\gamma\left[\exp(\beta\mathcal{A}_t^N)\right]\mathbb{1}_{n_t^N > M_\epsilon}\right]\right\rangle < \epsilon^2,$$

and we can conclude using Markov's inequality. □

Proof of Theorem 5.1. Throughout the proof, we keep $t > 0$ fixed. Let $\delta > 0$ be given, and let $s \in (0, s_\delta)$ such that $t + s \in \mathbb{Q}$. By Lemma 5.9, we have for $u > 0$,

$$\begin{aligned} Q\big(Z^t < e^{-u}\big) &= Q\Big(Z^t < e^{-u}; |Z^{t+s} - Z^t| > \delta\big) + Q\big(Z^t < e^{-u};\ |Z^{t+s} - Z^t| \le \delta\Big) \\ &\le Q\Big(|Z^{t+s} - Z^t| > \delta\Big) + Q\Big(Z^{t+s} < e^{-u} + \delta\Big) < \delta + Q\Big(Z^{t+s} < e^{-w_\delta}\Big), \end{aligned} \tag{5.11}$$

where $w_\delta := -\ln(\delta + e^{-u})$. Since $t + s \in \mathbb{Q}$, we may choose $M_\delta \in \mathbb{N}$ according to Lemma 5.10. Consider the event

$$A := \left\{\omega \,:\, e^{-\frac{\beta^2}{2}(t+s)}\mathbf{E}_\tau\left[\mathbf{E}_\gamma\left[\exp(\beta\mathcal{A}_0^{t+s})\right]\mathbb{1}_{n_{0,t+s}\le M_\delta}\right] < e^{-w_\delta}\right\},$$

where $\mathbf{E}_\tau$ denotes expectation with respect to $(\tau_j)_{j\in\mathbb{N}}$.

As $\big\{\omega \,:\, Z^{t+s} < e^{-w_\delta}\big\} \subset A$, we have

$$Q\big(Z^{t+s} < e^{-w_\delta}\big) \le Q(A). \tag{5.12}$$

For a fixed realization ω of the disorder $\big(W^z\big)_{z\in\mathbb{Z}^d}$ and for $k \in \mathbb{N}_0$, we define the map

$$\begin{aligned} &\varphi_k^\omega(t_1, \dots, t_{k+1}) := \\ &\mathbb{1}_T e^{-\frac{\beta^2}{2}(t+s)}\mathbf{E}_\gamma\left[\exp\left(\beta\sum_{i=0}^{k-1}\left(W^{\gamma_i}_{\sum_{j=1}^{i+1}t_j} - W^{\gamma_i}_{\sum_{j=1}^{i}t_j}\right) + \beta\left(W^{\gamma_k}_{t+s} - W^{\gamma_k}_{\sum_{j=1}^{k}t_j}\right)\right)\right]. \end{aligned}$$

where $T := (0,\infty)^{k+1} \cap \{\sum_{j=1}^{k} t_j < t+s \leq \sum_{j=1}^{k+1} t_j\}$. Path continuity of Brownian motion implies that the functions $(\varphi_k^\omega)_{k\in\mathbb{N}_0}$ are bounded for Q-almost every ω. Moreover, the set of discontinuities of φ_k^ω has measure zero with respect to the law of $(\tau_1,\ldots,\tau_{k+1})$. Thus, by the Portemanteau Theorem (see, e.g. [Kle08, Theorem 13.16]), Q-almost surely,

$$\lim_{N\to\infty} \mathbf{E}\left[\varphi_k^\omega(\tau_1^N,\ldots,\tau_{k+1}^N)\right] = \mathbf{E}\left[\varphi_k^\omega(\tau_1,\ldots,\tau_{k+1})\right], \quad k \in \mathbb{N}_0.$$

In particular,

$$Q\left(\bigcap_{k=0}^{M_\delta}\left\{\omega : \lim_{N\to\infty} \mathbf{E}[\varphi_k^\omega(\tau_1^N,\ldots,\tau_{k+1}^N)] = \mathbf{E}[\varphi_k^\omega(\tau_1,\ldots,\tau_{k+1})]\right\}\right) = 1.$$

As

$$\begin{aligned}&\bigcap_{k=0}^{M_\delta}\left\{\omega : \lim_{N\to\infty} \mathbf{E}\left[\varphi_k^\omega(\tau_1^N,\ldots,\tau_{k+1}^N)\right] = \mathbf{E}\left[\varphi_k^\omega(\tau_1,\ldots,\tau_{k+1})\right]\right\}\\ &\subset \bigcap_{k=0}^{M_\delta}\bigcup_{N=1}^{\infty}\bigcap_{j=N}^{\infty}\left\{\omega : \left|\mathbf{E}[\varphi_k^\omega(\tau_1^j,\ldots,\tau_{k+1}^j)] - \mathbf{E}[\varphi_k^\omega(\tau_1,\ldots,\tau_{k+1})]\right| < \frac{\delta}{M_\delta+1}\right\},\end{aligned}$$

there is $N_\delta \in \mathbb{N}$ such that for all $N \geq N_\delta$,

$$Q\left(B_N\right) > 1-\delta,$$

where

$$B_N := \bigcap_{j=N}^{\infty}\bigcap_{k=0}^{M_\delta}\left\{\omega : \left|\mathbf{E}[\varphi_k^\omega(\tau_1^j,\ldots,\tau_{k+1}^j)] - \mathbf{E}[\varphi_k^\omega(\tau_1,\ldots,\tau_{k+1})]\right| < \frac{\delta}{M_\delta+1}\right\}.$$

Consider $N > N_\delta$ such that $(t+s)N \in \mathbb{N}$. Assume further that N is so large that the conclusion of Lemma 5.10 holds, i.e.

$$Q\left(e^{-\frac{\beta^2}{2}(t+s)}\mathbf{E}_{\tau^N}\left[\mathbf{E}_\gamma\left[\exp(\beta\mathcal{A}_{t+s}^N)\right]\mathbb{1}_{n_{t+s}^N > M_\delta}\right] > \delta\right) < \delta.$$

Since $Q(B_N^c) < \delta$, we have

$$Q(A) = Q(A\cap B_N) + Q(A\cap B_N^c) \leq Q(A\cap B_N) + \delta. \tag{5.13}$$

For $\omega \in B_N$, we have

$$e^{-\frac{\beta^2}{2}(t+s)}\left|\mathbf{E}_\tau\left[\mathbb{1}_{\{n_{t+s}\le M_\delta)\}}\mathbf{E}_\gamma\left[e^{\beta\mathcal{A}_0^{t+s}}\right]\right]-\mathbf{E}_{\tau^N}\left[\mathbb{1}_{n_{t+s}^N\le M_\delta}\mathbf{E}_\gamma\left[e^{\beta\mathcal{A}_{t+s}^N}\right]\right]\right|$$
$$\le\sum_{k=0}^{M_\delta}\left|\mathbf{E}_\tau[\varphi_k^\omega(\tau_1,\dots,\tau_{k+1})]-\mathbf{E}_{\tau^N}[\varphi_k^\omega(\tau_1^N,\dots,\tau_{k+1}^N)]\right|<(M_\delta+1)\frac{\delta}{M_\delta+1}=\delta.$$

Accordingly,

$$A\cap B_N\subset C:=\left\{e^{-\frac{\beta^2}{2}(t+s)}\mathbf{E}_{\tau^N}\left[\mathbb{1}_{n_{t+s}^N\le M_\delta}\mathbf{E}_\gamma\left[e^{\beta\mathcal{A}_{t+s}^N}\right]\right]<e^{-v_\delta}\right\},$$

where $v_\delta := -\ln(\delta+e^{-w_\delta})$. Let

$$D:=\left\{e^{-\frac{\beta^2}{2}(t+s)}\mathbf{E}_{\tau^N}\left[\mathbf{E}_\gamma\left[e^{\beta\mathcal{A}_{t+s}^N}\right]\mathbb{1}_{n_{t+s}^N>M_\delta}\right]>\delta\right\}.$$

By Lemma 5.10,

$$Q(A\cap B_N)\le Q(C)=Q(C\cap D)+Q(C\cap D^c)<\delta+Q(Z_{t+s}^N<e^{-y_\delta}),\tag{5.14}$$

where $y_\delta := -\ln(\delta+e^{-v_\delta})$. By Theorem 5.3, there is $c>0$, independent of t and N, such that

$$Q(Z_{t+s}^N<e^{-y_\delta})\le ce^{-y_\delta^2/c}.$$

Combining this estimate with the estimates (5.11)-(5.14), we get

$$Q\big(Z^t<e^{-u}\big)\le 3\delta+ce^{-y_\delta^2/c}.$$

Since, in addition, $\lim_{\delta\searrow 0}(3\delta+ce^{-y_\delta^2/c})=ce^{-u^2/c}$, we obtain the desired estimate. □

chapter 6

Global Solutions to the Semi-Discrete Stochastic Heat Equation

Contents

In this final chapter, we address the problem of global solutions to the ***semidiscrete stochastic heat equation***:

$$\partial_t u(y,t) = \Delta_y u(y,t) + \beta u(y,t)\dot{W}_t^y, \quad y \in \mathbb{Z}^d, \tag{sSHE}$$

where $u = u(x,t)$ is a scalar function on the semi-discrete spacetime $\mathbb{Z}^d \times \mathbb{R}$, where Δ_y is the discrete Laplacian given by

$$\Delta_y u(y,t) = \frac{1}{2d} \sum_{z\in\mathbb{Z}^d : \|y-z\|_1 = 1} \big(u(y,t) - u(z,t)\big),$$

and $\dot{W}_t^y$ is the white noise associated with W_t^y, and where $\beta > 0$ is the coupling constant.

Our primary focus is studying the Cauchy problem for (sSHE) for initial conditions with subexponential growth in space. Let $f : \mathbb{Z}^d \to (0,\infty)$ be any function of subexponential growth in space; namely, such that for a sufficiently small $\epsilon \in (0,1)$, it satisfies

$$\lim_{\|x\|\to\infty} \frac{|\ln(f(x))|}{\|x\|^{1-\epsilon}} = 0. \tag{6.1}$$

We fix $s \in \mathbb{R}$ and consider the following Cauchy problem for the semidiscrete stochastic heat equation (sSHE), also known as the ***Parabolic Anderson Model*** (PAM):

$$\begin{cases} \partial_t u(y,t) = \Delta_y u(y,t) + \beta u(y,t)\dot{W}_t^y, \quad y \in \mathbb{Z}^d,\ t > s, \\ u(y,s) = f(y), \quad y \in \mathbb{Z}^d. \end{cases} \tag{PAM}$$

We emphasise that most studies of the PAM consider bounded or even localised initial data, whereas the initial data considered in this thesis is in a much more general class of functions with subexponential growth.

It was shown in [CM94] that the solution to the Cauchy problem (PAM), if interpreted as an integral equation in the sense of Itô, is given by

$$u_f^s(y,t) := \sum_{x\in\mathbb{Z}^d} f(x) Z_{x,s}^{y,t}, \quad t \geq s. \tag{6.2}$$

Note that $u_f^s(y,s) = f(y)$. In the special case that $s = 0$, we usually write u_f instead of u_f^0. Observe that for the specific choice of initial data $f \equiv 1$, we obtain a solution to the Cauchy problem (PAM) given by the partition function $Z_s^{y,t}$:

$$u_1^s(y,t) = \sum_{x\in\mathbb{Z}^d} Z_{x,s}^{y,t} = Z_s^{y,t}. \tag{6.3}$$

This result can be viewed as a Feynman–Kac formula for the semidiscrete parabolic Anderson problem. We can think of any solution $u_f^s(y,t)$ to (PAM) given by (6.2) as being local in time in the sense that it is defined only for $t > s$. The primary focus in this thesis is instead the analysis of solutions which are global in time in the sense that we now define.

For any time $s \in \mathbb{R}$, let $\theta_s : \Omega \to \Omega$ be the ***Wiener shift*** defined by

$$\theta_s\big(\omega(x,t)\big) := \omega(x,t+s) - \omega(x,s),$$

for all $(x,t) \in \mathbb{Z}^d \times \mathbb{R}$; i.e., every path $\omega(x,\cdot)$ is shifted by s to the left along the time axis and normalized to equal 0 at time $t = 0$. The probability measure Q is invariant with respect to $(\theta_s)_{s\in\mathbb{R}}$, in the sense that for every $s \in \mathbb{R}$ and for every $A \in \mathcal{F}$, one has $Q(\theta_s(A)) = Q(A)$.

Definition 6.1

Let $\Omega' \in \mathcal{F}$ such that $Q(\Omega') = 1$ and $\theta_t(\Omega') = \Omega'$ for all $t \in \mathbb{R}$. A measurable map $Z : \mathbb{Z}^d \times \mathbb{R} \times \Omega' \to \mathbb{R}$ is called a ***global stationary solution*** to (sSHE) if:

(1) For every $y \in \mathbb{Z}^d$, $s, t \in \mathbb{R}$ with $s < t$, and $\omega \in \Omega'$,

$$Z(y,t,\omega) = \sum_{x \in \mathbb{Z}^d} Z(x,s,\omega) Z_{x,s}^{y,t}(\omega);$$

(2) For every $y \in \mathbb{Z}^d$, $t \in \mathbb{R}$, and $\omega \in \Omega'$, we have $Z(y,t,\omega) = Z(y,0,\theta_t\omega)$.

Recall from Theorem 3.1 that $Z_{-\infty}^{y,t}$ the limit of $Z_s^{y,t}(\omega)$ as $s \to -\infty$ exists and is positive Q-almost surely for every $(y,t) \in \mathbb{Z}^d \times \mathbb{R}$. Thanks to Lemma 3.12 (after taking Remark 3.2 into account), the limiting partition function $Z_{-\infty}^{y,t}$ therefore defines a particular global stationary solution to (sSHE). Our theorem is as follows.

Proposition 6.2

There is an $\mathcal{F}$-measurable subset $\Omega^{\mathrm{sol}} \subset \Omega^+$ with $Q(\Omega^{\mathrm{sol}}) = 1$ such that the function

$$\begin{aligned} \mathbb{Z}^d \times \mathbb{R} \times \Omega^{\mathrm{sol}} &\to \mathbb{R} \\ (y,t,\omega) &\mapsto Z_{-\infty}^{y,t}(\omega) \end{aligned} \tag{6.4}$$

is a global stationary solution to (sSHE).

The main result of this thesis is that, up to normalisation at the origin $0 \in \mathbb{Z}^d$, this global solution $Z_{-\infty}^{y,t}(\omega)$ is unique, implying in particular that the solutions to (PAM) have a rather weak dependence on the initial data.

Theorem 6.3

Let Z be a global stationary solution to (sSHE) *which, Q-almost surely, has subexponential growth in space and satisfies $Z(0,t,\omega) \neq 0$. Then, for β sufficiently small and Q-almost surely for all $(y,t) \in \mathbb{Z}^d \times \mathbb{R}$,*

$$\frac{Z(y,t,\omega)}{Z(0,t,\omega)} = \frac{Z_{-\infty}^{y,t}(\omega)}{Z_{-\infty}^{0,t}(\omega)}.$$

Explicitly, by Z having subexponential growth in space we mean that, for all t and almost every $\omega \in \Omega$, we have

$$\lim_{\|x\| \to \infty} \frac{|\ln(Z(x,t,\omega))|}{\|x\|^{1-\epsilon}} = 0.$$

6.1. Attraction to the Particular Global Solution

The first major step towards Theorem 6.3 is a result which says that the particular global stationary solution $Z_{-\infty}^{y,t}$ from (6.4) attracts in a certain sense solutions to the Cauchy problem (PAM) with any subexponentially growing initial data f. For any $c > 0$ and $\epsilon \in (0, 1)$, we define the set

$$\mathcal{L}_{c,\epsilon} := \left\{ f : \mathbb{Z}^d \to (0, \infty) \ : \ \left| \ln(f(x)) \right| \leq c\|x\|^{1-\epsilon}, \ \forall x \in \mathbb{Z}^d \right\} \tag{6.5}$$

Note that this condition implies $f(0) = 1$, and is equivalent to

$$e^{-c\|x\|^{1-\epsilon}} \leq f(x) \leq e^{c\|x\|^{1-\epsilon}}, \quad \forall x \in \mathbb{Z}^d.$$

Then the following theorem is the central result in this section.

