# Multiplication of generalized affine Grassmannian slices and comultiplication of shifted Yangians

by

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# Abstract

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Given a semisimple algebraic group $G$, shifted Yangians are quantizations of certain generalized slices in $G((t^{-1}))$. In this thesis, we work with these generalized slices and the shifted Yangians in the simply-laced case.

Using a presentation of antidominantly shifted Yangians inspired by the work of Levendorskii, we show the existence of a family of comultiplication maps between shifted Yangians. We include a proof that these maps quantize natural multiplications of generalized slices.

On the commutative level, we define a Hamiltonian action on generalized slices, and show a relationship between them via Hamiltonian reduction. This relationship is established by constructing an explicit inverse to a multiplication map between slices.

Finally, we conjecture that the above relationship lifts to the Yangian level. We prove this conjecture for sufficiently dominantly shifted Yangians, and for the $\mathfrak{sl}_2$-case.

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# Contents

# Chapter 1

# Introduction

Given a simple finite-dimensional Lie algebra $\mathfrak{g}$, the Yangian $Y(\mathfrak{g})$ is a prominent object in mathematics and physics. Historically, the Yangian $Y(\mathfrak{gl}_n)$ first appears in the work of Faddeev on the *inverse scattering method* (see, for instance, [FT]). In his paper [Dr], Drinfeld coined the term *Yangian* and defined $Y(\mathfrak{g})$ as a canonical deformation quantization of $U(\mathfrak{g}[t])$, the universal enveloping algebra of the current algebra $\mathfrak{g}[t]$.

In this thesis, we look at a certain family of algebras called the shifted Yangians, denoted by $Y_\mu$, parametrized by coweights $\mu$ of $\mathfrak{g}$. When $\mu = 0$, $Y_0$ is the usual Yangian.

Moreover, thoughout the thesis, let us assume that we are in the simply-laced case.

## 1.1 Motivation and Setting

A central theme of this work is the story of quantization. The general idea is that the geometry of symplectic resolutions is intimately related to the representation theory of their quantizations. In this thesis, we will only deal with filtered quantizations. Let us start with a quick reminder.

### 1.1.1 Filtered quantization

Let $A$ be graded algebra over a field $k$. An $\mathbb{N}$-*filtered deformation* of $A$ is an $\mathbb{N}$-filtered algebra $\tilde{A} = \bigcup_{n\geq 0} \tilde{A}_n$, $\tilde{A}_n \subseteq \tilde{A}_{n+1}$, together with an isomorphism $\mathrm{gr}(\tilde{A}) \simeq A$ of graded algebras. Recall that $\mathrm{gr}\,\tilde{A} = \bigoplus_n \tilde{A}_n/\tilde{A}_{n-1}$, with $\tilde{A}_{-1} := \{0\}$, is the associated graded algebra of $\tilde{A}$.

If $A$ is commutative, we refer to $\tilde{A}$ as a *deformation quantization* of $A$. A filtered deformation $\tilde{A}$ of $A$ gives rise to a Poisson structure on $A$ via the isomorphism $\mathrm{gr}\,\tilde{A} \simeq A$. More precisely, given $y_1 \in \tilde{A}_i$ and $y_2 \in \tilde{A}_j$,

$$\{y_1 + \tilde{A}_{i-1}, y_2 + \tilde{A}_{j-1}\} := y_1 y_2 - y_2 y_1 \bmod \tilde{A}_{i+j-2}. \tag{1.1}$$

Given an affine algebraic variety $X$ over a field $k$, a *quantization* of $X$ is a deformation quantization of its coordinate ring $k[X]$.

### 1.1.2 Basic motivation: The Nilpotent Cone

The basic example of interest to us involves the nilpotent cone $\mathcal{N}$ of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ together with its Springer resolution $T^*(G/B)$ where $\mathfrak{g}$ is the Lie algebra of $G$ and $B$ is a Borel subgroup of $G$. The nilpotent cone consists of elements of $\mathfrak{g}$ which act nilpotently on every representation of $G$. Identifying $\mathcal{N}$ with a subvariety of $\mathfrak{g}^*$ via the Killing form, $\mathcal{N}$ inherits a Poisson structure coming from the Poisson-Lie structure of $\mathfrak{g}^*$.

The nilpotent cone is quantized by a family of algebras $A_\xi = U(\mathfrak{g})/Z_\xi$ where $Z_\xi$ is a central ideal of $U(\mathfrak{g})$, labelled by certain parameters $\xi$. In fact, via Beilinson-Bernstein localization theorem, $A_\xi = D^\xi(G/B)$, i.e., twisted differential operators on the flag variety $G/B$. Moreover, the isomorphisms $\mathrm{gr}(A_\xi) \simeq \mathbb{C}[\mathcal{N}]$ come from the PBW isomorphism $\mathrm{gr}(U\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$.

### 1.1.3 Affine Grassmannian slices and dominantly shifted Yangians

Our story starts with the study of slices in the affine Grassmannian, initiated in [KWWY]. Let $G$ be a complex semisimple algebraic group. Consider the affine Grassmannian $\mathrm{Gr} = G((t^{-1}))/G[t]$. Any coweight $\lambda$ of $G$ can be thought of as a $\mathbb{C}((t^{-1}))$-point of $G$. Denote by $t^\lambda$ the image of this point in Gr. Denote by $G_1[[t^{-1}]]$ the kernel of the evaluation map $G[[t^{-1}]] \longrightarrow G$ at infinity.

For a pair of dominant coweights $\lambda$ and $\mu$ with $\mu \leq \lambda$, we have the spaces $\mathrm{Gr}^\lambda = G[t]t^\lambda$ and $\mathrm{Gr}_\mu = G_1[[t^{-1}]]t^{w_0\mu}$ where $w_0$ is the longest element of the Weyl group of $G$. Consider also the slice $\overline{\mathrm{Gr}}^{\lambda}_{\mu} = \overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu$. Under the geometric Satake correspondence, the intersection homology of $\overline{\mathrm{Gr}}^{\lambda}_{\mu}$ is identified with the $\mu$-weight space of the irreducible $G^\vee$-representation of highest weight $\lambda$, where $G^\vee$ is the Langlands dual group. These slices carry natural Poisson structures described as follows. Given $x(t), y(t) \in \mathfrak{g}((t^{-1}))$, the bilinear form

$$\langle x(t), y(t)\rangle := \mathrm{Res}_{t=0}\, \kappa(x(t), y(t)),$$

where $\kappa$ is the Killing form on $\mathfrak{g}$, is invariant and nondegenerate. The spaces $\mathfrak{g}((t^{-1})), t^{-1}\mathfrak{g}[[t^{-1}]]$ and $\mathfrak{g}[t]$ form a Manin triple, which gives rise to Poisson-Lie structure on $G((t^{-1}))$ with Poisson subgroups $G_1[[t^{-1}]]$ and $G[t]$. This in turn induces a Poisson structure on the affine Grassmannian. By a general result about Manin triples and Poisson-Lie groups (first obtained by Mirković), see [KWWY, Thm. 2.5], the subvarieties $\mathrm{Gr}^\lambda_\mu = \mathrm{Gr}^\lambda \cap \mathrm{Gr}_\mu$ are symplectic leaves.

We have a natural map $G_1[[t^{-1}]] \longrightarrow \mathrm{Gr}_\mu$, $g \mapsto gt^\mu$. Thus, $\mathbb{C}[\mathrm{Gr}_\mu] \subseteq \mathbb{C}[G_1[[t^{-1}]]]$. To quantize these spaces, [KWWY] define families of shifted Yangians $Y_\mu$, and truncated shifted Yangians $Y^\lambda_\mu$. The shifted Yangians $Y_\mu$ are defined in [KWWY] as certain subalgebras of the usual Yangian $Y_0$, while $Y^\lambda_\mu$ are certain subquotients. By [KWWY, Thm. 3.12 and 4.8], $Y_\mu$ and $Y^\lambda_\mu$ quantize $\mathrm{Gr}_\mu$ and $\overline{\mathrm{Gr}}^{\lambda}_{\mu}$ respectively. More precisely, similar to the nilpotent cone situation, [KWWY] also shows that there is a family $Y_\mu(\mathbf{c})$ quantizing $\mathrm{Gr}_\mu$, and that there is a family $Y^\lambda_\mu(\mathbf{c})$ quantizing $\overline{\mathrm{Gr}}^{\lambda}_{\mu}$ where $\mathbf{c}$ range over a certain set of parameters. Their definition is inspired by the work of Brundan-Kleshchev, [BK], which introduces the shifted Yangians for $\mathfrak{gl}_n$. Now, in order to study the geometry of slices, it is natural to study the representation theory of these (truncated) shifted Yangians (see [KTWWY1]).

### 1.1.4 Generalized affine Grassmannian slices and shifted Yangians

In this thesis, we will be working with a generalization of the setting of the last paragraph. In [BFN], the authors introduce *generalized slices* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ which makes sense even in the case where $\mu$ is not dominant.

More explicitly, consider a pair of coweights $\lambda, \mu$ such that $\lambda$ is dominant and that $\mu \leq \lambda$. Given $T \subseteq B \subseteq G$ where $T$ is a torus and $B$ is Borel, let $U$ be the unipotent radical of $B$ (similarly, let $B_-$ be the opposite Borel and $U_-$ its unipotent radical). Consider the spaces $G[t]t^{\lambda}G[t]$, $\mathcal{W}_{\mu} = U_1[[t^{-1}]]T_1[[t^{-1}]]t^{\mu}U_{1,-}[[t^{-1}]]$, and $\overline{\mathcal{W}}^{\lambda}_{\mu} := \overline{G[t]t^{\lambda}G[t]} \cap \mathcal{W}_{\mu}$. When $\lambda$ and $\mu$ are dominant, one has that $\mathcal{W}_{\mu} \simeq \mathrm{Gr}_{\mu}$ and that $\overline{\mathrm{Gr}^{\lambda}_{\mu}} \simeq \overline{\mathcal{W}}^{\lambda}_{\mu}$.

The space $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is of dimension $\langle 2\rho, \lambda - \mu \rangle$ where $2\rho$ is the sum of positive roots. The definition given in [BFN, 2(ii)] describes $\mathcal{W}_{\mu}$ as a moduli space of the following data:

(a) a $G$-bundle $\mathcal{P}$,

(b) a trivialization $\sigma : \mathcal{P}_{\mathrm{triv}}|_{\hat{\mathbb{P}}^1_{\infty}} \xrightarrow{\sim} \mathcal{P}|_{\hat{\mathbb{P}}^1_{\infty}}$, where $\hat{\mathbb{P}}^1_{\infty}$ denotes a formal neighbourhood around $\infty$ in $\mathbb{P}^1$.

(c) a $B$-structure $\phi$ on $\mathcal{P}$ of degree $w_0\mu$ having fiber $B_- \subseteq G$ at $\infty \in \mathbb{P}^1$, with respect to the trivialization $\sigma$ of $\mathcal{P}$ at $\infty \in \mathbb{P}^1$.

The subvariety $\overline{\mathcal{W}^{\lambda}_{\mu}}$ is defined as being cut out by the condition that $\sigma$ extends as a rational trivialization with a unique pole at $0 \in \mathbb{P}^1$, and the order of the pole of $\sigma$ at $0 \in \mathbb{P}^1$ is $\leq \lambda$. In the setting of [BFN], the space $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is the Coulomb branch of a 3d $\mathcal{N} = 4$ SUSY gauge theory, with the corresponding Higgs branch being the Nakajima quiver variety $\mathcal{M}_0(\lambda, \mu)$. When $\lambda = 0$, $\overline{\mathcal{W}}^{0}_{\mu}$ is a space of based maps from $\mathbb{P}^1$ to $G/B$.

As candidates for quantizations, [BFN, Appendix B] provides definitions for $Y_{\mu}$ and $Y^{\lambda}_{\mu}$ for arbitrary $\mu$. The quantization question is addressed in [FKPRW]. Given any splitting $\mu = \nu_1 + \nu_2$, [FKPRW, Section 5.4] defines a filtration $F_{\nu_1,\nu_2}Y_{\mu}$ for $Y_{\mu}$.

**Theorem 1.1.4.1.** [FKPRW, Thm .5.15] *Suppose* $\mu = \nu_1 + \nu_2$. *Then* $\mathrm{gr}^{F_{\nu_1,\nu_2}} Y_{\mu} \simeq \mathbb{C}[\mathcal{W}_{\mu}]$.

Note that this theorem endows $\mathcal{W}_{\mu}$ with a Poisson structure, given by (1.1). For dominant $\mu$, this is the same as the natural Poisson structure coming from the aforementioned Manin triple. The above isormorphism also endows the subvariety $\overline{\mathcal{W}}^{\lambda}_{\mu}$ of $\mathcal{W}_{\mu}$ with a Poisson structure.

The BFN construction of the Coulomb branch also yields a defomation quantization $\mathcal{A}^{\mathrm{sph}}$. In [BFN, Cor. B.28] for dominant $\mu$ and in [We, Cor. 3.10] for general $\mu$, it has been established that $\mathcal{A}^{\mathrm{sph}}_{\hbar} \simeq Y^{\lambda}_{\mu}$.

## 1.2 Multiplication of generalized slices and coproducts for shifted Yangians

### 1.2.1 Multiplication of slices

There is natural family of maps between slices. Given any coweights $\mu_1, \mu_2$, one defines a map

$$m_{\mu_1,\mu_2} : \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}, \quad (g_1, g_2) \mapsto \pi_{\mu_1+\mu_2}(g_1 g_2)$$

where $\pi_\mu$ is the identification

$$\pi_\mu : U[t]\backslash U((t^{-1}))T_1[[t^{-1}]]t^\mu U^-((t^{-1}))/U^-[t] \longrightarrow \mathcal{W}_\mu.$$

These multiplication maps are not associative (see Remark 2.6.3.11). In the case of slices, [BFN, 2(vi)] constructs multiplication morphisms

$$m^{\lambda_1,\lambda_2}_{\mu_1,\mu_2} : \overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} \longrightarrow \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}.$$

Comparing the constructions of [BFN, 2(vi) and 2(xi)], $m_{\mu_1,\mu_2}$ restricts to $m^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}$.

The following result was conjectured in [FKPRW, Conj. 5.20]. We will include its proof, communicated to us by Alex Weekes, in section 2.6.3.

**Theorem 1.2.1.1.** (Thm 2.6.3.9) *The map $m_{\mu_1,\mu_2}$ is Poisson.*

### 1.2.2 Coproducts of shifted Yangians

Chapter 2 provides the basic information on the shifted Yangians $Y_\mu$ for arbitrary $\mu$. We mention the PBW theorem for $Y_\mu$ (see Theorem 2.2.0.2), which can also be found in [FKPRW]. However, most of the chapter is devoted to the generalization of the following result, stated without proof in [KT] and proved by Guay-Nakajima-Wendlandt in [GNW, Thm 4.1].

**Theorem 1.2.2.1.** [GNW, Thm 4.1] *There exists a coproduct $\Delta_{0,0} : Y_0 \longrightarrow Y_0 \otimes Y_0$, where $Y_0$ is the usual Yangian.*

In the paragraph before [FKPRW, Prop. 5.19], the authors explained why the coproduct $\Delta_{0,0}$ quantizes the multiplication $m_{0,0}$ in $G_1[[t^{-1}]]$. Our version is the following result.

**Theorem 1.2.2.2.** (Thm 2.3.3.1) *For arbitrary coweights $\mu_1$ and $\mu_2$, there exists a coproduct map $\Delta_{\mu_1,\mu_2} : Y_{\mu_1+\mu_2} \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$.*

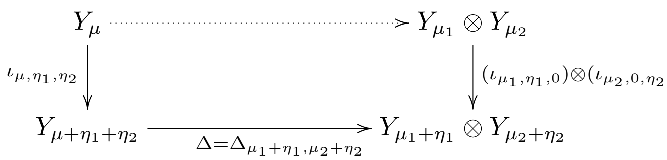
The existence of such maps is not surprising, as there should be a non-commutative version of the multiplication maps $m_{\mu_1,\mu_2}$. More precisely, using the fact that $m_{\mu_1,\mu_2}$ is Poisson, one has the following result.

**Theorem 1.2.2.3.** [FKPRW, Prop. 5.21] *$m_{\mu_1,\mu_2}$ is the classical limit of $\Delta_{\mu_1,\mu_2}$.*

Let us briefly explain our proof of the existence of $\Delta_{\mu_1,\mu_2}$. A typical approach when dealing with shifted Yangians is to try to reduce the proof to the antidominant case, or to the case of the usual Yangian $Y_0$. A reason for this is that we have natural shift embeddings $Y_\mu \longrightarrow Y_{\mu'}$ when $\mu' \leq \mu$ and that the structure of $Y_\mu$ is “simpler” when $\mu$ is antidominant.

Given a splitting $\mu = \mu_1 + \mu_2$ where $\mu, \mu_1, \mu_2$ are antidominant, we write down an alternative presentation for $Y_\mu$ in section 2.3.1. This presentation generalizes the Levendorskii presentation for the usual Yangian, with almost exactly the same proof (see [L1]). It also shows that $Y_\mu$ is finitely generated. It allows us to prove the existence of the coproduct $\Delta_{\mu_1,\mu_2}$ by checking only finitely many relations, and by using the existence of the coproduct $\Delta_{0,0} : Y_0 \longrightarrow Y_0 \otimes Y_0$ of the usual Yangian.

For arbitrary $\mu = \mu_1 + \mu_2$, the general case is done by embedding into the antidominant case. The following diagram can be found in the proof of Theorem 2.3.3.1.




where $\mu_1 + \eta_1$ and $\mu_2 + \eta_2$ are antidominant. The two vertical maps are shift embeddings.

## 1.3 A Hamiltonian reduction of $\mathcal{W}_\mu$

Denote by $I$ the set of simple roots of $\mathfrak{g}$. Fix $i \in I$. Recall that, for a Poisson variety $X$ equipped with a Hamiltonian $\mathbb{G}_a$-action together with moment map $\Phi : X \longrightarrow \mathbb{C}$, the reduction $\Phi^{-1}(1)/\mathbb{G}_a$ is denoted by $X/\!\!/_1 \mathbb{G}_a$. The goal of Chapter 3 is to show that one can obtain $\mathcal{W}_{\mu+\alpha_i}$ as a Hamiltonian reduction of $\mathcal{W}_\mu$.

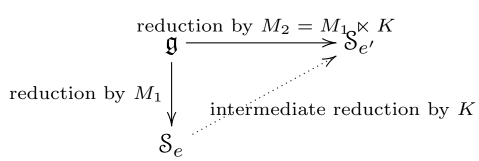
### 1.3.1 Some motivation

One of our original motivations is that such reduction would entail a relation between modules of $Y_\mu^\lambda$ and those of $Y_{\mu+\alpha_i}^\lambda$. It could lead to some categorification result, in the spirit of [KTWWY2].

A more concrete motivation is the work Morgan, [Mor], on quantum Hamiltonian reduction of $W$-algebras. Let $e$ be a nilpotent element of $\mathfrak{g}$. Under the adjoint action of $G$ on $\mathfrak{g}$, consider the nilpotent orbit $\mathcal{O}_e := G \cdot e$. By the Jacobson-Morozov theorem, one can complete $e$ to an $\mathfrak{sl}_2$-triple $\{e, h, f\}$. Corresponding to this triple, there is a natural transverse slice to $\mathcal{O}_e$, $\mathcal{S}_e := e + \ker(\operatorname{ad} f)$, known as the Slodowy slice.

Identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ via the Killing form, one can view $\mathcal{O}_e$ as lying inside $\mathfrak{g}^*$. The advantage is that $\mathfrak{g}^*$ has a natural Poisson structure coming from the Lie bracket of $\mathfrak{g}$. By a result of Gan-Ginzburg (see [GG], [Mor, Prop. 2.3.3]), the Slodowy slice inherits a Poisson structure from $\mathfrak{g}^*$. The co-adjoint action of $G$ on $\mathfrak{g}^*$ is Hamiltonian, with comoment map $\mu^* : \mathfrak{g} \longrightarrow \mathbb{C}[\mathfrak{g}^*]$, $x \mapsto x$. Given a correct choice of subgroup $M \subseteq G$ acting on $\mathfrak{g}^*$ by the co-adjoint action, the Slodowy slice can be realized as a Hamiltonian reduction, $\mathcal{S}_e \simeq \mu^{-1}(e)/M$ (see [GG]).

Our interest comes from the following diagram, which exhibits reduction by stages for Slodowy slices (see [Mor, Section 3.4]). Whenever $\mathcal{O}_e \subseteq \overline{\mathcal{O}}_{e'}$,




for some groups $M_1, M_2$, and $K$. The dotted arrow says that one can obtain $\mathcal{S}_{e'}$ as a reduction of $\mathcal{S}_e$.

Why should we expect a similar relation between $\mathcal{W}_\mu$ and $\mathcal{W}_{\mu+\alpha_i}$? This is explained in the work of Rowe (see [Ro, Chapter 5]). Let $G = GL_n$. For $\lambda, \mu$ dominant, there is a $\mathbb{G}_a$-action on $\mathrm{Gr}_\mu^\lambda$, $a \cdot L = x_i^+(a)L$ where $x_i^+ : \mathbb{G}_a \longrightarrow G$ denotes the exponential map into the "upper triangular" part of $G$.

Let $d$ be the height of $\mu$ where $\mu$ is dominant. There is a subspace $M_\mu \subseteq \mathfrak{gl}_d$ called the *Mirković -Vybornov slice*. Via the Mirković-Vybornov isomorphism, one obtains $\mathrm{Gr}_\mu^{d\omega_1} \simeq M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}$. So, we can transport the $\mathbb{G}_a$-action from $\mathrm{Gr}_\mu^{d\omega_1}$ onto $M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}$. To tie it in with the work of Morgan, the coweight $\mu$ gives rise to a Slodowy slice $S_\mu$ (see [Mor, Section 3.4.2]). Moreover, [Ro, Prop. 5.3.5] shows that $M_\mu \simeq S_\mu$. Additionally, by [Ro, Th. 5.3.6], the transported $\mathbb{G}_a$-action on $M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}$ is the same as the $\mathbb{G}_a$-action on $S_\mu \cap \mathcal{N}_{\mathfrak{gl}_d}$ described in [Mor]. Therefore, putting together [Ro, Thm 5.3.6] and the reduction result in [Mor], one has that $\mathrm{Gr}_\mu^\lambda /\!\!/_1 \mathbb{G}_a \simeq \mathrm{Gr}_{\mu+\alpha}^\lambda$ in the case where both $\mu$ and $\mu + \alpha$ are dominant.

### 1.3.2 Our Hamiltonian reduction

Section 3.2.1 is devoted to defining a Hamiltonian $\mathbb{G}_a$-action on $\mathcal{W}_\mu$. Given $r \in \mathbb{G}_a$ and $g \in \mathcal{W}_\mu$, the action is defined by $r \cdot g = \pi_\mu(x_i(r)g)$.

**Proposition 1.3.2.1.** (Prop. 3.2.1.10). *The action defined above is Hamiltonian with moment map* $\Phi_i : \mathcal{W}_\mu \longrightarrow \mathbb{C}$, $uht^\mu u_- \mapsto \Delta^{(1)}_{s_i w_0 \omega_{i^*}, w_0 \omega_{i^*}}(u)$.

Note that $w_0$ is the longest element of the Weyl group, $i^*$ is defined by $\alpha_{i^*} = -w_0\alpha_i$, and $\Delta^{(1)}_{s_i w_0 \omega_{i^*}, w_0 \omega_{i^*}}(g)$ is the coefficient of $t^{-1}v_{s_i w_0 \omega_{i^*}}$ in $gv_{w_0\omega_{i^*}}$ $(g \in G((t^{-1})))$ . To prove that the action is a Poisson action, it is worth mentioning that we employ the same techniques used to prove that $m_{\mu_1,\mu_2}$ is Poisson. Another observation is that this action restricts to an action on the slice $\overline{\mathcal{W}}_\mu^\lambda$.

In Section 3.2.2, we show that $\mathcal{W}_{\mu+\alpha_i}$ and $\overline{\mathcal{W}}_{\mu+\alpha_i}^\lambda$ can be obtained as Hamiltonian reductions of $\mathcal{W}_\mu$ and $\overline{\mathcal{W}}_\mu^\lambda$ respectively. Let $m$ be the map $m_{-\alpha_i,\mu+\alpha_i}$ restricted to $\overline{\mathcal{W}}_{-\alpha_i}^0 \times \mathcal{W}_{\mu+\alpha_i}$. Our approach is to define an explicit inverse of $m$. Observe that $\overline{\mathcal{W}}_{-\alpha_i}^0 \simeq \mathbb{C} \times \mathbb{C}^\times = T^*(\mathbb{C}^\times)$. Under this identification, the Poisson structure on $\overline{\mathcal{W}}_{-\alpha_i}^0$ is given by $\{c, a\} = c$ where $c$ is the coordinate function of $\mathbb{C}^\times$, and $a$ is that of $\mathbb{C}$. We define a $\mathbb{G}_a$-action on $\overline{\mathcal{W}}_{-\alpha_i}^0 \times \mathcal{W}_{\mu+\alpha_i}$ by acting only on the first factor as follows: $r \cdot (a, c) = (a + rc, c)$.

**Theorem 1.3.2.2.** (Thm 3.2.2.6 and Prop. 3.2.2.7) $m : \overline{\mathcal{W}}_{-\alpha_i}^0 \times \mathcal{W}_{\mu+\alpha_i} \longrightarrow \Phi_i^{-1}(\mathbb{C}^\times)$ *is a* $\mathbb{G}_a$*-equivariant Poisson isomorphism.*

Specializing at the level set $\Phi_i^{-1}(1)$ and taking reduction, we obtain

**Theorem 1.3.2.3.** *There is a Poisson isomorphism* $\mathcal{W}_\mu /\!\!/_1 \mathbb{G}_a \simeq \mathcal{W}_{\mu+\alpha_i}$.

**Corollary 1.3.2.4.** *There is a Poisson isomorphism* $\overline{\mathcal{W}}_\mu^\lambda /\!\!/_1 \mathbb{G}_a \simeq \overline{\mathcal{W}}_{\mu+\alpha_i}^\lambda$.

## 1.4 Toward quantum Hamiltonian reduction

Chapter 4 discusses the lifting of the isomorphism

$$m : \overline{\mathcal{W}}_{-\alpha_i}^0 \times \mathcal{W}_{\mu+\alpha_i} \longrightarrow \Phi_i^{-1}(\mathbb{C}^\times)$$

of Theorem 3.2.2.6 to the Yangian level. We know that the left-hand side is quantized by $Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$. It remains to make sense of the right-hand side.

In section 4.1, we prove that one can localize $Y_\mu$ at $E_i^{(1)}$.

**Theorem 1.4.0.1.** (Thm 4.1.0.5) *The set* $\{(E_i^{(1)})^k : k \in \mathbb{N}\} \subseteq Y_\mu$ *satisfies the right Ore condition.*

Denote the corresponding localization by $Y_\mu[(E_i^{(1)})^{-1}]$. Given any splitting $\mu = \nu_1+\nu_2$, recall that we have a filtration $F_{\nu_1,\nu_2}$ on $Y_\mu$. Now, following [S, 12.3], using the fact that $Y_\mu$ is a domain, we can put a filtration on $Y_\mu[(E_i^{(1)})^{-1}]$ as follows: given $x \in Y_\mu, s \in S = \{(E_i^{(1)})^n : n \in \mathbb{N}\}$, we specify $\deg(xs) = \deg(x) - \deg(s)$ where deg denotes the filtered degree.

The following result is a special case of a statement on localization of filtered rings.

**Proposition 1.4.0.2.** ([LR, I, 3.2], [S, Thm 6.6]). $\operatorname{gr} Y_\mu[(E_i^{(1)})^{-1}] \simeq \mathbb{C}[\Phi_i^{-1}(\mathbb{C}^\times)]$.

For our purpose, we do not need to fully define the truncated shifted Yangians $Y^\lambda_\mu$. For a general definition, we refer to [BFN, Appendix B]. In our case, the algebra $Y^0_{-\alpha_i}$ is generated by elements $A_i^{(1)}, (E_i^{(1)})^{\pm 1}$ with relation $[E_i^{(1)}, A_i^{(1)}] = E_i^{(1)}$. Moreover, $Y^0_{-\alpha_i} \simeq D(\mathbb{C}^\times)$, the algebra of differential operators on $\mathbb{C}^\times$.

Consider the following composite map

$$Y_\mu \longrightarrow Y_{-\alpha_i} \otimes Y_{\mu+\alpha_i} \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$$

where the first map is the comultiplication map $\Delta_{-\alpha_i,\mu+\alpha_i}$ and the second map is the projection in the first component. Since $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1$, by universal property of localization, we obtain a map $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1})] \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ extending the composite map.

In Section 4.2, we discuss our partial result at lifting the isomorphism of Theorem 3.2.2.6 to the Yangian level. In other words, we want to show that $\tilde{\Delta}$ is an isomorphism. We have some partial results in this direction.

Our approach involves working with filtrations of $Y_\mu$, as defined in [FKPRW, Section 5.4]. We would like to invoke the following lemma.

**Lemma 1.4.0.3.** *Let* $\phi : A \longrightarrow B$ *be a map of* $\mathbb{Z}$*-filtered algebras with increasing filtrations. Assume that all involved filtrations are exhaustive, i.e.,* $A = \bigcup_n A$ *and* $B = \bigcup_n B_n$*. Additionally, assume that* $\bigcap_n A_n = \{0\}$*. Denote by* $\operatorname{gr}\phi : \operatorname{gr} A \longrightarrow \operatorname{gr} B$ *the induced map on associated graded level.*

*(1) If* $\operatorname{gr}\phi$ *is injective, so is* $\phi$*.*

*(2) Suppose that* $A_n = \{0\}$ *for all* $n < 0$*. If* $\operatorname{gr}\phi$ *is surjective, so is* $\phi$*.*

By the previous lemma, $\tilde{\Delta}$ is injective. The main obstacle for surjectivity is that filtrations on $Y_\mu$ are not bounded below in general. However, we have the following partial result.