Theorem 6.4

For β sufficiently small, the following holds: for every $y \in \mathbb{Z}^d$ and for every $c > 0$, $\epsilon \in (0, 1)$, we have

$$\sup_{f \in \mathcal{L}_{c,\epsilon}} \left| \frac{u_f(y,t)}{u_f(0,t)} - \frac{Z_{-\infty}^{y,t}}{Z_{-\infty}^{0,t}} \right| \xrightarrow[t \to \infty]{} 0 \quad \textit{in probability.} \tag{6.6}$$

Before jumping into the proof of this theorem, we we briefly sketch the proof strategy and present the main ingredients. The first step is to fix $0 < \sigma < 1$ sufficiently close to 1, and for $f \in \mathcal{L}_{c,\epsilon}$, rewrite the solution u_f as a sum of two terms

$$u_f(y,t) = \sum_{\|x\| \leq t^\sigma} f(x) Z_{x,0}^{y,t} + \sum_{\|x\| > t^\sigma} f(x) Z_{x,0}^{y,t} \tag{6.7}$$

corresponding to terms inside and outside of the ball of radius t^σ respectively.

The contribution of the second term to $u_f(y,t)$ is negligible: its expectation with respect to the noise can be roughly bounded from above by

$$\sum_{\|x\| > t^\sigma} e^{c\|x\|^{1-\epsilon}} p_t^{y-x} \lesssim \sum_{\|x\| > t^\sigma} e^{c\|x\|^{1-\epsilon} - \kappa \frac{\|x\|^2}{t}},$$

for some $\kappa > 0$. The expression on the righthand side is negligible provided that σ is sufficiently close to 1 (specifically, $\sigma > \frac{1}{1+\epsilon}$).

The dominant contribution to $u_f(y,t)$ then comes from the first term on the righthand side of (6.7). To deal with it, we apply the factorization formula for the partition function $Z^{y,t}_{x,s}$ obtained in Theorem 4.1.

Using the representation in (4.1), we have

$$\begin{aligned}\sum_{\|x\|\le t^\sigma} f(x)Z^{y,t}_{x,0} &= \sum_{\|x\|\le t^\sigma} f(x)p_t^{y-x}\left(Z^{\infty}_{x,0}Z^{y,t}_{-\infty}+\delta^{y,t}_{x,0}\right) \\ &= \sum_{\|x\|\le t^\sigma} f(x)p_t^{y-x}Z^{\infty}_{x,0}Z^{y,t}_{-\infty} + \sum_{\|x\|\le t^\sigma} f(x)p_t^{y-x}\delta^{y,t}_{x,0}.\end{aligned} \tag{6.8}$$

Our aim is to show that the first term in the second line of (6.8) dominates the sum. To prove this fact, we use the smallness of $\delta^{y,t}_{x,0}$ implied by (4.2). However, we must also make sure that $Z^{y,t}_{-\infty}$ does not become too small for typical realizations of the noise. For this the main ingredient is Theorem 5.1, which implies that for $\theta > 0$, with high probability,

$$Z^{y,t}_{-\infty}Z^{\infty}_{x,0} \ge t^{-\theta}, \quad \forall x : \|x\| \le t^\sigma.$$

This allows us to conclude that the second term in (6.8) is indeed negligible compared to the first, and therefore

$$\frac{u_f(y_1,t)}{u_f(y_2,t)} \approx \frac{Z^{y_1,t}_{-\infty}\sum_{\|x\|\le t^\sigma} f(x)p_t^{y_1-x}Z^{\infty}_{x,0}}{Z^{y_2,t}_{-\infty}\sum_{\|x\|\le t^\sigma} f(x)p_t^{y_2-x}Z^{\infty}_{x,0}}.$$

Finally, for large t, the ratio $p_t^{y_1-x}/p_t^{y_2-x}$ is close to 1, uniformly in x (see Lemma 2.13), so the righthand side is roughly $Z^{y_2,t}_{-\infty}/Z^{y_1,t}_{-\infty}$, which is what we want to show. In the next section we give the formal proof of Theorem 1.7.

6.1.1. Proof of Theorem 1.7

Fix $c > 0$ and $\epsilon \in (0,1)$ and et $f \in \mathcal{L}_{c,\epsilon}$. Fix, once and for all, $\sigma \in (0,1)$ such that $\sigma > \frac{1}{1+\epsilon}$, and note that $\sigma > 1/2$ because $\epsilon < 1$. For any $y \in \mathbb{Z}^d$ and $t > 0$, we use the factorization formula from Theorem 4.1 to write $u_f(y,t)$ as a sum of three terms, which we denote by $\Sigma_1(f)$, $\Sigma_2(f)$, and $\Sigma_3(f)$:

$$\begin{aligned}u_f(y,t) &= \sum_{\|x\|\le t^\sigma} f(x)p_t^{y-x}Z^{\infty}_{x,0}Z^{y,t}_{-\infty} + \sum_{\|x\|\le t^\sigma} f(x)p_t^{y-x}\delta^{y,t}_{x,0} + \sum_{\|x\|> t^\sigma} f(x)Z^{y,t}_{x,0} \\ &=: \Sigma_1(f)+\Sigma_2(f)+\Sigma_3,(f)\end{aligned} \tag{6.9}$$

Then we factorize u_f as follows:

$$u_f = \Sigma_1(f)\left(1+\frac{\Sigma_2(f)}{\Sigma_1(f)}\right)\left(1+\frac{\Sigma_3(f)}{\Sigma_1(f)+\Sigma_2(f)}\right).$$

Proving Theorem 1.7 is equivalent to show that $u_f(y,t)/u_f(0,t)$ converges uniformly in probability to $Z^{y,t}_{-\infty}/Z^{0,t}_{-\infty}$. Therefore, we need to show the following proposition.

Proposition 6.5

For every $\epsilon > 0$ and every $\delta > 0$, there exists $T := T(\varepsilon,\delta)$, depending on ϵ and δ, such that

$$Q\left(\left|\frac{u_f(y,t)}{u_f(0,t)} - \frac{Z^{y,t}_{-\infty}}{Z^{0,t}_{-\infty}}\right| > \varepsilon\right) < \delta$$

for all $f \in \mathcal{L}_{c,\epsilon}$ if $t > T$.

Let

$$B_f(y,t) := \frac{\Sigma_3(f)}{\Sigma_1(f)+\Sigma_2(f)},\quad C_f(y,t) := \frac{\Sigma_2(f)}{\Sigma_1(f)},\quad \text{and}\quad D_f(y,t) := \Sigma_1(f),$$

then

$$\frac{u_f(y_1,t)}{u_f(y_2,t)} - \frac{Z^{y_1,t}_{-\infty}}{Z^{y_2,t}_{-\infty}} = \frac{Z^{y_1,t}_{-\infty}}{Z^{y_2,t}_{-\infty}}\left(\frac{(1+C_f(y,t))\,(1+B_f(y,t))\,D_f(y,t)}{(1+C_f(0,t))\,(1+B_f(0,t))\,D_f(0,t)} - 1\right).$$

Since Q-almost surely $Z^{y,t}_{-\infty} > 0$ by Theorem 3.1, the ratio $Z^{y_1,t}_{-\infty}/Z^{y_2,t}_{-\infty}$ is finite Q-almost surely.

From Lemma 2.13, for any $\zeta > 0$, there exists $T(\zeta)$ such that $1-\zeta < p_t^{y_1-x}/p_t^{y_2-x} < 1+\zeta$. Hence,

$$1-\zeta < \frac{\sum_{\|x\|\le t^\sigma} f(x)p_t^{y_1-x}Z^{\infty}_{x,0}}{\sum_{\|x\|\le t^\sigma} f(x)p_t^{y_2-x}Z^{\infty}_{x,0}} < 1+\zeta$$

for all $t \ge T(\delta)$ and $f \in \mathcal{L}_{(c,\epsilon)}$. Notice that $T(\zeta)$ is completely determined by the simple symmetric random walk. It only depends on y and is in particular independent of the initial condition f and of the noise as long as $\omega \in \Omega^+$. Thus, $D_f(y,t))/D_f(0,t)$ converges to 1 uniformly.

Hence, Proposition 6.5 will follow if we show that $B_f(y,t)$ and $C_f(y,t)$ converge to 0 uniformly in probability, namely

Lemma 6.6

Let β be small so that the conclusions of Theorems 4.1 and 5.1 hold. Then for every $\varepsilon > 0$ and every $\delta > 0$, there exists $T := T(\varepsilon, \delta)$ such that

$$\textbf{(a)} \qquad Q(|B_f(y,t)| > \varepsilon) < \delta, \quad \textit{for all } f \in \mathcal{L}_{c,\epsilon} \textit{ if } t > T. \tag{6.10}$$

and

$$\textbf{(b)} \qquad Q(|C_f(y,t)| > \varepsilon) < \delta, \quad \textit{for all } f \in \mathcal{L}_{c,\epsilon} \textit{ if } t > T. \tag{6.11}$$

6.1.2. Proof of Lemma 6.6, Part (a)

For $t > 0$, define the event

$$A(t) := \bigcup_{x:\|x\|\le t^\sigma} \{Z^{y,t}_{-\infty} Z^{\infty}_{x,0} < t^{-\theta}\}.$$

Theorem 5.1 implies that there is $c > 0$ such that

$$Q(Z^{\infty}_{0,0} < e^{-u}) \le c e^{-u^2/c}, \quad u > 0.$$

Thus,

$$Q(A(t)) \le \sum_{\|x\|\le t^\sigma} Q(Z^{y,t}_{-\infty} Z^{\infty}_{x,0} < t^{-\theta}) \lesssim t^{d\sigma} Q(Z^{y,t}_{-\infty} < t^{-\frac{\theta}{2}}) \lesssim t^{d\sigma - \frac{\theta^2}{4c}\ln^2(t)},$$

which tends to 0 as $t \to \infty$. In addition, by Theorem 4.1, there is $\theta > 0$ such that

$$\lim_{t\to\infty} t^\theta \sup_{\|x\|\le t^\sigma} \langle |\delta^{y,t}_{x,0}| \rangle = 0,$$

see also Remark 6.7 below. Fix $\varepsilon, \delta > 0$, and let $\tau > 0$ be so large that for any $t \ge \tau$,

$$Q(A(t)) < \frac{\delta}{2} \quad \text{and} \quad t^\theta \sup_{\|x\|\le t^\sigma} \langle |\delta^{y,t}_{x,0}| \rangle < \frac{\varepsilon\delta}{2}.$$

Then, with $A(t)^c$ denoting the complement of $A(t)$, we have for $t \geq \tau$,

$$Q\left(|C_t^f(y)| > \varepsilon\right) \leq \frac{\delta}{2} + \varepsilon^{-1} \int_{A(t)^c} \sum_{\|x\| \leq t^\sigma} \frac{f(x)|\delta_{x,0}^{y,t}(\omega)| p_t^{y-x}}{\sum_{\|z\| \leq t^\sigma} f(z) p_t^{y-z} Z_{-\infty}^{y,t}(\omega) Z_{z,0}^{\infty}(\omega)} Q(d\omega)$$
$$\leq \frac{\delta}{2} + \varepsilon^{-1} \sum_{\|x\| \leq t^\sigma} \frac{f(x) p_t^{y-x}}{\sum_{\|z\| \leq t^\sigma} f(z) p_t^{y-z}} t^\theta \int_{A(t)^c} |\delta_{x,0}^{y,t}(\omega)| \, Q(d\omega)$$
$$\leq \frac{\delta}{2} + \varepsilon^{-1} t^\theta \sup_{\|x\| \leq t^\sigma} \langle |\delta_{x,0}^{y,t}| \rangle < \delta.$$

□

Remark 6.7

Notice that since Theorem 4.1 holds for any $\sigma \in (0,1)$, in particular for $\hat{\sigma} \in (\sigma, 1)$, we have that for $\|x - y\| < t^{\hat{\sigma}}$, the error term $\delta_{x,0}^{y,t}$ verifies

$$\lim_{t \to \infty} t^\theta \sup_{\|x-y\| \leq t^{\hat{\sigma}}} \langle |\delta_{x,0}^{y,t}| \rangle = 0. \tag{6.12}$$

If we choose $\widehat{\sigma} \in (\sigma, 1)$, then for t sufficiently large, we have that

$$\{x \in \mathbb{Z}^d : \|x\| \leq t^\sigma\} \subset \{x \in \mathbb{Z}^d : \|x - y\| \leq t^{\widehat{\sigma}}\},$$

so

$$\sup_{x:\|x\| \leq t^\sigma} \langle |\delta_{x,0}^{y,t}| \rangle \leq \sup_{x:\|x-y\| \leq t^{\widehat{\sigma}}} \langle |\delta_{x,0}^{y,t}| \rangle.$$

Hence, (6.12) implies that

$$\lim_{t \to \infty} t^\theta \sup_{\|x\| \leq t^\sigma} \langle |\delta_{x,0}^{y,t}| \rangle = 0.$$

6.1.3. Proof of Lemma 6.6, Part (b)

We first show the following lemma.

Lemma 6.8

For any $\beta > 0$, we have

$$\lim_{t \to \infty} \left\langle \left| \sum_{\|x\| > t^\sigma} e^{c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t} \right| \right\rangle = 0.$$

Proof. It is easy to see from the definition of partition function $Z^{y,t}_{x,0}$ that $\left\langle Z^{y,t}_{x,0}\right\rangle = p^{y-x}_t$. Therefore,

$$\left\langle\left|\sum_{\|x\|>t^\sigma} e^{c\|x\|^{1-\epsilon}} Z^{y,t}_{x,0}\right|\right\rangle = \sum_{\|x\|>t^\sigma} e^{c\|x\|^{1-\epsilon}} p^{y-x}_t. \tag{6.13}$$

Recall that the continuous-time transition probability p^{y-x}_t satisfies

$$p^{y-x}_t = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!} q^{y-x}_n, \tag{6.14}$$

where $q^z_j := \mathbf{P}(\gamma_j = z|\gamma_0 = 0)$ is the transition probability for a discrete-time simple symmetric random walk $(\gamma_j)_{j\in\mathbb{N}_0}$ on $\mathbb{Z}^d$. If t is sufficiently large, we obtain the estimate

$$\|y-x\|_1^2 \geq \|y-x\|^2 \geq \frac{1}{2}\|x\|^2 > \frac{1}{2}t^{2\sigma} \quad \text{for all } x \text{ such that } \|x\| > t^\sigma, \tag{6.15}$$

so $q^{y-x}_n = 0$ for all $n \leq 2^{-1/2}t^\sigma$ and $x \in \mathbb{Z}^d$ such that $\|x\| > t^\sigma$. Furthermore, for any fixed $n > 2^{-1/2}t^\sigma$, we have $q^{y-x}_n = 0$ if $\|x\| \geq 2n$. Using these two observations together with (6.14) we can rewrite the righthand side of (6.13) as

$$\sum_{n>2^{-1/2}t^\sigma} e^{-t}\frac{t^n}{n!} \sum_{t^\sigma<\|x\|<2n} e^{c\|x\|^{1-\epsilon}} q^{y-x}_n. \tag{6.16}$$

It is well known that there are constants $c, \kappa > 0$ such that

$$q^z_j \leq c e^{-\kappa\|z\|^2/j}, \quad j \in \mathbb{N},\ z \in \mathbb{Z}^d. \tag{6.17}$$

Using the obvious estimate

$$q^z_j \leq \mathbf{P}\left(\max_{0\leq i\leq j} \|\gamma_i\| \geq \|z\|\right),$$

this can for instance be derived from Proposition 2.1.2 in [LL10]. On account of (6.17) and (6.15), the expression in (6.16), for large t, is bounded above by a constant times

$$\sum_{n>2^{-1/2}t^\sigma} e^{-t}\frac{t^n}{n!} \sum_{t^\sigma<\|x\|<2n} e^{c\|x\|^{1-\epsilon}-\kappa\frac{\|x\|^2}{2n}}.$$

Fix $\xi \in (1, \sigma(1+\epsilon))$ and split the sum above into

$$\sum_{2^{-1/2}t^\sigma<n\leq t^\xi} e^{-t}\frac{t^n}{n!} Y_n(t) + \sum_{n>t^\xi} e^{-t}\frac{t^n}{n!} Y_n(t), \tag{6.18}$$

where

$$Y_n(t) := \sum_{t^\sigma < \|x\| < 2n} e^{c\|x\|^{1-\epsilon} - \kappa \frac{\|x\|^2}{2n}}.$$

For $n \le t^\xi$ and $x \in \mathbb{Z}^d$ such that $\|x\| > t^\sigma$, we have

$$c\|x\|^{1-\epsilon} - \kappa \frac{\|x\|^2}{2n} \le \|x\|^2 \left(ct^{-\sigma(1+\epsilon)} - \frac{\kappa}{2} t^{-\xi} \right) \le -\|x\|^2 \frac{\kappa}{4} t^{-\xi}$$

provided that t is sufficiently large. This yields for $n \le t^\xi$ and t large enough

$$Y_n(t) \le \sum_{t^\sigma < \|x\| < 2n} e^{-\frac{\kappa}{4} \frac{\|x\|^2}{t^\xi}}.$$

Since

$$\left| \{x \in \mathbb{Z}^d : \|x\| \le r\} \right| = O(r^d),$$

we have for $n \le t^\xi$ the estimate

$$\left| \left\{ x \in \mathbb{Z}^d : t^\sigma < \|x\| < 2n \right\} \right| \lesssim t^{\xi d}.$$

As a result,

$$Y_n(t) \lesssim t^{\xi d} e^{-\frac{\kappa}{4} t^{2\sigma - \xi}} \xrightarrow[t \to \infty]{} 0. \tag{6.19}$$

Since the upper bound in (6.19) does not depend on n, we also have

$$\lim_{t \to \infty} \sum_{2^{-1/2} t^\sigma < n \le t^\xi} e^{-t} \frac{t^n}{n!} Y_n(t) = 0.$$

To deal with the second sum in (6.18), fix $\xi' > 1$ and notice that

$$Y_n(t) \le \sum_{t^\sigma < \|x\| < 2n} e^{c\|x\|^{1-\epsilon}} \lesssim n^d e^{c(2n)^{1-\epsilon}} \lesssim (\xi')^n.$$

The second sum in (6.18) is therefore bounded above by a constant times

$$\sum_{n > t^\xi} e^{-t} \frac{(\xi' t)^n}{n!}. \tag{6.20}$$

Using the tail estimate

$$\sum_{n=k}^{\infty} \frac{s^n}{n!} \leq \frac{s^k}{k!} \sum_{n=k}^{\infty} \left(\frac{s}{k}\right)^{n-k} = \frac{s^k}{k!} \frac{1}{1-\frac{s}{k}}, \quad k > s$$

and Stirling's formula, we see that (6.20) is bounded above by a constant times

$$t^{-\frac{\xi}{2}} e^{-t} \left(\frac{e\xi' t}{\lfloor t^{\xi} \rfloor}\right)^{\lfloor t^{\xi} \rfloor} \xrightarrow[t\to\infty]{} 0.$$

This completes the proof. □

Lemma 6.9

Let β be so small that the conclusion of Theorem 5.1 holds. Then, for every $\delta > 0$ there exist $u > 0$ and $T_1 > 0$ such that for all $t \geq T_1$,

$$Q\left(\sum_{\|x\| \leq t^{\sigma}} Z_{x,0}^{y,t} < e^{-u}\right) < \delta.$$

Proof. For $u > 2\ln 2$, we have

$$\begin{aligned} Q\left(\sum_{\|x\| \leq t^{\sigma}} Z_{x,0}^{y,t} < e^{-u}\right) &= Q\left(Z_0^{y,t} - \sum_{\|x\| > t^{\sigma}} Z_{x,0}^{y,t} < e^{-u}\right) \\ &\leq Q\left(Z_0^{y,t} < 2e^{-u}\right) + Q\left(\sum_{\|x\| > t^{\sigma}} Z_{x,0}^{y,t} > e^{-u}\right) \\ &\leq Q\left(Z_0^{y,t} < e^{-\frac{u}{2}}\right) + Q\left(\sum_{\|x\| > t^{\sigma}} Z_{x,0}^{y,t} > e^{-u}\right). \end{aligned} \tag{6.21}$$

Theorem 5.1 and Markov's inequality imply that the third line of (6.21) is less than

$$ce^{-\frac{u^2}{4c}} + e^{u} \left\langle \sum_{\|x\| > t^{\sigma}} Z_{x,0}^{y,t} \right\rangle = ce^{-\frac{u^2}{4c}} + e^{u} \sum_{\|x\| > t^{\sigma}} p_t^{y-x}$$

for some $c > 0$ that does not depend on u or t. Let u be so large that $ce^{-\frac{u^2}{4c}} < \frac{\delta}{2}$. Since $\sum_{\|x\| > t^{\sigma}} p_t^{y-x} \longrightarrow 0$ as $t \to \infty$, there exists $T_1 > 0$ such that for all $t \geq T_1$, we have $e^{u} \sum_{\|x\| > t^{\sigma}} p_t^{y-x} < \frac{\delta}{2}$.

□

PROOF OF (6.10): For any $f \in \mathcal{L}_{(c,\epsilon)}$, we have:

$$0 \leq \frac{\sum_{\|x\|>t^\sigma} f(x) Z_{x,0}^{y,t}}{\sum_{\|x\|\leq t^\sigma} f(x) Z_{x,0}^{y,t}} = \frac{\sum_{\|x\|>t^\sigma} e^{ct^{\sigma(1-\epsilon)}} f(x) Z_{x,0}^{y,t}}{\sum_{\|x\|\leq t^\sigma} e^{ct^{\sigma(1-\epsilon)}} f(x) Z_{x,0}^{y,t}} \leq \frac{\sum_{\|x\|>t^\sigma} e^{2c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t}}{\sum_{\|x\|\leq t^\sigma} Z_{x,0}^{y,t}}.$$

Let $\varepsilon, \delta > 0$. By Lemma 6.9, there are $u, T_1 > 0$ such that for every $t \geq T_1$,

$$Q\left(\sum_{\|x\|\leq t^\sigma} Z_{x,0}^{y,t} < e^{-u} \right) < \frac{\delta}{2}.$$

And by Lemma 6.8, there is $T \geq T_1$ such that for all $t \geq T$,

$$\left\langle \left| \sum_{\|x\|>t^\sigma} e^{2c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t} \right| \right\rangle < \frac{\delta \varepsilon e^{-u}}{2}.$$

For $t \geq T$, we have therefore

$$\begin{aligned}
& Q\left(\frac{\sum_{\|x\|>t^\sigma} e^{2c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t}}{\sum_{\|x\|\leq t^\sigma} Z_{x,0}^{y,t}} > \varepsilon \right) \\
\leq & Q\left(\sum_{\|x\|\leq t^\sigma} Z_{x,0}^{y,t} < e^{-u} \right) + Q\left(\sum_{\|x\|>t^\sigma} e^{2c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t} > \varepsilon e^{-u} \right) \\
< & \frac{\delta}{2} + \frac{e^u}{\varepsilon} \left\langle \left| \sum_{\|x\|>t^\sigma} e^{2c\|x\|^{1-\epsilon}} Z_{x,0}^{y,t} \right| \right\rangle < \delta.
\end{aligned}$$

This completes the proof of Lemma 6.6. □

6.2. Uniqueness of Invariant Probability Measures

After the attraction result Theorem 6.4, the second major step towards proving Theorem 6.3 goes through the theory of random dynamical systems; namely, we show uniqueness of physical invariant probability measures of a certain skew product that can be naturally associated with the (sSHE). In this section, we introduce this skew product and briefly recall some basic notions about random dynamical systems.

6.2.1. Preliminaries

Recall from (6.5) the sets $\mathcal{L}_{c,\epsilon}$. Define a set

$$\mathcal{L} := \bigcup_{\substack{c>0 \\ \epsilon\in(0,1)}} \mathcal{L}_{c,\epsilon}, \tag{6.22}$$

which can be thought of as the set of functions $f : \mathbb{Z}^d \to (0,\infty)$ of subexponential asymptotic growth and decay, normalized by imposing $f(0) = 1$. Notice that $\mathcal{L}$ is exactly the set of functions $f : \mathbb{Z}^d \to (0,\infty)$ that satisfy $f(0) = 1$ as well as the condition in (6.1). Note also that for any global stationary solution Z to sSHE from Theorem 1.6 (namely, with subexponential growth and with $Z(0,t,\omega) \neq 0$) the quotient $Z(y,t,\omega)/Z(0,t,\omega)$ is an element of $\mathcal{L}$.

Although the set of functions of subexponential growth has the structure of a vector space, the set $\mathcal{L}$ is not a vector space due to the requirement that every $f \in \mathcal{L}$ must satisfy $f(0) = 1$. However, $\mathcal{L}$ can be equipped with the structure of a metric space and the corresponding Borel σ-field $\mathcal{B}(\mathcal{L})$ using the metric

$$d(f,g) := \sum_{x\in\mathbb{Z}^d} e^{-\|x\|^2} |f(x) - g(x)|.$$

For $\omega \in \Omega$, $s,t \in \mathbb{R}$ such that $s < t$, and $f \in \mathcal{L}$, we define

$$L_\omega^{s,t} f(y) := \frac{\sum_{x\in\mathbb{Z}^d} f(x) Z_{x,s}^{y,t}(\omega)}{\sum_{x\in\mathbb{Z}^d} f(x) Z_{x,s}^{0,t}(\omega)}, \quad y \in \mathbb{Z}^d.$$

Lemma 6.10

The set $\mathcal{L}$ is Q-almost surely invariant under the dynamics induced by L, i.e. for Q-almost every $\omega \in \Omega$ the following holds: for every $f \in \mathcal{L}$ and for every $s,t \in \mathbb{R}$ such that $s < t$, we have $L_\omega^{s,t} f \in \mathcal{L}$.

Proof. We begin by defining some auxiliary functions and sets. For $x,y \in \mathbb{Z}^d$, let $g(x,y) := (1 + \|x\|)^{d+2}(1 + \|y\|)^{d+2}$. For $M \in \mathbb{N}$, let

$$\Omega_M^* := \Big\{ \omega \in \Omega : \exists R_M(\omega) > 0 \text{ such that } \forall x,y \in \mathbb{Z}^d \text{ with } \max\{\|x\|,\|y\|\} > R_M(\omega) : \\ \sup_{s,t\in(-M,M),s<t} Z_{x,s}^{y,t}(\omega) \le (p_{2M}^{y-x})^{\frac{1}{2}} g(x,y), \inf_{s,t\in(-M,M),s<t} Z_{x,s}^{x,t}(\omega) \ge g(x,x)^{-1} \Big\}.$$

Finally, set $\Omega^* := \bigcap_{M\in\mathbb{N}} \Omega_M^*$. We first show that $Q(\Omega_M^*) = 1$ for every $M \in \mathbb{N}$, and hence $Q(\Omega^*) = 1$.

Fix $M \in \mathbb{N}$. For $x, y \in \mathbb{Z}^d$, consider the events

$$E'_M(x,y) := \left\{ \omega \in \Omega : \sup_{s,t\in(-M,M), s<t} Z^{y,t}_{x,s}(\omega) > (p^{y-x}_{2M})^{\frac{1}{2}} g(x,y) \right\},$$
$$E''_M(x) := \left\{ \omega \in \Omega : \inf_{s,t\in(-M,M), s<t} Z^{x,t}_{x,s}(\omega) < g(x,x)^{-1} \right\}.$$

By the first Borel–Cantelli lemma, $Q(\Omega^*_M) = 1$ will follow once we show that

$$\sum_{x,y\in\mathbb{Z}^d} Q(E'_M(x,y)) + \sum_{x\in\mathbb{Z}^d} Q(E''_M(x)) < \infty.$$

By Markov's inequality,

$$Q(E'_M(x,y)) \le \frac{(p^{y-x}_{2M})^{-\frac{1}{2}} \langle \sup_{s,t\in(-M,M), s<t} Z^{y,t}_{x,s} \rangle}{g(x,y)}.$$

We will now show that the expression in the numerator is bounded by a constant that only depends on M and β. For fixed $\omega \in \Omega$ and $s, t \in (-M, M)$ such that $s < t$, we can estimate

$$\begin{aligned} Z^{y,t}_{x,s}(\omega) <& \frac{1}{p^0_{s+M}} \int \mathbb{1}_{\eta_s=x,\eta_t=y} e^{\beta \mathcal{A}^t_s(\eta,\omega)}\, \mathbf{P}_{x,-M}(d\eta) \\ \le& \left(\frac{p^{y-x}_{t-s}}{p^0_{s+M}} \right)^{\frac{1}{2}} \left(\int e^{2\beta|\mathcal{A}^t_s(\eta,\omega)|}\, \mathbf{P}_{x,-M}(d\eta) \right)^{\frac{1}{2}}, \end{aligned} \tag{6.23}$$

where the integral is taken over possible realizations of η, and where the Cauchy–Schwarz inequality was used. For any $s \in (-M, M)$, we have

$$p^0_{s+M} \ge e^{-(s+M)} \ge e^{-2M}. \tag{6.24}$$

Moreover, for every $z \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\frac{d}{dr} p^z_r \ge 0, \quad \forall r \in (0, \|z\|_1).$$

To see this, recall that for every $z \in \mathbb{Z}^d$ and $r \ge 0$,

$$p^z_r = e^{-r} \sum_{n=0}^{\infty} \frac{r^n}{n!} q^z_n.$$

Differentiating the expression on the right with respect to r yields

$$e^{-r}\left(-q_0^z+\sum_{n=1}^{\infty}q_n^z\left(\frac{r^{n-1}}{(n-1)!}-\frac{r^n}{n!}\right)\right).$$

Since $z\neq 0$, we have $q_0^z=0$. For every $n\geq r$,

$$\frac{r^n}{n!}=\frac{r}{n}\cdot\frac{r^{n-1}}{(n-1)!}\leq\frac{r^{n-1}}{(n-1)!}.$$

And for $n<r<\|z\|_1$, we have $q_n^z=0$. This proves the claim. For every $x,y\in\mathbb{Z}^d$ such that $\|y-x\|_1>2M$ and $s,t\in(-M,M)$ with $s<t$, we have thus

$$p_{t-s}^{y-x}\leq p_{2M}^{y-x}.$$

Hence, for every $x,y\in\mathbb{Z}^d$ and for every $s,t\in(-M,M)$ with $s<t$,

$$p_{t-s}^{y-x}\leq\mathbb{1}_{\|y-x\|_1>2M}p_{2M}^{y-x}+\mathbb{1}_{\|y-x\|_1\leq 2M}.$$

Together with (6.23) and (6.24), this yields

$$\begin{aligned}&(p_{2M}^{y-x})^{-\frac{1}{2}}Z_{x,s}^{y,t}(\omega)\\ \leq&e^{M}(1+\max_{z\in\mathbb{Z}^d:\|z\|_1\leq 2M}(p_{2M}^z)^{-\frac{1}{2}})\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}.\end{aligned}\tag{6.25}$$

From Lemma 3.10, we have that

$$\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}<h(M),\tag{6.26}$$

where $h(M)=2\cdot\sqrt{2}e^{516M\beta^2}\exp(2M(\sqrt{2}-1))$. Hence, together with (6.25),

$$Q(E'_M(x,y))\leq\frac{H(M)}{g(x,y)},$$

where

$$H(M):=e^{M}\left(1+\max_{z\in\mathbb{Z}^d:\|z\|_1\leq 2M}(p_{2M}^z)^{-\frac{1}{2}}\right)h(M).$$

Furthermore, by Markov's inequality, an estimate similar to (6.24), and Jensen's inequality,

$$Q(E''_M(x)) \leq \frac{\langle \sup_{s,t\in(-M,M),s<t}(1/Z_{x,s}^{x,t})\rangle}{g(x,x)} \tag{6.27}$$
$$\leq e^{\beta^2 M+2M}\frac{\langle \sup_{s,t\in(-M,M),s<t}\mathbf{E}_{x,s}^{x,t}e^{-\beta\mathcal{A}_s^t}\rangle}{g(x,x)}.$$

For fixed $\omega \in \Omega$ and fixed $s,t \in (-M,M)$ such that $s<t$, we have

$$\mathbf{E}_{x,s}^{x,t}e^{-\beta\mathcal{A}_s^t(\cdot,\omega)} \leq \left(p_{s+M}^0 p_{t-s}^0\right)^{\frac{1}{2}}\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}$$
$$\leq \left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\omega)|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}},$$

and we have already established that

$$\left\langle \sup_{s,t\in(-M,M),s<t}\left(\int e^{2\beta|\mathcal{A}_s^t(\eta,\dot{)}|}\,\mathbf{P}_{x,-M}(d\eta)\right)^{\frac{1}{2}}\right\rangle \leq h(M).$$

Hence, the expression on the right-hand side of (6.27) is bounded from above by

$$\frac{e^{\beta^2 M+2M}h(M)}{g(x,x)}.$$

Then

$$\sum_{x,y\in\mathbb{Z}^d} Q(E'_M(x,y)) \leq H(M)\left(\sum_{x\in\mathbb{Z}^d}\frac{1}{(1+\|x\|)^{d+2}}\right)^2 < \infty$$

and

$$\sum_{x\in\mathbb{Z}^d} Q(E''_M(x)) \leq e^{\beta^2 M+2M}h(M)\sum_{x\in\mathbb{Z}^d}\frac{1}{(1+\|x\|)^{2(d+2)}} < \infty.$$

This shows that $Q(\Omega_M^*)=1$, and thus $Q(\Omega^*)=1$.