**Theorem 1.4.0.4.** (Thm. 4.2.0.6) *If there exists a coweight* $\nu$ *such that the filtrations* $F_{\nu,\mu-\nu}Y_\mu$*,* $F_{\nu,-\alpha_i-\nu}Y^0_{-\alpha_i}$*, and* $F_{\alpha_i+\nu,\mu-\nu}$ *are non-negative, then* $\tilde{\Delta} : Y_\mu \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ *is an isomorphism.*

It is worth noting that the previous theorem implies that $\tilde{\Delta}$ is an isomorphism for sufficiently dominant $\mu$. We can push the argument a little bit further with the following commutative diagram:

$$\begin{array}{ccc} Y_{\mu+\eta} & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\eta+\alpha_i} \\ {\scriptstyle \iota_{\mu+\eta,0,-\eta}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Id} \otimes \iota_{\mu+\eta+\alpha_i,0,-\eta}} \\ Y_\mu & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i} \end{array}$$

One can pick a dominant $\eta$ such that the top arrow is an isomorphism. The advantage of this is that the vertical maps can be quite easily described. These maps, called shift embeddings, will fix $E_j^{(r)}$ and shift $F_j^{(r)}$ to $F_j^{(r+\langle \eta,\alpha_j \rangle)}$. In fact, in the case of $\mathfrak{sl}_2$, using this idea and the description of the coproduct in the antidominant case, we have the following.

**Proposition 1.4.0.5.** (Prop. 4.2.0.9) *If $\mathfrak{g} = \mathfrak{sl}_2$, then $\tilde{\Delta}$ is an isomorphism.*

We also believe that the following conjecture holds.

**Conjecture 1.4.0.6.** *For all $\mathfrak{g}$ and for any coweight $\mu$ of $\mathfrak{g}$, $\tilde{\Delta} : Y_\mu \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ is an isomorphism.*

## 1.5 Some notations

We write down some more frequently used notation. Let $G$ be a simply-laced algebraic group.

For a simple root $\alpha_i$, we write $\alpha_{i^*} = -w_0\alpha_i$ where $w_0$ is the longest element of the Weyl group. For any weight $\omega$, we write $\omega^* = -w_0\omega$.

Denote by $\varphi_i : SL_2((t^{-1})) \longrightarrow G((t^{-1}))$ the map induced by the inclusion $SL_2 \longrightarrow G$ corresponding to the root $\alpha_i$. Let $x_i : \mathbb{C}((t^{-1})) \longrightarrow G((t^{-1}))$ be the exponential map into the "lower triangular" part of $\varphi_i\big(SL_2((t^{-1}))\big)$. Likewise, let $x_i^+$ be the corresponding map into the "upper triangular" part.

Let $V$ be a representation of $G$, let $v \in V$, and let $\beta \in V^*$. The "matrix coefficient" $\mathcal{D}_{\beta,v}$ is a function on $G$ defined by $\mathcal{D}_{\beta,v}(g) = \langle \beta, gv \rangle$. Let $W$ be the Weyl group of $G$. With the standard Chevalley generators $e_i, h_i, f_i$ of $\mathfrak{g}$, we define a lift of $W$ by $\overline{s_i} = \exp(f_i)\exp(-e_i)\exp(f_i)$. For a dominant weight $\omega$ and for $w_1, w_2 \in W$, define $\mathcal{D}_{w_1\omega,w_2\omega} := \langle \overline{w_1} v_{-\omega}, \overline{w_2} v_\omega \rangle$ where $v_\omega$ is the highest weight vector for the highest weight representation $V(\omega)$ and $v_{-\omega}$ is the lowest weight vector for the lowest weight dual representation $V(\omega)^*$. We extend these definitions to $G((t^{-1}))$. More precisely, for $g \in G((t^{-1}))$,

$$\mathcal{D}_{\beta,v}(g) = \sum_{s\in\mathbb{Z}} \mathcal{D}^{(s)}_{\beta,v}(g)t^{-s}.$$

For $f \in \mathbb{C}((t^{-1}))$, denote by $\underline{f}$ the principal part of $f$, i.e., for $f = \sum_{n\in\mathbb{Z}} f_n t^n$, $\underline{f} = \sum_{n<0} f_n t^n$.

# Chapter 2

# Shifted Yangians and their classical limits

The majority of results in this chapter can be found in [FKPRW]. There are a few exceptions, notably the results in 2.6.3, communicated to us by Alex Weekes.

## 2.1 Basic definition

Following [BFN, Appendix B], we introduce a family of algebras known as the shifted Yangians. These algebras are our main objects of study.

Let $\mathfrak{g}$ be a simply-laced simple Lie algebra of finite type. Denote by $\{\alpha_i\}_{i\in I}$ the simple roots of $\mathfrak{g}$. We write $\alpha_i \cdot \alpha_j$ for the usual inner product of these simple roots.

**Definition 2.1.0.1.** The *Cartan doubled Yangian* $Y_\infty := Y_\infty(\mathfrak{g})$ is defined to be the $\mathbb{C}$–algebra with generators $E_i^{(q)}, F_i^{(q)}, H_i^{(p)}$ for $i \in I$, $q > 0$ and $p \in \mathbb{Z}$, with relations

$$[H_i^{(p)}, H_j^{(q)}] = 0, \tag{2.1}$$

$$[E_i^{(p)}, F_j^{(q)}] = \delta_{ij} H_i^{(p+q-1)}, \tag{2.2}$$

$$[H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] = \frac{\alpha_i \cdot \alpha_j}{2}(H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}), \tag{2.3}$$

$$[H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] = -\frac{\alpha_i \cdot \alpha_j}{2}(H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}), \tag{2.4}$$

$$[E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] = \frac{\alpha_i \cdot \alpha_j}{2}(E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}), \tag{2.5}$$

$$[F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] = -\frac{\alpha_i \cdot \alpha_j}{2}(F_i^{(p)} F_j^{(q)} + F_j^{(p)} F_i^{(q)}), \tag{2.6}$$

$$i \neq j, N = 1 - \alpha_i \cdot \alpha_j \Rightarrow \mathrm{sym}[E_i^{(p_1)}, [E_i^{(p_2)}, \cdots [E_i^{(p_N)}, E_j^{(q)}] \cdots]] = 0, \tag{2.7}$$

$$i \neq j, N = 1 - \alpha_i \cdot \alpha_j \Rightarrow \mathrm{sym}[F_i^{(p_1)}, [F_i^{(p_2)}, \cdots [F_i^{(p_N)}, F_j^{(q)}] \cdots]] = 0. \tag{2.8}$$

We denote by $Y_\infty^>, Y_\infty^\geq$ the subalgebras of $Y_\infty$ generated by the $E_i^{(q)}$ (resp. $E_i^{(q)}$ and $H_i^{(p)}$). Similarly, we denote by $Y_\infty^<, Y_\infty^\leq$ the subalgebras generated by the $F_i^{(q)}$ (resp. $F_i^{(q)}, H_i^{(p)}$). Also, denote by $Y_\infty^=$ the subalgebra generated by the $H_i^{(p)}$.

**Definition 2.1.0.2.** For any coweight $\mu$, the shifted Yangian $Y_\mu$ is defined to be the quotient of $Y_\infty$ by the relations $H_i^{(p)} = 0$ for all $p < -\langle \mu, \alpha_i \rangle$ and $H_i^{(-\langle \mu, \alpha_i \rangle)} = 1$.

*Remark* 2.1.0.3. When $\mu = 0$, the algebra $Y = Y_0$ is the usual Yangian. The above generators and the above relations correspond to the Drinfeld presentation of $Y$.

We can relate these algebras in a natural way, via "shift homomorphisms".

**Proposition 2.1.0.4.** [FKPRW, Prop 3.8] *Let $\mu$ be a coweight, and $\mu_1, \mu_2$ be antidominant coweights. Then there exists a homomorphism $\iota_{\mu,\mu_1,\mu_2} : Y_\mu \longrightarrow Y_{\mu+\mu_1+\mu_2}$ defined by*

$$H_i^{(r)} \mapsto H_i^{(r - \langle \mu_1 + \mu_2, \alpha_i \rangle)}, \quad E_i^{(r)} \mapsto E_i^{(r - \langle \mu_1, \alpha_i \rangle)}, \quad F_i^{(r)} \mapsto F_i^{(r - \langle \mu_2, \alpha_i \rangle)}. \tag{2.9}$$

*Proof.* This is immediate from the definition of shifted Yangians. □

*Remark* 2.1.0.5. In [KWWY], for $\mu$ dominant, the shifted Yangian $Y_\mu$ is realized as a subalgebra of the usual Yangian $Y_0$ and not as a quotient of $Y_\infty$. In our setting, the shift map $\iota_{\mu,-\mu,0}$ corresponds to the natural inclusion $Y_\mu \longrightarrow Y_0$ in [KWWY].

Next, let us introduce the following elements of the shifted Yangians, similar to certain elements of the usual Yangian considered by Levendorskii in [L1].

**Definition 2.1.0.6.** Set $S_i^{(-\langle \mu, \alpha_i \rangle + 1)} = H_i^{(-\langle \mu, \alpha_i \rangle + 1)}$ and

$$S_i^{(-\langle \mu, \alpha_i \rangle + 2)} = H_i^{(-\langle \mu, \alpha_i \rangle + 2)} - \tfrac{1}{2}\big(H_i^{(-\langle \mu, \alpha_i \rangle + 1)}\big)^2 \tag{2.10}$$

For $r \geq 1$, it is not hard to check that

$$[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_j^{(r)}] = (\alpha_i \cdot \alpha_j) E_j^{(r+1)},$$
$$[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_j^{(r)}] = -(\alpha_i \cdot \alpha_j) F_j^{(r+1)}.$$

Note that these elements play the role of "raising operators", allowing us to obtain higher $E_i^{(r)}$ and $F_i^{(r)}$ from $E_i^{(1)}$ and $F_i^{(1)}$.

**Lemma 2.1.0.7.** [FKPRW, Lem 3.11] *Let $\mu$ be an antidominant coweight. As a unital associative algebra, $Y_\mu$ is generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + 1)} = H_i^{(-\langle \mu, \alpha_i \rangle + 1)}, S_i^{(-\langle \mu, \alpha_i \rangle + 2)} = H_i^{(-\langle \mu, \alpha_i \rangle + 2)} - \frac{1}{2}(H_i^{(-\langle \mu, \alpha_i \rangle + 1)})^2$. Alternatively, $Y_\mu$ is also generated by $E_i^{(1)}, F_i^{(1)}, H_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$). In particular, $Y_\mu$ is finitely generated.*

*Proof.* For the first assertion, it is enough to show that $E_i^{(r)}, F_i^{(r)} H_i^{(s)}$ lie in the subalgebra generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$) for all $r \geq 1, s \geq -\langle \mu, \alpha_i \rangle + 1$. This is clear since $E_i^{(r)} = \frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, E_i^{(r-1)}]$, $F_i^{(r)} = -\frac{1}{2}[S_i^{(-\langle \mu, \alpha_i \rangle + 2)}, F_i^{(r-1)}]$ for all $r \geq 2$ and since $H_i^{(s)} = [E_i^{(1)}, F_i^{(s)}]$ for all $s \geq -\langle \mu, \alpha_i \rangle + 1$.

The second assertion follows immediately from the first since the subalgebra generated by $E_i^{(1)}, F_i^{(1)}, S_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$) is contained in the subalgebra generated by the $E_i^{(1)}, F_i^{(1)}$ and $H_i^{(-\langle \mu, \alpha_i \rangle + k)}$ ($k = 1, 2$). □

## 2.2 PBW theorem

In this section, we describe the PBW theorem for shifted Yangians, generalizing the case of ordinary Yangian (due to Levendorskii in [L2]), and the case of dominantly shifted Yangians [KWWY, Prop 3.11].

**Definition 2.2.0.1.** Let $\beta$ be a positive root, and pick any decomposition $\beta = \alpha_{i_1} + \ldots + \alpha_{i_l}$ into simple roots so that the element $[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{l-1}}, e_{i_l}] \cdots]$ is a non-zero element of the root space $\mathfrak{g}_\beta$. Consider also $q > 0$ and a decomposition $q + l - 1 = q_1 + \ldots + q_l$ into positive integers. Then we define a corresponding element of $Y_\infty$:

$$E_\beta^{(q)} := [E_{i_1}^{(q_1)}, [E_{i_2}^{(q_2)}, \ldots [E_{i_{l-1}}^{(q_{l-1})}, E_{i_l}^{(q_l)}] \cdots]. \tag{2.11}$$

This element $E_\beta^{(q)}$, called a *PBW variable*, depends on the choices above. However, we will fix such a choice for each $\beta$ and $q$.

Similarly, we define PBW variable $F_\beta^{(q)}$ for each positive root $\beta$ and each $q > 0$.

For each positive root $\beta$ and $q > 0$, consider elements $E_\beta^{(q)}, F_\beta^{(q)} \in Y_\mu$ defined as images under $Y_\infty \twoheadrightarrow Y_\mu$ of the elements of $Y_\infty$ in Definition 2.2.0.1. Choose a total order on the set of PBW variables

$$\left\{E_\beta^{(q)} : \beta \in \Delta^+, q > 0\right\} \cup \left\{F_\beta^{(q)} : \beta \in \Delta^+, q > 0\right\} \cup \left\{H_i^{(p)} : i \in I, p > -\langle \mu, \alpha_i \rangle\right\} \tag{2.12}$$

In the case $\mu = 0$, by [L2], ordered monomials in these PBW variables form a basis of $Y = Y_0$.

For simplicity we will assume that we have chosen a block order with respect to the three subsets above, i.e. ordered monomials have the form $EFH$.

**Theorem 2.2.0.2.** [FKPRW, Cor. 3.15] *For $\mu$ arbitrary, the set of ordered monomials in PBW variables form a PBW basis for $Y_\mu$ over $\mathbb{C}$.*

## 2.3 Coproducts of shifted Yangians

In this section, we describe a family of coproducts for shifted Yangians. Namely, for any decomposition $\mu = \mu_1 + \mu_2$, we establish the existence of a homomorphism

$$\Delta_{\mu_1,\mu_2} : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}. \tag{2.13}$$

This generalizes the coproduct for the ordinary Yangian $Y = Y_0$.

### 2.3.1 Levendorskii presentation

Let $\mu$ be an antidominant coweight. We follow Levendorskii's approach in [L1] for the ordinary Yangian to define a new presentation for $Y_\mu$.

Fix a decomposition $\mu = \mu_1 + \mu_2$ where the $\mu_i$'s are antidominant coweights.

Denote by $Y_{\mu_1,\mu_2}$ the algebra generated by: $S_i^{(-\langle\mu,\alpha_i\rangle+1)}$, $S_i^{(-\langle\mu,\alpha_i\rangle+2)}$, $E_i^{(r)}$ $(1 \le r \le -\langle\mu_1,\alpha_i\rangle+2)$, $F_i^{(r)}$ $(1 \le r \le -\langle\mu_2,\alpha_i\rangle+2)$ for all $i \in I$, with the following relations:

$$[S_i^{(k)}, S_j^{(l)}] = 0; \tag{2.14}$$

$$[S_i^{(-\langle\mu,\alpha_i\rangle+1)}, E_j^{(r)}] = (\alpha_i \cdot \alpha_j) E_j^{(r)},\ \ 1 \le r \le \langle\mu_1,\alpha_j\rangle + 1; \tag{2.15}$$

$$[S_i^{(-\langle\mu,\alpha_i\rangle+1)}, F_j^{(r)}] = -(\alpha_i \cdot \alpha_j) F_j^{(r)},\ \ 1 \le r \le \langle\mu_2,\alpha_j\rangle + 1; \tag{2.16}$$

$$[S_i^{(-\langle\mu,\alpha_i\rangle+2)}, E_j^{(r)}] = (\alpha_i \cdot \alpha_j) E_j^{(r+1)},\ \ 1 \le r \le \langle\mu_1,\alpha_j\rangle + 1; \tag{2.17}$$

$$[S_i^{(-\langle\mu,\alpha_i\rangle+2)}, F_j^{(r)}] = -(\alpha_i \cdot \alpha_j) F_j^{(r+1)},\ \ 1 \le r \le \langle\mu_2,\alpha_j\rangle + 1; \tag{2.18}$$

$$[E_i^{(r)}, F_j^{(s)}] = \begin{cases} 0 & i \neq j \\ 0 & i = j, r+s < -\langle\mu,\alpha_i\rangle + 1 \\ 1 & i = j, r+s = -\langle\mu,\alpha_i\rangle + 1 \\ S_i^{(-\langle\mu,\alpha_i\rangle+1)} & i = j, r+s = -\langle\mu,\alpha_i\rangle + 2 \\ S_i^{(-\langle\mu,\alpha_i\rangle+2)} + \frac{1}{2}\big(S_i^{(-\langle\mu,\alpha_i\rangle+1)}\big)^2 & i = j, r+s = -\langle\mu,\alpha_i\rangle + 3 \end{cases} \tag{2.19}$$

$$[E_i^{(r+1)}, E_j^{(s)}] = [E_i^{(r)}, E_j^{(s+1)}] + \frac{\alpha_i \cdot \alpha_j}{2}(E_i^{(r)} E_j^{(s)} + E_j^{(s)} E_i^{(r)}); \tag{2.20}$$

$$[F_i^{(r+1)}, F_j^{(s)}] = [F_i^{(r)}, F_j^{(s+1)}] - \frac{\alpha_i \cdot \alpha_j}{2}(F_i^{(r)} F_j^{(s)} + F_j^{(s)} F_i^{(r)}); \tag{2.21}$$

$$\mathrm{ad}(E_i^{(1)})^{1-(\alpha_i\cdot\alpha_j)}(E_j^{(1)}) = 0; \tag{2.22}$$

$$\mathrm{ad}(F_i^{(1)})^{1-(\alpha_i\cdot\alpha_j)}(F_j^{(1)}) = 0; \tag{2.23}$$

$$\big[S_i^{(-\langle\mu,\alpha_i\rangle+2)}, [E_i^{(-\langle\mu_1,\alpha_i\rangle+2)}, F_i^{(-\langle\mu_2,\alpha_i\rangle+2)}]\big] = 0. \tag{2.24}$$

For $r \ge 2$ and $s \ge 1$, set

$$\begin{aligned} E_i^{(r)} &= \frac{1}{2}[S_i^{(-\langle\mu,\alpha_i\rangle+2)}, E_i^{(r-1)}]; \\ F_i^{(r)} &= -\frac{1}{2}[S_i^{(-\langle\mu,\alpha_i\rangle+2)}, F_i^{(r-1)}]; \\ H_i^{(s)} &= [E_i^{(1)}, F_i^{(s)}]. \end{aligned}$$

*Remark* 2.3.1.1. Note that $H_i^{(s)} = 0$ if $s < -\langle\mu,\alpha_i\rangle$ and $H_i^{(-\langle\mu,\alpha_i\rangle)} = 1$.

We sketch the proof of the following theorem.

**Theorem 2.3.1.2.** *There exists an isomorphism $Y_\mu \longrightarrow Y_{\mu_1,\mu_2}$ of unital associative algebras given by*

$$E_i^{(r)} \mapsto E_i^{(r)}, F_i^{(r)} \mapsto F_i^{(r)}, H_i^{(s)} \mapsto H_i^{(s)},$$

*for $r \ge 1$ and $s \ge -\langle\mu,\alpha_i\rangle + 1$.*

*Sketch of proof.* One can check directly with finitely many computations that the relations of $Y_\mu$ imply the relations of $Y_{\mu_1,\mu_2}$. So, one has to show that the elements $E_i^{(r)}, F_i^{(r)}$ and $H_i^{(s)}$ of $Y_{\mu_1,\mu_2}$ satisfy the relations introduced by Definitions 2.1.0.1 and 2.1.0.2.

Using $S_i^{(-\langle\mu,\alpha_i\rangle+2)}$ together with relations (2.22) and (2.23), one can show relations (2.22) and (2.23) with $E_i^{(1)}$ and $F_i^{(1)}$ replaced by $E_i^{(-\langle\mu,\alpha_i\rangle+1)}$ and $F_i^{(-\langle\mu,\alpha_i\rangle+1)}$ respectively. In fact, more generally, relations (2.7) and (2.8) can be proved using the Levendorskii relations in the same way as in [L1, page 11].

Then, note that the subalgebra $Y = \langle E_i^{(-\langle \mu_1,\alpha_i\rangle+r)}, S_i^{(-\langle \mu,\alpha_i\rangle+r)}, F_i^{(-\langle \mu,\alpha_i\rangle+r)} : i \in I, r = 1,2\rangle$ of $Y_{\mu_1,\mu_2}$ has precisely the relations given by Levendorskii in [L1], meaning that it is isomorphic to the usual Yangian. So, all relations hold for high enough $E$, $H$ and $F$. Most importantly, relation (2.1) holds.

Omitting relations (2.19) and (2.24), the subagebras $\langle H_i^{(s)}, E_i^{(1)} : s \geq -\langle \mu,\alpha_i\rangle + 1\rangle$ and $\langle H_i^{(s)}, F_i^{(1)} : s \geq -\langle \mu,\alpha_i\rangle + 1\rangle$, with the remaining relations, are isomorphic to the positive (resp. negative) Borel Yangians, which means that all relations (except (2.2)) hold.

Now, we know that (2.2) holds for high enough $E$ and $F$ (by existence of the usual Yangian $Y$ in $Y_{\mu_1,\mu_2}$), it also holds for low enough $E$ and $F$ by (2.19). We can approach the case where we have a low $E$ and a high $F$ by induction as follows. Suppose that $r \leq -\langle \mu,\alpha_i\rangle$ and $s = -\langle \mu,\alpha_j\rangle + 3$, we have that

$$\begin{aligned} 0 = [S_j^{(-\langle \mu,\alpha_j\rangle+2)}, [E_i^{(r)}, F_j^{(s-1)}]] &= a_{ij}[E_i^{(r+1)}, F_j^{(s-1)}] + 2[F_j^{(s)}, E_i^{(r)}] \\ &= a_{ij}\delta_{ij}H_i^{(r+s-1)} - 2[E_i^{(r)}, F_j^{(s)}] \end{aligned}$$

Thus, $[E_i^{(r)}, F_j^{(s)}] = \delta_{ij}H_i^{(r+s-1)}$, as desired. Using the same argument, the result follows by induction on $s$. For high $E$ and low $F$, we swap $r$ and $s$ and use the same induction argument. □

*Remark* 2.3.1.3. Via the isomorphism of the previous theorem, the generators $S_i^{(-\langle \mu,\alpha_i\rangle+1)}$, $S_i^{(-\langle \mu,\alpha_i\rangle+2)}$ of $Y_{\mu_1,\mu_2}$ correspond precisely to the elements of $Y_\mu$ introduced in Definition 2.1.0.6.

### 2.3.2 Coproduct in the antidominant case

Let $\mu, \mu_1$, and $\mu_2$ be antidominant coweights. We wish to define a map

$$\Delta_{\mu_1,\mu_2} = \Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}.$$

When $\mu_1 = \mu_2 = 0$, the existence of a coproduct is stated without proof in [KT] and proved by Guay-Nakajima-Wendlandt in [GNW, Thm 4.1].

**Theorem 2.3.2.1.** [GNW, Thm 4.1] *There exists a coproduct* $\Delta_{0,0} : Y_0 \longrightarrow Y_0 \otimes Y_0$.

We define $\Delta_{\mu_1,\mu_2}$ on generators as follows.

$$\begin{aligned} \Delta(E_i^{(r)}) &= E_i^{(r)} \otimes 1, \;\; 1 \leq r \leq -\langle \mu_1,\alpha_i\rangle; \\ \Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}) &= E_i^{(-\langle \mu_1,\alpha_i\rangle+1)} \otimes 1 + 1 \otimes E_i^{(1)}; \\ \Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+2)}) &= E_i^{(-\langle \mu_1,\alpha_i\rangle+2)} \otimes 1 + 1 \otimes E_i^{(2)} + S_i^{(-\langle \mu_1,\alpha_i\rangle+1)} \otimes E_i^{(1)} \\ &\quad - \sum_{\gamma>0} F_\gamma^{(1)} \otimes [E_i^{(1)}, E_\gamma^{(1)}]; \end{aligned}$$

$$\begin{aligned}
\Delta(F_i^{(r)}) &= 1 \otimes F_i^{(r)}, \;\; 1 \le r \le -\langle \mu_2, \alpha_i \rangle; \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle +1)}) &= 1 \otimes F_i^{(-\langle \mu_2, \alpha_i \rangle +1)} + F_i^{(1)} \otimes 1; \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle +2)}) &= 1 \otimes F_i^{(-\langle \mu_2, \alpha_i \rangle +2)} + F_i^{(2)} \otimes 1 + F_i^{(1)} \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle +1)} \\
&\quad + \sum_{\gamma > 0} [F_i^{(1)}, F_\gamma^{(1)}] \otimes E_\gamma^{(1)}; \\
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +1)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle +1)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle +1)}; \\
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +2)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle +2)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle +2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_\gamma^{(1)} \otimes E_\gamma^{(1)}.
\end{aligned}$$

*Remark* 2.3.2.2. When $\mu = \mu_1 = \mu_2 = 0$, it is not hard to see that $\Delta_{0,0}$ agrees with the usual coproduct, and hence is well-defined.

For any antidominant coweights $\mu_1, \mu_2$, recall the shift maps $\iota_{0,\mu_1,0}$ and $\iota_{0,0,\mu_2}$ from Proposition 2.1.0.4. It is not hard to see that, for $k = 1, 2$,

$$\begin{aligned}
\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_i^{(k)}), \\
\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle +k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(E_i^{(k)}), \\
\Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle +k)}) &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(F_i^{(k)}).
\end{aligned}$$

**Theorem 2.3.2.3.** *$\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ is a well-defined map.*

*Proof.* We have to check that $\Delta$ preserves the defining relations. By Theorem 2.3.1.2 it suffices to check the relations (2.14) – (2.24).

First, we check relation (2.14). For $1 \le k, l \le 2$,

$$\begin{aligned}
&[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +k)}), \Delta(S_j^{(-\langle \mu, \alpha_j \rangle +l)})] = \\
&= [(\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_i^{(k)}), (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(S_j^{(l)})] = 0.
\end{aligned}$$

We check relation (2.15). For $1 \le r \le -\langle \mu_1, \alpha_j \rangle$,

$$[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +1)}), \Delta(E_j^{(r)})] = [S_i^{(-\langle \mu_1, \alpha_i \rangle +1)}, E_j^{(r)}] \otimes 1 = (\alpha_i \cdot \alpha_j) \Delta(E_j^{(r)}).$$

For $r = -\langle \mu_1, \alpha_j \rangle + 1$,

$$\begin{aligned}
[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle +1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle +1)})] &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}([S_i^{(1)}, E_j^{(1)}]) \\
&= (\alpha_i \cdot \alpha_j)(\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(E_j^{(1)}) \\
&= (\alpha_i \cdot \alpha_j) \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle +1)}).
\end{aligned}$$

The proof of relation (2.16) is similar to that of relation (2.15).

We check relation (2.17). For $1 \le r \le -\langle \mu_1, \alpha_j \rangle$,

$$[\Delta(S_i^{(-\langle \mu,\alpha_i\rangle+2)}), \Delta(E_j^{(r)})] = [S_i^{(-\langle \mu_1,\alpha_i\rangle+2)}, E_j^{(r)}] \otimes 1 + \sum_{\gamma>0} \langle \alpha_i, \gamma\rangle [E_j^{(r)}, F_\gamma^{(1)}] \otimes E_\gamma^{(1)}$$
$$= (\alpha_i \cdot \alpha_j) E_j^{(r+1)} \otimes 1 + \sum_{\gamma>0} \langle \alpha_i, \gamma\rangle [E_j^{(r)}, F_\gamma^{(1)}] \otimes E_\gamma^{(1)}.$$

Note that if $r < -\langle \mu_1, \alpha_i\rangle$, then $[E_j^{(r)}, F_l^{(1)}] = 0$ for all $l$. Then, by induction, $[E_j^{(r)}, F_\gamma^{(1)}] = 0$ for all $\gamma > 0$. The result follows in this case. If $r = -\langle \mu_1, \alpha_i\rangle$, then $[E_j^{(r)}, F_i^{(1)}] = \delta_{ij} 1$. Then, by induction, $[E_j^{(r)}, F_\gamma^{(1)}] = 0$ for all $\gamma$ of height greater than or equal to 2. The second summand becomes $(\alpha_i \cdot \alpha_j) 1 \otimes E_j^{(1)}$. Hence, the result follows.

For $r = -\langle \mu_1, \alpha_j\rangle + 1$,

$$[\Delta(S_i^{(-\langle \mu,\alpha_i\rangle+2)}), \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+1)})] = (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}([S_i^{(2)}, E_j^{(1)}])$$
$$= (\alpha_i \cdot \alpha_j)(\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0}(E_j^{(2)})$$
$$= (\alpha_i \cdot \alpha_j) \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+2)}).$$

Similarly, $\Delta$ preserves relation (2.18).

Next, we check relation (2.19). If $1 \le r \le -\langle \mu_1, \alpha_i\rangle$ and $1 \le s \le -\langle \mu_2, \alpha_j\rangle$, then

$$[\Delta(E_i^{(r)}), \Delta(F_j^{(s)})] = [E_i^{(r)} \otimes 1, 1 \otimes F_j^{(s)}] = 0.$$

For $r = -\langle \mu_1, \alpha_i\rangle + 1$ and $1 \le s \le -\langle \mu_2, \alpha_j\rangle$,

$$[\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}), \Delta(F_j^{(s)})] = 1 \otimes [E_i^{(1)}, F_j^{(s)}] = \delta_{ij} 1 \otimes H_i^{(s)}.$$

The result follows for this case.

The case where $1 \ge r \le -\langle \mu_1, \alpha_i\rangle$ and $s = -\langle \mu_2, \alpha_j\rangle + 1$ is similar.