For every $\omega\in\Omega$, we have

$$Z_{x,s}^{y,t}(\omega) > 0, \quad \forall x,y\in\mathbb{Z}^d,\ \forall s,t\in(-M,M), s<t.$$

Moreover, by Lemma 3.10, for all $x,y\in\mathbb{Z}^d$,

$$\left\langle \sup_{s,t\in(-M,M),s<t} Z_{x,s}^{y,t}\right\rangle < \infty.$$

As a result, we have for Q-almost every $\omega \in \Omega$

$$0 < Z_{x,s}^{y,t}(\omega) < \infty \quad \forall x, y \in \mathbb{Z}^d,\ s, t \in (-M, M), s < t.$$

By subtracting from Ω_M^* a set of measure 0 and calling the resulting set still Ω_M^*, we can then assume without loss of generality that for every $\omega \in \Omega^*$

$$0 < Z_{x,s}^{y,t}(\omega) < \infty \tag{6.28}$$

for all $x, y \in \mathbb{Z}^d$, $M \in \mathbb{N}$, and $s, t \in (-M, M)$ such that $s < t$.

Fix $\omega \in \Omega^*$, $f \in \mathcal{L}$, and $s, t \in \mathbb{R}$ such that $s < t$. We need to show that $L_\omega^{s,t} f \in \mathcal{L}$, i.e. we need to show that there exist $\tilde{c} > 0$ and $\tilde{\epsilon} \in (0, 1)$, depending on ω, u, s, and t, such that

$$e^{-\tilde{c}\|y\|^{1-\tilde{\epsilon}}} \leq L_\omega^{s,t} f(y) \leq e^{\tilde{c}\|y\|^{1-\tilde{\epsilon}}}, \quad \forall y \in \mathbb{Z}^d.$$

Since $f \in \mathcal{L}$, there exist $c > 0$ and $\epsilon \in (0, 1)$ such that $e^{-c\|x\|^{1-\epsilon}} \leq f(x) \leq e^{c\|x\|^{1-\epsilon}}$ for all $x \in \mathbb{Z}^d$. Let $M \in \mathbb{N}$ be so large that $s, t \in (-M, M)$. Since $\omega \in \Omega_M^*$, we can estimate the numerator of $L_\omega^{s,t} f(y)$ for $y \in \mathbb{Z}^d$ such that $\|y\| > R_M(\omega)$ as follows:

$$\sum_{x \in \mathbb{Z}^d} f(x) Z_{x,s}^{y,t}(\omega) \leq \sum_{x \in \mathbb{Z}^d} e^{c\|x\|^{1-\epsilon}} (p_{2M}^{y-x})^{\frac{1}{2}} g(x, y). \tag{6.29}$$

Next we write

$$p_{2M}^{y-x} = e^{-2M} \sum_{n=0}^{\infty} \frac{(2M)^n}{n!} q_n^{y-x}$$

and note that

$$(p_{2M}^{y-x})^{\frac{1}{2}} < \sum_{n=0}^{\infty} \left(\frac{(2M)^n}{n!} q_n^{y-x} \right)^{\frac{1}{2}}.$$

The transition probability q_n^{y-x} can only be positive if $\|y - x\|_1 \leq n$. If $\|x\| > \|y\| + n$, we have

$$\|y - x\|_1 \geq \|y - x\| \geq \|x\| - \|y\| > n,$$

so $q_n^{y-x} = 0$. It follows that the expression on the right-hand side of (6.29) is bounded from above by

$$(1 + \|y\|)^{d+2} \sum_{n=0}^{\infty} \left(\frac{(2M)^n}{n!} \right)^{\frac{1}{2}} Y_n, \tag{6.30}$$

where

$$Y_n := \sum_{x:\|x\|\le\|y\|+n} \left(q_n^{y-x}\right)^{\frac{1}{2}} e^{c\|x\|^{1-\epsilon}}(1+\|x\|)^{d+2}.$$

We split the expression from (6.30) in two:

$$(1+\|y\|)^{d+2} \sum_{0\le n\le\|y\|} \left(\frac{(2M)^n}{n!}\right)^{\frac{1}{2}} Y_n \tag{6.31}$$

$$+(1+\|y\|)^{d+2} \sum_{n>\|y\|} \left(\frac{(2M)^n}{n!}\right)^{\frac{1}{2}} Y_n. \tag{6.32}$$

For $0 \le n \le \|y\|$, we have the estimate

$$\begin{aligned}
Y_n \le& \sum_{x:\|x\|\le 2\|y\|} \left(q_n^{y-x}\right)^{\frac{1}{2}} e^{c\|x\|^{1-\epsilon}}(1+\|x\|)^{d+2}\\
\le& e^{c2^{1-\epsilon}\|y\|^{1-\epsilon}}(1+2\|y\|)^{d+2} \sum_{x:\|x\|\le 2\|y\|} \left(q_n^{y-x}\right)^{\frac{1}{2}}\\
\lesssim& e^{c2^{1-\epsilon}\|y\|^{1-\epsilon}}(1+2\|y\|)^{d+2}\|y\|^d.
\end{aligned}$$

If we set

$$m(a) := (1+a)^{d+2}(1+2a)^{d+2}a^d,$$

we can bound the expression in (6.31) from above by a constant times

$$\sum_{n=0}^{\infty} \left(\frac{(2M)^n}{n!}\right)^{\frac{1}{2}} m(\|y\|)e^{c2^{1-\epsilon}\|y\|^{1-\epsilon}}.$$

Here it is important to notice that $\sum(\frac{(2M)^n}{n!})^{\frac{1}{2}}$ is finite and independent of y, and that $m(\|y\|)$ is polynomial in $\|y\|$.

Suppose now that $n > \|y\|$. Then

$$\begin{aligned}
Y_n \le& \sum_{x:\|x\|\le 2n} \left(q_n^{y-x}\right)^{\frac{1}{2}} e^{c\|x\|^{1-\epsilon}}(1+\|x\|)^{d+2}\\
\le& e^{c2^{1-\epsilon}n^{1-\epsilon}}(1+2n)^{d+2} \sum_{x:\|x\|\le 2n} \left(q_n^{y-x}\right)^{\frac{1}{2}}\\
\lesssim& e^{c2^{1-\epsilon}n^{1-\epsilon}}(1+2n)^{d+2}n^d.
\end{aligned}$$

The expression in (6.32) is hence bounded from above by a constant times

$$\sum_{n=0}^{\infty}\left(\frac{(2M)^n}{n!}\right)^{\frac{1}{2}} e^{c2^{1-\epsilon}n^{1-\epsilon}}(1+2n)^{d+2}n^d(1+\|y\|)^{d+2},$$

where one should note that the series converges. We have thus shown that there exists a polynomial p, with coefficients depending only on M, β, and d, such that

$$\sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{y,t}(\omega) \le p(\|y\|)e^{c2^{1-\epsilon}\|y\|^{1-\epsilon}}$$

for all $y \in \mathbb{Z}^d$ with $\|y\| > R_M(\omega)$. In light of (6.28), it is then possible to choose $\tilde{c}_1 > 0$ so large that

$$\sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{y,t}(\omega) \le \sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{0,t}(\omega)e^{\tilde{c}_1\|y\|^{1-\epsilon}}, \quad \forall y \in \mathbb{Z}^d,$$

and thus

$$L_\omega^{s,t}f(y) \le e^{\tilde{c}_1\|y\|^{1-\epsilon}}, \quad \forall y \in \mathbb{Z}^d.$$

The proof of the lower bound for $L_\omega^{s,t}f$ is simpler: Since $\omega \in \Omega_M^1$, we have for every $y \in \mathbb{Z}^d$ with $\|y\| > R_M(\omega)$

$$\sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{y,t}(\omega) \ge \sum_{x\in\mathbb{Z}^d} e^{-c\|x\|^{1-\epsilon}} Z_{x,s}^{y,t}(\omega) \ge e^{-c\|y\|^{1-\epsilon}} Z_{y,s}^{y,t}(\omega) \ge \frac{e^{-c\|y\|^{1-\epsilon}}}{g(y,y)}.$$

Then, again by virtue of (6.28), we can choose $\tilde{c}_2 > 0$ so large that

$$\sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{y,t}(\omega) \ge \sum_{x\in\mathbb{Z}^d} f(x)Z_{x,s}^{0,t}(\omega)e^{-\tilde{c}_2\|y\|^{1-\epsilon}}, \quad \forall y \in \mathbb{Z}^d,$$

and hence we have for $\tilde{c} := \max\{\tilde{c}_1, \tilde{c}_2\}$ the estimate

$$e^{-\tilde{c}\|y\|^{1-\epsilon}} \le L_\omega^{s,t}f(y) \le e^{\tilde{c}\|y\|^{1-\epsilon}}, \quad \forall y \in \mathbb{Z}^d.$$

□

Lemma 6.11

There is a set $\widetilde{\Omega} \subset \Omega$ *with* $Q(\widetilde{\Omega}) = 1$ *that satisfies the following conditions:*

(1) $\widetilde{\Omega}$ *is invarian t under* θ_s *for every* $s \in \mathbb{R}$*, i.e.* $\theta_s(\widetilde{\Omega}) = \widetilde{\Omega}$ *for every* $s \in \mathbb{R}$*;*

(2) *For every* $\omega \in \widetilde{\Omega}$*,* $f \in \mathcal{L}$*, and* $s, t \in \mathbb{R}$ *such that* $s < t$*, we have* $L_\omega^{s,t} f \in \mathcal{L}$*;*

(3) *For every* $\omega \in \widetilde{\Omega}$*,*

$$\lim_{s \to -\infty} Z_s^{x,0}(\omega) = Z_{-\infty}^{x,0}(\omega) > 0, \quad \forall x \in \mathbb{Z}^d;$$

(4) *For every* $\omega \in \widetilde{\Omega}$*, the function* $x \mapsto \widetilde{Z}_{-\infty}^{x,0}(\omega) := Z_{-\infty}^{x,0}(\omega)/Z_{-\infty}^{0,0}(\omega)$ *is an element of* $\mathcal{L}$*.*

Proof. In order to prove this lemma it is enough to find a set $\widetilde{\Omega} \subset \Omega^+ \cap \Omega^L$ such that $\widetilde{Z}_{\infty}^{x,0}(\omega) \in \mathcal{L}$ for every $\omega \in \widetilde{\Omega}$. Let $\widetilde{\Omega}$ be the set given by

$$\widetilde{\Omega} := \left\{ \omega \in \Omega^+ \cap \Omega^L \ : \ \begin{array}{c} \exists R(\omega) > 0 \text{ s.t } \forall x \in \mathbb{Z}^d \text{ with } \|x\| > R(\omega) : \\ (1 + \|x\|)^{-(d+2)} \leq Z_{-\infty}^{x,0}(\omega) \leq (1 + \|x\|)^{(d+2)} \end{array} \right\}. \tag{6.33}$$

Define the sets

$$E_1(x) := \{\omega \in \Omega^+ \cap \Omega^L \ : \ Z_{-\infty}^{x,0}(\omega) > (1 + \|x\|)^{(d+2)}\}$$
$$E_2(x) := \{\omega \in \Omega^+ \cap \Omega^L \ : \ Z_{-\infty}^{x,0}(\omega) < (1 + \|x\|)^{-(d+2)}\},$$

and let $E(x) = E_1(x) \cup E_2(x)$.

By the Borel-Cantelli Lemma, in order to show that $Q(\widetilde{\Omega}) = 1$, it is enough to show that $\sum_{x \in \mathbb{Z}^d} Q(E(x)) < \infty$. Indeed, notice that by Markov's inequality we have

$$Q(E_1(x)) \leq \frac{\langle Z_{-\infty}^{x,0} \rangle}{(1 + \|x\|)^{d+2}} = \frac{\langle Z_{-\infty}^{0,0} \rangle}{(1 + \|x\|)^{d+2}} = \frac{1}{(1 + \|x\|)^{d+2}}.$$

Similarly, using again Markov's inequality together with the fact that all negative moments exists by Remark 5.2,

$$Q(E_2(x)) \leq \frac{\left\langle \frac{1}{Z_{-\infty}^{x,0}} \right\rangle}{(1 + \|x\|)^{d+2}} = \frac{\left\langle \frac{1}{Z_{-\infty}^{0,0}} \right\rangle}{(1 + \|x\|)^{d+2}} = \frac{C}{(1 + \|x\|)^{d+2}}.$$

for some $C > 0$.

Hence,

$$\sum_{x\in\mathbb{Z}^d} Q(E(x)) \le \sum_{x\in\mathbb{Z}^d} \frac{C+1}{(1+\|x\|)^{d+2}} < \infty,$$

which completes the proof of the lemma. □

6.2.2. Invariant Measure for the Skew Product

Lemma 6.12

The map $\Phi : [0,\infty) \times \widetilde{\Omega} \times \mathcal{L} \to \mathcal{L}$, *given by*

$$(t,\omega,f) \mapsto \Phi^t_\omega f := L^{0,t}_\omega f,$$

defines a cocycle; i.e., for all $s,t \ge 0$ *and all* $\omega \in \widetilde{\Omega}$,

$$\Phi^{s+t}_\omega = \Phi^t_{\theta_s\omega} \circ \Phi^s_\omega.$$

Proof. It is not hard to see that Φ is a $(\mathcal{B}([0,\infty)) \otimes \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L}), \mathcal{B}(\mathcal{L}))$-measurable map, where $\mathcal{B}([0,\infty))$ is the Borel σ-field on $[0,\infty)$ and $\widetilde{\mathcal{F}}$ is the restriction of $\mathcal{F}$ to $\widetilde{\Omega}$. On $\widetilde{\Omega} \times \mathcal{L}$, we define the skew product

$$\Theta^t(\omega,f) := (\theta_t(\omega), \Phi^t_\omega f), \quad t \ge 0.$$

If either s or t are zero, the result follows immediately. Let $t,s >$, then

$$\begin{aligned}
\Phi^t_{\theta_s\omega} \circ \Phi^s_\omega u(y) =& \frac{\sum_{x\in\mathbb{Z}^d} \Phi^s_\omega u(x) Z^{y,t}_{x,0}(\theta_s\omega)}{\sum_{x\in\mathbb{Z}^d} \Phi^s_\omega u(x) Z^{0,t}_{x,0}(\theta_s\omega)} \\
=& \frac{\sum_{x\in\mathbb{Z}^d} \frac{\sum_{z\in\mathbb{Z}^d} u(z) Z^{x,s}_{z,0}(\omega)}{\sum_{z\in\mathbb{Z}^d} u(x) Z^{0,s}_{z,0}(\omega)} Z^{y,t}_{x,0}(\theta_s\omega)}{\sum_{x\in\mathbb{Z}^d} \frac{\sum_{z\in\mathbb{Z}^d} u(z) Z^{x,s}_{z,0}(\omega)}{\sum_{z\in\mathbb{Z}^d} u(x) Z^{0,s}_{z,0}(\omega)} Z^{0,t}_{x,0}(\theta_s\omega)} \\
=& \frac{\sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} u(z) Z^{x,s}_{z,0}(\omega) Z^{y,t}_{x,0}(\theta_s\omega)}{\sum_{x\in\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} u(z) Z^{x,s}_{z,0}(\omega) Z^{0,t}_{x,0}(\theta_s\omega)} \\
=& \frac{\sum_{z\in\mathbb{Z}^d} u(z) \sum_{x\in\mathbb{Z}^d} Z^{x,s}_{z,0}(\omega) Z^{y,t+s}_{x,s}(\omega)}{\sum_{z\in\mathbb{Z}^d} u(z) \sum_{x\in\mathbb{Z}^d} Z^{x,s}_{z,0}(\omega) Z^{0,t+s}_{x,s}(\omega)} \\
=& \frac{\sum_{z\in\mathbb{Z}^d} u(z) Z^{y,t+s}_{z,0}(\omega)}{\sum_{z\in\mathbb{Z}^d} u(z) Z^{0,t+s}_{z,0}(\omega)} \\
=& \Phi^{t+s}_\omega u(y)
\end{aligned}$$

□

Definition 6.13

An ***invariant probability measure*** for the skew product $(\Theta^t)_{t\geq 0}$ is a probability measure μ on $(\widetilde{\Omega}\times\mathcal{L}, \widetilde{\mathcal{F}}\otimes\mathcal{B}(\mathcal{L}))$ such that

(1) μ has marginal $\widetilde{Q}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$, where $\widetilde{Q}$ is the restriction of Q to $\widetilde{\Omega}$;

(2) $\mu((\Theta^t)^{-1}(\cdot)) = \mu(\cdot), \quad \forall t\geq 0.$

If μ is an invariant probability measure for $(\Theta^t)_{t\geq 0}$, then there exists a family $(\mu^\omega)_{\omega\in\widetilde{\Omega}}$ of probability measures on $(\mathcal{L}, \mathcal{B}(\mathcal{L}))$, so-called sample measures, such that for every $A\in\widetilde{\mathcal{F}}\otimes\mathcal{B}(\mathcal{L})$,

$$\mu(A) = \int_{\widetilde{\Omega}} \mu^\omega(A_\omega)\, Q(d\omega),$$

where $A_\omega := \{f\in\mathcal{L} : (\omega, f)\in A\}$.