Consider the case where $r = -\langle \mu_1, \alpha_i\rangle + 2$ and $1 \le s \le -\langle \mu_2, \alpha_j\rangle$,

$$[\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+2)}), \Delta(F_j^{(s)})] =$$
$$= 1 \otimes [E_i^{(2)}, F_j^{(s)}] + S_i^{(-\langle \mu_1,\alpha_i\rangle+1)} \otimes [E_i^{(1)}, F_j^{(s)}] - \sum_{\gamma>0} F_\gamma^{(1)} \otimes [[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}]$$
$$= \delta_{ij} 1 \otimes H_i^{(s+1)} + \delta_{ij} S_i^{(-\langle \mu_1,\alpha_i\rangle+1)} \otimes H_i^{(s)} - \sum_{\gamma>0} F_\gamma^{(1)} \otimes [[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}].$$

Note that

$$[[E_i^{(1)}, E_\gamma^{(1)}], F_j^{(s)}] = [E_i^{(1)}, [E_\gamma^{(1)}, F_j^{(s)}]] + [E_\gamma^{(1)}, [F_j^{(s)}, E_i^{(1)}]].$$

Since $s \le -\langle \mu_2, \alpha_j\rangle$, by induction, $[E_\gamma^{(1)}, F_j^{(s)}] \in \mathbb{C}1$. Hence, $[E_i^{(1)}, [E_\gamma^{(1)}, F_j^{(s)}]] = 0$. Again, since $s \le -\langle \mu_2, \alpha_j\rangle$, $[F_j^{(s)}, E_i^{(1)}] = \delta_{ij} H_j^{(s)} \in \mathbb{C}1$. So, $[E_\gamma^{(1)}, [F_j^{(s)}, E_i^{(1)}]] = 0$. Hence, the last sum is 0. Moreover, it is straightforward to check that the first two summands are consistent with the relation.

The case where $1 \le r \le -\langle \mu_1, \alpha_i\rangle$ and $s = -\langle \mu_2, \alpha_j\rangle + 2$ is similar.

Next, for $1 \leq k, l \leq 2$ not both equal to 2, we have that

$$\begin{aligned}[\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + k)}), \Delta(F_j^{(-\langle \mu_2, \alpha_j \rangle + l)})] &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2})\Delta_{0,0}([E_i^{(k)}, F_j^{(l)}]) \\ &= \delta_{ij}(\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2})\Delta_{0,0}(H_i^{(k+l-1)}) \\ &= \delta_{ij}\Delta(H_i^{(-\langle \mu, \alpha_i \rangle + k + l - 1)}).\end{aligned}$$

Next, we check relation (2.20).

First, consider the case where $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $1 \leq s < -\langle \mu_1, \alpha_j \rangle$. Then, we have

$$\begin{aligned}&[\Delta(E_i^{(r+1)}), \Delta(E_j^{(s)})] = [E_i^{(r+1)}, E_j^{(s)}] \otimes 1 = \\ &= \Big([E_i^{(r)}, E_j^{(s+1)}] + \frac{\alpha_i \cdot \alpha_j}{2}(E_i^{(r)}E_j^{(s)} + E_j^{(s)}E_i^{(r)})\Big) \otimes 1 \\ &= [E_i^{(r)} \otimes 1, E_j^{(s+1)} \otimes 1] + \frac{\alpha_i \cdot \alpha_j}{2}\big((E_i^{(r)} \otimes 1)(E_j^{(s)} \otimes 1) + (E_j^{(s)} \otimes 1)(E_i^{(r)} \otimes 1)\big).\end{aligned}$$

Consider the case where $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle$.

$$\begin{aligned}&[\Delta(E_i^{(r+1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle)})] - [\Delta(E_i^{(r)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] \\ &= ([E_i^{(r+1)}, E_j^{(-\langle \mu_1, \alpha_j \rangle)}] - [E_i^{(r)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}]) \otimes 1 \\ &= \frac{\alpha_i \cdot \alpha_j}{2}\big((E_i^{(r)} \otimes 1)(E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1) + (E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1)(E_j^{(r)} \otimes 1)\big).\end{aligned}$$

The case where $r = -\langle \mu_1, \alpha_i \rangle$ and $1 \leq s < -\langle \mu_1, \alpha_j \rangle$ is similar

Next, consider the case $1 \leq r < -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle + 1$.

$$\begin{aligned}&[\Delta(E_i^{(r+1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle) + 1})] - [\Delta(E_i^{(r)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)})] \\ &= [E_i^{(r+1)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes 1 - [E_i^{(r)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 2)}] \otimes 1 - [E_i^{(r)}, S_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}] \otimes E_j^{(1)} \\ &+ \sum_{\gamma > 0}[E_i^{(r)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}] \\ &= \frac{\alpha_i \cdot \alpha_j}{2}(E_i^{(r)}E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)} + E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}E_i^{(r)}) \otimes 1 + (\alpha_i \cdot \alpha_j)E_i^{(r)} \otimes E_j^{(1)} \\ &+ \sum_{\gamma > 0}[E_i^{(r)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}].\end{aligned}$$

Since $r < -\langle \mu_1, \alpha_i \rangle$, by induction, $[E_i^{(r)}, F_\gamma^{(1)}] = 0$ for all $\gamma > 0$. The current case follows. The proof for $r = -\langle \mu_1, \alpha_i \rangle + 1$ and $s < -\langle \mu_1, \alpha_j \rangle$ is similar.

Next, let us look at the case where $r = -\langle \mu_1, \alpha_i \rangle$ and $s = -\langle \mu_1, \alpha_j \rangle$.

$$\begin{aligned}&[\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle + 1)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle)})] - [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle)}), \Delta(E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)})] \\ &= ([E_i^{(-\langle \mu_1, \alpha_i \rangle)}, E_j^{(-\langle \mu_1, \alpha_j \rangle)}] - [E_i^{(-\langle \mu_1, \alpha_i \rangle)}, E_j^{(-\langle \mu_1, \alpha_j \rangle + 1)}]) \otimes 1 \\ &= \frac{\alpha_i \cdot \alpha_j}{2}\big((E_i^{(-\langle \mu_1, \alpha_i \rangle)} \otimes 1)(E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1) + (E_j^{(-\langle \mu_1, \alpha_j \rangle)} \otimes 1)(E_i^{(-\langle \mu_1, \alpha_i \rangle)} \otimes 1)\big).\end{aligned}$$

Next, for $r = -\langle \mu_1, \alpha_i\rangle$ and $s = -\langle \mu_1, \alpha_j\rangle + 1$.

$$\begin{aligned}
&[\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}), \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle)+1})] - [\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle)}), \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+2)})] = \\
&= [E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}, E_j^{(-\langle \mu_1,\alpha_j\rangle+1)}] \otimes 1 + 1 \otimes [E_i^{(1)}, E_j^{(1)}] - [E_i^{(-\langle \mu_1,\alpha_i\rangle)}, E_j^{(-\langle \mu_1,\alpha_j\rangle+2)}] \otimes 1 \\
&- [E_i^{(-\langle \mu_1,\alpha_i\rangle)}, S_j^{(-\langle \mu_1,\alpha_j\rangle+1)}] \otimes E_j^{(1)} + \sum_{\gamma>0} [E_i^{(-\langle \mu_1,\alpha_i\rangle)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}] \\
&= \frac{\alpha_i \cdot \alpha_j}{2} \big( E_i^{(-\langle \mu_1,\alpha_i\rangle)} E_j^{-\langle \mu_1,\alpha_j\rangle+1)} \otimes 1 + E_j^{(-\langle \mu_1,\alpha_j\rangle+1)} E_i^{(-\langle \mu_1,\alpha_i\rangle+1)} \otimes 1 \\
&+ 2E_i^{(-\langle \mu_1,\alpha_i\rangle)} \otimes E_j^{(1)} \big) + 1 \otimes [E_i^{(1)}, E_j^{(1)}] + \sum_{\gamma>0} [E_i^{(-\langle \mu_1,\alpha_i\rangle)}, F_\gamma^{(1)}] \otimes [E_j^{(1)}, E_\gamma^{(1)}].
\end{aligned}$$

Note that $[E_i^{(-\langle \mu_1,\alpha_i\rangle)}, F_l^{(1)}] \in \mathbb{C}1$. So, if $\gamma$ is of height greater than or equal to 2, then $[E_i^{(-\langle \mu_1,\alpha_i\rangle)}, F_\gamma^{(1)}] = 0$ by induction. Hence, the only term that survives in the last summand is $1 \otimes [E_j^{(1)}, E_i^{(1)}]$, and we are done. The case where $r = -\langle \mu_1, \alpha_i\rangle + 1$ and $s = -\langle \mu_1, \alpha_j\rangle$ is totally analogous.

Lastly, consider the case $r = -\langle \mu_1, \alpha_i\rangle + 1$ and $s = -\langle \mu_1, \alpha_j\rangle + 1$.

$$\begin{aligned}
&[\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+2)}), \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle)+1})] - [\Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}), \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+2)})] = \\
&= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0} \big( [E_i^{(2)}, E_j^{(1)}] - [E_i^{(1)} E_j^{(2)}] \big) \\
&= \frac{\alpha_i \cdot \alpha_j}{2} (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2}) \Delta_{0,0} \big( E_i^{(1)} E_j^{(1)} + E_j^{(1)} E_i^{(1)} \big) \\
&= \frac{\alpha_i \cdot \alpha_j}{2} \big( \Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}) \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+1)}) + \Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+1)}) \Delta(E_i^{(-\langle \mu_1,\alpha_i\rangle+1)}) \big).
\end{aligned}$$

Relation (2.21) can be checked in the same fashion.

We now check relation (2.22). Set $N = 1 - \alpha_i \cdot \alpha_j$. First, if $1 \le -\langle \mu_1, \alpha_i\rangle$ and $1 \le -\langle \mu_1, \alpha_j\rangle$, then

$$\mathrm{ad}(\Delta(E_i^{(1)}))^N (\Delta(E_j^{(1)})) = \mathrm{ad}(E_i^{(1)} \otimes 1)^N (E_j^{(1)} \otimes 1) = \big( (\mathrm{ad}\, E_i^{(1)})^N (E_j^{(1)}) \big) \otimes 1 = 0.$$

For $1 \le -\langle \mu_1, \alpha_i\rangle$ and $1 = -\langle \mu_1, \alpha_j\rangle + 1$,

$$\begin{aligned}
\mathrm{ad}(\Delta(E_i^{(1)}))^N (\Delta(E_j^{(1)})) &= \mathrm{ad}(E_i^{(1)} \otimes 1)^N (E_j^{(1)} \otimes 1 + 1 \otimes E_j^{(1)}) \\
&= \big( (\mathrm{ad}\, E_i^{(1)})^N (E_j^{(1)}) \big) \otimes 1 = 0,
\end{aligned}$$

since $E_i^{(1)} \otimes 1$ commutes with $1 \otimes E_j^{(1)}$.

Next, suppose $1 = -\langle \mu_1, \alpha_i\rangle + 1$. Since $[E_i^{(1)} \otimes 1, 1 \otimes E_i^{(1)}] = 0$,

$$\begin{aligned}
(\mathrm{ad}(\Delta(E_i^{(1)})))^N &= (\mathrm{ad}(E_i^{(1)} \otimes 1) + \mathrm{ad}(1 \otimes E_i^{(1)}))^N \\
&= \sum_{l=0}^{N} \binom{N}{l} \mathrm{ad}(E_i^{(1)} \otimes 1)^i\, \mathrm{ad}(1 \otimes E_i^{(1)})^{N-i}.
\end{aligned}$$

Now, if $1 \leq -\langle \mu_1, \alpha_j \rangle$, then

$$\begin{aligned}(\mathrm{ad}(\Delta(E_i^{(1)})))^N(\Delta(E_j^{(1)}) &= \sum_{l=0}^{N} \binom{N}{l} \mathrm{ad}(E_i^{(1)} \otimes 1)^i \, \mathrm{ad}(1 \otimes E_i^{(1)})^{N-i}(E_j^{(1)} \otimes 1) \\ &= \mathrm{ad}(E_i^{(1)})^N(E_j^{(1)}) \otimes 1 = 0.\end{aligned}$$

If $1 = -\langle \mu_1, \alpha_j \rangle + 1$, then

$$\begin{aligned}\mathrm{ad}(\Delta(E_i^{(1)}))^N(\Delta(E_j^{(1)})) &= \sum_{l=0}^{N} \binom{N}{l} \mathrm{ad}(E_i^{(1)} \otimes 1)^i \, \mathrm{ad}(1 \otimes E_i^{(1)})^{N-i}(E_j^{(1)} \otimes 1 + 1 \otimes E_j^{(1)}) \\ &= \mathrm{ad}(E_i^{(1)})^N(E_j^{(1)}) \otimes 1 + 1 \otimes \mathrm{ad}(E_i^{(1)})^N(E_j^{(1)}) = 0.\end{aligned}$$

The proof for (2.23) is similar to that of (2.22).

Finally, we check relation (2.24).

$$\begin{aligned}&\big[\Delta(S_i^{(-\langle \mu, \alpha_i \rangle+2)}), [\Delta(E_i^{(-\langle \mu_1, \alpha_i \rangle+2)}), \Delta(F_i^{(-\langle \mu_2, \alpha_i \rangle+2)})]\big] \\ &= (\iota_{0,\mu_1,0} \otimes \iota_{0,0,\mu_2})\Delta_{0,0}\big([S_i^{(2)}, [E_i^{(2)}, F_i^{(2)}]\big) = 0.\end{aligned}$$

This proves that $\Delta$ is well-defined. $\square$

By Lemma 2.1.0.7, we have the next result.

**Lemma 2.3.2.4.** *The coproduct $\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ is uniquely determined by the following*

$$\begin{aligned}\Delta(E_i^{(1)}) &= E_i^{(1)} \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes E_i^{(1)}; \\ \Delta(F_i^{(1)}) &= \delta_{\langle \mu_2, \alpha_i \rangle, 0} F_i^{(1)} \otimes 1 + 1 \otimes F_i^{(1)}; \\ \Delta(S_i^{(-\langle \mu, \alpha_i \rangle+1)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle+1)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle+1)}; \\ \Delta(S_i^{(-\langle \mu, \alpha_i \rangle+2)}) &= S_i^{(-\langle \mu_1, \alpha_i \rangle+2)} \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2, \alpha_i \rangle+2)} - \sum_{\gamma > 0} \langle \alpha_i, \gamma \rangle F_\gamma^{(1)} \otimes E_\gamma^{(1)}.\end{aligned}$$

**Proposition 2.3.2.5.** *Let $\mu = \mu_1 + \mu_2 + \mu_3$ where the $\mu_i$'s are antidominant coweights. Then, we have the following commutative diagram*

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1, \mu_2+\mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} \\ \Big\downarrow{\scriptstyle \Delta_{\mu_1+\mu_2, \mu_3}} & & \Big\downarrow{\scriptstyle 1 \otimes \Delta_{\mu_2, \mu_3}} \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow[\Delta_{\mu_1, \mu_2} \otimes 1]{} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} \end{array}$$

*Proof.* By Lemma 2.3.2.4, it is enough to check the commutativity on $S_i^{(-\langle \mu, \alpha_i \rangle+k)}$ $(k = 1, 2)$, $E_i^{(1)}$ and $F_i^{(1)}$.

$$\begin{aligned}(1 \otimes \Delta_{\mu_2, \mu_3})\Delta_{\mu_1, \mu_2+\mu_3}(E_i^{(1)}) &= E_i^{(1)} \otimes 1 \otimes 1 + \delta_{\langle \mu_1, \alpha_i \rangle, 0} 1 \otimes E_i^{(1)} \otimes 1 \\ &\quad + \delta_{\langle \mu_1, \alpha_i \rangle, 0} \delta_{\langle \mu_2, \alpha_i \rangle, 0} 1 \otimes 1 \otimes E_i^{(1)},\end{aligned}$$

$$(\Delta_{\mu_1,\mu_2} \otimes 1)\Delta_{\mu_1+\mu_2,\mu_3}(E_i^{(1)}) = E_i^{(1)} \otimes 1 \otimes 1 + \delta_{\langle \mu_1,\alpha_i\rangle,0} 1 \otimes E_i^{(1)} \otimes 1 \\ + \delta_{\langle \mu_1+\mu_2,\alpha_i\rangle} 1 \otimes 1 \otimes E_i^{(1)}.$$

The result follows for $E_i^{(1)}$ since $\delta_{\langle \mu_1+\mu_2,\alpha_i\rangle,0} = \delta_{\langle \mu_1,\alpha_i\rangle,0}\delta_{\langle \mu_2,\alpha_i\rangle,0}$. The computation for $F_i^{(1)}$ is totally analogous. The computation for $S_i^{(-\langle \mu,\alpha_i\rangle+1)}$ is straightforward.

Finally, we have that

$$\begin{aligned}
&(1 \otimes \Delta_{\mu_2,\mu_3})\Delta_{\mu_1,\mu_2+\mu_3}(S_i^{(-\langle \mu,\alpha_i\rangle+2)}) = \\
&= 1 \otimes S_i^{(-\langle \mu_2,\alpha_i\rangle+2)} \otimes 1 + 1 \otimes 1 \otimes S_i^{(-\langle \mu_3,\alpha_i\rangle+2)} - \sum_{\beta>0} \langle \alpha_i,\beta\rangle 1 \otimes F_\beta^{(1)} \otimes E_\beta^{(1)}) \\
&\quad + S_i^{(-\langle \mu_1,\alpha_i\rangle+2)} \otimes 1 \otimes 1 - \sum_{\gamma>0} \langle \alpha_i,\gamma\rangle F_\gamma^{(1)} \otimes \Delta_{\mu_2,\mu_3}(E_\gamma^{(1)}), \\
&(\Delta_{\mu_1,\mu_2} \otimes 1)\Delta_{\mu_1+\mu_2,\mu_3}(S_i^{(-\langle \mu,\alpha_i\rangle+2)}) = \\
&= S_i^{(-\langle \mu_1,\alpha_i\rangle+2)} \otimes 1 \otimes 1 + 1 \otimes S_i^{(-\langle \mu_2,\alpha_i\rangle+2)} \otimes 1 - \sum_{\beta>0} \langle \alpha_i,\beta\rangle F_\beta^{(1)} \otimes E_\beta^{(1)} \otimes 1 \\
&\quad + 1 \otimes 1 \otimes S_i^{(-\langle \mu_3,\alpha_i\rangle+2)} - \sum_{\gamma>0} \langle \alpha_i,\gamma\rangle \Delta_{\mu_1,\mu_2}(F_\gamma^{(1)}) \otimes E_\gamma^{(1)}.
\end{aligned}$$

For a positive root $\gamma = \sum_i n_i\alpha_i$, we can show by induction that $\Delta_{\mu_2,\mu_3}(E_\gamma^{(1)}) = E_\gamma^{(1)} \otimes 1 + C_\gamma 1 \otimes E_\gamma^{(1)}$ and that $\Delta_{\mu_1,\mu_2}(F_\gamma^{(1)}) = 1 \otimes F_\gamma^{(1)} + C_\gamma F_\gamma^{(1)} \otimes 1$ where $C_\gamma = \prod_i \delta_{\langle \mu_2,\alpha_i\rangle,0}^{n_i}$. The result follows. □

### 2.3.3 The coproduct in the general case

**Theorem 2.3.3.1.** *Let $\mu = \mu_1 + \mu_2$ where $\mu, \mu_1, \mu_2$ are arbitrary coweights. There exists a coproduct $\Delta_{\mu_1,\mu_2} : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ such that, for all antidominant coweights $\eta_1, \eta_2$, the following diagram is commutative*

$$\begin{array}{ccc}
Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2}} & Y_{\mu_1} \otimes Y_{\mu_2} \\
{\scriptstyle \iota_{\mu,\eta_1,\eta_2}} \downarrow & & \downarrow {\scriptstyle (\iota_{\mu_1,\eta_1,0})\otimes(\iota_{\mu_2,0,\eta_2})} \\
Y_{\mu+\eta_1+\eta_2} & \xrightarrow[\Delta_{\mu_1+\eta_1,\mu_2+\eta_2}]{} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2}
\end{array}$$

*Proof.* First, we need to define the map $\Delta_{\mu_1,\mu_2}$. Let $\eta_1, \eta_2$ be antidominant coweights such that $\mu_1 + \eta_1$ and $\mu_2 + \eta_2$ are also antidominant. We see that $\mu + \eta_1 + \eta_2$ is also antidominant.

Consider the diagram

$$\begin{array}{ccc}
Y_\mu & & Y_{\mu_1} \otimes Y_{\mu_2} \\
{\scriptstyle \iota_{\mu,\eta_1,\eta_2}} \downarrow & & \downarrow {\scriptstyle (\iota_{\mu_1,\eta_1,0})\otimes(\iota_{\mu_2,0,\eta_2})} \\
Y_{\mu+\eta_1+\eta_2} & \xrightarrow[\Delta=\Delta_{\mu_1+\eta_1,\mu_2+\eta_2}]{} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2}
\end{array}$$

In order to define $\Delta_{\mu_1,\mu_2}$, we need to show that

$$\Delta(\iota_{\mu,\eta_1,\eta_2}(Y_\mu)) \subseteq (\iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2,0,\eta_2})(Y_{\mu_1} \otimes Y_{\mu_2}).$$

Note that $Y^{\leq}_{\mu_1+\eta_1} \otimes Y^{\geq}_{\mu_2+\eta_2} \subseteq \iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2,0,\eta_2}(Y_{\mu_1} \otimes Y_{\mu_2})$.

First, for $r \geq 1$, we claim that

$$\begin{aligned}\Delta(E_i^{(r)}) &\in E_i^{(r)} \otimes 1 + Y^{\leq}_{\mu_1+\eta_1} \otimes Y^{>}_{\mu_2+\eta_2},\\ \Delta(F_i^{(r)}) &\in 1 \otimes F_i^{(r)} + Y^{<}_{\mu_1+\eta_1} \otimes Y^{\geq}_{\mu_2+\eta_2}.\end{aligned}$$

We prove the claim for $E$, the proof for $F$ is similar. We proceed by induction.

If $1 \leq -\langle \mu_1 + \eta_1, \alpha_i \rangle$, then it is clear since $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1$.

If $0 = \langle \mu_1 + \eta_1, \alpha_i \rangle$, then it is also clear since $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1 + 1 \otimes E_i^{(1)}$ and since $1 \otimes E_i^{(1)} \in Y^{\leq}_{\mu_1+\eta_1} \otimes Y^{>}_{\mu_2+\eta_2}$.

The induction step follows from the fact that $\Delta$ is a homomorphism and the fact that $[S_i^{(-\langle \mu+\eta_1+\eta_2,\alpha_i\rangle+2)}, E_i^{(r)}] = 2E_i^{(r+1)}$. This proves the claim.

Note that $\iota_{\mu,\eta_1,\eta_2}(Y_\mu)$ is generated by $E_i^{(r)}(r > -\langle \eta_1, \alpha_i \rangle)$, $F_i^{(s)}(s > -\langle \eta_2, \alpha_i \rangle)$ and $H_i^{(t)}(t > -\langle \mu + \eta_1 + \eta_2, \alpha_i \rangle)$.

Applying the claim for $r > -\langle \eta_1, \alpha_i \rangle$, we get $\Delta(E_i^{(r)}) \in (\iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2,0,\eta_2})(Y_{\mu_1} \otimes Y_{\mu_2})$ since $E_i^{(r)} \otimes 1 = (\iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2,0,\eta_2})(E_i^{(r+\langle \eta_1,\alpha_i\rangle)} \otimes 1)$.

Similarly, we obtain $\Delta(F_i^{(r)}) \in (\iota_{\mu_1,\eta_1,0} \otimes \iota_{\mu_2,0,\eta_2})(Y_{\mu_1} \otimes Y_{\mu_2})$ for $s > -\langle \eta_2, \alpha_i \rangle$.

Finally, for $t > -\langle \mu + \eta_1 + \eta_2, \alpha_i \rangle$,

$$\begin{aligned}\Delta(H_i^{(t)}) &= [\Delta(E_i^{(t)}), \Delta(F_i^{(1)})]\\ &\in [E_i^{(t)} \otimes 1, Y^{<}_{\mu_1+\eta_1} \otimes Y^{\geq}_{\mu_2+\eta_2}] + [Y^{\leq}_{\mu_1+\eta_1} \otimes Y^{>}_{\mu_2+\eta_2}, 1 \otimes F_i^{(1)}]\\ &\subseteq Y^{\leq}_{\mu_1+\eta_1} \otimes Y^{\geq}_{\mu_2+\eta_2}.\end{aligned}$$

Therefore, we have a coproduct $\Delta_{\mu_1,\mu_2} : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$.

Next, we show that $\Delta_{\mu_1\mu_2}$ is independent of the choice of $\eta_1, \eta_2$, i.e., for all $\eta_1, \eta_2$ such that $\mu_1+\eta_1, \mu_2+\eta_2$ are antidominant, the diagram in the statement of the theorem is commutative. To see this, let $\eta_1', \eta_2'$ be another such pair of coweights. Consider the diagram

$$\begin{array}{ccc} Y_\mu & & Y_{\mu_1} \otimes Y_{\mu_2} \\ \downarrow{\scriptstyle \iota_{\mu,\eta_1,\eta_2}} & & \downarrow{\scriptstyle (\iota_{\mu_1,\eta_1,0})\otimes(\iota_{\mu_2,0,\eta_2})} \\ Y_{\mu+\eta} & \xrightarrow[\Delta_{\mu_1+\eta_1,\mu_2+\eta_2}]{} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2} \\ \downarrow{\scriptstyle \iota_{\mu+\eta,\eta_1',\eta_2'}} & & \downarrow{\scriptstyle (\iota_{\mu_1+\eta_1,\eta_1',0})\otimes(\iota_{\mu_2+\eta_2,0,\eta_2'})} \\ Y_{\mu+\eta+\eta'} & \xrightarrow[\Delta_{\mu_1+\eta_1+\eta_1',\mu_2+\eta_2+\eta_2'}]{} & Y_{\mu_1+\eta_1+\eta_1'} \otimes Y_{\mu_2+\eta_2+\eta_2'} \end{array}$$

One has that $\iota_{\mu+\eta,\eta_1',\eta_2'} \circ \iota_{\mu,\eta_1,\eta_2} = \iota_{\mu,\eta_1+\eta_1',\eta_2+\eta_2'}$, and $\iota_{\mu_1+\eta_1,\eta_1',0} \circ \iota_{\mu_1,\eta_1,0} = \iota_{\mu,\eta_1+\eta_1',0}$, and $\iota_{\mu_2+\eta_2,0,\eta_2'} \circ \iota_{\mu_2,0,\eta_2} = \iota_{\mu_2,0,\eta_2+\eta_2'}$. Moreover, it is not hard to check, on generators, that the lower square commutes.

Therefore, the choice of $\Delta_{\mu_1,\mu_2}$ is the same for the pairs of coweights $(\eta_1, \eta_2)$ and $(\eta_1 + \eta_1', \eta_2 + \eta_2')$. By swapping the roles of $\eta$ and $\eta'$ in the above, the choice of $\Delta_{\mu_1,\mu_2}$ is also the same for the pairs $(\eta_1', \eta_2')$ and $(\eta_1 + \eta_1', \eta_2 + \eta_2')$.

Finally, we check that the diagram in the statement of the theorem commutes for any pair of antidominant coweights $\eta_1, \eta_2$. Let $\eta_1', \eta_2'$ be antidominant coweights such that $\mu_k + \eta_k + \eta_k'$ $(k = 1, 2)$ are antidominant. Consider the diagram

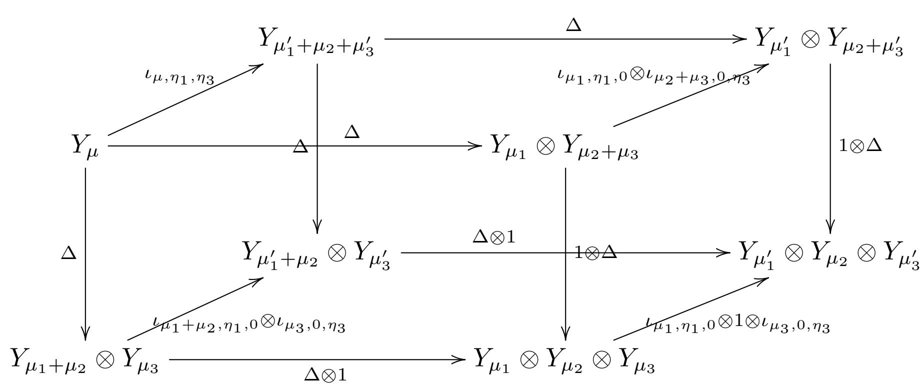
$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2}} & Y_{\mu_1} \otimes Y_{\mu_2} \\ \downarrow{\scriptstyle \iota_{\mu,\eta_1,\eta_2}} & & \downarrow{\scriptstyle (\iota_{\mu_1,\eta_1,0})\otimes(\iota_{\mu_2,0,\eta_2})} \\ Y_{\mu+\eta} & \xrightarrow[\Delta_{\mu_1+\eta_1,\mu_2+\eta_2}]{} & Y_{\mu_1+\eta_1} \otimes Y_{\mu_2+\eta_2} \\ \downarrow{\scriptstyle \iota_{\mu+\eta,\eta_1',\eta_2'}} & & \downarrow{\scriptstyle (\iota_{\mu_1+\eta_1,\eta_1',0})\otimes(\iota_{\mu_2+\eta_2,0,\eta_2'})} \\ Y_{\mu+\eta+\eta'} & \xrightarrow[\Delta_{\mu_1+\eta_1+\eta_1',\mu_2+\eta_2+\eta_2'}]{} & Y_{\mu_1+\eta_1+\eta_1'} \otimes Y_{\mu_2+\eta_2+\eta_2'} \end{array}$$

One has that $\iota_{\mu+\eta,\eta_1',\eta_2'} \circ \iota_{\mu,\eta_1,\eta_2} = \iota_{\mu,\eta_1+\eta_1',\eta_2+\eta_2'}$, and $\iota_{\mu_1+\eta_1,\eta_1',0} \circ \iota_{\mu_1,\eta_1,0} = \iota_{\mu,\eta_1+\eta_1',0}$ and $\iota_{\mu_2+\eta_2,0,\eta_2'} \circ \iota_{\mu_2,0,\eta_2} = \iota_{\mu_2,0,\eta_2+\eta_2'}$. The outer square and the lower square are commutative. Since $\iota_{\mu_1+\eta_1,\eta_1',0} \otimes \iota_{\mu_2+\eta_2,0,\eta_2'}$ is injective, we see that the upper square is also commutative. □

**Proposition 2.3.3.2.** *Suppose that $\mu = \mu_1 + \mu_2 + \mu_3$ where $\mu_2$ is antidominant. Then, the following diagram is commutative:*

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2+\mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} \\ \downarrow{\scriptstyle \Delta_{\mu_1+\mu_2,\mu_3}} & & \downarrow{\scriptstyle 1\otimes\Delta_{\mu_2,\mu_3}} \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow[\Delta_{\mu_1,\mu_2}\otimes 1]{} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} \end{array}$$

*Proof.* Let $\eta_1, \eta_3$ be antidominant coweights such that $\mu_1' = \mu_1 + \eta_1$ and $\mu_3' = \mu_3 + \eta_3$ are antidominant. Consider the diagram




We have the commutativity of all faces of this cube except for that of the front face

$$\begin{array}{ccc} Y_\mu & \xrightarrow{\Delta_{\mu_1,\mu_2+\mu_3}} & Y_{\mu_1} \otimes Y_{\mu_2+\mu_3} \\ \downarrow{\scriptstyle \Delta_{\mu_1+\mu_2,\mu_3}} & & \downarrow{\scriptstyle 1\otimes\Delta_{\mu_2,\mu_3}} \\ Y_{\mu_1+\mu_2} \otimes Y_{\mu_3} & \xrightarrow[\Delta_{\mu_1,\mu_2}\otimes 1]{} & Y_{\mu_1} \otimes Y_{\mu_2} \otimes Y_{\mu_3} \end{array}$$

Using the commutativity of the other faces and injectivity of shift maps, we see that the above square also commutes. □

*Remark* 2.3.3.3. In general, the coproducts are not coassociative. More precisely, when $\mu_2$ is not antidominant, the diagram from Proposition 2.3.3.2 does not commute. We delay the argument of this fact until Remark 2.6.3.11.

## 2.4 Filtrations of shifted Yangians

We begin with some generalities on filtrations, which can be found in [FKPRW, 5.1].

Given a $\mathbb{C}$-algebra $A$ with an increasing $\mathbb{Z}$-filtration $F^\bullet A$, $A$ is said to be *almost commutative* if its associated graded $\operatorname{gr}^F A = \bigoplus_n F^n A/F^{n-1}A$ is commutative. Note that we use the notations $A_n$ and $F^n A$ interchangeably if the filtration in question is clear.

The filtration $F^\bullet A$ is said to be *exhaustive* if $A = \bigcup_n F^n A$, is said to be *separated* if $\bigcap_n F^n A = \{0\}$, and is said to *admit an expansion* if there exists a filtered vector space isomorphism $\operatorname{gr}^F A \simeq A$.

Given filtered algebras $F^\bullet A$ and $F^\bullet B$, one can define a filtration on $A \otimes B$ as follows, $F^n(A\otimes B) = \bigoplus_{n=k+l} F^k A \otimes F^l B$. If $F^\bullet A$ and $F^\bullet B$ admit expansions, then $\operatorname{gr}(A\otimes B) \simeq \operatorname{gr} A \otimes \operatorname{gr} B$.

Returning to our setting, given a splitting $\mu = \nu_1 + \nu_2$, define a filtration $F_{\nu_1,\nu_2} Y_\mu$ as follows

$$\deg E_\alpha^{(q)} = \langle \nu_1, \alpha\rangle + q,\ \deg F_\beta^{(q)} = \langle \nu_2, \beta\rangle + q,\ \deg H_i^{(p)} = \langle \mu, \alpha_i\rangle + p \tag{2.25}$$

Define the filtered piece $F^k_{\nu_1,\nu_2} Y_\mu$ to be the span of all ordered monomials in PBW variables with total degree at most $k$.

**Proposition 2.4.0.1.** [FKPRW, Prop 5.7] *The filtration $F_{\nu_1,\nu_2} Y_\mu$ is an algebra filtration, is independent of the choice of PBW variables, and is independent of the order of the variables used to form monomials. The algebra $Y_\mu$ is almost commutative, i.e., $\operatorname{gr}^{F_{\nu_1,\nu_2}} Y_\mu$ is a commutative (polynomial) algebra.*

**Proposition 2.4.0.2.** *For any splitting $\mu = \nu_1 + \nu_2$, $F_{\nu_1,\nu_2} Y_\mu$ is exhaustive, separated, and admits an expansion.*

*Proof.* These follow from the fact that $Y_\mu$ admits a PBW basis, by Theorem 2.2.0.2. The expansion map is given by $\operatorname{gr} Y_\mu \longrightarrow Y_\mu$, $\bar{m} \mapsto m$, where $m$ is a monomial of degree $k$ in PBW variables. This map is well-defined by degree consideration. □

**Proposition 2.4.0.3.** *Let $\mu, \mu_k, \nu_k$ $(k = 1, 2)$ be such that $\mu = \mu_1 + \mu_2 = \nu_1 + \nu_2$. Then the map $\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ respects the filtrations $F_{\nu_1,\nu_2} Y_\mu$, $F_{\nu_1,\mu_1-\nu_1} Y_{\mu_1}$, and $F_{\mu_2-\nu_2,\nu_2} Y_{\mu_2}$.*

*Proof.* First, consider the case where $\mu, \mu_1, \mu_2$ are antidominant. Consider the definition of $\Delta$ on the Levendorskii presentation. One can check that $\Delta$ respects filtrations on Levendorskii generators by inspecting degrees. For example, we have that

$$\Delta(E_j^{(-\langle \mu_1,\alpha_j\rangle+2)}) = E_j^{(-\langle \mu_1,\alpha_j\rangle+2)} \otimes 1 + 1 \otimes E_j^{(2)} + S_j^{(-\langle \mu_1,\alpha_j\rangle+1)} \otimes E_i^{(1)} - \sum_{\gamma>0} F_\gamma^{(1)} \otimes [E_j^{(1)}, E_\gamma^{(1)}].$$

$$
\begin{aligned}
\deg(E_j^{(-\langle\mu_1,\alpha_j\rangle+2)}) &= \langle\nu_1,\alpha_j\rangle - \langle\mu,\alpha_j\rangle + 2 = \langle\nu_1-\mu_1,\alpha_j\rangle + 2, \\
\deg(E_j^{(-\langle\mu_1,\alpha_j\rangle+2)} \otimes 1) &= \langle\mu_1,\alpha_j\rangle - \langle\mu_1,\alpha_j\rangle + 2 = \langle\nu_1-\mu_1,\alpha_j\rangle + 2, \\
\deg(1 \otimes E_j^{(2)}) &= \langle\mu_2-\nu_2,\alpha_j\rangle + 2 = \langle\nu_1-\mu_1,\alpha_j\rangle + 2, \\
\deg(S_j^{(-\langle\mu_1,\alpha_j\rangle+1)} \otimes E_i^{(1)}) &= \langle\mu_1,\alpha_j\rangle - \langle\mu_1,\alpha_j\rangle + 1 + \langle\nu_1-\mu_1,\alpha_j\rangle + 1 = \langle\nu_1-\mu_1,\alpha_j\rangle + 2, \\
\deg(F_\gamma^{(1)} \otimes [E_j^{(1)}, E_\gamma^{(1)}]) &= \langle\mu_1-\nu_1,\gamma\rangle + 1 + \langle\mu_2-\nu_2,\gamma+\alpha_j\rangle + 1 \\
&= \langle\mu_1-\nu_1,\gamma\rangle + 1 + \langle\nu_1-\mu_1,\gamma+\alpha_j\rangle + 1 = \langle\nu_1-\mu_1,\alpha_j\rangle + 2.
\end{aligned}
$$

Since these filtrations admit expansions, we have that $\operatorname{gr}(Y_{\mu_1}\otimes Y_{\mu_2}) \simeq \operatorname{gr} Y_{\mu_1}\otimes \operatorname{gr} Y_{\mu_2}$. Hence, by Theorem 2.6.2.1, $Y_{\mu_1}\otimes Y_{\mu_2}$ is almost commutative. The higher $E_j^{(r)}, F_j^{(r)}, H_j^{(r)}$ and the PBW variables $E_\beta^{(r)}, F_\beta^{(r)}$ are all obtained from commutators. Thus, $\Delta$ respects their degrees since $Y_{\mu_1}\otimes Y_{\mu_2}$ is almost commutative. To be more precise,

$$
\begin{aligned}
\deg\Delta(E_j^{(-\langle\mu_1,\alpha_j\rangle+3)}) &= \deg\Delta([S_j^{(-\langle\mu,\alpha_j\rangle+2)}, E_j^{(-\langle\mu_1,\alpha_j\rangle+2)}]) \\
&= \deg\Delta(S_j^{(-\langle\mu,\alpha_j\rangle+2)}) + \deg\Delta(E_j^{(-\langle\mu_1,\alpha_j\rangle+2)}) - 1 \\
&= 2 + \langle\nu_1-\mu_1,\alpha_j\rangle + 2 - 1 \\
&= \langle\nu_1-\mu_1,\alpha_j\rangle + 3,
\end{aligned}
$$

where the second equality uses the almost commutativity of $Y_{\mu_1}\otimes Y_{\mu_2}$. The degrees of the higher $E_j^{(r)}$'s are obtained by induction. Similarly, one can show that $\Delta$ respects the degrees of $F_j^{(r)}, H_j^{(r)}, E_\beta^{(r)}, F_\beta^{(r)}$. Hence, $\Delta$ respects filtrations in the case where $\mu, \mu_1, \mu_2$ are antidominant.

For the general case, let $\eta$ be dominant such that $\mu-\eta$ is antidominant. Let $\eta_1, \eta_2$ be coweights such that $\eta = \eta_1 + \eta_2$. There is a commutative diagram

$$
\begin{array}{ccc}
F_{\nu_1,\nu_2}Y_\mu & \longrightarrow & F_{\nu_1,\mu_1-\nu_1}Y_{\mu_1} \otimes F_{\mu_2-\nu_2,\nu_2}Y_{\mu_2} \\
\downarrow{\scriptstyle \iota_{\mu,\eta_1,\eta_2}} & & \downarrow{\scriptstyle \iota_{\mu_1,\eta_1,0}\otimes\iota_{\mu_2,0,\eta_2}} \\
F_{\nu_1-\eta_1,\nu_2-\eta_2}Y_{\mu-\eta} & \longrightarrow & F_{\nu_1-\eta_1,\mu_1-\nu_1}Y_{\mu_1-\eta_1} \otimes F_{\mu_2-\nu_2,\nu_2-\eta_2}Y_{\mu_2-\eta_2}
\end{array}
$$

Since the vertical maps and the bottom maps respect filtrations, so is the top map. □

## 2.5 GLKO generators

In some situations, it is more convenient to work with the series

$$
H_j(t) = t^{\langle\mu,\alpha_j\rangle} + \sum_{r\geq 1} H_j^{(-\langle\mu,\alpha_j\rangle+r)} t^{\langle\mu,\alpha_j\rangle-r}, \quad E_j(t) = \sum_{r\geq 1} E_j^{(r)} t^{-r}, \quad F_j(t) = \sum_{r\geq 1} F_j^{(r)} t^{-r}. \tag{2.26}
$$

Following [GLKO], [KWWY], and [FTs], we can change the Cartan generators of $Y_\mu$ as follows. For $j \in I$, define $A_j(t)$ by the following equation

$$
t^{-\langle\mu,\alpha_j\rangle} H_j(t) = \frac{\prod_{k\neq j} A_k(t - \frac{(\alpha_j+\alpha_k,\alpha_k)}{2})^{-a_{jk}}}{A_j(t)A_j(t-\frac{(\alpha_j,\alpha_j)}{2})} \tag{2.27}
$$

*Remark* 2.5.0.1. The existence of the $A_j$'s follows from the same proof as [GLKO, Lemma 2.1].

Moreover, the series $A_j$'s all start at 1.

*Remark* 2.5.0.2. Alternatively, one can define the $A_j$'s with a slightly different convention.

$$H_j(t) = \frac{\prod_{k\neq j} A_k(t - \frac{(\alpha_j+\alpha_k,\alpha_k)}{2})^{-a_{jk}}}{A_j(t)A_j(t - \frac{(\alpha_j,\alpha_j)}{2})}.$$

Under this definition, the series $A_j$ starts at $t^{\langle \mu, w_0\omega_{j^*}\rangle}$. It would give a "nicer" (unshifted) version of Theorem 2.6.2.1. However, to be consistent with the literature, we will not use this convention.

Similarly to [GLKO], we can also define some other elements of $Y_\mu$ as follows.

$$B_j(t) = E_j(t)A_j(t), \quad C_j(t) = F_j(t)A_j(t). \tag{2.28}$$

**Proposition 2.5.0.3.** [FTs, Thm 6.6] *We have the following relations*

$$[A_j(s), A_k(t)] = 0 \tag{2.29}$$

$$[A_j(s), B_k(t)] = [A_j(s), C_k(t)] = 0, \quad j \neq k, \tag{2.30}$$

$$[B_j(s), B_k(t)] = [C_j(s), C_k(t)] = 0, \quad a_{jk} = 0, 2 \tag{2.31}$$

$$(s-t)[A_j(s), B_j(t)] = B_j(s)A_j(t) - B_j(t)A_j(s), \tag{2.32}$$

$$(s-t)[A_j(s), C_j(t)] = A_j(s)C_j(t) - A_j(t)C_j(s). \tag{2.33}$$

*Proof.* These relations take place only in the upper (or lower) Borel Yangians. So they follow from the same argument as in [GLKO]. □

*Remark* 2.5.0.4. In [FTs], we can find other relations like in [GLKO], and more. In their work, it is worth noting that relations between $B_j$'s and $C_k$'s only hold in the antidominant case.

## 2.6 Relation to geometry

### 2.6.1 The variety $\mathcal{W}_\mu$

Let $G$ be a simply-laced algebraic group with Lie algebra $\mathfrak{g}$. Let $T$ be a maximal torus of $G$, $B$ a Borel subgroup, and $B^-$ the opposite Borel subgroup. Consider $U$ (resp. $U^-$) the unipotent radical of $B$ (resp. $B^-$).

For any algebraic group $H$, denote by $H_1[[t^{-1}]]$ the kernel of the evaluation $H((t^{-1})) \longrightarrow H$ at $t^{-1} = 0$.

Let $\mu : \mathbb{G}_m \longrightarrow T$ be any coweight. Any coweight can be thought of as a $\mathbb{C}((t^{-1}))$-point. Denote by $t^\mu$ its image in $G((t^{-1}))$. The space of our interest is the infinite type scheme

$$\mathcal{W}_\mu := U_1[[t^{-1}]]T_1[[t^{-1}]]t^\mu U_1^-[[t^{-1}]]. \tag{2.34}$$

The inclusion $U_1[[t^{-1}]] \longrightarrow U((t^{-1}))$ gives rise to the isomorphism $U_1[[t^{-1}]] \simeq U((t^{-1}))/U[t]$.

So, there is an isomorphism

$$\pi_\mu : U[t]\backslash U((t^{-1}))T_1[[t^{-1}]]t^\mu U^-((t^{-1}))/U^-[t] \longrightarrow \mathcal{W}_\mu.$$

*Remark* 2.6.1.1. There is a family of related spaces, called generalized slices. Given a dominant coweight $\lambda$ of $G$, consider the space $G[t]t^\lambda G[t]$. For any coweight $\mu$, the corresponding generalized slice is $\overline{\mathcal{W}}^\lambda_\mu = \mathcal{W}_\mu \cap \overline{G[t]t^\lambda G[t]}$. A useful fact about these slices that we will need later is that $\bigcup_\lambda \overline{\mathcal{W}}^\lambda_\mu$ is dense in $\mathcal{W}_\mu$, we will prove this in Proposition 2.6.1.6.

*Remark* 2.6.1.2. We shall briefly discuss the topic of loop spaces. Consider the $\mathbb{C}$-algebra $A = \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle$ where the $f_i$'s are polynomials in $x_j$'s. Let $X = \operatorname{Spec} A$. Let $d \geq 0$. The formal loop spaces $X[t]$, $X[t]_{\leq d}$ and $X[[t]]$ are defined in terms of their functor of points. For any $\mathbb{C}$-algebra $R$,

$$\begin{aligned} X[t](R) &= X(R[t]) = \hom_{\mathbb{C}}(A, R[t]) = \{(x_1(t), \ldots, x_n(t)) \in \mathbb{C}[t]^n : f_i(x_j(t)) = 0\}, \\ X[t]_{\leq d}(R) &= \{(x_1(t), \ldots, x_n(t)) \in \mathbb{C}[t]^n : f_i(x_j(t)) = 0, \deg(x_j(t)) \leq d\} \subseteq X[t](R), \\ X[[t]](R) &= X(R[[t]]) = \hom_{\mathbb{C}}(A, R[[t]]) = \{(x_1(t), \ldots, x_n(t)) \in \mathbb{C}[[t]]^n : f_i(x_j(t)) = 0\}. \end{aligned}$$

We write $x_j(t) = \sum_r x_j^{(r)} t^r$. Consider the polynomial ring $\mathbb{C}[x_j^{(r)} : 1 \leq j \leq n, r > 0]$. Following [Fr, 3.4.2], define a derivation $T$ on this polynomial ring by $T(x_j^{(r)}) = r x_j^{(r+1)}$. For $1 \leq i \leq m$, let $\tilde{f}_i \in \mathbb{C}[x_j^{(r)} : 1 \leq j \leq n, r > 0]$ be the same polynomials as $f_i$, with $x_j$ replaced by $x_j^{(1)}$. Next, set

$$B = \mathbb{C}[x_j^{(r)} : 1 \leq n \leq j, r \geq 0]/\langle T^k \tilde{f}_i : 1 \leq i \leq m, k \geq 0\rangle.$$

**Lemma 2.6.1.3.** [Fr, 3.4.2] $X[[t]] = \operatorname{Spec} B$.

*Remark* 2.6.1.4. Let $I_d = \langle x_j^{(r)} : r > d\rangle \subseteq B$. Then

$$X[t]_{\leq d} = \operatorname{Spec} B/I_d, \quad \text{and} \quad X[t] = \varinjlim X[t]_{\leq d}.$$

**Lemma 2.6.1.5.** *For any affine variety $X$, $X[t]$ is dense in $X[[t]]$.*

*Proof.* We need to show that $\bigcap_d I_d = \{0\}$. Note that $B$ is a graded ring by setting $\deg(x_j^{(r)}) = r$, and that $I_d$ is homogeneous and lies in degree greater than $d$. Therefore, $\bigcap_d I_d$ is also homegeneous and lies in degree greater than $d$ for all $d$. Therefore, $\bigcap_d I_d = \{0\}$. □

**Proposition 2.6.1.6.** $\bigcup_\lambda \overline{\mathcal{W}}^\lambda_\mu$ *is dense in* $\mathcal{W}_\mu$.

*Proof.* We have that $\bigcup_\lambda G[t]t^\lambda G[t] = G[t, t^{-1}]$. Thus, $\bigcup_\lambda \overline{\mathcal{W}}^\lambda_\mu = \mathcal{W}_\mu \cap G[t, t^{-1}]$. We see that union contains $\mathcal{W}^{\text{pol}}_\mu = U_1[t^{-1}]T_1[t^{-1}]t^\mu U_1^-[t^{-1}]$, the polynomial version of $\mathcal{W}_\mu$.

We know that $\mathcal{W}_\mu$ is isomorphic to $U_1[[t^{-1}]] \times T_1[[t^{-1}]] \times U_1^-[[t^{-1}]]$ via the map $uht^\mu u^- \mapsto (u, h, u^{-1})$. Similarly, $\mathcal{W}^{\text{pol}}_\mu$ is isomorphic to $U_1[t^{-1}] \times T_1[t^{-1}] \times U_1^-[t^{-1}]$. The inclusion $\mathcal{W}^{\text{pol}}_\mu \subseteq \mathcal{W}_\mu$ is compatible with that of the corresponding products. Our claim then follows from the previous lemma. □

### 2.6.2 Poisson structure of $\mathcal{W}_\mu$ and quantizations

Consider a splitting $\mu = \nu_1 + \nu_2$. Consider the filtration $F_{\nu_1,\nu_2} Y_\mu$ of $Y_\mu$, defined earlier.

**Theorem 2.6.2.1.** [FKPRW, Thm 5.15] . *There exists an isomorphism of graded algebras* $\mathrm{gr}^{F_{\nu_1,\nu_2}} Y_\mu \simeq \mathbb{C}[\mathcal{W}_\mu]$. *Moreover, in terms of the GLKO generators, the isomorphism is given by*

$$\begin{aligned}A_j(t) &\mapsto t^{-\langle \mu, w_0\omega_{j^*}\rangle} \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(t)\\ B_j(t) &\mapsto t^{-\langle \mu, w_0\omega_{j^*}\rangle} \mathcal{D}_{s_j w_0\omega_{j^*}, w_0\omega_{j^*}}(t)\\ C_j(t) &\mapsto t^{-\langle \mu, w_0\omega_{j^*}\rangle} \mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(t).\end{aligned}$$

Recall that the functions $\mathcal{D}_{\omega_1,\omega_2}$ were defined in the introduction. While we will not explain the proof of the above theorem, we can have a closer look to see that the generators $E_j^{(r)}, F_j^{(r)}, H_j^{(r)}$ of $Y_\mu$ are closely related to the Gauss decomposition of $\mathcal{W}_\mu$. By abuse of notation, let us denote the images of these generators in $\mathbb{C}[\mathcal{W}_\mu]$ by the same names.

*Remark* 2.6.2.2. Let $g = uht^\mu u_- \in \mathcal{W}_\mu$. Then, under the isomorphism of Theorem 2.6.2.1,

$$\begin{aligned}E_j(g) &= \mathcal{D}_{s_j w_0\omega_{j^*}, w_0\omega_{j^*}}(u),\\ F_j(g) &= \mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(u_-),\\ H_j(g) &= \alpha_j(ht^\mu),\end{aligned}$$

where the last line means that we take the projection with respect to the simple root $\alpha_j$.

The next lemma contains some useful relations analogous to the ones found in [GLKO] and [KTWWY1, Section 5.3]. In fact, these relations follow from the definition of GLKO generators and Theorem 2.6.2.1. However, we provide explicit computations on commutative level.

**Lemma 2.6.2.3.** *In* $\mathcal{W}_\mu$, *we have the following: for* $j \in I$, $\mathcal{D}_{s_j w_0\omega_{j^*}, w_0\omega_{j^*}} = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} E_j$, $\mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}} = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} F_j$, $\mathcal{D}_{s_j w_0\omega j^* j, s_j w_0\omega_{j^*}} = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} H_j + E_j \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} F_j$, *and* $H_j = \frac{\prod_{k\sim j} \mathcal{D}_{w_0\omega_{k^*}, w_0\omega_{k^*}}}{\mathcal{D}^2_{w_0\omega_{j^*}, w_0\omega_{j^*}}}$ *where* $k \sim j$ *means that* $k$ *is connected to* $j$ *on the Dynkin diagram.*

*Proof.* Write $g = uht^\mu u_- \in \mathcal{W}_\mu$.

For the $E$ case,

$$\begin{aligned}uht^\mu u_- v_{w_0\omega_{j^*}} &= u(w_0\omega_{j^*}(h) t^{\langle \mu, w_0\omega_{j^*}\rangle} v_{w_0\omega_{j^*}})\\ &= w_0\omega_{j^*}(h) t^{\langle \mu, w_0\omega_{j^*}\rangle}(v_{w_0\omega_{j^*}} + \mathcal{D}_{w_0 s_{j^*}\omega_{j^*}, w_0\omega_{j^*}}(u) v_{w_0 s_{j^*}\omega_{j^*}} + \cdots).\end{aligned}$$

We see that $\mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g) = (w_0\omega_{j^*}(h) t^{\langle \mu, w_0\omega_{j^*}\rangle}$. Hence,

$$\mathcal{D}_{s_j w_0\omega_{j^*}, w_0\omega_{j^*}}(g) = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g) \mathcal{D}_{s_j w_0\omega_{j^*}, w_0\omega_{j^*}}(u) = (\mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} E_j)(g).$$

For the $F$ case, we have that

$$\begin{aligned}uht^\mu u_- v_{s_j w_0\omega_{j^*}} &= uht^\mu(v_{s_j w_0\omega_{j^*}} + \mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(u_-) v_{w_0\omega_{j^*}})\\ &= u((s_j w_0\omega_{j^*})(h) t^{\langle \mu, s_j w_0\omega_{j^*}\rangle} v_{s_j w_0\omega_{j^*}} +\\ &\qquad + (w_0\omega_{j^*})(h) t^{\langle \mu, w_0\omega_{j^*}\rangle} \mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(u_-) v_{w_0\omega_{j^*}})).\end{aligned}$$

Hence, $\mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(g) = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g) \mathcal{D}_{w_0\omega_{j^*}, s_j w_0\omega_{j^*}}(u_-) = \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}} F_j(g)$.

Next, for the $H$ case,

$$H_j(g) = \alpha_j(ht^\mu) = (\sum_k (-a_{jk}) w_0\omega_{k^*})(ht^\mu) = \prod_k (w_0\omega_{k^*})(ht^\mu)^{-a_{jk}}$$
$$= \prod_k \mathcal{D}_{w_0\omega_{k^*}, w_0\omega_{k^*}}(g)^{-a_{jk}} = \frac{\prod_{k\sim j} \mathcal{D}_{w_0\omega_{k^*}, w_0\omega_{k^*}}(g)}{\mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g)^2}.$$

where $a_{jk} = \langle \alpha_j, \alpha_k \rangle$ is an entry of the Cartan matrix.

Lastly, we have that

$$\begin{aligned}
uht^\mu u_- v_{s_j w_0 \omega_{j^*}} &= uht^\mu (v_{s_j w_0 \omega_{j^*}} + \mathcal{D}_{w_0\omega_{j^*}, s_j w_0 \omega_{j^*}}(u_-) v_{w_0\omega_{j^*}}) \\
&= u((s_j w_0 \omega_{j^*})(h) t^{\langle \mu, s_j w_0 \omega_{j^*} \rangle} v_{s_j w_0 \omega_{j^*}} + \\
&\quad + (w_0\omega_{j^*})(h) t^{\langle \mu, w_0\omega_{j^*} \rangle} \mathcal{D}_{w_0\omega_{j^*}, s_j w_0 \omega_{j^*}}(u_-) v_{w_0\omega_{j^*}}) \\
&= (s_j w_0 \omega_{j^*})(h) t^{\langle \mu, s_j w_0 \omega_{j^*} \rangle} v_{s_j w_0 \omega_{j^*}} + \cdots + \\
&\quad + \mathcal{D}_{s_j w_0 \omega_{j^*}, w_0\omega_{j^*}}(u)(w_0\omega_{j^*})(h) t^{\langle \mu, w_0\omega_{j^*} \rangle} \mathcal{D}_{w_0\omega_{j^*}, s_j w_0 \omega_{j^*}}(u_-) v_{w_0 s_{j^*} \omega_{j^*}}) + \cdots \\
\mathcal{D}_{s_j w_0 \omega_{j^*}, s_j w_0 \omega_{j^*}} &= (s_j w_0 \omega_{j^*})(h) t^{\langle \mu, s_j w_0 \omega_{j^*} \rangle} + \\
&\quad + \mathcal{D}_{s_j w_0 \omega_{j^*}, w_0\omega_{j^*}}(u)(w_0\omega_{j^*})(h) t^{\langle \mu, w_0\omega_{j^*} \rangle} \mathcal{D}_{w_0\omega_{j^*}, s_j w_0 \omega_{j^*}}(u_-) \\
&= \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g) H_j(g) + E_j(g) \mathcal{D}_{w_0\omega_{j^*}, w_0\omega_{j^*}}(g) F_j(g).
\end{aligned}$$

This concludes the proof. □

**Lemma 2.6.2.4.** *In $\mathbb{C}[\mathcal{W}_\mu]$, we have that*

$$\{E_i^{(1)}, A_i(t)\} = B_i(t) \tag{2.35}$$
$$\{E_i^{(1)}, B_j(t)\} = \mathcal{D}_{s_j w_0 \omega_{j^*} + \alpha_i, w_0 \omega_{j^*}}(t), \quad a_{ij} = -1. \tag{2.36}$$

*Proof.* The first equation follows Proposition 2.5.0.3. The second equation takes place in the positive Borel Yangian, so it follows from [KTWWY1, Section 5.2]. □

**Proposition 2.6.2.5.** [FKPRW, Lemma 5.17]*. Let $\mu$ be an antidominant coweight. Then the classical shifted Yangian* $\operatorname{gr} Y_\mu \simeq \mathbb{C}[\mathcal{W}_\mu]$ *is generated by $E_i^{(1)}, F_i^{(1)}, H_i^{(-\langle \mu, \alpha_i \rangle + 1)}$, and $H_i^{(-\langle \mu, \alpha_i \rangle + 2)}$ as a Poisson algebra.*

To compare to [KWWY], recall that $\mathfrak{g}((t^{-1})), \mathfrak{g}[t]$ and $t^{-1}\mathfrak{g}[[t^{-1}]]$ form a Manin triple, which gives rise to a Poisson structure on $G((t^{-1})))$ with $G[t]$ and $G_1[[t^{-1}]]$ as Poisson subgroups. In our case, Theorem 2.6.2.1 endows $\mathcal{W}_\mu$ with a Poisson structure via equation (1.1).

Comparing [KWWY, Thm 3.9] and the proof of [FKPRW, Thm 5.15], one has the following.

**Theorem 2.6.2.6.** [KWWY, Thm 3.9] *The Poisson structure on $\mathcal{W}_0 = G_1[[t^{-1}]]$ given by Theorem* 2.6.2.1 *is the same as the structure given by the Manin triple.*

### 2.6.3 Multiplication maps between the varieties $\mathcal{W}_\mu$

In this section, we look at some natural maps between the varieties $\mathcal{W}_\mu$, namely the multiplication maps and the shift maps. We will show that these maps are Poisson with respect to the

Poisson structure given by Theorem 2.6.2.1. The employed method is also going to be useful in the next chapter where we will show that a certain $\mathbb{G}_a$-action on $\mathcal{W}_\mu$ is Poisson.

**Definition 2.6.3.1.** Let $\mu_1, \mu_2$ be antidominant coweights. We define the shift maps $\iota_{\mu,\mu_1,\mu_2} : \mathcal{W}_{\mu+\mu_1+\mu_2} \longrightarrow \mathcal{W}_\mu$, $g \mapsto \pi_\mu(t^{-\mu_1} g t^{-\mu_2})$.