Theorem 6.14

The skew product $(\Theta^t)_{t\geq 0}$ admits a unique invariant probability measure whose sample measures are given by

$$\mu^\omega(\cdot) = \delta_{y\mapsto \widetilde{Z}^{y,0}_{-\infty}(\omega)}(\cdot), \quad \omega\in\widetilde{\Omega}.$$

Remark 6.15

One can define a Markov semigroup $(\mathbf{P}^t)_{t\geq 0}$ on $\mathcal{L}$ by setting

$$\mathbf{P}^t(f, F) := \widetilde{Q}\left(\left\{\omega\in\widetilde{\Omega} : \Phi^t_\omega f\in F\right\}\right), \quad f\in\mathcal{L},\; F\in\mathcal{B}(\mathcal{L}).$$

By Ledrappier-Young, there is a one-to-one correspondence between invariant probability measures for $(\mathbf{P}^t)_{t\geq 0}$ and so-called physical invariant probability measures for the skew product $(\Theta^t)_{t\geq 0}$. The latter are invariant probability measures μ for $(\Theta^t)_{t\geq 0}$ with sample measures (μ^ω) such that $\omega\mapsto\mu^\omega$ is measurable with respect to $\sigma(W^y_u : u\leq 0, y\in\mathbb{Z}^d)$. It is easy to see that the unique invariant probability measure from Theorem 6.14 is physical. Therefore, $(\mathbf{P}^t)_{t\geq 0}$ admits a unique invariant probability measure given by

$$\int_{\widetilde{\Omega}} \delta_{x\mapsto \widetilde{Z}^{x,0}_{-\infty}(\omega)}(\cdot)\widetilde{Q}(d\omega).$$

Proof. We first show that the measure given by

$$\mu(A) = \int_{\widetilde{\Omega}} \mu^{\omega}(A_{\omega})\, Q(d\omega),$$

where $\mu^{\omega}(\cdot) = \delta_{y \mapsto \widetilde{Z}^{y,0}_{-\infty}(\omega)}(\cdot)$ and $A_{\omega} := \{f \in \mathcal{L} : (\omega, f) \in A\}$ is an invariant probability measure for the skew product $(\Theta^t)_{t\geq 0}$.

EXISTENCE. Fix $t > 0$. Let $A \in \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L})$. We write $A = A_1 \times A_2$ where $A_1 \in \widetilde{\mathcal{F}}$ and $A_2 \in \mathcal{B}(\mathcal{L})$. Then,

$$\begin{aligned}
\mu((\Theta^t)^{-1}A) &= \mu(\{(\omega, u) \in \widetilde{\Omega} \times \mathcal{L} \;:\; \Theta^t(\omega, u) \in A\}) \\
&= \mu(\{(\omega, u) \in \widetilde{\Omega} \times \mathcal{L} \;:\; (\theta_t(\omega), \Phi^t_{\omega} u) \in A\}) \\
&= \mu^{\omega}(\{u \in \mathcal{L} \;:\; (\theta_t(\omega), \Phi^t_{\omega} u) \in A\})\widetilde{Q}(d\omega) \\
&= \int_{\widetilde{\Omega}} \mathbb{1}_{\{\theta_t(\omega) \in A_1\}} \mathbb{1}_{\{\Phi^t_{\omega} \widetilde{Z}^{x,0}_{-\infty} \in A_2\}} \widetilde{Q}(d\omega)
\end{aligned}$$

Notice that

$$\begin{aligned}
\{\omega \;:\; \Phi^t_{\omega} \widetilde{Z}^{x,0}_{-\infty} \in A_2\} &= \left\{\omega \;:\; \frac{\sum_{y \in \mathbb{Z}^d} Z^{y,0}_{-\infty}(\omega) Z^{x,t}_{y,0}(\omega)}{\sum_{y \in \mathbb{Z}^d} Z^{y,0}_{-\infty}(\omega) Z^{0,t}_{y,0}(\omega)} \in A_2\right\} \\
&= \left\{\omega \;:\; \frac{Z^{x,t}_{-\infty}(\omega)}{Z^{0,t}_{-\infty}(\omega)} \in A_2\right\} \\
&= \left\{\omega \;:\; \widetilde{Z}^{x,0}_{-\infty}(\theta_t(\omega)) \in A_2\right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu((\Theta^t)^{-1}A) &= \int_{\widetilde{\Omega}} \mathbb{1}_{\{\theta_t(\omega) \in A_1\}} \mathbb{1}_{\{\widetilde{Z}^{x,0}_{-\infty}(\theta_t(\omega)) \in A_2\}} \\
&= \int_{\widetilde{\Omega}} \mathbb{1}_{\{\omega \in A_1\}} \mathbb{1}_{\{\widetilde{Z}^{x,0}_{-\infty}(\omega) \in A_2\}} \\
&= \int_{\widetilde{\Omega}} \mu^{\omega}(\{u \in \mathcal{L} \;:\; (\omega, u) \in A\}) \widetilde{\Omega}(d\omega) \\
&= \mu(A)
\end{aligned}$$

UNIQUENESS. Let μ be an invariant probability measure for $(\Theta^t)_{t\geq 0}$. Then there exists a family

$(\mu^\omega)_{\omega\in\widetilde{\Omega}}$ of probability measures on $(\mathcal{L},\mathcal{B}(\mathcal{L}))$ such that

$$\mu(A) = \int_{\widetilde{\Omega}} \mu^\omega(A_\omega)\,\widetilde{Q}(d\omega).$$

To simplify notation, let $\mathcal{Z}(\omega)$ be the element of $\mathcal{L}$ defined by $\mathcal{Z}(\omega)[y] := \widetilde{Z}^{y,0}_{-\infty}(\omega)$. We need to show that for $\widetilde{Q}$-a.e. $\omega \in \widetilde{\Omega}$,

$$\mu^\omega = \delta_{\mathcal{Z}(\omega)}.$$

Let $\delta > 0$ be arbitrary but fixed. We will show that

$$\widetilde{Q}(\{\omega \in \widetilde{\Omega} : \mu^\omega = \delta_{\mathcal{Z}(\omega)}\}) > 1-\delta.$$

If $F : \mathcal{L} \to \mathbb{R}$ is a bounded measurable function and if μ is a probability measure on $(\mathcal{L},\mathcal{B}(\mathcal{L}))$, we set

$$\mu F := \int_{\mathcal{L}} F(u)\,\mu(du).$$

Let $\text{Lip}^1_b(\mathcal{L})$ denote the space of bounded Lipschitz continuous functions $F : \mathcal{L} \to \mathbb{R}$ for which there are $L, b > 0$ such that $L + b \le 1$ and $|F(u)| \le b$, $|F(u) - F(v)| \le Ld(u,v)$ for all $u, v \in \mathcal{L}$. We have

$$\{\omega \in \widetilde{\Omega} : \mu^\omega F = \delta_{\mathcal{Z}(\omega)}F\ \forall F \in \text{Lip}^1_b(\mathcal{L})\} = \{\omega \in \widetilde{\Omega} : \mu^\omega = \delta_{\mathcal{Z}(\omega)}\}$$

by the Portemanteau Theorem [Kle08, Theorem 13.16]. It is therefore enough to show that

$$\widetilde{Q}(\{\omega \in \widetilde{\Omega} : \mu^\omega F = \delta_{\mathcal{Z}(\omega)}F\ \forall F \in \text{Lip}^1_b(\mathcal{L})\}) > 1-\delta.$$

For $V \subset \mathbb{Z}^d$, let

$$\pi_V : \mathcal{L} \to \mathcal{L},\ u \mapsto \pi_V(u),$$

where

$$\pi_V(u)[x] := u(x)\mathbb{1}_{x\in V} + \mathbb{1}_{x\notin V}.$$

Let $(V_m)_{m\ge1}$ be an enumeration of all finite subsets of $\mathbb{Z}^d$.

Claim 6.1

Suppose that there is a family of subsets of $\widetilde{\Omega}$, $(\hat{\Omega}_{n,m})_{n,m\in\mathbb{N}}$, *such that the following holds:*

(1) $\widetilde{Q}(\hat{\Omega}_{n,m}) > 1 - 2^{-(n+m)}\delta$ *for all* $n, m \in \mathbb{N}$;

(2) *For every* $F \in \mathit{Lip}_b^1(\mathcal{L})$ *and* $\omega \in \hat{\Omega}_{n,m}$,

$$|\mu^\omega(F \circ \pi_{V_m}) - \delta_{\mathcal{Z}(\omega)}(F \circ \pi_{V_m})| < \frac{1}{n}.$$

Then, we have

$$\widetilde{Q}(\{\omega \in \widetilde{\Omega} : \mu^\omega F = \delta_{\mathcal{Z}(\omega)} F \ \forall F \in \mathit{Lip}_b^1(\mathcal{L})\}) > 1 - \delta.$$

PROOF: Since, by (1),

$$\widetilde{Q}(\bigcap_{n,m\in\mathbb{N}} \hat{\Omega}_{n,m}) > 1 - \delta,$$

it is enough to show that

$$\bigcap_{n,m\in\mathbb{N}} \hat{\Omega}_{n,m} \subset \{\omega \in \widetilde{\Omega} : \mu^\omega F = \delta_{\mathcal{Z}(\omega)} F \ \forall F \in \mathrm{Lip}_b^1(\mathcal{L})\}.$$

Fix $\omega \in \bigcap_{n,m\in\mathbb{N}} \hat{\Omega}_{n,m}$. Let $F \in \mathrm{Lip}_b^1(\mathcal{L})$. Since $\omega \in \bigcap_{n,m\in\mathbb{N}} \hat{\Omega}_{n,m}$, we have for every finite subset V of $\mathbb{Z}^d$

$$|\mu^\omega(F \circ \pi_V) - \delta_{\mathcal{Z}(\omega)}(F \circ \pi_V)| = 0. \tag{6.34}$$

Let $(\tilde{V}_k)_{k\geq 1}$ be an increasing sequence (w.r.t. set inclusion) of finite subsets of $\mathbb{Z}^d$ such that $\bigcup_{k\geq 1} \tilde{V}_k = \mathbb{Z}^d$. We claim that $\pi_{\tilde{V}_k}$ converges pointwise to the identity on $\mathcal{L}$. To see this, let $u \in \mathcal{L}$. Then,

$$d(\pi_{\tilde{V}_k}(u), u) = \sum_{x\in\mathbb{Z}^d} e^{-\|x\|^2} |\pi_{\tilde{V}_k}(u)[x] - u(x)| = \sum_{x\notin\tilde{V}_k} e^{-\|x\|^2} |u(x) - 1| \to 0$$

as $k \to \infty$, as the series converges. Since F is continuous, it follows that $F \circ \pi_{\tilde{V}_k}$ converges pointwise to F. Since F is bounded, we have by bounded convergence

$$\lim_{k\to\infty} \int_{\mathcal{L}} F(\pi_{\tilde{V}_k}(u))\, \mu^\omega(du) = \int_{\mathcal{L}} F(u)\, \mu^\omega(du).$$

The claim then follows from (6.34). □

We still need to show that the assumption of the previous lemma holds, i.e. there is a family of subsets of $\widetilde{\Omega}$, $(\hat{\Omega}_{n,m})_{n,m\in\mathbb{N}}$, such that

(1) $\widetilde{Q}(\hat{\Omega}_{n,m}) > 1 - 2^{-(n+m)}\delta$ for all $n, m \in \mathbb{N}$;

(2) For every $F \in \mathrm{Lip}_b^1(\mathcal{L})$ and $\omega \in \hat{\Omega}_{n,m}$,

$$|\mu^\omega(F \circ \pi_{V_m}) - \delta_{\mathcal{Z}(\omega)}(F \circ \pi_{V_m})| < \frac{1}{n}.$$

Fix $n, m \in \mathbb{N}$. We choose $c_0 > 0$ and $\epsilon_0 \in (0, 1)$ so large that

$$\Omega(n, c_0, \epsilon_0) := \{\omega \in \widetilde{\Omega} : \mu^\omega(\mathcal{L}_{c_0,\epsilon_0}) > 1 - \frac{1}{4n}\}$$

has positive $\widetilde{Q}$-measure.

By Theorem 1.7, for every $x \in V_m$, we have

$$\lim_{t\to\infty} \widetilde{Q}(\sup_{u\in\mathcal{L}_{c_0,\epsilon_0}} |\Phi^t_{\theta_{-t}\omega} u[x] - \mathcal{Z}(\omega)[x]| > \frac{1}{2n|V_m|}) = 0,$$

and hence also

$$\lim_{t\to\infty} \sum_{x\in V_m} \widetilde{Q}(\{ \sup_{u\in\mathcal{L}_{c_0,\epsilon_0}} |\Phi^t_{\theta_{-t}\omega} u[x] - \mathcal{Z}(\omega)[x]| > \frac{1}{2n|V_m|}\}) = 0.$$

Thus, for every $k \geq 1$, there is $t_k > 0$ such that

$$\sum_{x\in V_m} \widetilde{Q}(\{ \sup_{u\in\mathcal{L}_{c_0,\epsilon_0}} |\Phi^{t_k}_{\theta_{-t_k}\omega} u[x] - \mathcal{Z}(\omega)[x]| \geq \frac{1}{2n|V_m|}\}) < \frac{1}{|V_m|} 2^{-(k+n+m)}\delta,$$

and we may assume that $t_k \nearrow \infty$.