As $\mu_1, \mu_2$ are antidominant, $t^{-\mu_1} U_1[[t^{-1}]] t^{\mu_1} \subseteq U_1[[t^{-1}]]$ and $t^{\mu_2} U_{1,-}[[t^{-1}]] t^{-\mu_2} \subseteq U_{1,-}[[t^{-1}]]$. Hence, the shift map $\iota_{\mu,\mu_1,\mu_2}$ is well-defined. The following lemma is part of [FKPRW, Thm 5.15].

**Lemma 2.6.3.2.** [FKPRW, Thm 5.15] *The isomorphism of Theorem 2.6.2.1 is compatible with shift maps on both sides.*

*Remark* 2.6.3.3. In the context of the previous lemma, the shift map between the $\operatorname{gr} Y_\mu$'s comes from the shift homomorphism of Proposition 2.1.0.4. Moreover, both types of shift maps are denoted $\iota_{\mu,\mu_1,\mu_2}$ to emphasize the fact that they are compatible with each other by the previous lemma.

**Definition 2.6.3.4.** For any coweights $\mu_1$ and $\mu_2$, we define the multiplication map $m_{\mu_1,\mu_2} : \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}$ as $(g_1, g_2) \mapsto \pi_{\mu_1+\mu_2}(g_1 g_2)$ where $\pi_\mu$ is as in Section 2.6.1.

For this definition to make sense, one has to check that $g_1 g_2 \in U((t^{-1})) T_1[[t^{-1}]] t^\mu U^-((t^{-1}))$.

Write $g_1 = u_1 h_1 t^{\mu_1} u_1^-$, $g_2 = u_2 h_2 t^{\mu_2} u_2^-$. Since $h_1 u_1^- u_2 h_2 \in G_1[[t^{-1}]]$, $h_1 u_1^- u_2 h_2 = u_3 h_3 u_3^-$ where $u_i \in U_1[[t^{-1}]]$, $h_i \in T_1[[t^{-1}]]$, and $u_i^- \in U_1^-[[t^{-1}]]$. So,

$$\begin{aligned} g_1 g_2 &= u_1 t^{\mu_1} (h_1 u_1^- u_2 h_2) t^{\mu_2} u_2^- \\ &= u_1 t^{\mu_1} u_3 h_3 u_3^- t^{\mu_2} u_2^- \\ &= u_1 (t^{\mu_1} u_3 t^{-\mu_1}) h_3 t^\mu (t^{-\mu_2} u_3^- t^{\mu_2}) u_2^-, \end{aligned}$$

which lies in $U((t^{-1})) T_1[[t^{-1}]] t^\mu U^-((t^{-1}))$.

**Lemma 2.6.3.5.** [FKPRW, Lemma 5.11] *Let $\mu_1, \mu_2$ be any coweights and let $\nu_1, \nu_2$ be antidominant coweights. The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{W}_{\mu_1+\nu_1} \times \mathcal{W}_{\mu_2+\nu_2} & \xrightarrow{m_{\mu_1+\nu_1+\mu_2+\nu_2}} & \mathcal{W}_{\mu_1+\mu_2+\nu_1+\nu_2} \\ \downarrow{\scriptstyle \iota_{\mu_1,\nu_1,0} \times \iota_{\mu_2,0,\nu_2}} & & \downarrow{\scriptstyle \iota_{\mu_1+\mu_2,\nu_1,\nu_2}} \\ \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} & \xrightarrow[m_{\mu_1+\mu_2}]{} & \mathcal{W}_{\mu_1+\mu_2} \end{array}$$

*i.e., the shift maps and the multiplications maps are compatible.*

**Proposition 2.6.3.6.** *The shift homomorphisms are Poisson and are compatible with the shift homomorphisms of Proposition 2.1.0.4. If $\lambda$ is dominant such that $\lambda \geq \mu$ and $\lambda + \mu_1 + \mu_2$ is dominant, then the shift homomorphism $\iota_{\mu,\mu_1,\mu_2}$ restricts to a map $\overline{\mathcal{W}}^{\lambda+\mu_1+\mu_2}_{\mu+\mu_1+\mu_2} \longrightarrow \overline{\mathcal{W}}^\lambda_\mu$. The restriction map is Poisson and birational.*

*Proof.* The first claim follows Lemma 2.6.3.2 and the fact that shift maps between the associated graded algebras are Poisson. Since $-\mu_1$ and $-\mu_2$ are dominant, for any $g \in \overline{\mathcal{W}}^{\lambda+\mu_1+\mu_2}_{\mu+\mu_1+\mu_2}$, we have that

$$t^{-\mu_1} g t^{-\mu_2} \in \overline{G[t] t^{-\mu_1} G[t]} \cdot \overline{G[t] t^{\lambda+\mu_1+\mu_2} G[t]} \cdot \overline{G[t] t^{-\mu_2} G[t]} \subseteq \overline{G[t] t^\lambda G[t]}$$

Now, since $\overline{G[t]t^\lambda G[t]}$ is invariant under multiplication by $U^{\pm}[t]$, we see that $\iota_{\mu,\mu_1.\mu_2}$ restricts to a map $\overline{\mathcal{W}}^{\lambda+\mu_1+\mu_2}_{\mu+\mu_1+\mu_2} \longrightarrow \overline{\mathcal{W}}^{\lambda}_{\mu}$. This restriction is also Poisson.

For birationality, the result follows from [BFN, Rem. 3. 11]. $\square$

Next, we want to show the multiplication is Poisson. We will use the following lemma several times.

**Lemma 2.6.3.7.** *Let $X_i, Y_i$ $(i = 1, 2)$ be irreducible affine Poisson varieties. Suppose that we have a commutative diagram*

$$\begin{CD} X_1 @>{f_1}>> Y_1 \\ @V{i}VV @VV{j}V \\ X_2 @>>{f_2}> Y_2 \end{CD}$$

*such that the vertical maps are birational and Poisson. Then the top arrow is Poisson if and only if the bottom arrow is Poisson.*

*Proof.* Since $i$ is birational, there exist open sets $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ such that $U_1 \simeq U_2$. By commutativity of the diagram, we see that $f_2|_{U_2} = j \circ f_1 \circ (i^{-1}|_{U_2})$. If $f_1$ is Poisson, we see that $f_2|_{U_2}$ is Poisson. Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}[Y_2] & \xrightarrow{f_2^*} & \mathbb{C}[X_2] \\ & \searrow^{f_2|_{U_2}^*} & \downarrow \\ & & \mathbb{C}[U_2] \end{array}$$

We see that the vertical map is injective and the diagonal map is Poisson. Thus, $f_2$ is also Poisson.

Since $j$ is birational, there exist open sets $V_1 \subseteq Y_1$ and $V_2 \subseteq Y_2$ such that $V_1 \simeq V_2$. Let $U_1' = f_1^{-1}(V_1)$. By commutativity of the diagram, $(f_2 \circ i)(U_1') \subseteq V_2$. We see that $f_1|_{U_1'} = j^{-1} \circ f_2 \circ (i|_{U_1'})$. Thus, if $f_2$ is Poisson, so is $f_1|_{U_1'}$. Therefore, $f_1$ is also Poisson by the same reasoning as above. $\square$

**Proposition 2.6.3.8.** *The multiplication map $m_{\mu_1,\mu_2} : \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}$ restricts to a map $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} \longrightarrow \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$. Moreover, the restricted map is Poisson.*

*Proof.* The first claim follows from comparing the constructions of [BFN, 2(vi) and 2(xi)]. For the second claim, first consider the case where $\mu_1 = \mu_2 = 0$. We know that $\mathcal{W}_0 = G_1[[t^{-1}]]$ is a Poisson algebraic group. The map $m_{0,0}$ is precisely the group multiplication in $G_1[[t^{-1}]]$. Hence, it is Poisson, and so are its restrictions.

First, suppose that $\mu_1, \mu_2$ are dominant. If $\lambda_1 \geq \mu_1$ and $\lambda_2 \geq \mu_2$, consider $\nu_1 = -\mu_1$, $\nu_2 = -\mu_2$. We have the following slice version of Lemma 2.6.3.5.

$$\begin{CD} \overline{\mathcal{W}}^{\lambda_1-\mu_1}_0 \times \overline{\mathcal{W}}^{\lambda_2-\mu_2}_0 @>>> \overline{\mathcal{W}}^{\lambda_1+\lambda_2-\mu_1-\mu_2}_0 \\ @V{\iota_{\mu_1,-\mu_1,0} \times \iota_{\mu_2,0,-\mu_2}}VV @VV{\iota_{\mu_1+\mu_2,-\mu_1,-\mu_2}}V \\ \overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} @>>> \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2} \end{CD}$$

By Proposition 2.6.3.6, the two vertical arrows are Poisson and birational. Since the top arrow is Poisson, by Lemma 2.6.3.7, the bottom arrow is also Poisson, proving this case.

Next, suppose that $\mu_1$ and $\mu_2$ are arbitrary. We can choose choose $\nu_1, \nu_2$ antidominant such that $\mu_1 - \nu_1, \mu_2 - \nu_2$ are dominant. Now we have the following slice version of Lemma 2.6.3.5.

$$\begin{array}{ccc}
\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} & \longrightarrow & \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2} \\
{\scriptstyle \iota_{\mu_1-\nu_1,\nu_1,0} \times \iota_{\mu_2-\nu_2,0,\nu_2}} \downarrow & & \downarrow {\scriptstyle \iota_{\mu_1+\mu_2-\nu_1-\nu_2,\nu_1,\nu_2}} \\
\overline{\mathcal{W}}^{\lambda_1-\nu_1}_{\mu_1-\nu_1} \times \overline{\mathcal{W}}^{\lambda_2-\nu_2}_{\mu_2-\nu_2} & \longrightarrow & \overline{\mathcal{W}}^{\lambda_1+\lambda_2-\nu_1-\nu_2}_{\mu_1+\mu_2-\nu_1-\nu_2}
\end{array}$$

The bottom arrow is Poisson by our previous case. Therefore, by Lemma 2.6.3.7, the top arrow is also Poisson. □

**Theorem 2.6.3.9.** [FKPRW, Conjecture 5.20] $m_{\mu_1,\mu_2} : \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}$ *is Poisson.*

*Proof.* Let $f, g \in \mathbb{C}[\mathcal{W}_{\mu_1+\mu_2}]$. Let $\Delta$ be the corresponding comultiplication. We need to show that $h := \Delta(\{f, g\}) - \{\Delta(f), \Delta(g)\} = 0$. By Proposition 2.6.1.6, it suffices to show that the restriction of $h$ on each $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2}$ is zero.

Let $I$ be the ideal of $\overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$ and let $J$ be the ideal of $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2}$. We see that these are Poisson ideals. Since $\Delta(I) \subseteq J$ and since the restriction maps are Poisson,

$$\begin{aligned}
0 + J &= \Delta(\{f + I, g + I\}) - \{\Delta(f + I), \Delta(g + I)\} + J \\
&= \Delta(\{f, g\}) - \{\Delta(f), \Delta(g)\} + J \\
&= h + J.
\end{aligned}$$

Therefore, the multiplication is Poisson. □

As a natural next step, we discuss how the coproduct map $\Delta_{\mu_1,\mu_2}$ quantizes the multiplication map $m_{\mu_1,\mu_2}$.

For any coweights $\mu_1, \mu_2$, the multiplication $m_{\mu_1,\mu_2} : \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}$ gives rise to a comultiplication map $\Delta^1_{\mu_1,\mu_2} : \mathbb{C}[\mathcal{W}_{\mu_1+\mu_2}] \longrightarrow \mathbb{C}[\mathcal{W}_{\mu_1}] \otimes \mathbb{C}[\mathcal{W}_{\mu_2}]$.

On the other hand, the coproduct $Y_{\mu_1+\mu_2} \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$ is compatible with the filtrations of Proposition 2.4.0.3. It gives rise to a map $\operatorname{gr} Y_{\mu_1+\mu_2} \longrightarrow \operatorname{gr} Y_{\mu_1} \otimes \operatorname{gr} Y_{\mu_2}$. Under the isomorphism of Theorem 2.6.2.1, we obtain a map $\Delta^2_{\mu_1,\mu_2} : \mathbb{C}[\mathcal{W}_{\mu_1+\mu_2}] \longrightarrow \mathbb{C}[\mathcal{W}_{\mu_1}] \otimes \mathbb{C}[\mathcal{W}_{\mu_2}]$.

Assuming the result of Theorem 2.6.3.9, [FKPRW] proves the following:

**Theorem 2.6.3.10.** [FKPRW, Prop. 5.21] *For arbitrary coweights $\mu_1, \mu_2$, the two maps $\Delta^1_{\mu_1,\mu_2}$ and $\Delta^2_{\mu_1,\mu_2}$ agree.*

The next remark shows that multiplication between the $\mathcal{W}_\mu$'s are not associative. By the previous theorem, it will also justify Remark 2.3.3.3.

*Remark* 2.6.3.11. Multiplications between the $\mathcal{W}_\mu$'s are not associative, i.e.

$$m_{\mu_1+\mu_2,\mu_3} \circ (m_{\mu_1,\mu_2}, 1) \neq m_{\mu_1,\mu_2+\mu_3} \circ (1, m_{\mu_2,\mu_3}).$$

Consider the case where $G = PGL_2$. Let $\mu_1 = \mu_2 = \alpha/2$ and $\mu_3 = 0$. Let

$$x = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} \in \mathcal{W}_{\mu_1}$$
$$y = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{W}_{\mu_2}$$
$$z = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \in \mathcal{W}_{\mu_3}$$

We will show that $\pi_\alpha(\pi_\alpha(xy)z) \neq \pi_\alpha(x\pi_\alpha(yz))$.

$$\begin{aligned} xy &= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t^{-1} & 1+t^{-2} \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1}(1+t^{-2})^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1+t^{-2})^{-1} & 0 \\ 0 & 1+t^{-2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1}(1+t^{-2})^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & (1+t^{-2})^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^2(1+t^{-2})^{-1} & 0 \\ 0 & 1+t^{-2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (1+t^{-2})^{-1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & (1+t^{-2})^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^2(1+t^{-2})^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (1+t^{-2})^{-1} & 1 \end{pmatrix} \end{aligned}$$

where the last equality follows from the fact that we are in $PGL_2((t^{-1}))$. Therefore, since $1 + t^{-2} = \sum_{n\geq 0}(-1)^n t^{-2n}$,

$$\pi_\alpha(xy) = \begin{pmatrix} 1 & \sum_{n\geq 1}(-1)^n t^{-2n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^2(1+t^{-2})^{-2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sum_{n\geq 1}(-1)^n t^{-2n} & 1 \end{pmatrix}.$$

On the other hand,

$$yz = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1}+1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\pi_{\alpha/2}(yz) = y$.

We have to show that $\pi_\alpha(\pi_\alpha(xy)z) \neq \pi_\alpha(x\pi_{\alpha/2}(yz)) = \pi_\alpha(xy)$. By uniqueness of Gauss decomposition, it is sufficient to compare the lower triangular part. We need to compute the lower triangular part of $\pi_\alpha(\pi_\alpha(xy)z)$. For brevity, denote $p = \sum_{n\geq 1}(-1)^n t^{-2n}$.

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & t^{-1} \\ p & 1+t^{-1}p \end{pmatrix} \\ &= \begin{pmatrix} 1 & t^{-1}(1+t^{-1}p)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1+t^{-1}p)^{-1} & 0 \\ 0 & 1+t^{-1}p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p(1+t^{-1}p)^{-1} & 1 \end{pmatrix} \end{aligned}$$

We see that $p(1+t^{-1}p)^{-1} \in t^{-1}\mathbb{C}[[t^{-1}]]$. The result follows since $p(1+t^{-1}p)^{-1} \neq p$.

# Chapter 3

# On a certain Hamiltonian reduction: the commutative level

## 3.1 Some generalities on Hamiltonian $\mathbb{G}_a$-actions

Let $X$ be an affine Poisson variety together with a (left) $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \times X \longrightarrow X$, preserving the Poisson structure (i.e., a Poisson action).

For any affine algebraic group $G$, a $G$-action on an affine variety $X$ is equivalent to a comodule structure $\mathbb{C}[X] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[X]$. So, in our case where $G = \mathbb{G}_a$, we may consider the coaction $\rho^* : \mathbb{C}[X] \longrightarrow \mathbb{C}[y] \otimes \mathbb{C}[X]$, $f \mapsto \sum_n y^n f_n$.

**Definition 3.1.0.1.** The $\mathbb{G}_a$-action $\rho$ is said to be *Hamiltonian* if there exists $g \in \mathbb{C}[X]$ such that $\{g, f\} = f_1$. This function $g \in \mathbb{C}[X]$ is called the *moment map* of the action $\rho$.

Recall that the algebra $\mathbb{C}[y]$ is a coalgebra with comultiplication $\Delta_{\mathbb{C}[y]} : \mathbb{C}[y] \longrightarrow \mathbb{C}[u] \otimes \mathbb{C}[v]$, $y \mapsto u \otimes 1 + 1 \otimes v$ and counit $\varepsilon : \mathbb{C}[y] \longrightarrow \mathbb{C}$, $y \mapsto 0$.

**Lemma 3.1.0.2.** *Let $A$ be a $\mathbb{C}$-algebra. Let $g \in A$ be such that $\{g, -\}$ is locally nilpotent. The map $\rho_A : A \longrightarrow \mathbb{C}[y] \otimes A$, $a \mapsto \sum_{n \geq 0} \frac{1}{n!} y^n \{g, -\}^n(a)$ gives $A$ a comodule structure over the coalgebra $\mathbb{C}[y]$.*

*Proof.* First, let us check that $(\Delta_{\mathbb{C}[y]} \otimes \mathrm{Id}_A) \circ \rho_A = (\mathrm{Id}_A \otimes \rho_A) \circ \rho_A$. For $a \in A$,

$$\begin{aligned}
(\Delta_{\mathbb{C}[y]} \otimes \mathrm{Id}_A) \circ \rho_A(a) &= (\mathrm{Id}_A \otimes \Delta_{\mathbb{C}[y]})(\sum_{n \geq 0} \frac{1}{n!} y^n \{g, -\}^n(a)) \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n!} u^k v^{n-k} \{g, -\}^n(a) \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} u^k v^{n-k} \{g, -\}^n(a),
\end{aligned}$$

$$
\begin{aligned}
(\mathrm{Id}_A \otimes \rho_A) \circ \rho_A(a) &= (\mathrm{Id}_A \otimes \rho_A)(\sum_{n\geq 0} \frac{1}{k!} y^k \{g,-\}^k(a)) \\
&= \sum_{k\geq 0} \frac{1}{k!} u^k \rho_A(\{g,-\}^k(a)) \\
&= \sum_{k\geq 0} \frac{1}{k!} \sum_{i\geq 0} \frac{1}{i!} u^k v^i \{g,-\}^i(\{g,-\}^k(a)) \\
&= \sum_{k\geq 0} \sum_{i\geq 0} \frac{1}{k!i!} u^k v^i \{g,-\}^{k+i}(a) \\
&= \sum_{k\geq 0} \sum_{n\geq 0} \frac{1}{k!(n-k)!} u^k v^{n-k} \{g,-\}^n(a),
\end{aligned}
$$

where the last equality is obtained by reindexing $n = k + i$. By reordering the sum, we see that $(\Delta_{\mathbb{C}[y]} \otimes \mathrm{Id}_A) \circ \rho_A = (\mathrm{Id}_A \otimes \rho_A) \circ \rho_A$.

Next, let us check that $(\varepsilon \otimes \mathrm{Id}_A) \circ \rho = \mathrm{Id}_A$. For $a \in A$,

$$
((\varepsilon \otimes \mathrm{Id}_A) \circ \rho)(a) = (\varepsilon \otimes \mathrm{Id}_A)(\sum_{n\geq 0} \frac{1}{n!} y^n \{g,-\}^n(a)) = a.
$$

Therefore, $\rho_A$ gives $A$ a comodule structure over $\mathbb{C}[y]$. □

*Remark* 3.1.0.3. In the case where $A = \mathbb{C}[X]$, if $g \in \mathbb{C}[X]$ is such that $\{g,-\}$ is locally nilpotent, then the (co)action defined in Lemma 3.1.0.2 is Hamiltonian.

For the coaction of Lemma 3.1.0.2, we see that the higher terms $a_n = \{g,-\}^n(a)$ depend closely on $a_1 = \{g, a\}$. In general, this must indeed be the case. We have the following proposition.

**Proposition 3.1.0.4.** *Let $A$ be an algebra. Let $\rho : A \longrightarrow A \otimes \mathbb{C}[y]$, $a \mapsto \sum_{n\geq 0} a_n y^n$ be a comodule structure. Then, for $a \in A$, $a_{l+1} = \frac{1}{l+1}(a_1)_l$. Consider the function $\varphi : A \longrightarrow A, a \mapsto a_1$. For $a \in A$, $\varphi^n(a) = n!a_n$ for $n \in \mathbb{N}$. In other words., $\rho$ is uniquely determined by $\varphi$.*

*Proof.* The coidentity condition, $(\mathrm{Id} \otimes \varepsilon) \circ \rho = \mathrm{Id}$, tells us that $a_0 = a$. Thus, $a_0$ is completely independent of the rest. For the other $a_i$'s, we use the cocompatibility condition.

$$
\begin{aligned}
((\rho \otimes \mathrm{Id}) \circ \rho)(a) &= \sum_{n\geq 0} \rho(a_n) y^n = \sum_{n\geq 0} \sum_{k\geq 0} (a_n)_k u^k v^n, \\
((\mathrm{Id} \otimes \Delta_{\mathbb{C}[y]}) \circ \rho)(a) &= \sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} a_n u^i v^{n-i}
\end{aligned}
$$

For all $l \geq 0$, we see that $a_{l+1} = \frac{1}{l+1}(a_1)_l$.

It follows that $\varphi(x)_l = (l+1)x_{l+1}$ for all $x \in A$. Let $a \in A$. For $x = \varphi^k(a)$, we have that $\varphi^{k+1}(a)_l = (l+1)\varphi^k(a)_{l+1}$.

$$
\varphi^n(a) = \varphi^{n-1}(a)_1 = 2\varphi^{n-2}(a)_2 = 3!\varphi^{n-3}(a)_3 = \cdots = n!\varphi^{n-n}(a)_n = n!a_n.
$$

This concludes the proof. □

**Theorem 3.1.0.5.** *The following are equivalent:*

*(1) a Hamiltonian $\mathbb{G}_a$-action $\rho$ on $X$ with moment map $g \in \mathbb{C}[X]$,*

*(2) $g \in \mathbb{C}[X]$ with $\{g, -\}$ locally nilpotent.*

*Proof.* The (2)⇒(1) direction is simply Lemma 3.1.0.2. For (1)⇒(2), we need to show that $\{g, -\}$ is locally nilpotent. For any $f \in \mathbb{C}[X]$, write $\rho^*(f) = \sum_n y^n f_n \in \mathbb{C}[y] \otimes \mathbb{C}[X]$. By definition of a Hamiltonian action, $\{g, f\} = f_1$. Hence, by the previous lemma, $\{g, -\}^n(f) = n! f_n$. Since $f_n = 0$ for large enough $n$, we conclude that $\{g, -\}$ is locally nilpotent. □

## 3.2 On a certain Hamiltonian reduction for $\mathcal{W}_\mu$

### 3.2.1 The action

Let $\mu$ be any coweight. Fix $i \in I$. Define an action of $\mathbb{G}_a$ on $\mathcal{W}_\mu$ by $r \cdot g = \pi_\mu(x_i(r)g)$, where $x_i$ is defined in 1.5 and $\pi_\mu$ is as in Section 2.6.1. We need to make sense of this by showing that $x_i(r)g \in U((t^{-1}))T_1[[t^{-1}]]t^\mu U_-((t^{-1}))$.

From now on, unless stated otherwise, we write $\mathcal{D}$ for $\mathcal{D}_{s_i w_0 \omega_{i^*}, w_0 \omega_{i^*}}$.

**Lemma 3.2.1.1.** *Let $g \in G((t^{-1}))$. Then $g \in U((t^{-1}))T_1[[t^{-1}]]t^\mu U^-((t^{-1}))$ if and only if, for all dominant weights $\omega$ of $G$, $\mathcal{D}_{w_0\omega, w_0\omega}(g) \in t^{\langle w_0\omega, \mu\rangle} + t^{\langle w_0\omega, \mu\rangle - 1}\mathbb{C}[[t^{-1}]]$.*

*Proof.* Let $g = uht^\mu u_- \in U((t^{-1}))T_1[[t^{-1}]]t^\mu U^-((t^{-1}))$. For a dominant weight $\omega$,

$$\mathcal{D}_{w_0\omega, w_0\omega}(g) = \langle v^*_{w_0\omega}, uht^\mu u_- v_{w_0\omega}\rangle = \langle v^*_{w_0\omega}, uht^\mu v_{w_0\omega}\rangle$$

Since $ht^\mu v_{w_0\omega} = (w_0\omega)(h)t^{\langle w_0\omega, \mu\rangle} v_{w_0\omega}$ and since $(w_0\omega)(h) \in 1 + t^{-1}\mathbb{C}[[t^{-1}]]$, $\mathcal{D}_{w_0\omega, w_0\omega}(g) \in t^{\langle w_0\omega, \mu\rangle} + t^{\langle w_0\omega, \mu\rangle - 1}\mathbb{C}[[t^{-1}]]$.

On the other hand, suppose that $\mathcal{D}_{w_0\omega, w_0\omega}(g) \in t^{\langle w_0\omega, \mu\rangle} + t^{\langle w_0\omega, \mu\rangle - 1}\mathbb{C}[[t^{-1}]]$ for all $\omega$. Since it is nonzero, this means that $g$ has a Gauss decomposition, i.e., $g \in U((t^{-1}))T((t^{-1}))U_-((t^{-1}))$. From the assumption on $\mathcal{D}_{w_0\omega, w_0\omega}(g)$ for highest weights $\omega$, we see that the Cartan part has to lie in $T_1[[t^{-1}]]t^\mu$. □

*Remark* 3.2.1.2. In Lemma 3.2.1.1, instead of working with all dominant weights $\omega$, it is enough to work with the fundamental weights $\omega_j$.

**Lemma 3.2.1.3.** *Let $g \in \mathcal{W}_\mu$, $r \in \mathbb{G}_a$. Then $x_i(r)g \in U((t^{-1}))T_1[[t^{-1}]]t^\mu U_-((t^{-1}))$.*

*Proof.* Write $g = uht^\mu u_-$. We have that

$$\begin{aligned}\mathcal{D}_{w_0\omega_j, w_0\omega_j}(x_i(r)g) &= \langle v^*_{w_0\omega_j}, x_i(r)uht^\mu u_- v_{w_0\omega_j}\rangle \\ &= \langle v^*_{w_0\omega_j}, x_i(r)u(t^{\langle \mu, w_0\omega_j\rangle} w_0\omega_j(h) v_{w_0\omega_j})\rangle.\end{aligned}$$

If $j \neq i^*$, we see that $\mathcal{D}_{w_0\omega_j, w_0\omega_j}(x_i(r)g) = t^{\langle \mu, w_0\omega_j\rangle} w_0\omega_j(h)$.

If $j = i^*$, write $uv_{w_0\omega_{i^*}} = v_{w_0\omega_{i^*}} + \mathcal{D}(u)v_{s_i w_0\omega_{i^*}} + \cdots$. Thus, we have that $\mathcal{D}_{w_0\omega_{i^*}, w_0\omega_{i^*}}(x_i(r)g) = (1 + r\mathcal{D}(u))t^{\langle \mu, w_0\omega_j\rangle} w_0\omega_j(h)$.

From both cases and Lemma 3.2.1.1, $x_i(r)g \in U((t^{-1}))T_1[[t^{-1}]]t^\mu U^-((t^{-1}))$. □

We have a useful lemma, which tells us more about the product $x_i(r)g$.

**Lemma 3.2.1.4.** *Let $g \in \mathcal{W}_\mu$, $r \in \mathbb{G}_a$. Let $n \in U[t], n_- \in U_-[t]$ be such that $\pi_\mu(x_i(r)g) = nx_i(r)gn_-$. Then, $n = 1$.*

*Proof.* Write $g = uht^\mu u_-$. For $j \in I$, we have that

$$\begin{aligned} x_i(r)uht^\mu u_- v_{w_0\omega_{j^*}} &= x_i(r)ut^{\langle \mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)v_{w_0\omega_{j^*}} \\ &= x_i(r)t^{\langle \mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)v_\lambda) \\ &= t^{\langle \mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)(v_\lambda + rv_{\lambda-\alpha_i})) \\ &= t^{\langle \mu, w_0\omega_{j^*}\rangle}A_j(g)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)(v_\lambda + rv_{\lambda-\alpha_i})), \end{aligned}$$

where $v_{\lambda-\alpha_i} = 0$ if $\lambda - \alpha_i$ is not a weight of the irreducible representation of lowest weight $w_0\omega_{j^*}$.

On the other hand, let us write $x_i g = \hat{u}\hat{h}t^\mu \hat{u}_-$ where $\hat{u}_\pm \in U_\pm((t^{-1}))$. Then,

$$x_i g v_{w_0\omega_{j^*}} = (w_0\omega_{j^*})(\hat{h})t^{\langle w_0\omega_{j^*}, \mu\rangle}(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(\hat{u})v_\lambda).$$

Comparing the two different forms of $x_i g v_{w_0\omega_{j^*}}$, we obtain

$$\mathcal{D}_{\lambda, w_0\omega_{j^*}}(\hat{u}) = \frac{w_0\omega_{j^*}(h)}{w_0\omega_{j^*}(\hat{h})}(\mathcal{D}_{\lambda, w_0\omega_{j^*}}(u) + r\mathcal{D}_{\lambda+\alpha_i, w_0\omega_{j^*}}(u)) \in t^{-1}\mathbb{C}[[t^{-1}]],$$

for $\lambda > w_0\omega_{j^*}$. Therefore, $n = 1$. □

**Lemma 3.2.1.5.** *For $u \in U((t^{-1}))$, $\mathcal{D}(u) = -\mathcal{D}_{\omega_i, s_i\omega_i}(u)$.*

*Proof.* First, since $u$ is upper triangular, $uv_{s_i\omega_i} = v_{s_i\omega_i} + \mathcal{D}_{\omega_i, s_i\omega_i}(u)v_{\omega_i}$. We have a similar equation for $u^{-1} \in U((t^{-1}))$. Now, applying $u^{-1}$ on both sides of the above equation, $u^{-1}v_{s_i\omega_i} = v_{s_i\omega_i} - \mathcal{D}_{\omega_i, s_i\omega_i}(u)v_{\omega_i}$. We obtain $\mathcal{D}_{\omega_i, s_i\omega_i}(u) = -\mathcal{D}_{\omega_i, s_i\omega_i}(u^{-1})$.