Since transformation θ_{-1} is measure-preserving on $\widetilde{\Omega}$ and it is also mixing, by Lemma 7.6,

$$\widetilde{Q}\left\{\{\omega \in \widetilde{\Omega} : \exists j \in \mathbb{N} \text{ such that } \theta_{-t_j}(\omega) \in \Omega(n, c_0, \epsilon_0)\}\right\} = 1.$$

Define

$$\hat{\Omega}_{n,m} := \bigcap_{k\geq 1} \bigcap_{x\in V_m} \{\omega \in \widetilde{\Omega} : \sup_{u\in\mathcal{L}_{c_0,\epsilon_0}} |\Phi^{t_k}_{\theta_{-t_k}\omega} u[x] - \mathcal{Z}(\omega)[x]| < \frac{1}{2n|V_m|}\}$$
$$\cap \{\omega \in \widetilde{\Omega} : \exists j \in \mathbb{N} \text{ such that } \theta_{-t_j}(\omega) \in \Omega(n, c_0, \epsilon_0)\}.$$

We have

$$
\begin{aligned}
\widetilde{Q}(\hat{\Omega}_{n,m}) \geq & \widetilde{Q}(\{\omega \in \widetilde{\Omega} : \exists j \in \mathbb{N} \text{ such that } \theta_{-t_j}(\omega) \in \Omega(n, c_0, \epsilon_0)\}) \\
& - \sum_{k=1}^{\infty} \sum_{x \in V_m} \widetilde{Q}(\omega : \sup_{u \in \mathcal{L}_{c_0,\epsilon_0}} |\Phi^{t_k}_{\theta_{-t_k}\omega} u[x] - \mathcal{Z}(\omega)[x]| \geq \frac{1}{2n|V_m|}\}) \\
\geq & 1 - 2^{-(n+m)}\delta.
\end{aligned}
$$

Fix $F \in \mathrm{Lip}^1_b(\mathcal{L})$ and $\omega \in \hat{\Omega}_{n,m}$. Set $t := t_j$, where j depends on ω. Then

$$
\begin{aligned}
& |\mu^{\omega}(F \circ \pi_{V_m}) - \delta_{\mathcal{Z}(\omega)}(F \circ \pi_{V_m})| \\
= & \left| \mu^{\theta_{-t}\omega}(F \circ \pi_{V_m} \circ \Phi^t_{\theta_{-t}\omega}) - F(\pi_{V_m}(\mathcal{Z}(\omega))) \right| \\
= & \Big| \int_{\mathcal{L}_{c_0,\epsilon_0}} F(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u)) \, \mu^{\theta_{-t}\omega}(du) \\
& + \int_{\mathcal{L} \setminus \mathcal{L}_{c_0,\epsilon_0}} F(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u)) \, \mu^{\theta_{-t}\omega}(du) - F(\pi_{V_m}(\mathcal{Z}(\omega))) \Big| \\
\leq & \int_{\mathcal{L}_{c_0,\epsilon_0}} \left| F(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u)) - F(\pi_{V_m}(\mathcal{Z}(\omega))) \right| \, \mu^{\theta_{-t}\omega}(du) \\
& + \int_{\mathcal{L} \setminus \mathcal{L}_{c_0,\epsilon_0}} \left| F(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u)) - F(\pi_{V_m}(\mathcal{Z}(\omega))) \right| \, \mu^{\theta_{-t}\omega}(du).
\end{aligned}
$$

Since $F \in \mathrm{Lip}^1_b(\mathcal{L})$,

$$
\int_{\mathcal{L} \setminus \mathcal{L}_{c_0,\epsilon_0}} \left| F(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u)) - F(\pi_{V_m}(\mathcal{Z}(\omega))) \right| \, \mu^{\theta_{-t}\omega}(du) \leq 2\mu^{\theta_{-t}\omega}(\mathcal{L} \setminus \mathcal{L}_{c_0,\epsilon_0}).
$$

Notice that $\theta_{-t}\omega \in \Omega(n, c_0, \epsilon_0)$, so $\mu^{\theta_{-t}\omega}(\mathcal{L} \setminus \mathcal{L}_{c_0,\epsilon_0}) \leq \frac{1}{4n}$. Moreover, the first integral is bounded from above by

$$
\sup_{u \in \mathcal{L}_{c_0,\epsilon_0}} d(\pi_{V_m}(\Phi^t_{\theta_{-t}\omega} u), \pi_{V_m}(\mathcal{Z}(\omega))) \leq \sum_{x \in V_m} \sup_{u \in \mathcal{L}_{c_0,\epsilon_0}} |\Phi^t_{\theta_{-t}\omega} u[x] - \mathcal{Z}(\omega)[x]| < \frac{1}{2n}.
$$

This completes the proof of Theorem 6.14. □

6.3. Uniqueness of Global Solutions to the Semi-Discrete Stochastic Heat Equation

Now we are ready to prove our main theorem, Theorem 6.3. Let $\widehat{\Omega} \subset \widetilde{\Omega}$ such that $Q(\widehat{\Omega}) = 1$ and

$$y \mapsto \frac{Z(y,t,\omega)}{Z(0,t,\omega)} \in \mathcal{L}, \quad \forall t \in \mathbb{R},\ \omega \in \widehat{\Omega}.$$

For every $\omega \in \widehat{\Omega}$, let ν^ω be the probability measure on $(\mathcal{L}, \mathcal{B}(\mathcal{L}))$ defined by

$$\nu^\omega(\cdot) := \delta_{\mathcal{Y}(\omega)}(\cdot),$$

where $\mathcal{Y}(\omega)$ is the element of $\mathcal{L}$ defined by $\mathcal{Y}(\omega)[y] := Z(y,0,\omega)/Z(0,0,\omega)$. As in the proof of Theorem 6.14, let $\mathcal{Z}(\omega)$ be the element of $\mathcal{L}$ defined by $\mathcal{Z}(\omega)[y] := \widetilde{Z}^{y,0}_{-\infty}(\omega)$. For $A \in \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L})$, set

$$\nu(A) := \int_{\widehat{\Omega}} \nu^\omega(A_\omega)\, Q(d\omega),$$

where $A_\omega = \{f \in \mathcal{L} : (\omega, f) \in A\}$. Since Z is a global stationary solution to sSHE, one can proceed as in the proof of Theorem 6.14 to show that ν is an invariant probability measure for the skew product $(\Theta^t)_{t\geq 0}$. The uniqueness part of Theorem 6.14 then implies that

$$\nu(A) = \int_{\widetilde{\Omega}} \delta_{\mathcal{Z}(\omega)}(A_\omega)\, Q(d\omega), \quad A \in \widetilde{\mathcal{F}} \otimes \mathcal{B}(\mathcal{L}).$$

Consider the set

$$A := \{(\omega, f) \in \widetilde{\Omega} \times \mathcal{L} : \mathcal{Z}(\omega) = f\}.$$

For every $\omega \in \widetilde{\Omega}$, $A_\omega = \{\mathcal{Z}(\omega)\}$. Hence

$$\int_{\widehat{\Omega}} \delta_{\mathcal{Y}(\omega)}(\{\mathcal{Z}(\omega)\})\, Q(d\omega) = 1,$$

which implies that $\mathcal{Y} = \mathcal{Z}$ Q-almost surely. And since Q is invariant under each shift θ_t, there is a set Ω' of full Q-measure such that $\mathcal{Y}(\omega) = \mathcal{Z}(\omega)$ for every $\omega \in \Omega'$, and $\theta_r(\Omega') = \Omega'$ for every $r \in \mathbb{Z}$. Fix $\omega \in \Omega'$. As Z and $Z^{\cdot,\cdot}_{-\infty}$ are global stationary solutions to sSHE (see Proposition 6.2), we have for every $y \in \mathbb{Z}^d$ and $t > 0$

$$\begin{aligned}\frac{Z(y,t,\omega)}{Z(0,t,\omega)} &= \frac{\sum_{x\in\mathbb{Z}^d} Z(x,0,\omega) Z_{x,0}^{y,t}(\omega)}{\sum_{x\in\mathbb{Z}^d} Z(x,0,\omega) Z_{x,0}^{0,t}(\omega)} \\ &= \frac{\sum_{x\in\mathbb{Z}^d} \mathcal{Y}(\omega)[x] Z_{x,0}^{y,t}(\omega)}{\sum_{x\in\mathbb{Z}^d} \mathcal{Y}(\omega)[x] Z_{x,0}^{0,t}(\omega)} = \frac{\sum_{x\in\mathbb{Z}^d} \mathcal{Z}(\omega)[x] Z_{x,0}^{y,t}(\omega)}{\sum_{x\in\mathbb{Z}^d} \mathcal{Z}(\omega)[x] Z_{x,0}^{0,t}(\omega)} = \widetilde{Z}_{-\infty}^{y,t}(\omega).\end{aligned}$$

Finally, for every $t \leq 0$ there is $r \in \mathbb{Z}$ such that $t + r > 0$, so we obtain for every $y \in \mathbb{Z}^d$

$$\frac{Z(y,t,\omega)}{Z(0,t,\omega)} = \frac{Z(y,t+r,\theta_{-r}\omega)}{Z(0,t+r,\theta_{-r}\omega)} = \widetilde{Z}_{-\infty}^{y,t+r}(\theta_{-r}\omega) = \widetilde{Z}_{-\infty}^{y,t}(\omega).$$

chapter 7

Appendix

Contents

7.1. Building Blocks: Proof of Lemma 4.12

Let $\sigma \in (\frac{3}{4}, 1)$, $\sigma \in (0, \frac{1}{2}(1-\sigma))$, $\mu \in (-\infty, -1)$, $\nu \in (\frac{1}{2}, 1)$, and $\nu_1 \in (\nu^{-1} - 1, 1)$. Let $\chi_1(t)$ be the smallest even integer $\geq t(1-t^{-\sigma})$, and let $\chi_2(t)$ be the largest odd integer $\leq t(1+t^{-\sigma})$. Recall

$$J(t) = \left\{ n \in \mathbb{N} : \left| \frac{n}{t} - 1 \right| < 1 - \nu \right\}, \qquad \text{and} \qquad K(t) = \{ l \in \mathbb{N} : \chi_1(t) \leq l \leq \chi_2(t) \}.$$

Before proving Lemma 4.12, we derive a proposition that will allow us to streamline the proof.

Proposition 7.1

For $\beta > 0$ sufficiently small, the following statements hold.

(B1) *There is $\tilde{\rho} > 0$ such that for every $\rho \in (0, \tilde{\rho}]$, $c \geq 0$,*

$$\lim_{t\to\infty} e^{(\rho-1)t} \sum_{l \notin J(t)} e^{ct^{\mu}l} \frac{t^l}{l!} \sum_{1 \leq r \leq l+1} r\alpha^r A(t,l,r) = 0.$$

(B2) *There is $\tilde{\rho} > 0$ such that for every $\rho \in (0, \tilde{\rho}]$,*

$$\lim_{t\to\infty} e^{(\rho-1)t} \sum_{l \in J(t)} \frac{t^l}{l!} \sum_{\nu_1 l \leq r \leq l+1} r\alpha^r A(t,l,r) = 0.$$

(B3) *We have*

$$\lim_{t\to\infty} e^{t^{\sigma}} e^{-t} \sum_{l \in J(t) \setminus K(t)} \frac{t^l}{l!} \sum_{1 \leq r \leq l+1} r\alpha^r A(t,l,r) = 0.$$

Proof. Let us first show (B1). Fix $\delta \in (0,1)$, $\kappa \in \left(e^{1-\frac{1}{\delta}}, \delta\right)$, $\hat{\kappa} \in \left(e^{1-\frac{1}{\nu}}, \nu\right)$, and $\nu_0 \in (0, \frac{\delta}{2})$. Let $\beta > 0$ be so small that

$$e^{1-\beta^2} > \left(\frac{e}{\kappa}\right)^{\delta} \vee \left(\frac{e}{\hat{\kappa}}\right)^{\nu}, \quad \beta^2 < \nu_0 \alpha^{-1}.$$

Then, let $\tilde{\rho} > 0$ be so small that

$$e^{1-\beta^2-\tilde{\rho}} > \left(\frac{e}{\kappa}\right)^{\delta} \vee \left(\frac{e}{\hat{\kappa}}\right)^{\nu}.$$

For fixed $\rho \in (0, \tilde{\rho}]$ and $c \geq 0$, we decompose

$$e^{(\rho-1)t} \sum_{l \notin J(t)} e^{ct^{\mu} l} \frac{t^l}{l!} \sum_{1 \leq r \leq l+1} r\alpha^r A(t,l,r)$$

into

$$\sum_{0 \leq l \leq \nu_0 t} Y_l(t) + \sum_{\nu_0 t < l \leq \nu t} Y_l(t) + \sum_{l \geq (2-\nu)t} Y_l(t), \tag{7.1}$$

where

$$Y_l(t) = e^{(\rho-1)t} \frac{t^l}{l!} e^{ct^{\mu} l} \sum_{1 \leq r \leq l+1} r\alpha^r A(t,l,r).$$

For $0 \leq l \leq \nu t$,

$$e^{ct^{\mu} l} \leq e^{c\nu t^{\mu+1}}.$$

As $\mu + 1 < 0$, $\exp(c\nu t^{\mu+1})$ stays bounded, so we disregard the factors $\exp(ct^{\mu} l)$ in the first two sums in (7.1). Lemma 4.6 implies for $t \geq \beta^{-2} \vee 2$

$$Y_l(t) \lesssim \beta^{-2} e^{-(1-\beta^2-\rho)t} (l+1)^2 e^{ct^{\mu} l} \sum_{1 \leq r \leq l+1} r(\alpha\beta^2)^r \frac{t^{l+r}}{(l+r)!}. \tag{7.2}$$

Then by Lemma 4.12,

$$Y_l(t) \lesssim e^{\lfloor \delta t \rfloor} (t+1)^5 \frac{t^{\lfloor \delta t \rfloor}}{\sqrt{2\pi \lfloor \delta t \rfloor} \lfloor \delta t \rfloor^{\lfloor \delta t \rfloor}} \leq \frac{(t+1)^3}{\sqrt{2\pi\kappa t}} \left(\frac{e}{\kappa}\right)^{\delta t}.$$

Since

$$e^{1-\beta^2-\rho} > \left(\frac{e}{\kappa}\right)^{\delta},$$

we have

$$\lim_{t \to \infty} \sum_{0 \leq l \leq \nu_0 t} Y_l(t) = 0.$$

Next, observe that for $l > \nu_0 t$ and $1 \le r \le l+1$,

$$\frac{t^r}{(l+1)\dots(l+r)} \le \left(\frac{t}{l}\right)^r \le \nu_0^{-r}. \tag{7.3}$$

From this, (7.2), and $\beta^2 < \nu_0 \alpha^{-1}$, we infer the estimate

$$\sum_{\nu_0 t < l \le \nu t} Y_l(t) \lesssim \beta^{-2} e^{-(1-\beta^2-\rho)t} \sum_{\nu_0 t < l \le \nu t} (l+1)^2 \frac{t^l}{l!}.$$

As

$$e^{1-\beta^2-\rho} > \left(\frac{e}{\hat{\kappa}}\right)^\nu,$$

we have similarly, again with the help of Stirling's formula,

$$\lim_{t\to\infty} \sum_{\nu_0 t < l \le \nu t} Y_l(t) = 0.$$

From (7.3) and $\beta^2 < \nu_0 \alpha^{-1}$, we may also infer

$$\begin{aligned}\sum_{l \ge (2-\nu)t} Y_l(t) &\lesssim \beta^{-2} e^{-(1-\beta^2-\rho)t} \sum_{l \ge (2-\nu)t} (l+1)^2 \frac{(te^{ct^\mu})^l}{l!} \\ &\lesssim \beta^{-2} t^2 e^{-(1-\beta^2-\rho)t} \sum_{l \ge (2-\nu)t-2} \frac{(te^{ct^\mu})^l}{l!}.\end{aligned}$$

Let $\rho_2 = 2 - \nu$ and $\rho_1 \in \left(e^{1-\frac{1}{\rho_2}}, \rho_2\right)$. Then,

$$\rho_1 t \exp(ct^\mu) < \lfloor (2-\nu)t - 2 \rfloor < \rho_2 t \exp(ct^\mu)$$

for t sufficiently large, and Lemma 7.4 gives

$$\lim_{t\to\infty} \sum_{l \ge (2-\nu)t} Y_l(t) = 0.$$

Now, we show (B2). Notice that $f(t) = \lfloor \nu(1+\nu_1)t \rfloor$, $\rho_2 = \nu(1+\nu_1)$, and $\rho_1 \in \left(e^{1-\frac{1}{\rho_2}}, \rho_2\right)$ satisfy the conditions in Lemma 7.4. Therefore, we can choose $\lambda > 0$ so small that

$$\lim_{t\to\infty} e^{(\lambda-1)t} \sum_{n=f(t)}^{\infty} \frac{t^n}{n!} = 0. \tag{7.4}$$

Let $\beta > 0$ be so small that $\beta^2 < \lambda \wedge \alpha^{-1}$, and let $\tilde{\rho} \in (0, (\lambda \wedge \alpha^{-1}) - \beta^2)$. Fix $\rho \in (0, \tilde{\rho}]$. Lemma 4.6 implies for $\beta^2 < 1$ and $t \geq \beta^{-2} \vee 2$

$$e^{(\rho-1)t} \sum_{l \in J(t)} \frac{t^l}{l!} \sum_{\nu_1 l \leq r \leq l+1} r\alpha^r A(t,l,r) \tag{7.5}$$
$$\lesssim \beta^{-2} e^{-(1-\beta^2-\rho)t} \sum_{l \in J(t)} (l+1)^2 \sum_{\nu_1 l \leq r \leq l+1} r(\alpha\beta^2)^r \frac{t^{l+r}}{(l+r)!}.$$

For $l > \nu t$ and $\nu_1 l \leq r \leq l+1$, we have $l + r \geq (1+\nu_1)l > (1+\nu_1)\nu t > t$, where we used that $\nu_1 > \nu^{-1} - 1$. Hence,

$$\frac{t^{l+r}}{(l+r)!} \leq \frac{t^{\lceil (1+\nu_1)l \rceil}}{\lceil (1+\nu_1)l \rceil !}. \tag{7.6}$$

Since $\beta^2 < \alpha^{-1}$, the expression in the second line of (7.5) is thus less than a constant times

$$\beta^{-2} e^{-(1-\beta^2-\rho)t} \sum_{l \in J(t)} (l+1)^2 \frac{t^{\lceil (1+\nu_1)l \rceil}}{\lceil (1+\nu_1)l \rceil !} \lesssim \beta^{-2} e^{-(1-\beta^2-\rho)t} t^2 \sum_{l > \nu(1+\nu_1)t} \frac{t^l}{l!}.$$

Since $\beta^2 + \rho < \lambda$, the right side tends to 0 on account of (7.4).

Finally, we show (B3). In light of (B2), it is enough to show

$$\lim_{t\to\infty} e^{t^\sigma} e^{-t} \sum_{l \in J(t)\setminus K(t)} \frac{t^l}{l!} \sum_{1 \leq r < \nu_1 l} r\alpha^r A(t,l,r) = 0.$$

By Lemma 4.7, we have for β so small that $\alpha(2-\nu)\psi < 1$ the estimate

$$\sum_{l \in J(t)\setminus K(t)} \frac{t^l}{l!} \sum_{1 \leq r < \nu_1 l} r\alpha^r A(t,l,r) \lesssim \sum_{l \in J(t)\setminus K(t)} \frac{t^l}{l!}.$$

Fix $\tilde{\sigma} \in (0, \sigma)$. By Stirling's formula,

$$e^{t^\sigma} e^{-t} \sum_{0 \leq l < \chi_1(t)} \frac{t^l}{l!} \lesssim e^{t^\sigma} e^{-t} \sum_{0 \leq l \leq (1-t^{-\tilde{\sigma}})t} \frac{t^l}{l!} \lesssim t e^{t^\sigma - t} \frac{e^{\lceil (1-t^{-\tilde{\sigma}})t \rceil} t^{\lceil (1-t^{-\tilde{\sigma}})t \rceil}}{\lceil (1-t^{-\tilde{\sigma}})t \rceil^{\lceil (1-t^{-\tilde{\sigma}})t \rceil}}. \tag{7.7}$$

If we set

$$r(t) = \frac{1}{2}\left(e^{1-\frac{1}{1-t^{-\tilde{\sigma}}}} + \left(1 - t^{-\tilde{\sigma}}\right)\right),$$

we have

$$e^{1-\frac{1}{1-t^{-\tilde{\sigma}}}} < r(t) < 1 - t^{-\tilde{\sigma}}.$$

By L'Hospital's rule,

$$\begin{aligned}\lim_{t\to\infty} t\left(1-t^{-\tilde{\sigma}}-r(t)\right) =& \frac{\tilde{\sigma}}{2}\lim_{t\to\infty} t^{1-\tilde{\sigma}}\left(\frac{e^{1-\frac{1}{1-t^{-\tilde{\sigma}}}}}{(1-t^{-\tilde{\sigma}})^2}-1\right)\\ =& \frac{\tilde{\sigma}^2}{2(\tilde{\sigma}-1)}\lim_{t\to\infty} t^{1-2\tilde{\sigma}}\frac{e^{1-\frac{1}{1-t^{-\tilde{\sigma}}}}}{(1-t^{-\tilde{\sigma}})^4}\left(2t^{-\tilde{\sigma}}-1\right)=\infty,\end{aligned}$$

where we used that $\tilde{\sigma}<\frac{1}{2}$. Then, for t sufficiently large,

$$\lceil(1-t^{-\tilde{\sigma}})t\rceil \geq (1-t^{-\tilde{\sigma}})t-\frac{t}{2}\left(1-t^{-\tilde{\sigma}}-r(t)\right)=\frac{t}{2}\left(1-t^{-\tilde{\sigma}}+r(t)\right)>tr(t).$$

Thus, the right side of (7.7) is less than a constant times

$$te^{t^{\sigma}-t}\left(\frac{e}{r(t)}\right)^{(1-t^{-\tilde{\sigma}})t}.$$

It remains to check that the term above converges to 0 as $t\to\infty$, or equivalently

$$\begin{aligned}&\lim_{t\to\infty}\ln\left(te^{t^{\sigma}-t}\left(\frac{e}{r(t)}\right)^{(1-t^{-\tilde{\sigma}})t}\right)\\ &=\lim_{t\to\infty}\left(\ln(t)-t\left(t^{-\tilde{\sigma}}-t^{\sigma-1}+(1-t^{-\tilde{\sigma}})\ln(r(t))\right)\right)=-\infty.\end{aligned}$$

To simplify notation, we set

$$\iota(t)=\frac{1}{1-t^{-\tilde{\sigma}}},\quad e(t)=e^{1-\iota(t)}.$$

For fixed $\varphi\in(\sigma,1-2\tilde{\sigma})$, L'Hospital's rule implies

$$\begin{aligned}&\lim_{t\to\infty} t^{-\varphi}t\left(t^{-\tilde{\sigma}}-t^{\sigma-1}+(1-t^{-\tilde{\sigma}})\ln(r(t))\right)\\ =&\frac{\tilde{\sigma}}{\varphi-1}\lim_{t\to\infty}t^{1-\tilde{\sigma}-\varphi}\left(\frac{1-t^{-\tilde{\sigma}}+e(t)\iota(t)^2(1-t^{-\tilde{\sigma}})}{1-t^{-\tilde{\sigma}}+e(t)}-1+\frac{1-\sigma}{\tilde{\sigma}}t^{\tilde{\sigma}+\sigma-1}+\ln(r(t))\right)\\ =&\frac{\tilde{\sigma}^2}{(\varphi-1)(\tilde{\sigma}+\varphi-1)}\lim_{t\to\infty}t^{1-2\tilde{\sigma}-\varphi}\left(\frac{2+2e(t)\iota(t)^2}{1-t^{-\tilde{\sigma}}+e(t)}+\frac{1-t^{-\tilde{\sigma}}}{(1-t^{-\tilde{\sigma}}+e(t))^2}\right.\\ &\cdot\left(\left(1-t^{-\tilde{\sigma}}+e(t)\right)\left(e(t)\iota(t)^4-2e(t)\iota(t)^3\right)-\left(1+e(t)\iota(t)^2\right)^2\right)\\ &\left.+\frac{1-\sigma}{\tilde{\sigma}^2}(\tilde{\sigma}+\sigma-1)t^{2\tilde{\sigma}+\sigma-1}\right)=\frac{\tilde{\sigma}^2}{2(\varphi-1)(\tilde{\sigma}+\varphi-1)}\lim_{t\to\infty}t^{1-2\tilde{\sigma}-\varphi}=\infty.\end{aligned}$$

If t is so large that $t^{-\varphi}t\left(t^{-\tilde{\sigma}}-t^{\sigma-1}+(1-t^{-\tilde{\sigma}})\ln(r(t))\right)\geq 1$, we have

$$\ln(t)-t\left(t^{-\tilde{\sigma}}-t^{\sigma-1}+(1-t^{-\tilde{\sigma}})\ln(r(t))\right)\leq\left(\ln(t)-t^{\varphi}\right),$$

which tends to $-\infty$ as $t \to \infty$. Stirling's formula and the tail estimate

$$\sum_{n=k}^{\infty} \frac{t^n}{n!} \le \frac{t^k}{k!} \frac{1}{1-\frac{t}{k}}, \quad k > t,$$

yield in addition

$$e^{t^\sigma} e^{-t} \sum_{l > \chi_2(t)} \frac{t^l}{l!} \lesssim e^{t^\sigma} e^{-t} \sum_{l \ge (1+t^{-\tilde{\sigma}})t} \frac{t^l}{l!} \lesssim e^{t^\sigma - t} \frac{e^{\lceil (1+t^{-\tilde{\sigma}})t \rceil} t^{\lceil (1+t^{-\tilde{\sigma}})t \rceil}}{((1+t^{-\tilde{\sigma}})t)^{\lceil (1+t^{-\tilde{\sigma}})t \rceil}} \cdot \frac{1}{1-\frac{1}{1+t^{-\tilde{\sigma}}}}$$
$$\lesssim t^{\tilde{\sigma}} e^{t^\sigma - t} \left(\frac{e}{1+t^{-\tilde{\sigma}}}\right)^{(1+t^{-\tilde{\sigma}})t}.$$

To complete the proof of (B3), let us now show that

$$\lim_{t\to\infty} \ln\left(t^{\tilde{\sigma}} e^{t^\sigma - t} \left(\frac{e}{1+t^{-\tilde{\sigma}}}\right)^{(1+t^{-\tilde{\sigma}})t} \right) = -\infty.$$

For $\varphi \in (\sigma, 1-2\tilde{\sigma})$, we have by virtue of L'Hospital's rule

$$\begin{aligned}
&\lim_{t\to\infty} t^{-\varphi} t \left((1+t^{-\tilde{\sigma}}) \ln(1+t^{-\tilde{\sigma}}) - t^{-\tilde{\sigma}} - t^{\sigma-1}\right) \\
=&\frac{\tilde{\sigma}}{1-\varphi} \lim_{t\to\infty} \left(t^{1-\tilde{\sigma}-\varphi} \ln(1+t^{-\tilde{\sigma}}) + \frac{\sigma-1}{\tilde{\sigma}} t^{\sigma-\varphi} \right) \\
=&\frac{\tilde{\sigma}}{1-\varphi} \lim_{t\to\infty} t^{1-\tilde{\sigma}-\varphi} \ln(1+t^{-\tilde{\sigma}}) \\
=&\frac{\tilde{\sigma}^2}{(\varphi-1)(\tilde{\sigma}+\varphi-1)} \lim_{t\to\infty} \frac{t^{1-2\tilde{\sigma}-\varphi}}{1+t^{-\tilde{\sigma}}} = \infty.
\end{aligned}$$

In particular, we can choose t so large that $t^{-\varphi} t \left((1+t^{-\tilde{\sigma}}) \ln(1+t^{-\tilde{\sigma}}) - t^{-\tilde{\sigma}} - t^{\sigma-1}\right) \ge \tilde{\sigma}$, in which case, as $t \to \infty$, we have:

$$\tilde{\sigma} \ln(t) - t((1+t^{-\tilde{\sigma}}) \ln(1+t^{-\tilde{\sigma}}) - t^{-\tilde{\sigma}} - t^{\sigma-1}) \le \tilde{\sigma}(\ln(t) - t^{\varphi}) \to -\infty \qquad \square$$

7.1.1. Proof of Lemma 4.12

(A0) is an immediate consequence of (A1) since

$$\sum_{0 \le r < \nu_1 l - 1} (r+1) \alpha^r A(t - 2t^{\xi_1}, l, r+1) < 1.$$

We now prove (A1). We split the proof into two parts. First, we show that

$$\limsup_{t\to\infty} e^{-\beta^2 t^{\xi_1}} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{0\le r<\nu_1 l-1} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) < \infty. \tag{7.8}$$

For $l \in J(t-2t^{\xi_1})$ and $0 \le r < \nu_1 l - 1$, we have by Lemma 4.7 that $A(t-2t^{\xi_1}, l, r+1) \lesssim ((2-\nu)\psi)^r$. Moreover,

$$\frac{\mathbf{P}^\bullet(l)}{\mathbf{P}^\bullet(l+1)} = \frac{(t-2t^{\xi_1})^l (l+1)!}{(t-2t^{\xi_1})^{l+1} l!} = \frac{l+1}{t-2t^{\xi_1}},$$

which is bounded in t. Hence,

$$\begin{aligned}
&\sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{0\le r<\nu_1 l-1} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1)\\
&\lesssim \sum_{\substack{l\in J(t-2t^{\xi_1}),\\ l\equiv \|y\|_1}} q^y_l \mathbf{P}^\bullet(l) + \sum_{\substack{l\in J(t-2t^{\xi_1}),\\ l\equiv \|y\|_1+1}} q^y_{l+1} \mathbf{P}^\bullet(l) \lesssim p^y_{t-2t^{\xi_1}}.
\end{aligned}$$

By Lemma 2.12,

$$\frac{p^y_{t-2t^{\xi_1}}}{p^y_t} \lesssim e^{\beta^2 t^{\xi_1}}$$

for t sufficiently large. This implies (7.8). The statement in (A1) will then follow from

$$\lim_{t\to\infty} \sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{\nu_1 l-1\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1) = 0. \tag{7.9}$$

Let $\tilde\sigma \in (\sigma, 1)$. We have

$$\begin{aligned}
&\sup_{\|y\|\le t^\sigma} \frac{1}{p_t^y} \sum_{l\in J(t-2t^{\xi_1})} q^y_{\iota(y,l)} \mathbf{P}^\bullet(l) \sum_{\nu_1 l-1\le r\le l} (r+1)\alpha^r A(t-2t^{\xi_1}, l, r+1)\\
\lesssim & e^{(t-2t^{\xi_1})^{\tilde\sigma}} e^{-(t-2t^{\xi_1})} \sum_{l\in J(t-2t^{\xi_1})} \frac{(t-2t^{\xi_1})^l}{l!} \sum_{\nu_1 l\le r\le l+1} r\alpha^r A(t-2t^{\xi_1}, l, r),
\end{aligned}$$

and the expression above tends to 0 as $t\to\infty$ by (B2) of Proposition 7.1.

To prove (A2), it is enough to replace $t-2t^{\xi_1}$ with t^{ξ_1} in (7.8) and to drop the terms $e^{-\beta^2 t^{\xi_1}}$ and $q^y_{\iota(y,l)}/p^y_t$. With respect to (A3), replace $t-2t^{\xi_1}$ with t^{ξ_1} in the proof of (7.9), and notice that (B2) of Proposition 7.1 also lets us deal with the additional factors t^θ and $e^{\beta^2 t^{\xi_1}}$ provided that $\beta^2 \le \tilde\rho$.

With regard to (A4) , we have for $\theta, c > 0$, $\beta^2 < \tilde{\rho}$, and $\mu := (\sigma - 1)/\xi_1 < -1$

$$\begin{aligned} & t^\theta e^{\beta^2 t^{\xi_1}} \sum_{l \notin J(t^{\xi_1})} e^{ct^{\sigma-1} l} \mathbf{P}^-(l) \sum_{0 \le r \le l} (r+1)\alpha^r A(t^{\xi_1}, l, r+1) \\ \lesssim & e^{(\tilde{\rho}-1)t^{\xi_1}} \sum_{l \notin J(t^{\xi_1})} e^{c(t^{\xi_1})^\mu l} \frac{(t^{\xi_1})^l}{l!} \sum_{1 \le r \le l+1} r\alpha^r A(t^{\xi_1}, l, r), \end{aligned}$$

and the expression above tends to 0 as $t \to \infty$ by (B1) of Proposition 7.1.