On the other hand, since $w_0\omega_{i^*} = -\omega_i$,

$$\mathcal{D}(u) = \langle v_{-s_i w_0\omega_{i^*}}, uv_{w_0\omega_{i^*}}\rangle = \langle v_{s_i\omega_i}, uv_{-\omega_i}\rangle = \langle u^{-1}v_{s_i\omega_i}, v_{-\omega_i}\rangle = \mathcal{D}_{\omega_i, s_i\omega_i}(u^{-1}).$$

The result follows from comparing the two equations. □

**Proposition 3.2.1.6.** *The expression $r \cdot g = \pi_\mu(x_i(r)g)$ defines an action of $\mathbb{G}_a$ on $\mathcal{W}_\mu$.*

*Proof.* Let $a, r \in \mathbb{C}$, let $g = uht^\mu u_- \in \mathcal{W}_\mu$.

$$a \cdot (r \cdot g) = \pi_\mu(x_i(a)\pi_\mu(x_i(r)g)) = \hat{n}x_i(a)\tilde{n}x_i(r)g\tilde{n}_-\hat{n}_-.$$

for some $\hat{n}, \tilde{n} \in U[t]$, $\hat{n}_-, \tilde{n}_- \in U_-[t]$. It suffices to show that $x_i(a)\tilde{n}x_i(-a) \in U[t]$, i.e., we can commute $x_i(a)$ and $\tilde{n}$.

Let $\omega_j$ be a fundamental weight. If $i \neq j$, then $x_i(a)\tilde{n}x_i(-a)v_{\omega_j} = v_{\omega_j}$. If $i = j$, then we have that

$$\begin{aligned} x_i(a)\tilde{n}x_i(-a)v_{\omega_i} &= x_i(a)\tilde{n}(v_{\omega_i} - av_{s_i\omega_i}) \\ &= x_i(a)\big((1 - a\mathcal{D}_{\omega_i,s_i\omega_i}(\tilde{n}))v_{\omega_j} - av_{s_i\omega_i}\big) \\ &= (1 - a\mathcal{D}_{\omega_i,s_i\omega_i}(\tilde{n}))v_{\omega_i} + a(1 - a\mathcal{D}_{s_i\omega_i,\omega_i}(\tilde{n}))v_{s_i\omega_i}) - av_{s_i\omega_i} \\ &= (1 - a\mathcal{D}_{\omega_i,s_i\omega_i}(\tilde{n}))v_{\omega_i} - a^2\mathcal{D}_{\omega_i,s_i\omega_i}(\tilde{n})v_{s_i\omega_i}. \end{aligned}$$

By Lemma 3.2.1.5, we only need to show that $\mathcal{D}(\tilde{n}) = 0$, which is true by Lemma 3.2.1.4. □

Fix $r \in \mathbb{G}_a$, denote by $\varphi_r$ the map $\mathcal{W}_\mu \longrightarrow \mathcal{W}_\mu$, $g \mapsto \pi_\mu(x_i(r)g)$.

**Proposition 3.2.1.7.** *For $g \in \mathcal{W}_0 = G_1[[t^{-1}]]$, $\varphi_r(g) = x_i(r)gx_i(r)^{-1}$.*

*Proof.* Since $G_1[[t^{-1}]]$ is a normal subgroup of $G[[t^{-1}]]$, we see that $x_i(r)gx_i(r)^{-1} \in G_1[[t^{-1}]]$. By definition of $\pi_0$ and by uniqueness of Gauss decomposition, we see that $\varphi_r(g) = x_i(r)gx_i(r)^{-1}$. □

For antidominant coweights $\nu_1, \nu_2$, recall the shift map $\iota_{\mu,\nu_1,\nu_2} : \mathcal{W}_{\mu+\nu_1+\nu_2} \longrightarrow \mathcal{W}_\mu$, $g \mapsto \pi_\mu(t^{-\nu_1}gt^{-\nu_2})$.

**Lemma 3.2.1.8.** *For $\nu$ antidominant, the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W}_{\mu+\nu} & \xrightarrow{\varphi_r} & \mathcal{W}_{\mu+\nu} \\ \downarrow{\scriptstyle \iota_{\mu,0,\nu}} & & \downarrow{\scriptstyle \iota_{\mu,0,\nu}} \\ \mathcal{W}_\mu & \xrightarrow[\varphi_r]{} & \mathcal{W}_\mu \end{array}$$

*Moreover, $\varphi_r$ restricts to a map $\overline{\mathcal{W}}^\lambda_\mu \longrightarrow \overline{\mathcal{W}}^\lambda_\mu$, the corresponding diagram of slices also commutes.*

$$\begin{array}{ccc} \mathcal{W}^{\lambda+\nu}_{\mu+\nu} & \xrightarrow{\varphi_r} & \mathcal{W}^{\lambda+\nu}_{\mu+\nu} \\ \downarrow{\scriptstyle \iota_{\mu,0,\nu}} & & \downarrow{\scriptstyle \iota_{\mu,0,\nu}} \\ \mathcal{W}^\lambda_\mu & \xrightarrow[\varphi_r]{} & \mathcal{W}^\lambda_\mu \end{array}$$

*Proof.* Let $g \in \mathcal{W}_{\mu+\nu}$. Then, there are appropriate elements $n, \tilde{n}, n', n'' \in U[t]$, $n_-, \tilde{n}_-, n'_-, n''_- \in U_-[t]$ such that

$$\begin{aligned} \iota_{\mu,0,\nu} \circ \varphi_r(g) &= \tilde{n}nx_i(r)gn_-t^{-\nu}\tilde{n}_-, \\ \varphi_r \circ \iota_{\mu,0,\nu}(g) &= n''x_i(r)n'gt^{-\nu}n'_-n''_- \end{aligned}$$

We note that $n' = 1$ as the element is shifted from the right. Since $t^\nu U_-[t]t^{-\nu} \in U_-[t]$ as $\nu$ is antidominant, we see that both of the above equalities compute $\pi_\mu(x_i(r)gt^{-\nu})$.

For the second claim, note that $\mathcal{W}^\lambda$ is invariant under multiplication by $U^\pm[t]$. The result follows. □

**Proposition 3.2.1.9.** *The map $\varphi_r : \mathcal{W}_\mu \longrightarrow \mathcal{W}_\mu$, $g \mapsto \pi_\mu(x_i(r)g))$, is Poisson, i.e., $\mathbb{G}_a$ acts on $\mathcal{W}_\mu$ by Poisson automorphisms.*

*Proof.* We begin by showing that the restricted maps $\varphi_r : \overline{\mathcal{W}}_\mu^\lambda \longrightarrow \overline{\mathcal{W}}_\mu^\lambda$ are Poisson. Since $G((t^{-1}))$ is a Poisson algebraic group, conjugation by a group element is Poisson. By Proposition 3.2.1.7, $\varphi_r : \mathcal{W}_0 \longrightarrow \mathcal{W}_0$ and its restricted maps are Poisson.

First, suppose that $\mu$ is dominant. For $\lambda \geq \mu$, by Lemma 3.2.1.8, the following diagram is commutative.

$$\begin{array}{ccc} \overline{\mathcal{W}}_0^{\lambda-\mu} & \xrightarrow{\varphi_r} & \overline{\mathcal{W}}_0^{\lambda-\mu} \\ \downarrow{\scriptstyle \iota_{\mu,0,-\mu}} & & \downarrow{\scriptstyle \iota_{\mu,0,-\mu}} \\ \overline{\mathcal{W}}_\mu^{\lambda} & \xrightarrow[\varphi_r]{} & \overline{\mathcal{W}}_\mu^{\lambda} \end{array}$$

Since the top arrow is Poisson, by Lemma 2.6.3.7, so is the bottom. If $\mu$ is arbitrary, let $\nu$ be dominant such that $\mu - \nu$ and $\lambda - \nu$ are dominant. The following lemma is commutative.

$$\begin{array}{ccc} \overline{\mathcal{W}}_\mu^{\lambda} & \xrightarrow{\varphi_r} & \overline{\mathcal{W}}_\mu^{\lambda} \\ \downarrow{\scriptstyle \iota_{\mu-\nu,0,-\nu}} & & \downarrow{\scriptstyle \iota_{\mu-\nu,0,-\nu}} \\ \overline{\mathcal{W}}_{\mu-\nu}^{\lambda-\nu} & \xrightarrow[\varphi_r]{} & \overline{\mathcal{W}}_{\mu-\nu}^{\lambda-\nu} \end{array}$$

Since the bottom arrow is Poisson by the dominant case, so is the top arrow.

Next, to show that $\varphi_r$ is Poisson, we need to show that, for $f, g \in \mathbb{C}[\mathcal{W}_\mu]$, $h := \varphi_r(\{f, g\}) - \{\varphi_r(f), \varphi_r(g)\} = 0$. By Proposition 2.6.1.6, it suffices to show that the restriction of $h$ on each $\overline{\mathcal{W}}_\mu^\lambda$ is zero. This is indeed the case following the same computation of Theorem 2.6.3.9.

More precisely, let $I$ be the ideal of $\overline{W}_\mu^\lambda$. Since $\varphi_r^*(I) \subseteq I$ and since the restricted maps are Poisson by the first part,

$$\begin{aligned} 0 + I &= \mathcal{D}(\{f + I, g + I\}) - \{\mathcal{D}(f + I), \mathcal{D}(g + I)\} + I \\ &= \mathcal{D}(\{f, g\}) - \{\mathcal{D}(f), \mathcal{D}(g)\} + I \\ &= h + I. \end{aligned}$$

Thus, $h|_{\overline{\mathcal{W}}_\mu^\lambda} = 0$. Therefore, $h = 0$. □

Recall the $\mathbb{C}((t^{-1}))$-valued functions $E_j, F_j$ and $H_j$ defined after statement of Theorem 2.6.2.1. For $g = uht^\mu u_- \in \mathcal{W}_\mu$,

$$\begin{aligned} E_j(g) &= \mathcal{D}_{s_j w_0 \omega_{j^*}, w_0 \omega_{j^*}}(u) \\ F_j(g) &= \mathcal{D}_{w_0 \omega_{j^*}, s_j w_0 \omega_{j^*}}(u_-) \\ H_j(g) &= \alpha_j(ht^\mu), \end{aligned}$$

where the right-hand side of last equation means the projection corresponding to the root $\alpha_j$.

Consider the element $E_i^{(1)} \in \mathbb{C}[\mathcal{W}_\mu]$. Denote by $\{-,-\}$ the Poisson bracket on $\mathcal{W}_\mu$. From the structure theory of $\operatorname{gr} Y_\mu$, the operator $\{E_i^{(1)}, -\}$ is locally nilpotent. Hence, by Lemma 3.1.0.2, the comodule structure $\mathbb{C}[\mathcal{W}_\mu] \longrightarrow \mathbb{C}[y] \otimes \mathbb{C}[\mathcal{W}_\mu], f \mapsto \sum_n y^n \{E_i^{(1)}, -\}^n(f)$ defines a Hamiltonian $\mathbb{G}_a$-action on $\mathcal{W}_\mu$.

In the next proposition, we will show that this action given by $E_i^{(1)}$ is the same as the action given in Proposition 3.2.1.6.

**Proposition 3.2.1.10.** *The action of Proposition* 3.2.1.6 *coincides with that of Lemma* 3.1.0.2*. In particular, it is Hamiltonian with moment map* $\Phi_i : \mathcal{W}_\mu \longrightarrow \mathbb{C}$, $uht^\mu u_- \mapsto \mathcal{D}^{(1)}(u)$.

*Proof.* Since the action of Proposition 3.2.1.6 is Poisson by Proposition 3.2.1.9, it is enough to show that the action by $\varphi_r$ agrees with the action of Lemma 3.1.0.2 on Poisson generators. More precisely, for a generator $S$ of $\mathbb{C}[\mathcal{W}_\mu]$, we would like to show that $\varphi_r^*(S) = \{E_i^{(1)}, S\}$. We will use the generators $B_j, A_j$ and $F_j$ of $\mathbb{C}[\mathcal{W}_\mu]$.

For $g = uht^\mu u_- \in \mathcal{W}_\mu$, we write $\varphi_r(g) = nx_i(r)gn_-$ for $n \in U[t], n_- \in U_-[t]$. By Lemma 3.2.1.4, we already have that $n = 1$. In fact, let us repeat the computation of Lemma 3.2.1.4 here. For $j \in I$, we have that

$$\begin{aligned} x_i(r)uht^\mu u_- v_{w_0\omega_{j^*}} &= x_i(r)ut^{\langle\mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)v_{w_0\omega_{j^*}} \\ &= x_i(r)t^{\langle\mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)v_\lambda) \\ &= t^{\langle\mu, w_0\omega_{j^*}\rangle}(w_0\omega_{j^*})(h)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)(v_\lambda + rv_{\lambda-\alpha_i})) \\ &= t^{\langle\mu, w_0\omega_{j^*}\rangle}A_j(g)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)(v_\lambda + rv_{\lambda-\alpha_i})), \end{aligned}$$

where $v_{\lambda-\alpha_i} = 0$ if $\lambda - \alpha_i$ is not a weight of the irreducible representation of lowest weight $w_0\omega_{j^*}$.

We see that $\varphi_r^*(A_j(t)) = A_j(g)(1 + r\delta_{i^*j}\mathcal{D}_{s_jw_0\omega_{j^*}, w_0\omega_{j^*}}(u))$. For $j \neq i$, the coefficient of $r$ is 0. For $j = i$, the coefficient of $r$ is $A_i(t)E_i(g) = B_i(g)$.

Also, $\varphi_r^*(B_j)(g) = A_j(g)(\mathcal{D}_{s_jw_0\omega_{j^*}, w_0\omega_{j^*}}(u) + r\mathcal{D}_{s_jw_0\omega_{j^*}+\alpha_i, w_0\omega_{j^*}}(u))$.

If $j = i$, then $s_iw_0\omega_{i^*}$ is a weight, $s_iw_0\omega_{i^*} + \alpha_i$ is not. The $r$-coefficient of $\varphi_r^*(B_j)(g)$ is 0.

If $\langle\alpha_i, \alpha_j\rangle = 0$, then $s_jw_0\omega_{j^*} + \alpha_i$ is not a weight. Hence, the $r$-coefficient of $\varphi_r^*(B_j)(g)$ is 0.

If $\langle\alpha_i, \alpha_j\rangle = -1$, then $s_jw_0\omega_{j^*} + \alpha_i$ is a weight. So, the $r$-coefficient of $\varphi_r^*(B_j)(g)$ is $\mathcal{D}_{s_jw_0\omega_{j^*}+\alpha_i, w_0\omega_{j^*}}(u)$, agreeing with Lemma 2.6.2.4.

Next, we wish to compute $\varphi_r^*(C_j)(g)$.

$$\begin{aligned} &x_i(r)uht^\mu u_-n_-v_{s_jw_0\omega_{j^*}} \\ &= x_i(r)u\big(w_0\omega_{j^*}(h)t^{\langle\mu, w_0\omega_{j^*}\rangle}\mathcal{D}_{w_0\omega_{j^*}, s_jw_0\omega_{j^*}}(u_-n_-)v_{w_0\omega_{j^*}} + s_jw_0\omega_{j^*}(h)t^{\langle\mu, s_jw_0\omega_{j^*}\rangle}v_{s_jw_0\omega_{j^*}}\big) \\ &= x_i(r)\Big(w_0\omega_{j^*}(h)t^{\langle\mu, w_0\omega_{j^*}\rangle}\mathcal{D}_{w_0\omega_{j^*}, s_jw_0\omega_{j^*}}(u_-n_-)\big(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)v_\lambda\big) + \\ &+ (s_jw_0\omega_{j^*})(h)t^{\langle\mu, s_jw_0\omega_{j^*}\rangle}\big(v_{s_jw_0\omega_{j^*}} + \sum_{\gamma > s_jw_0\omega_{j^*}} \mathcal{D}_{\gamma, s_jw_0\omega_{j^*}}(u)v_\gamma\big)\Big) \\ &= t^{\langle\mu, w_0\omega_{j^*}\rangle}A_j(g)F_j(u_-n_-)(v_{w_0\omega_{j^*}} + \sum_{\lambda > w_0\omega_{j^*}} \mathcal{D}_{\lambda, w_0\omega_{j^*}}(u)(v_\lambda + rv_{\lambda-\alpha_i}) \\ &+ t^{\langle\mu, w_0\omega_{j^*}\rangle}A_j(g)H_j(g)\big(v_{s_jw_0\omega_{j^*}} + rv_{s_jw_0\omega_{j^*}-\alpha_i} + \sum_{\gamma > s_jw_0\omega_{j^*}} \mathcal{D}_{\gamma, s_jw_0\omega_{j^*}}(u)(v_\gamma + rv_{\gamma-\alpha_i})\big). \end{aligned}$$

Thus,

$$\varphi_r^*(C_j)(g) = A_j(g)F_j(u_-n_-)(1 + r\mathcal{D}_{w_0\omega_{j^*}+\alpha_i, w_0\omega_{j^*}}(u)) + A_j(g)H_j(g)\delta_{ij}r$$

Again, by the relations of Lemma 2.6.2.3,

$$\varphi_r^*(F_j)(g) = \frac{F_j(u_-n_-)(1 + r\mathcal{D}_{w_0\omega_{j^*}+\alpha_i, w_0\omega_{j^*}}(u)) + H_j(g)\delta_{ij}r}{(1 + r\delta_{ij}\mathcal{D}_{s_jw_0\omega_{j^*}, w_0\omega_{j^*}}(u))}$$

If $j \neq i$, $w_0\omega_{j^*} + \alpha_i$ is not a weight,

$$\varphi_r^*(F_j)(g) = F_j(u_-n_-).$$

Hence, its $r$-coefficient is 0. If $j = i$,

$$\varphi_r^*(F_i)(g) = F_i(u_-) + F_i(n_-) + rH_i(g)(1 + rE_i(g))^{-1}.$$

We see that the coefficient of $r$ is $\underline{H_i(g)}$ since $n_-$ is chosen to satisfy, in particular, that $\varphi_r^*(F_i)(g) \in t^{-1}\mathbb{C}[[t^{-1}]]$. □

### 3.2.2 How to relate $\mathcal{W}_\mu$ to $\mathcal{W}_{\mu+\alpha_i}$

Our goal of this section is to construct an isomorphism $\Phi_i^{-1}(\mathbb{C}^\times) \simeq \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$.

Consider the slice $\overline{\mathcal{W}}^0_{-\alpha_i}$. This space is quite nice, it is isomorphic to $\mathbb{C} \times \mathbb{C}^\times$ with Poisson structure given by $\{c, a\} = c$ where $a$ is the $\mathbb{C}$-coordinate function and $c$ is the $\mathbb{C}^\times$-coordinate function. There is a Hamiltonian $\mathbb{G}_a$-action on it given by $r \cdot g = x_i(r)g$ for $g \in \overline{\mathcal{W}}^0_{-\alpha_i}$, or $r \cdot (a, c) = (a + rc, c)$ for $(a, c) \in \mathbb{C} \times \mathbb{C}^\times$.

**Lemma 3.2.2.1.** *$G = SL_2$, $\overline{\mathcal{W}}^0_{-\alpha} = \{\begin{pmatrix} 0 & c \\ -c^{-1} & t+a \end{pmatrix} : c \in \mathbb{C}^\times, a \in \mathbb{C}\} \simeq \mathbb{C} \times \mathbb{C}^\times$. Moreover, if*

$$g = \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ht^{-1} & 0 \\ 0 & h^{-1}t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \in \mathcal{W}^0_{-\alpha},$$

*then $c = E^{(1)}$, $t + a = E^{(1)}E^{-1} = h^{-1}t$ and $F = -(E^{(1)})^{-2}E$.*

*Proof.* This can be proved by a straightforward computation. □

**Lemma 3.2.2.2.** *The elements of $\overline{\mathcal{W}}^0_{-\alpha_i}$ are precisely the elements of $G((t^{-1}))$ of the form $\alpha_i(c)x_i(c(t+a))s_i^{-1}$ where $c \in \mathbb{C}^\times$.*

*Proof.* We know that the result is true for the case $G = SL_2((t^{-1}))$. More precisely,

$$\begin{pmatrix} 0 & c \\ -c^{-1} & t+a \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c(t+a) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall the map $\varphi_i$ introduced in Section 1.5. We want to show that $\varphi_i(\overline{\mathcal{W}}^0_{-\alpha}) = \overline{\mathcal{W}}^0_{-\alpha_i}$. Since $\varphi_i$ is a homomorphism, $\varphi_i(\overline{\mathcal{W}}^0_{-\alpha}) \subseteq \overline{\mathcal{W}}^0_{-\alpha_i}$.

Let $g = uht^{-\alpha_i}u_- \in \overline{\mathcal{W}}^0_{-\alpha_i}$. Consider the lowest weight vector $v_{w_0\lambda}$ for a representation $V$.

$$gv_{w_0\lambda} = uht^{-\alpha_i}u_-v_{w_0\lambda} = (w_0\lambda)(h)t^{\langle w_0\lambda, -\alpha_i\rangle}uv_{w_0\lambda} \in V[t]$$

If $\lambda = -w_0\omega_j$ where $j \neq i$, then $t^{\langle w_0\lambda, -\alpha_i\rangle} = 1$. But since $w_0\lambda(h) \in 1 + t^{-1}\mathbb{C}[[t^{-1}]]$ and $uv_{w_0\lambda} \in V_1[[t^{-1}]]$, we have that $w_0\lambda(h) = 1$ and $uv_{w_0\lambda} = v_{w_0\lambda}$. Thus, $h \in \varphi_i(SL_2((t^{-1}))$. Since $u$ leaves $v_{w_0\lambda}$ invariant where $\lambda = -w_0\omega_j$ for all $j \neq i$, it lies in each of the parabolic subgroups $P_j = \{g \in G((t^{-1})) : gv_{-w_0\omega_j} = v_{-w_0\omega_j}\}$ for $j \neq i$. We know that $\bigcap_{j\neq i} P_j \cap U((t^{-1})) = \varphi_i(SL_2((t^{-1}))) \cap U((t^{-1}))$. Thus, $u \in \varphi_i(SL_2((t^{-1})))$.

Now, by a symmetric argument on $g^{-1}$, we see that $u_- \in \varphi_i(SL_2((t^{-1})))$. Therefore, $g \in \varphi_i(\mathcal{W}^0_{-\alpha})$. □

**Lemma 3.2.2.3.** *Let $g_1 \in \overline{\mathcal{W}}^0_{-\alpha_i}$ and $g_2 \in \mathcal{W}_{\mu+\alpha_i}$. Then, $\Phi_i(m_{-\alpha_i,\mu+\alpha_i}(g_1, g_2)) = \Phi_i(g_1)$. Consequently, $m_{-\alpha_i,\mu+\alpha_i}(\overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}) \subseteq \Phi_i^{-1}(\mathbb{C}^\times)$.*

*Proof.* Let $u_1t^{-\alpha_i}h_1u_1^- \in \overline{\mathcal{W}}^0_{-\alpha_i}$, $u_2h_2t^{\mu+\alpha_i}u_2^- \in \mathcal{W}_{\mu+\alpha_i}$ where $u_i \in U_1[[t^{-1}]]$, $h_i \in T_1[[t^{-1}]]$, and $u_i^- \in U_1^-[[t^{-1}]]$. Write $h_1u_1^-u_2h_2 = uhu^-$. The product becomes

$$u_1(t^{-\alpha_i}ut^{\alpha_i})ht^\mu(t^{-(\mu+\alpha_i)}u^-t^{\mu+\alpha_i})u_2^-.$$

We see that $\mathcal{D}(t^{-\alpha_i}ut^{\alpha_i}) \in t^{-2}\mathbb{C}[[t^{-1}]]$. Thus, $\mathcal{D}^{(1)}(u_1t^{-\alpha_i}ut^{\alpha_i}) = \mathcal{D}^{(1)}(u_1) + \mathcal{D}^{(1)}(t^{-\alpha_i}ut^{\alpha_i}) = \mathcal{D}^{(1)}(u_1) \neq 0$. Therefore, the product lies in $\Phi_i^{-1}(\mathbb{C}^\times)$. □

Thus, we may write the map $m = m_{-\alpha_i,\mu+\alpha_i} : \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i} \longrightarrow \Phi_i^{-1}(\mathbb{C}^\times)$.

Next, we wish to define the inverse to $m$. First, consider the map

$$\xi : \Phi_i^{-1}(\mathbb{C}^\times) \longrightarrow \overline{\mathcal{W}}^0_{-\alpha_i}, \quad g = uht^\mu u_- \mapsto \alpha_i(\Phi_i(u))x_i(\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))s_i^{-1}.$$

We define a candidate for the inverse as follows,

$$\psi : \Phi_i^{-1}(\mathbb{C}^\times) \longrightarrow \overline{\mathcal{W}}^0_{-\alpha_i} \times G((t^{-1})), \quad g \mapsto (\xi(g), \pi_{\mu+\alpha_i}(\xi(g)^{-1}g)).$$

**Lemma 3.2.2.4.** *The image of $\psi$ lies in $\overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$.*

*Proof.* Write $g = uht^\mu u_- \in \Phi_i^{-1}(\mathbb{C}^\times)$. We will use Lemma 3.2.1.1 in order to show that, for $g \in \Phi_i^{-1}(\mathbb{C}^\times)$,

$$\xi(g)^{-1}g = s_ix_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}g$$

lies in $U((t^{-1}))T_1[[t^{-1}]]t^{\mu+\alpha_i}U^-((t^{-1}))$. Let $\omega_j$ be a fundamental weight.

$$\begin{aligned}
&\mathcal{D}_{w_0\omega_j,w_0\omega_j}(s_ix_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}g)\\
=&\mathcal{D}_{s_iw_0\omega_j,w_0\omega_j}(x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}g)\\
=&\langle v^*_{s_iw_0\omega_j}, x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}uht^\mu u_-v_{w_0\omega_j}\rangle\\
=&\langle v^*_{s_iw_0\omega_j}, x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}u(w_0\omega_j)(h)t^{\langle w_0\omega_j,\mu\rangle}v_{w_0\omega_j}\rangle
\end{aligned}$$

Consider the case where $j = i^*$. We have that

$$uv_{w_0\omega_{i^*}} = v_{w_0\omega_{i^*}} + \mathcal{D}(u)v_{s_iw_0\omega_{i^*}} + \cdots$$

Thus,

$$\begin{aligned} &x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}(v_{w_0\omega_{i^*}} + \mathcal{D}(u)v_{s_iw_0\omega_{i^*}} + \cdots) \\ &= x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))(\Phi_i(u)v_{w_0\omega_{i^*}} + \Phi_i(u)^{-1}\mathcal{D}(u)v_{s_iw_0\omega_{i^*}} + \cdots) \\ &= \mathcal{D}(u)\underline{\mathcal{D}(u)^{-1}}v_{w_0\omega_{i^*}} + \Phi_i(u)^{-1}\mathcal{D}(u)v_{s_iw_0\omega_{i^*}} + \cdots \end{aligned}$$

So, the coefficient of $v_{s_iw_0\omega_{i^*}}$ is $\Phi_i(u)^{-1}\mathcal{D}(u)(w_0\omega_{i^*})(h)t^{\langle w_0\omega_{i^*},\mu\rangle}$. Hence, since $g \in \Phi_i^{-1}(\mathbb{C}^\times)$, $\mathcal{D}_{w_0\omega_{i^*},w_0\omega_{i^*}}(\xi(g)^{-1}g)$ is of the form $t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle} + t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle-1}\mathbb{C}[[t^{-1}]]$.

Consider the case where $j \neq i^*$. So, $s_iw_0\omega_j = w_0\omega_j$. Also,

$$\langle v^*_{w_0\omega_j}, x_i(-\Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(u))^{-1}(v_{w_0\omega_j} + \cdots)\rangle = 1$$

Hence, in this case, we also get the desired result. □

Therefore, we may write the map $\psi : \Phi_i^{-1}(\mathbb{C}^\times) \longrightarrow \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$.

**Lemma 3.2.2.5.** *For $y = \pi_{\mu+\alpha_i}\big(s_ix_i(-\Phi_i(g)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(g))^{-1}g\big)$. Let $n \in U[t]$ and $n_- \in U^-[t]$ be such that*

$$y = n\big(s_ix_i(-\Phi_i(g)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}}))\alpha_i(\Phi_i(g))^{-1}g\big)n_-$$

*Then $\mathcal{D}(n) = 0$.*

*Proof.* This immediately follows from the computations in the previous lemma. □

**Theorem 3.2.2.6.** *The maps $\psi$ and $m$ are inverses of each other.*

*Proof.* First, we show that $m \circ \psi = \mathrm{Id}$. Let $g = uht^\mu u_- \in \Phi_i^{-1}(\mathbb{C}^\times)$. Let $n, n_-$ be the same as in the previous lemma. We need to show that $\pi_\mu(\xi(g)n\xi(g)^{-1}gn_-) = g$. To do so, we prove that $\xi(g)n\xi(g)^{-1} \in U[t]$. Write $y = \Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}})$ for simplicity.

$$\begin{aligned} \xi(g)n\xi(g)^{-1} &= \alpha_i(\Phi_i(u))x_i(y)s_i^{-1}ns_ix_i(-y)\alpha_i(\Phi_i(u))^{-1} \\ &= \alpha_i(\Phi_i(u))s_i^{-1}x_i^+(-y)nx_i^+(y)s_i\alpha_i(\Phi_i(u))^{-1} \end{aligned}$$

Now, as $\mathcal{D}(n) = 0$ by the previous lemma, we claim that

$$s_i^{-1}x_i^+(-y)nx_i^+(y)s_i \in U[t].$$

To prove the claim, write $y = \Phi_i(u)^2(\mathcal{D}(u)^{-1} - \underline{\mathcal{D}(u)^{-1}})$ for simplicity. Consider the highest weight vector $v_{\omega_i}$

$$\begin{aligned} s_i^{-1}x_i^+(-y)nx_i^+(y)s_iv_{\omega_i} &= s_i^{-1}x_i^+(-y)nx_i^+(y)v_{s_i\omega_i} \\ &= s_i^{-1}x_i^+(-y)n(v_{s_i\omega_i}+yv_{\omega_i}) \\ &= s_i^{-1}x_i^+(-y)(v_{s_i\omega_i}+yv_{\omega_i}) \\ &= s_i^{-1}(v_{s_i\omega_i}) = v_{\omega_i}, \end{aligned}$$

where the third equality uses the fact that $\mathcal{D}_{\omega_i,s_i\omega_i}(n) = -\mathcal{D}(n) = 0$. We also see that $s_i^{-1}x_i^+(-y)nx_i^+(y)s_iv_{\omega_i}$ fixes the other highest weight vectors $v_{\omega_j}$ for $j \neq i$. Thus, $\xi(g)n\xi(g)^{-1} \in U[t]$. Hence, $m \circ \psi = \mathrm{Id}$.

Now, we show that $\psi \circ m = \mathrm{Id}$. Let $\big(\alpha_i(c)x_i(c(t+a))s_i^{-1}, g\big) \in \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$ where $g = uht^{\mu+\alpha_i}u_-$. Consider the Gauss decomposition $\pi(\alpha_i(c)x_i(c(t+a))s_i^{-1}g) = \hat{u}\hat{h}t^{\mu}\hat{u}_-$. We claim that

$$\mathcal{D}(\hat{u}) = c(-c^{-1}\mathcal{D}(u) + (t+a))^{-1}.$$

Let $n \in U[t]$, $n_- \in U_-[t]$ be such that $\pi(\alpha_i(c)x_i(c(t+a))s_i^{-1}g) = n\alpha_i(c)x_i(c(t+a))s_i^{-1}gn_-$. On one hand,

$$\mathcal{D}(\hat{u}\hat{h}t^{\mu}\hat{u}_-) = (w_0\omega_{i^*})(\hat{h})\mathcal{D}(\hat{u})t^{\langle w_0\omega_{i^*},\mu\rangle}.$$

On the other hand,

$$\begin{aligned} &\mathcal{D}(n\alpha_i(c)x_i(c(t+a))s_i^{-1}uht^{\mu+\alpha_i}u_-n_-) = \\ &= \Big(c + \mathcal{D}(n)\big(-c^{-1}\mathcal{D}(u) + (t+a)\big)\Big)t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle}(w_0\omega_{i^*}(h)), \end{aligned}$$

which lies in $t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle} + t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle-1}\mathbb{C}[[t^{-1}]]$. Thus, we have

$$\mathcal{D}(n\alpha_i(c)x_i(c(t+a))s_i^{-1}uht^{\mu+\alpha_i}u_-n_-) = ct^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle}(w_0\omega_{i^*}(h)).$$

Therefore, we deduce that

$$\mathcal{D}(\hat{u}) = \frac{ct^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle}(w_0\omega_{i^*}(h))}{t^{\langle w_0\omega_{i^*},\mu\rangle}(w_0\omega_{i^*})(\hat{h})} = \frac{ct^{-1}(w_0\omega_{i^*}(h))}{(w_0\omega_{i^*})(\hat{h})},$$

since $\langle w_0\omega_{i^*},\alpha_i\rangle = \langle w_0^2\omega_{i^*},\alpha_{i^*}\rangle = -1$. To prove the claim, it remains to find the relationship between $(w_0\omega_{i^*})(h)$ and $(w_0\omega_{i^*})(\hat{h})$.

Write the Gauss decomposition $\alpha_i(c)x_i(c(t+a))s_i^{-1} = wkt^{-\alpha_i}w_- \in \mathcal{W}_{-\alpha_i}$. Note that the product $wkt^{-\alpha_i}w_-u$ lies in $U((t^{-1}))T_1[[t^{-1}]]t^{-\alpha_i}U^-((t^{-1}))$ since $u \in U_1[[t^{-1}]] \subseteq \mathcal{W}_0$. Thus, we may consider the decomposition $wkt^{-\alpha_i}w_-u = w'k't^{-\alpha_i}w'_-$. The Gauss decomposition of the product $\alpha_i(c)x_i(c(t+a))s_i^{-1}g$ is computed as follows.

$$\begin{aligned} \alpha_i(c)x_i(c(t+a))s_i^{-1}g &= (wkt^{-\alpha_i}w_-u)ht^{\mu+\alpha_i}u^- \\ &= w'k't^{-\alpha_i}w'_-ht^{\mu+\alpha_i}u^- \\ &= w'k'ht^{\mu}w''_-u^-, \end{aligned}$$

where the last step is obtained by moving $w'_-$ pass $h$, i.e., $w'_-h = hw''_-$. By uniqueness of Gauss decomposition, we see that $\hat{h} = k'h$, and so $(w_0\omega_{i^*})(\hat{h}) = (w_0\omega_{i^*})(k')(w_0\omega_{i^*})(h)$.

Lastly, let us compute $(w_0\omega_{i^*})(k')$. On one hand, we have that

$$\mathcal{D}_{w_0\omega_{i^*},w_0\omega_{i^*}}(\alpha_i(c)x_i(c(t+a))s_i^{-1}u) = -c^{-1}\mathcal{D}(u) + (t+a).$$

On the other hand,

$$\mathcal{D}_{w_0\omega_{i^*},w_0\omega_{i^*}}(w'k't^{-\alpha_i}w'_-) = (w_0\omega_{i^*})(k')t^{\langle w_0\omega_{i^*},-\alpha_i\rangle} = (w_0\omega_{i^*})(k')t.$$

Hence, $(w_0\omega_{i^*})(k') = \frac{-c^{-1}\mathcal{D}(u)+(t+a)}{t}$. Therefore,

$$\mathcal{D}(\hat{u}) = \frac{ct^{-1}(w_0\omega_{i^*}(h))}{(w_0\omega_{i^*})(\hat{h})} = \frac{ct^{-1}(w_0\omega_{i^*}(h))}{(w_0\omega_{i^*})(h)(-c^{-1}\mathcal{D}(u)+(t+a))t^{-1}} = c(-c^{-1}\mathcal{D}(u)+(t+a))^{-1},$$

as claimed.

This means that $\Phi_i(\hat{u}) = c$. So, the $\overline{\mathcal{W}}^0_{-\alpha_i}$-component of $\psi(\pi(\alpha_i(c)x_i(c(t+a))s_i^{-1}g))$ is precisely $\alpha_i(c)x_i(c(t+a))s_i^{-1}$. We have that

$$\begin{aligned}&s_ix_i(-c(t+a))\alpha_i(c)^{-1}\pi(\alpha_i(c)x_i(c(t+a))s_i^{-1}g)\\=&x_i^+(c(t+a))s_i^{-1}\alpha_i(c)^{-1}n\alpha_i(c)s_ix_i^+(-c(t+a))gn_-\end{aligned}$$

Since $\mathcal{D}(n) = 0$, the above product lies in $U[t]$ by similar arguments as before. Therefore, $\psi\circ\pi = \mathrm{Id}$. □

Next, consider the $\mathbb{G}_a$-action on $\overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$ acting solely on the first component by $r\cdot x := x_i(r)x$. Under the identification $\overline{\mathcal{W}}^0_{-\alpha_i} \simeq \mathbb{C}\times\mathbb{C}^\times$, the action is $r\cdot(a,c) = (a+rc,c)$.

**Proposition 3.2.2.7.** *$m$ is $\mathbb{G}_a$-equivariant.*

*Proof.* Let $(x,g) \in \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$. For $r\in\mathbb{G}_a$, we have to show that

$$\pi_\mu(x_i(r)xg) = \pi_\mu(x_i(r)\pi_\mu(xg)).$$

In other words, there exist $n, n_-, \tilde{n}, \tilde{n}_-, \hat{n}, \hat{n}_- \in N[t]$ such that

$$nx_i(r)xgn_- = \hat{n}x_i(r)\tilde{n}xg\tilde{n}_-\hat{n}_-.$$

It suffices to show that $x_i(r)\tilde{n}x_i(-r) \in N[t]$. For this, it is enough to prove that $\mathcal{D}(\tilde{n}) = 0$.

Write $\hat{g} = xg = \hat{u}\hat{h}t^\mu\hat{u}_- = xuht^\mu u_-$. We wish to compute $\mathcal{D}(\hat{u})$. This is the same as in the proof of the previous theorem. More precisely, from the two descriptions of $\hat{g}$,

$$\begin{aligned}\mathcal{D}(\hat{g}) &= t^{\langle w_0\omega_{i^*},\mu+\alpha_i\rangle}w_0\omega_{i^*}(h)c\\ \mathcal{D}(\hat{g}) &= \langle v^*_{w_0\omega_{i^*}}, \hat{u}\hat{h}t^\mu\hat{u}_-v_{w_0\omega_{i^*}}\rangle = t^{\langle w_0\omega_{i^*},\mu\rangle}w_0\omega_{i^*}(\hat{h})\mathcal{D}(\hat{u}).\end{aligned}$$

Thus, since $\langle w_0\omega_{i^*}, \alpha_i\rangle = -1$,

$$\mathcal{D}(\hat{u}) = \frac{t^{\langle w_0\omega_{i^*}, \mu+\alpha_i\rangle} w_0\omega_{i^*}(h)c}{t^{\langle w_0\omega_{i^*}, \mu\rangle} w_0\omega_{i^*}(\hat{h})} \in t^{-1}\mathbb{C}[[t^{-1}]].$$

This means that $\mathcal{D}(\tilde{n}) = 0$, concluding the proof. $\square$

In fact, in the proof of the previous proposition, $\mathcal{D}(\hat{u}) = c(-c^{-1}\mathcal{D}(u) + (t+a))^{-1}$, as computed previously.

Having the equivariant isomorphism $m : \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i} \longrightarrow \Phi_i^{-1}(\mathbb{C}^\times)$, we also obtain an isomorphism on the level of slices.

**Corollary 3.2.2.8.** *Given a dominant coweight $\lambda$, $m$ restricts to a $\mathbb{G}_a$-equivariant isomorphism $m_\lambda : \Phi_i^{-1}(\mathbb{C}^\times) \cap \overline{G[t]t^\lambda G[t]} \longrightarrow \overline{\mathcal{W}}^0_{-\alpha_i} \times \overline{\mathcal{W}}^\lambda_{\mu+\alpha_i}$.*

*Proof.* This follows from the fact that the maps $m$ and $\pi_{\mu+\alpha_i}$ preserve slices. $\square$

We arrive at the desired reduction result.

**Theorem 3.2.2.9.** $\Phi_i^{-1}(\mathbb{C}^\times)/\mathbb{G}_a \simeq \mathbb{C}^\times \times \mathcal{W}_{\mu+\alpha_i}$.

*Proof.* Recall that the $\mathbb{G}_a$-action on the right-hand side only acts on the first factor. The action on the first factor is given by $r \cdot (a, c) = (a + rc, c)$. So $\overline{\mathcal{W}}^0_{-\alpha_i}/\mathbb{G}_a = \mathbb{C}^\times$. $\square$

**Corollary 3.2.2.10.** $(\Phi_i^{-1}(\mathbb{C}^\times) \cap \overline{G[t]t^\lambda G[t]})/\mathbb{G}_a \longrightarrow \mathbb{C}^\times \times \overline{\mathcal{W}}^\lambda_{\mu+\alpha_i}$.

### 3.2.3 An explicit computational example

For $G = SL_n$, the result of Theorem 3.2.2.6 can be done by explicit matrix computations. In this section, we provide some calculations to illustrate Theorem 3.2.2.6 in the case of $G = SL_2$

Let $\mu$ be an arbitrary coweight. An element $g$ of $W_\mu$ is of the form

$$g = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$$

Then, we see that

$$\xi(g) = \begin{pmatrix} 0 & e^{(1)} \\ -(e^{(1)})^{-1} & e^{(1)}(e^{-1} - \underline{e^{-1}}) \end{pmatrix}, \qquad \xi(g)^{-1} = \begin{pmatrix} e^{(1)}(e^{-1} - \underline{e^{-1}}) & -e^{(1)} \\ (e^{(1)})^{-1} & 0 \end{pmatrix}$$

For ease of notation, let us write $e^{(1)} = u$. We would like to compute $\pi_{\mu+\alpha}(\xi(g)^{-1}g)$.

$$\begin{aligned}\begin{pmatrix} u(e^{-1} - \underline{e^{-1}}) & -u \\ u^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} u(e^{-1} - \underline{e^{-1}}) & -u\underline{e^{-1}}e \\ u^{-1} & u^{-1}e \end{pmatrix} \\ &= \begin{pmatrix} 1 & -u^2\underline{e^{-1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ue^{-1} & 0 \\ 0 & u^{-1}e \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-1} & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ e^{-1} & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} &= \begin{pmatrix} h & 0 \\ e^{-1}h & h^{-1} \end{pmatrix} \\ &= \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-1}h^2 & 1 \end{pmatrix}\end{aligned}$$

If $h \in t^n + t^{n-1}\mathbb{C}[[t^{-1}]]$, we see that $ue^{-1}h \in t^{n+1} + t^n\mathbb{C}[[t^{-1}]]$. Hence, $\xi(g)^{-1}g \in U((t^{-1}))T_1[[t^{-1}]]t^{\mu+\alpha}U_-((t^{-1}))$. Therefore,

$$\pi_{\mu+\alpha}(\xi(g)^{-1}g) = \begin{pmatrix} 1 & -u^2\underline{e^{-1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ue^{-1}h & 0 \\ 0 & u^{-1}eh^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \underline{e^{-1}h^2} + f & 1 \end{pmatrix} \tag{3.1}$$

Next, we compute $\pi_\mu(\xi(g)\pi_{\mu+\alpha}(\xi(g)^{-1}g))$.

$$\begin{aligned}\begin{pmatrix} 0 & u \\ -u^{-1} & u(e^{-1} - \underline{e^{-1}}) \end{pmatrix} \begin{pmatrix} 1 & -u^2\underline{e^{-1}} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & u \\ -u^{-1} & ue^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1}e & 0 \\ 0 & ue^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-2}e & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ -u^{-2}e & 1 \end{pmatrix} \begin{pmatrix} ue^{-1}h & 0 \\ 0 & u^{-1}eh^{-1} \end{pmatrix} &= \begin{pmatrix} ue^{-1}h & 0 \\ 0 & u^{-1}eh^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{-1}h^2 & 1 \end{pmatrix}\end{aligned}$$

Thus,

$$\xi(g)\pi_{(n+1)\alpha}(\xi(g)^{-1}g) = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{-1}h^2 + \underline{e^{-1}h^2} + f & 1 \end{pmatrix}$$

Since $-e^{-1}h^2 + \underline{e^{-1}h^2} \in \mathbb{C}[t]$, we see that $\pi_\mu(\xi(g)\pi_{\mu+\alpha}(\xi(g)^{-1}g)) = g$. This shows that $m \circ \psi = \mathrm{Id}$.

Next, we would like to show that $\psi \circ m = \mathrm{Id}$. Consider

$$y = \begin{pmatrix} 0 & u \\ -u^{-1} & t+a \end{pmatrix} \in \mathcal{W}^0_{-\alpha}, \qquad g = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \in \mathcal{W}_{\mu+\alpha}.$$

We have that

$$\begin{pmatrix} 0 & u \\ -u^{-1} & t+a \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & u \\ -u^{-1} & -u^{-1}e + t + a \end{pmatrix}$$

For ease of notation, let us write $p = t + a - u^{-1}e$. Thus,

$$\begin{aligned}\begin{pmatrix}0 & u\\ -u^{-1} & t+a\end{pmatrix}\begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix} &= \begin{pmatrix}0 & u\\ -u^{-1} & p\end{pmatrix}\\ &= \begin{pmatrix}1 & up^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^{-1} & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}1 & 0\\ -u^{-1}p^{-1} & 1\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ -u^{-1}p^{-1} & 1\end{pmatrix}\begin{pmatrix}h & 0\\ 0 & h^{-1}\end{pmatrix} &= \begin{pmatrix}h & 0\\ 0 & h^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -u^{-1}p^{-1}h^2 & 1\end{pmatrix}\end{aligned}$$

If $h \in t^n + t^{n-1}\mathbb{C}[[t^{-1}]]$, we see that $p^{-1}h \in t^{n-1} + t^{n-2}\mathbb{C}[[t^{-1}]]$. Thus,

$$m(y, g) \in U((t^{-1}))T_1[[t^{-1}]]t^{\mu}U_-((t^{-1})).$$

Therefore

$$m(x, g) = \begin{pmatrix}1 & up^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^{-1}h & 0\\ 0 & ph^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\underline{u^{-1}p^{-1}h^2} + f & 1\end{pmatrix} \tag{3.2}$$

Now, we see that $(up^{-1})^{(1)} = u$. Since $u(u^{-1}p - \underline{u^{-1}p}) = p - \underline{p} = t + a$, $\xi(m(y, g)) = y$. We compute $\xi(m(y, g))^{-1}m(y, g)$.

$$\begin{aligned}\begin{pmatrix}p-\underline{p} & -u\\ u^{-1} & 0\end{pmatrix}\begin{pmatrix}1 & up^{-1}\\ 0 & 1\end{pmatrix} &= \begin{pmatrix}p-\underline{p} & -up^{-1}\underline{p}\\ u^{-1} & p^{-1}\end{pmatrix}\\ &= \begin{pmatrix}1 & -u\underline{p}\\ 0 & 1\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & p^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ u^{-1}p & 1\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ u^{-1}p & 1\end{pmatrix}\begin{pmatrix}p^{-1}h & 0\\ 0 & ph^{-1}\end{pmatrix} &= \begin{pmatrix}p^{-1}h & 0\\ 0 & ph^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ u^{-1}p^{-1}h^2 & 1\end{pmatrix}\end{aligned}$$

Since $-u\underline{p} = -u(-u^{-1}e) = e$, we have that

$$\xi(m(y, g))^{-1}m(y, g) = \begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}\begin{pmatrix}h & 0\\ 0 & h^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ u^{-1}p^{-1}h^2 - \underline{u^{-1}p^{-1}h^2} + f & 1\end{pmatrix}.$$

Since $u^{-1}p^{-1}h^2 - \underline{u^{-1}p^{-1}h^2} \in \mathbb{C}[t]$, we see that $\psi(m(y, g)) = (y, g)$, and that $\psi \circ m = \mathrm{Id}$.

Next, we will show that $m$ is $\mathbb{G}_a$-equivariant. Consider

$$y = \begin{pmatrix}0 & u\\ -u^{-1} & t+a\end{pmatrix} \in \mathcal{W}^0_{-\alpha}, \qquad g = \begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}\begin{pmatrix}h & 0\\ 0 & h^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix} \in \mathcal{W}_{\mu}.$$

We need to show that, for $r \in \mathbb{G}_a$, $r \cdot m(y, g) = m(r \cdot (y, g))$. In other words,

$$\pi_{\mu}(x_{\alpha}(r)\pi_{\mu}(yg)) = \pi_{\mu}(\pi_{-\alpha}(x_{\alpha}(r)y)g) = \pi_{\mu}(x_{\alpha}(r)yg),$$

where the second equality is clear since

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ -u^{-1} & t+a \end{pmatrix} = \begin{pmatrix} 0 & u \\ -u^{-1} & t+a+ru \end{pmatrix}.$$

We already have part of the computation done in (3.2). We first compute the left-hand side.

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 1 & up^{-1} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & up^{-1} \\ r & rup^{-1}+1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & u(ru+p)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (rup^{-1}+1)^{-1} & 0 \\ 0 & rup^{-1}+1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(rup^{-1}+1)^{-1} & 1 \end{pmatrix} \end{aligned}$$

Hence, writing $q = rup^{-1}+1$, we obtain

$$x_\alpha(r)\pi_\mu(yg) = \begin{pmatrix} 1 & up^{-1}q^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^{-1}p^{-1}h & 0 \\ 0 & qph^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ rq^{-1}p^{-2}h^2 - \underline{u^{-1}p^{-1}h^2} + f & 1 \end{pmatrix}$$

Additionally,

$$x_\alpha(r)yg = \begin{pmatrix} 1 & up^{-1}q^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^{-1}p^{-1}h & 0 \\ 0 & qph^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ rq^{-1}p^{-2}h^2 - u^{-1}p^{-1}h^2 + f & 1 \end{pmatrix}$$

Since $\underline{f+g} = \underline{f} + \underline{g}$ for all $f, g \in \mathbb{C}((t^{-1}))$, we see that $x_\alpha \pi_\mu(xg)$ and $x_\alpha xg$ have the same image under $\pi_\mu$. Therefore, $m$ is $\mathbb{G}_a$-equivariant.

**An alternate description for $\psi$ in the $\mathfrak{sl}_2$-case**

Write $\mu = m\alpha$. An alternate description for $\mathcal{W}_\mu$ is

$$\{\begin{pmatrix} d & b \\ c & a \end{pmatrix} : a \in t^m + t^{m-1}\mathbb{C}[[t^{-1}]], \mathrm{val}(b), \mathrm{val}(c) < m, ad - bc = 1\}.$$

where $\mathrm{val}(f)$ denotes the largest interger $n$ such that the coefficient of $t^n$ in $f$ is nonzero.

Write $a(t) = t^m + a^{(1)}t^{m-1} + \cdots$, $b(t) = b^{(1)}t^{m-1} + b^{(2)}t^{m-2} + \cdots$, $c(t) = c^{(1)}t^{m-1} + c^{(2)}t^{m-2} + \cdots$. We would like to describe $\psi(g)$ in terms of $a, b, c, d$. The Gauss form in this description is

$$g = \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}.$$

We see that $(b/a)^{(1)} = b^{(1)}$. For $g \in \Phi^{-1}(\mathbb{C}^\times)$, recall that $b^{(1)} \neq 0$.

$$\xi(g) = \begin{pmatrix} 0 & b^{(1)} \\ -(b^{(1)})^{-1} & b^{(1)}((b/a)^{-1} - \underline{(b/a)^{-1}}) \end{pmatrix}, \qquad \xi(g)^{-1} = \begin{pmatrix} b^{(1)}((b/a)^{-1} - \underline{(b/a)^{-1}}) & -b^{(1)} \\ (b^{(1)})^{-1} & 0 \end{pmatrix}$$

Replacing $e$ by $b/a$, $h$ by $a^{-1}$, and $f$ by $c/a$ in (3.1), we obtain

$$\begin{aligned}\pi_{\mu+\alpha}(\xi(g)^{-1}g) &= \begin{pmatrix}1 & -(b^{(1)})^2\underline{a/b} \\ 0 & 1\end{pmatrix}\begin{pmatrix}b^{(1)}b^{-1} & 0 \\ 0 & (b^{(1)})^{-1}b\end{pmatrix}\begin{pmatrix}1 & 0 \\ \underline{(ab)^{-1}}+c/a & 1\end{pmatrix} \\ &= \begin{pmatrix}b^{(1)}b^{-1}-b^{(1)}(\underline{a/b})b(\underline{(ab)^{-1}}+c/a) & -b^{(1)}(\underline{a/b})b \\ (b^{(1)})^{-1}b(\underline{(ab)^{-1}}+c/a) & (b^{(1)})^{-1}b\end{pmatrix}.\end{aligned}$$

# Chapter 4

# Quantum Hamiltonian reduction

In this chapter, we discuss the lifting of the isomorphism $\Phi_i^{-1}(\mathbb{C}^\times) \simeq \overline{\mathcal{W}}^0_{-\alpha_i} \times \mathcal{W}_{\mu+\alpha_i}$ to the Yangian level. This should be expected since the multiplication of the right-hand side is Poisson, and we would anticipate a quantization at the Yangian level. We know that a quantization of the right-hand side is $Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$. So, our first task is to find a quantization of $\mathbb{C}[\Phi_i^{-1}(\mathbb{C}^\times)]$. It corresponds to a localization of $Y_\mu$.

## 4.1 A localization for $Y_\mu$

Recall that, if $S$ is a multiplicative set of a ring $A$, then $S$ is said to satisfy the right Ore condition if $aS \cap sA \neq \emptyset$ for all $a \in A$ and $s \in S$.

**Lemma 4.1.0.1.** [S, Lem. 6.6]. *Let $A$ be a ring. Fix $r \in A$, set $S = \{r^n : n \geq 0\}$. Let $\{x_j : j \in J\}$ be a generating set for $A$. Suppose that, for all $n \geq 0$ and $j \in J$, $x_j S \cap r^n A \neq \emptyset$. Then, for all $a \in A$ and $n \in \mathbb{N}$, $aS \cap r^n A \neq \emptyset$.*

*Proof.* Let us start the proof with the additional assumption that the Ore condition holds for monomials in $x_j$'s. Every element $a \in A$ is of the form $\sum_{k=1}^{l} a_k$ where each $a_k$ is a monomial in $x_j$'s. By our assumption, for $1 \leq k \leq l$ and $n \geq 0$, there exist $m_k \in \mathbb{N}, b_k \in A$ such that $a_k r^{m_k} = r^n b_k$. Let $M = \max\{m_1, \ldots, m_l\}$. We have that $ar^M = r^n \sum_k b_k r^{M-m_k}$.

Thus, we need to justify our assumption by proving that the right Ore condition holds for monomials in $x_j$'s. By the assumption of the lemma, for all $n \geq 0, j \in J$, $x_j S \cap r^n A \neq \emptyset$. A monomial in the $x_j$'s is of the form $x_1 \cdots x_k$. Let $n \geq 0$, there exist $m_1 \geq 0$ and $y_1 \in A$ such that $r^n y_1 = x_1 r^{m_1}$. Now, according to the condition on $x_2$ and $m_1$, there exist $m_2 \geq 0, y_2 \in A$ such that $r^{m_1} y_2 = x_2 r^{m_2}$. Repeating this process until we have obtained $m_1, \ldots, m_k$, and $y_1, \ldots, y_k$ satisfying the conditions $r^{m_j} y_{j+1} = x_{j+1} r^{m_{j+1}}$. We see that

$$x_1 \cdots x_k r^{m_k} = x_1 \cdots x_{k-1} r^{m_{k-1}} y_m = \cdots = r^n y_1 \cdots y_m.$$

Therefore, $x_1 \cdots x_k S \cap r^n A \neq \emptyset$. ☐

Let $S = \{(E_i^{(1)})^k : k \in \mathbb{N}\} \subseteq Y_\mu$. Let us start with proving that a subset of the Yangian generators satisfies the equation of the right Ore condition.

**Lemma 4.1.0.2.** *We have the following formulas.*

*(a) For all* $n \geq 1$, $E_i^{(2)}(E_i^{(1)})^n = (E_i^{(1)})^n(nE_i^{(1)} + E_i^{(2)})$,

*(b) If* $a_{ij} = -1$, *for all* $r \geq 1$ *and* $n \geq 1$, $E_j^{(r)}(E_i^{(1)})^{n+1} = (E_i^{(1)})^n((n+1)E_j^{(r)}E_i^{(1)} - nE_i^{(1)}E_j^{(r)})$,

*(c) If* $a_{ij} = 0$, *then* $E_j^{(r)}(E_i^{(1)})^n = (E_i^{(1)})^n E_j^{(r)}$,

*(d) For all* $j \in I$ *and* $n \geq 1$, $H_j^{(-\langle\mu,\alpha_j\rangle+1)}(E_i^{(1)})^n = (E_i^{(1)})^n(na_{ij} + H_j^{(-\langle\mu,\alpha_j\rangle+1)})$,

*(e) For all* $j \neq i$ *and* $r \geq 1$, $F_j(E_i^{(1)})^n = (E_i^{(1)})^n F_j^{(r)}$,

*(f) For all* $n \geq 1$ *and* $r \geq 1$, $F_i^{(r)}(E_i^{(1)})^n = (E_i^{(1)})^n F_i^{(r)} - \sum_{k=0}^{n-1}(E_i^{(1)})^k H_i^{(r)}(E_i^{(1)})^{n-1-k}$.

*Proof.* (a) When $n = 1$, $[E_i^{(2)}, E_i^{(1)}] - [E_i^{(1)}, E_i^{(2)}] = 2(E_i^{(1)})^2$. Thus, rearranging the equation, we obtain $E_i^{(2)}E_i^{(1)} = E_i^{(1)}(E_i^{(2)} + E_i^{(1)})$. Assuming the result for some $n \geq 1$, we want to show the case $n+1$.

$$\begin{aligned}
E_i^{(2)}(E_i^{(1)})^{n+1} &= E_i^{(2)}(E_i^{(1)})^n E_i^{(1)} = (E_i^{(1)})^n(nE_i^{(1)} + E_i^{(2)})E_i^{(1)} \\
&= (E_i^{(1)})^n(n(E_i^{(1)})^2 + E_i^{(2)}E_i^{(1)}) \\
&= (E_i^{(1)})^n((n+1)(E_i^{(1)})^2 + E_i^{(1)}E_i^{(2)}) \\
&= (E_i^{(1)})^{n+1}((n+1)E_i^{(1)} + E_i^{(2)}).
\end{aligned}$$

(b) We fix $r$ and proceed by induction on $n$. When $n = 1$, using the Serre's relation,

$$\begin{aligned}
0 = [E_i^{(1)}, [E_i^{(1)}, E_j^{(r)}]] &= E_i^{(1)}[E_i^{(1)}, E_j^{(r)}] - [E_i^{(1)}, E_j^{(r)}]E_i^{(1)} \\
&= (E_i^{(1)})^2 E_j^{(r)} - 2E_i^{(1)}E_j^{(r)}E_i^{(1)} + E_j^{(r)}(E_i^{(1)})^2.
\end{aligned}$$

Thus, we have that

$$E_j^{(r)}(E_i^{(1)})^2 = E_i^{(1)}(2E_j^{(r)}E_i^{(1)} - E_i^{(1)}E_j^{(r)}),$$

proving the base case. We proceed onto the induction step.

$$\begin{aligned}
E_j^{(r)}(E_i^{(1)})^{n+2} &= E_j^{(r)}(E_i^{(1)})^{n+1}E_j^{(1)} = (E_i^{(1)})^n((n+1)E_j^{(r)}E_i^{(1)} - nE_i^{(1)}E_j^{(r)})E_i^{(1)} \\
&= (E_i^{(1)})^n((n+1)E_j^{(r)}(E_i^{(1)})^2 - nE_i^{(1)}E_j^{(r)}E_i^{(1)}) \\
&= (E_i^{(1)})^n(2(n+1)E_i^{(1)}E_j^{(r)}E_i^{(1)} - (n+1)(E_i^{(1)})^2E_j^{(r)} - nE_i^{(1)}E_j^{(r)}E_i^{(1)}) \\
&= (E_i^{(1)})^n((n+2)E_j^{(r)}E_i^{(1)} - (n+1)E_i^{(1)}E_j^{(r)}) \\
&= (E_i^{(1)})^{n+1}((n+2)E_i^{(1)}E_j^{(r)}E_i^{(1)} - (n+1)(E_i^{(1)})^2E_j^{(r)}).
\end{aligned}$$

(c) This is clear since $[E_j^{(r)}, E_i^{(1)}] = 0$.