Instead of proving (A5), we will show the following stronger statement:

$$\lim_{t \to \infty} t^\theta e^{t^\sigma} \sum_{l \notin K(t - 2t^{\xi_1})} \mathbf{P}^\bullet(l) \sum_{0 \le r \le l} (r+1)\alpha^r A(t - 2t^{\xi_1}, l, r+1) = 0, \quad \theta > 0. \tag{7.10}$$

Notice that for $\theta > 0$ and $\tilde{\sigma} \in (\sigma, 1)$,

$$\begin{aligned} & t^\theta e^{t^\sigma} \sum_{l \notin K(t-2t^{\xi_1})} \mathbf{P}^\bullet(l) \sum_{0 \le r \le l} (r+1)\alpha^r A(t - 2t^{\xi_1}, l, r+1) \\ \lesssim & e^{(\tilde{\rho}-1)(t-2t^{\xi_1})} \sum_{l \notin J(t-2t^{\xi_1})} \frac{(t-2t^{\xi_1})^l}{l!} \sum_{1 \le r \le l+1} r\alpha^r A(t-2t^{\xi_1}, l, r) \\ & + e^{(t-2t^{\xi_1})^{\tilde{\sigma}}} e^{-(t-2t^{\xi_1})} \sum_{l \in J(t-2t^{\xi_1}) \setminus K(t-2t^{\xi_1})} \frac{(t-2t^{\xi_1})^l}{l!} \sum_{1 \le r \le l+1} r\alpha^r A(t-2t^{\xi_1}, l, r). \end{aligned}$$

and the term above tends to 0 as $t \to \infty$ by (B1) and (B3) of Proposition 7.1.

Finally, (A8) is an immediate consequence of (7.10), completing the proof of Lemma 4.12.

7.2. Calculus Estimates

Lemma 7.2

There is $c > 0$ such that for any $n \in \mathbb{N}$, $l \in \mathbb{N}_0$, and $M > 0$,

$$\sum_{\substack{n_1 + \ldots + n_{l+1} = n, \\ n_1, \ldots, n_{l+1} \ge M}} \prod_{j=1}^{l+1} n_j^{-\frac{d}{2}} \le \frac{c^l}{M^{l(\frac{d}{2}-1)}} n^{-\frac{d}{2}}. \tag{7.11}$$

Here, $n_1, \ldots, n_{l+1}$ are always integers.

Proof. We choose $c > 2^d \max\left\{\zeta(\frac{d}{2}), (\frac{d}{2}-1)^{-1}\right\}$, where ζ is the Riemann zeta function, and prove the statement by induction. In the base case $l = 0$, the left side of (4.37) becomes $n^{-\frac{d}{2}} = \frac{c^0}{M^0} n^{-\frac{d}{2}}$, so we even have equality. In the induction step, suppose that (4.37) holds for some $l \in \mathbb{N}_0$. Then,

$$\sum_{\substack{n_1+\ldots+n_{l+2}=n,\\ n_1,\ldots,n_{l+2}\geq M}} \prod_{j=1}^{l+2} n_j^{-\frac{d}{2}} = \sum_{\substack{n'+n_{l+2}=n,\\ n',n_{l+2}\geq M}} \Bigg(\sum_{\substack{n_1+\ldots+n_{l+1}=n',\\ n_1,\ldots,n_{l+1}\geq M}} \prod_{j=1}^{l+1} n_j^{-\frac{d}{2}} \Bigg) n_{l+2}^{-\frac{d}{2}}. \tag{7.12}$$

For any n',

$$\sum_{\substack{n_1+\ldots+n_{l+1}=n',\\ n_1,\ldots,n_{l+1}\geq M}} \prod_{j=1}^{l+1} n_j^{-\frac{d}{2}} \leq \frac{c^l}{M^{l(\frac{d}{2}-1)}} (n')^{-\frac{d}{2}}$$

by induction hypothesis. Hence, the right side of (7.12) is bounded from above by

$$\frac{c^l}{M^{l(\frac{d}{2}-1)}} \sum_{\substack{n'+n_{l+2}=n,\\ n',n_{l+2}\geq M}} (n')^{-\frac{d}{2}} n_{l+2}^{-\frac{d}{2}}. \tag{7.13}$$

We have

$$\sum_{\substack{n'+n_{l+2}=n,\\ n',n_{l+2}\geq M}} (n')^{-\frac{d}{2}} n_{l+2}^{-\frac{d}{2}} \leq 2 \sum_{\substack{n'+n_{l+2}=n,\\ n'\geq n_{l+2}\geq M}} (n')^{-\frac{d}{2}} n_{l+2}^{-\frac{d}{2}}.$$

If $n' + n_{l+2} = n$ and $n' \geq n_{l+2}$, it follows that $n' \geq \frac{n}{2}$, so the right side is less than

$$2^{\frac{d}{2}+1} n^{-\frac{d}{2}} \sum_{n_{l+2}\geq M} n_{l+2}^{-\frac{d}{2}}. \tag{7.14}$$

If $M \geq 2$, we have

$$\sum_{n_{l+2}\geq M} n_{l+2}^{-\frac{d}{2}} \leq \int_{\frac{M}{2}}^{\infty} x^{-\frac{d}{2}}\, dx = \frac{2^{\frac{d}{2}-1}}{\frac{d}{2}-1} M^{1-\frac{d}{2}}.$$

If $M < 2$,

$$\sum_{n_{l+2}\geq M} n_{l+2}^{-\frac{d}{2}} \leq \zeta(\tfrac{d}{2}) < \zeta(\tfrac{d}{2}) 2^{\frac{d}{2}-1} M^{1-\frac{d}{2}}.$$

The expression in (7.14) is therefore less than

$$2^d n^{-\frac{d}{2}} \max\left\{(\zeta(\tfrac{d}{2}), (\tfrac{d}{2}-1)^{-1}\right\} M^{1-\frac{d}{2}} < c M^{1-\frac{d}{2}} n^{-\frac{d}{2}}.$$

Combining this estimate with (7.13) yields

$$\sum_{\substack{n_1+\ldots+n_{l+2}=n,\\ n_1,\ldots,n_{l+2}\geq M}} \prod_{j=1}^{l+2} n_j^{-\frac{d}{2}} \leq \frac{c^{l+1}}{M^{(l+1)(\frac{d}{2}-1)}} n^{-\frac{d}{2}}.$$

□

In the proof of Lemma 3.7, we use the following auxiliary result that corresponds to Inequality (3.29) in [Kif97].

Lemma 7.3

There is a constant $c > 0$, depending only on the dimension d, such that for any $r \in \mathbb{N}$,

$$\sum_{0<i_1<\ldots<i_r<n} i_1^{-\frac{d}{2}} (i_2 - i_1)^{-\frac{d}{2}} \ldots (i_r - i_{r-1})^{-\frac{d}{2}} (n - i_r)^{-\frac{d}{2}} \leq c^r n^{-\frac{d}{2}}, \quad n \geq r + 1. \tag{7.15}$$

Proof. We prove the statement by induction. By Jensen's inequality,

$$\left(\frac{1}{i} + \frac{1}{n-i}\right)^{\frac{d}{2}} \leq 2^{\frac{d}{2}-1} \left(\frac{1}{i^{\frac{d}{2}}} + \frac{1}{(n-i)^{\frac{d}{2}}}\right), \quad 1 \leq i < n.$$

For $r = 1$, the left side of (7.15) becomes

$$\begin{aligned}\sum_{i=1}^{n-1} \frac{1}{i^{\frac{d}{2}}} \cdot \frac{1}{(n-i)^{\frac{d}{2}}} &= \frac{1}{n^{\frac{d}{2}}} \sum_{i=1}^{n-1} \left(\frac{1}{i} + \frac{1}{n-i}\right)^{\frac{d}{2}} \\ &\lesssim \frac{1}{n^{\frac{d}{2}}} \sum_{i=1}^{n-1} \left(\frac{1}{i^{\frac{d}{2}}} + \frac{1}{(n-i)^{\frac{d}{2}}}\right) \lesssim \frac{1}{n^{\frac{d}{2}}}.\end{aligned}$$

In the induction step, assume that (7.15) holds for some $r \in \mathbb{N}$. Then, for $n \geq r + 2$,

$$\begin{aligned}&\sum_{0<i_1<\ldots<i_{r+1}<n} i_1^{-\frac{d}{2}} (i_2 - i_1)^{-\frac{d}{2}} \ldots (i_{r+1} - i_r)^{-\frac{d}{2}} (n - i_{r+1})^{-\frac{d}{2}} \\ &= \sum_{i_{r+1}=r+1}^{n-1} (n - i_{r+1})^{-\frac{d}{2}} \sum_{0<i_1<\ldots i_r<i_{r+1}} i_1^{-\frac{d}{2}} (i_2 - i_1)^{-\frac{d}{2}} \ldots (i_{r+1} - i_r)^{-\frac{d}{2}} \\ &\leq c^r \sum_{i_{r+1}=1}^{n-1} (n - i_{r+1})^{-\frac{d}{2}} \cdot i_{r+1}^{-\frac{d}{2}} \leq c^{r+1} n^{-\frac{d}{2}}.\end{aligned}$$

□

Lemma 7.4

Let $f(t)$ be an integer-valued function such that there are $\rho_2 > \rho_1 > 1$ for which $e^{\rho_2 - 1} < \rho_1^{\rho_2}$ and $\rho_1 t < f(t) < \rho_2 t$ for all t sufficiently large. Then, we have for $\lambda > 0$ sufficiently small

$$\lim_{t\to\infty} e^{(\lambda-1)t} \sum_{n=f(t)}^{\infty} \frac{t^n}{n!} = 0.$$

Proof. Using the tail estimate

$$\sum_{n=k}^{\infty} \frac{t^n}{n!} \le \frac{t^k}{k!} \sum_{n=k}^{\infty} \left(\frac{t}{k}\right)^{n-k} = \frac{t^k}{k!} \frac{1}{1-\frac{t}{k}}, \quad k > t$$

and Stirling's formula, we have

$$e^{(\lambda-1)t} \sum_{n=f(t)}^{\infty} \frac{t^n}{n!} \lesssim t^{-\frac{1}{2}} e^{(\lambda-1)t} \left(\frac{e}{\rho_1}\right)^{\rho_2 t}.$$

From this we obtain the desired convergence for $\lambda < 1 - \rho_2(1 - \ln(\rho_1))$. □

7.3. Some Ergodic Theory results

Lemma 7.5

The shift θ is mixing; in particular, it is ergodic.

Proof. For $x \in \mathbb{Z}^d$, interpret W^x as the coordinate map that assigns to $\omega \in \Omega$ the function $\omega^x \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. Each W^x is then a random variable taking on values in the space $\mathcal{C}(\mathbb{R}, \mathbb{R})$, and the collection of random variables $(W^x)_{x\in\mathbb{Z}^d}$ is i.i.d. Let $\sigma(W^x : x \in \mathbb{Z}^d)$ be the smallest σ-algebra with respect to which the random variables $(W^x)_{x\in\mathbb{Z}^d}$ are measurable. We need to show that for any $A, B \in \sigma(W^x : x \in \mathbb{Z}^d)$,

$$\lim_{n\to\infty} Q(A \cap \theta^{-n}(B)) = Q(A)Q(B). \tag{7.16}$$

It is enough to verify (7.16) for all A, B in some π-system that generates $\sigma(W^x : x \in \mathbb{Z}^d)$, see for instance [Dur10]. Such a generating π-system is given by

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \sigma(W^{x_j} : 1 \le j \le k),$$

where $(x_j)_{j\geq 1}$ is an enumeration of $\mathbb{Z}^d$. Let $A, B \in \mathcal{A}$. Then, there is $k \in \mathbb{N}$ such that

$$A, B \in \sigma(W^{x_j} : 1 \leq j \leq k).$$

Let $N \in \mathbb{N}$ be so large that

$$\{x_j : \ 1 \leq j \leq k\} \cap \{x_j + ne_1 : \ 1 \leq j \leq k\} = \varnothing, \quad n \geq N.$$

Then, for $n \geq N$, the random variables $W^{x_1}, \ldots, W^{x_k}, W^{x_1+ne_1}, \ldots, W^{x_k+ne_1}$ are independent, and so are the σ-algebras $\sigma(W^{x_j} : 1 \leq j \leq k)$ and $\sigma(W^{x_j+ne_1} : 1 \leq j \leq k)$. As $A \in \sigma(W^{x_j} : 1 \leq j \leq k)$, $\theta^{-n}(B) \in \sigma(W^{x_j+ne_1} : 1 \leq j \leq k)$ and as θ is μ-preserving, we have

$$Q(A \cap \theta^{-n}(B)) = Q(A)Q(\theta^{-n}(B)) = Q(A)Q(B).$$ □

Lemma 7.6

Let T be a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$, and assume in addition that T is mixing. Moreover, let $(t_i)_{i\geq 1}$ be a sequence of positive integers that increases to ∞, and let $A \in \mathcal{F}$ with $\mathbf{P}(A) > 0$. Then, for $\mathbf{P}$-almost every $\omega \in \Omega$, there is $i \in \mathbb{N}$ such that $T^{t_i}(\omega) \in A$.

Proof. Suppose by contradiction that there exits a set D such that $\mathbf{P}(D) > 0$ and

$$T^{t_i}(\omega) \in A^c \text{ for all } i \in \mathbb{N} \text{ and all } \omega \in D.$$

Let $\beta := \mathbf{P}(A^c) = 1 - \mathbf{P}(A) < 1$, and let $k \in \mathbb{N}$ be such that $\beta^k < \mathbf{P}(D)$.
Let $C_1 := T^{-t_1}(A^c)$, then $\mathbf{P}(C_1) = \mathbf{P}(A^c)$ because T is measure-preserving.
Let $\varepsilon > 0$. Since T is mixing, there exists $N_1 \in \mathbb{N}$ such that

$$\mathbf{P}(T^{-n}(A^c) \cap C_1) < \mathbf{P}(A^c)\mathbf{P}(C_1) + \varepsilon < \mathbf{P}(A^c)^2 + \varepsilon \text{ for all } n > N_1.$$

Let $i_1 \in \mathbb{N}$ such that $t_{i_1} > N_1$, then

$$\mathbf{P}(T^{-t_{i_1}}(A^c) \cap C_1) < \mathbf{P}(A^c)^2 + \varepsilon = \beta^2 + \varepsilon.$$

Now let $C_2 := T^{-t_{i_1}}(A^c) \cap C_1$. Then, there exists $N_2 \in \mathbb{N}$ such that

$$\mathbf{P}(T^{-n}(A^c) \cap C_2) < \mathbf{P}(A^c)P(C_2) + \varepsilon < \beta(\beta^2 + \varepsilon) + \varepsilon \text{ for all } n > N_2.$$

So taking $i_2 \in \mathbb{N}$ such that $t_{i_2} > N_2$, we can define a set $C_3 := T^{-t_{i_2}}(A^c) \cap C_2$ such that for N_3 sufficiently large

$$\mathbf{P}(T^{-n}A^c \cap C_3) < \mathbf{P}(A^c)P(C_3) + \varepsilon < \beta\Big(\beta(\beta^2 + \varepsilon) + \varepsilon\Big) + \varepsilon = \beta^4 + \varepsilon(\beta^2 + \beta + 1) \text{ for all } n > N_3.$$

Continuing in this fashion, we can construct $t_{i_1}, \ldots t_{i_k}$ and a set

$$C_k := \bigcap_{j=1}^{k} T^{-t_{i_j}}(A^c)$$

for which

$$\mathbf{P}(C_k) = \beta^k + \varepsilon \left(\sum_{m=0}^{k-2} \beta^m \right).$$

However,

$$D = \bigcap_{i=1}^{\infty} T^{-t_i}(A^c) \subseteq \bigcap_{j=1}^{k} T^{-t_{i_j}}(A^c) = C_k.$$

Therefore, $\mathbf{P}(D) \leq \beta^k + \varepsilon \left(\sum_{m=0}^{k-2} \beta^m \right)$ for all $\varepsilon > 0$. Thus, $\mathbf{P}(D) \leq \beta^k$, contradicting our previous assumption that $\beta^k < \mathbf{P}(D)$. □

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