(d) We also proceed by induction on $n$. When $n = 1$. $[H_j^{(-\langle\mu,\alpha_j\rangle+1)}, E_i^{(1)}] = a_{ij}E_i^{(1)}$.

Rearranging the equation, we obtain $H_j^{(-\langle\mu,\alpha_j\rangle+1)}E_i^{(1)} = E_i^{(1)}(a_{ij} + H_j^{(-\langle\mu,\alpha_j\rangle+1)})$.

$$\begin{aligned} H_j^{(-\langle\mu,\alpha_j\rangle+1)}(E_i^{(1)})^{n+1} &= (E_i^{(1)})^n(na_{ij} + H_j^{(-\langle\mu,\alpha_j\rangle+1)})E_i^{(1)} \\ &= (E_i^{(1)})^n(na_{ij}E_i^{(1)} + a_{ij}E_i^{(1)} + E_i^{(1)}H_j^{(-\langle\mu,\alpha_j\rangle+1)}) \\ &= (E_i^{(1)})^{n+1}((n+1)a_{ij} + H_j^{(-\langle\mu,\alpha_j\rangle+1)}). \end{aligned}$$

(e) This part is clear since $F_j^{(r)}$ commutes with $E_i^{(1)}$.

(f) Let $r$ be arbitrary. We proceed by induction on $n$. Since $[E_i^{(1)}, F_i^{(r)}] = H_i^{(r)}$ the base case $n = 1$ is clear. Assuming the result for $n$, consider the case $n+1$.

$$\begin{aligned} F_i^{(r)}(E_i^{(1)})^{n+1} &= \Big((E_i^{(1)})^nF_i^{(r)} - \sum_{k=0}^{n-1}(E_i^{(1)})^kH_i^{(r)}(E_i^{(1)})^{n-1-k}\Big)E_i^{(1)} \\ &= (E_i^{(1)})^{n+1}F_i^{(r)} - (E_i^{(1)})^nH_i^{(r)} - \sum_{k=0}^{n-1}(E_i^{(1)})^kH_i^{(r)}(E_i^{(1)})^{n-k} \\ &= (E_i^{(1)})^{n+1}F_i^{(r)} - \sum_{k=0}^{n}(E_i^{(1)})^kH_i^{(r)}(E_i^{(1)})^{n-k}. \end{aligned}$$

□

*Remark* 4.1.0.3. According to the previous lemma, by proving some simple formulas, we see that $E_i^{(1)}, E_i^{(2)}, E_j^{(r)}(j \neq i, r \geq 1), H_j^{(-\langle\mu,\alpha_j\rangle+1)}, F_j^{(r)}(j \neq i, r \geq 1)$ satisfy the equation of the right Ore condition with respect to $S$. Moreover, by Lemma 4.1.0.1, the elements of the subalgebra generated by these generators also satisfy the right Ore condition with respect to $S$. Using this, we will prove the Ore condition for the other generators.

**Lemma 4.1.0.4.** *We have the following*

*(a) For $r \geq 2$, there exists $x_r = x'_r + E_i^{(r)}(E_i^{(1)})^{r-3}$ where $x'_r \in \langle E_i^{(k)} : 1 \leq k \leq r-1\rangle$ such that*

$$E_i^{(r)}(E_i^{(1)})^{r-2} = E_i^{(1)}x_r. \tag{4.1}$$

*(b) For $r \geq 1$, there exists $y_r = y'_r + H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^{r-1}$ where $y'_r \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle\mu,\alpha_j\rangle+l)} : 1 \leq l \leq r-1\rangle$ such that*

$$H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^r = E_i^{(1)}y_r. \tag{4.2}$$

*Proof.* (a) We prove this claim by induction on $r$. The base case $r = 2$ is clearly true. Assume the claim holds for some $r \geq 2$. For the case $r+1$, consider the relation $[E_i^{(r+1)}, E_i^{(1)}] - [E_i^{(r)}, E_i^{(2)}] = E_i^{(r)}E_i^{(1)} + E_i^{(1)}E_i^{(r)}$. Rearranging the equation, we obtain

$$E_i^{(r+1)}E_i^{(1)} = (E_i^{(r)}E_i^{(1)} + E_i^{(1)}E_i^{(r)}) + E_i^{(1)}E_i^{(r+1)} + E_i^{(r)}E_i^{(2)} - E_i^{(2)}E_i^{(r)}.$$

Multiplying both sides of the above equation by $(E_i^{(1)})^{r-2}$ on the right gives

$$\begin{aligned}E_i^{(r+1)}(E_i^{(1)})^{r-1} &= E_i^{(r)}(E_i^{(1)})^{r-1} + E_i^{(1)}E_i^{(r)}(E_i^{(1)})^{r-2} + E_i^{(1)}E_i^{(r+1)}(E_i^{(1)})^{r-2} \\ &\quad + E_i^{(r)}E_i^{(2)}(E_i^{(1)})^{r-2} - E_i^{(2)}E_i^{(r)}(E_i^{(1)})^{r-2} \\ &= E_i^{(1)}x_rE_i^{(1)} + (E_i^{(1)})^2x_r + E_i^{(1)}E_i^{(r+1)}(E_i^{(1)})^{r-2} \\ &\quad + E_i^{(r)}((r-2)(E_i^{(1)})^{r-1} + (E_i^{(1)})^{r-2}E_i^{(2)}) - E_i^{(1)}(E_i^{(1)} + E_i^{(2)})x_r \\ &= E_i^{(1)}x_rE_i^{(1)} + (E_i^{(1)})^2x_r + E_i^{(1)}E_i^{(r+1)}(E_i^{(1)})^{r-2} \\ &\quad + (r-2)E_i^{(1)}x_rE_i^{(1)} + E_i^{(1)}x_rE_i^{(2)} - (E_i^{(1)})^2x_r - E_i^{(1)}E_i^{(2)}x_r \\ &= (r+1)E_i^{(1)}x_rE_i^{(1)} + E_i^{(1)}E_i^{(r+1)}(E_i^{(1)})^{r-2} + E_i^{(1)}[x_r, E_i^{(2)}].\end{aligned}$$

Set $x'_{r+1} = (r+1)x_rE_i^{(1)} + [x_r, E_i^{(2)}]$. We see that $y_{r+1} \in \langle E_i^{(k)} : 1 \leq k \leq r\rangle$, proving (a).

(b) We proceed by induction on $r$. When $r = 1$, the result follows from part (d) of Lemma 4.1.0.2. Assuming that the result holds for some $r \geq 1$. For the $r+1$ case, consider the relation

$$[H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}, E_i^{(1)}] - [H_j^{(-\langle\mu,\alpha_j\rangle+r)}, E_i^{(2)}] = \frac{a_{ij}}{2}(H_j^{(-\langle\mu,\alpha_j\rangle+r)}E_i^{(1)} + E_i^{(1)}H_j^{(-\langle\mu,\alpha_j\rangle+r)})$$

Rearranging the equation and multiply both sides on the right by $(E_i^{(1)})^r$, we obtain

$$\begin{aligned}H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}(E_i^{(1)})^{r+1} &= E_i^{(1)}H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}(E_i^{(1)})^r + [H_j^{(-\langle\mu,\alpha_j\rangle+r)}, E_i^{(2)}](E_i^{(1)})^r \\ &\quad + \frac{a_{ji}}{2}(H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^{r+1} + E_i^{(1)}H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^r) \\ &= E_i^{(1)}H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}(E_i^{(1)})^r + [H_j^{(-\langle\mu,\alpha_j\rangle+r)}, E_i^{(2)}](E_i^{(1)})^r \\ &\quad + \frac{a_{ji}}{2}(E_i^{(1)}y_rE_i^{(1)} + (E_i^{(1)})^2y_r)\end{aligned}$$

Now, we have that

$$\begin{aligned}H_j^{(-\langle\mu,\alpha_j\rangle+r)}E_i^{(2)}(E_i^{(1)})^r &= H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^r(rE_i^{(1)} + E_i^{(2)}) \\ &= E_i^{(1)}y_r(rE_i^{(1)} + E_i^{(2)}) \\ E_i^{(2)}H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^r &= E_i^{(2)}E_i^{(1)}y_r = E_i^{(1)}(E_i^{(1)} + E_i^{(2)})y_r.\end{aligned}$$

Setting $y'_{r+1} = \frac{a_{ji}}{2}(x_rE_i^{(1)} + E_i^{(1)}x_r) + y_r(rE_i^{(1)} + E_i^{(2)}) - (E_i^{(1)} + E_i^{(2)})y_r$. Since $y_{r+1} \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle\mu,\alpha_j\rangle+l)} : 1 \leq l \leq r\rangle$, we are done. □

**Theorem 4.1.0.5.** *The set $S = \{(E_i^{(1)})^n : n \geq 0\} \subseteq Y_\mu$ satisfies the right Ore condition.*

*Proof.* By Lemma 4.1.0.1, it remains to show the right Ore condition for the remaining generators $E_i^{(r)} (r \geq 3), H_j^{(-\langle\mu,\alpha_j\rangle+s)} (j \in I, s \geq 2), F_i^{(t)} (t \geq 1)$, i.e., for a generator $x$, for all $n \geq 0$, there exist $m \geq 0$ and $y \in Y_\mu$ such that

$$x(E_i^{(1)})^m = (E_i^{(1)})^n y. \tag{4.3}$$

For each $n \geq 1$, we prove the existence of equation (4.3) for $E_i^{(r)}$, $r \geq 2$. We prove this by induction on $r$ and on $n$. The strategy to prove our statement is to prove for a fixed $r$ and all $n$, before moving onto $r+1$ and all $n$.

More precisely, we claim that, for $r \geq 2$, and for $n \geq 0$, there exist $a_n \in \langle E_i^{(l)} : 1 \leq l \leq r-1 \rangle$, $m_n, k_n \geq 0$ such that

$$E_i^{(r)}(E_i^{(1)})^{m_n} = (E_i^{(1)})^n(a_n + E_i^{(r)}(E_i^{(1)})^{k_n}). \tag{4.4}$$

Consider the base case $r = 2$. For all $n \geq 0$, the existence of equation (4.4) follows from part (a) of Lemma 4.1.0.2. Assume our new claim holds for numbers between 1 and r and all $n \geq 0$. Consider the case $r+1$. We prove existence of equation (4.4) by induction on $n$. The base case $n = 0$ is clear. Assume the existence of (4.4) for $r+1$ and some $n \geq 1$, i.e.,

$$E_i^{(r+1)}(E_i^{(1)})^{m_n} = (E_i^{(1)})^n(a_n + E_i^{(r+1)}(E_i^{(1)})^{k_n}), \tag{4.5}$$

for some $a_n \in \langle E_i^{(l)} : 1 \leq l \leq r \rangle$. Consider the case $n+1$.

Now, $a_n \in \langle E_i^{(l)} : 1 \leq l \leq r \rangle$ and $E_i^{(l)}(1 \leq l \leq r)$ satisfy the right Ore condition by induction hypothesis. By Lemma 4.1.0.1 applied to the subalgebra $\langle E_i^{(l)} : 1 \leq l \leq r \rangle$, $a_n$ satisfies the right Ore condition equation, i.e., equation (4.3) holds for $x = a_n$ and all natural numbers. Thus, there exist $p \geq 0$ and $a'_n \in \langle E_i^{(l)} : 1 \leq l \leq r \rangle$ such that $a_n(E_i^{(1)})^p = E_i^{(1)} a'_n$. Set $M = \max\{p, r-2\}$.

Multiply both sides of equation (4.5) on the right by $(E_i^{(1)})^M$, we obtain

$$\begin{aligned} E_i^{(r+1)}(E_i^{(1)})^{m_n+M} &= (E_i^{(1)})^n(a_n(E_i^{(1)})^p(E_i^{(1)})^{M-p} + E_i^{(r+1)}(E_i^{(1)})^{r-1}(E_i^{(1)})^{M-(r-1)+k_n}) \\ &= (E_i^{(1)})^{n+1}(a'_n(E_i^{(1)})^{M-p} + (x'_{r+1} + E_i^{(r+1)}(E_i^{(1)})^{r-2})(E_i^{(1)})^{M-(r-1)+k_n}), \end{aligned}$$

where the second equality uses part (a) of Lemma 4.1.0.4.

Set $a_{n+1} = a'_n(E_i^{(1)})^{M-p} + x'_{r+1}(E_i^{(1)})^{M-(r-1)+k_n}$. Since $x'_{r+1} \in \langle E_i^{(l)} : 1 \leq l \leq r \rangle$, we are done.

Next, we work with the $H$'s in the exact same manner. We claim that, for $r \geq 1$ and $n \geq 0$, there exists $a_n \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle \mu, \alpha_j \rangle + l)} : 1 \leq l \leq r-1 \rangle$, $m_n, k_n \geq 0$ such that

$$H_j^{(-\langle \mu, \alpha_j \rangle + r)}(E_i^{(1)})^{m_n} = (E_i^{(1)})^n(a_n + H_j^{(-\langle \mu, \alpha_j \rangle + r)}(E_i^{(1)})^{k_n}). \tag{4.6}$$

Consider the case $r = 1$. For all $n \geq 0$, the result follows from part (d) of Lemma 4.1.0.2. Assume the result holds for numbers between 1 and $r$. Consider the case $r+1$, we prove the existence of equation (4.6) by induction on $n$. The case $n = 0$ is clear. For our induction hypothesis, for some $n \geq 0$,

$$H_j^{(-\langle \mu, \alpha_j \rangle + r+1)}(E_i^{(1)})^{m_n} = (E_i^{(1)})^n(a_n + H_j^{(-\langle \mu, \alpha_j \rangle + r+1)}(E_i^{(1)})^{k_n}), \tag{4.7}$$

where $a_n \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle \mu, \alpha_j \rangle + l)} : 1 \leq l \leq r \rangle$.

By induction hypothesis, the set $S = \{(E_i^{(1)})^n : n \geq 0\}$ satisfies the right Ore condition in the subalgebra $a_n \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle \mu, \alpha_j \rangle + l)} : 1 \leq l \leq r \rangle$. So, equation 4.3 holds for $a_n$ and all natural numbers. Thus, there exists $p \geq 0$ and $a'_n \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle \mu, \alpha_j \rangle + l)} : 1 \leq l \leq r \rangle$ such that $a_n(E_i^{(1)})^p = E_i^{(1)} a'_n$.

Set $M = \max\{p, r+1\}$. Multiply both sides of (4.7) on the right by $(E_i^{(1)})^M$, we obtain

$$\begin{aligned} &H_j^{(-\langle\mu,\alpha_j\rangle+r)}(E_i^{(1)})^{m_n+M} = (E_i^{(1)})^n(a_n(E_i^{(1)})^{p+M-p} + H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}(E_i^{(1)})^{r+1+M-r-1+k_n}) \\ &= (E_i^{(1)})^{n+1}(a'_n(E_i^{(1)})^{M-p} + (y'_{r+1} + H_j^{(-\langle\mu,\alpha_j\rangle+r+1)}(E_i^{(1)})^r)(E_i^{(1)})^{M-r-1+k_n}) \end{aligned}$$

where the second equality uses part (b) of Lemma 4.1.0.4.

Set $a_{n+1} = a'_n(E_i^{(1)})^{M-p} + y'_{r+1}(E_i^{(1)})^{M-r-1+k_n}$. Since $y'_{r+1} \in \langle E_i^{(1)}, E_i^{(2)}, H_j^{(-\langle\mu,\alpha_j\rangle+l)} : 1 \le l \le r\rangle$, we are done.

Next, let us work on $F_i^{(r)}$. By part (f) of Lemma 4.1.0.2, for $n \ge 1$,

$$F_i^{(r)}(E)_i^{(1)})^n = (E_i^{(1)})^n F_i^{(r)} - \sum_{k=0}^{n-1}(E_i^{(1)})^k H_i^{(r)}(E_i^{(1)})^{n-1-k}. \tag{4.8}$$

Now, by what we have shown for the $H_j$'s, there exists $p$ and $z$ such that $H_i^{(r)}(E_i^{(1)})^p = (E_i^{(1)})^n z$. Using this property, we get the desired result by mutiplying both sides of (4.8) on the right by $(E_i^{(1)})^p$. □

Therefore, it makes sense to talk about $Y_\mu[(E_i^{(1)})^{-1}]$. Given any splitting $\mu = \nu_1 + \nu_2$, we have a filtration $F_{\nu_1,\nu_2}$ on $Y_\mu$. Now, following [S, 12.3], we can put a filtration on $Y_\mu[(E_i^{(1)})^{-1}]$ as follows. Since $Y_\mu$ is a domain (by PBW theorem), given $x \in Y_\mu, s \in S = \{(E_i^{(1)})^n : n \in \mathbb{N}\}$, we specify the degree $\deg(xs) = \deg(x) - \deg(s)$.

**Proposition 4.1.0.6.** $\operatorname{gr} Y_\mu[(E_i^{(1)})^{-1}] \simeq \mathbb{C}[\Phi_i^{-1}(\mathbb{C}^\times)]$.

*Proof.* This is a special case of a general statement on localization of filtered rings (see [LR, II,3.2], [S, Prop. 12.5]). □

Recall from the introduction that the algebra $Y^0_{-\alpha_i}$ is generated by elements $A_i^{(1)}, (E_i^{(1)})^{\pm 1}$ with the relation $[E_i^{(1)}, A_i^{(1)}] = E_i^{(1)}$.

**Proposition 4.1.0.7.** *There exists a map* $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1}] \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$.

*Proof.* Consider $\Delta : Y_\mu \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$. We see that $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1$. Since $E_i^{(1)}$ is invertible in $Y^0_{-\alpha_i}$, $\tilde{\Delta}$ exists by universal property of localization. □

## 4.2 Lifting the isomorphism

We discuss our attempt at lifting the isomorphism of Theorem 3.2.2.6 to the Yangian level. The crux of our approach involves filtrations of $Y_\mu$.

Recall that, for coweights $\nu_1, \nu_2$ such that $\mu = \nu_1 + \nu_2$, there exists a filtration $F_{\nu_1,\nu_2}Y_\mu$

$$\deg E_\alpha^{(q)} = \langle\nu_1, \alpha\rangle + q, \ \deg F_\beta^{(q)} = \langle\nu_2, \beta\rangle + q, \ \deg H_i^{(p)} = \langle\mu, \alpha_i\rangle + p.$$

**Lemma 4.2.0.1.** *Consider the filtrations* $F_{\nu,\mu-\nu}Y_\mu, F_{\nu,-\alpha_i-\nu}Y^0_{-\alpha_i}, F_{\alpha_i+\nu,\mu-\nu}Y_{\mu+\alpha_i}$. *Then* $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1}] \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ *respect these filtrations*

*Proof.* This follows from Proposition 2.4.0.3. □

We would like to use the following lemma.

**Lemma 4.2.0.2.** *Let $\phi : A \longrightarrow B$ be a map of $\mathbb{Z}$-filtered algebras with increasing filtrations. Assume that all involved filtrations are exhaustive, i.e., $A = \bigcup_n A$ and $B = \bigcup_n B_n$. Additionally, assume that the filtration on $A$ is separated, i.e., $\bigcap_n A_n = \{0\}$. Denote by $\operatorname{gr}\phi : \operatorname{gr} A \longrightarrow \operatorname{gr} B$ the induced map on the associated graded level.*

*(1) If $\operatorname{gr}\phi$ is injective, so is $\phi$.*

*(2) Suppose that $A_n = \{0\}$ for all $n < 0$. If $\operatorname{gr}\phi$ is surjective, so is $\phi$.*

*Proof.* (1) Assume that $\operatorname{gr}\phi$ is surjective. Suppose that $\phi(a) = 0$. Assume that $a \neq 0$. Since $\bigcap_n A_n = \{0\}$, there exists $d$ such that $a \in A_d$, and $a \notin A_{d-1}$. For $\bar{a} \in A_d/A_{d-1}$, since $\phi(a) = 0$, $\operatorname{gr}\phi(\bar{a}) = \overline{\phi(a)} = 0$. Since $\operatorname{gr}\phi$ is injective, $\bar{a} = 0$. This means that $a \in A_{d-1}$, a contradiction. Hence, $a = 0$.

(2) Assume that $A_n = \{0\}$ for all $n < 0$. We prove by induction on $d$ that $\phi : A_d \longrightarrow B_d$ is surjective. Suppose that $b \in B_0$. Since $\operatorname{gr}\phi$ is surjective, there exists $x \in \operatorname{gr}(A)$ such that $(\operatorname{gr}\phi)(x) = \bar{b}$. Since $\operatorname{gr}\phi$ is a map of graded spaces and $\bar{b} \in B_0$, $x$ lies in the graded piece of degree 0, which is $A_0$.

Suppose the result holds for all $b \in B_d$. Suppose that $b \in B_{d+1}$. There exists $a + A_d \in A_{d+1}/A_d$ such that $\overline{\phi(a)} = \bar{b}$. Thus, $\phi(a) - b \in B_d$, i.e., $\phi(a) - b = b_d$ for some $b_d \in B_d$. By induction hypothesis, there exists $a_d \in A_d$ such that $b_d = \phi(a_d)$. Therefore, $b = \phi(a - a_d)$. □

*Remark* 4.2.0.3. By Theorem 3.2.2.6, the previous lemma shows that $\tilde{\Delta}$ is injective.

The obstacle for surjectivity of $\tilde{\Delta}$ is that filtrations for $Y_\mu$ are not bounded below in general. However, if one can find $\nu$ (in the context of Lemma 4.2.0.1) such that all of the involved filtrations are non-negative, then we can lift the isomorphism of Theorem 3.2.2.6 to the Yangian level. This leads us to the next result.

**Lemma 4.2.0.4.** *Suppose that there exists a coweight $\nu$ such that*

*(i) $\langle \nu + \omega_i, \alpha_i \rangle = 0$,*

*(ii) for all positive roots $\beta$, $\langle \nu, \beta \rangle \geq -1$,*

*(iii) for all positive roots $\beta$, $\langle \mu - \nu, \beta \rangle \geq -1$,*

*Then, the filtrations $F_{\nu,\mu-\nu}Y_\mu, F_{\nu,-\alpha_i-\nu}Y^0_{-\alpha_i}, F_{\alpha_i+\nu,\mu-\nu}Y_{\mu+\alpha_i}$ are non-negative. Moreover, $E_i^{(1)}$ has filtered degree zero.*

*Proof.* We inspect the degrees of $H_j^{(r)}, E_\beta^{(r)}, F_\beta^{(r)}$ in these algebras. In the case of $F_{\nu,\mu-\nu}Y_\mu$,

$$\begin{aligned}
\deg(H_j^{(r)}) &= \langle \mu, \alpha_j \rangle + r \geq \langle \mu, \alpha_j \rangle - \langle \mu, \alpha_j \rangle + 1 \geq 1,\\
\deg(E_\beta^{(r)}) &= \langle \nu, \beta \rangle + r \geq r - 1 \geq 0,\\
\deg(F_\beta^{(r)}) &= \langle \mu - \nu, \beta \rangle + r \geq r - 1 \geq 0.
\end{aligned}$$

In the case of $Y^0_{-\alpha_i}$, since $A_i^{(1)}, (E_i^{(1)})^{\pm 1}$ generate the algebra, it is enough to look at the case $\beta = \alpha_i$.

$$\deg(E_i^{(1)}) = \langle \nu, \alpha_i \rangle + 1 = \langle -\omega_i + \gamma, \alpha_i \rangle + 1 = -1 + 1 = 0,$$
$$\deg(A_i^{(1)}) = \deg(H_i^{(3)}) = \langle -\alpha_i, \alpha_i \rangle + 3 = 1.$$

In the case of $Y_{\mu+\alpha_i}$, we have that

$$\deg(H_j^{(r)}) = \langle \mu + \alpha_i, \alpha_j \rangle + r \geq \langle \mu + \alpha_i, \alpha_j \rangle - \langle \mu + \alpha_i, \alpha_j \rangle + 1 \geq 1,$$
$$\deg(F_\beta^{(r)}) = \langle \mu - \nu, \beta \rangle + r \geq r - 1 \geq 0,$$
$$\deg(E_\beta^{(r)}) = \langle \alpha_i + \nu, \beta \rangle + r = \langle s_i(\nu), \beta \rangle + r = \langle \nu, s_i(\beta) \rangle + r.$$

If $\beta = \alpha_i$, then $\langle \nu, s_i(\beta) \rangle = -1$. If $\beta \neq \alpha_i$, then $s_i(\beta)$ is a positive root not equal to $\alpha_i$, and so $\langle \nu, s_i(\beta) \rangle \geq -1$. Thus, $\deg(E_\beta^{(r)}) \geq 0$. □

*Remark* 4.2.0.5. Suppose that $\mu$ is dominant, we see that $\nu = -\omega_i$ satisfies the conditions of the previous lemma. So, corresponding filtrations are non-negative.

**Proposition 4.2.0.6.** *If the conditions of Lemma 4.2.0.4 hold, then* $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1}] \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ *is an isomorphism.*

Now, we can push the argument a little further with the following lemma.

**Lemma 4.2.0.7.** *For a dominant coweight* $\eta$*, the following diagram is commutative.*

$$\begin{array}{ccc} Y_{\mu+\eta} & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\eta+\alpha_i} \\ \downarrow{\scriptstyle \iota_{\mu+\eta,0,-\eta}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \iota_{\mu+\eta+\alpha_i,0,-\eta}} \\ Y_\mu & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i} \end{array}$$

*Proof.* This is a consequence of Theorem 2.3.3.1 □

One can always choose a sufficiently dominant $\eta$ such that the conditions of Lemma 4.2.0.4 hold for $\mu + \eta$ and some $\nu$. Thus, we have the following corollary.

**Corollary 4.2.0.8.** *The image of* $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1}] \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ *contains the subalgebra*

$$Y^0_{-\alpha_i} \otimes \langle E_\beta^{(r)}, H_j^{(s)}, F_\beta^{(k)} : r \geq 1, s \geq -\langle \mu + \alpha_i, \alpha_j \rangle, s \geq N_\beta \rangle,$$

*where* $N_\beta$ *is a sufficiently large positive integer.*

*Proof.* Let us choose $\eta$ such that $\mu + \eta$ is dominant. By Remark 4.2.0.5, with $\nu = -\omega_i$, the filtrations

$$F_{\nu,\mu+\eta-\nu} Y_\mu, F_{\nu,-\alpha_i-\nu} Y^0_{-\alpha_i}, F_{\alpha_i+\nu,\mu+\eta-\nu} Y_{\mu+\eta+\alpha_i}.$$

are non-negative. Consider the commutative diagram

$$\begin{array}{ccc} Y_{\mu+\eta} & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\eta+\alpha_i} \\ \downarrow{\scriptstyle \iota_{\mu+\eta,0,-\eta}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \iota_{\mu+\eta+\alpha_i,0,-\eta}} \\ Y_{\mu} & \longrightarrow & Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i} \end{array}$$

By Proposition 4.2.0.6, the top arrow is an isomorphism. The result follows from inspecting the image of the map $\mathrm{Id} \otimes \iota_{\mu+\eta+\alpha_i,0,-\eta}$. □

**Proposition 4.2.0.9.** *Let $\mathfrak{g} = \mathfrak{sl}_2$. For any coweight $\mu$, $\tilde{\Delta} : Y_\mu[(E_i^{(1)})^{-1}] \longrightarrow Y^0_{-\alpha} \otimes Y_{\mu+\alpha}$ is an isomorphism.*

*Proof.* If $\mu \geq -\alpha$, then taking $\nu = -\omega$, we see that the conditions of Lemma 4.2.0.4 are satisfied, and we are done. Consider the case where $\mu < -\alpha$. Set $\eta = -\alpha - \mu$. Consider the commutative diagram

$$\begin{array}{ccc} Y_{-\alpha} & \longrightarrow & Y^0_{-\alpha} \otimes Y_0 \\ \downarrow{\scriptstyle \iota_{-\alpha,0,-\eta}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \iota_{0,0,-\eta}} \\ Y_{\mu} & \longrightarrow & Y^0_{-\alpha} \otimes Y_{\mu+\alpha} \end{array}$$

Since $-\alpha$ is antidominant, we have an explicit description of $Y_{-\alpha} \longrightarrow Y^0_{-\alpha} \otimes Y_0$ on Levendorskii generators. For $1 \leq r \leq -\langle \mu+\alpha, \alpha \rangle$, $\Delta(F^{(r)}) = 1 \otimes F^{(r)}$. Since $\iota_{0,0,-\alpha-\mu}(F^{(s)}) = F^{(s+\langle \mu+\alpha,\alpha \rangle)}$ for $s \geq 1$. So, $1 \otimes F^{(r)}$ lies in the image of $\tilde{\Delta} : Y_\mu \longrightarrow Y^0_{-\alpha} \otimes Y_{\mu+\alpha}$ for all $r \geq 1$. □

**Conjecture 4.2.0.10.** *For all $\mathfrak{g}$ and for any coweight $\mu$ of $\mathfrak{g}$, $\tilde{\Delta} : Y_\mu \longrightarrow Y^0_{-\alpha_i} \otimes Y_{\mu+\alpha_i}$ is an isomorphism.*

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